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Abstract

Quantum dissipation is studied for a discrete system that linearly interacts with a reservoir of harmonic oscillators at thermal equilibrium. Initial correlations between system and reservoir are assumed to be absent. The dissipative dynamics as determined by the unitary evolution of system and reservoir is described by a Kraus map consisting of an infinite number of matrices. For all Laplace-transformed Kraus matrices exact solutions are constructed in terms of continued fractions that depend on the pair correlation functions of the reservoir. By performing factorizations in the Kraus map a perturbation theory is set up that conserves in arbitrary perturbative order both positivity and probability of the density matrix. The latter is determined by an integral equation for a bitemporal matrix and a finite hierarchy for Kraus matrices. In the lowest perturbative order this hierarchy reduces to one equation for one Kraus matrix. Its solution is given by a continued fraction of a much simpler structure as compared to the non-perturbative case. In the lowest perturbative order our non-Markovian evolution equations are applied to the damped Jaynes–Cummings model. From the solution for the atomic density matrix it is found that the atom may remain in the state of maximum entropy for a significant time span that depends on the initial energy of the radiation field.

Keywords: quantum dissipation, non-Markovian dynamics, Kraus maps

(Some figures may appear in colour only in the online journal)

1. Introduction

The Wigner–Weisskopf theory of natural linewidth [1] is one of the earliest descriptions of a quantum system that exchanges energy with an electromagnetic radiation field. The role of

the open system is played by a two-level atom which spontaneously emits a photon by making a transition from the excited state $|2\rangle$ of energy ω_2 to the ground state $|1\rangle$ of energy ω_1. The atomic decay is modeled with the help of an exponential function. Furthermore, in the interaction Hamiltonian between atom and electromagnetic field counter-rotating terms are discarded. Owing to these simplifications, the lifetime of the excited atomic state can be readily expressed in terms of the coupling constant $g(\omega)$ between atom and field mode of frequency ω. A mathematically rigorous treatment of the Wigner–Weisskopf atom is furnished in [2].

For times smaller than $|g(\omega_2 - \omega_1)|^{-2}$ the behaviour of the two-level atom is substantially influenced by transient effects. Consequently, the atomic evolution in time t cannot be described on the basis of exponential functions. Hence, the atomic density matrix $\rho_a(t)$ cannot be written as a Markovian map $\exp(Lt)\rho_a(0)$, with a generator L independent of time. Indeed, still dropping counter-rotating terms one determines the lower diagonal element of the atomic density matrix for arbitrary times as

$$
\langle 2|\rho_a(t)|2\rangle = \left| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp\left(-i\omega t\right) \left[\omega - \int_0^{\infty} d\omega' \frac{|g(\omega')|^2}{\omega + \omega_2 - \omega_1 - \omega' + i0} \right]^{-1} \right|^2,
\tag{1}
$$

where the purely imaginary number i0 lies above and infinitesimally close to the real axis. The standard result (1), which is valid at zero temperature of the radiation field, has been known long since and has been featured in textbooks [3, 4].

Despite its advanced age extensions of (1) are scarce. In [5] radiative decay of a multi-level atom was studied. However, the assumption of weak coupling between atom and electromagnetic field was made. Then the dissipative dynamics can be generated from a Markovian map. If one stays outside the weak-coupling limit, a dynamical map with constant generator L does not exist. For this regime extensions of (1) to such relevant cases as an electromagnetic continuum of finite temperature or a decaying multi-level atom have not been published in the literature. The present work aims at filling up these gaps in our knowledge of quantum dynamics.

In realizing our goal, previous findings might give us a clue as to what shape extensions of (1) should possess. More specifically, in [6] dissipative quantum evolution was studied in a separable Hilbert space that was coupled to a reservoir of zero temperature through an interaction free of counter-rotating terms. For this simplified setting continued fractions made their appearance in the density matrix of the open system. Moreover, the solution for the density matrix was found to reproduce (1) upon choosing a Hilbert space of dimension two. We thus may surmise that continued fractions will constitute an effective tool in expanding on the fundamental formula (1). Previous proposals for studying quantum dissipation outside the weak-coupling limit include employment of the Feshbach projection method [7] and use of one-dimensional projection operators [8]. The derivation of exact results outside the weak-coupling limit is a problem of long standing [9, 10] and has turned out to be a very hard task [11].

Inspired by the foregoing considerations, we are going to devise a finite-temperature theory of non-Markovian quantum dynamics on the basis of matrix continued fractions. Our treatment is founded on the following three assumptions: (i) the Hilbert space of the open quantum system is separable, i.e. it is spanned by a countable number of ket vectors; (ii) initial correlations between the open quantum system and the surrounding reservoir are absent, so that the initial composite density operator for system and reservoir factorizes; in reality, initial correlations may influence the dissipative dynamics significantly [12–15]; (iii) the reservoir consists of a continuum of harmonic oscillators that are initially at thermal equilibrium and

that linearly interact with the system potentials. This choice is indispensable to our treatment, because it allows us to factorize all higher-order correlation functions of the reservoir with the help of Wick's theorem. Thus the influence of the reservoir is completely accounted for by a set of pair correlation functions.

Adopting the setting (i)–(iii), we show in section 2 that an extension of the Markovian map $\exp(Lt)\rho_S$ exists that is valid outside the regime of weak coupling. This non-Markovian map takes on the form of $\mathcal{T}\exp[L(t)]\rho_S$, with $\mathcal{T}$ a time-ordering prescription and ρ_S the initial state of the open system. The pair correlation functions of the reservoir enter through the time-dependent generator $L(t)$. In the limit of weak coupling the latter reduces to the standard generator that is known from the literature [16]. Upon carefully carrying out the time-ordering prescription, our non-Markovian map reduces to an infinite expansion in terms of Kraus matrices [17].

Next, by suitably enlarging the set of Kraus matrices and performing an integral transformation of Laplace type, a closed hierarchy of nonlinear equations is constructed. For reason of clarity, this job is divided into two parts. In section 3.1 all of the necessary definitions and notations are gathered, while in section 3.2 the derivation of the nonlinear hierarchy is carried out. Its iteration yields exact solutions for the Kraus matrices in terms of infinite continued fractions. The actual execution of this last iterative process is technically demanding and therefore deferred to another paper [18].

In section 4, we attempt to reduce the technical barriers to our formalism by adopting a perturbative approach. Truncating the infinite set of Kraus matrices and factorizing Kraus matrices of any order exceeding the truncation parameter, we arrive at a finite set of integral equations for modelling non-Markovian evolution of an open quantum system. The perturbative density matrix complies with conservation of both positivity and probability, for any value of the truncation parameter.

In the lowest perturbative order, the complete dissipative dynamics can be described in terms of one Kraus matrix only. The solution for the latter is given by a continued fraction possessing a much simpler structure than its counterpart of the exact case. Therefore, applications of the theory can be worked out analytically now. This is demonstrated in section 5, where we find the non-Markovian density matrix of the resonant Jaynes–Cummings model. We introduce radiative damping by coupling the two-level atom to a transverse electromagnetic continuum of zero temperature through an interaction that does not contain counter-rotating contributions.

In addition to the setting (i)–(iii), we furthermore assume that all of the infinite continued fractions occurring in this paper are convergent. We recall that the use of continued fractions in quantum optics goes back to early work on the Rabi model by Schweber [19] and Swain [20]. A few years ago, these treatments have been reviewed and put on a sound mathematical basis by Braak [21]. Other interesting applications of continued fractions to quantum optics include, for instance, representation of a perturbative series for an anharmonic oscillator [22], solution of master equations in phase space [23], and solution of a semiclassical master equation for an atom in a strong electromagnetic field [24, 25].

Since the study of open quantum systems is a subject with a decades-long history, the continued-fraction approach is only one among many methods that came to light over the years. Numerous books and reviews, for example [26–35], appeared on Markovian master equations, non-Markovian master equations of integro-differential, time-convolutionless, and time-discrete type, as well as stochastic evolution equations, projection-operator techniques, path-integral procedures, field-theoretic formalisms, and of course numerical endeavours. Evidently, the foregoing enumeration is not meant to be exhaustive.

2. Kraus map for the density matrix

The evolution in time t of a quantum system S that exchanges energy with a thermal reservoir R is completely described by the density matrix $\rho_S(t)$, given by

$$\rho_S(t) = \text{Tr}_R \left[\exp(\mathrm{i}H_0 t) \exp(-\mathrm{i}H_{SR} t) \rho_{SR} \exp(\mathrm{i}H_{SR} t) \exp(-\mathrm{i}H_0 t) \right], \tag{2}$$

where the interaction picture has been adopted. The initial state of system and reservoir is denoted as ρ_{SR}. The Hamiltonian H_{SR} governing the unitary dynamics of system and reservoir is equal to $H_0 + H_1$. The free Hamiltonian H_0 is equal to the sum of the Hamiltonian H_S of the system and the Hamiltonian H_R of the reservoir. The Hamiltonian H_1, which describes the interaction between system and reservoir, can be expanded as

$$H_1 = \sum_{\alpha} V_{\alpha} \otimes U_{\alpha}. \tag{3}$$

In view of assumption (i) of the introduction, the index α takes on a countable number of values. The system potential V_{α} and the reservoir potential U_{α} evolve as

$$V_{\alpha}(t) = \exp(\mathrm{i}H_S t) V_{\alpha} \exp(-\mathrm{i}H_S t),$$

$$U_{\alpha}(t) = \exp(\mathrm{i}H_R t) U_{\alpha} \exp(-\mathrm{i}H_R t). \tag{4}$$

Note that the potentials figuring in (3) need not be self-adjoint. In turning (2) into a Kraus map we let us be guided by the treatment presented in [36].

Assumption (ii) of the Introduction allows us to factorize the initial state ρ_{SR} of system and reservoir as $\rho_S \otimes \rho_R$, where ρ_S denotes the initial state $\rho_S(t=0)$ of the system. The evolution of the reservoir starts from the thermal state $\rho_R = \exp(-\beta H_R)/Z$ of temperature β^{-1}, with Z equal to $\text{Tr}_R[\exp(-\beta H_R)]$. We scale all Hamiltonians as well as β^{-1} with Planck's constant.

Upon expanding the unitary evolution operators of (2) as

$$\exp(\mathrm{i}H_0 t) \exp(-\mathrm{i}H_{SR} t) = 1_S \otimes 1_R$$
$$+ \sum_{m=1}^{\infty} \sum_{\alpha_1 \cdots \alpha_m} (-\mathrm{i})^m \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{m-1}} \mathrm{d}t_m V_{\alpha_1}(t_1) \cdots V_{\alpha_m}(t_m) \otimes U_{\alpha_1}(t_1) \cdots U_{\alpha_m}(t_m), \tag{5}$$

we meet reservoir correlation functions of arbitrary order. Owing to assumption (iii) of the Introduction, all of these can be factorized by means of Wick's theorem. This gives rise to three types of pair correlation functions, viz.

$$c_{\alpha_1 \alpha_2}(t_1, t_2) = \text{Tr}_R \left[U_{\alpha_1}(t_1) U_{\alpha_2}(t_2) \rho_R \right],$$
$$c_{\alpha_1 \alpha_2}^{(+)}(t_1, t_2) = c_{\alpha_1 \alpha_2}(t_1, t_2) \theta(t_1 - t_2) + c_{\alpha_2 \alpha_1}(t_2, t_1) \theta(t_2 - t_1), \tag{6}$$
$$c_{\alpha_1 \alpha_2}^{(-)}(t_1, t_2) = c_{\alpha_1 \alpha_2}(t_1, t_2) \theta(t_2 - t_1) + c_{\alpha_2 \alpha_1}(t_2, t_1) \theta(t_1 - t_2),$$

where $\theta(t)$ denotes the Heaviside step function, i.e. $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$.

The Wick factorization permits us to generate the dynamics (2) with the help of the non-Markovian map [36, 37]

$$\rho_S(t) = \mathcal{T} \exp[L(t)] \rho_S. \tag{7}$$

The superoperator $L(t)$ comes out as

$$L(t)\rho_S = K^{(+)}(t)\rho_S + \rho_S K^{(-)}(t) + \sum_{\alpha\beta} \int_0^t \mathrm{d}u \int_0^t \mathrm{d}v\, c_{\beta\alpha}(v, u)\, V_{\alpha}(u)\rho_S V_{\beta}(v),$$

$$K^{(\eta)}(t) = -\tfrac{1}{2} \sum_{\alpha\beta} \int_0^t du \int_0^t dv \, c_{\alpha\beta}^{(\eta)}(u,v) \, \mathcal{T}_\eta \left\{ V_\alpha(u) V_\beta(v) \right\} . \tag{8}$$

The prescription $\mathcal{T}$ orders products of system potentials $\{V_\alpha(t)\}_\alpha$ according to

$$\mathcal{T} \left\{ \prod_{i=1}^m V_{\alpha_i}(t_i) \rho_S \prod_{j=1}^n V_{\alpha_j'}(t_j') \right\} = \mathcal{T}_+ \left\{ \prod_{i=1}^m V_{\alpha_i}(t_i) \right\} \rho_S \mathcal{T}_- \left\{ \prod_{j=1}^n V_{\alpha_j'}(t_j') \right\},$$

$$\mathcal{T}_+ \left\{ \prod_{i=1}^m V_{\alpha_i}(t_i) \right\} = V_{\alpha_1}(t_1) \cdots V_{\alpha_m}(t_m),$$

$$\mathcal{T}_- \left\{ \prod_{j=1}^n V_{\alpha_j'}(t_j') \right\} = V_{\alpha_n'}(t_n') \cdots V_{\alpha_1'}(t_1'), \tag{9}$$

where the inequalities $t_1 > \cdots > t_m$ and $t_1' > \cdots > t_n'$ are assumed.

By introducing the Kraus matrices

$$W_q^{(\eta)}(t; t_1, \cdots, t_q)_{\alpha_1 \cdots \alpha_q} = \mathcal{T}_\eta \left\{ \exp[K^{(\eta)}(t)] \prod_{i=1}^q V_{\alpha_i}(t_i) \right\}, \tag{10}$$

we can cast the map (7) into the Kraus format [17]

$$\rho_S(t) = W_0^{(+)}(t) \rho_S W_0^{(-)}(t) + \sum_{q=1}^\infty \sum_{\alpha_1 \cdots \alpha_q} \sum_{\alpha_1' \cdots \alpha_q'} \int_0^t dt_1 \cdots \int_0^{t_{q-1}} dt_q \int_0^t dt_1' \cdots \int_0^{t_{q-1}'} dt_q'$$

$$\times W_q^{(+)}(t; t_1, \cdots, t_q)_{\alpha_1 \cdots \alpha_q} \rho_S W_q^{(-)}(t; t_1', \cdots, t_q')_{\alpha_1' \cdots \alpha_q'} \sum_{PQ} \frac{1}{q!} \prod_{k=1}^q c_{\alpha_{Q(k)}' \alpha_{P(k)}}(t_{Q(k)}', t_{P(k)}). \tag{11}$$

On the right-hand side we sum over all permutations P and Q of the integers $\{1, \ldots, q\}$. In order to get access to the dissipative dynamics described by (11) one must find a way to cope with the time-ordering operator appearing in (10).

Starting from a Kato identity [38], one shows that the Kraus matrices $\{W_q^{(+)}\}_{q \geqslant 0}$ satisfy the infinite hierarchy

$$W_q^{(+)}(t; t_1, \cdots, t_q)_{\alpha_1 \cdots \alpha_q} = V_{\alpha_1}(t_1) W_{q-1}^{(+)}(t_1; t_2, \cdots, t_q)_{\alpha_2 \cdots \alpha_q}$$

$$- \sum_{j=1}^{q+1} \sum_{\alpha\beta} \int_{t_1}^t du \int_{t_j}^{t_{j-1}} dv \, c_{\alpha\beta}(u,v) V_\alpha(u)$$

$$\times W_{q+1}^{(+)}(u; t_1, \cdots, t_{j-1}, v, t_j, \cdots, t_q)_{\alpha_1 \cdots \alpha_{j-1} \beta \alpha_j \cdots \alpha_q}, \tag{12}$$

with $t > t_1 > \cdots > t_q > 0$ and $q \geqslant 0$. In evaluating the boundaries of the integral over v one has to choose $t_0 = u$ and $t_{q+1} = 0$. If q equals zero on the right-hand side of (12), one must replace the contribution $V_{\alpha_1}(t_1) W_{q-1}^{(+)}$ by the unit matrix and the time t_1 by zero. The Kraus matrices $\{W_q^{(-)}\}_{q \geqslant 0}$ can be obtained by taking the adjoint of (10) for the case $\eta = +$ and replacing the potentials $\{V_{\alpha_i}(t_i)\}_{i=1}^q$ by their adjoints.

Obviously, the effectiveness of the Kraus approach critically depends on the possibility of explicitly computing Kraus matrices. In the next section we shall show how the exact solution of the Kraus hierarchy (12) can be constructed.

3. Exact theory

3.1. Definitions and conventions

Before entering into the computation of Kraus matrices, we first propose some definitions and notational conventions. The eigenvalues and eigenstates of the system's Hamiltonian H_S are denoted as $\{\omega_k\}_{k\geqslant 1}$ and $\{|k\rangle\}_{k\geqslant 1}$. The latter provide an orthonormal basis for the Hilbert space of S. The representation of system potentials and reservoir potentials in terms of the states $\{|k\rangle\}_{k\geqslant 1}$ reads

$$\alpha \to (kl)\,, \quad V_\alpha(t) \to |k\rangle\langle l|\exp[\mathrm{i}\omega_{(kl)}t]\,, \quad U_\alpha(t) \to U_{(kl)}(t)\,. \tag{13}$$

For differences between energy eigenvalues the notation $\omega_{(kl)} = \omega_k - \omega_l$ is used.

Multi-indices of matrices are abbreviated as

$$K_q^n = (k_{n+1}k_{n+2}\cdots k_q)\,, \quad K_p^m k K_q^n = (k_{m+1}k_{m+2}\cdots k_p k k_{n+1}k_{n+2}\cdots k_q)\,, \tag{14}$$

with the special case $K_q = K_q^0$. The adjoint of a multi-index matrix M is defined as $(M^\dagger)_{K_q L_q} = M^*_{L_q K_q}$. For the elements of the multi-index unit matrix the notation $\delta_{K_q L_q} = \prod_{s=1}^q \delta_{k_s l_s}$ is employed.

Time arguments of multivariate functions are abbreviated as

$$T_q^n = t_{n+1}, t_{n+2}, \cdots, t_q\,, \quad T_p^m, t, T_q^n = t_{m+1}, t_{m+2}, \cdots, t_p, t, t_{n+1}, t_{n+2}, \cdots, t_q\,, \tag{15}$$

with the special case $T_q = T_q^0$. If a fixed time t is substracted from all variables of T_q^n we use the notation

$$T_q^n - t = t_{n+1} - t, t_{n+2} - t, \cdots, t_q - t\,. \tag{16}$$

Repeated integrals over the variables T_q^n are expressed as

$$\int_s^t \mathrm{d}T_q^n = \int_s^t \mathrm{d}t_{n+1} \int_s^{t_{n+1}} \mathrm{d}t_{n+2} \cdots \int_s^{t_{q-1}} \mathrm{d}t_q\,, \tag{17}$$

where on the left-hand side the condition $t > t_{n+1} > t_{n+2} > \cdots > t_q > s$ is always in force.

In the setting (13) the reservoir pair correlation function $c_{\alpha\beta}(t,s)$ takes on the generic form $c_{(kl)(mn)}(t,s)$. Laplace transformation of the latter function happens via the standard prescription $\hat{c}_{(kl)(mn)}(y) = -\mathrm{i}\int_0^\infty \mathrm{d}t\exp(\mathrm{i}yt)c_{(kl)(mn)}(t,0)$, with Im$y$ positive. The inverse transform reads $c_{(kl)(mn)}(t,0) = \mathrm{i}(2\pi)^{-1}\int_C \mathrm{d}y\exp(-\mathrm{i}yt)\hat{c}_{(kl)(mn)}(y)$, with the contour C determined by $-\infty < \mathrm{Re}y < \infty$ and Im$y > 0$ fixed.

For the Kraus matrices we switch over to the representation (13) via the definition

$$\langle k_1|W_q^{(+)}(t;t_1,\cdots,t_q)_{(l_1 k_2)\cdots(l_q k_{q+1})}|l_{q+1}\rangle = W_q(t;T_q)_{K_{q+1}L_{q+1}}\,. \tag{18}$$

In (10) the transition $\alpha_j \to (l_j k_{j+1})$ has been made for $1 \leqslant j \leqslant q$. In the new representation the Kraus matrices obey the initial condition $W_q(t=0; T_q=0)_{K_{q+1}L_{q+1}} = \delta_{K_{q+1}L_{q+1}}$.

In order to examine Kraus matrices in Laplace representation we define the following Laplace transform of a multivariate matrix M of time arguments t and T_q:

$$\check{M}(z; Z_q)_{K_{q+1}L_{q+1}}$$
$$= (-\mathrm{i})^{q+1} \int_0^\infty \mathrm{d}t \int_0^t \mathrm{d}T_q \exp[\mathrm{i}zt - \mathrm{i}\omega_{k_1}t + \mathrm{i}Z_q \cdot T_q] M(t; T_q)_{K_{q+1}L_{q+1}}, \tag{19}$$

where $\mathrm{Im}z$ and $\mathrm{Im}z_i$ must be positive, with $1 \leqslant i \leqslant q$. Since the Laplace variable z is shifted by an amount of $-\omega_{k_1}$, the right-hand side of (19) differs from the standard prescription for a Laplace transform. The definition of multiple Laplace variables such as Z_q is found from (15) by replacing all uppercase letters T by Z and all lowercase letters t by z. For the choice $q = 0$ the integral over T_q must be omitted.

The inverse of the transformation (19) contains a multiple complex integral over the single variable z and multiple variable Z_q. This integral is evaluated by means of repeated integration. By convention, the transform (19) of the Kraus matrix W_{-1} is equal to unity, i.e. $\check{W}_{-1} = 1$. By performing in (19) the integrals in reverse order and by making use of the initial condition for the Kraus matrices we can evaluate the transformed Kraus matrices for $|z|$ and $|z_i|$ large, with $1 \leqslant i \leqslant q$. This leads to the asymptotic result

$$\check{W}_q(z; Z_q)_{K_{q+1}L_{q+1}} \simeq \prod_{n=0}^{q} (Z_n^+ - \omega_{k_1})^{-1} \delta_{K_{q+1}L_{q+1}}, \tag{20}$$

where $\mathrm{Im}z$ and $\mathrm{Im}z_i$ must be positive. In (19) and (20) the shorthand notation

$$Z_q \cdot T_q = \sum_{i=1}^{q} z_i t_i, \quad Z_n^+ = z + z_1 + z_2 + \cdots + z_n, \tag{21}$$

is employed.

Last, for $q = 1, 2, 3, \ldots$ we introduce a superoperator Δ_q, which acts on an arbitrary matrix $\phi(s, s')$ according to the rule

$$\Delta_q[\phi](t, t'; t_1, t_1'; s, s')_{(k_1 l_1)(k_1' l_1')} = \sum_{K_{q+1}^1 L_{q+1}^1 K_{q+1}'^1 L_{q+1}'^1} \int_s^{t_1} \mathrm{d}T_q^1 \int_{s'}^{t_1'} \mathrm{d}T'_q^1$$
$$\times W_q(t - s; T_q - s)_{K_{q+1}L_{q+1}} \phi(s, s')_{(l_{q+1})(l_{q+1}')} W_q^\dagger(t' - s'; T'_q - s')_{L_{q+1}'K_{q+1}'} \tag{22}$$
$$\times \sum_{PQ} \frac{1}{q!} \prod_{h=1}^{q} \{Q(h)' \, P(h)\}.$$

We sum over all permutations P and Q of the integers $\{1, 2, \ldots, q\}$. The summand contains a product of reservoir pair correlation functions. With the help of the notation

$$\{m' \, n\} = c_{(k_{m+1}' l_m')(l_n k_{n+1})}(t_m', t_n) \tag{23}$$

this product has been put into a concise form.

3.2. Exact Kraus matrices in terms of continued fractions

Our program for solving the Kraus hierarchy (12) consists of three parts. In part one we represent the evolution equations generating the dissipative dynamics, i.e. the Kraus map (11) and the Kraus hierarchy (12), with the help of the eigenstates of the system's Hamiltonian. This representation has the advantage that we dispose of the system potentials (4). In part two we make the transfer to Laplace representation so as to remove the time integrals from the

Kraus hierarchy. Part one and part two of our program are fairly straightforward and lead to a three-term recursion relation for the Kraus matrices. Its solution is very lengthy and virtually untractable.

Therefore, our program is in need of a nontrivial third part. We embed the Kraus matrices in a larger set of matrices, to be called matrix ratios. By executing a series of technical maneuvers in the three-term recursion relation for the Kraus matrices, we acquire an infinite but closed hierarchy for the matrix ratios. Construction of the iterative solution of the new hierarchy gives rise to matrix continued fractions. Explicit derivation of exact expressions for the Kraus matrices is a separate job that is not completed in the present treatment.

With the preparations of section 3.1 made, the Kraus map (11) for the evolution of $\rho_S(t)$ can be put into the form

$$\langle k_1 | \rho_S(t) | k_1' \rangle = \sum_{l_1 l_1'} W_0(t)_{(k_1)(l_1)} \langle l_1 | \rho_S(0) | l_1' \rangle W_0^\dagger(t)_{(l_1')(k_1')}$$

$$+ \sum_{q=1}^{\infty} \sum_{l_1 l_1'} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_1' \; \Delta_q[\psi](t, t; t_1, t_1'; 0, 0)_{(k_1 l_1)(k_1' l_1')} \; , \quad (24)$$

with the initial condition $\psi(0,0)_{(l_1)(l_1')} = \langle l_1 | \rho_S(0) | l_1' \rangle$ and $\rho_S(t=0)$ the initial state of S. Use has been made of the eigenstates $\{\omega_k\}_{k \geqslant 1}$ of the Hamiltonian H_S and the transition (13) for system potentials. In this representation the Kraus hierarchy (12) appears as

$$W_q(t; T_q)_{K_{q+1} L_{q+1}} = \delta_{k_1 l_1} \exp[\mathrm{i}\omega_{(k_1 k_2)} t_1] W_{q-1}(t_1; T_q^1)_{K_{q+1}^1 L_{q+1}^1}$$

$$- \sum_{j=1}^{q+1} \sum_{klm} \int_{t_1}^t \mathrm{d}u \int_{t_j}^{t_{j-1}} \mathrm{d}v \exp[\mathrm{i}\omega_{(k_1 k)} u] \quad (25)$$

$$\times W_{q+1}(u; T_{j-1}, v, T_q^{j-1})_{(kK_j^1 m K_{q+1}^j)(L_{j-1} l L_{q+1}^{j-1})} c_{(k_1 k)(lm)}(u, v) ,$$

with $t > t_1 > t_2 > \cdots > t_q > 0$ and $q \geqslant 0$. In evaluating the boundaries of the integral over v one has to make the choices $t_0 = u$ and $t_{q+1} = 0$. For the choice $q = 0$ one must set t_1 equal to zero.

Part one of our program being completed, we turn the hierarchy (25) into semi-algebraic form by carrying out the Laplace transformation (19). The transformed Kraus hierarchy reads

$$\check{W}_q(z; Z_q)_{K_{q+1} L_{q+1}} = \delta_{k_1 l_1} (z - \omega_{k_1})^{-1} \check{W}_{q-1}(z + z_1; Z_q^1)_{K_{q+1}^1 L_{q+1}^1}$$

$$- \sum_{j=1}^{q+1} \sum_{klm} \int_C \frac{\mathrm{d}y}{2\pi \mathrm{i}} (z - \omega_{k_1})^{-1} \check{W}_{q+1}(z - y; Z_{j-1}, y, Z_q^{j-1})_{(kK_j^1 m K_{q+1}^j)(L_{j-1} l L_{q+1}^{j-1})} \hat{c}_{(k_1 k)(lm)}(y). \quad (26)$$

The contour C must be parametrized as $-\infty < \mathrm{Re}\, y < \infty$, with $\mathrm{Im}\, y$ fixed and $\mathrm{Im}\, z > \mathrm{Im}\, y > 0$. Note that $\check{W}_{q-1}$, $\check{W}_q$ and $\check{W}_{q+1}$ appear in the same relation. This three-term recurrence gives rise to a bulky solution of little practical value.

We now embark on the third part of our program. Repeated employment of (26) delivers the more general hierarchy

$$\check{W}_q(z;Z_q)_{K_{q+1}L_{q+1}} = \delta_{K_nL_n}\prod_{s=1}^{n}(Z_{s-1}^+ - \omega_{k_s})^{-1}\check{W}_{q-n}(Z_n^+;Z_q^n)_{K_{q+1}^nL_{q+1}^n}$$

$$- \sum_{p=-1}^{n-2}\sum_{j=p+2}^{q+1}\sum_{klm}\int_C\frac{\mathrm{d}y}{2\pi\mathrm{i}}\,\delta_{K_{p+1}L_{p+1}}\prod_{s=1}^{p+2}(Z_{s-1}^+ - \omega_{k_s})^{-1} \tag{27}$$

$$\times \check{W}_{q-p}(Z_{p+1}^+ - y;Z_{j-1}^{p+1},y,Z_q^{j-1})_{(kK_j^{p+2}mK_{q+1}^j)(L_{j-1}^{p+1}lL_{q+1}^{j-1})}\hat{c}_{(k_{p+2}k)(lm)}(y),$$

where the conditions $q \geqslant 0$ and $0 \leqslant n \leqslant q+1$ must be satisfied. As it turns out, the hierarchy (26) does not allow for further treatment on the basis of continued fractions, whereas the extended hierarchy (27) does.

To see this in detail we introduce the following matrix ratio:

$$R_{q,n}(z;Z_q)_{K_{q+1}L_{q+1}}$$
$$= \sum_{M_{q+1}}\check{W}_q(z;Z_q)_{K_{q+1}M_{q+1}}\delta_{M_nL_n}\check{W}_{q-n}^{-1}(Z_n^+;Z_q^n)_{M_{q+1}^nL_{q+1}^n}, \tag{28}$$

with $q \geqslant 0$ and $0 \leqslant n \leqslant q+1$. The imaginary parts of the Laplace variables z and $\{z_i\}_{i=1}^q$ must be chosen sufficiently large. Then the determinant $|\check{W}_{q-n}(Z_n^+;Z_q^n)|$ differs from zero, as follows from (20) and a continuity argument. Therefore, the right-hand side of (28) is well-defined in the entire domain that is needed for carrying out inverse Laplace transformation. Note that for $n = 0$ the matrix ratio reduces to $R_{q,0}(z;Z_q)_{K_{q+1}L_{q+1}} = \delta_{K_{q+1}L_{q+1}}$. The equality

$$R_{q,q+1}(z;Z_q)_{K_{q+1}L_{q+1}} = \check{W}_q(z;Z_q)_{K_{q+1}L_{q+1}} \tag{29}$$

is a consequence of the convention $\check{W}_{-1} = 1$. It shows that the set of Kraus matrices is completely contained in the larger family of matrix ratios.

Upon multiplying (27) from the right by $\check{W}_q^{-1}$, the unit matrix appears on the left-hand side, whereas in the first contribution on the right-hand side the inverse of the matrix ratio (28) is created. The second term on the right-hand side can be restated in terms of matrix ratios if we insert the identity $\sum_{M_{q+1}}G^{-1}(z;Z_q)_{K_{q+1}M_{q+1}}G(z;Z_q)_{M_{q+1}L_{q+1}} = \delta_{K_{q+1}L_{q+1}}$, with the intermediate matrix G chosen as

$$G(z;Z_q)_{K_{q+1}L_{q+1}} = \delta_{K_jL_j}\check{W}_{q-j}(Z_j^+;Z_q^j)_{K_{q+1}^jL_{q+1}^j}. \tag{30}$$

Rearrangement of dummy variables brings us to the key result

$$R_{q,n}^{-1}(z;Z_q)_{K_{q+1}L_{q+1}} = \delta_{K_{q+1}L_{q+1}}\prod_{s=1}^{n}(Z_{s-1}^+ - \omega_{k_s}) + \sum_{p=-1}^{n-2}\sum_{j=p+2}^{q+1}\sum_{M_{q+1}}\sum_{klm}\int_C\frac{\mathrm{d}y}{2\pi\mathrm{i}}\delta_{K_{p+1}M_{p+1}}$$

$$\times \prod_{s=p+3}^{n}(Z_{s-1}^+ - \omega_{k_s})R_{q-p,j-p}(Z_{p+1}^+ - y;Z_{j-1}^{p+1},y,Z_q^{j-1})_{(kK_j^{p+2}mK_{q+1}^j)(M_{j-1}^{p+1}lM_{q+1}^{j-1})}$$

$$\times R_{q,j}^{-1}(z;Z_q)_{M_{q+1}L_{q+1}}\hat{c}_{(k_{p+2}k)(lm)}(y), \tag{31}$$

with the constraint $0 \leqslant n \leqslant q+1$. For $n = 0$ the right-hand side of (31) reduces to the unit matrix, in line with definition (28).

Matrix inversion on both sides of (31), followed by iteration and the choice $n = q + 1$, furnishes the exact Kraus matrices. That is to say, the Kraus matrices emanating from the Schrödinger equation for system and reservoir in the setting (i)–(iii) of the Introduction. As planned, our solution is built up from matrix continued fractions. It constitutes a direct extension of the standard formula (1) to the case of a multi-level system interacting with a reservoir of finite temperature. However, execution of the afore-mentioned program requires a lot of technical effort and is therefore shifted to another paper [18]. In the present treatment, we shall focus on the derivation of an approximate continued-fraction solution possessing a much simpler structure and allowing for analytical work. This is the subject of the next section.

Before closing we solve the Kraus hierarchy for a damped two-level atom at zero temperature. We couple the atom to the annihilation and creation operators $b(\omega)$ and $b^{\dagger}(\omega)$ of the radiation field via the interaction Hamiltonian $|1\rangle\langle 2| \otimes \int_0^{\infty} \mathrm{d}\omega g^*(\omega) b^{\dagger}(\omega) + \text{h.c.}$, where counter-rotating contributions have been dropped, as well as the diagonal reservoir potentials $U_{(11)}$ and $U_{(22)}$. Then all correlation functions $c_{(kl)(mn)}$ equal zero, except for the choice $k = n = 2, l = m = 1$. Furthermore, the atom is assumed to start from the excited state so that the initial condition $\rho_S = |2\rangle\langle 2|$ is in force. Now the matrix $\check{W}_{q+1}$ figuring on the right-hand side of the Kraus hierarchy (26), with $K_{q+1} = (12\cdots 22)$, $L_{q+1} = (11\cdots 12)$ and $q \geqslant 0$, can be expressed in terms of the matrix $\check{W}_q$, with $K_q = (2\cdots 22)$ and $L_q = (1\cdots 12)$. It then appears that $\check{W}_q$ is vanishing for $q \geqslant 1$. Hence, in (24) only the term with W_0 survives. From the Kraus hierarchy (26) we obtain the solution

$$\check{W}_0(z)_{(2)(2)} = \left[z - \omega_2 + \int_C \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{\hat{c}_{(21)(12)}(y)}{z - y - \omega_1} \right]^{-1}. \tag{32}$$

The contour C must satisfy the condition $\mathrm{Im}z > \mathrm{Im}y > 0$. At zero temperature the Laplace transformed correlation function is found as

$$\hat{c}_{(21)(12)}(y) = \int_0^{\infty} \mathrm{d}\omega |g(\omega)|^2 / (y - \omega). \tag{33}$$

Upon elaborating the first contribution of (24) with the help of (32) and (33) we recover the well-known result (1).

4. Perturbation theory

In developing a sound perturbation theory, we should safeguard the properties $\rho_S(t) \geqslant 0$ and $\mathrm{Tr}\rho_S(t) = 1$ of the exact density matrix. Only then it is guaranteed that the perturbative dynamics does not exhibit unphysical artefacts. Furthermore, we should work in arbitrary perturbative order N, the limit $N \to \infty$ being capable of generating the exact dynamics. This permits us to control errors coming along with the perturbative approach. Realization of our perturbative program will be achieved in three steps: (i) truncation of the Kraus hierarchy; (ii) factorization of Kraus matrices of order N and higher; (iii) factorization of the permutations P and Q figuring in (22).

First, we truncate the Kraus hierarchy through discarding the second term on the right-hand side of (25) for $q = N$ and replacing $\delta_{k_1 l_1}$ by $W_0(t - t_1)_{(k_1)(l_1)}$. Thus the truncation prescription is given by the factorization

$$W_N(t; T_N)_{K_{N+1} L_{N+1}} \to \exp[\mathrm{i}\omega_{(k_1 k_2)} t_1] W_0'(t - t_1)_{(k_1)(l_1)} W_{N-1}'(t_1; T_N^1)_{K_{N+1}^1 L_{N+1}^1}, \tag{34}$$

where the ordering $t > t_1 > t_2 > \cdots > t_N > 0$ is in force and a prime is used to denote perturbative Kraus matrices[1]. The truncation parameter N defines the order of the perturbation theory and takes on the values $N = 1, 2, 3, \ldots$. In [36] it was shown that (34) is exact in the asymptotic regime of large times.

Second, for Kraus matrices of higher order we perform the factorization

$$
\begin{aligned}
W_{q+pN}(t; T_{q+pN})_{K_{q+pN+1}L_{q+pN+1}} &\to \exp(\mathrm{i}\omega_{(k_1 k_{q+2})} t_{q+1}) W'_q(t - t_{q+1}; T_q - t_{q+1})_{K_{q+1}L_{q+1}} \\
&\times \prod_{j=1}^{p} \Big[\exp(\mathrm{i}\omega_{(k_{q+(j-1)N+2} k_{q+jN+2})} t_{q+jN+1}) \\
&\times W'_{N-1}(t_{q+(j-1)N+1} - t_{q+jN+1}; T^{q+(j-1)N+1}_{q+jN} - t_{q+jN+1})_{K^{q+(j-1)N+1}_{q+jN+1} L^{q+(j-1)N+1}_{q+jN+1}} \Big], \quad (35)
\end{aligned}
$$

with $q = 0, 1, 2, \ldots, N - 1$ and $p = 1, 2, 3, \ldots$. We make use of the convention $t_{q+pN+1} = 0$. The right-hand side of (35) is completely determined by the Kraus matrices $\{W'_q\}_{q=0}^{N-1}$. The latter can be obtained by solving the truncated Kraus hierarchy. This closed set of N equations corresponds to the choices $q = 0, 1, 2, \ldots, N - 1$ in (25). Note that (35) reduces to (34) under the choices $q = 0$ and $p = 1$, so the first step is consistent with the second step. Last, we point out that factorization of the adjoint Kraus matrices happens by taking the complex conjugate of (35).

Third, having fixed all perturbative Kraus matrices, we commence the construction of a perturbative series for the density matrix. To that end, (35) is substituted into (24) and the sum over q is divided into bunches of N terms. Next, P and Q are factorized such that any coupling between different sectors of the perturbative series is eliminated. Recalling Wick's theorem, we see that this transition comes down to discarding reservoir correlation functions with more than $2N$ time arguments. In other words, memory effects induced by the reservoir are taken into account up to a certain degree only.

Implementation in (24) of the factorizations as described above provides us with the perturbative density matrix $\rho_S^{(N)}(t)$ we are after. The expansion for $\rho_S^{(N)}(t)$ can be generated with the help of the identity

$$
\begin{aligned}
\langle k_1 | \rho_S^{(N)}(t) | k_1' \rangle &= \sum_{l_1 l_1'} W'_0(t)_{(k_1)(l_1)} \langle l_1 | \rho_S(0) | l_1' \rangle W_0'^{\dagger}(t)_{(l_1')(k_1')} \\
&+ \sum_{q=1}^{N-1} \sum_{l_1 l_1'} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_1' \, \Delta'_q[\psi](t, t; t_1, t_1'; 0, 0)_{(k_1 l_1)(k_1' l_1')} \\
&+ \sum_{l_1 l_1'} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \exp[\mathrm{i}\omega_{k_1} s - \mathrm{i}\omega_{k_1'} s'] \\
&\quad \times W'_0(t - s)_{(k_1)(l_1)} \chi^{(N)}(s, s')_{(l_1)(l_1')} W_0'^{\dagger}(t - s')_{(l_1')(k_1')} \\
&+ \sum_{q=1}^{N-1} \sum_{l_1 l_1'} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \int_s^t \mathrm{d}t_1 \int_{s'}^t \mathrm{d}t_1' \exp[\mathrm{i}\omega_{k_1} s - \mathrm{i}\omega_{k_1'} s'] \\
&\quad \times \Delta'_q[\chi^{(N)}](t, t; t_1, t_1'; s, s')_{(k_1 l_1)(k_1' l_1')}.
\end{aligned} \quad (36)
$$

The bitemporal matrix $\chi^{(N)}$ satisfies the integral equation

---

[1] In [36] a tilde was used to distinguish between exact and perturbative Kraus matrices. This notation becomes awkward for the Laplace transform $\breve{W}_0$.

$$\chi^{(N)}(t_1, t_1')_{(l_1)(l_1')} = \sum_{k_1 k_1'} \exp\left[-i\omega_{k_1} t_1 + i\omega_{k_1'} t_1'\right] \Delta_N'[\psi](t_1, t_1'; t_1, t_1'; 0, 0)_{(k_1 l_1)(k_1' l_1')}$$

$$+ \sum_{k_1 k_1'} \int_0^{t_1} ds \int_0^{t_1'} ds' \exp\left[-i\omega_{k_1}(t_1 - s) + i\omega_{k_1'}(t_1' - s')\right]$$

$$\times \Delta_N'[\chi^{(N)}](t_1, t_1'; t_1, t_1'; s, s')_{(k_1 l_1)(k_1' l_1')}, \tag{37}$$

with $N = 1, 2, 3, \ldots$ and the initial condition $\psi(0, 0)_{(l_1)(l_1')} = \langle l_1 | \rho_S(0) | l_1' \rangle$, as in (24). Last, the superoperators Δ_q' and Δ_N' must be obtained from (22) through replacement of all Kraus matrices by their primed counterparts.

The set (36) and (37) determines the evolution of the density matrix of the open quantum system S in arbitrary perturbative order N. For the case of $N = 1$ we carry out the additional transformation

$$\xi(t, t')_{(k)(k')} = \sum_{ll'} W_0'(t)_{(k)(l)} \langle l | \rho_S(0) | l' \rangle W_0'^\dagger(t')_{(l')(k')}$$

$$+ \sum_{ll'} \int_0^t ds \int_0^{t'} ds' \exp[i\omega_k s - i\omega_{k'} s'] W_0'(t - s)_{(k)(l)} \chi^{(1)}(s, s')_{(l)(l')} W_0'^\dagger(t' - s')_{(l')(k')}. \tag{38}$$

Then the representation of the lowest order perturbative density matrix simplifies to $\langle k | \rho_S^{(1)}(t) | k' \rangle = \xi(t, t)_{(k)(k')}$. The shifted bitemporal matrix ξ satisfies the evolution equation

$$\xi(t, t')_{(k)(k')} = \sum_{ll'} W_0'(t)_{(k)(l)} \langle l | \rho_S(0) | l' \rangle W_0'^\dagger(t')_{(l')(k')}$$

$$+ \sum_{ll'mm'} \int_0^t ds \int_0^{t'} ds' \exp[i\omega_{(km)} s + i\omega_{(m'k')} s'] W_0'(t - s)_{(k)(l)}$$

$$\times \xi(s, s')_{(m)(m')} W_0'^\dagger(t' - s')_{(l')(k')} c_{(m'l')(lm)}(s', s), \tag{39}$$

whereas the Kraus hierarchy reduces to the single equation

$$W_0'(t)_{(k)(l)} = \delta_{kl} - \sum_{mnp} \int_0^t du \int_0^u dv \exp[i\omega_{(km)} u + i\omega_{(mp)} v]$$

$$\times W_0'(u - v)_{(m)(n)} W_0'(v)_{(p)(l)} c_{(km)(np)}(u, v). \tag{40}$$

The dissipative dynamics governed by (39) and (40) was first derived in [6].

After employment of the Laplace transform (19), the matrix $\check{W}_0'(z)$ appears both on the left-hand side and on the right-hand side of (40). Therefore, we are invited to multiply this equation from the right by $\check{W}_0'^{-1}(z)$. In view of (20) the afore-mentioned inverse matrix exists for Imz sufficiently large. It is found to obey

$$\check{W}_0'^{-1}(z)_{(k)(l)} = (z - \omega_k)\delta_{kl} + \sum_{mn} \int_C \frac{dy}{2\pi i} \hat{c}_{(km)(nl)}(y) \check{W}_0'(z - y)_{(m)(n)}, \tag{41}$$

with Im$z >$ Im$y > 0$. This is a finite-temperature identity which is valid in the presence of counter-rotating contributions in the interaction Hamiltonian for system and reservoir. Solution of (41) happens by matrix inversion on both sides and subsequent iteration. The ensuing matrix continued fraction possesses a much more orderly structure than its exact counterpart originating from (31). At the same time, (41) is not an ad hoc result since it is the

low-end product of a full-fledged perturbation theory that is tied up with the exact dynamics through the limit of $N \to \infty$.

The analytic properties of $\check{W}_0'(z)$ can be explored by temporarily assuming a discrete reservoir with energy eigenvalues $\{\mu_j\}_j$ and energy eigenstates $\{|r_j\rangle\}_j$. Thus the expansion $H_R = \sum_j \mu_j |r_j\rangle\langle r_j|$ can be utilized. It then appears that $\check{W}_0'(z)$ is analytic for $\mathrm{Im}\,z \neq 0$ and that this matrix possesses simple poles $\{z = x_s\}_s$ on the real axis. The foregoing statement can be proved by applying induction to the iterative solution for $\check{W}_0'(z)$. In doing so, (41) should be replaced by the matrix identity

$$\check{W}_0'^{-1}(z) = E(z) - \sum_{jj's} (z - x_s + \mu_j - \mu_{j'})^{-1} C^{jj'} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x_s} \check{W}_0'^{-1}(x_s) \right)^{-1} \cdot C^{jj'\,\dagger},$$

(42)

where the dot means matrix multiplication. The matrices $E(z)_{kl}$ and $C_{kl}^{jj'}$ are defined as $(z - \omega_k)\delta_{kl}$ and $\langle r_j | \rho_R | r_j \rangle^{1/2} \langle r_j | U_{(kl)} | r_{j'} \rangle$, respectively.

While carrying out the above-mentioned induction proof, one verifies in each iterative order that the inverse matrix on the right-hand side of (42) is strictly positive. Hence, if $\mathrm{Im}\,z$ differs from zero the imaginary part of the standard quadratic form $< v, \check{W}_0'^{-1}(z)v >$ differs from zero as well, for arbitrary vector $v \neq 0$. Consequently, all poles of the matrix $\check{W}_0'(z)$ lie on the real axis and inverse Laplace transformation can be performed on the basis of the contour $z = \omega + \mathrm{i}0$, with ω real. As one takes a continuum limit for the reservoir, this important conclusion remains valid, albeit that the poles on the real axis merge together into a branch cut.

Returning to the case of arbitrary perturbative order, we are now going to verify that both positivity and probability are conserved for the perturbative density matrix. By construction, the expansion for $\rho_S^{(N)}(t)$ has the Kraus form $\sum_j V_j \rho_S V_j^\dagger$, where the matrices $\{V_j\}_j$ need not be specified. As a consequence, the property $\rho_S^{(N)}(t) \geqslant 0$ is obvious. In contrast, conservation of probability is not manifest for $\rho_S^{(N)}(t)$ and indeed the proof requires some effort. In appendix A the result

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{k_1} \langle k_1 | \rho_S^{(N)}(t) | k_1 \rangle = 0$$

(43)

is established, with $N = 1, 2, 3, \ldots$.

Let us again focus on the lowest perturbative order $N = 1$ and reconsider the damped two-level atom at zero temperature. Choosing the interaction Hamiltonian as specified in section 3.2, one recognizes that the solution of (41) for $k = l = 2$ coincides with (32). Hence, the lowest order perturbation theory is capable of reproducing the exact evolution (1). Moreover, for a system of arbitrary dimension this theory is also capable of reproducing the Markovian map $\rho_S(t) = \exp(Lt)\rho_S(0)$. To that end, the evolution equations (39) and (40) must be solved in the limit of large time and weak coupling between system and reservoir. As demonstrated in [6], for the constant generator L the standard [16] form originating from the exact dynamics (11) is obtained.

For the damped two-level atom of section 3.2 the above-mentioned Markovian map provides us with the Wigner–Weisskopf evolution

$$\begin{aligned}
\langle 2|\rho_S(t)|2\rangle &= 1 - \langle 1|\rho_S(t)|1\rangle = \exp(-2\gamma t)\langle 2|\rho_S(0)|2\rangle, \\
\langle 2|\rho_S(t)|1\rangle &= \langle 1|\rho_S(t)|2\rangle^* = \exp(-\gamma t + \mathrm{i}\bar{\omega}t)\langle 2|\rho_S(0)|1\rangle,
\end{aligned}$$

(44)

where we have defined the damping constant $\gamma = \pi |g(\omega_{(21)})|^2$ and the level shift $\bar{\omega} = \mathcal{P} \int_0^\infty d\omega |g(\omega)|^2 / (\omega - \omega_{(21)})$. In the field of quantum communication the evolution (44) is cast into the form of an amplitude-damping channel [39], given by

$$\rho_S(t) = M(t)\rho_S(0)M(t)^\dagger + N(t)\rho_S(0)N(t)^\dagger . \tag{45}$$

The nonzero elements of the Kraus matrices read $\langle 1|M(t)|1\rangle = 1$, $\langle 2|M(t)|2\rangle = \exp(-\gamma t + i\bar{\omega}t)$, and $\langle 1|N(t)|2\rangle = [1 - \exp(-2\gamma t)]^{1/2}$.

In the next section, we shall discuss an application of the dissipative quantum theory resting on the evolution equations (39) and (40). We shall demonstrate that density matrices can be computed with relative ease and that quantum evolutions can be investigated by analytical means.

5. Jaynes–Cummings model with non-Markovian damping

As system S we choose a two-level atom with excited state $|2\rangle_a$ of energy $\omega_{a,2}$ and ground state $|1\rangle_a$ of energy $\omega_{a,1}$, as well as a single electromagnetic mode with states $|n\rangle_f = (n!)^{-1/2} a^{\dagger\,n} |0\rangle_f$ of energies $n\omega_f$ ($n = 0, 1, 2, \ldots$). Here $|0\rangle_f$ denotes the vacuum state of the field mode and $a^\dagger$ the photon creation operator. The atom is assumed to be on resonance with the field mode, so the statement $\omega_f = \omega_{a,2} - \omega_{a,1} > 0$ holds true. Exchange of energy between atom and field mode happens through the Jaynes–Cummings interaction $f |1\rangle_a\,{}_a\langle 2| \otimes a^\dagger + \text{h.c.}$, with f real and positive.

For the orthonormal energy eigenstates and energy eigenvalues of S we find

$$|\epsilon, n\rangle = \nu_n(|1\rangle_a \otimes |n+1\rangle_f + \epsilon |2\rangle_a \otimes |n\rangle_f) ,$$
$$\Omega_{\epsilon,n} = \omega_{a,2}(n+1) - \omega_{a,1}n + \epsilon f(n+1)^{1/2} , \tag{46}$$

with $\epsilon = -1, +1$, $n = -1, 0, 1, \ldots$, $\nu_n = \delta_{n,-1} + \theta_n / \sqrt{2}$, and $|-1\rangle_f \equiv 0$. One has $|-1, -1\rangle = |+1, -1\rangle$, so for $n = -1$ we drop the choice of $\epsilon = -1$. In summations a prime appears to indicate this. By θ_n the discrete theta function is meant, i.e. $\theta_n = 1$ for n nonnegative and $\theta_n = 0$ for n negative. The eigenstates $\{|\epsilon, n\rangle\}$ satisfy a completeness relation, so they span the system's Hilbert space. We assume that the inequality $\omega_f > f$ is true. Then one has the property $\Omega_{\epsilon,n} > \Omega_{+1,-1}$ for arbitrary $(\epsilon, n) \neq (+1, -1)$, which expresses the fact that the energy of the state $|+1, -1\rangle$ is lowest.

The above model describes an immobile atom in a cavity that selects a privileged field mode. Radiative damping is introduced by coupling the atom to a continuum of transverse electromagnetic modes. These are (de-)excited by the annihilation and creation operators $b(\omega)$ and $b^\dagger(\omega)$. The transverse electromagnetic continuum is kept at zero temperature. Furthermore, cavity damping and collisional damping are not taken into consideration. Discarding counter-rotating contributions, we can model the interaction between system and reservoir as

$$H_1 = |1\rangle_a\,{}_a\langle 2| \otimes \int_0^\infty d\omega\, g^*(\omega) b^\dagger(\omega) + \text{h.c.},$$
$$= \sum_{\epsilon_1, n_1, \epsilon_2, n_2}{}' |\epsilon_1, n_1\rangle\langle \epsilon_2, n_2| \otimes U_{(\epsilon_1, n_1)(\epsilon_2, n_2)} ,$$
$$U_{(\epsilon_1, n_1)(\epsilon_2, n_2)} = \tfrac{1}{\sqrt{2}} \epsilon_2 \nu_{n_1} \theta_{n_2} \delta_{n_1+1, n_2} \int_0^\infty d\omega\, g^*(\omega) b^\dagger(\omega)$$
$$+ \tfrac{1}{\sqrt{2}} \epsilon_1 \nu_{n_2} \theta_{n_1} \delta_{n_1, n_2+1} \int_0^\infty d\omega\, g(\omega) b(\omega) . \tag{47}$$

14

As announced, a prime is used to exclude the state $|-1,-1\rangle$ from the summation.

We factorize the initial state of S and choose the privileged mode to be in a number state of p photons. Then the matrix elements of $\rho_S(0)$ read

$$
\langle \epsilon_1, n_1 | \rho_S(0) | \epsilon_2, n_2 \rangle = \nu_{n_1} \nu_{n_2} (\rho_{a,11} \delta_{n_1,n_2} \delta_{n_1+1,p} + \epsilon_1 \rho_{a,21} \delta_{n_1,n_2+1} \delta_{n_1,p}
$$
$$
+ \epsilon_2 \rho_{a,12} \delta_{n_1+1,n_2} \delta_{n_2,p} + \epsilon_1 \epsilon_2 \rho_{a,22} \delta_{n_1,n_2} \delta_{n_1,p}) ,
\tag{48}
$$

with $\rho_{a,kl} = {}_a\langle k | \rho_a(0) | l \rangle_a$ and $\rho_a(0)$ the initial atomic state. The pair correlation functions come out as

$$
c_{((\epsilon_1,n_1)(\epsilon_2,n_2))((\epsilon_3,n_3)(\epsilon_4,n_4))}(t,0) = \tfrac{1}{2} \epsilon_1 \epsilon_4 \theta_{n_1} \theta_{n_4} \nu_{n_2} \nu_{n_3} \delta_{n_1,n_2+1} \delta_{n_3+1,n_4}
$$
$$
\times \int_0^\infty \mathrm{d}\omega |g(\omega)|^2 \exp(-\mathrm{i}\omega t) .
\tag{49}
$$

We are ready now to construct the solution of (39).

Making use of the Laplace transform

$$
\check{\xi}(z,z')_{(\epsilon_1,n_1)(\epsilon_1',n_1')}
$$
$$
= \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}t' \exp(\mathrm{i}zt - \mathrm{i}\Omega_{\epsilon_1,n_1}t - \mathrm{i}z't' + \mathrm{i}\Omega_{\epsilon_1',n_1'}t') \xi(t,t')_{(\epsilon_1,n_1)(\epsilon_1',n_1')} ,
\tag{50}
$$

with $\mathrm{Im}z$ and $-\mathrm{Im}z'$ positive, we can cast (39) into an algebraic form, given by

$$
\check{\xi}(z,z')_{(\epsilon_1,n_1+1)(\epsilon_1',n_1'+1)}
$$
$$
= \sideset{}{'}\sum_{\epsilon_2,n_2,\epsilon_2',n_2'} \check{W}_0'(z)_{(\epsilon_1,n_1+1)(\epsilon_2,n_2)} \langle \epsilon_2, n_2 | \rho_S(0) | \epsilon_2', n_2' \rangle \bar{W}_0'(z')_{(\epsilon_1',n_1'+1)(\epsilon_2',n_2')}
$$
$$
+ \sideset{}{'}\sum_{\epsilon_2,n_2,\epsilon_2',n_2',\epsilon_3,\epsilon_3'} \tfrac{1}{2} \nu_{n_2} \nu_{n_2'} \epsilon_3 \epsilon_3' \int_0^\infty \mathrm{d}\omega_1 |g(\omega_1)|^2 \check{W}_0'(z)_{(\epsilon_1,n_1+1)(\epsilon_2,n_2)} \bar{W}_0'(z')_{(\epsilon_1',n_1'+1)(\epsilon_2',n_2')}
$$
$$
\times \check{\xi}(z+\omega_1, z'+\omega_1)_{(\epsilon_3,n_2+1)(\epsilon_3',n_2'+1)} ,
\tag{51}
$$

with $\mathrm{Im}z$ and $-\mathrm{Im}z'$ positive. For the transform $\check{W}_0'$ we find from (41) and (49) the relation

$$
\check{W}_0'^{-1}(z)_{(\epsilon_1,n_1)(\epsilon_2,n_2)} = (z - \Omega_{\epsilon_1,n_1}) \delta_{\epsilon_1,\epsilon_2} \delta_{n_1,n_2}
$$
$$
- \sideset{}{'}\sum_{\epsilon_3 \epsilon_4} \tfrac{1}{2} \epsilon_1 \epsilon_2 \theta_{n_1} \theta_{n_2} \nu_{n_1-1} \nu_{n_2-1} \int_0^\infty \mathrm{d}\omega |g(\omega)|^2 \check{W}_0'(z-\omega)_{(\epsilon_3,n_1-1)(\epsilon_4,n_2-1)} .
\tag{52}
$$

The definition

$$
\bar{W}_0'(z)_{(\epsilon_1,n_1)(\epsilon_2,n_2)} = \left[\check{W}_0'(z^*)_{(\epsilon_1,n_1)(\epsilon_2,n_2)} \right]^*
\tag{53}
$$

specifies the outcome of transforming the adjoint Kraus matrix.

The atomic density matrix $\rho_a(t)$ must be computed from

$$
{}_a\langle k | \rho_a(t) | l \rangle_a = \sum_{n=0}^\infty {}_a\langle k | \otimes {}_f\langle n | \rho_S^{(1)}(t) | l \rangle_a \otimes | n \rangle_f .
\tag{54}
$$

If the matrix elements on the right-hand side are represented in terms of the states (46), the iterative solution of (51) can be exploited. For the probability that the atom is in the excited state at time t we then arrive at the result

$$
\begin{aligned}
{}_a\langle 2|\,\rho_a(t)|2\rangle_a &= \sum_{r=0}^{\infty} \sideset{}{'}\sum_{\{\epsilon_s,\epsilon_s'\}_{s=1}^{2r+2}} \sum_{\{n_s,n_s'=-1\}_{s=1}^{r+2}} \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\int_0^{\infty}\mathrm{d}\omega_1\cdots\int_0^{\infty}\mathrm{d}\omega_r \\
&\quad\times \prod_{s=0}^{r}\Big[\check{W}_0'(\omega+\omega_s^+ +\mathrm{i}0)_{(\epsilon_{2s+1},n_{s+1}+1)(\epsilon_{2s+2},n_{s+2})}\,\bar{W}_0'(\omega'+\omega_s^+ -\mathrm{i}0)_{(\epsilon_{2s+1}',n_{s+1}'+1)(\epsilon_{2s+2}',n_{s+2}')}\Big] \\
&\quad\times \Big(\tfrac{1}{2}\Big)^{r+1}\delta_{n_1,n_1'}\epsilon_1\epsilon_1'\prod_{s=1}^{r}\Big(\epsilon_{2s+1}\epsilon_{2s+1}'\nu_{n_{s+1}}\nu_{n_{s+1}'}|g(\omega_s)|^2\Big)\,\langle\epsilon_{2r+2},n_{r+2}|\rho_S(0)|\epsilon_{2r+2}',n_{r+2}'\rangle \\
&\quad\times \exp(-\mathrm{i}\omega t+\mathrm{i}\omega' t+\mathrm{i}\Omega_{\epsilon_1,n_1+1}t-\mathrm{i}\Omega_{\epsilon_1',n_1+1}t)\,,
\end{aligned}
\tag{55}
$$

where the abbreviation $\omega_s^+ = \omega_1+\omega_2+\cdots+\omega_s$ is employed. Contours have been laid in accordance with the prescription derived in the previous section. The matrix element of the initial state $\rho_S(0)$ can be computed from (48), whereas the matrices $\check{W}_0'$ and $\bar{W}_0'$ give rise to continued fractions after inversion and subsequent iteration of (52). Obviously, the diagonal element ${}_a\langle 1|\,\rho_a(t)|1\rangle_a$ follows from conservation of trace. Finally, the solution for the off-diagonal element ${}_a\langle 2|\,\rho_a(t)|1\rangle_a$ can be found in the same manner as discussed above.

In [36] it was argued that for large times $\rho_S(t)$ converges to the ground state. Such asymptotic behaviour gives rise to the limit

$$
\lim_{t\to\infty}\rho_a(t) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\,,
\tag{56}
$$

the verification of which is indeed possible with the help of our solution for $\rho_a(t)$. To see how things work out, we recall the conclusions of the previous section on the analytic properties of the perturbative Kraus matrix. From this material it may be inferred that the matrix elements $\check{W}_0'(\omega+\mathrm{i}0)_{(\epsilon_1,n_1+1)(\epsilon_2,n_2)}$ are smooth and bounded. By an argument of Riemann–Lebesgue type one subsequently proves that the matrix elements ${}_a\langle 2|\,\rho_a(t)|2\rangle_a$ and ${}_a\langle 2|\,\rho_a(t)|1\rangle_a$ tend to zero for large times. The asymptotic behaviour of the other diagonal matrix element is determined by the Kraus matrix $\check{W}_0'(\omega+\mathrm{i}0)_{(+1,-1)(+1,-1)}$. It contains a pole at $\omega = \Omega_{+1,-1}$ which generates a contribution that does not depend on time. In view of trace conservation this contribution must be equal to one.

The asymptotic decay of (55) can be made more explicit upon taking the weak-coupling limit $g\to\lambda\tilde{g}$, $t\to\tilde{t}/\lambda^{\alpha}$, and $\lambda\to 0$, with $\alpha>2$. By scaling in (55) as $\omega\to\lambda^2\tilde{\omega}+\Omega_{\epsilon_1,n_1+1}$ and $\omega'\to\lambda^2\tilde{\omega}'+\Omega_{\epsilon_1',n_1+1}$ we eliminate fast Rabi oscillations from the exponential factor. Next, the forms $\lambda^2\check{W}_0'(\lambda^2\tilde{\omega}+\Omega_{\epsilon_1,n_1+1}+\mathrm{i}0)_{(\epsilon_1,n_1+1)(\epsilon_2,n_2)}$ and $\lambda^2\bar{W}_0'(\lambda^2\tilde{\omega}'+\Omega_{\epsilon_1',n_1+1}-\mathrm{i}0)_{(\epsilon_1',n_1+1)(\epsilon_2',n_2')}$ can be evaluated with the help of (53) and the limit

$$
\begin{aligned}
\lim_{\lambda\to 0}&\lambda^2\check{W}_0'(\lambda^2\tilde{\omega}+\Omega_{\epsilon_1,n_1}+\mathrm{i}0)_{(\epsilon_1,n_1)(\epsilon_2,n_2)} \\
&= \delta_{\epsilon_1,\epsilon_2}\delta_{n_1,n_2}\left[\tilde{\omega}+\sideset{}{'}\sum_{\epsilon}\tfrac{1}{2}\theta_{n_2}\nu_{n_2-1}^2\int_0^{\infty}\mathrm{d}\omega\frac{|\tilde{g}(\omega)|^2}{\omega+\Omega_{\epsilon,n_2-1}-\Omega_{\epsilon_2,n_2}-\mathrm{i}0}\right]^{-1},
\end{aligned}
\tag{57}
$$

where the prime indicates that terms containing $\Omega_{-1,-1}$ must be excluded from the summation. One can verify the above result through iterating (52) and dropping terms of order λ^2. Inverting the right-hand side of (52) via the expansion $(A+B)^{-1}=A^{-1}-A^{-1}BA^{-1}+\cdots$, one recognizes that (57) is indeed a diagonal matrix.

For $n_2\geqslant 0$ and λ small, the Kraus matrix (57) generates exponential decay to zero in (55). This is due to a pole lying below the real axis of the complex $\tilde{\omega}$ plane. To ascertain the location of this dissipative pole one should recognize that the difference $\Omega_{\epsilon_2,n_2}-\Omega_{\epsilon,n_2-1}=\omega_f+\epsilon_2 f(n_2+1)^{1/2}-\epsilon f(n_2)^{1/2}$ is positive for $n_2\geqslant 0$, $\epsilon=-1$, and
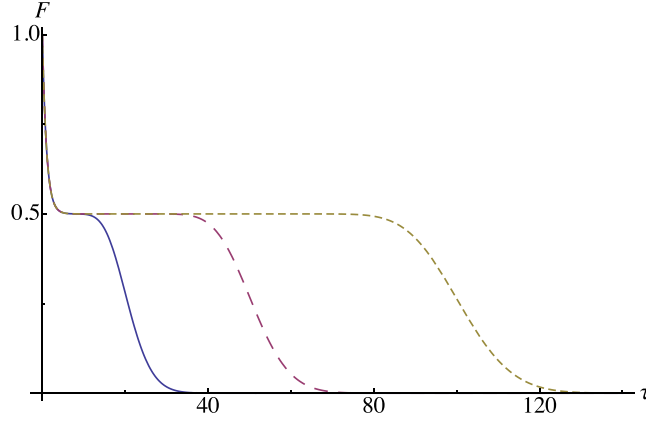
Figure 1. Plot of evolution function $F(\tau)$ defined in (58) for $\rho_{a,22} = 1$ and $p = 20$ (—), $p = 50$ (– – –), and $p = 100$ (- - -).

$\omega_f > f$. The afore-mentioned dissipative pole is also found upon taking the weak-coupling limit of the matrix element $_a\langle 2| \rho_a(t)|1\rangle_a$. Therefore, in the weak-coupling regime the evolution of the atomic density matrix is in tune with the limit (56).

In appendix B we demonstrate that for small coupling parameter g and large initial photon number p the evolution (55) can be approximated as

$$_a\langle 2| \rho_a(t)|2\rangle_a \simeq F(\tau) \equiv \rho_{a,22} \exp(-\tau) + \tfrac{1}{2} \exp(-\tau) \sum_{r=1}^{p} (1 - \rho_{a,11}\delta_{r,p}) \frac{\tau^r}{r!} \ , \tag{58}$$

with the scaled time given by $\tau = \pi|g(\omega_f)|^2 t/2$. Surprisingly, for times of order $p/(\pi|g(\omega_f)|^2)$ the existence of the limit (56) is not reflected in the atomic evolution at all. Figure 1 shows that during the afore-mentioned time span the diagonal element (58) takes on the value of 1/2. Only for times larger than $p/(\pi|g(\omega_f)|^2)$ exponential decay to the ground state sets in.

From (44) we see that the lifetime τ_e of the excited level of the free atom is equal to $1/(2\gamma)$. Since ω_f equals $\omega_{(21)}$ the plateau with $F(\tau) = 1/2$ thus appears at times of order $2p\tau_e$. The initial photon number p may be chosen arbitrarily large, so there is good reason to believe that in the presence of initial correlations between system and reservoir the behaviour depicted in figure 1 will remain intact. This assertion is based on the expectation that assumption (ii) of the Introduction affects the evolution of the system up to a certain number of lifetimes τ_e and not at all times.

As the off-diagonal elements of the atomic density matrix are found to decay to zero for times of order $|g(\omega_f)|^{-2}$, the plateau of figure 1 with $_a\langle 2| \rho_a(t)|2\rangle_a = 1/2$ corresponds to the state of maximum entropy. The atomic propensity towards maximum entropy is corroborated by the limit

$$\lim_{\lambda \to 0} \rho_a(g = \lambda\tilde{g}, p = \tilde{p}/\lambda^\beta, t = \tilde{t}/\lambda^\alpha) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \ , \tag{59}$$

where the variables $\tilde{g}$, $\tilde{p}$, and $\tilde{t}$ must be kept constant, with $\tilde{p}/\lambda^\beta$ being of integer value. Furthermore, the conditions $2 < \alpha < \beta + 2$ and $0 < \beta < 4/3$ must be satisfied. A proof of the above limit is outlined in appendix B.

In this section, we have obtained a solution of non-Markovian character for the Jaynes–Cummings model with atomic damping by a transverse radiation field. Working at zero

temperature, we have uncovered an analytical structure consisting of products of matrix continued fractions. Conservation of positivity and probability has been taken care of from the very outset. The non-Markovian evolution has been examined by performing analytical work, that is to say, by taking suitable limits. Both the ground state and the state of maximum entropy have been identified as attractor of the dissipative dynamics.

6. Conclusion

One of the main objectives of the theory of open quantum systems consists of describing, predicting and understanding experimental observations. A fundamental and well-known method to fulfil this ambition comprises the derivation and analysis of master equations for density operators. Unfortunately, it has become apparent, in particular from the recent literature [7, 8, 11], that the mathematics involved is very complicated, giving rise to virtually insurmountable technical barriers if one wishes to obtain exact non-Markovian results. For that reason, we decided some time ago to shift our attention from master equations to Kraus maps for evolutions of density matrices, opting for the system-reservoir setting as outlined in the Introduction.

In the absence of initial correlations between system and reservoir, any dissipative evolution of a state ρ_S in time t is governed by the Kraus map $\sum_{j=1}^{\infty} W_j(t)\rho_S W_j^{\dagger}(t)$ [29]. Evaluation of the Kraus matrices $\{W_j(t)\}_j$ requires us to solve a time-ordering problem [36], a most complex task [40]. Indeed, as it turns out in section 3 the exact solution for each Kraus matrix is given by an infinite continued fraction of a disturbingly difficult structure. This outcome most likely impedes any possibility of analytically performing the infinite summation figuring in the above Kraus map. We therefore corroborate conclusions from the recent articles cited above indicating that an exact description of dissipative dynamics by a finite set of evolution equations probably lies beyond our reach.

In the Kraus map partial summation of terms becomes feasible as soon as one resorts to perturbation theory. This happens by carrying out factorizations such that the expansion for the density matrix is chopped up into bunches of N terms. Owing to the self-similarity of the ensuing perturbative expansion, the possibility of deriving a finite set of evolution equations does exist now for arbitrary perturbative order. As shown in section 4, this set consists of a finite Kraus hierarchy as well as an integral equation for a bitemporal matrix. The latter gives us direct access to the perturbative density matrix, the positivity and trace of which are conserved in time. In the lowest perturbative order, the dissipative dynamics depends on one Kraus matrix only. Its Laplace transform has neat analytical properties and can be represented by a much simpler continued fraction as compared to the non-perturbative case. As discussed in [36], the exact density matrix and the perturbative density matrix coincide for large times and arbitrary order N.

The work presented in section 3 demonstrates that for short times, when transients from the initial state and memory effects from the reservoir still have a large impact, exact evaluation of dissipative Kraus maps will demand excessive efforts. In fact, one may doubt whether the concept of reduced dynamics, i.e. embedding all equations of motion in the system's Hilbert space, is the optimal starting-point for studying early stages of dissipative dynamics. At the same time, our caveat does not imply that system-reservoir theory should be completely abandoned. On the contrary, outside the short-time regime the perturbative tools developed in section 4 can be put to use. This opens up the possibility of thoroughly examining non-Markovian evolutions towards thermal equilibrium by analytical means.

To illustrate the last remark, we show in section 5 that the lowest order perturbative density matrix of the Jaynes–Cummings model with non-Markovian radiative damping can be readily computed. In order to keep formulas as concise as possible, we assume zero temperature and discard counter-rotating contributions in the Hamiltonian. The evolution of the two-level atom can be analytically explored, namely by taking suitable asymptotic limits. We predict that the atom may remain in the state of maximum entropy for a significant time span that depends on the initial energy of the resonant radiation field as well as the energy loss to the transverse radiation field.

Within the framework of Markovian dynamics, it has been shown [41] that the role of the state of maximum entropy as intermediate attractor occurs for a large class of initial states, including the case that the radiation field starts from a coherent state. Hence, experimental observation of the plateau depicted in figure 1 might be feasible. A setup is required for which an atomic evolution can be monitored during a number of lifetimes of the excited level of the free atom that is of the same order as twice the initial average photon number of the radiation field.

Appendix A. Conservation of probability

We set out to prove that the derivative $\sum_{k_1} \mathrm{d}\langle k_1|\rho_S^{(N)}(t)|k_1\rangle/\mathrm{d}t$ as determined by (36) is vanishing. The proof for the case $N = 1$ was given in [6], so we assume $N \geqslant 2$.

We shall need partial derivatives of the Kraus matrices $\{W'_q\}_{q=0}^{N-1}$ with respect to t. From (25) we obtain the identity

$$
\frac{\partial}{\partial t} W'_q(t; T_q)_{K_{q+1}L_{q+1}}
$$
$$
= -\sum_{j=1}^{q+1} \sum_{klm} \int_{t_j}^{t_{j-1}} \mathrm{d}u \exp[\mathrm{i}\omega_{(k_1k)}t] W'_{q+1}(t; T_{j-1}, u, T_q^{j-1})_{(kK_j^1 mK_{q+1})(L_{j-1}lL_{q+1}^{j-1})} c_{(k_1k)(lm)}(t,u) \ ,
$$
(A.1)

where $t_0 = t$ and $t_{q+1} = 0$ must be substituted. Here and in the following, for W'_N the factorized form (34) must be inserted.

For the time being, we focus on the first and second contribution on the right-hand side of (36). The time derivative that must be evaluated is given by

$$
D_1 = \frac{\partial}{\partial t} \sum_{k_1 k'_1 l_1 l'_1} W'_0(t)_{(k_1)(l_1)} \langle l_1|\rho_S(0)|l'_1\rangle W'^{\dagger}_0(t)_{(l'_1)(k'_1)} \delta_{k_1 k'_1}
$$
$$
+ \frac{\partial}{\partial t} \sum_{q=1}^{N-1} \sum_{k_1 k'_1 l_1 l'_1} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t'_1 \ \Delta'_q[\psi](t,t;t_1,t'_1;0,0)_{(k_1 l_1)(k'_1 l'_1)} \delta_{k_1 k'_1} \ .
$$
(A.2)

The first term on the right-hand side can be handled with the help of the choice $q = 0$ in (A.1). In order to avoid any ambiguity in applying definition (22), traces are taken by performing a sum over k_1 and k'_1 with $\delta_{k_1 k'_1}$ as a weight. Defining

$$
D_2 = \sum_{q=1}^{N-1} \sum_{k_1 k'_1 l_1 l'_1} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t'_1 \ \frac{\partial}{\partial t} \Delta'_q[\psi](t,t;t_1,t'_1;0,0)_{(k_1 l_1)(k'_1 l'_1)} \delta_{k_1 k'_1} \ ,
$$

$$D_3 = \sum_{q=2}^{N-1} \sum_{k_1 k_1' l_1 l_1'} \int_0^t dt_1' \ \Delta_q'[\psi](t,t;t_1,t_1';0,0)_{(k_1 l_1)(k_1' l_1')}\Big|_{t_1=t} \ \delta_{k_1 k_1'} \ ,$$

$$D_4 = \sum_{q=2}^{N-1} \sum_{k_1 k_1' l_1 l_1'} \int_0^t dt_1 \ \Delta_q'[\psi](t,t;t_1,t_1';0,0)_{(k_1 l_1)(k_1' l_1')}\Big|_{t_1'=t} \ \delta_{k_1 k_1'} \ , \qquad \text{(A.3)}$$

we can express the result for the time derivative (A.2) as $D_1 = D_2 + D_3 + D_4$.

Upon employing (A.1) in D_2 one obtains

$$D_2 = -\sum_{q=1}^{N-1} \sum_{K_{q+2} L_{q+2} K_{q+2}' L_{q+2}'} \int_0^t dT_{q+1} \int_0^t dT_{q+1}'^1$$

$$\times \ W_{q+1}'(t;T_{q+1})_{K_{q+2} L_{q+2}} \psi(0,0)_{(l_{q+2})(l_{q+2}')} W_{q+1}'^\dagger(t;t,T_{q+1}'^1)_{L_{q+2}' K_{q+2}'} \quad \text{(A.4)}$$

$$\times \sum_{PQ} \sum_{j=1}^{q+1} \frac{1}{q!} \prod_{h=1}^q \{\tilde{Q}(h)' \ \tilde{P}(h)\} \ \{1' \ j\}|_{t_1'=t} \ \delta_{k_1 k_1'} + \text{c.c.} \ ,$$

where one must substitute $\tilde{P}(h) = P(h) + \sum_{i=j}^q \delta_{P(h),i}$ and $\tilde{Q}(h) = Q(h) + 1$. The symbols P and Q denote permutations of the integers $\{1,2,\ldots,q\}$. Variables have been transformed as $u \to t_j$, $m \to k_{j+1}$, $l \to l_j$, $t_i \to t_{i+1}, k_{i+1} \to k_{i+2}, l_i \to l_{i+1}$ for $i = j, j+1, \ldots, q$ and $t_i' \to t_{i+1}', k_{i+1}' \to k_{i+2}', l_i' \to l_{i+1}'$ for $i = 1,2,\ldots,q$. Also, the shifts $k \to l_1'$, $l_{q+1} \to l_{q+2}$, and $l_{q+1}' \to l_{q+2}'$ have been performed. Identity (25) has been applied to $W_{q+1}'^\dagger$, with t_1' set equal to t.

In (A.4) a combinatorial sum over products of correlation functions is performed. Employing Kronecker deltas we extend the permutations P en Q with the elements $P(q+1)$ and $Q(q+1)$, respectively. The ensuing expression can be simplified by means of the identity

$$\sum_{PQ} \sum_{j=1}^{q+1} \prod_{h=1}^q \{\tilde{Q}(h)' \ \tilde{P}(h)\}\{1' \ j\} \delta_{Q(q+1),q+1} \delta_{P(q+1),q+1}$$

$$= \frac{1}{q+1} \sum_{PQ} \prod_{h=1}^{q+1} \{Q(h)' \ P(h)\} \ , \qquad \text{(A.5)}$$

which can be verified by inspection. On both sides of (A.5) we sum over all permutations P and Q of the integers $\{1,2,\ldots,q+1\}$.

Owing to (A.5) we can phrase (A.4) in the following concise manner

$$D_2 = -\sum_{q=2}^{N} \sum_{k_1 k_1' l_1 l_1'} \int_0^t dt_1 \ \Delta_q'[\psi](t,t;t_1,t_1';0,0)_{(k_1 l_1)(k_1' l_1')}\Big|_{t_1'=t} \ \delta_{k_1 k_1'} + \text{c.c.} \ .$$

$$\text{(A.6)}$$

Note that first one should perform all permutations contained in Δ_q' and then make the substitution $t_1' = t$. In the c.c. term one should make the substitution $t_1 = t$.

Except for the term with $q = N$ the right-hand side of (A.6) cancels out against the sum of D_3 and D_4. Therefore, the desired derivative D_1 attains the form

$$D_1 = -\sum_{k_1 k_1' l_1 l_1'} \int_0^t dt_1 \ \Delta_N'[\psi](t,t;t_1,t_1';0,0)_{(k_1 l_1)(k_1' l_1')}\Big|_{t_1'=t} \ \delta_{k_1 k_1'} + \text{c.c.} \ . \quad \text{(A.7)}$$

In making Δ_N' explicit one should employ the truncation prescription (34).

The derivatives of the third and fourth term on the right-hand side of (36) can be treated in a similar manner as explained above. Abbreviating these contributions as

$$
\begin{aligned}
D_5 = &\frac{\partial}{\partial t} \sum_{k_1 k'_1 l_1 l'_1} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \exp[\mathrm{i}\omega_{k_1} s - \mathrm{i}\omega_{k'_1} s'] \\
&\times W'_0(t-s)_{(k_1)(l_1)} \chi^{(N)}(s,s')_{(l_1)(l'_1)} W_0'^{\dagger}(t-s')_{(l'_1)(k'_1)} \delta_{k_1 k'_1} \\
&+ \frac{\partial}{\partial t} \sum_{q=1}^{N-1} \sum_{k_1 k'_1 l_1 l'_1} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \int_s^t \mathrm{d}t_1 \int_{s'}^t \mathrm{d}t'_1 \exp[\mathrm{i}\omega_{k_1} s - \mathrm{i}\omega_{k'_1} s'] \\
&\times \Delta'_q[\chi^{(N)}](t,t;t_1,t'_1;s,s')_{(k_1 l_1)(k'_1 l'_1)} \delta_{k_1 k'_1},
\end{aligned}
\tag{A.8}
$$

we obtain

$$
\begin{aligned}
D_5 = &\sum_{l_1 l'_1} \int_0^t \mathrm{d}s' \exp[\mathrm{i}\omega_{l_1}(t-s')]\chi^{(N)}(t,s')_{(l_1)(l'_1)} W_0'^{\dagger}(t-s')_{(l'_1)(l_1)} \\
&- \sum_{k_1 k'_1 l_1 l'_1} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \int_s^t \mathrm{d}t_1 \exp[\mathrm{i}\omega_{k_1} s - \mathrm{i}\omega_{k'_1} s'] \\
&\times \Delta'_N[\chi^{(N)}](t,t;t_1,t'_1;s,s')_{(k_1 l_1)(k'_1 l'_1)} \Big|_{t'_1=t} \delta_{k_1 k'_1} + \text{c.c.} .
\end{aligned}
\tag{A.9}
$$

The complex conjugate of the two terms on the right-hand side must be added and in the second complex conjugate $t_1 = t$ must be chosen.

Upon combining definition (22) with prescription (34) we see that on the right-hand side of (A.9) evolution equation (37) can be invoked. Then (A.9) simplifies to the form

$$
D_5 = \sum_{k_1 k'_1 l_1 l'_1} \int_0^t \mathrm{d}t_1 \ \Delta'_N[\psi](t,t;t_1,t'_1;0,0)_{(k_1 l_1)(k'_1 l'_1)} \Big|_{t'_1=t} \delta_{k_1 k'_1} + \text{c.c.} .
\tag{A.10}
$$

Addition of (A.7) and (A.10) yields zero. This completes the proof.

Appendix B. Limit of maximum entropy

In demonstrating the validity of the limit (59) we shall interchange limits and infinite sums without proof. A Markovian counterpart of (59) was rigorously proved in [41].

By performing the scaling $g \to \lambda\tilde{g}$, $p \to \tilde{p}/\lambda^{\beta}$, $t \to \tilde{t}/\lambda^{\alpha}$, $\omega \to \lambda^2\tilde{\omega} + \Omega_{\epsilon_1,n_1+1}$, and $\omega' \to \lambda^2\tilde{\omega}' + \Omega_{\epsilon'_1,n_1+1}$, with $\alpha, \beta > 0$, we eliminate fast Rabi oscillations from the exponential factor of (55) and pave the way for use of (57). With the help of the identity

$$
\Omega_{\epsilon,n-1} - \Omega_{\epsilon',n} = -\omega_f + \epsilon f(n)^{1/2} - \epsilon' f(n+1)^{1/2}
\tag{B.1}
$$

the denominator of (57) can be elaborated.

Upon inserting (48) into (55) one recognizes that integer n is of order $\tilde{p}/\lambda^{\beta}$. For the choice $\epsilon = \epsilon'$ the right-hand side of (B.1) converges to $-\omega_f$ as λ becomes small. In contrast, the choice $\epsilon = -\epsilon'$ makes a contribution to (57) that decays as $\lambda^{\beta/2}$. For large n and n' we thus arrive at

$$
\lim_{\lambda \to 0} \lambda^2 \check{W}'_0(\lambda^2\tilde{\omega} + \Omega_{\epsilon,n} + \mathrm{i}0)_{(\epsilon,n)(\epsilon',n')} = \delta_{\epsilon,\epsilon'} \delta_{n,n'} (\tilde{\omega} + \mathrm{i}\Gamma + \bar{\omega}_f)^{-1} ,
\tag{B.2}
$$

with Γ and $\bar{\omega}_f$ given by $\pi|\tilde{g}(\omega_f)|^2/4$ and $(1/4)\mathcal{P}\int_0^\infty d\omega|\tilde{g}(\omega)|^2/(\omega-\omega_f)$, respectively.

Transforming integral dummies as $\omega_s^+ \to \lambda^2\tilde{\omega}_s + \Omega_{\epsilon_{2s+1},n_{s+1}+1} - \Omega_{\epsilon_1,n_1+1}$ for $1 \leqslant s \leqslant r$, we encounter the forms $\lambda^2\check{W}_0'(\lambda^2\tilde{\omega} + \lambda^2\tilde{\omega}_s + \Omega_{\epsilon_{2s+1},n_{s+1}+1} + i0)_{(\epsilon_{2s+1},n_{s+1}+1)(\epsilon_{2s+2},n_{s+2})}$, with $0 \leqslant s \leqslant r$ and $\tilde{\omega}_0 = 0$. For small λ, these can be computed on the basis of (B.2), whereafter the conditions $\epsilon_{2s+1} = \epsilon_{2s+2}$ and $n_{s+1} + 1 = n_{s+2}$ appear, with $0 \leqslant s \leqslant r$.

By (53) and (B.2) the matrix $\lambda^2\bar{W}_0'(\lambda^2\tilde{\omega}' + \Omega_{\epsilon_1',n_1+1} - i0)_{(\epsilon_1',n_1+1)(\epsilon_2',n_2')}$ gives rise to the conditions $\epsilon_1' = \epsilon_2'$ and $n_1 + 1 = n_2'$, with λ small. Hence, the $r = 0$ term of (55) converges to $\rho_{a,22}\exp[-2\Gamma\tilde{t}\lambda^{2-\alpha}]$. For $r > 0$ we meet matrices $\bar{W}_0'$ containing the sum

$$\Phi_s = \Omega_{\epsilon_1',n_1+1} - \Omega_{\epsilon_1,n_1+1} - \Omega_{\epsilon_{2s+1}',n_{s+1}'+1} + \Omega_{\epsilon_{2s+1},n_{s+1}+1} \; , \tag{B.3}$$

with $1 \leqslant s \leqslant r$. As long as Φ_s decays slower than λ^2 for $\lambda \to 0$, the limit

$$\lim_{\lambda\to 0} \lambda^2\bar{W}_0'(\Phi_s + \lambda^2\tilde{\omega}' + \lambda^2\tilde{\omega}_s + \Omega_{\epsilon_{2s+1}',n_{s+1}'+1} - i0)_{(\epsilon_{2s+1}',n_{s+1}'+1)(\epsilon_{2s+2}',n_{s+2}')} = 0 \tag{B.4}$$

is true. Hence, finite contributions to (55) arise only if Φ_s decays faster than λ^2.

Obviously, the condition $\Phi_s = 0$ is sufficient. As ω_f and f are independent, it is equivalent to $n_{s+1} = n_{s+1}'$ and $(\epsilon_1 - \epsilon_1')(n_1 + 2)^{1/2} = (\epsilon_{2s+1} - \epsilon_{2s+1}')(n_{s+1} + 2)^{1/2}$, with $1 \leqslant s \leqslant r$. For $\epsilon_1 \neq \epsilon_1'$ the last relation gives $n_1 = n_{s+1}$, an outcome that is contradictory to the identity $n_1 + 1 = n_2$ derived earlier. Hence, for $1 \leqslant s \leqslant r$ we find $n_{s+1} = n_{s+1}'$ and $\epsilon_{2s+1} = \epsilon_{2s+1}'$. One verifies that for $\beta < 4$ these choices are not only sufficient but necessary as well. Once Φ_s has disappeared, the Kronecker deltas of (B.2) give rise to the equalities $\epsilon_{2s+1}' = \epsilon_{2s+2}'$ and $n_{s+1}' + 1 = n_{s+2}'$, with $1 \leqslant s \leqslant r$.

Using (48) we find that for $r > 0$ the initial matrix element of (55) reduces to

$$\langle\epsilon_{2r+2},n_{r+2}|\rho_S(0)|\epsilon_{2r+2}',n_{r+2}'\rangle = \tfrac{1}{2}(\rho_{a,11}\delta_{n_{r+2},p-1} + \rho_{a,22}\delta_{n_{r+2},p}) \; . \tag{B.5}$$

For $1 \leqslant s \leqslant r + 2$ and $\rho_{a,22} = 1$ we must choose $n_s = n_s' = p + s - r - 2$, so that the condition $r \leqslant p$ emerges. Of course, for $\rho_{a,11} = 1$ the replacement $p \to p - 1$ must be carried out in n_s and n_s'. For small λ the coupling constants of (55) attain the form

$$\tilde{g}(\omega_s) = \tilde{g}(\omega_f - \epsilon_{2s-1}f[n_1 + s + 1]^{1/2} + \epsilon_{2s+1}f[n_1 + s + 2]^{1/2}) \; . \tag{B.6}$$

Since n_1 is of order $\tilde{p}/\lambda^\beta$, the argument of $\tilde{g}$ diverges as $\lambda^{-\beta/2}$. From the assumption $\tilde{g}(\omega) \to 0$ for $\omega \to \infty$ it thus follows that the choice $\epsilon_{2s-1} = \epsilon_{2s+1}$ is compulsory. Then for all coupling constants the finite value $\tilde{g}(\omega_f)$ is found.

For the integrals over $\{\tilde{\omega}_s\}_{s=1}^r$ the domain of integration is determined by

$$\lambda^2\tilde{\omega}_s > \lambda^2\tilde{\omega}_{s-1} - \Omega_{\epsilon_{2s+1},n_{s+1}+1} + \Omega_{\epsilon_{2s-1},n_s+1} \; , \tag{B.7}$$

with $1 \leqslant s \leqslant r$ and $\tilde{\omega}_0 = 0$. For λ tending to zero (B.7) boils down to $\tilde{\omega}_s > \tilde{\omega}_{s-1} - \omega_f/\lambda^2$ or $\tilde{\omega}_s > -\infty$. Hence, all integrals over $\tilde{\omega}_s$ can be computed with the help of the residue theorem. For the remaining integrals over $\tilde{\omega}$ and $\tilde{\omega}'$ the transformation $x = \tilde{\omega} - \tilde{\omega}'$ can be performed. If λ is small this brings us to

$$\begin{aligned}
{}_a\langle 2|\, \rho_a(g=\lambda\tilde{g}\,,p=\tilde{p}/\lambda^\beta\,,t=\tilde{t}/\lambda^\alpha)|2\rangle_a &\to \rho_{a,22}\exp[-2\Gamma\tilde{t}\lambda^{2-\alpha}] \\
&+ \frac{i}{4\pi}\sum_{r=1}^p (1 - \rho_{a,11}\delta_{r,p})\int_{-\infty}^\infty dx\exp[-ix\tilde{t}\lambda^{2-\alpha}]\frac{(2i\Gamma)^r}{(x+2i\Gamma)^{r+1}} \; .
\end{aligned} \tag{B.8}$$

For $\alpha > 2$ use of the residue theorem leads to the evolution (58) appearing in the main text. Upon combining a standard formula [42] with the condition $\alpha < \beta + 2$ we obtain from (B.8) the desired result of 1/2 for λ small.

Performing the same scaling as above, one shows that $_a\langle 2|\, \rho_a(t)|1\rangle_a$ converges to zero in the limit (59). In (B.4) the sum Φ_s must be exchanged for

$$\Phi'_s = \Omega_{\epsilon'_1, n_1+1} - \Omega_{\epsilon_1, n_1} - \Omega_{\epsilon'_{2s+1}, n'_{s+1}+1} + \Omega_{\epsilon_{2s+1}, n_{s+1}+1} \ , \tag{B.9}$$

with $1 \leqslant s \leqslant r$. All Kraus matrices $\overline{W}'_0$ figuring in $_a\langle 2|\, \rho_a(t)|1\rangle_a$ tend to zero if Φ'_s decays slower than λ^2 for $\lambda \to 0$. From (46) one obtains $\Phi'_s = (\epsilon'_1 - \epsilon_1)\mathcal{O}(\lambda^{\beta/2}) + \mathcal{O}(\lambda^{3\beta/2})$. Under the choice $\epsilon'_1 = \epsilon_1$ the condition $\beta < 4/3$ is found.

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