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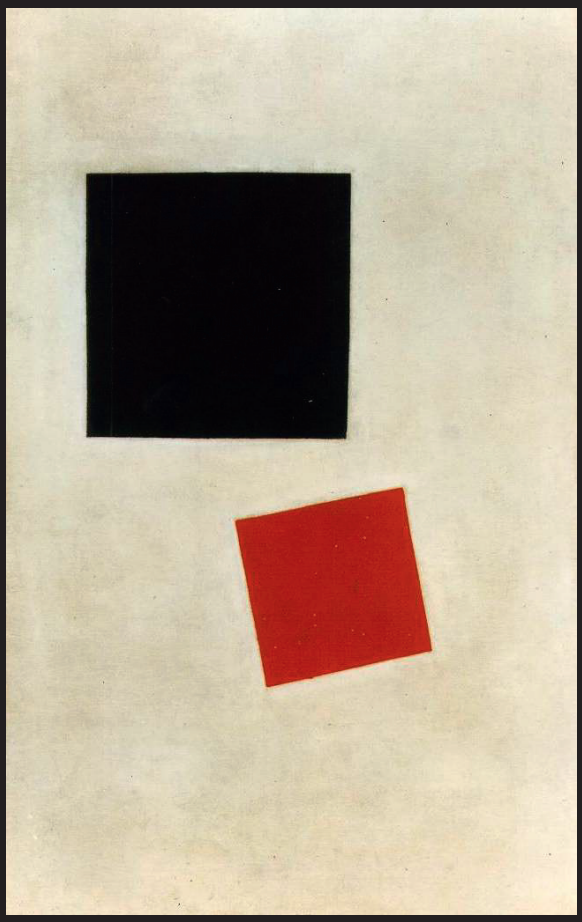

Nikolaos Rekatsinas

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica

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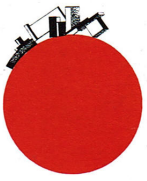
The illustration of the cover, and that of the dedication page is composed with parts of *Lissitzky*'s book for children of all ages, *Pro dva kvadrata* (About two squares).

«Και πρώτα απ' όλα τι εννοούμε λέγοντας παιδεία; Την πληροφορία, την τεχνική, το δίπλωμα εξειδίκευσης που εξασφαλίζει γάμο, αυτοκίνητο κι ακίνητο, με πληρωμή την πλήρη υποταγή του εξασφαλισθέντος ή την πνευματική και ψυχική διάπλαση ενός ελεύθερου ανθρώπου, με τεχνική αναθεώρησης κι ονειρικής δομής, με αγωνία απελευθέρωσης και με διαθέσεις μιας ιπτάμενης φυγής προς τ' άστρα;»

Μάνος Χατζιδάκις



"But first of all, what do we mean by education? The information, the technique, or the diploma of specialisation, being potentially the guarantee of a marriage, movable and immovable property, that comes along with the price of full subordination of the person seeking this kind of security? Or, the spiritual and mental structure of a free man who practices the art of revision, and composed by the very essence of dreams craves after the agony of liberation for a flying escape to the stars?"

Manos Hadjidakis

Contents

1. Introduction

In principle, partial differential equations (boundary value problems) emanate from the mathematical description of phenomena or processes appearing in a great variety of topics in physics, mechanics, engineering, chemistry, biology, or even finance. The complexity of the resulting mathematical problems does not allow for an *analytical solution* following classical analytical techniques. Thus, the development of (advanced) numerical approximation techniques for the solution of these problems has been, in modern times, the dominant approach to meet the ever-rising demands of mathematical applications to real-world problems.

1.1 The numerical solution of operator equations

A *boundary value problem* can typically be formulated as a well-posed *operator equation* in the setting of Hilbert spaces. This means, one may rewrite the problem equivalently as

$$Au = f, \tag{1.1.1}$$

where $A : \mathscr{H} \to \mathscr{K}'$ is a *boundedly invertible* linear[1] operator, i.e. $A \in \mathcal{L}\mathrm{is}(\mathscr{H}, \mathscr{K}')$, with $\mathscr{H}$, $\mathscr{K}$ being Hilbert spaces over $\mathbb{R}$, usually Sobolev spaces on a domain $\Omega \subset \mathbb{R}^n$, and right hand side $f \in \mathscr{K}'$. Moreover, a correspondence of the operator A with a *bilinear* form $a(\cdot, \cdot)$ on $\mathscr{H} \times \mathscr{K}$ can be achieved by the relation $a(u, v) = (Au)(v)$, so that (1.1.1) is equivalent to

$$a(u, v) = f(v), \quad \forall v \in \mathscr{K}. \tag{1.1.2}$$

Considering finite dimensional subspaces $\mathscr{H}_h \subset \mathscr{H}, \mathscr{K}_h \subset \mathscr{K}$, called the *trial* and *test* spaces, respectively, an approximate solution of (1.1.2) is to find $u_h \in \mathscr{H}_h$ such that

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in \mathscr{K}_h. \tag{1.1.3}$$

This u_h is a so-called *(Petrov-) Galerkin* solution which satisfies the *orthogonality* condition $a(u - u_h, v_h) = 0, \ \forall v_h \in \mathscr{K}_h$.

[1] For simplicity, in this introduction we consider linear operators only although the results extend to certain classes of non-linear operators as well.

In the case when $\mathscr{K} = \mathscr{H}$, $\mathscr{K}_h = \mathscr{H}_h$, and $a(\cdot,\cdot)$ is *elliptic*, i.e., when the symmetric part $\frac{1}{2}(A + A') \in \mathcal{L}\mathrm{is}(\mathscr{H}, \mathscr{H}')$, this Petrov-Galerkin solution, then called *Galerkin* solution, exists and it is a quasi-best approximation to $u \in \mathscr{H}$ from $\mathscr{H}_h$.

When the subspace $\mathscr{H}_h$ is chosen as the space of all piecewise polynomial functions in $\mathscr{H}$ of some specified degree $d-1$ w.r.t. to a partition of Ω into n-simplices we speak about a *finite element method*. Often quasi-uniform partitions are considered. Solving for subsequent uniform dyadically refined meshes, i.e., with a decreasing mesh size h, a basic question on *convergence* is how does the error of the approximation to u reduces as $h \to 0$.

As an indicative example to the latter, if $\mathscr{H}$ is equal to $H^m(\Omega)$, or is a subspace of that space, as with an elliptic boundary value problem of order $2m$, and u is approximated by piecewise polynomials of degree $d-1$ with $d > m$, then if $u \in H^d(\Omega)$, one has an error of order $N^{-\frac{d-m}{n}}$, where $N := \dim \mathscr{H}_h$. In other words,

$$\|u - u_h\|_{H^m(\Omega)} \lesssim N^{-\frac{d-m}{n}} \|u\|_{H^d(\Omega)}, \tag{1.1.4}$$

where '$\lesssim$' denotes inequality modulo some constant factor. The fact that the *rate of convergence* $\frac{d-m}{n}$ is inversely proportional to n is known as the *curse of dimensionality*.

In order to realise the convergence rate $\frac{d-m}{n}$, we assumed that $u \in H^d(\Omega)$. In many cases this condition is violated because non-smooth parts of the boundary (corners, edges), or non-smooth coefficients in the partial differential operator, or a non-smooth forcing function f cause singularities in the solution. In these cases, the consequences of reduced regularity of the solution can be circumvented by enabling *appropriate local refinements* of the partition that underlies the trial space. Since a priori the solution is unknown, the question is how to find such partitions.

Adaptive finite element methods perform a repetition of the following loop: Given a partition, determine the Galerkin solution associated to the current partition. Compute an a posteriori error estimator, being a sum of local error indicators associated to the individual simplices. Refine those simplices that are associated to the largest indicators, and repeat. This adaptive method has been shown to give the best possible convergence rate s in terms of the number of simplices N among essentially all possible partitions of the domain into simplices. Moreover, under mild conditions, this rate s is equal to the best possible rate $\frac{d-m}{n}$ known for smooth solutions. These results, however, hold true for *elliptic problems* only.

Our focus in this thesis will be on *adaptive wavelet methods* for solving operator equations as originally introduced in [CDD01, CDD02] and further developed in, e.g., [GHS07, Ste14]. In particular since we consider wavelets from finite element spaces, these methods have quite some similarities to adaptive finite element methods. An important difference is, however, that proofs of optimality of adaptive wavelet methods will *not* be restricted to elliptic problems.

1.1.1 Adaptive Wavelet Galerkin Methods

Considering the operator equation (1.1.1) in the previous section, we equip the (separable) Hilbert spaces $\mathscr{H}, \mathscr{K}$ with Riesz bases $\Psi^{\mathscr{H}} = \{\psi^{\mathscr{H}}_\lambda : \lambda \in \vee_{\mathscr{H}}\}$ and $\Psi^{\mathscr{K}} =$

$\{\psi_\lambda^{\mathscr{K}} : \lambda \in \vee_{\mathscr{K}}\}$, respectively.

The collection $\Psi^{\mathscr{H}}$ being a Riesz basis for $\mathscr{H}$ (and similarly $\Psi^{\mathscr{K}}$ being one for $\mathscr{K}$) means that the *analysis operator* $\mathcal{F}_{\mathscr{H}} : g \mapsto [g(\psi_\lambda^{\mathscr{H}})]_{\lambda\in\vee_{\mathscr{H}}}$ is boundedly invertible from the dual $\mathscr{H}'$ to the sequence space $\ell_2(\vee_{\mathscr{H}})$, i.e., it is in $\mathcal{L}\mathrm{is}(\mathscr{H}', \ell_2(\vee_{\mathscr{H}}))$, so that its adjoint, known as the *synthesis operator*, $\mathcal{F}'_{\mathscr{H}} : \mathbf{v} \mapsto \mathbf{v}^\top\Psi^{\mathscr{H}} := \sum_{\lambda\in\vee_{\mathscr{H}}} v_\lambda\psi_\lambda^{\mathscr{H}} \in \mathcal{L}\mathrm{is}(\ell_2(\vee_{\mathscr{H}}), \mathscr{H})$.

We will employ Riesz bases of wavelet type. Considering $\Psi^{\mathscr{H}}$, for some nested sequence $V_0 \subset V_1 \subset \cdots \subset \mathscr{H}$, with $\mathrm{clos}_{\mathscr{H}}(\cup_{j\geq 0}V_j) = \mathscr{H}$, the basis will be a union of a basis for V_0 and bases for suitable complements spaces $V_{j+1} \ominus V_j$, for $j = 0, 1, \ldots$.

Writing the unknown solution u of (1.1.1) as $u = \mathcal{F}'_{\mathscr{H}}\mathbf{u}$, and by pre-'multiplying' the equation with the invertible operator $\mathcal{F}_{\mathscr{K}}$, we arrive at the *equivalent* problem

$$\mathcal{F}_{\mathscr{K}} A\mathcal{F}'_{\mathscr{H}}\mathbf{u} = \mathcal{F}_{\mathscr{K}} f, \tag{1.1.5}$$

being a countable collection of coupled scalar equations, where the bi-infinite *stiffness matrix*

$$\mathbf{A} := \mathcal{F}_{\mathscr{K}} A\mathcal{F}'_{\mathscr{H}} = [a(\psi_\mu^{\mathscr{H}}, \psi_\lambda^{\mathscr{K}})]_{(\lambda,\mu)\in\vee_{\mathscr{K}}\times\vee_{\mathscr{H}}} \in \mathcal{L}\mathrm{is}(\ell_2(\vee_{\mathscr{H}}), \ell_2(\vee_{\mathscr{K}})),$$

and infinite *load vector* $\mathbf{f} := \mathcal{F}_{\mathscr{K}} f = [f(\psi_\lambda^{\mathscr{K}})]_{\lambda\in\vee_{\mathscr{K}}} \in \ell_2(\vee_{\mathscr{K}})$.

To describe the *adaptive wavelet Galerkin method* (**awgm**) for solving (1.1.5), we may restrict ourselves to the case that $\vee := \vee_{\mathscr{H}} = \vee_{\mathscr{K}}$, and $\mathbf{A} = \mathbf{A}^\top$ is *positive definite*. Indeed, if this initially does not hold, then the following can be applied to the *normal* equations

$$\mathbf{A}^\top\mathbf{A}\mathbf{u} = \mathbf{A}^\top\mathbf{f}$$

which always do satisfy these assumptions, although from a quantitative point of view this might be less attractive.

Like the adaptive finite element method, the **awgm** consists of a repetition of the following three steps. Given a finite $\Lambda \subset \vee$, *solve* $\mathbf{u}_\Lambda \in \ell_2(\Lambda) \subset \ell_2(\vee)$ from the Galerkin system

$$\mathbf{A}|_{\Lambda\times\Lambda}\mathbf{u}_\Lambda = \mathbf{f}|_\Lambda.$$

The matrix $\mathbf{A}|_{\Lambda\times\Lambda}$ is symmetric, positive definite, and its condition number is bounded by that of $\mathbf{A}$. Next *compute* the residual $\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda$, whose support is contained in $\vee\setminus\Lambda$. Since $\mathbf{A}$ is boundedly invertible, it holds that $\|\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda\|_{\ell_2(\vee)} \eqsim \|\mathbf{u} - \mathbf{u}_\Lambda\|_{\ell_2(\vee)} (\eqsim \|u - \mathbf{u}_\Lambda^\top\Psi_{\mathscr{H}}\|_{\mathscr{H}})$, meaning that $\|\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda\|_{\ell_2(\vee)}$ can serve as an a posteriori error estimator. Therefore, as the third step, *collect* those indices from $\mathbf{f} - \mathbf{A}\mathbf{u}_\Lambda$ that correspond to its largest entries and add them to Λ, and repeat.

With $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subset \vee$ denoting the sequence generated by this **awgm**, it has been shown that

$$(\|u - \mathbf{u}_{\Lambda_i}^\top\Psi_{\mathscr{H}}\| \eqsim)\, \|\mathbf{u} - \mathbf{u}_{\Lambda_i}\|_{\ell_2(\vee)} \lesssim (\#\Lambda_i)^{-s}$$

for the *best possible* rate $s > 0$ among all such sequences.

The algorithm as presented is not realisable since it involves the evaluation of the generally infinitely supported residuals $\mathbf{f} - \mathbf{A}\mathbf{u}_{\Lambda_i}$. It has, however, been shown

that the optimal rate is still realized with an *approximate evaluation of the residual* within a fixed, sufficiently small relative tolerance. For wavelets that are sufficiently smooth, and have sufficiently many vanishing moments, it has been shown that such an approximate evaluation can be performed at linear cost. Additionally, by iteratively solving the arising Galerkin systems within a fixed, sufficiently small relative tolerance, one arrives at an algorithm that has *optimal computational complexity*.

1.1.2 Contributions from this thesis

1.1.2.1 FOSLS

The routine for the approximate evaluation of the residual that has been developed is based on the observation that although generally each column of $\mathbf{A}$ has infinitely many unknowns, this stiffness matrix is close to being sparse. This in the sense that for the difference in 'levels' of wavelets ψ_λ and ψ_μ tending to infinity, the corresponding entry $\mathbf{A}_{\lambda\mu} = a(\psi_\mu, \psi_\lambda)$ tends to zero, with a rate that depends on the order of the operator, the smoothness of the wavelets, and the number of their vanishing moments. In the approximate matrix-vector multiplication routine, known as the **apply**-routine, columns of $\mathbf{A}$ are approximated within tolerances that depend on the modulus of the corresponding entry in the input vector, meaning that its outcome depends *nonlinearly* on this vector. Although this routine gives rise to a qualitatively satisfactory approximate residual evaluation routine, in a quantitative sense it has turned out to be quite expensive.

Chapter 2 is devoted to the construction of an optimally converging **awgm** whilst avoiding the use of **apply**. The idea is not to split the residual into two terms, the application of the operator to the current approximate solution and the right-hand side, but to approximate it as a whole. By doing so, one benefits from the fact that the residual gets small when the iteration proceeds, whereas the aforementioned terms do not, meaning that approximating each of them separately within a small relative tolerance does not suffice.

Approximating a residual as a whole requires that both its terms can be represented in a common dictionary. Considering continuous piecewise polynomial wavelets and an operator of second order as we will do, the operator applied to a finite linear combination of wavelets yields a distribution, which does not permit the suggested approach.

Therefore, by introducing the flux ∇u as an additional separate variable, we write our second order PDE as a system of PDEs of first order. W.r.t. suitable Hilbert spaces, again denoted as $\mathscr{H}$ and $\mathscr{K}$, the resulting operator $\mathscr{H} \to \mathscr{K}'$, denoted as $\vec{A}$, is a homeomorphism with its range, generally being a proper subspace of $\mathscr{K}'$. Consequently, we consider the well-posed *first order system least squares* (FOSLS) problem of finding $\operatorname{argmin}_{\vec{u}\in\mathscr{H}} \|\vec{A}\vec{u} - \vec{f}\|_{\mathscr{K}'}$. Its solution can be found by solving the Euler-Lagrange equations

$$\langle \vec{A}\vec{v}, \vec{A}\vec{u} - \vec{f}\rangle_{\mathscr{K}'} = 0 \quad \forall \vec{v} \in \mathscr{H}.$$

Since for $g \in \mathscr{K}'$ the dual norm $\|g\|_{\mathscr{K}'}$ can generally not be evaluated, by equipping $\mathscr{K}$ with a (wavelet) Riesz basis $\Psi^{\mathscr{K}}$, we replace it by the *equivalent* norm

$$\|g(\psi_\lambda^{\mathscr{K}})_{\lambda\in\vee_{\mathscr{K}}}\|_{\ell_2(\vee_{\mathscr{K}})}.$$

For background information on the FOSLS approach we refer to [CLMM94, BG09].

The resulting operator equation is now ready for the application of the **awgm**. Regardless of the original PDE, its stiffness matrix is always symmetric and positive definite, so that it is not needed to form normal equations (which actually has already taken place on the 'continuous level'). The reformulation as a first order system allows a proper approximate residual evaluation scheme that now depends *linearly* on the current approximate solution, and that is quantitatively more efficient than by using the **apply** routine. A necessary, but mild condition for this scheme to be applicable is the restriction to subsets $\Lambda \subset \vee$ that form *trees*, meaning that if $\lambda \in \Lambda$ then its natural parent on the next coarser level is included in Λ as well. The whole approach applies equally well to semi-linear PDEs.

Our theoretical findings are supported by numerical experiments.

1.1.2.2 Quadratic wavelets on general polygons

The lowest order of the wavelet basis for the flux variable that is applicable turns out to be two, i.e., piecewise linears. Consequently, to approximate the original variable we need a wavelet basis consisting of piecewise quadratics. In view of applications for solving parabolic PDEs that will be discussed below, we need wavelets that, properly scaled, form a Riesz basis for Sobolev spaces with smoothness index s for s ranging from 1 to -1. Moreover, we need such wavelets on general polygons in $\mathbb{R}^2$.

Available constructions of such wavelets are rare, and they yield either a basis with disappointingly large condition numbers ([DS99c]), or wavelets with very large supports ([NS09]). In Chapter 3, we construct continuous piecewise quadratic wavelets on general polygons that generate Riesz bases for Sobolev spaces with smoothness index $|s| < \frac{3}{2}$, that have two vanishing moments, and that are given as linear combinations of at most 13 nodal basis functions. The condition numbers are of the order 90 for $s = 1$ and 40 for $s = -1$.

1.1.2.3 Adaptive wavelet solution of parabolic evolutionary PDEs

In the recent years there is a growing interest in simultaneous space-time solvers for evolutionary PDEs. The reasons are three-fold. First, standard time-stepping schemes are not very suited to approximate singularities that are local in space and time. Secondly, one aims to use the additional time dimension to enhance the possibilities for a massive parallelization. Thirdly, the product structure of the space-time cylinder allows for a reduction in computational complexity and storage requirements by employing tensor product approximation (in a non-adaptive setting known as *sparse grids*).

Our investigations start with a well-posed simultaneous space-time variational formulation of a parabolic PDE. Then in order to apply our new approximate residual evaluation scheme, we rewrite this problem as a well-posed FOSLS formation. The arising Hilbert spaces are Bochner spaces, or intersections of those. The natural construction of Riesz bases for such spaces is by forming tensor products of temporal and spatial wavelet bases. The application of tensor product bases has the advantage of an effective dimension reduction. The whole time evolution can be approximated at a rate of approximating one stationary problem.

A price to be paid for the use of tensor-product wavelets, however, is that the residual evaluation is more complex. Other than with the usual 'isotropic' wavelets, one cannot transform to a locally single-scale basis to perform the matrix-vector multiplication, since this would destroy the computational complexity. Restricting to index sets Λ that satisfy a double-tree constraint, we design a proper approximate residual evaluation scheme using the unidirectional principle that, in a non-adaptive setting, has been introduced in [BZ96]. The resulting **awgm** solves the parabolic PDE at the best possible rate in linear complexity, albeit at the expense of a rather complicated implementation.

The theoretical findings are illustrated by numerical results that we consider as being very competitive.

1.1.2.4 Details on implementation

In Chapter 5 we provide information on our implementation of the **awgm**. We give details on the wavelet bases that have been used, and describe the implementation of the approximate residual evaluation routine in the isotropic- and tensor-product wavelet case. Information is provided about data structures that have been used, and a number of relevant routines are given in pseudo code.

References

The content of Chapters 2–4, with only small adaptations to increase readability, is essentially that of the following papers. The work presented in these papers is jointly produced with my thesis advisor Rob Stevenson, and two of these are currently under review at scholarly journals.

- N.Rekatsinas and R.P. Stevenson (2018). An optimal adaptive wavelet method for First Order System Least Squares. *Numer. Math.*, doi: 10.1007/s00211-018-0961-7.
- N.Rekatsinas and R.P. Stevenson (2018). A quadratic finite element wavelet Riesz basis. *Int. J. Wavelets Multiresolut. Inf. Process.*, doi: 10.1142/S0219691318500339.
- N.Rekatsinas and R.P. Stevenson (2018). An optimal adaptive tensor product wavelet solver of a space-time FOSLS formulation of parabolic evolution problems. *Submitted.*

2. An optimal adaptive wavelet method for first order system least squares

In this chapter, it is shown that *any* well-posed 2nd order PDE can be reformulated as a well-posed first order least squares system. This system will be solved by an adaptive wavelet solver in optimal computational complexity. The applications that are considered are second order elliptic PDEs with general inhomogeneous boundary conditions, and the stationary Navier-Stokes equations.

2.1 Introduction

In this chapter, a wavelet method is constructed for the optimal adaptive solution of *stationary* PDEs. We develop a general procedure to write *any* well-posed 2nd order PDE as a well-posed first order least squares system. The (natural) least squares formulations contain dual norms, that, however, impose no difficulties for a wavelet solver. The advantages of the first order least squares system formulation are two-fold.

Firstly, regardless of the original problem, the least squares problem is symmetric and positive definite, which opens the possibility to develop an optimal adaptive solver. The obvious use of the least-squares functional as an a posteriori error estimator, however, is not known to yield a convergent method (see, however, [CP15] for an alternative for Poisson's problem). As we will see, the use of the (approximate) residual in wavelet coordinates as an a posteriori error estimator does give rise to an optimal adaptive solver.

Secondly, as we will discuss in more detail in the following subsections, the optimal application of a wavelet solver to a first order system reformulation allows for a simpler, and quantitatively more efficient approximate residual evaluation than with the standard formulation of second order. Moreover, it applies equally well to semi-linear equations, as e.g. the stationary Navier-Stokes equations, and it applies to wavelets that have only *one* vanishing moment.

The approach to apply the wavelet solver to a well-posed first order least squares system reformulation also applies to *time-dependent* PDEs in simultaneous space-time variational formulations, as parabolic problems or instationary Navier-Stokes equations. With those problems, the wavelet bases consist of tensor products of temporal

and spatial wavelets. Consequently, they require a different procedure for the approximate evaluation of the residual in wavelet coordinates, which will be the topic of a forthcoming chapter.

2.1.1 Adaptive wavelet schemes, and the approximate residual evaluation

Adaptive wavelet schemes can solve well-posed linear and nonlinear operator equations at the best possible rate allowed by the basis in linear complexity ([CDD01, CDD02, XZ03, CDD03a, Urb09, Ste09, Ste14]). Schemes with those properties will be called *optimal*. The schemes can be applied to PDEs, which we study in this work, as well as to integral equations ([DHS07]).

There are two kinds of adaptive wavelet schemes. One approach is to apply some convergent iterative method to the infinite system in wavelet coordinates, with decreasing tolerances for the inexact evaluations of residuals ([CDD02, CDD03a]). These schemes rely on the application of coarsening to achieve optimal rates.

The other approach is to solve a sequence of Galerkin approximations from spans of nested sets of wavelets. The (approximate) residual in wavelet coordinates of the current approximation is used as an a posteriori error estimator to make an optimal selection of the wavelets to be added to form the next set ([CDD01]). With this scheme, that is studied in the current work, the application of coarsening can be avoided ([XZ03, GHS07]), and it turns out to be quantitatively more efficient. This approach is restricted to PDOs whose Fréchet derivatives are symmetric and positive definite (compact perturbations can be added though, see [Gan08]).

A key computational ingredient of both schemes is the approximate evaluation of the residual in wavelet coordinates. Let us discuss this for a linear operator equation $Au = f$, with, for some separable Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, for convenience over $\mathbb{R}$, $f \in \mathscr{K}'$ and $A \in \mathcal{L}\mathrm{is}(\mathscr{H}, \mathscr{K}')$ (i.e., $A \in \mathcal{L}(\mathscr{H}, \mathscr{K}')$ and $A^{-1} \in \mathcal{L}(\mathscr{K}', \mathscr{H})$).

Equipping $\mathscr{H}$ and $\mathscr{K}$ with *Riesz bases* $\Psi^{\mathscr{H}}$, $\Psi^{\mathscr{K}}$, formally viewed as column vectors, $Au = f$ can be equivalently written as a bi-infinite system of coupled scalar equations $\mathbf{A}\mathbf{u} = \mathbf{f}$, where $\mathbf{f} = f(\Psi^{\mathscr{K}})$ is the infinite 'load vector', $\mathbf{A} = (A\Psi^{\mathscr{H}})(\Psi^{\mathscr{K}})$ is the infinite 'stiffness' or system matrix, and $u = \mathbf{u}^\top \Psi^{\mathscr{H}}$.

Here we made use of following notations:

Notation 2.1.1. For countable collections of functions Σ and Υ, we write $g(\Sigma) = [g(\sigma)]_{\sigma\in\Sigma}$, $M(\Sigma)(\Upsilon) = [M(\sigma)(\upsilon)]_{\upsilon\in\Upsilon,\sigma\in\Sigma}$, and $\langle \Upsilon, \Sigma\rangle = [\langle \upsilon, \sigma\rangle]_{\upsilon\in\Upsilon,\sigma\in\Sigma}$, assuming g, M, and $\langle\,,\,\rangle$ are such that the expressions at the right-hand sides are well-defined.

The space of square summable vectors of reals indexed over a countable index set $\vee$ will be denoted as $\ell_2(\vee)$ or simply as ℓ_2. The norm on this space will be simply denoted as $\|\cdot\|$.

As a consequence of $A \in \mathcal{L}\mathrm{is}(\mathscr{H}, \mathscr{K}')$, we have that $\mathbf{A} \in \mathcal{L}\mathrm{is}(\ell_2, \ell_2)$. For the moment, let us additionally assume that $\mathbf{A}$ is symmetric and positive definite, as when $\mathscr{K} = \mathscr{H}$, $(Au)(v) = (Av)(u)$ and $(Au)(u) \gtrsim \|u\|^2_{\mathscr{H}}$ $(u, v \in \mathscr{H})$. If this is not the case, then the following can be applied to the normal equations $\mathbf{A}^\top\mathbf{A}\mathbf{u} = \mathbf{A}^\top\mathbf{f}$.

For the finitely supported approximations $\tilde{\mathbf{u}}$ to $\mathbf{u}$ that are generated inside the adaptive wavelet scheme, the residual $\mathbf{r} = \mathbf{f} - \mathbf{A}\tilde{\mathbf{u}}$ has to be approximated within a sufficiently small relative tolerance. The resulting scheme has been shown to converge with the best possible rate: Whenever $\mathbf{u}$ *can* be approximated at rate s, i.e. $\mathbf{u} \in \mathcal{A}^s$, meaning that for any $N \in \mathbb{N}$ there *exists* a vector of length N that approximates $\mathbf{u}$ within tolerance $\mathcal{O}(N^{-s})$, the approximations produced by the scheme converge with this rate s. Moreover, the scheme has *linear computational complexity* under the *cost condition* that

$$\text{the approximate residual evaluation within an (absolute) tolerance } \varepsilon \gtrsim \|\mathbf{r}\| \text{ requires not more than } \mathcal{O}(\varepsilon^{-1/s} + \#\operatorname{supp}\tilde{\mathbf{u}}) \text{ operations.} \tag{2.1.1}$$

The lower bound on ε reflects the fact that inside the adaptive scheme, it is never needed to approximate a residual more accurately than within a sufficiently small, but fixed relative tolerance. The validity of (2.1.1) will require additional properties of $\Psi^{\mathscr{H}}$ and $\Psi^{\mathscr{K}}$ in addition to being Riesz bases. For that reason we consider wavelet bases.

The *standard way* to approximate the residual within tolerance ε is to approximate both $\mathbf{f}$ and $\mathbf{A}\tilde{\mathbf{u}}$ *separately* within tolerance $\varepsilon/2$. Under reasonable assumptions, $\mathbf{f}$ can be approximated within tolerance $\varepsilon/2$ by a vector of length $\mathcal{O}(\varepsilon^{-1/s})$.

For the approximation of $\mathbf{A}\tilde{\mathbf{u}}$, it is used that, thanks to the properties of the wavelets as having vanishing moments, each column of $\mathbf{A}$, although generally infinitely supported, can be well approximated by finitely supported vectors. In the approximate matrix-vector multiplication routine introduced in [CDD01], known as the **APPLY**-routine, the accuracy with which a column is approximated is judiciously chosen *depending* on the size of the corresponding entry in the input vector $\tilde{\mathbf{u}}$. It has been shown to realise a tolerance $\varepsilon/2$ at cost $\mathcal{O}(\varepsilon^{-1/s}|\tilde{\mathbf{u}}|_{\mathcal{A}^s}^{1/s} + \#\operatorname{supp}\tilde{\mathbf{u}})$, for any s in some range $(0, s^*]$. For wavelets that have *sufficiently many* vanishing moments, this range was shown to include the range of $s \in (0, s_{\max}]$ for which, in view of the order of the wavelets, $\mathbf{u} \in \mathcal{A}^s$ can possibly be expected (cf. [Ste04]). Using that for the approximations $\tilde{\mathbf{u}}$ to $\mathbf{u}$ that are generated inside the adaptive wavelet scheme, it holds that $|\tilde{\mathbf{u}}|_{\mathcal{A}^s} \lesssim |\mathbf{u}|_{\mathcal{A}^s}$, in those cases the cost condition is satisfied, and so the adaptive wavelet scheme is optimal.

The **APPLY**-routine, however, is quite difficult to implement. Note, in particular, that its outcome depends *nonlinearly* on the input vector $\tilde{\mathbf{u}}$. Furthermore, in experiments, the routine turns out to be quantitatively expensive. Finally, although it has been generalized to certain classes of semi-linear PDEs, in those cases it has not been shown that $s^* \geq s_{\max}$, meaning that for nonlinear problems the issue of optimality is actually open.

2.1.2 An alternative for the **APPLY** routine

A main goal of this chapter is to develop a quantitatively efficient alternative for the **APPLY**-routine, that, moreover, gives rise to provable optimal adaptive wavelet

schemes for classes of *semi-linear* PDEs, and applies to wavelets with only *one vanishing moment*. As an introduction, we consider the model problem of Poisson's equation in one space dimension $\begin{cases} -u'' &= f \text{ on } (0,1), \\ u &= 0 \text{ at } \{0,1\}, \end{cases}$ that, in standard variational form, reads as finding $u \in \mathscr{H} := H^1_0(0,1)$ such that

$$\langle u', v' \rangle_{L_2(0,1)} = \langle f, v \rangle_{L_2(0,1)}, \quad (v \in \mathscr{K} := H^1_0(0,1)),$$

where, by identifying $L_2(0,1)'$ with $L_2(0,1)$ and using that $H^1_0(0,1) \hookrightarrow L_2(0,1)$ is dense, $\langle\, , \rangle_{L_2(0,1)}$ is also used to denote the duality pairing on $H^{-1}(0,1) \times H^1_0(0,1)$. We consider piecewise polynomial, locally supported wavelet Riesz bases $\Psi^{\mathscr{H}}$ and $\Psi^{\mathscr{K}}$ for $H^1_0(0,1)$. Let us exclusively consider *admissible* approximations $\tilde{\mathbf{u}}$ to $\mathbf{u}$ in the sense that their finite supports form *trees*, meaning that if $\lambda \in \operatorname{supp} \tilde{\mathbf{u}}$, then then there exists a $\mu \in \operatorname{supp} \tilde{\mathbf{u}}$, whose level is one less than the level of λ, and $\operatorname{meas}(\operatorname{supp} \psi^{\mathscr{H}}_\lambda \cap \operatorname{supp} \psi^{\mathscr{H}}_\mu) > 0$. It is known that the approximation classes $\mathcal{A}^s$ become only 'slightly' smaller by this restriction to tree approximation compared to unconstrained approximation (cf. [CDDD01]). What is more, the restriction to tree approximation seems mandatory anyway to construct an optimal algorithm for nonlinear operators. The benefit of tree approximation is that $\tilde{u} := \tilde{\mathbf{u}}^\top \Psi^{\mathscr{H}}$ has an alternative, 'single-scale' representation as a piecewise polynomial w.r.t. a partition $\mathcal{T}_1$ of $(0,1)$ with $\#\mathcal{T}_1 \lesssim \#\operatorname{supp} \tilde{\mathbf{u}}$.

For the moment, let us make the *additional assumption* that $\Psi^{\mathscr{H}}$ is selected inside $H^2(0,1)$. Then, for an admissible $\tilde{\mathbf{u}}$, with its support denoted as $\Lambda^{\mathscr{H}}$, integration-by-parts shows that

$$\mathbf{r} := \mathbf{f} - \mathbf{A}\tilde{\mathbf{u}} = \langle \Psi^{\mathscr{K}}, f + \tilde{u}'' \rangle_{L_2(0,1)},$$

where $\tilde{u}''$ is piecewise polynomial w.r.t. $\mathcal{T}_1$. If $\mathbf{u} \in \mathcal{A}^s$, then for any $\varepsilon > 0$ there exists a piecewise polynomial f_ε w.r.t. a partition $\mathcal{T}_2$ of (0,1) into $\mathcal{O}(\varepsilon^{-1/s})$ subintervals such that $\|f - f_\varepsilon\|_{H^{-1}(0,1)} \le \varepsilon$. [1] The term $f - f_\varepsilon$ is commonly referred to as *data oscillation*.

The function $f_\varepsilon + \tilde{u}''$ is piecewise polynomial w.r.t. the smallest common refinement $\mathcal{T}$ of $\mathcal{T}_1$ and $\mathcal{T}_2$. Thanks to this piecewise smoothness of $f_\varepsilon + \tilde{u}''$ w.r.t. $\mathcal{T}$, and the property of $\psi^{\mathscr{K}}_\lambda$ having *one* vanishing moment, $|\langle \psi^{\mathscr{K}}_\lambda, f_\varepsilon + \tilde{u}'' \rangle_{L_2(0,1)}|$ is decreasing as function of the minimal difference of the *level* of $\psi^{\mathscr{K}}_\lambda$ and that of any subinterval in $\mathcal{T}$ that has non-empty intersection with $\operatorname{supp} \psi^{\mathscr{K}}_\lambda$. Here with the level of $\psi^{\mathscr{K}}_\lambda$ or that of an interval ω, we mean an $\ell \in \mathbb{N}_0$ such that $2^{-\ell} \eqsim \operatorname{diam}(\operatorname{supp} \psi^{\mathscr{K}}_\lambda)$ or $2^{-\ell} \eqsim \operatorname{diam} \omega$, respectively. In particular, given a constant $\varsigma > 0$, there exists a constant k, such that by dropping all λ for which the aforementioned minimal level difference exceeds k, the remaining indices form a tree $\Lambda^{\mathscr{K}}$ with $\#\Lambda^{\mathscr{K}} \lesssim \#\mathcal{T} \lesssim \varepsilon^{-1/s} + \#\Lambda^{\mathscr{H}}$ (dependent on k), and (see Prop. 2.9.1)

$$\begin{aligned} \|\mathbf{f}_\varepsilon - \mathbf{A}\tilde{\mathbf{u}} - (\mathbf{f}_\varepsilon - \mathbf{A}\tilde{\mathbf{u}})|_{\Lambda^{\mathscr{K}}}\| \le \varsigma \|f_\varepsilon + \tilde{u}''\|_{H^{-1}(0,1)} &\le \varsigma \|f + \tilde{u}''\|_{H^{-1}(0,1)} + \varsigma\varepsilon \\ &\lesssim \varsigma \|u - \tilde{u}\|_{H^1(0,1)} + \varsigma\varepsilon, \end{aligned}$$

[1] Indeed, for an admissible $\bar{\mathbf{u}}$ with $\|\mathbf{u} - \bar{\mathbf{u}}\| \le \varepsilon$ and $\#\operatorname{supp} \bar{\mathbf{u}} \lesssim \varepsilon^{-1/s}$, take $f_\varepsilon = -\bar{u}''$ and use $\|f + \bar{u}''\|_{H^{-1}(\Omega)} \eqsim \|u - \bar{u}\|_{H^1(0,1)} \eqsim \|\mathbf{u} - \bar{\mathbf{u}}\|$.

and so, using $\|\mathbf{r}\| \eqsim \|u - \tilde{u}\|_{H^1(0,1)}$ and $\|\mathbf{f} - \mathbf{f}_\varepsilon\| \eqsim \|f - f_\varepsilon\|_{H^{-1}(0,1)}$,

$$\|\mathbf{r} - \mathbf{r}|_{\Lambda^{\mathscr{K}}}\| \lesssim \varsigma\|\mathbf{r}\| + \varepsilon.$$

Note that for ς being sufficiently small, and so k sufficiently large, by taking ε suitably the approximate residual will meet any accuracy that is required in the cost condition (2.1.1).

By selecting 'single scale' collections $\Phi^{\mathscr{H}}$ and $\Phi^{\mathscr{K}}$ with $\operatorname{span}\Phi^{\mathscr{H}} \supseteq \operatorname{span}\Psi^{\mathscr{H}}|_{\Lambda^{\mathscr{K}}}$ and $\operatorname{span}\Phi^{\mathscr{K}} \supseteq \operatorname{span}\Psi^{\mathscr{K}}|_{\Lambda^{\mathscr{K}}}$, and $\#\Phi^{\mathscr{H}} \lesssim \#\Lambda^{\mathscr{H}}$ and $\#\Phi^{\mathscr{K}} \lesssim \#\Lambda^{\mathscr{K}}$, this approximate residual $\mathbf{r}|_{\Lambda^{\mathscr{K}}}$ can be computed in $\mathcal{O}(\Lambda^{\mathscr{K}})$ operations as follows: First express $\tilde{u}$ in terms of $\Phi^{\mathscr{H}}$ by applying a multi-to-single scale transformation to $\tilde{\mathbf{u}}$, then apply to this representation the sparse stiffness matrix $\langle(\Phi^{\mathscr{K}})', (\Phi^{\mathscr{H}})'\rangle_{L_2(0,1)}$, subtract $\langle\Phi^{\mathscr{K}}, f\rangle_{L_2(0,1)}$, and finally apply the transpose of the multi-to-single scale transformation involving $\Psi^{\mathscr{K}}|_{\Lambda^{\mathscr{K}}}$ and $\Phi^{\mathscr{K}}$. This approximate residual evaluation thus satisfies the cost condition for optimality, it is relatively easy to implement, and it is observed to be quantitatively much more efficient.

It furthermore generalizes to semi-linear operators, in any case for nonlinear terms that are multivariate polynomials in u and derivatives of u. Indeed, as an example, suppose that instead of $-u'' = f$ the equation reads as $-u''+u^3 = f$. Then the residual is given by $\langle\Psi^{\mathscr{K}}, f+\tilde{u}''-\tilde{u}^3\rangle_{L_2(0,1)}$. Since $f_\varepsilon+\tilde{u}''-\tilde{u}^3$ is a piecewise polynomial w.r.t. $\mathcal{T}$, the same arguments shows that $\langle\Psi^{\mathscr{K}}, f+\tilde{u}''-\tilde{u}^3\rangle_{L_2(0,1)}\big|_{\Lambda^{\mathscr{K}}}$ is a valid approximate residual.

The essential idea behind our approximate residual evaluation is that, after the replacement of f by f_ε, the different terms that constitute the residual are expressed in a *common* dictionary, before the residual, *as a whole*, is integrated against $\Psi^{\mathscr{K}}$. In our simple one-dimensional example this was possible by selecting $\Psi^{\mathscr{H}} \subset H^2(0,1)$, so that the operator could be applied to the wavelets in strong, or more precisely, mild sense, meaning that the result of the application lands in $L_2(0,1)$. It required piecewise smooth, globally C^1-wavelets. Although the same approach applies in more dimensions, there, except on product domains, the construction of C^1-wavelet bases is cumbersome. For that reason, our approach will be to write a PDE of second order as a *system of PDEs of first order*. It will turn out that there are several possibilities to do so.

2.1.3 A common first order system least squares formulation

To introduce ideas, let us again consider the model problem of Poisson's equation in one dimension. By introducing the additional unknown $\theta = u'$, for given $f \in L_2(0,1)$ this PDE can be written as the first order system of finding $(u,\theta) \in H^1_0(0,1)\times H^1(0,1)$ such that

$$\vec{H}(u,\theta) := (-\theta' - f, -u' + \theta) = \vec{0} \quad \text{in } L_2(0,1)\times L_2(0,1).$$

The corresponding homogeneous operator[2] $\vec{H}^{\mathrm{h}} := (v,\eta) \mapsto (-\eta', -v'+\eta)$ belongs to $\mathcal{L}\mathrm{is}(H^1_0(0,1)\times H^1(0,1), L_2(0,1)\times L_2(0,1))$ (cf. [Ste13, (proof of) Thm. 3.1]). To arrive at a symmetric and positive definite system, we consider the least squares problem of solving

$$\operatorname*{argmin}_{(u,\theta)\in H^1_0(0,1)\times H^1(0,1)} \|\vec{H}(u,\theta)\|^2_{L_2(0,1)\times L_2(0,1)}.$$

Its solution solves the Euler-Lagrange equations

$$\langle \vec{H}^h(v,\eta), \vec{H}(u,\theta)\rangle_{L_2(0,1)\times L_2(0,1)} = 0 \quad ((v,\eta)\in H^1_0(0,1)\times H^1(0,1)).$$

which in this setting are known as the *normal equations*.

To these normal equations we apply the adaptive wavelet scheme, so with '$\mathscr{H}$'='$\mathscr{K}$'$= H^1_0(0,1)\times H^1(0,1)$, ('$A$'$(u,\theta))(v,\eta) := \langle \vec{H}^h(v,\eta), \vec{H}^h(u,\theta)\rangle_{L_2(0,1)\times L_2(0,1)}$ and right-hand side 'f'$(v,\eta) := \langle f, -\eta'\rangle_{L_2(0,1)}$. From $\vec{H}^{\mathrm{h}}$ being a homeomorphism with its range, i.e.,

$$\|\vec{H}^{\mathrm{h}}(v,\eta)\|_{L_2(0,1)\times L_2(0,1)} \eqsim \|(v,\eta)\|_{H^1_0(0,1)\times H^1(0,1)},$$

being a consequence of $\vec{H}^{\mathrm{h}}$ being even boundedly invertible between the full spaces, it follows that the bilinear form is bounded, symmetric, and coercive. After equipping $H^1_0(0,1)$ and $H^1(0,1)$ with wavelet Riesz bases $\Psi^{H^1_0}$ and Ψ^{H^1}, for admissible $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\theta}}$, with $\tilde{u} := \tilde{\mathbf{u}}^\top \Psi^{H^1_0}$ and $\tilde{\theta} := \tilde{\boldsymbol{\theta}}^\top \Psi^{H^1}$ the residual reads as

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} \langle (\Psi^{H^1_0})', \tilde{u}' - \tilde{\theta}\rangle_{L_2(0,1)} \\ \langle (\Psi^{H^1})', \tilde{\theta}' + f\rangle_{L_2(0,1)} + \langle \Psi^{H^1}, \tilde{\theta} - \tilde{u}'\rangle_{L_2(0,1)} \end{bmatrix}. \tag{2.1.2}$$

The construction of an approximate residual follows the same lines as described before for the standard variational formulation.[3] The functions $\tilde{u}'$, $\tilde{\theta}$, $\tilde{\theta}'$ are piecewise polynomials w.r.t. a partition $\mathcal{T}_1$ of $(0,1)$ into $\mathcal{O}(\#\operatorname{supp}\tilde{\mathbf{u}} + \#\operatorname{supp}\tilde{\boldsymbol{\theta}})$ subintervals. If $(\mathbf{u},\boldsymbol{\theta})\in\mathcal{A}^s$, then there exists a piecewise polynomial f_ε w.r.t. a partition $\mathcal{T}_2$ of $(0,1)$ into $\mathcal{O}(\varepsilon^{-1/s})$ subintervals such that $\|f - f_\varepsilon\|_{L_2(0,1)} \le \varepsilon$. Thanks to the piecewise smoothness of $\tilde{u}' - \tilde{\theta}$ and $\tilde{\theta}' + f_\varepsilon$, there exist trees $\Lambda^{H^1_0}$ and Λ^{H^1}, with $\#\Lambda^{H^1_0} + \#\Lambda^{H^1} \lesssim \#\mathcal{T}_1 + \#\mathcal{T}_2$ (dependent on ς), such that

$$\left\| \mathbf{r} - \begin{bmatrix} \mathbf{r}_1|_{\Lambda^{H^1_0}} \\ \mathbf{r}_2|_{\Lambda^{H^1}} \end{bmatrix} \right\| \lesssim \varsigma(\|\tilde{u}' - \tilde{\theta}\|_{L_2(0,1)} + \|\tilde{\theta}' + f_\varepsilon\|_{L_2(0,1)}) + \varepsilon \lesssim \varsigma\|\mathbf{r}\| + \varepsilon.$$

Since the approximate residual can be evaluated in $\mathcal{O}(\#\Lambda^{H^1_0}\cup\Lambda^{H^1})$ operations, we conclude that it satisfies the cost condition (2.1.1) for optimality of the adaptive wavelet scheme.

[2] For general non-affine $\vec{H}$, $\vec{H}^{\mathrm{h}}$ should be read as the Fréchet derivative $D\vec{H}(u,\theta)$.

[3] Actually, in the current setting its analysis is more straightforward, because the residuals are measured in $L_2(0,1)$ instead of in $H^{-1}(0,1)$.

Remark 2.1.2. Recall that, as always with least squares formulations, the same results are valid when lower order, possibly *non-symmetric* terms are added to the second order PDE, as long as the standard variational formulation remains well-posed. Furthermore, as we will discuss, least squares formulations allows to handle *inhomogeneous boundary conditions*. Finally, as we will see, the approach of reformulating a 2nd order PDE as a first order least squares problem, and then optimally solving the normal equations applies to *any* well-posed PDE, not necessarily being elliptic.

In [CS15] we applied the adaptive wavelet scheme to a least squares formulation of the current, common type. *Disadvantages* of this formulation are that (i) it requires that $f \in L_2(0,1)$, instead of $f \in H^{-1}(0,1)$ as allowed in the standard variational formulation. Related to that, and more importantly, for a semi-linear equation $-u'' + N(u) = f$, (ii) it is needed that N maps $H^1_0(0,1)$ into $L_2(0,1)$, instead of into $H^{-1}(0,1)$. Finally, with the generalization of this least squares formulation to more than one space dimensions, (iii) the space $H^1(0,1)$ for θ reads as $H(\mathrm{div};\Omega)$. In [CS15], for two-dimensional connected polygonal domains Ω, we managed to construct a wavelet Riesz basis for $H(\mathrm{div};\Omega)$. This construction, however, relied on the fact that, in two dimensions, any divergence-free function is the curl of an H^1-function. To the best of our knowledge, wavelet Riesz bases for $H(\mathrm{div};\Omega)$ for non-product domains in three and more dimensions have not been constructed.

In the next subsection, we describe a prototype of a least-squares formulation with which these disadvantages (i)–(iii) are avoided.

2.1.4 A seemingly unpractical least squares formulation

The first order system least squares formulation that will be studied in this chapter reads, for the model problem, as follows: Again we introduce $\theta = u'$, but now consider the first order system of finding $(u,\theta) \in H^1_0(0,1) \times L_2(0,1)$ such that

$$\vec{H}(u,\theta) := (D'\theta + f, -u' + \theta) = \vec{0} \quad \text{in } H^{-1}(0,1) \times L_2(0,1).$$

where $(D'\theta)(v) := -\langle \theta, v' \rangle_{L_2(0,1)}$, i.e., $D'\theta$ is the distributional derivative of θ.

In 'primal' mixed form, this system reads as

$$\langle \theta, v' \rangle_{L_2(0,1)} + \langle \theta - u', \eta \rangle_{L_2(0,1)} = \langle f, v \rangle_{L_2(0,1)} \quad ((v,\eta) \in H^1_0(0,1) \times L_2(0,1)).$$

The corresponding homogeneous operator $\vec{H}^{\mathrm{h}}$ is in $\mathcal{L}\mathrm{is}(H^1_0(0,1) \times L_2(0,1), H^{-1}(0,1) \times L_2(0,1))$, and the least squares problem reads as solving

$$\operatorname*{argmin}_{(u,\theta) \in H^1_0(0,1) \times L_2(0,1)} \|\vec{H}(u,\theta)\|^2_{H^{-1}(0,1) \times L_2(0,1)}, \tag{2.1.3}$$

with normal equations reading as

$$\begin{aligned} 0 &= \langle \vec{H}^h(v,\eta), \vec{H}(u,\theta) \rangle_{H^{-1}(0,1) \times L_2(0,1)} \\ &= \langle D'\eta, D'\theta - f \rangle_{H^{-1}(0,1)} + \langle -v' + \theta, -u' + \theta \rangle_{L_2(0,1)} \end{aligned} \tag{2.1.4}$$

$((v,\eta) \in H^1_0(0,1) \times L_2(0,1))$.

In the terminology from [BG09], our current least squares problem is identified as being unpractical because of the appearance of the dual norm. To deal with this, as in [DKS02], we select *some* wavelet Riesz basis $\Psi^{\hat{H}^1_0}$ for $H^1_0(0,1)$, and replace the norm on $H^{-1}(0,1)$ in (2.1.3) by the *equivalent* norm defined by $\|g(\Psi^{\hat{H}^1_0})\|$ for $g \in H^{-1}(\Omega)$. Correspondingly, in (2.1.4) we replace the inner product $\langle g,h\rangle_{H^{-1}(0,1)}$ by $g(\Psi^{\hat{H}^1_0})^\top h(\Psi^{\hat{H}^1_0})$, so that the resulting normal equations read as finding $(u,\theta) \in H^1_0(0,1)\times L_2(0,1)$ such that

$$\langle \eta, (\Psi^{\hat{H}^1_0})'\rangle_{L_2(0,1)}\big\{\langle (\Psi^{\hat{H}^1_0})', \theta\rangle_{L_2(0,1)} - \langle \Psi^{\hat{H}^1_0}, f\rangle_{L_2(0,1)}\big\} + \langle -v'+\eta, -u'+\theta\rangle_{L_2(0,1)} = 0$$

for all $(v,\eta) \in H^1_0(0,1)\times L_2(0,1)$.

To apply the adaptive wavelet scheme to these normal equations, we equip $H^1_0(0,1)$ and $L_2(0,1)$ with wavelet Riesz bases $\Psi^{H^1_0}$ and Ψ^{L_2}, respectively. When these bases have order $p+1$ and p, the best possible convergence rate $s_{\max}$ will be equal to p (p/n on an n-dimensional domain). Note that the order of the basis $\Psi^{\hat{H}^1_0}$ is irrelevant.

For approximations $(\tilde{u},\tilde{\theta}) = (\tilde{\mathbf{u}}^\top \Psi^{H^1}, \tilde{\boldsymbol{\theta}}^\top \Psi^{L_2})$ for admissible $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\theta}}$, the residual $\mathbf{r}$ of $(\tilde{\mathbf{u}},\tilde{\boldsymbol{\eta}})$ reads as

$$\begin{bmatrix} \langle (\Psi^{H^1_0})', \tilde{u}' - \tilde{\theta}\rangle_{L_2(0,1)} \\ \langle \Psi^{L_2}, (\Psi^{\hat{H}^1_0})'\rangle_{L_2(0,1)}\big\{\langle (\Psi^{\hat{H}^1_0})', \tilde{\theta}\rangle_{L_2(0,1)} - \langle \Psi^{\hat{H}^1_0}, f\rangle_{L_2(0,1)}\big\} + \langle \Psi^{L_2}, \tilde{\theta} - \tilde{u}'\rangle_{L_2(0,1)} \end{bmatrix}$$

that, under the *additional condition* that $\Psi^{L_2} \subset H^1(0,1)$, and thus $\tilde{\theta} \in H^1(0,1)$, is equal to

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} = \begin{bmatrix} \langle (\Psi^{H^1_0})', \tilde{u}' - \tilde{\theta}\rangle_{L_2(0,1)} \\ -\langle \Psi^{L_2}, (\Psi^{\hat{H}^1_0})'\rangle_{L_2(0,1)} \langle \Psi^{\hat{H}^1_0}, \tilde{\theta}' + f\rangle_{L_2(0,1)} + \langle \Psi^{L_2}, \tilde{\theta} - \tilde{u}'\rangle_{L_2(0,1)} \end{bmatrix}. \quad (2.1.5)$$

This last step is essential because it allows us, after the replacement of f by a piecewise polynomial f_ε, to express $\tilde{\theta}' + f_\varepsilon$ in a common dictionary. The additional condition is satisfied by piecewise polynomial, globally *continuous* wavelets, which are available on general domains in multiple dimensions.

In view of the previous discussions, to describe the approximate residual evaluation, it suffices to consider the term $\langle \Psi^{L_2}, \mathbf{z}^\top (\Psi^{\hat{H}^1_0})'\rangle_{L_2(0,1)}$ with $\mathbf{z} := \langle \Psi^{\hat{H}^1_0}, f + \tilde{\theta}'\rangle_{L_2(0,1)}$. The by now familiar approach is applied twice: The function $\tilde{\theta}'$ is piecewise polynomial w.r.t. a partition $\mathcal{T}_1$ into $\mathcal{O}(\#\operatorname{supp}\tilde{\boldsymbol{\theta}})$ subintervals. If $(\mathbf{u},\boldsymbol{\theta}) \in \mathcal{A}^s$, then there exists a piecewise polynomial f_ε w.r.t. a partition $\mathcal{T}_2$ of (0,1) into $\mathcal{O}(\varepsilon^{-1/s})$ subintervals such that $\|f - f_\varepsilon\|_{H^{-1}(0,1)} \le \varepsilon$. Consequently, there exists a tree $\Lambda^{\hat{H}^1_0}$ with $\#\Lambda^{\hat{H}^1_0} \lesssim \#\operatorname{supp}\tilde{\boldsymbol{\theta}} + \varepsilon^{-1/s}$ (dependent on ς) such that $\|\mathbf{z} - \mathbf{z}|_{\Lambda^{\hat{H}^1_0}}\| \lesssim \varsigma\|f + \tilde{\theta}'\|_{H^{-1}(0,1)} + \varepsilon$. The function $\tilde{z} := \mathbf{z}|^\top_{\Lambda^{\hat{H}^1_0}} \Psi^{\hat{H}^1_0}$ is piecewise polynomial w.r.t. a partition $\mathcal{T}_2$ of (0,1) into $\mathcal{O}(\#\Lambda^{\hat{H}^1_0})$ subintervals. Consequently, exists a tree $\Lambda^{L_2}_1$ with $\#\Lambda^{L_2}_1 \lesssim \#\Lambda^{\hat{H}^1_0}$ (dependent on ς) such that $\|\langle \Psi^{L_2}, \tilde{z}'\rangle_{L_2(0,1)} - \langle \Psi^{L_2}, \tilde{z}'\rangle_{L_2(0,1)}\big|_{\Lambda^{L_2}_1}\| \le \varsigma\|\tilde{z}\|_{H^1(0,1)} \lesssim$

$\varsigma(\|\mathbf{z}\|+\|\mathbf{z}-\mathbf{z}|_{\Lambda^{\hat{H}^1_0}}\|) \lesssim \varsigma(\|f+\tilde{\theta}'\|_{H^{-1}(0,1)}+\varepsilon)$. Combining this with the approximations for the other two terms that constitute the residual, we infer that there exist trees $\Lambda^{H^1_0}$ and Λ^{L_2} with $\#\Lambda^{H^1_0}+\#\Lambda^{L_2} \lesssim \#\operatorname{supp}\#\tilde{\mathbf{u}}+\#\operatorname{supp}\tilde{\boldsymbol{\theta}}+\varepsilon^{-1/s}$ (dependent on ς), such that

$$\Big\|\mathbf{r}-\begin{bmatrix}\mathbf{r}_1|_{\Lambda^{H^1_0}}\\ \tilde{\mathbf{r}}_2|_{\Lambda^{L_2}}\end{bmatrix}\Big\| \lesssim \varsigma(\|\tilde{u}'-\tilde{\theta}\|_{L_2(0,1)}+\|\tilde{\theta}'+f_\varepsilon\|_{H^{-1}(0,1)})+\varepsilon \lesssim \varsigma\|\mathbf{r}\|+\varepsilon,$$

where $\tilde{\mathbf{r}}_2$ is constructed from $\mathbf{r}_2$ by replacing $\mathbf{z}$ by $\mathbf{z}|_{\Lambda^{\hat{H}^1_0}}$. This approximate residual evaluation satisfies the cost condition (2.1.1) for optimality.

As we will see in Sect. 2.2, the advantage of the current construction of a first order system least squares problem is that it applies to *any* well-posed (semi-linear) second order PDE. The two instances of the spaces $H^1_0(0,1)$ represent the trial and test spaces in the standard variational formulation, and well-posedness of the latter implies well-posedness of the least squares formulation. The additional space $L_2(0,1)$ reads in general as an L_2-type space. So, in particular, with this formulation, $H(\mathrm{div})$-spaces do not enter. The price to be paid is that (2.1.5) is somewhat more complicated than (2.1.2), and that therefore its approximation is somewhat more costly to compute.

Remark 2.1.3. The more popular 'dual' mixed formulation of our model problem reads as finding $(u,\theta)\in L_2(0,1)\times H^1(0,1)$ such that $\langle-\theta',v\rangle_{L_2(0,1)}+\langle\theta,\eta\rangle_{L_2(0,1)}+\langle u,\eta'\rangle_{L_2(0,1)}=\langle f,v\rangle_{L_2(0,1)}$ $((v,\eta)\in L_2(0,1)\times H^1(0,1))$. The resulting least squares formulation has the combined disadvantages of both other formulations that we considered. It requires $f\in L_2(0,1)$, possibly nonlinear terms should map into $L_2(0,1)$, in more than one dimension the space $H^1(0,1)$ reads as an $H(\mathrm{div})$-space, and one of the norms involved in the least squares minimalisation is a dual norm.

Remark 2.1.4. With the aim to avoid both a dual norm in the least squares minimalisation, and $H(\mathrm{div})$ or other vectorial Sobolev spaces as trial spaces, in our first investigations of this least squares approach in [Ste14], we considered the 'extended div-grad' first order system least squares formulation studied in [CMM97a]. A sufficient and necessary ([Ste13]), but restrictive condition for its well-posedness is H^2-regularity of the homogeneous boundary value problem.

2.1.5 Layout of the chapter

In Sect. 2.2, a general procedure is given to reformulate *any* well-posed semi-linear 2nd order PDE as a well-posed first order least squares problem. As we will see, this procedure gives an effortless derivation of well-posed first order least squares formulations of elliptic 2nd order PDEs, and that of the stationary Navier-Stokes equations. The arising dual norm can be replaced by the equivalent ℓ_2-norm of a functional in wavelet coordinates.

In Sect. 2.3, we recall properties of the adaptive wavelet Galerkin method (**awgm**). Operator equations of the form $F(z)=0$, where, for some Hilbert space $\mathscr{H}$, $F:\mathscr{H}\to\mathscr{H}'$ and $DF(z)$ is symmetric and positive definite, are solved by the **awgm** at the best possible rate from a Riesz basis for $\mathscr{H}$. Furthermore, under a condition on the

cost of the approximate residual evaluations, the method has optimal computational complexity.

In the short Sect. 2.4, it is shown that the **awgm** applies to the normal equations that result from the first order least squares problems as derived in Sect. 2.2.

In Sect. 2.5, we apply the **awgm** to a first order least squares formulation of a semi-linear 2nd order elliptic PDE with general inhomogeneous boundary conditions. Under a mild condition on the wavelet basis for the trial space, the efficient approximate residual evaluation that was outlined in Sect. 2.1.4 applies, and it satisfies the cost condition, so that the **awgm** is optimal. Wavelet bases that satisfy the assumptions are available on general polygonal domains. Some technical results needed for this section are given in Appendix 2.9.

In Sect. 2.6 the findings from Sect. 2.5 are illustrated by numerical results.

In Sect. 2.7, we consider the so-called velocity–pressure–velocity gradient and the velocity–pressure–vorticity first order system formulations of the stationary Navier-Stokes equations. Results analogously to those demonstrated for the elliptic problem will be shown to be valid here as well.

2.2 Reformulation of a semi-linear second order PDE as a first order system least squares problem

In an abstract framework, we give a procedure to write semi-linear second order PDEs, that have well-posed standard variational formulations, as a well-posed first order system least squares problems. A particular instance of this approach has been discussed in Sect. 2.1.4.

For some separable Hilbert spaces $\mathscr{U}$ and $\mathscr{V}$, for convenience over $\mathbb{R}$, consider a differentiable mapping

$$G : \mathscr{U} \supset \mathrm{dom}(G) \to \mathscr{V}'.$$

Remark 2.2.1. In applications G is the operator associated to a variational formulation of a PDO with trial space $\mathscr{U}$ and test space $\mathscr{V}$.

For $\mathscr{T}$ being another separable Hilbert space, let

$$G = G_0 + G_1 G_2, \text{ where } G_1 \in \mathcal{L}(\mathscr{T}, \mathscr{V}'),\ G_2 \in \mathcal{L}(\mathscr{U}, \mathscr{T}), \tag{2.2.1}$$

i.e., G_1 and G_2 are bounded linear operators.

Remark 2.2.2. In applications, as those discussed in Sect. 2.5 and 2.7, $G_1 G_2$ will be a factorization of the leading second order part of the PDO (possibly modulo terms that vanish at the solution, cf. Sect. 2.7.2) into a product of first order PDOs.

Obviously, u solves $\boxed{G(u) = 0}$ if and only if it is the first component of the solution (u, θ) of

$$\boxed{\vec{H}(u, \theta) := (G_0(u) + G_1\theta, \theta - G_2 u) = \vec{0},}$$

where $\vec{H} : \mathscr{U} \times \mathscr{T} \supset \mathrm{dom}(G) \times \mathscr{T} = \mathrm{dom}(\vec{H}) \to \mathscr{V}' \times \mathscr{T}$.

The following lemma shows that well-posedness of the original formulation implies that of the reformulation as a system.

Lemma 2.2.3. *Let $DG(u) \in \mathcal{L}(\mathscr{U}, \mathscr{V}')$ be a homeomorphism with its range, meaning that $\|DG(u)v\|_{\mathscr{V}'} \eqsim \|v\|_{\mathscr{U}}$ $(v \in \mathscr{U})$. Then*

$$D\vec{H}(u, \theta) = \begin{bmatrix} DG_0(u) & G_1 \\ -G_2 & I \end{bmatrix} \in \mathcal{L}(\mathscr{U} \times \mathscr{T}, \mathscr{V}' \times \mathscr{T})$$

is a homeomorphism with its range, being $\{(f, g) \in \mathscr{V}' \times \mathscr{T} : f - G_1 g \in \mathrm{ran}\, DG(u)\}$. In particular, with $r, R > 0$ such that $r\|v\|_{\mathscr{U}} \le \|DG(u)(v)\|_{\mathscr{V}'} \le R\|v\|_{\mathscr{U}}$, it holds that

$$\|(v, \eta)\|^2_{\mathscr{U}\times\mathscr{T}} \le \Big[\big(\tfrac{1+\|G_2\|}{r}\big)^2 + \big(1 + \tfrac{1+\|G_1\|(1+\|G_2\|)}{r}\big)^2\Big] \Big\| D\vec{H}(u, \theta) \begin{bmatrix} v \\ \eta \end{bmatrix} \Big\|^2_{\mathscr{V}'\times\mathscr{T}}, \tag{2.2.2}$$

$$\Big\| D\vec{H}(u, \theta) \begin{bmatrix} v \\ \eta \end{bmatrix} \Big\|^2_{\mathscr{V}'\times\mathscr{T}} \le \big((R + (1+\|G_1\|)\|G_2\|)^2 + (1+\|G_1\|)^2\big)\|(v, \eta)\|^2_{\mathscr{U}\times\mathscr{T}}. \tag{2.2.3}$$

Remark 2.2.4. Since $\operatorname{ran} DG(u) = \mathscr{V}'$ iff $\operatorname{ran} D\vec{H}(u,\theta) = \mathscr{V}' \times \mathscr{T}$, in particular we have that $DG(u) \in \mathcal{L}\mathrm{is}(\mathscr{U}, \mathscr{V}')$ implies that $D\vec{H}(u,\theta) \in \mathcal{L}\mathrm{is}(\mathscr{U} \times \mathscr{T}, \mathscr{V}' \times \mathscr{T})$.

Proof. We have $D\vec{H}(u,\theta)\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} DG_0(u)v + G_1\eta \\ \eta - G_2 v \end{bmatrix} = \begin{bmatrix} DG(u)v + G_1(\eta - G_2 v) \\ \eta - G_2 v \end{bmatrix}$ by $DG_0(\cdot) = DG(\cdot) - G_1 G_2$. So

$$\operatorname{ran} D\vec{H}(u,\theta) \subseteq \{(f,g) \in \mathscr{V}' \times \mathscr{T} \colon f - G_1 g \in \operatorname{ran} DG(u)\}. \tag{2.2.4}$$

By estimating $\|D\vec{H}(u,\theta)\begin{bmatrix} v \\ \eta \end{bmatrix}\|_{\mathscr{V}' \times \mathscr{T}} \leq \|DG(u)v + G_1(\eta - G_2 v)\|_{\mathscr{V}'} + \|\eta - G_2 v\|_{\mathscr{T}}$, one easily arrives at (2.2.3).

For (f,g) being in the set at the right-hand side in (2.2.4), consider the system

$$D\vec{H}(u,\theta)\begin{bmatrix} v \\ \eta \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \iff \begin{cases} DG_0(u)v + G_1\eta = f \\ \eta - G_2 v = g \end{cases} \iff \begin{cases} DG(u)v = f - G_1 g \\ \eta = g + G_2 v \end{cases}.$$

This system has a unique solution, so that the $\subseteq$-symbol in (2.2.4) reads as an equality sign, and $r\|v\|_{\mathscr{U}} \leq \|f\|_{\mathscr{V}'} + \|G_1\|\|g\|_{\mathscr{T}}$ and $\|\eta\|_{\mathscr{T}} \leq \|g\|_{\mathscr{T}} + \|G_2\|\|v\|_{\mathscr{U}}$. By estimating $\|(v,\eta)\|_{\mathscr{U}\times\mathscr{T}} \leq \|v\|_{\mathscr{U}} + \|\eta\|_{\mathscr{T}}$ one easily arrives at (2.2.2). □

In the following, we will always assume that

(i) there exists a solution u of $G(u) = 0$;

(ii) G is two times continuously Fréchet differentiable in a neighborhood of u;

(iii) $DG(u) \in \mathcal{L}(\mathscr{U}, \mathscr{V}')$ is a homeomorphism with its range.

Then

(a) $(u,\theta) = (u, G_2 u)$ solves $\vec{H}(u,\theta) = \vec{0}$;

(b) $\vec{H}$ is two times continuously Fréchet differentiable in a neighborhood of (u,θ);

(c) $D\vec{H}(u,\theta) \in \mathcal{L}(\mathscr{U} \times \mathscr{T}, \mathscr{V}' \times \mathscr{T})$ is a homeomorphism with its range,

the latter by Lemma 2.2.3. In summary, when the equation $G(u) = 0$ is *well-posed* ((i)-(iii) are valid), then so is $\vec{H}(u,\theta) = \vec{0}$ ((a)-(c) are valid), and solving one equation is equivalent to solving the other.

Remark 2.2.5. Actually, one might dispute whether these equations should be called well-posed when $\operatorname{ran} DG(u) \subsetneq \mathscr{V}'$ and so $\operatorname{ran} D\vec{H}(u,\theta) \subsetneq \mathscr{V}' \times \mathscr{T}$. In any case, under conditions (i)–(iii), and so (a)–(c), the corresponding least-squares problems and resulting (nonlinear) normal equations *are* well-posed, as we will see next.

A solution (u,θ) of $\vec{H}(u,\theta) = \vec{0}$ is a minimizer of the least squares functional

$$\boxed{Q(u,\theta) := \tfrac{1}{2}\|\vec{H}(u,\theta)\|^2_{\mathscr{V}'\times\mathscr{T}}.}$$

In particular, it holds that

Lemma 2.2.6. *For $\vec{H} : \mathscr{U} \times \mathscr{T} \supset \mathrm{dom}(\vec{H}) \to \mathscr{V}' \times \mathscr{T}$, and $\vec{H}$ being Fréchet differentiable at a root (u, θ), property* (c) *is equivalent to the property that for $(\tilde{u}, \tilde{\theta}) \in \mathscr{U} \times \mathscr{T}$ in a neighbourhood of (u, θ),*

$$Q(\tilde{u}, \tilde{\theta}) \eqsim \|u - \tilde{u}\|^2_{\mathscr{U}} + \|\theta - \tilde{\theta}\|^2_{\mathscr{T}}.$$

Proof. This is a consequence of $\vec{H}(\tilde{u}, \tilde{\theta}) = D\vec{H}(u, \theta)(\tilde{u} - u, \tilde{\theta} - \theta) + o(\|\tilde{u} - u\|_{\mathscr{U}} + \|\tilde{\theta} - \theta\|_{\mathscr{T}})$. □

A minimizer (u, θ) of Q is a solution of the Euler-Lagrange equations

$$\boxed{DQ(u, \theta)(v, \eta) = \left\langle D\vec{H}(u, \theta) \begin{bmatrix} v \\ \eta \end{bmatrix}, \vec{H}(u, \theta) \right\rangle_{\mathscr{V}' \times \mathscr{T}} = 0 \quad ((v, \eta) \in (\mathscr{U} \times \mathscr{T}))}, \tag{2.2.5}$$

that, in this setting, are usually called (nonlinear) *normal equations*. Using (a)-(b), one computes that

$$D^2Q(u, \theta)(v_1, \eta_1)(v_2, \eta_2) = \left\langle D\vec{H}(u, \theta) \begin{bmatrix} v_1 \\ \eta_1 \end{bmatrix}, D\vec{H}(u, \theta) \begin{bmatrix} v_2 \\ \eta_2 \end{bmatrix} \right\rangle_{\mathscr{V}' \times \mathscr{T}}.$$

We conclude the following: Under the assumptions (a)–(c), it holds that

(1) DQ is a mapping from a subset of a separable Hilbert space, viz. $\mathscr{U} \times \mathscr{T}$, to its dual;

(2) there exists a solution of $DQ(u, \theta) = 0$ (viz. any solution of $\vec{H}(u, \theta) = 0$);

(3) DQ is continuously Fréchet differentiable in a neighborhood of (u, θ);

(4) $0 < D^2Q(u, \theta) = D^2Q(u, \theta)' \in \mathcal{L}\mathrm{is}(\mathscr{U} \times \mathscr{T}, (\mathscr{U} \times \mathscr{T})')$.

As a consequence of the last property, one infers that in a neighborhood of (u, θ), $DQ(u, \theta) = 0$ has exactly one solution.

In view of the above findings, in order to solve $G(u) = 0$, for a G that satisfies (i)–(iii), we are going to solve the (nonlinear) normal equations $DQ(u, \theta) = 0$. A major advantage of DQ over G is that its derivative is symmetric and coercive.

A concern, however, is whether, for given $(u, \theta), (v, \eta) \in \mathscr{U} \times \mathscr{T}$, $DQ(u, \theta)(v, \eta)$ as given by (2.2.5) is evaluable. We will think of the inner product on $\mathscr{T}$ as being evaluable. In our applications, $\mathscr{T}$ will be of the form $L_2(\Omega)^N$. To deal with the dual norm on $\mathscr{V}'$, we equip $\mathscr{V}$ with a *Riesz basis*

$$\Psi^{\mathscr{V}} = \{\psi^{\mathscr{V}}_\lambda : \lambda \in \vee_{\mathscr{V}}\},$$

meaning that the *analysis operator*

$$\mathcal{F}_{\mathscr{V}} : g \mapsto [g(\psi^{\mathscr{V}}_\lambda)]_{\lambda \in \vee_{\mathscr{V}}} \in \mathcal{L}\mathrm{is}(\mathscr{H}, \ell_2(\vee_{\mathscr{V}})),$$

and so its adjoint, known as the *synthesis operator*,

$$\mathcal{F}'_{\mathscr{V}} : \mathbf{v} \mapsto \mathbf{v}^\top \Psi^{\mathscr{V}} := \sum_{\lambda \in \vee_{\mathscr{V}}} v_\lambda \psi_\lambda^{\mathscr{V}} \in \mathcal{L}\mathrm{is}(\ell_2(\vee_{\mathscr{V}}), \mathscr{V}').$$

In the definition of the least squares functional Q, and consequently in that of DQ, we now *replace* the standard dual norm on $\mathscr{V}'$ by the *equivalent* norm $\|\mathcal{F}_{\mathscr{V}} \cdot \|_{\ell_2(\vee_{\mathscr{V}})}$. Then in view of the definition of $\vec{H}$ and the expression for $D\vec{H}$, we obtain that

$$\boxed{\begin{aligned} &DQ(u,\theta)(v,\eta) = \\ &(DG_0(u)v + G_1\eta)(\Psi^{\mathscr{V}})^\top (G_0(u) + G_1\theta)(\Psi^{\mathscr{V}}) + \langle \eta - G_2 v, \theta - G_2 u\rangle_{\mathscr{T}}, \end{aligned}} \tag{2.2.6}$$

where (1)–(4) are still valid.

Remark 2.2.7. We refer to [BLP97] for an alternative approach to solve least square problems that involves dual norms.

To solve the obtained (nonlinear) normal equations $DQ(u,\theta) = 0$ we are going to apply the adaptive wavelet Galerkin method (**awgm**). Note that the definition of $DQ(u,\theta)(v,\eta)$ still involves an infinite sum over $\vee_{\mathscr{V}}$ that later, inside the solution process, is going to be replaced by a finite one.

2.3 The adaptive wavelet Galerkin method (**awgm**)

In this section, we summarize findings about the **awgm** from [Ste14, CS15]. Let

(I) $F : \mathscr{H} \supset \mathrm{dom}(F) \to \mathscr{H}'$, with $\mathscr{H}$ being a separable Hilbert space;

(II) $F(z) = 0$;

(III) F be continuously differentiable in a neighborhood of z;

(IV) $0 < DF(z) = DF(z)' \in \mathcal{L}\mathrm{is}(\mathscr{H}, \mathscr{H}')$.

In our applications, the triple $(F, \mathscr{H}, z)$ will read as $(DQ, \mathscr{U} \times \mathscr{T}, (u,\theta))$, so that (I)-(IV) are guaranteed by (1)–(4).

Let $\Psi = \{\psi_\lambda : \lambda \in \vee\}$ be a *Riesz basis* for $\mathscr{H}$, with analysis operator $\mathcal{F} : g \mapsto [g(\psi_\lambda)]_{\lambda\in\vee} \in \mathcal{L}\mathrm{is}(\mathscr{H}, \ell_2(\vee))$, and so synthesis operator $\mathcal{F}' : \mathbf{v} \mapsto \mathbf{v}^\top\Psi := \sum_{\lambda\in\vee} v_\lambda\psi_\lambda \in \mathcal{L}\mathrm{is}(\ell_2(\vee), \mathscr{H}')$. For any $\Lambda \subset \vee$, we set

$$\ell_2(\Lambda) := \{\mathbf{v} \in \ell_2(\vee) \colon \mathrm{supp}\,\mathbf{v} \subset \Lambda\}.$$

For satisfying the forthcoming Condition 2.3.5 that concerns the computational cost, it will be relevant that Ψ is a basis of *wavelet* type.

Writing $z = \mathcal{F}'\mathbf{z}$, and with

$$\mathbf{F} := \mathcal{F}F\mathcal{F}' \colon \ell_2(\vee) \to \ell_2(\vee),$$

an *equivalent* formulation of $F(z) = 0$ is given by

$$\mathbf{F}(\mathbf{z}) = 0.$$

We are going to approximate $\mathbf{z}$, and so z, by a sequence of Galerkin approximations from the spans of increasingly larger sets of wavelets, which sets are created by an adaptive process. Given $\Lambda \subset \vee$, the Galerkin approximation $\mathbf{z}_\Lambda$, or equivalently, $z_\Lambda := \mathbf{z}_\Lambda^\top \Psi$, are the solutions of $\langle \mathbf{F}(\mathbf{z}_\Lambda), \mathbf{v}_\Lambda \rangle_{\ell_2(\vee)} = 0$ ($\mathbf{v}_\Lambda \in \ell_2(\Lambda)$), i.e., $\mathbf{F}(\mathbf{z}_\Lambda)|_\Lambda = 0$, and $F(z_\Lambda)(v_\Lambda) = 0$ ($v_\Lambda \in \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda\}$), respectively. These solutions exist uniquely when $\inf_{\tilde{\mathbf{z}}_\Lambda \in \ell_2(\Lambda)} \|\mathbf{z} - \tilde{\mathbf{z}}_\Lambda\|$ is sufficiently small ([PR94, Ste14]).

In order to be able to construct efficient algorithms, in particular when F is non-affine, it will be needed to consider only sets Λ from a certain subset of all finite subsets of $\vee$. In our applications, this collection of so-called *admissible* Λ will consist of (Cartesian products of) finite *trees*. For the moment, it suffices when the collection of admissible sets is such that the union of any two admissible sets is again admissible.

To provide a benchmark to evaluate our adaptive algorithm, for $s > 0$, we define the nonlinear *approximation class*

$$\mathcal{A}^s := \Big\{ \mathbf{z} \in \ell_2(\vee) \colon \|\mathbf{z}\|_{\mathcal{A}^s} := \sup_{\varepsilon > 0} \varepsilon \times \min \big\{ (\#\Lambda)^s \colon \Lambda \text{ is admissible}, \inf_{\tilde{\mathbf{z}} \in \ell_2(\Lambda)} \|\mathbf{z} - \tilde{\mathbf{z}}\| \le \varepsilon \big\} < \infty \Big\}. \tag{2.3.1}$$

A vector $\mathbf{z}$ is in $\mathcal{A}^s$ if and only if there exists a sequence of admissible $(\Lambda_i)_i$, with $\lim_{i\to\infty} \#\Lambda_i = \infty$, such that $\sup_i \inf_{\mathbf{z}_i \in \ell_2(\Lambda_i)} (\#\Lambda_i)^s \|\mathbf{z} - \mathbf{z}_i\| < \infty$. This means that $\mathbf{z}$ *can* be approximated in $\ell_2(\vee)$ at rate s by vectors supported on admissible sets, or, equivalently, z can be approximated in $\mathscr{H}$ at rate s from spaces of type $\operatorname{span}\{\psi_\lambda \colon \lambda \in \Lambda, \Lambda \text{ is admissible}\}$.

The adaptive wavelet Galerkin method (**awgm**) defined below produces a sequence of increasingly more accurate Galerkin approximations $\mathbf{z}_\Lambda$ to $\mathbf{z}$. The, generally, infinite residual $\mathbf{F}(\mathbf{z}_\Lambda)$ is used as an a posteriori error estimator. A motivation for the latter is given by the following result.

Lemma 2.3.1. *For $\|\mathbf{z} - \tilde{\mathbf{z}}\|$ sufficiently small, it holds that $\|\mathbf{F}(\tilde{\mathbf{z}})\| \eqsim \|\mathbf{z} - \tilde{\mathbf{z}}\|$.*

Proof. With $\tilde{z} = \tilde{\mathbf{z}}^\top \Psi$, it holds that $\|\mathbf{F}(\tilde{\mathbf{z}})\| \eqsim \|F(\tilde{z})\|_{\mathscr{H}'}$. From (II)-(III), we have $F(\tilde{z}) = DF(z)(\tilde{z} - z) + o(\|\tilde{z} - z\|_{\mathscr{H}})$. The proof is completed by $\|DF(z)(\tilde{z} - z)\|_{\mathscr{H}'} \eqsim \|\tilde{z} - z\|_{\mathscr{H}}$ by (IV). □

This a posteriori error estimator guides an appropriate enlargement of the current set Λ using a bulk chasing strategy, so that the sequence of approximations converge with the best possible rate to $\mathbf{z}$. To arrive at an implementable method, that is even of optimal computational complexity, both the Galerkin solution and its residual are allowed to be computed inexactly within sufficiently small relative tolerances.

Algorithm 2.3.2 (awgm).

% Let $0 < \mu_0 \le \mu_1 < 1$, $\delta, \gamma > 0$ *be constants,* $\Lambda_0 \subset \vee$ *be admissible,*
% and $\mathbf{z}_{\Lambda_0} \in \ell_2(\Lambda_0)$. *Let* $\mathbf{Z}$ *be a neighborhood of* $\mathbf{z} \in \ell_2(\vee)$.

`for` $i = 0, 1, \ldots$ `do`

(R) $\zeta := \frac{2\delta}{1+\delta}\|\mathbf{r}_{i-1}\|$. % (*Read* $\|\mathbf{r}_{-1}\|$ *as some scalar* $\eqsim \|\mathbf{z}\|$.)
`do` $\zeta := \zeta/2$; *Compute* $\mathbf{r}_i \in \ell_2(\vee)$ *such that* $\|\mathbf{r}_i - \mathbf{F}(\mathbf{z}_{\Lambda_i})\| \le \zeta$.
`until` $\zeta \le \frac{\delta}{1+\delta}\|\mathbf{r}_i\|$.

(B) *Determine an admissible* $\Lambda_{i+1} \supset \Lambda_i$ *with* $\|\mathbf{r}_i|_{\Lambda_{i+1}}\| \ge \mu_0\|\mathbf{r}_i\|$ *such that* $\#(\Lambda_{i+1} \setminus \Lambda_i) \lesssim \#(\tilde{\Lambda} \setminus \Lambda_i)$ *for any admissible* $\tilde{\Lambda} \supset \Lambda_i$ *with* $\|\mathbf{r}_i|_{\tilde{\Lambda}}\| \ge \mu_1\|\mathbf{r}_i\|$.

(G) *Compute* $\mathbf{z}_{\Lambda_{i+1}} \in \ell_2(\Lambda_{i+1}) \cap \mathbf{Z}$ *with* $\|\mathbf{F}(\mathbf{z}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}\| \le \gamma\|\mathbf{r}_i\|$.

`endfor`

In step (R), by means of a loop in which an absolute tolerance is decreased, the true *residual* $\mathbf{F}(\mathbf{z}_{\Lambda_i})$ is approximated within a relative tolerance δ. In step (B), *bulk chasing* is performed on the approximate residual. The idea is to find a smallest admissible $\Lambda_{i+1} \supset \Lambda_i$ with $\|\mathbf{r}_i|_{\Lambda_{i+1}}\| \ge \mu_0\|\mathbf{r}_i\|$. In order to be able to find an implementation that is of linear complexity, the condition of having a truly smallest Λ_{i+1} has been relaxed. Finally, in step (G), a sufficiently accurate approximation of the *Galerkin* solution w.r.t. the new set Λ_{i+1} is determined.

Convergence of the adaptive wavelet Galerkin method, with the *best possible rate*, is stated in the following theorem.

Theorem 2.3.3 ([Ste14, Thm. 3.9])**.** *Let* μ_1, γ, δ, $\inf_{\mathbf{v}_{\Lambda_0} \in \ell_2(\Lambda_0)} \|\mathbf{z} - \mathbf{v}_{\Lambda_0}\|$, $\|\mathbf{F}(\mathbf{z}_{\Lambda_0})|_{\Lambda_0}\|$, *and the neighborhood* $\mathbf{Z}$ *of the solution* $\mathbf{z}$ *all be sufficiently small. Then, for some* $\alpha = \alpha[\mu_0] < 1$, *the sequence* $(\mathbf{z}_{\Lambda_i})_i$ *produced by* **awgm** *satisfies*

$$\|\mathbf{z} - \mathbf{z}_{\Lambda_i}\| \lesssim \alpha^i \|\mathbf{z} - \mathbf{z}_{\Lambda_0}\|.$$

If, for whatever $s > 0$, $\mathbf{z} \in \mathcal{A}^s$, *then* $\#(\Lambda_{i+1} \setminus \Lambda_0) \lesssim \|\mathbf{z} - \mathbf{z}_{\Lambda_i}\|^{-1/s}$.

The *computation* of the approximate Galerkin solution $\mathbf{z}_{\Lambda_{i+1}}$ can be implemented by performing the simple fixed point iteration

$$\mathbf{z}^{(j+1)}_{\Lambda_{i+1}} = \mathbf{z}^{(j)}_{\Lambda_{i+1}} - \omega\mathbf{F}(\mathbf{z}^{(j)}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}. \tag{2.3.2}$$

Taking $\omega > 0$ to be a sufficiently small constant and starting with $\mathbf{z}^{(0)}_{\Lambda_{i+1}} = \mathbf{z}_{\Lambda_i}$, a fixed number of iterations suffices to meet the condition $\|\mathbf{F}(\mathbf{z}^{(j+1)}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}\| \le \gamma\|\mathbf{r}_i\|$. This holds also true when each of the $\mathbf{F}()|_{\Lambda_{i+1}}$ evaluations is performed within an absolute tolerance that is a sufficiently small fixed multiple of $\|\mathbf{r}_i\|$.

Optimal *computational* complexity of the **awgm** –meaning that the work to obtain an approximation within a given tolerance $\varepsilon > 0$ can be bounded on some constant multiple of the bound on its support length from Thm. 2.3.3,– is guaranteed under the

following two conditions concerning the cost of the "bulk chasing" process, and that of the approximate residual evaluation, respectively. Indeed, apart from some obvious computations, these are the only two tasks that have to be performed in **awgm**.

Condition 2.3.4. The determination of Λ_{i+1} in Algorithm 2.3.2 is performed in $\mathcal{O}(\#\operatorname{supp}\mathbf{r}_i + \#\Lambda_i)$ operations.

In case of unconstrained approximation, i.e., any finite $\Lambda \subset \vee$ is admissible, this condition is satisfied by collecting the largest entries in modulus of $\mathbf{r}_i$, where, to avoid a suboptimal complexity, an exact sorting should be replaced by an approximate sorting based on binning. With tree approximation, the condition is satisfied by the application of the so-called *Thresholding Second Algorithm* from [BD04]. We refer to [Ste14, §3.4] for a discussion.

To understand the second condition, that in the introduction was referred to as the cost condition (2.1.1), note that inside the **awgm** it is never needed to approximate a residual more accurately than within a sufficiently small, but fixed relative tolerance.

Condition 2.3.5. For a sufficiently small, fixed $\varsigma > 0$, there exists a neighborhood $\mathbf{Z}$ of the solution $\mathbf{z}$ of $\mathbf{F}(\mathbf{z}) = 0$, such that for all admissible $\Lambda \subset \vee$, $\tilde{\mathbf{z}} \in \ell_2(\Lambda) \cap \mathbf{Z}$, and any $\varepsilon > 0$, there exists an $\mathbf{r} \in \ell_2(\vee)$ with

$$\|\mathbf{F}(\tilde{\mathbf{z}}) - \mathbf{r}\| \leq \varsigma \|\mathbf{F}(\tilde{\mathbf{z}})\| + \varepsilon,$$

that one can compute in $\mathcal{O}(\varepsilon^{-1/s} + \#\Lambda)$ operations. Here $s > 0$ is such that $\mathbf{z} \in \mathcal{A}^s$.

Under both conditions, the **awgm** has optimal computational complexity:

Theorem 2.3.6. *In the setting of Theorem 2.3.3, and under Conditions 2.3.4 and 2.3.5, not only $\#\mathbf{z}_{\Lambda_i}$, but also the number of arithmetic operations required by* **awgm** *for the computation of $\mathbf{z}_{\Lambda_i}$ is $\mathcal{O}(\|\mathbf{z} - \mathbf{z}_{\Lambda_i}\|^{-1/s})$.*

2.4 Application to normal equations

As discussed in Sect. 2.2, we will apply the **awgm** to the (nonlinear) normal equations $DQ(u, \theta) = 0$, with DQ from (2.2.6). That is, we apply the findings collected in the previous section for the general triple $(F, \mathscr{H}, z)$ now reading as $(DQ, \mathscr{U} \times \mathscr{T}, (u, \theta))$.

For $\Psi^{\mathscr{U}} = \{\psi^{\mathscr{U}}_\lambda : \lambda \in \vee_{\mathscr{U}}\}$ and $\Psi^{\mathscr{T}} = \{\psi^{\mathscr{T}}_\lambda : \lambda \in \vee_{\mathscr{T}}\}$ being Riesz bases for $\mathscr{U}$ and $\mathscr{T}$, respectively, we equip $\mathscr{U} \times \mathscr{T}$ with Riesz basis

$$\Psi = \{\psi_\lambda : \lambda \in \vee := \vee_{\mathscr{U}} \cup \vee_{\mathscr{T}}\} := (\Psi^{\mathscr{U}}, 0_{\mathscr{T}}) \cup (0_{\mathscr{U}}, \Psi^{\mathscr{T}}) \tag{2.4.1}$$

(w.l.o.g. we assume that $\vee_{\mathscr{U}} \cap \vee_{\mathscr{T}} = \emptyset$). With $\mathcal{F} \in \mathcal{L}\mathrm{is}(\mathscr{U} \times \mathscr{T}, \ell_2(\vee))$ being the corresponding analysis operator, and $D\mathbf{Q} := \mathcal{F} DQ \mathcal{F}'$, for $[\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top \in \ell_2(\vee)$, and with $(\tilde{u}, \tilde{\theta}) := [\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]\Psi$, we have

$$\begin{aligned} D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) = &\begin{bmatrix} DG_0(\tilde{u})(\Psi^{\mathscr{U}})(\Psi^{\mathscr{V}})^\top \\ G_1(\Psi^{\mathscr{T}})(\Psi^{\mathscr{V}})^\top \end{bmatrix} \big(G_0(\tilde{u}) + G_1\tilde{\theta}\big)(\Psi^{\mathscr{V}}) \\ &+ \left\langle \begin{bmatrix} -G_2(\Psi^{\mathscr{U}}) \\ \Psi^{\mathscr{T}} \end{bmatrix}, \tilde{\theta} - G_2\tilde{u} \right\rangle_{\mathscr{T}}. \end{aligned} \tag{2.4.2}$$

In this setting, using Lemma 2.3.1, Condition 2.3.5 can be reformulated as follows:

Condition 2.3.5*. For a sufficiently small, fixed $\varsigma > 0$, there exists a neighborhood of the solution (u, θ) of $DQ(u, \theta) = 0$, such that for all admissible $\Lambda \subset \vee$, all $[\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top \in \ell_2(\Lambda)$, with $(\tilde{u}, \tilde{\theta}) := [\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]\Psi$ being in this neighborhood, and any $\varepsilon > 0$, there exists an $\mathbf{r} \in \ell_2(\vee)$ with

$$\|D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) - \mathbf{r}\| \leq \varsigma(\|u - \tilde{u}\|_{\mathscr{U}} + \|\theta - \tilde{\theta}\|_{\mathscr{T}}) + \varepsilon,$$

that one can compute in $\mathcal{O}(\varepsilon^{-1/s} + \#\Lambda)$ operations, where $s > 0$ is such that $[\mathbf{u}^\top, \theta^\top]^\top \in \mathcal{A}^s$.

To verify this condition, we will use the additional property, i.e. on top of (1)–(4), that $\|u - \tilde{u}\|_{\mathscr{U}} + \|\theta - \tilde{\theta}\|_{\mathscr{T}} \eqsim \|G_0(\tilde{u}) - G_1\tilde{\theta}\|_{\mathscr{V}'} + \|\tilde{\theta} - G_2\tilde{u}\|_{\mathscr{T}}$, which is provided by Lemma 2.2.6.

2.5 Semi-linear 2nd order elliptic PDE

We apply the solution method outlined in Sect. 2.2-2.4 to the example of a semi-linear 2nd order elliptic PDE with general (inhomogeneous) boundary conditions. The main task will be to verify Condition 2.3.5*.

2.5.1 Reformulation as a first order system least squares problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Gamma_N \cup \Gamma_D = \partial\Omega$ with $\mathrm{meas}(\Gamma_N \cap \Gamma_D) = 0$, $\mathrm{meas}(\Gamma_D) > 0$ when $\Gamma_D \neq \emptyset$, and $A : \Omega \to \mathbb{R}^{n\times n}_{\mathrm{symm}}$ with $\xi^\top A(\cdot)\xi \eqsim \|\xi\|^2$ ($\xi \in \mathbb{R}^n$, a.e.). We set

$$\boxed{\mathscr{U} := H^1(\Omega)} \text{ or, in case } \mathrm{meas}(\Gamma_D) = 0, \text{ possibly } \boxed{\mathscr{U} := H^1(\Omega)/\mathbb{R}},$$

and

$$\boxed{\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 := \{u \in \mathscr{U} : u|_{\Gamma_D} = 0\} \times H^{-\frac{1}{2}}(\Gamma_D)}.$$

For $N : \mathscr{U} \supset \mathrm{dom}(N) \to \mathscr{V}_1'$, $f \in \mathscr{V}_1'$, $g \in H^{\frac{1}{2}}(\Gamma_D)$, and $h \in H^{-\frac{1}{2}}(\Gamma_N)$, we consider the semi-linear boundary value problem

$$\left\{\begin{array}{rl} -\operatorname{div} A\nabla u + N(u) = f & \text{on } \Omega, \\ u = g & \text{on } \Gamma_D, \\ A\nabla u \cdot \mathbf{n} = h & \text{on } \Gamma_N, \end{array}\right. \tag{2.5.1}$$

that in standard variational form reads as finding $u \in \mathscr{U}$ such that

$$(Gu)(v) := \int_\Omega A\nabla u \cdot \nabla v_1 + (N(u) - f)v_1 dx - \int_{\Gamma_N} hv_1\, ds + \int_{\Gamma_D} (u - g)v_2\, ds = 0$$

$(v = (v_1, v_2) \in \mathscr{V})$.

We assume that this variational problem has a solution u, and that G, i.e., N, is two times continuously Fréchet differentiable in a neighborhood of u, and $DG(u) \in \mathcal{L}(\mathscr{U}, \mathscr{V}')$ is a homeomorphism with its range, i.e., we *assume* that

$$G : \mathscr{U} \supset \mathrm{dom}(G) \to \mathscr{V}' \text{ satisfies (i) – (iii)}$$

formulated in Sect. 2.2.

Remarks 2.5.1. Because $\mathscr{U} \to H^{\frac{1}{2}}(\Gamma_D) \colon u \mapsto u|_{\partial\Omega}$ is surjective, from [Ste13, Thm. 2.1] it follows that condition (iii) is satisfied when

$$L := w \mapsto \Big(v_1 \mapsto \int_\Omega A\nabla w \cdot \nabla v_1 + DN(u)(w)v_1 \, dx\Big) \in \mathcal{L}\mathrm{is}(\mathscr{V}_1, \mathscr{V}_1')$$

(actually, L being a homeomorphism with its range is already sufficient).

By writing $g = u_0|_{\Gamma_D}$ for some $u_0 \in \mathscr{U}$, one infers that for linear N, existence of a (unique) solution u, i.e. (i), follows from $L \in \mathcal{L}\mathrm{is}(\mathscr{V}_1, \mathscr{V}_1')$. For $g = 0$, the conditions of N being monotone and locally Lipschitz are sufficient for having a (unique) solution u. Relaxed conditions on N suffice to have a (locally unique) solution. We refer to [BS11].

Using the framework outlined in Sect. 2.2, we write this second order elliptic PDE as a first order system least squares problem. Putting $\boxed{\mathscr{T} = L_2(\Omega)^n}$, we define

$$G_1 \in \mathcal{L}(\mathscr{T}, \mathscr{V}'), \qquad G_2 \in \mathcal{L}(\mathscr{U}, \mathscr{T}),$$

by

$$G_2 u = A\nabla u, \qquad (G_1\vec{\theta})(v_1, v_2) = \int_\Omega \vec{\theta} \cdot \nabla v_1 \, dx.$$

The results from Sect. 2.2 show that the solution u can be found as the first component of the minimizer $(u, \vec{\theta}) \in \mathscr{U} \times \mathscr{T}$ of

$$\begin{aligned} Q(u, \vec{\theta}) := \tfrac{1}{2}\Big(\big\| v_1 \mapsto \int_\Omega \vec{\theta} \cdot \nabla v_1 + (N(u) - f)v_1 \, dx - \int_{\Gamma_N} v_1 h \, ds \big\|^2_{\mathscr{V}_1'} + \\ \|\vec{\theta} - A\nabla u\|^2_{L_2(\Omega)^n} + \|u - g\|^2_{\mathscr{V}_2'} \Big), \end{aligned} \tag{2.5.2}$$

being the solution of the normal equations $DQ(u, \vec{\theta}) = 0$, and furthermore, that these normal equations are well-posed in the sense that they satisfy (1)–(4).

To deal with the 'unpractical' norm on $\mathscr{V}'$, as in Sect. 2.1.4, at the end of Sect. 2.2, and in Sect. 2.4, we equip $\mathscr{V}_1$ and $\mathscr{V}_2$ with wavelet *Riesz bases*

$$\Psi^{\mathscr{V}_1} = \{\psi_\lambda^{\mathscr{V}_1} \colon \lambda \in \vee_{\mathscr{V}_1}\}, \quad \Psi^{\mathscr{V}_2} = \{\psi_\lambda^{\mathscr{V}_2} \colon \lambda \in \vee_{\mathscr{V}_2}\},$$

and replace, in the definition of Q, the norms on their duals by the equivalent norms defined by $\|g(\Psi^{\mathscr{V}_1})\|$ or $\|g(\Psi^{\mathscr{V}_2})\|$, for $g \in \mathscr{V}_1'$ or $g \in \mathscr{V}_2'$, respectively.

Next, after equiping $\mathscr{U}$ and $\mathscr{T}$ with *Riesz bases*

$$\Psi^{\mathscr{U}} = \{\psi^{\mathscr{U}}_\lambda : \lambda \in \vee_{\mathscr{U}}\}, \quad \Psi^{\mathscr{T}} = \{\psi^{\mathscr{T}}_\lambda : \lambda \in \vee_{\mathscr{T}}\},$$

and so $\mathscr{U} \times \mathscr{T}$ with $\Psi = (\Psi^{\mathscr{U}}, 0_{\mathscr{T}}) \cup (0_{\mathscr{U}}, \Psi^{\mathscr{T}})$, we apply the **awgm** to the resulting system

$$D\mathbf{Q}([\mathbf{u}^\top, \boldsymbol{\theta}^\top]^\top) = \Big\langle \begin{bmatrix} A\nabla\Psi^{\mathscr{U}} \\ -\Psi^{\mathscr{T}} \end{bmatrix}, A\nabla u - \vec{\theta} \Big\rangle_{L_2(\Omega)^n} + \begin{bmatrix} \langle \Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_2} \rangle_{L_2(\Gamma_D)} \\ 0_{\vee_{\mathscr{T}}} \end{bmatrix} \langle \Psi^{\mathscr{V}_2}, u - g \rangle_{L_2(\Gamma_D)} +$$
$$\begin{bmatrix} \langle DN(u)\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1} \rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}}, \nabla\Psi^{\mathscr{V}_1} \rangle_{L_2(\Omega)^n} \end{bmatrix} \Big\{ \big\langle \Psi^{\mathscr{V}_1}, N(u) - f \big\rangle_{L_2(\Omega)} - \big\langle \Psi^{\mathscr{V}_1}, h \big\rangle_{L_2(\Gamma_N)} + \big\langle \nabla\Psi^{\mathscr{V}_1}, \vec{\theta} \big\rangle_{L_2(\Omega)^n} \Big\} = 0,$$

where $(u, \vec{\theta}) := [\mathbf{u}^\top, \boldsymbol{\theta}^\top]\Psi$.

To express the three terms in $v \mapsto \langle v, N(u) - f\rangle_{L_2(\Omega)} - \langle v, h\rangle_{L_2(\Gamma_N)} + \langle \nabla v, \vec{\theta}\rangle_{L_2(\Omega)^n} \in \mathscr{V}_1'$ w.r.t. one dictionary of functions on Ω and one dictionary of functions on Γ_N, similarly to Sect. 2.1.4 we impose the additional, but in applications easily realisable condition that

$$\Psi^{\mathscr{T}} \subset H(\mathrm{div}; \Omega). \tag{2.5.3}$$

Then for finitely supported approximations $[\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top$ to $[\mathbf{u}^\top, \boldsymbol{\theta}^\top]^\top$, for $(\tilde{u}, \vec{\tilde{\theta}}) := [\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]\Psi \in \mathscr{U} \times H(\mathrm{div}; \Omega)$, we have

$$\boxed{\begin{aligned} D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) = {} & \Big\langle \begin{bmatrix} A\nabla\Psi^{\mathscr{U}} \\ -\Psi^{\mathscr{T}} \end{bmatrix}, A\nabla\tilde{u} - \vec{\tilde{\theta}} \Big\rangle_{L_2(\Omega)^n} + \begin{bmatrix} \langle \Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_2} \rangle_{L_2(\Gamma_D)} \\ 0_{\vee_{\mathscr{T}}} \end{bmatrix} \langle \Psi^{\mathscr{V}_2}, \tilde{u} - g \rangle_{L_2(\Gamma_D)} + \\ & \begin{bmatrix} \langle DN(\tilde{u})\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1} \rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}}, \nabla\Psi^{\mathscr{V}_1} \rangle_{L_2(\Omega)^n} \end{bmatrix} \Big\{ \big\langle \Psi^{\mathscr{V}_1}, N(\tilde{u}) - f - \mathrm{div}\, \vec{\tilde{\theta}} \big\rangle_{L_2(\Omega)} + \big\langle \Psi^{\mathscr{V}_1}, \vec{\tilde{\theta}} \cdot \mathbf{n} - h \big\rangle_{L_2(\Gamma_N)} \Big\}, \end{aligned}} \tag{2.5.4}$$

where we used the vanishing traces of $v \in \mathscr{V}_1$ at Γ_D, to write $\langle \nabla v, \vec{\tilde{\theta}}\rangle_{L_2(\Omega)^n}$ as $\langle v, -\mathrm{div}\, \vec{\tilde{\theta}}\rangle_{L_2(\Omega)} + \langle v, \vec{\tilde{\theta}} \cdot \mathbf{n}\rangle_{L_2(\Gamma_N)}$.

Each of the terms $A\nabla\tilde{u} - \vec{\tilde{\theta}}$, $\tilde{u} - g$, $N(\tilde{u}) - f - \mathrm{div}\, \vec{\tilde{\theta}}$, and $\vec{\tilde{\theta}} \cdot \mathbf{n} - h$ correspond, in strong form, to a term of the least squares functional, and therefore their norms can be bounded by a multiple of the norm of the residual, which is the basis of our approximate residual evaluation. In order to verify Condition 2.3.5*, we have to collect some assumptions on the wavelets, which will be done in the next subsection.

Remark 2.5.2. If $\Gamma_D = \emptyset$, then obviously (2.5.4) should be read without the second term involving $\Psi^{\mathscr{V}_2}$. If $\Gamma_D \neq \emptyset$ and *homogeneous* Dirichlet boundary conditions are prescribed on Γ_D, i.e., $g = 0$, it is simpler to select $\mathscr{U} = \mathscr{V}_1 = \{u \in H^1(\Omega) \colon u|_{\Gamma_D} = 0\}$, and to omit integral over Γ_D in the definition of G, so that again (2.5.4) should be read without the second term involving $\Psi^{\mathscr{V}_2}$.

2.5.2 Wavelet assumptions and definitions

We formulate conditions on $\Psi^{\mathscr{V}_1}$, $\Psi^{\mathscr{V}_2}$, $\Psi^{\mathscr{U}}$, and $\Psi^{\mathscr{T}}$, in addition to being Riesz bases for $\mathscr{V}_1$, $\mathscr{V}_2$, $\mathscr{U}$, and $\mathscr{T}$, respectively.

Recalling that $\mathscr{T} = \mathscr{T}_1 \times \cdots \times \mathscr{T}_n$, we select $\Psi^{\mathscr{T}}$ of canonical form

$$(\Psi^{\mathscr{T}_1}, 0_{\mathscr{T}_2}, \dots, 0_{\mathscr{T}_n}) \cup \cdots \cup (0_{\mathscr{T}_1}, \dots, 0_{\mathscr{T}_{n-1}}, \Psi^{\mathscr{T}_n}),$$

where $\Psi^{\mathscr{T}_q} = \{\psi_\lambda^{\mathscr{T}_q} \colon \lambda \in \vee_{\mathscr{T}_q}\}$ is a Riesz basis for $\mathscr{T}_q$ (with $\vee_{\mathscr{T}_{q'}} \cap \vee_{\mathscr{T}_{q''}} = \emptyset$ when $q' \neq q''$).

For $* \in \{\mathscr{U}, \mathscr{T}_1, \dots, \mathscr{T}_n, \mathscr{V}_1\}$, we collect a number of (standard) assumptions, (w_1)–(w_4), on the scalar-valued wavelet collections $\Psi^* = \{\psi_\lambda^* \colon \lambda \in \vee_*\}$ on Ω. Corresponding assumptions on the wavelets $\Psi^{\mathscr{V}_2}$ on Γ_D will be formulated at the end of this subsection. To each $\lambda \in \vee_*$, we associate a value $|\lambda| \in \mathbb{N}_0$, which is called the *level* of λ. We will assume that the elements of Ψ^* have *one vanishing moment*, and are *locally supported, piecewise polynomial* of some degree m, w.r.t. *dyadically nested partitions* in the following sense:

- (w_1) There exists a collection $\mathcal{O}_\Omega := \{\omega \colon \omega \in \mathcal{O}_\Omega\}$ of closed polytopes, such that, with $|\omega| \in \mathbb{N}_0$ being the *level* of ω, $\operatorname{meas}(\omega \cap \omega') = 0$ when $|\omega| = |\omega'|$ and $\omega \neq \omega'$; for any $\ell \in \mathbb{N}_0$, $\bar{\Omega} = \cup_{|\omega|=\ell} \omega$; $\operatorname{diam} \omega \eqsim 2^{-|\omega|}$; and ω is the union of ω' for some ω' with $|\omega'| = |\omega| + 1$. We call ω the *parent* of its *children* ω'. Moreover, we assume that the $\omega \in \mathcal{O}_\Omega$ are uniformly *shape regular*, in the sense that they satisfy a uniform Lipschitz condition, and $\operatorname{meas}(F_\omega) \eqsim \operatorname{meas}(\omega)^{\frac{n-1}{n}}$ for F_ω being any facet of ω.

- (w_2) $\operatorname{supp} \psi_\lambda^*$ is contained in a connected union of a uniformly bounded number of ω's with $|\omega| = |\lambda|$, and restricted to each of these ω's is ψ_λ^* a polynomial of degree m.

- (w_3) Each ω is intersected by the supports of a uniformly bounded number of ψ_λ^*'s with $|\lambda| = |\omega|$.

- (w_4) $\int_\Omega \psi_\lambda^* \, dx = 0$, possibly with the exception of those λ with $\operatorname{dist}(\operatorname{supp} \psi_\lambda^*, \Gamma_D) \lesssim 2^{-|\lambda|}$, or with $|\lambda| = 0$.

Generally, the polynomial degree m will be different for the different bases, but otherwise fixed. The collection $\mathcal{O}_\Omega$ is shared among all bases. Note that the conditions of $\Psi^{\mathscr{U}}$ being a basis for $\mathscr{U}$, and to consist of piecewise polynomials, implies that $\mathscr{U} \subset C(\bar{\Omega})$. Wavelets of in principle arbitrary order that satisfy these assumptions can be found in e.g. [DS99c, NS09].

Definition 2.5.3. A collection $\mathcal{T} \subset \mathcal{O}_\Omega$ such that $\overline{\Omega} = \cup_{\omega \in \mathcal{T}} \omega$, and for $\omega_1 \neq \omega_2 \in \mathcal{T}$, $\operatorname{meas}(\omega_1 \cap \omega_2) = 0$ will be called a *tiling*. With $\mathcal{P}_m(\mathcal{T})$, we denote the space of piecewise polynomials of degree m w.r.t. $\mathcal{T}$. The smallest common refinement of tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ is denoted as $\mathcal{T}_1 \oplus \mathcal{T}_2$.

To be able to find, in linear complexity, a representation of a function, given as linear combination of wavelets, as a piecewise polynomial w.r.t. a tiling –mandatory for an efficient evaluation of nonlinear terms–, we will impose a tree constraint on the underlying set of wavelet indices. A similar approach was followed earlier in [DSX00, CDD03b, XZ05, BU08, Vor09].

Definition 2.5.4. To each $\lambda \in \vee_*$ with $|\lambda| > 0$, we associate one $\mu \in \vee_*$ with $|\mu| = |\lambda| - 1$ and $\operatorname{meas}(\operatorname{supp}\psi^*_\lambda \cap \operatorname{supp}\psi^*_\mu) > 0$. We call μ the *parent* of λ, and so λ a *child* of μ.

To each $\lambda \in \vee_*$, we associate some neighbourhood $\mathcal{S}(\psi^*_\lambda)$ of $\operatorname{supp}\psi^*_\lambda$, with diameter $\lesssim 2^{-|\lambda|}$, such that $\mathcal{S}(\psi^*_\lambda) \subset \mathcal{S}(\psi^*_\mu)$ when λ is a child of μ.

We call a finite $\Lambda \subset \vee_*$ a *tree*, if it contains all $\lambda \in \vee_*$ with $|\lambda| = 0$, as well as the parent of any $\lambda \in \Lambda$ with $|\lambda| > 0$.

Note that we now have tree structures on the set $\mathcal{O}_\Omega$ of polytopes, and as well as on the wavelet index sets $\vee_*$. We trust that no confusion will arise when we speak about parents or children.

For some collections of wavelets, as the Haar or more generally, Alpert wavelets ([Alp93]), it suffices to take $\mathcal{S}(\psi^*_\lambda) := \operatorname{supp}\psi^*_\lambda$. The next result shows that, thanks to (w_1)-(w_2), a suitable neighbourhood $\mathcal{S}(\psi^*_\lambda)$ as meant in Definition 2.5.4 always exists.

Lemma 2.5.5. *With $C := \sup_{\lambda\in\vee_*} 2^{|\lambda|}\operatorname{diam}\operatorname{supp}\psi^*_\lambda$, a valid choice of $\mathcal{S}(\psi^*_\lambda)$ is given by $\{x \in \Omega\colon \operatorname{dist}(x, \operatorname{supp}\psi^*_\lambda) \le C2^{-|\lambda|}\}$.*

Proof. For $\mu, \lambda \in \vee_*$ with $|\mu| = |\lambda| - 1$ and $\operatorname{meas}(\operatorname{supp}\psi^*_\lambda \cap \operatorname{supp}\psi^*_\mu) > 0$, and $x \in \Omega$ with $\operatorname{dist}(x, \operatorname{supp}\psi^*_\lambda) \le C2^{-|\lambda|}$, it holds that $\operatorname{dist}(x, \operatorname{supp}\psi^*_\mu) \le \operatorname{dist}(x, \operatorname{supp}\psi^*_\lambda) + \operatorname{diam}(\operatorname{supp}\psi^*_\lambda) \le C2^{-|\mu|}$. □

A proof of the following proposition, as well as an algorithm to apply the multi-to-single-scale transformation that is mentioned, is given in [Ste14, §4.3].

Proposition 2.5.6. *Given a tree $\Lambda \subset \vee_*$, there exists a tiling $\mathcal{T}(\Lambda) \subset \mathcal{O}_\Omega$ with $\#\mathcal{T}(\Lambda) \lesssim \#\Lambda$ such that $\operatorname{span}\{\psi^*_\lambda\colon \lambda \in \Lambda\} \subset \mathcal{P}_m(\mathcal{T}(\Lambda))$. Moreover, equipping $\mathcal{P}_m(\mathcal{T}(\Lambda))$ with a basis of functions, each of which supported in ω for one $\omega \in \mathcal{T}(\Lambda)$, the representation of this embedding, known as the multi- to single-scale transform, can be applied in $\mathcal{O}(\#\Lambda)$ operations.*

The benefit of the definition of $\mathcal{S}(\psi^*_\lambda)$ appears from the following lemma.

Lemma 2.5.7. *Let $\overline{\Omega} = \Sigma_0 \supseteq \Sigma_1 \supseteq \cdots$. Then*

$$\Lambda^* := \big\{\lambda \in \vee_*\colon \operatorname{meas}(\mathcal{S}(\psi^*_\lambda) \cap \Sigma_{|\lambda|}) > 0\big\}$$

is a tree.

Proof. The set Λ^* contains all $\lambda \in \vee_*$ with $|\lambda| = 0$, as well as, by definition of $\mathcal{S}(\cdot)$, the parent of any $\lambda \in \Lambda^*$ with $|\lambda| > 0$. □

In Proposition 2.5.6 we saw that for each tree Λ there exists a tiling $\mathcal{T}(\Lambda)$, with $\#\mathcal{T}(\Lambda) \lesssim \#\Lambda$, such that $\operatorname{span}\{\psi^*_\lambda\colon \lambda \in \Lambda\} \subset \mathcal{P}_m(\mathcal{T}(\Lambda))$. Conversely, in the following, given a tiling $\mathcal{T}$, and a constant $k \in \mathbb{N}_0$, we construct a tree $\Lambda^*(\mathcal{T}, k)$ with $\#\Lambda^*(\mathcal{T}, k) \lesssim \#\mathcal{T}$ (dependent on k) such that a kind of reversed statements hold: In Appendix 2.9, statements of type $\lim_{k\to\infty}\sup_{0\neq g\in\mathcal{P}_m(\mathcal{T})} \frac{\|\langle\Psi^*, g\rangle_{L_2(\Omega)}|_{\vee_*\setminus\Lambda^*(\mathcal{T},k)}\|}{\|g\|_{*'}} = 0$ will be shown,

meaning that for any tolerance there exist a k such that for any $g \in \mathcal{P}_m(\mathcal{T})$ the relative error in dual norm in the best approximation from the span of the corresponding dual wavelets with indices in $\Lambda^*(\mathcal{T},k)$ is less than this tolerance.

Definition 2.5.8. Given a tiling $\mathcal{T} \subset \mathcal{O}_\Omega$, let $t(\mathcal{T}) \subset \mathcal{O}_\Omega$ be its enlargement by adding all ancestors of all $\omega \in \mathcal{T}$. Given a $k \in \mathbb{N}_0$, we set

$$\Lambda^*(\mathcal{T},k) := \big\{\lambda \in \vee_*\colon \operatorname{meas}\big(\mathcal{S}(\psi^*_\lambda) \cap \bigcup_{\{\omega\in t(\mathcal{T})\colon\, |\omega|=\max(|\lambda|-k,0)\}} \omega\big) > 0\big\}.$$

Proposition 2.5.9. *The set $\Lambda^*(\mathcal{T},k)$ is a tree, and $\#\Lambda^*(\mathcal{T},k) \lesssim \#\mathcal{T}$ (dependent on $k \in \mathbb{N}_0$).*

Proof. The first statement follows from Lemma 2.5.7. Since the number of children of any $\omega \in \mathcal{O}_\Omega$ is uniformly bounded, it holds that $\#t(\mathcal{T}) \lesssim \#\mathcal{T}$, and so $\#\Lambda^*(\mathcal{T},k) \lesssim \#\mathcal{T}$ as a consequence of the wavelets being locally supported. □

Example 2.5.10. Let $\Psi = \{\psi_\lambda\colon \lambda \in \vee\}$ be the collection of Haar wavelets on $\Omega = (0,1)$, i.e., the union of the function $\psi_{0,0} \equiv 1$, and, for $\ell \in \mathbb{N}$ and $k = 0,\dots,2^{\ell-1}-1$, the functions $\psi_{\ell,k} := 2^{\frac{\ell-1}{2}}\psi(2^{\ell-1}(\cdot)-k)$, where $\psi \equiv 1$ on $[0,\frac12]$ and $\psi \equiv -1$ on $(\frac12,1]$. Writing $\lambda = (\ell,k)$, we set $|\lambda| = \ell$. The parent of λ with $|\lambda| > 0$ is μ with $|\mu| = |\lambda|-1$ and $\operatorname{supp}\psi_\lambda \subset \operatorname{supp}\psi_\mu$, and $\mathcal{S}(\psi_\lambda) := \operatorname{supp}\psi_\lambda$.

Let $\mathcal{O}_\Omega$ be the union, for $\ell \in \mathbb{N}_0$ and $k = 0,\dots,2^\ell-1$, of the intervals $2^\ell[k,k+1]$ to which we assign the level ℓ.

Now as an example let $\Lambda \subset \vee$ be the set $\{(0,0),(1,0),(2,0),(3,0)\}$, which is a tree in the sense of Definition 2.5.4. It corresponds to the solid parts in the left picture in Figure 2.1. The (minimal) tiling $\mathcal{T}(\Lambda)$ as defined in Proposition 2.5.6 is given by

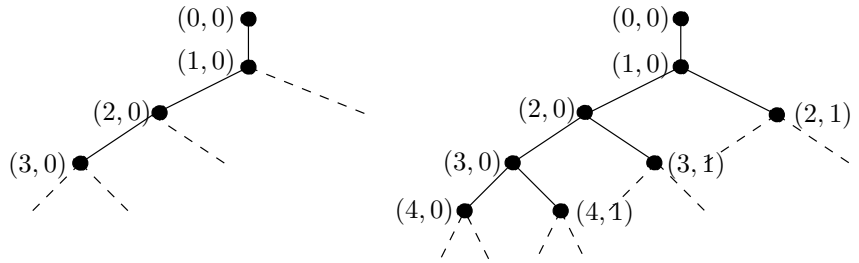


Figure 2.1: The index set of the Haar basis given as an infinite binary tree, and its subsets $\Lambda = \{(0,0),(1,0),(2,0),(3,0)\}$ (left) and $\Lambda(\mathcal{T}(\Lambda),1)$ (right).

$\{[0,\frac18],[\frac18,\frac14],[\frac14,\frac12],[\frac12,1]\}$.

Conversely, taking $\mathcal{T} := \mathcal{T}(\Lambda)$, the set $\Lambda(\mathcal{T},1) \subset \vee$ as defined in Definition 2.5.8 is given by $\{(0,0),(1,0),(2,0),(2,1),(3,0),(3,1),(4,0),(4,1)\}$ and is illustrated in the right picture in Figure 2.1.

Definition 2.5.11. Recalling from (2.4.1) and the first lines of this subsection that the Riesz basis Ψ for $\mathscr{U} \times \mathscr{T}$ is of canonical form

$$\Psi = (\Psi^{\mathscr{U}}, 0_{\mathscr{T}_1}, \dots, 0_{\mathscr{T}_n}) \cup (0_{\mathscr{U}}, \Psi^{\mathscr{T}_1}, 0_{\mathscr{T}_2}, \dots, 0_{\mathscr{T}_n}) \cup \cdots \cup (0_{\mathscr{U}}, 0_{\mathscr{T}_1}, \dots, 0_{\mathscr{T}_{n-1}}, \Psi^{\mathscr{T}_n}),$$

with index set $\vee := \vee_{\mathscr{U}} \cup \vee_{\mathscr{T}_1} \cup \cdots \cup \vee_{\mathscr{T}_n}$, we call $\Lambda \subset \vee$ *admissible* when each of $\Lambda \cap \vee_{\mathscr{U}}, \Lambda \cap \vee_{\mathscr{T}_1}, \dots, \Lambda \cap \vee_{\mathscr{T}_n}$ are trees. The tiling $\mathcal{T}(\Lambda)$ is defined as the smallest common refinement $\mathcal{T}(\Lambda \cap \vee_{\mathscr{U}}) \oplus \mathcal{T}(\Lambda \cap \vee_{\mathscr{T}_1}) \oplus \cdots \oplus \mathcal{T}(\Lambda \cap \vee_{\mathscr{T}_n})$. Conversely, given a tiling $\mathcal{T} \subset \mathcal{O}_\Omega$ and a $k \in \mathbb{N}_0$, we define the admissible set $\Lambda(\mathcal{T}, k) \subset \vee$ by $\Lambda^{\mathscr{U}}(\mathcal{T}, k) \cup \Lambda^{\mathscr{T}_1}(\mathcal{T}, k) \cup \cdots \cup \Lambda^{\mathscr{T}_n}(\mathcal{T}, k)$.

Remark 2.5.12. Let $\Psi^{\mathscr{U}}$ be a wavelet basis for $\mathscr{U}$ of order $d_{\mathscr{U}} > 1$ (i.e., all wavelets $\psi^{\mathscr{U}}_\lambda$ up to level ℓ span all piecewise polynomials in $\mathscr{U}$ of degree $d_{\mathscr{U}} - 1$ w.r.t. $\{\omega\colon \omega \in \mathcal{O}_\Omega, |\omega| = \ell\}$), and similarly, for $1 \le q \le n$, let $\Psi^{\mathscr{T}_q}$ be a wavelet basis for $\mathscr{T}_q$ of order $d_{\mathscr{T}} > 0$. Recalling the definition of an approximation class given in (2.3.1), a sufficiently smooth solution $(u, \vec{\theta})$ is in $\mathcal{A}^s$ for $s = s_{\max} := \min(\frac{d_{\mathscr{U}}-1}{n}, \frac{d_{\mathscr{T}}}{n})$, whereas on the other hand membership of $\mathcal{A}^s$ for $s > s_{\max}$ cannot be expected under whatever smoothness condition.

For $s \le s_{\max}$, a sufficient and 'nearly' necessary condition for $(u, \vec{\theta}) \in \mathcal{A}^s$ is that $(u, \vec{\theta}) \in B^{sn+1}_{p,\tau}(\Omega) \times B^{sn}_{p,\tau}(\Omega)^n$ for $\frac{1}{p} < s + \frac{1}{2}$ and arbitrary $\tau > 0$, see [CDDD01]. This mild smoothness condition in the 'Besov scale' has to be compared to the condition $(u, \vec{\theta}) \in H^{sn+1}(\Omega) \times H^{sn}(\Omega)^n$ that is necessary to obtain a rate s with approximation from the spaces of type $\operatorname{span}\{\psi^{\mathscr{U}}_\lambda\colon |\lambda| \le L\} \times \prod_{q=1}^n \operatorname{span}\{\psi^{\mathscr{T}_q}_\lambda\colon |\lambda| \le L\}$.

We pause to add *three more assumptions* on our PDE: We assume that w.r.t. the coarsest possible tiling $\{\omega\colon \omega \in \mathcal{O}_\Omega, |\omega| = 0\}$ of $\bar{\Omega}$,

$$A \text{ is piecewise polynomial,} \tag{2.5.5}$$

$$N(u) \text{ is a partial differential operator of at most first order, with coefficients that are piecewise polynomials in } u \text{ and } x, \text{ and} \tag{2.5.6}$$

$$\overline{\Gamma}_N, \text{ and so } \overline{\Gamma}_D, \text{ is the union of facets of } \omega \in \mathcal{O}_\Omega \text{ with } |\omega| = 0. \tag{2.5.7}$$

Remark 2.5.13. The subsequent analysis can easily be generalized to A being piecewise smooth. With some more efforts other nonlinear terms N can be handled as well. For example, for $N(u)$ of the form $n_1(u)u$, it will be needed that for some $m \in \mathbb{N}$, for each $\omega \in \mathcal{O}_\Omega$, there exists a subspace $V_\omega \subset H^1_0(\omega)$ with $\|\cdot\|_{H^1(\omega)} \lesssim 2^{|\omega|}\|\cdot\|_{L_2(\omega)}$ on V_ω, and

$$\|N(p_1) + p_2\|_{L_2(\omega)} \lesssim \sup_{0 \ne v \in V_\omega} \frac{\langle N(p_1) + p_2, v\rangle_{L_2(\omega)}}{\|v\|_{L_2(\omega)}} \quad (p_1, p_2 \in \mathcal{P}_m(\omega))$$

(cf. proof of Lemma 2.9.2).

Finally in this subsection we formulate our assumptions on the wavelet Riesz basis $\Psi^{\mathscr{V}_2} = \{\psi^{\mathscr{V}_2}_\lambda\colon \lambda \in \vee_{\mathscr{V}_2}\}$ for $H^{-\frac{1}{2}}(\Gamma_D)$. We assume that it satisfies the assumptions (w_2) and (w_3) with $\mathcal{O}_\Omega$ reading as

$$\mathcal{O}_{\Gamma_D} = \{\omega \cap \Gamma_D\colon \omega \in \mathcal{O}_\Omega, \operatorname{meas}_{n-1}(\omega \cap \Gamma_D) > 0\}.$$

Furthermore, we impose that

$$\|\psi_\lambda^{\mathscr{V}_2}\|_{H^{-1}(\Gamma_D)} \lesssim 2^{-|\lambda|/2}. \tag{2.5.8}$$

which, for biorthogonal wavelets, is essentially is (w_4) (cf. [DS99c, lines following (A.2)]). (To relax the smoothness conditions on Γ_D needed for the definition of $H^{-1}(\Gamma_D)$, one can replace (2.5.8) by $\|\psi_\lambda^{\mathscr{V}_2}\|_{H^s(\Gamma_D)} \lesssim 2^{|\lambda|(s+\frac{1}{2})}$ for some $s \in [-1,-\frac{1}{2})$).

The definition of a boundary tiling $\mathcal{T}_{\Gamma_D} \subset \mathcal{O}_{\Gamma_D}$ is similar to Definition 2.5.3. Also similar to the corresponding preceding definitions are that of a tree $\Lambda \subset \vee_{\mathscr{V}_2}$, and of the boundary tiling $\mathcal{T}_{\Gamma_D}(\Lambda) \subset \mathcal{O}_{\Gamma_D}$ for $\Lambda \subset \vee_{\mathscr{V}_2}$ being a tree. Conversely, for a boundary tiling $\mathcal{T}_{\Gamma_D} \subset \mathcal{O}_{\Gamma_D}$ and $k \in \mathbb{N}_0$, for $* \in \{\mathscr{U}, \mathscr{V}_2\}$ we define the tree

$$\Lambda^*(\mathcal{T}_{\Gamma_D}, k) := \big\{\lambda \in \vee_*\colon \operatorname{meas}_{n-1}(\mathcal{S}(\psi_\lambda^*) \cap \bigcup_{\{\omega \in \overline{\mathcal{T}}_{\Gamma_D}\colon |\omega| = \max(|\lambda|-k,0)\}} \omega) > 0\big\}.$$

2.5.3 An appropriate approximate residual evaluation

Given an admissible $\Lambda \subset \vee$, $[\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top \in \ell_2(\Lambda)$ with $(\tilde{u}, \vec{\tilde{\theta}}) := [\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]\Psi$ sufficiently close to $(u, \vec{\theta})$, and an $\varepsilon > 0$, our approximate evaluation of $D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top)$, given in (2.5.4), is built in the following steps, where $k \in \mathbb{N}_0$ is a sufficiently large constant:

(s1) Find a tiling $\mathcal{T}(\varepsilon) \subset \mathcal{O}_\Omega$, such that

$$\inf_{(g_\varepsilon, f_\varepsilon, \vec{h}_\varepsilon) \in \mathcal{P}_m(\mathcal{T}(\varepsilon)\cap\Gamma_D)\cap C(\Gamma_D) \times \mathcal{P}_m(\mathcal{T}(\varepsilon)) \times \mathcal{P}_m(\mathcal{T}(\varepsilon))^n} \Big(\|g - g_\varepsilon\|_{H^{\frac{1}{2}}(\Gamma_D)} + \|v_1 \mapsto \int_\Omega (f - f_\varepsilon) v_1 \, dx + \int_{\Gamma_N} (h - \vec{h}_\varepsilon \cdot \mathbf{n}) v_1 \, ds \|_{\mathscr{V}_1'} \Big) \le \varepsilon.$$

Set $\mathcal{T}(\Lambda, \varepsilon) := \mathcal{T}(\Lambda) \oplus \mathcal{T}(\varepsilon)$.

(s2) (a) Approximate

$$\mathbf{r}_1^{(\frac{1}{2})} := \langle \Psi^{\mathscr{V}_1}, N(\tilde{u}) - f - \operatorname{div} \vec{\tilde{\theta}} \rangle_{L_2(\Omega)} + \langle \Psi^{\mathscr{V}_1}, \vec{\tilde{\theta}} \cdot \mathbf{n} - h \rangle_{L_2(\Gamma_N)}$$

by

$$\tilde{\mathbf{r}}_1^{(\frac{1}{2})} := \mathbf{r}_1^{(\frac{1}{2})}|_{\Lambda^{\mathscr{V}_1}(\mathcal{T}(\Lambda,\varepsilon),k)}.$$

(b) With $\tilde{r}_1^{(\frac{1}{2})} := (\tilde{\mathbf{r}}_1^{(\frac{1}{2})})^\top \Psi^{\mathscr{V}_1}$, approximate

$$\mathbf{r}_1 = \begin{bmatrix} \mathbf{r}_{11} \\ \mathbf{r}_{12} \end{bmatrix} := \begin{bmatrix} \langle DN(\tilde{u})\Psi^{\mathscr{U}}, \tilde{r}_1^{(\frac{1}{2})} \rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}}, \nabla \tilde{r}_1^{(\frac{1}{2})} \rangle_{L_2(\Omega)^n} \end{bmatrix}$$

by $\tilde{\mathbf{r}}_1 = \begin{bmatrix} \tilde{\mathbf{r}}_{11} \\ \tilde{\mathbf{r}}_{12} \end{bmatrix} := \mathbf{r}_1|_{\Lambda(\mathcal{T}(\Lambda^{\mathscr{V}_1}(\mathcal{T}(\Lambda,\varepsilon),k)),k)}$.

(s3) Approximate

$$\mathbf{r}_2 := \left\langle \begin{bmatrix} -A\nabla\Psi^{\mathscr{U}} \\ \Psi^{\mathscr{T}} \end{bmatrix}, \vec{\tilde{\theta}} - A\nabla\tilde{u} \right\rangle_{L_2(\Omega)^n}$$

by $\tilde{\mathbf{r}}_2 := \mathbf{r}_2|_{\Lambda(\mathcal{T}(\Lambda),k)}$.

(s4) (a) Approximate $\mathbf{r}_3^{(\frac{1}{2})} := \langle \Psi^{\mathscr{V}_2}, \tilde{u} - g\rangle_{L_2(\Gamma_D)}$ by

$$\tilde{\mathbf{r}}_3^{(\frac{1}{2})} := \mathbf{r}_3^{(\frac{1}{2})}|_{\Lambda^{\mathscr{V}_2}(\mathcal{T}(\Lambda,\varepsilon)\cap\Gamma_D,k)}.$$

(b) With $\tilde{r}_3^{(\frac{1}{2})} := (\tilde{\mathbf{r}}_3^{(\frac{1}{2})})^\top \Psi^{\mathscr{V}_2}$, approximate $\mathbf{r}_3 := \begin{bmatrix} \langle \Psi^{\mathscr{U}}, \tilde{r}_3^{(\frac{1}{2})}\rangle_{L_2(\Gamma_D)} \\ 0_{\vee_{\mathscr{T}}} \end{bmatrix}$ by

$$\tilde{\mathbf{r}}_3 := \mathbf{r}_3|_{\Lambda(\mathcal{T}_{\Gamma_D}(\Lambda^{\mathscr{V}_2}(\mathcal{T}(\Lambda,\varepsilon)\cap\Gamma_D,k)),k)}.$$

Although (s2)–(s4) may look involved at first glance, the same kind of approximation is used at all instances. Each term in (2.5.4) consists essentially of a wavelet basis that is integrated against a piecewise polynomial, or more precisely, a function that can be sufficiently accurately approximated by a piecewise polynomial thanks to the control of the data oscillation by the refinement of the partition performed in (s1). In each of these terms, all wavelets are neglected whose levels exceed locally the level of the partition plus a constant k.

In the next theorem it is shown that this approximate residual evaluation satisfies the condition for optimality of the adaptive wavelet Galerkin method.

Theorem 2.5.14. *For an admissible $\Lambda \subset \vee$, $[\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top \in \ell_2(\Lambda)$ with $(\tilde{u}, \vec{\tilde{\theta}}) := [\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]\Psi$ sufficiently close to $(u, \vec{\theta})$, and an $\varepsilon > 0$, consider the steps* (s1)-(s4). *With $s > 0$ such that $[\mathbf{u}^\top, \boldsymbol{\theta}^\top]^\top \in \mathcal{A}^s$, let $\mathcal{T}(\varepsilon)$ from* (s1) *satisfy $\#\mathcal{T}(\varepsilon) \lesssim \varepsilon^{-1/s}$. Then*

$$\|D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) - (\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3)\| \lesssim 2^{-k/2}(\|u - \tilde{u}\|_{\mathscr{U}} + \|\vec{\theta} - \vec{\tilde{\theta}}\|_{\mathscr{T}}) + \varepsilon,$$

where the computation of $\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3$ requires $\mathcal{O}(\#\Lambda + \varepsilon^{-1/s})$ operations. So by taking k sufficiently large, Condition 2.3.5 is satisfied.*

Before giving the proof of this theorem, let us discuss matters related to step (s1). First of all, *existence* of tilings $\mathcal{T}(\varepsilon)$ as mentioned in the theorem is guaranteed. Indeed, because of $[\mathbf{u}^\top, \boldsymbol{\theta}^\top]^\top \in \mathcal{A}^s$, given $C > 0$, for any $\varepsilon > 0$ there exists a $[\tilde{\mathbf{u}}_\varepsilon^\top, \tilde{\boldsymbol{\theta}}_\varepsilon^\top]^\top \in \ell_2(\Lambda_\varepsilon)$, where Λ_ε is admissible and $\#\Lambda_\varepsilon \lesssim \varepsilon^{-1/s}$, such that $\|[\mathbf{u}^\top, \boldsymbol{\theta}^\top]^\top - [\tilde{\mathbf{u}}_\varepsilon^\top, \tilde{\boldsymbol{\theta}}_\varepsilon^\top]^\top\| \leq \varepsilon/C$. By taking a suitable C, from Lemma 2.2.6, (2.5.2) and (2.5.3) we infer that with $(\tilde{u}_\varepsilon, \vec{\tilde{\theta}}_\varepsilon) := [\tilde{\mathbf{u}}_\varepsilon^\top, \tilde{\boldsymbol{\theta}}_\varepsilon^\top]\Psi$,

$$\|\tilde{u}_\varepsilon - g\|_{H^{\frac{1}{2}}(\Gamma_D)} + \|v_1 \mapsto \int_\Omega (N(\tilde{u}_\varepsilon) - \operatorname{div}\vec{\tilde{\theta}}_\varepsilon - f)v_1\,dx + \int_{\Gamma_N} (\vec{\tilde{\theta}}_\varepsilon \cdot \mathbf{n} - h)v_1\,ds\|_{\mathscr{V}_1'} \leq \varepsilon.$$

Since $\tilde{u}_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\Lambda_\varepsilon)) \cap C(\Omega)$, $N(\tilde{u}_\varepsilon) - \operatorname{div}\vec{\tilde{\theta}}_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\Lambda_\varepsilon))$, $\vec{\tilde{\theta}}_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\Lambda_\varepsilon))^n$, and $\#\mathcal{T}(\Lambda_\varepsilon) \lesssim \#\Lambda_\varepsilon$, the tiling $\mathcal{T}(\varepsilon) := \mathcal{T}(\Lambda_\varepsilon)$ satisfies the assumptions.

Loosely speaking, this result can be rephrased by saying that if the solution of $D\mathbf{Q}([\mathbf{u}^\top, \boldsymbol{\theta}^\top]^\top) = 0$ is in $\mathcal{A}^s$, then so is the forcing function (f, g, h). This is not automatically true, cf. [CDN12] for a discussion in the adaptive finite element context, but in the current setting it is a consequence of the fact that, thanks to assumption (2.5.3), the first order partial differential operators apply to the wavelet bases Ψ^* for $* \in \{\mathscr{U}, \mathscr{T}_1, \dots, \mathscr{T}_n, \mathscr{V}_1, \mathscr{V}_2\}$ in 'mild' sense (the result of the application of each of these operators lands in L_2-space).

Knowing that a suitable $\mathcal{T}(\varepsilon)$ exists is different from knowing how to construct it. For our convenience thinking of $g = h = 0$, and so $\mathscr{U} = \mathscr{V}_1 = H^1_0(\Omega)$, assuming that $f \in L_2(\Omega)$ one has $\inf_{f_\varepsilon \in \mathcal{P}_m(\mathcal{T})} \|f - f_\varepsilon\|^2_{H^{-1}(\Omega)} \lesssim \operatorname{osc}(f, \mathcal{T})^2 := \sum_{\omega\in\mathcal{T}} \operatorname{diam}(\omega)^2 \inf_{f_\omega \in \mathcal{P}_m(\omega)} \|f - f_\omega\|^2_{L_2(\omega)}$. Ignoring quadrature issues, for any partition $\mathcal{T}$, $\operatorname{osc}(f, \mathcal{T})$ is computable. A quasi-minimal partition $\mathcal{T}(\varepsilon)$ such that $\operatorname{osc}(f, \mathcal{T}(\varepsilon)) \lesssim \varepsilon$ can be computed using the Thresholding Second Algorithm from [BD04]. Now the assumption to be added to Theorem 2.5.14 is that for such a partition, $\#\mathcal{T}(\varepsilon) \lesssim \varepsilon^{-1/s}$.

Note that it is nowhere needed to explicitly approximate the forcing functions by approximating their wavelet expansions.

Proof of Theorem 2.5.14. By construction we have

$$\begin{aligned} D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) - (\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3) = \begin{bmatrix} \langle DN(\tilde{u})\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}}, \nabla\Psi^{\mathscr{V}_1}\rangle_{L_2(\Omega)^n} \end{bmatrix} (\mathbf{r}_1^{(\frac12)} - \tilde{\mathbf{r}}_1^{(\frac12)}) \\ + \mathbf{r}_1 - \tilde{\mathbf{r}}_1 + \mathbf{r}_2 - \tilde{\mathbf{r}}_2 + \begin{bmatrix} \langle \Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_2}\rangle_{L_2(\Gamma_D)} \\ 0_{\vee_{\mathscr{T}}} \end{bmatrix} (\mathbf{r}_3^{(\frac12)} - \tilde{\mathbf{r}}_3^{(\frac12)}) + \mathbf{r}_3 - \tilde{\mathbf{r}}_3. \end{aligned}$$

From $\Psi^{\mathscr{U}}$, $\Psi^{\mathscr{T}}$, and $\Psi^{\mathscr{V}_1}$ being Riesz bases, and $N : \mathscr{U} \supset \operatorname{dom}(N) \to \mathscr{V}_1'$ being continuously differentiable at u, one infers that $\begin{bmatrix} \langle DN(\tilde{u})\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}}, \nabla\Psi^{\mathscr{V}_1}\rangle_{L_2(\Omega)^n} \end{bmatrix} \in \mathcal{L}(\ell_2(\vee_{\mathscr{V}_1}), \ell_2(\vee))$, with a norm that is bounded uniformly in $\tilde{u}$ from a neighbourhood of u. Similarly, from $u \mapsto u|_{\Gamma_D} \in \mathcal{L}(\mathscr{U}, H^{\frac12}(\Gamma_D))$, and $\Psi^{\mathscr{U}}$ and $\Psi^{\mathscr{V}}_2$ being Riesz bases for $\mathscr{U}$ and $H^{\frac12}(\Gamma_D)'$, respectively, we infer that $\begin{bmatrix} \langle \Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_2}\rangle_{L_2(\Gamma_D)} \\ 0 \end{bmatrix} \in \mathcal{L}(\ell_2(\vee_{\mathscr{V}_2}), \ell_2(\vee))$. We conclude that

$$\begin{aligned} &\|D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) - (\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3)\| \lesssim \\ &\qquad \|\mathbf{r}_1^{(\frac12)} - \tilde{\mathbf{r}}_1^{(\frac12)}\| + \|\mathbf{r}_1 - \tilde{\mathbf{r}}_1\| + \|\mathbf{r}_2 - \tilde{\mathbf{r}}_2\| + \|\mathbf{r}_3^{(\frac12)} - \tilde{\mathbf{r}}_3^{(\frac12)}\| + \|\mathbf{r}_3 - \tilde{\mathbf{r}}_3\|. \end{aligned} \tag{2.5.9}$$

We bound all terms at the right-hand side.

With f_ε, $\vec{h}_\varepsilon$ from (s1), using that $\Psi^{\mathscr{V}_1}$ is a Riesz basis, we have that

$$\begin{aligned} \|\mathbf{r}_1^{(\frac12)} - \tilde{\mathbf{r}}_1^{(\frac12)}\| \lesssim \varepsilon + \\ \Big\| \Big(\langle \Psi^{\mathscr{V}_1}, N(\tilde{u}) - f_\varepsilon - \operatorname{div}\vec{\tilde{\theta}}\rangle_{L_2(\Omega)} + \langle \Psi^{\mathscr{V}_1}, (\vec{\tilde{\theta}} - \vec{h}_\varepsilon)\cdot\mathbf{n}\rangle_{L_2(\Gamma_N)} \Big)\big|_{\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T}(\Lambda,\varepsilon),k)} \Big\|. \end{aligned} \tag{2.5.10}$$

From $N(\tilde{u}) - f_\varepsilon - \operatorname{div}\vec{\tilde{\theta}} \in \mathcal{P}_m(\mathcal{T}(\Lambda,\varepsilon))$, and $\vec{\tilde{\theta}} - \vec{h}_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\Lambda,\varepsilon))^n$, Proposition 2.9.1 shows that the norm at the right-hand side of (2.5.10) is $\lesssim 2^{-k}\big\|v_1 \mapsto \int_\Omega \vec{\tilde{\theta}}\cdot\nabla v_1 + (N(\tilde{u})-f_\varepsilon)v_1\,dx - \int_{\Gamma_N}\vec{h}_\varepsilon\cdot\mathbf{n}v_1\,ds\big\|_{\mathscr{Y}_1'}$. Again by using (s1), we infer that

$$\|\mathbf{r}_1^{(\frac12)} - \tilde{\mathbf{r}}_1^{(\frac12)}\| \lesssim \varepsilon + 2^{-k}\big\|v_1 \mapsto \int_\Omega \vec{\tilde{\theta}}\cdot\nabla v_1 + (N(\tilde{u})-f)v_1\,dx - \int_{\Gamma_N} hv_1\,ds\big\|_{\mathscr{Y}_1'}. \quad (2.5.11)$$

Thanks to $\nabla\tilde{r}_1^{(\frac12)} \in \mathcal{P}_m(\mathcal{T}(\Lambda^{\mathscr{Y}_1}(\mathcal{T}(\Lambda,\varepsilon),k)))^n$, an application of Proposition 2.9.4 (first estimate) shows that

$$\|\mathbf{r}_{12} - \tilde{\mathbf{r}}_{12}\| \lesssim 2^{-k/2}\|\tilde{r}_1^{(\frac12)}\|_{\mathscr{Y}_1} \eqsim 2^{-k/2}\|\tilde{\mathbf{r}}_1^{(\frac12)}\|.$$

Our assumptions on N show that $DN(\tilde{u})w$ is of the form $p_1(\tilde{u})w + \vec{p}_2(\tilde{u})\cdot\nabla w$ for some piecewise polynomials p_1 and p_2 in $\tilde{u}$ and x w.r.t. $\{\omega\colon \omega\in\mathcal{O}_\Omega, |\omega|=0\}$, where moreover $w\mapsto p_1(\tilde{u})w \in \mathcal{L}(\mathscr{U},\mathscr{Y}_1')$, $\vec{w}\mapsto\vec{p}_2(\tilde{u})\cdot\vec{w}\in\mathcal{L}(L_2(\Omega)^n,\mathscr{Y}_1')$, uniformly in $\tilde{u}$ in a neighborhood of $u\in\mathscr{U}$. Consequently, applications of Propositions 2.9.3-2.9.4 (second estimate) show that

$$\begin{aligned}\|\mathbf{r}_{11} - \tilde{\mathbf{r}}_{11}\| &\lesssim 2^{-k}\|p_1(\tilde{u})\tilde{r}_1^{(\frac12)}\|_{\mathscr{U}'} + 2^{-k/2}\|\vec{p}_2(\tilde{u})\tilde{r}_1^{(\frac12)}\|_{L_2(\Omega)^n}\\ &\lesssim 2^{-k/2}\|\tilde{r}_1^{(\frac12)}\|_{\mathscr{Y}_1} \eqsim 2^{-k/2}\|\tilde{\mathbf{r}}_1^{(\frac12)}\|.\end{aligned}$$

Now use that $\|\tilde{\mathbf{r}}_1^{(\frac12)}\| \le \|\mathbf{r}_1^{(\frac12)} - \tilde{\mathbf{r}}_1^{(\frac12)}\| + \|\mathbf{r}_1^{(\frac12)}\|$, and $\|\mathbf{r}_1^{(\frac12)}\| \lesssim \big\|v_1\mapsto\int_\Omega\vec{\tilde{\theta}}\cdot\nabla v_1 + (N(\tilde{u})-f)v_1\,dx - \int_{\Gamma_N}hv_1\,ds\big\|_{\mathscr{Y}_1'}$, to conclude that

$$\|\mathbf{r}_1 - \tilde{\mathbf{r}}_1\| \lesssim 2^{-k/2}\Big(\varepsilon + \big\|v_1\mapsto\int_\Omega\vec{\tilde{\theta}}\cdot\nabla v_1 + (N(\tilde{u})-f)v_1\,dx - \int_{\Gamma_N}hv_1\,ds\big\|_{\mathscr{Y}_1'}\Big). \quad (2.5.12)$$

Thanks to $\vec{\tilde{\theta}} - A\nabla\tilde{u} \in \mathcal{P}_m(\mathcal{T}(\Lambda))^n$ by (2.5.5), and $A^\top$ being piecewise polynomial, an application of Proposition 2.9.4 shows that

$$\|\mathbf{r}_2 - \tilde{\mathbf{r}}_2\| \lesssim 2^{-k/2}\|\vec{\tilde{\theta}} - A\nabla\tilde{u}\|_{L_2(\Omega)^n}. \quad (2.5.13)$$

With g_ε from (s1), using that $\Psi^{\mathscr{Y}_2}$ is a Riesz basis for $H^{-\frac12}(\Gamma_D)$, we have that

$$\|\mathbf{r}_3^{(\frac12)} - \tilde{\mathbf{r}}_3^{(\frac12)}\| \lesssim \varepsilon + \|\langle\Psi^{\mathscr{Y}_2}, \tilde{u} - g_\varepsilon\rangle_{L_2(\Gamma_D)}|_{\vee_{\mathscr{Y}_2}\setminus\Lambda^{\mathscr{Y}_2}(\mathcal{T}(\Lambda,\varepsilon)\cap\Gamma_D,k)}\|. \quad (2.5.14)$$

From $\tilde{u}|_{\Gamma_D} - g_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\Lambda,\varepsilon)\cap\Gamma_D)\cap C(\Gamma_D)$, Proposition 2.9.5 shows that the norm at the right-hand side of (2.5.14) is $\lesssim 2^{-k/2}\|\tilde{u} - g_\varepsilon\|_{H^{\frac12}(\Gamma_D)}$. Again by using (s1), we infer that

$$\|\mathbf{r}_3^{(\frac12)} - \tilde{\mathbf{r}}_3^{(\frac12)}\| \lesssim \varepsilon + 2^{-k/2}\|\tilde{u} - g\|_{H^{\frac12}(\Gamma_D)}. \quad (2.5.15)$$

Thanks to $\tilde{r}_3^{(\frac{1}{2})} \in \mathcal{P}_m(\mathcal{T}_{\Gamma_D}(\Lambda^{\mathscr{V}_2}(\mathcal{T}(\Lambda,\varepsilon)\cap\Gamma_D,k)))$, Proposition 2.9.6 shows that $\|\mathbf{r}_3-\tilde{\mathbf{r}}_3\| \lesssim 2^{-k/2}\|\tilde{r}_3^{(\frac{1}{2})}\|_{H^{-\frac{1}{2}}(\Gamma_D)} \eqsim 2^{-k/2}\|\tilde{\mathbf{r}}_3^{(\frac{1}{2})}\|$. Now use that $\|\tilde{\mathbf{r}}_3^{(\frac{1}{2})}\| \le \|\mathbf{r}_3^{(\frac{1}{2})} - \tilde{\mathbf{r}}_3^{(\frac{1}{2})}\| + \|\mathbf{r}_3^{(\frac{1}{2})}\|$, and $\|\mathbf{r}_3^{(\frac{1}{2})}\| \lesssim \|\tilde{u}-g\|_{H^{\frac{1}{2}}(\Gamma_D)}$ to conclude that

$$\|\mathbf{r}_3-\tilde{\mathbf{r}}_3\| \lesssim 2^{-k/2}\big(\varepsilon + \|\tilde{u}-g\|_{H^{\frac{1}{2}}(\Gamma_D)}\big). \tag{2.5.16}$$

By collecting the upper bounds (2.5.11)–(2.5.16) derived for all five terms at the right-hand side of (2.5.9), and by using Lemma 2.2.6 in combination with the least squares functional given in (2.5.2), the proof of the first statement is completed.

To bound the cost of the computations, we consider the computation of $\tilde{\mathbf{r}}_1^{(\frac{1}{2})}$. First, find a representation of $N(\tilde{u})-\operatorname{div}\vec{\tilde{\theta}}$ as an element of $\mathcal{P}_m(\mathcal{T}(\Lambda,\varepsilon))$ by applying multi- to single-scale transforms. For each tile $\omega \in \mathcal{T}(\Lambda^{\mathscr{V}_1}(\mathcal{T}(\Lambda,\varepsilon),k))$, and for ϕ running over some basis of $\mathcal{P}_m(\omega)$, compute $\langle \phi, N(\tilde{u})-f-\operatorname{div}\vec{\theta}\rangle_{L_2(\omega)}$. From this, compute $[\langle\psi_\lambda^{\mathscr{V}_1}, N(\tilde{u})-f-\operatorname{div}\vec{\tilde{\theta}}\rangle_{L_2(\Omega)}]_{\lambda\in\Lambda^{\mathscr{V}_1}(\mathcal{T}(\Lambda,\varepsilon),k)}$ by applying a transpose of a multi- to single-scale transform. Similar steps yield $[\langle\psi_\lambda^{\mathscr{V}_1}, \vec{\tilde{\theta}}\cdot\mathbf{n}-h\rangle_{L_2(\Gamma_N)}]_{\lambda\in\Lambda^{\mathscr{V}_1}(\mathcal{T}(\Lambda,\varepsilon),k)}$. The total cost involved in computing $\tilde{\mathbf{r}}_1^{(\frac{1}{2})}$ is bounded by a multiple of $\#\mathcal{T}(\Lambda,\varepsilon) \lesssim \#\Lambda + \varepsilon^{-1/s}$ operations.

Since fully analogous considerations apply to bounding the cost of the computations of $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_2$, $\tilde{\mathbf{r}}_3^{(\frac{1}{2})}$, and $\tilde{\mathbf{r}}_3$, the proof is completed. □

2.6 Numerical results

For $\Omega \subset \mathbb{R}^2$ being the L-shaped domain $(0,1)^2 \setminus [\frac{1}{2},1)^2$, we consider the semi-linear boundary value problem

$$\begin{cases} -\Delta u + u^3 = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6.1}$$

where, for simplicity, $f=1$ (to test our code we also tried some right hand sides corresponding to some fabricated polynomial solutions u). With $\mathscr{U}=H_0^1(\Omega)=\mathscr{V}$, $\mathscr{T}=L_2(\Omega)^2$, we applied the **awgm** (Algorithm 2.3.2), with $\mathbf{F}$ reading as $D\mathbf{Q}$, for the adaptive solution of $[\mathbf{u}^\top,\boldsymbol{\theta}^\top]^\top$ from

$$\begin{aligned} D\mathbf{Q}([\mathbf{u}^\top,\boldsymbol{\theta}^\top]^\top) = &\begin{bmatrix} \langle\partial_1\Psi^{\mathscr{U}},\partial_1 u-\theta_1\rangle_{L_2(\Omega)} + \langle\partial_2\Psi^{\mathscr{U}},\partial_2 u-\theta_2\rangle_{L_2(\Omega)} \\ \langle\Psi^{\mathscr{T}_1},\theta_1-\partial_1 u\rangle_{L_2(\Omega)} \\ \langle\Psi^{\mathscr{T}_2},\theta_2-\partial_2 u\rangle_{L_2(\Omega)} \end{bmatrix} \\ &+\begin{bmatrix} \langle\Psi^{\mathscr{U}},3u^2\Psi^{\mathscr{V}}\rangle_{L_2(\Omega)} \\ \langle\Psi^{\mathscr{T}_1},\partial_1\Psi^{\mathscr{V}}\rangle_{L_2(\Omega)} \\ \langle\Psi^{\mathscr{T}_2},\partial_2\Psi^{\mathscr{V}}\rangle_{L_2(\Omega)} \end{bmatrix}\big\langle\Psi^{\mathscr{V}},u^3-f-\operatorname{div}\vec{\theta}\big\rangle_{L_2(\Omega)} = \mathbf{0}, \end{aligned}$$

where $u := \mathbf{u}^\top\Psi^{\mathscr{U}}$, $\theta_i := \boldsymbol{\theta}_i^\top\Psi^{\mathscr{T}_i}$.

Here we equipped $\mathscr{T}_i$ $(i = 1, 2)$ with the continuous piecewise linear three-point wavelet basis from [Ste98b], the space $\mathscr{V}$ with the same basis (obviously scaled differently, and with homogeneous boundary conditions incorporated), and $\mathscr{U}$ with a newly developed continuous piecewise quadratic wavelet basis.

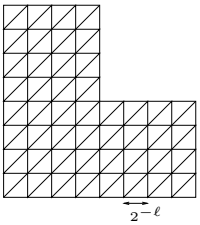


Figure 2.2: Meshes w.r.t. which the wavelets are piecewise polynomial.

These bases can be applied on any polygon, and they satisfy all assumptions (w_1)-(w_4). In particular all wavelets except possibly those 'near' the Dirichlet boundary have one vanishing moment. For each basis, to each wavelet that is not on the coarsest level we associate one parent on the next coarsest level according to Definition 2.5.4. For any $\ell \in \mathbb{N}_0$ the subsets of the bases consisting of all wavelets up to some level span exactly the space of continuous piecewise linears, continuous piecewise linears zero at $\partial\Omega$, or continuous piecewise quadratics zero at $\partial\Omega$, respectively, w.r.t. the subdivision of Ω as indicated in Figure 2.2.

On a bounded domain, the three-point basis has actually not been proven to be stable in $L_2(\Omega)$. Although alternative bases are available whose Riesz basis property has been proven, we opted for the three-point basis, because of its efficient implementation and because numerical results indicated that it is stable. In Figure 2.3, numerically computed condition numbers are given of sets of all wavelets up to some level.

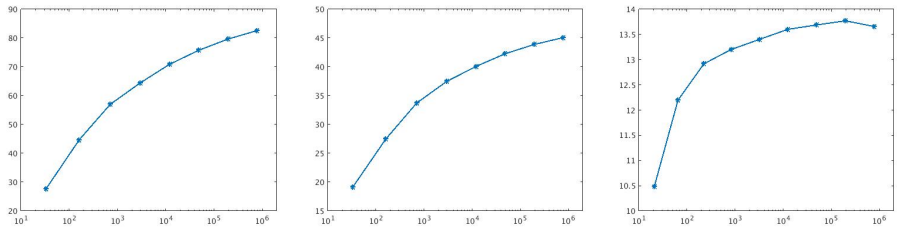


Figure 2.3: Condition numbers of $\langle \nabla\Psi_N^{\mathscr{U}}, \nabla\Psi_N^{\mathscr{U}} \rangle_{L_2(\Omega)^2}$, $\langle \nabla\Psi_N^{\mathscr{V}}, \nabla\Psi_N^{\mathscr{V}} \rangle_{L_2(\Omega)^2}$, and $\langle \Psi_N^{\mathscr{T}_i}, \Psi_N^{\mathscr{T}_i} \rangle_{L_2(\Omega)}$, where Ψ_N^* is the subset of all wavelets from Ψ^* up to some level, where N denotes its cardinality.

The continuous piecewise quadratic wavelets are biorthogonal ones with the 'dual multiresolution analysis' being the sequence of continuous piecewise linears, zero at the $\partial\Omega$, w.r.t. one additional level of refinement. Details of this basis construction are given in the next chapter.

We performed the approximate evaluation of $D\mathbf{Q}(\cdot)$ according to (s1)-(s4) and Theorem 2.5.14 in Sect. 2.5.3 with some simplifications because of the current homogeneous Dirichlet boundary conditions and sufficiently smooth right-hand side ((s1) and (s4) are void, and in (s2) the boundary term is void). Taking the parameter $k = 1$, it turns out that the approximate evaluation is sufficiently accurate to be used in Step (R) of **awgm** (so we do not perform a loop), as well in the simple fixed point iteration (2.3.2) with damping $\omega = \frac{1}{4}$ that we use for Step (G). We took the parameter γ in Step (G) equal to 0.15 (more precisely, for stopping the iteration we checked whether the norm of the approximate residual, restricted to Λ_{i+1}, is less or equal to $0.15\|\mathbf{r}_i\|$).

For the bulk chasing, i.e. Step (B), we simply collected the indices of the largest entries of the approximate residual $\mathbf{r}_i$ until the norm of the residual restricted to those indices is not less than $0.4\|\mathbf{r}_i\|$ (i.e. $\mu_1 = 0.4$), and then, after adding the indices from the current $\mathbf{\Lambda}_i$ to this set, we expand it to an admissible set (cf. Definition 2.5.11). Although this simple procedure is neither guaranteed to satisfy Condition 2.3.4 nor (B) for some constant $0 < \mu_0 \leq \mu_1$, we observed that it works satisfactory in practice.

In view of the orders 3 and 2 of the bases for $\mathscr{U}$ and $\mathscr{T}$, and the fact the PDE is posed in $n = 2$ space dimensions, the best possible convergence rate that can be expected is $\min(\frac{3-1}{2}, \frac{2-0}{2}) = 1$. In Figure 2.4, we show the norm of the approximate residual vs. the total number of wavelets underlying the approximation for $(u, \vec{\theta})$. The

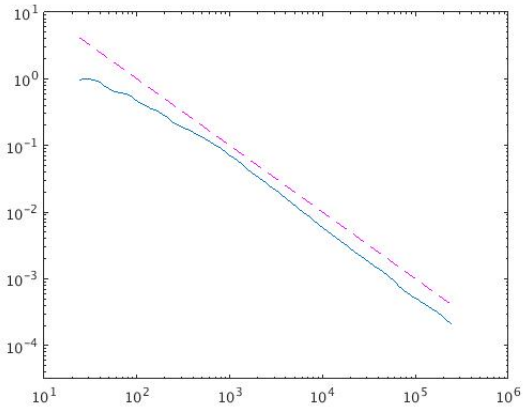


Figure 2.4: Norm of the (approximate) residual, normalized by the norm of the initial residual, generated by the **awgm** vs. the total number of wavelets. The dotted line indicates the best possible slope -1.

norm of the approximate residual is proportional to the $\mathscr{U} \times \mathscr{T}$-norm of the error in the approximation for $(u, \vec{\theta})$. We conclude that it decays with the best possible rate. Moreover, we observed that the computing times scale linearly with the number of unknowns. Throughout the iteration, the number of wavelets for the approximation for u is of the same order as the number of wavelets for the approximation for $\vec{\theta}$. The maximum level that is reached at the end of the computations is 26 for u and 28 for $\vec{\theta}$). An approximate solution is illustrated in Figure 2.5. Centers of the supports of the wavelets that were selected for the approximation for u are illustrated in Figure 2.6.

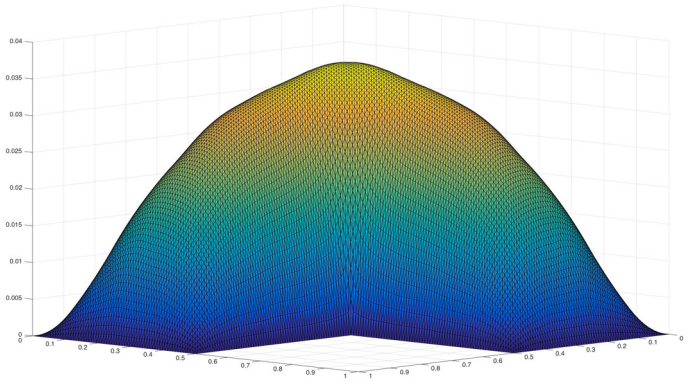


Figure 2.5: The approximation for u from the span of 202 wavelets.

Finally, in order to get an impression of the condition number of the bi-infinite linearized normal equations that eventually we are solving, we consider the Poisson equation, i.e. (2.6.1) without the u^3 term. We are interested in the spectral condition number of the 'system matrix' given by

$$\begin{bmatrix} \langle \nabla \Psi^{\mathscr{U}}, \nabla \Psi^{\mathscr{U}} \rangle_{L_2(\Omega)^2} & -\langle \partial_1 \Psi^{\mathscr{U}}, \Psi^{\mathscr{T}_1} \rangle_{L_2(\Omega)} & -\langle \partial_2 \Psi^{\mathscr{U}}, \Psi^{\mathscr{T}_2} \rangle_{L_2(\Omega)} \\ -\langle \Psi^{\mathscr{T}_1}, \partial_1 \Psi^{\mathscr{U}} \rangle_{L_2(\Omega)} & \langle \Psi^{\mathscr{T}_1}, \Psi^{\mathscr{T}_1} \rangle_{L_2(\Omega)} & 0 \\ -\langle \Psi^{\mathscr{T}_2}, \partial_2 \Psi^{\mathscr{U}} \rangle_{L_2(\Omega)} & 0 & \langle \Psi^{\mathscr{T}_1}, \Psi^{\mathscr{T}_1} \rangle_{L_2(\Omega)} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \langle \Psi^{\mathscr{T}_1}, \partial_1 \Psi^{\mathscr{V}} \rangle_{L_2(\Omega)} \langle \partial_1 \Psi^{\mathscr{V}}, \Psi^{\mathscr{T}_1} \rangle_{L_2(\Omega)} & \langle \Psi^{\mathscr{T}_1}, \partial_1 \Psi^{\mathscr{V}} \rangle_{L_2(\Omega)} \langle \partial_2 \Psi^{\mathscr{V}}, \Psi^{\mathscr{T}_2} \rangle_{L_2(\Omega)} \\ 0 & \langle \Psi^{\mathscr{T}_2}, \partial_2 \Psi^{\mathscr{V}} \rangle_{L_2(\Omega)} \langle \partial_1 \Psi^{\mathscr{V}}, \Psi^{\mathscr{T}_1} \rangle_{L_2(\Omega)} & \langle \Psi^{\mathscr{T}_2}, \partial_2 \Psi^{\mathscr{V}} \rangle_{L_2(\Omega)} \langle \partial_2 \Psi^{\mathscr{V}}, \Psi^{\mathscr{T}_2} \rangle_{L_2(\Omega)} \end{bmatrix} \tag{2.6.2}$$

To that end we numerically approximated the condition numbers of the finite square blocks of rows and columns with indices in Λ, with Λ running over the wavelet index sets that were created by the **awgm**. Even such a finite block cannot be evaluated exactly, because it still involves the infinite collection $\Psi^{\mathscr{V}}$. Given a Λ, we restricted this

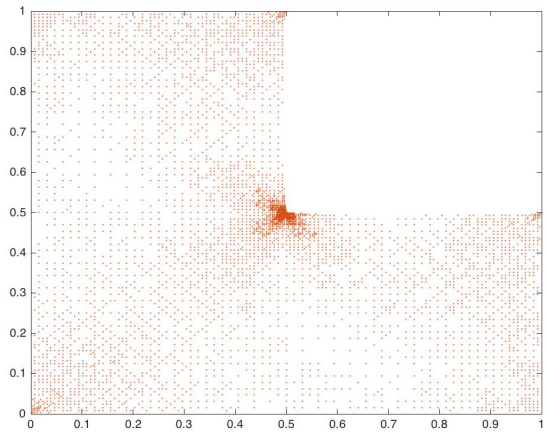


Figure 2.6: Centers of the supports of the first 10366 wavelets for the approximation for u that were selected by the **awgm**.

collection to the wavelets with indices in $\Lambda^{\mathscr{V}}(\mathcal{T}(\Lambda), k)$ as defined by Proposition 2.5.6 and Definition 2.5.8, where, as always, we take $k = 1$. The resulting matrix is exactly the one that we approximately invert in Step (G) by the fixed point iteration. The computed condition numbers are given in Figure 2.7. We performed the same computation also for $k = 2$, so with an enlarged set of wavelet indices from the basis $\Psi^{\mathscr{V}}$, and found nearly indistinguishable results. We may conclude that for $\Lambda \to \infty$, the given numbers give accurate approximations for the condition number of the matrix in (2.6.2).

2.7 Stationary Navier-Stokes equations

For $n \in \{2, 3, 4\}$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The stationary Navier-Stokes equations in velocity–pressure formulation and with no-slip boundary conditions are given by

$$\left\{ \begin{array}{rcll} -\nu \triangle \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p &=& \vec{f} & \text{on } \Omega \\ \operatorname{div} \vec{u} &=& g & \text{on } \Omega \\ \vec{u} &=& 0 & \text{on } \partial\Omega. \end{array} \right.$$

In order to obtain, in any case in the linear Stokes case, results that hold uniformly in $\nu > 0$, one may equip the spaces for velocities and pressure with ν-dependent norms. The equivalent, but notationally more convenient approach that we will follow is to keep the standard norms, but to make the substitutions $\breve{u} = \sqrt{\nu}\,\vec{u}$, $\breve{p} = \frac{1}{\sqrt{\nu}} p$,

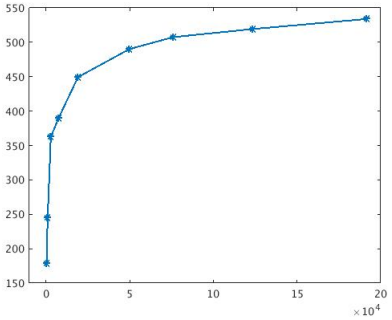


Figure 2.7: Condition numbers of the (approximate) Galerkin system matrices vs. $\#\Lambda$.

$\breve{\vec{f}} = \frac{1}{\sqrt{\nu}}\vec{f}$, and $\breve{g} = \sqrt{\nu}\, g$. For convenience dropping the $\breve{}$-accents, the equations for the new unknowns read as

$$\left\{\begin{array}{rcll} -\triangle\vec{u} + \nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u} + \nabla p &=& \vec{f} & \text{on } \Omega \\ \operatorname{div}\vec{u} &=& g & \text{on } \Omega \\ \vec{u} &=& 0 & \text{on } \partial\Omega. \end{array}\right.$$

In variational form they read as finding $(\vec{u}, p) \in \boxed{\mathscr{U} := H^1_0(\Omega)^n \times L_2(\Omega)/\mathbb{R}}$ such that for some $(\vec{f}, g) \in \mathscr{U}'$,

$$G(\vec{u}, p)(\vec{v}, q) := \int_\Omega \nabla\vec{u} : \nabla\vec{v} - p \operatorname{div}\vec{v} + \nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v} + q(\operatorname{div}\vec{u} - g) - \vec{f}\cdot\vec{v}\, dx = 0$$

$((\vec{v}, q) \in \boxed{\mathscr{V} := \mathscr{U}})$.

It is known that $G : \mathscr{U} \to \mathscr{U}'$, and that a solution $(\vec{u}, p)$ exists (see e.g. [GR79, Ch. IV]). Furthermore, G is two times differentiable with its second derivative being constant. We will *assume* that $DG(\vec{u}, p) \in \mathcal{L}(\mathscr{U}, \mathscr{U}')$ is a homeomorphism with its range, so that each of the conditions (i)–(iii) from Sect. 2.2 are satisfied. The latter is known to hold true, with its range being equal to $\mathscr{U}'$, when $\vec{f}$ is sufficiently small, in which case the solution $(\vec{u}, p)$ is also unique (e.g. see [GR79, Ch. IV]). For the linear case, so without the term $\nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u}$, thanks to our re-scaling, $DG(\vec{u}, p) = G \in \mathcal{L}\mathrm{is}(\mathscr{U}, \mathscr{U}')$, and is independent of ν.

Using the framework outlined in Sect. 2.2, we write this second order elliptic PDE as a first order system least squares problem. There are different possibilities to do so.

2.7.1 Velocity–pressure–velocity gradient formulation

With $\boxed{\mathscr{T} := L_2(\Omega)^{n^2}}$, we define

$$G_1 \in \mathcal{L}(\mathscr{T}, \mathscr{U}'), \qquad G_2 \in \mathcal{L}(\mathscr{U}, \mathscr{T}),$$

by

$$G_2(\vec{u}, p) = \nabla\vec{u}, \qquad (G_1\underline{\theta})(\vec{v}, q) = \int_\Omega \underline{\theta} : \nabla\vec{v}\, dx$$

The results from Sect. 2.2 show that the solution $(\vec{u}, p)$ can be found as the first components of the minimizer $(\vec{u}, p, \underline{\theta}) \in \mathscr{U} \times \mathscr{T}$ of

$$Q(\vec{u}, p, \underline{\theta}) := \tfrac{1}{2}\Big(\big\| \vec{v} \mapsto \int_\Omega \underline{\theta} : \nabla\vec{v} - p \operatorname{div}\vec{v} + \nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v} - \vec{f}\cdot\vec{v} \big\|^2_{H^{-1}(\Omega)^n} + \\ \|\operatorname{div}\vec{u} - g\|^2_{L_2(\Omega)} + \|\underline{\theta} - \nabla\vec{u}\|^2_{L_2(\Omega)^{n^2}} \Big), \tag{2.7.1}$$

and so as the solution of the normal equations $DQ(\vec{u}, p, \underline{\theta}) = 0$. Here we have used that on $H^1_0(\Omega)^n$, $\|\operatorname{div}\cdot\|_{(L_2(\Omega)/\mathbb{R})'} = \|\operatorname{div}\cdot\|_{L_2(\Omega)}$. Following [BG09], we call $\underline{\theta} = \nabla\vec{u}$ the velocity gradient. As follows from Sect. 2.2, these normal equations are well-posed in the sense that they satisfy (1)–(4). This gives us an alternative, effortless proof for [CMM97b, Thm. 3.1].

To deal with the 'unpractical' norm on $H^{-1}(\Omega)^n$, we equip $H^1_0(\Omega)^n$ with some wavelet Riesz basis

$$\Psi^{(\hat{H}^1_0)^n} = \{\psi_\lambda^{(\hat{H}^1_0)^n} : \lambda \in \vee_{(\hat{H}^1_0)^n}\},$$

and replace, in the definition of Q, the norm on its dual by the equivalent norm defined by $\|\vec{h}(\Psi^{(\hat{H}^1_0)^n})\|$ for $\vec{h} \in H^{-1}(\Omega)^n$.

Next, after equipping $* \in \{H^1_0(\Omega)^n, L_2(\Omega)/\mathbb{R}, L_2(\Omega)^{n^2}\}$ with a Riesz basis $\Psi^* = \{\psi^*_\lambda : \lambda \in \vee_*\}$, and so $H^1_0(\Omega)^n \times L_2(\Omega)/\mathbb{R} \times L_2(\Omega)^{n^2}$ with

$$\Psi := (\Psi^{(H^1_0)^n}, 0_{L_2/\mathbb{R}}, 0_{L_2^{n^2}}) \cup (0_{(H^1_0)^n}, \Psi^{L_2/\mathbb{R}}, 0_{L_2^{n^2}}) \cup (0_{(H^1_0)^n}, 0_{L_2/\mathbb{R}}, \Psi^{L_2^{n^2}}),$$

with index set $\vee := \vee_{(H^1_0)^n} \cup \vee_{L_2/\mathbb{R}} \cup \vee_{L_2^{n^2}}$, we apply the **awgm** to the resulting system

$$D\mathbf{Q}([\mathbf{u}^\top, \mathbf{p}^\top, \boldsymbol{\theta}^\top]^\top) = \begin{bmatrix} \langle \operatorname{div}\Psi^{(H^1_0)^n}, \operatorname{div}\vec{u} - g\rangle_{L_2(\Omega)} \\ 0_{\vee_{L_2/\mathbb{R}}} \\ 0_{\vee_{L_2^{n^2}}} \end{bmatrix} + \begin{bmatrix} \langle \nabla\Psi^{(H^1_0)^n}, \nabla\vec{u} - \underline{\theta}\rangle_{L_2(\Omega)^{n^2}} \\ 0_{\vee_{L_2/\mathbb{R}}} \\ \langle \Psi^{L_2^{n^2}}, \underline{\theta} - \nabla\vec{u}\rangle_{L_2(\Omega)^{n^2}} \end{bmatrix} + \\ \begin{bmatrix} \langle \frac{(\vec{u}\cdot\nabla)\Psi^{(H^1_0)^n} + (\Psi^{(H^1_0)^n}\cdot\nabla)\vec{u}}{\nu^{3/2}}, \Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)^n} \\ -\langle \Psi^{L_2/\mathbb{R}}, \operatorname{div}\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)} \\ \langle \Psi^{L_2^{n^2}}, \nabla\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)^{n^2}} \end{bmatrix} \Big\{ \langle \Psi^{(\hat{H}^1_0)^n}, \tfrac{(\vec{u}\cdot\nabla)\vec{u}}{\nu^{3/2}} - \vec{f}\rangle_{L_2(\Omega)^n} \\ + \langle \nabla\Psi^{(\hat{H}^1_0)^n}, \underline{\theta}\rangle_{L_2(\Omega)^{n^2}} - \langle \operatorname{div}\Psi^{(\hat{H}^1_0)^n}, p\rangle_{L_2(\Omega)} \Big\} = 0.$$

To express the three terms in $\vec{v} \mapsto \langle \vec{v}, \nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u} - \vec{f}\rangle_{L_2(\Omega)^n} + \langle \nabla\vec{v}, \underline{\theta}\rangle_{L_2(\Omega)^{n^2}} - \langle \operatorname{div}\vec{v}, p\rangle_{L_2(\Omega)} \in H^{-1}(\Omega)^n$ w.r.t. one dictionary, similarly to Sect. 2.1.4 we impose the additional, but in applications easily realizable conditions that

$$\Psi^{L_2/\mathbb{R}} \subset H^1(\Omega),\ \Psi^{L_2^{n^2}} \subset H(\operatorname{div};\Omega)^n. \tag{2.7.2}$$

Then for finitely supported approximations $[\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top$ to $[\mathbf{u}^\top, \mathbf{p}^\top, \boldsymbol{\theta}^\top]^\top$, for $(\tilde{\vec{u}}, \tilde{p}, \tilde{\underline{\theta}}) := [\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]\Psi \in H_0^1(\Omega)^n \times H^1(\Omega) \times H(\operatorname{div};\Omega)^n$, we have

$$\boxed{D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) = \begin{bmatrix} \langle \operatorname{div}\Psi^{(H_0^1)^n}, \operatorname{div}\tilde{\vec{u}} - g\rangle_{L_2(\Omega)} \\ 0_{\vee_{L_2/\mathbb{R}}} \\ 0_{\vee_{L_2^{n^2}}} \end{bmatrix} + \begin{bmatrix} \langle \nabla\Psi^{(H_0^1)^n}, \nabla\tilde{\vec{u}} - \tilde{\underline{\theta}}\rangle_{L_2(\Omega)^{n^2}} \\ 0_{\vee_{L_2/\mathbb{R}}} \\ \langle \Psi^{L_2^{n^2}}, \tilde{\underline{\theta}} - \nabla\tilde{\vec{u}}\rangle_{L_2(\Omega)^{n^2}} \end{bmatrix} + \begin{bmatrix} \langle \frac{(\tilde{\vec{u}}\cdot\nabla)\Psi^{(H_0^1)^n} + (\Psi^{(H_0^1)^n}\cdot\nabla)\tilde{\vec{u}}}{\nu^{3/2}}, \Psi^{(\hat{H}_0^1)^n}\rangle_{L_2(\Omega)^n} \\ -\langle \Psi^{L_2/\mathbb{R}}, \operatorname{div}\Psi^{(\hat{H}_0^1)^n}\rangle_{L_2(\Omega)} \\ -\langle \operatorname{div}\Psi^{L_2^{n^2}}, \Psi^{(\hat{H}_0^1)^n}\rangle_{L_2(\Omega)^n} \end{bmatrix} \langle \Psi^{(\hat{H}_0^1)^n}, \tfrac{(\tilde{\vec{u}}\cdot\nabla)\tilde{\vec{u}}}{\nu^{3/2}} - \vec{f} - \operatorname{div}\tilde{\underline{\theta}} + \nabla\tilde{p}\rangle_{L_2(\Omega)^n}.} \tag{2.7.3}$$

Each of the terms $\operatorname{div}\tilde{\vec{u}} - g$, $\nabla\tilde{\vec{u}} - \tilde{\underline{\theta}}$, $\nu^{-3/2}(\tilde{\vec{u}}\cdot\nabla)\tilde{\vec{u}} - \vec{f} - \operatorname{div}\tilde{\underline{\theta}} + \nabla\tilde{p}$ correspond, in strong form, to a term of the least squares functional, and therefore their norms can be bounded by a multiple of the norm of the residual, which is the basis of our approximate residual evaluation.

This approximate residual evaluation follows the same lines as with the elliptic problem from Sect. 2.5. Actually, things are easier here because we assume homogeneous boundary conditions. Selecting the Riesz bases for the Cartesian products $H_0^1(\Omega)^n$ and $L_2(\Omega)^{n^2}$ of canonical form, we assume that all scalar-valued bases Ψ^* for $* \in \{\hat{H}_0^1, H_0^1, L_2/\mathbb{R}, L_2\}$ satisfy the assumptions that were made in Sect. 2.5.2, in particular (w_1)–(w_4). Let $\Lambda := \operatorname{supp}[\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top$ be admissible, i.e., $\Lambda \cap \vee_*$ are trees.

(s1) Find a tiling $\mathcal{T}(\varepsilon) \subset \mathcal{O}_\Omega$, such that

$$\inf_{\vec{f}_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\varepsilon))^n,\, g_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\varepsilon))/\mathbb{R}} \|\vec{f} - \vec{f}_\varepsilon\|_{H^{-1}(\Omega)^n} + \|g - g_\varepsilon\|_{L_2(\Omega)} \le \varepsilon.$$

If $[\mathbf{u}^\top, \mathbf{p}^\top, \boldsymbol{\theta}^\top]^\top \in \mathcal{A}^s$, then such a tiling exists with $\#\mathcal{T}(\varepsilon) \lesssim \varepsilon^{-1/s}$. Set $\mathcal{T}(\Lambda, \varepsilon) := \mathcal{T}(\Lambda) \oplus \mathcal{T}(\varepsilon)$.

(s2) (a) Approximate $\mathbf{r}_1^{(\frac{1}{2})} := \langle \Psi^{(\hat{H}_0^1)^n}, \nu^{-3/2}(\vec{\tilde{u}}\cdot\nabla)\vec{\tilde{u}} - \vec{f} - \operatorname{div}\underline{\tilde{\theta}} + \nabla\tilde{p}\rangle_{L_2(\Omega)^n}$ by $\tilde{\mathbf{r}}_1^{(\frac{1}{2})} := \mathbf{r}_1^{(\frac{1}{2})}|_{\Lambda^{(\hat{H}_0^1)^n}(\mathcal{T}(\Lambda,\varepsilon),k)}$.

(b) With $\tilde{r}_1^{(\frac{1}{2})} := (\tilde{\mathbf{r}}_1^{(\frac{1}{2})})^\top \Psi^{(H_0^1)^n}$, approximate

$$\mathbf{r}_1 = \begin{bmatrix} \mathbf{r}_{11} \\ \mathbf{r}_{12} \\ \mathbf{r}_{13} \end{bmatrix} := \begin{bmatrix} \langle \frac{(\tilde{\vec{u}}\cdot\nabla)\Psi^{(H_0^1)^n} + (\Psi^{(H_0^1)^n}\cdot\nabla)\tilde{\vec{u}}}{\nu^{3/2}}, \tilde{r}_1^{(\frac{1}{2})}\rangle_{L_2(\Omega)^n} \\ -\langle \Psi^{L_2/\mathbb{R}}, \operatorname{div}\tilde{r}_1^{(\frac{1}{2})}\rangle_{L_2(\Omega)} \\ -\langle \operatorname{div}\Psi^{L_2^{n^2}}, \tilde{r}_1^{(\frac{1}{2})}\rangle_{L_2(\Omega)^n} \end{bmatrix}$$

by $\tilde{\mathbf{r}}_1 := \mathbf{r}_1|_{\Lambda(\mathcal{T}(\Lambda^{(H^1_0)^n}(\mathcal{T}(\Lambda,\varepsilon),k)),k)}$.

(s3) Approximate

$$\mathbf{r}_2 = \begin{bmatrix}\mathbf{r}_{21}\\ \mathbf{r}_{22}\\ \mathbf{r}_{23}\end{bmatrix} := \begin{bmatrix}\langle \operatorname{div} \Psi^{(H^1_0)^n}, \operatorname{div} \vec{\tilde{u}} - g\rangle_{L_2(\Omega)}\\ 0_{\vee_{L_{2/\mathbb{R}}}}\\ 0_{\vee_{L_2^{n^2}}}\end{bmatrix} \text{ by } \tilde{\mathbf{r}}_2 := \mathbf{r}_2|_{\Lambda(\mathcal{T}(\Lambda,\varepsilon),k)}$$

(s4) Approximate

$$\mathbf{r}_3 = \begin{bmatrix}\mathbf{r}_{31}\\ \mathbf{r}_{32}\\ \mathbf{r}_{33}\end{bmatrix} := \begin{bmatrix}\langle \nabla\Psi^{(H^1_0)^n}, \nabla\vec{\tilde{u}} - \underline{\tilde{\theta}}\rangle_{L_2(\Omega)^{n^2}}\\ 0_{\vee_{L_{2/\mathbb{R}}}}\\ \langle \Psi^{L_2^{n^2}}, \underline{\tilde{\theta}} - \nabla\vec{\tilde{u}}\rangle_{L_2(\Omega)^{n^2}}\end{bmatrix} \text{ by } \tilde{\mathbf{r}}_3 := \mathbf{r}_3|_{\Lambda(\mathcal{T}(\Lambda,\varepsilon),k)}.$$

The same arguments (actually a subset) that led to Theorem 2.5.14 show the following theorem.

Theorem 2.7.1. *For an admissible $\Lambda \subset \vee$, $[\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top \in \ell_2(\Lambda)$ with $(\vec{\tilde{u}}, \tilde{p}, \underline{\tilde{\theta}})$ sufficiently close to $(\vec{u}, p, \underline{\theta})$, and an $\varepsilon > 0$, consider the steps* (s1)-(s4). *With $s > 0$ such that $[\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top \in \mathcal{A}^s$, it holds that*

$$\begin{aligned}\|D\mathbf{Q}([\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\theta}}^\top]^\top) - (\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3)\| \\ \lesssim 2^{-k/2}(\|\vec{u} - \vec{\tilde{u}}\|_{H^1_0(\Omega)^n} + \|p - \tilde{p}\|_{L_2(\Omega)} + \|\underline{\theta} - \underline{\tilde{\theta}}\|_{L_2(\Omega)^{n^2}}) + \varepsilon,\end{aligned}$$

where the computation of $\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3$ requires $\mathcal{O}(\#\Lambda + \varepsilon^{-1/s})$ operations. So by taking k sufficiently large, Condition 2.3.5 is satisfied.*

We conclude that the **awgm** is an optimal solver for the stationary Navier-Stokes equations in the form $D\mathbf{Q}([\mathbf{u}^\top, \mathbf{p}^\top, \boldsymbol{\theta}^\top]^\top) = 0$ resulting from the velocity–pressure–velocity gradient formulation. Obviously, we cannot claim or even expect that this holds true uniformly in a vanishing viscosity parameter ν. This because in the limit already well-posedness of $DG(\vec{u}, p)$ cannot be expected.

2.7.2 Velocity–pressure–vorticity formulation

Restricting to $n \in \{2, 3\}$, we set $\boxed{\mathscr{T} := L_2(\Omega)^{2n-3}}$, and define

$$G_1 \in \mathcal{L}(\mathscr{T}, \mathscr{U}'), \qquad G_2 \in \mathcal{L}(\mathscr{U}, \mathscr{T}),$$

by

$$G_2(\vec{u}, p) = \operatorname{curl} \vec{u}, \qquad (G_1\vec{\omega})(\vec{v}, q) = \int_\Omega \vec{\omega} \cdot \operatorname{curl} \vec{v}\, dx$$

where for $n = 2$, curl should be read as the scalar-valued operator $\vec{v} \mapsto \partial_x v_2 - \partial_y v_1$. (and so $\vec{\omega} \cdot \operatorname{curl} \vec{u}$ as $\omega \operatorname{curl} \vec{u}$). The (formal) adjoint curl' equals curl for $n = 3$, whereas for $n = 2$ it is $v \mapsto [\partial_y v, -\partial_x v]^\top$.

Since a vector field in the current space $\mathscr{T}$ has $2n-3$ components, instead of n^2 as in the previous subsection, the first order system formulation studied in this subsection is more attractive. As we will see, later in its derivation it will be needed that $g = 0$, i.e., $\operatorname{div} \vec{u} = 0$.

Using that on $H^1_0(\Omega)^n \times H^1_0(\Omega)^n$, $\int_\Omega \nabla\vec{u} : \nabla\vec{v} - \operatorname{div}\vec{u}\operatorname{div}\vec{v} - \operatorname{curl}\vec{u}\cdot\operatorname{curl}\vec{v}\,dx = 0$, the results from Sect. 2.2 show that the solution $(\vec{u}, p)$ can be found as the first component of the solution in $\mathscr{U} \times \mathscr{T}$ of the system

$$\vec{H}_1(\vec{u}, p, \vec{\omega}) := \Big((\vec{v}, q) \mapsto \int_\Omega \vec{\omega}\cdot\operatorname{curl}\vec{v} + \operatorname{div}\vec{u}\operatorname{div}\vec{v} - p\operatorname{div}\vec{v} + \nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v} + q(\operatorname{div}\vec{u} - g) - \vec{f}\cdot\vec{v}\,dx, \vec{\omega} - \operatorname{curl}\vec{u}\Big) = \vec{0}$$

on $\mathscr{U}' \times \mathscr{T}$, being a minimizer of

$$Q_1(\vec{u}, p, \vec{\omega}) := \tfrac{1}{2}\Big(\big\|\vec{v} \mapsto \int_\Omega \vec{\omega}\cdot\operatorname{curl}\vec{v} + \operatorname{div}\vec{u}\operatorname{div}\vec{v} - p\operatorname{div}\vec{v} + \tfrac{(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v}}{\nu^{3/2}} - \vec{f}\cdot\vec{v}\,dx\big\|^2_{H^{-1}(\Omega)^n} + \|\operatorname{div}\vec{u} - g\|^2_{L_2(\Omega)} + \|\vec{\omega} - \operatorname{curl}\vec{u}\|^2_{L_2(\Omega)^{2n-3}}\Big).$$

The function $\vec{\omega} = \operatorname{curl}\vec{u}$ is known as the vorticity.

Since G satisfies (i)–(iii), $\vec{H}_1$ satisfies (a)–(c), and so by Lemma 2.2.6,

$$Q_1(\tilde{\vec{u}}, \tilde{p}, \tilde{\vec{\omega}}) \eqsim \|(\tilde{\vec{u}}, \tilde{p}, \tilde{\vec{\omega}}) - (\vec{u}, p, \vec{\omega})\|^2_{\mathscr{U}\times\mathscr{T}} \tag{2.7.4}$$

for $(\tilde{\vec{u}}, \tilde{p}, \tilde{\vec{\omega}})$ in a neighborhood of $(\vec{u}, p, \vec{\omega})$.

From here on, we assume that

$$g = 0,$$

so that the velocities component of the exact solution is divergence-free. This will allow us to get rid of the second order term $\nabla\operatorname{div}\vec{u}$ in the definition of $\vec{H}_1$. We define

$$\vec{H}_2(\vec{u}, p, \vec{\omega}) := \Big((\vec{v}, q) \mapsto \int_\Omega \vec{\omega}\cdot\operatorname{curl}\vec{v} - p\operatorname{div}\vec{v} + \tfrac{(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v}}{\nu^{3/2}} + q\operatorname{div}\vec{u} - \vec{f}\cdot\vec{v}\,dx, \vec{\omega} - \operatorname{curl}\vec{u}\Big)$$

with corresponding quadratic functional

$$Q_2(\vec{u}, p, \vec{\omega}) := \tfrac{1}{2}\Big(\big\|\vec{v} \mapsto \int_\Omega \vec{\omega}\cdot\operatorname{curl}\vec{v} - p\operatorname{div}\vec{v} + \tfrac{(\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v}}{\nu^{3/2}} - \vec{f}\cdot\vec{v}\,dx\big\|^2_{H^{-1}(\Omega)^n} + \|\operatorname{div}\vec{u}\|^2_{L_2(\Omega)} + \|\vec{\omega} - \operatorname{curl}\vec{u}\|^2_{L_2(\Omega)^{2n-3}}\Big).$$

Clearly the solution of $\vec{H}_1(\vec{u}, p, \vec{\omega}) = 0$ is a solution of $\vec{H}_2(\vec{u}, p, \vec{\omega}) = 0$ ((a)), and $\vec{H}_2$ is two times continuously differentiable ((b)). From $\|\vec{v} \mapsto \int_\Omega \operatorname{div}\vec{u}\operatorname{div}\vec{v}\,dx\|_{H^{-1}(\Omega)^n} \lesssim \|\operatorname{div}\vec{u}\|_{L_2(\Omega)}$, one infers that $Q_1 \lesssim Q_2$ by the triangle inequality, and analogously $Q_2 \lesssim Q_1$. Thanks to (2.7.4), an application of Lemma 2.2.6 shows that $\vec{H}_2$ satisfies also (c). We conclude that $\vec{H}_2(\vec{u}, p, \vec{\omega})$ is a well-posed first order system formulation

of $G(\vec{u},p)=0$, and consequently, that $(\vec{u},p,\vec{\omega})$ can be found by solving the normal equations $DQ_2(\vec{u},p,\vec{\omega})=0$, which are well-posed in the sense that they satisfy (1)–(4). This gives us an alternative, effortless proof of [CMM95, Thm. 2.1].

As usual, to deal with the 'unpractical' norm on $H^{-1}(\Omega)^n$, we equip $H^1_0(\Omega)^n$ with a wavelet Riesz basis

$$\Psi^{(\hat{H}^1_0)^n}=\{\psi_\lambda^{(\hat{H}^1_0)^n}\colon \lambda\in\vee_{(\hat{H}^1_0)^n}\},$$

and replace, in the definition of Q_2, the norm on its dual by the equivalent norm $\|\vec{g}(\Psi^{(\hat{H}^1_0)^n})\|$ for $\vec{g}\in H^{-1}(\Omega)^n$.

Next, after equipping $*\in\{H^1_0(\Omega)^n, L_2(\Omega)/\mathbb{R}, L_2(\Omega)^{2n-3}\}$ with Riesz basis $\Psi^*=\{\psi^*_\lambda\colon\lambda\in\vee_*\}$, and so $H^1_0(\Omega)^n\times L_2(\Omega)/\mathbb{R}\times L_2(\Omega)^{2n-3}$ with

$$\Psi:=(\Psi^{(H^1_0)^n},0_{L_2/\mathbb{R}},0_{L_2^{2n-3}})\cup(0_{(H^1_0)^n},\Psi^{L_2/\mathbb{R}},0_{L_2^{2n-3}})\cup(0_{(H^1_0)^n},0_{L_2/\mathbb{R}},\Psi^{L_2^{2n-3}})$$

with index set $\vee:=\vee_{(H^1_0)^n}\cup\vee_{L_2/\mathbb{R}}\cup\vee_{L_2^{2n-3}}$, we apply the **awgm** to the resulting system

$$\begin{aligned}D\mathbf{Q}_2([\mathbf{u}^\top,\mathbf{p}^\top,\boldsymbol{\omega}^\top]^\top)&=\begin{bmatrix}\langle\operatorname{div}\Psi^{(H^1_0)^n},\operatorname{div}\vec{u}\rangle_{L_2(\Omega)}\\0_{\vee_{L_2/\mathbb{R}}}\\0_{\vee_{L_2^{2n-3}}}\end{bmatrix}+\begin{bmatrix}\langle\operatorname{curl}\Psi^{(H^1_0)^n},\operatorname{curl}\vec{u}-\vec{\omega}\rangle_{L_2(\Omega)^{2n-3}}\\0_{\vee_{L_2/\mathbb{R}}}\\\langle\Psi^{L_2^{2n-3}},\vec{\omega}-\operatorname{curl}\vec{u}\rangle_{L_2(\Omega)^{2n-3}}\end{bmatrix}\\&+\begin{bmatrix}\langle\frac{(\vec{u}\cdot\nabla)\Psi^{(H^1_0)^n}+(\Psi^{(H^1_0)^n}\cdot\nabla)\vec{u}}{\nu^{3/2}},\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)^n}\\-\langle\Psi^{L_2/\mathbb{R}},\operatorname{div}\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)}\\\langle\Psi^{L_2^{2n-3}},\operatorname{curl}\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)^{2n-3}}\end{bmatrix}\Big\{\langle\Psi^{(\hat{H}^1_0)^n},\tfrac{(\vec{u}\cdot\nabla)\vec{u}}{\nu^{3/2}}-\vec{f}\rangle_{L_2(\Omega)^n}+\\&\qquad\langle\operatorname{curl}\Psi^{(\hat{H}^1_0)^n},\vec{\omega}\rangle_{L_2(\Omega)^{2n-3}}-\langle\operatorname{div}\Psi^{(\hat{H}^1_0)^n},p\rangle_{L_2(\Omega)}\Big\}=0.\end{aligned}$$

To express the three terms in $\vec{v}\mapsto\langle\vec{v},\nu^{-3/2}(\vec{u}\cdot\nabla)\vec{u}-\vec{f}\rangle_{L_2(\Omega)^n}+\langle\operatorname{curl}\vec{v},\vec{\omega}\rangle_{L_2(\Omega)^{2n-3}}-\langle\operatorname{div}\vec{v},p\rangle_{L_2(\Omega)}$ w.r.t. one dictionary, we impose the easily realizable conditions that

$$\Psi^{L_2/\mathbb{R}}\subset H^1(\Omega),\ \Psi^{L_2^{2n-3}}\subset H(\operatorname{curl};\Omega)$$

Then for finitely supported approximations $[\tilde{\mathbf{u}}^\top,\tilde{\mathbf{p}}^\top,\tilde{\boldsymbol{\omega}}^\top]^\top$ to $[\mathbf{u}^\top,\mathbf{p}^\top,\boldsymbol{\omega}^\top]^\top$, for $(\tilde{\vec{u}},\tilde{p},\tilde{\vec{\omega}}):=[\tilde{\mathbf{u}}^\top,\tilde{\mathbf{p}}^\top,\tilde{\boldsymbol{\omega}}^\top]\Psi\in H^1_0(\Omega)^n\times H^1(\Omega)\times H(\operatorname{curl}';\Omega)$, we have

$$\boxed{\begin{aligned}D\mathbf{Q}_2([\tilde{\mathbf{u}}^\top,\tilde{\mathbf{p}}^\top,\tilde{\boldsymbol{\omega}}^\top]^\top)=&\begin{bmatrix}\langle\operatorname{div}\Psi^{(H^1_0)^n},\operatorname{div}\tilde{\vec{u}}\rangle_{L_2(\Omega)}\\0_{\vee_{L_2/\mathbb{R}}}\\0_{\vee_{L_2^{2n-3}}}\end{bmatrix}+\begin{bmatrix}\langle\operatorname{curl}\Psi^{(H^1_0)^n},\operatorname{curl}\tilde{\vec{u}}-\tilde{\vec{\omega}}\rangle_{L_2(\Omega)^{2n-3}}\\0_{\vee_{L_2/\mathbb{R}}}\\\langle\Psi^{L_2^{2n-3}},\tilde{\vec{\omega}}-\nabla\tilde{\vec{u}}\rangle_{L_2(\Omega)^{2n-3}}\end{bmatrix}+\\&\begin{bmatrix}\langle\frac{(\tilde{\vec{u}}\cdot\nabla)\Psi^{(H^1_0)^n}+(\Psi^{(H^1_0)^n}\cdot\nabla)\tilde{\vec{u}}}{\nu^{3/2}},\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)^n}\\-\langle\Psi^{L_2/\mathbb{R}},\operatorname{div}\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)}\\\langle\Psi^{L_2^{2n-3}},\operatorname{curl}\Psi^{(\hat{H}^1_0)^n}\rangle_{L_2(\Omega)^{2n-3}}\end{bmatrix}\langle\Psi^{(\hat{H}^1_0)^n},\tfrac{(\tilde{\vec{u}}\cdot\nabla)\tilde{\vec{u}}}{\nu^{3/2}}-\vec{f}+\operatorname{curl}'\tilde{\vec{\omega}}+\nabla\tilde{p}\rangle_{L_2(\Omega)^n}.\end{aligned}}$$

The design of an approximate residual evaluation follows analogous steps as in the previous subsection. Equipping Cartesian products with bases of canonical form, and

assuming that the scalar-valued bases Ψ^* for $* \in \{\hat{H}^1_0, H^1_0, L_2/\mathbb{R}, L_2\}$ satisfy (w_1)–(w_4), and that $[\tilde{\mathbf{u}}^\top, \tilde{\mathbf{p}}^\top, \tilde{\boldsymbol{\omega}}^\top]^\top$ is supported on an admissible set, four steps fully analogous to (s1)–(s4) in the previous subsection define an approximation scheme that satisfies Condition 2.3.5*. We conclude that the **awgm** is an optimal solver for the stationary Navier-Stokes equations in the form $D\mathbf{Q}([\mathbf{u}^\top, \mathbf{p}^\top, \boldsymbol{\theta}^\top]^\top) = 0$ resulting from the velocity–pressure–vorticity formulation. Again, also here we cannot claim or even expect that this holds true uniformly in a vanishing viscosity parameter ν.

2.8 Conclusion

We have seen that a well-posed (system of) 2nd order PDE(s) can always be formulated as a well-posed 1st order least squares system. The arising dual norm(s) can be replaced by the equivalent ℓ_2-norm(s) of the wavelet coefficients of the functional. The resulting Euler-Lagrange equations, also known as the (nonlinear) normal equations, can be solved at the best possible rate by the adaptive wavelet Galerkin method. We developed a new approximate residual evaluation scheme that also for semi-linear problems satisfies the condition for optimal computational complexity, and that is quantitatively much more efficient than the usual **apply** scheme. Moreover, regardless of the order of the wavelets, it applies already to wavelet bases that have only one vanishing moment. As applications we discussed optimal solvers for first order least squares reformulations of 2nd order elliptic PDEs with inhomogeneous boundary conditions, and that of the stationary Navier-Stokes equations. In Chapter 4 we will apply this approach to time-evolution problems.

2.9 Appendix: Decay estimates

We collect a number of decay estimates that have been used in the proof of Theorem 2.5.14. Recall the definition of the spaces $\mathscr{U}$ and $\mathscr{V}$ given at the beginning of Sect. 2.5.1.

The following proposition and subsequent lemma have been used to bound $\|\mathbf{r}_1^{(\frac{1}{2})} - \tilde{\mathbf{r}}_1^{(\frac{1}{2})}\|$. The presence of the boundary integral and the fact that the upper bound that is given depends on the norm of g as a whole, and not on norms of g_1 and g_2 requires a non-standard treatment.

Proposition 2.9.1. *For a tiling $\mathcal{T} \subset \mathcal{O}_\Omega$, let $g \in \mathscr{V}_1'$ be of the form*

$$g(v) = \int_\Omega g_1 v \, dx + \int_{\Gamma_N} \vec{g}_2 \cdot \mathbf{n} v \, ds,$$

where $g_1 \in \mathcal{P}_m(\mathcal{T})$, $\vec{g}_2 \in \mathcal{P}_m(\mathcal{T})^n$. Then

$$\big\|g(\Psi^{\mathscr{V}_1})\big|_{\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)}\big\| \lesssim 2^{-k}\|g\|_{\mathscr{V}_1'}$$

(*uniform in $\mathcal{T}$ and g*).

Proof. Since by assumption (w_4), for $\lambda \in \vee_{\mathscr{V}_1}$ with $|\lambda| > 0$ either $\int_\Omega \psi_\lambda^{\mathscr{V}_1}\,dx = 0$ or $\operatorname{dist}(\operatorname{supp}\psi_\lambda^{\mathscr{V}}, \Gamma_D) \lesssim 2^{-|\lambda|}$, an application of Friedrich's or Poincaré's inequality shows that $\|\psi_\lambda^{\mathscr{V}_1}\|_{L_2(\Omega)} \lesssim 2^{-|\lambda|}|\psi_\lambda^{\mathscr{V}_1}|_{H^1(\Omega)} \eqsim 2^{-|\lambda|}$.

Since by (w_1)–(w_2), each descendant $\omega' \in \mathcal{O}_\Omega$ of $\omega \in \mathcal{T}$ with $|\omega'| > |\omega| + k$ is intersected by the supports of a uniformly bounded number of $\lambda \in \vee_{\mathscr{V}_1} \setminus \Lambda^{\mathscr{V}_1}(\mathcal{T}, k)$ with $|\lambda| = |\omega'|$, we have

$$\begin{aligned}\sum_{\lambda\in\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)} \Big|\int_\Omega g_1\psi_\lambda^{\mathscr{V}_1}\,dx\Big|^2 &\le \sum_{\lambda\in\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)}\sum_{\omega\in\mathcal{T}} 4^{-|\lambda|}\|g_1\|^2_{L_2(\omega\cap\operatorname{supp}\psi_\lambda^{\mathscr{V}_1})}\\ &\lesssim 4^{-k}\sum_{\omega\in\mathcal{T}} 4^{-|\omega|}\|g_1\|^2_{L_2(\omega)},\end{aligned} \tag{2.9.1}$$

A standard homogeneity argument shows that for $\omega \in \mathcal{T}$ and $v \in H^1(\omega)$, $\|v\|_{L_2(\partial\omega)} \lesssim 2^{-|\omega|/2}(|v|_{H^1(\omega)} + 2^{|\omega|}\|v\|_{L_2(\omega)})$, so that $\|\psi_\lambda^{\mathscr{V}_1}\|_{L_2(\Gamma_N)} \lesssim 2^{-|\lambda|/2}$. Writing $\partial\omega \cap \Gamma_N$ as $\partial\omega_N$, the arguments that led to (2.9.1) show that

$$\begin{aligned}\sum_{\lambda\in\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)} \Big|\int_{\Gamma_N} \vec{g}_2\cdot\mathbf{n}\psi_\lambda^{\mathscr{V}_1}\,ds\Big|^2 &\lesssim \sum_{\lambda\in\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)}\sum_{\omega\in\mathcal{T}} 2^{-|\lambda|}\|\vec{g}_2\cdot\mathbf{n}\|^2_{L_2(\omega_N\cap\operatorname{supp}\psi_\lambda^{\mathscr{V}_1})}\\ &\lesssim 2^{-k}\sum_{\omega\in\mathcal{T}} 2^{-|\omega|}\|\vec{g}_2\cdot\mathbf{n}\|^2_{L_2(\omega_N)}.\end{aligned} \tag{2.9.2}$$

By combining (2.9.1), (2.9.2) with Lemma 2.9.2, the proof is completed unless $\mathscr{U} = \mathscr{V}_1 = H^1(\Omega)/\mathbb{R}$. In the latter case, define $\bar{g}_1 := g_1 - \operatorname{meas}(\Omega)^{-1}g(\mathbb{1})$ and $\bar{g}(v) := \int_\Omega \bar{g}_1 v\,dx + \int_{\Gamma_N}\vec{g}_2\cdot\mathbf{n}v\,ds$. From $g(\Psi^{\mathscr{V}_1})\big|_{\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)} = \bar{g}(\Psi^{\mathscr{V}_1})\big|_{\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}(\mathcal{T},k)}$, and $\|g\|_{\mathscr{V}_1'} = \|\bar{g}\|_{\mathscr{V}_1'}$, applications of (2.9.1), (2.9.2) and that of Lemma 2.9.2 to $\bar{g}$ complete the proof in this case. □

Lemma 2.9.2. *In the situation of Proposition 2.9.1, with additionally $g(\mathbb{1}) = 0$ when $\mathscr{U} = \mathscr{V}_1 = H^1(\Omega)/\mathbb{R}$, it holds that*

$$\sum_{\omega\in\mathcal{T}} 4^{-|\omega|}\|g_1|_\omega\|^2_{L_2(\omega)} + 2^{-|\omega|}\|\vec{g}_2\cdot\mathbf{n}|_\omega\|^2_{L_2(\partial\omega\cap\Gamma_N)} \lesssim \|g\|^2_{\mathscr{V}'}.$$

Proof. Thanks to the uniform shape regularity condition, for any $\omega \in \mathcal{O}_\Omega$, there exists a $V_\omega \subset H^1_0(\omega)$ such that

$$\begin{aligned}\|v\|_{H^1(\omega)} &\lesssim 2^{|\omega|}\|v\|_{L_2(\omega)} \quad (v \in V_\omega),\\ \|p\|_{L_2(\omega)} &\lesssim \sup_{0\neq v\in V_\omega}\frac{\int_\omega pv\,dx}{\|v\|_{L_2(\omega)}} \quad (p \in \mathcal{P}_m(\omega)).\end{aligned}$$

For each $\omega \in \mathcal{T}$, select $v_\omega \in V_\omega$ with $\|g_1|_\omega\|_{L_2(\omega)}\|v_\omega\|_{L_2(\omega)} \lesssim \int_\omega g_1|_\omega v_\omega\,dx$, and $\|v_\omega\|_{L_2(\omega)} = 4^{-|\omega|}\|g_1|_\omega\|_{L_2(\omega)}$. Then, with $v = \sum_{\omega\in\mathcal{T}} v_\omega \in H^1_0(\Omega)$, we have

$\sum_{\omega\in\mathcal{T}} 4^{-|\omega|}\|g_1|_\omega\|^2_{L_2(\omega)} \lesssim \int_\Omega g_1 v\,dx$. By combining this with

$$\|v\|^2_{H^1(\Omega)} = \sum_{\omega\in\mathcal{T}} \|v_\omega\|^2_{H^1(\omega)} \lesssim \sum_{\omega\in\mathcal{T}} 4^{|\omega|}\|v_\omega\|^2_{L_2(\omega)} = \sum_{\omega\in\mathcal{T}} 4^{-|\omega|}\|g_1|_\omega\|^2_{L_2(\omega)},$$

and $g(v) = \int_\Omega g_1 v\,dx$, we arrive at $\sqrt{\sum_{\omega\in\mathcal{T}} 4^{-|\omega|}\|g_1|_\omega\|^2_{L_2(\omega)}} \lesssim \frac{g(v)}{\|v\|_{H^1(\Omega)}} \le \|g\|_{\mathcal{Y}'}$, with the last inequality being valid when $H^1_0(\Omega)\subset\mathcal{V}_1$.

Otherwise, when $\mathcal{V}_1 = H^1(\Omega)/\mathbb{R}$, we take $\bar v = v - \frac{\int_\Omega v\,dx}{\mathrm{meas}(\Omega)}\mathbb{1}\in\mathcal{V}$. Then $\|\bar v\|_{H^1(\Omega)}\lesssim \|v\|_{H^1(\Omega)}$, $g(v) = g(\bar v)$ by assumption, and so $\frac{g(v)}{\|v\|_{H^1(\Omega)}} \lesssim \frac{g(\bar v)}{\|\bar v\|_{H^1(\Omega)}} \le \|g\|_{\mathcal{Y}'}$.

For bounding the second term in the statement of the lemma, for a tile $\omega\in\mathcal{O}_\Omega$ we write $\partial\omega\cap\Gamma_N$ as $\partial\omega_N$. Thanks to the uniform shape regularity condition, for any $\omega\in\mathcal{O}_\Omega$ with $\mathrm{meas}(\partial\omega_N)>0$, there exists a $V_\omega\subset\{v\in H^1(\omega)\colon v|_{\partial\omega\setminus\partial\omega_N}=0\}$ such that

$$\begin{aligned}
&\|v\|_{H^1(\omega)} \lesssim 2^{|\omega|/2}\|v\|_{L_2(\partial\omega_N)} \quad (v\in V_\omega),\\
&\|\vec p\cdot\mathbf{n}\|_{L_2(\partial\omega_N)} \lesssim \sup_{0\neq v\in V_\omega} \frac{\int_{\partial\omega_N}\vec p\cdot\mathbf{n} v\,ds}{\|v\|_{L_2(\partial\omega_N)}} \quad (\vec p\in\mathcal{P}_m(\omega)^n),\\
&V_\omega \perp_{L_2(\omega)} \mathcal{P}_m(\omega).
\end{aligned}$$

For each $\omega\in\mathcal{T}$ with $\mathrm{meas}(\partial\omega_N)>0$, select $v_\omega\in V_\omega$ with $\|\vec g_2|_\omega\cdot\mathbf{n}\|_{L_2(\partial\omega_N)}\|v_\omega\|_{L_2(\partial\omega_N)} \lesssim \int_{\partial\omega_N}\vec g_2|_\omega\cdot\mathbf{n} v_\omega\,ds$, and $\|v_\omega\|_{L_2(\partial\omega_N)} = 2^{-|\omega|}\|\vec g_2\cdot\mathbf{n}|_\omega\|_{L_2(\partial\omega_N)}$. For the other $\omega\in\mathcal{T}$, set $v_\omega=0$. Then, for the function $v=\sum_{\omega\in\mathcal{T}} v_\omega\in\{w\in H^1(\Omega)\colon w|_{\Gamma_D}=0\}$, we have

$$\sum_{\omega\in\mathcal{T}} 2^{-|\omega|}\|\vec g_2\cdot\mathbf{n}|_\omega\|^2_{L_2(\partial\omega_N)} \lesssim \int_{\Gamma_N}\vec g_2\cdot\mathbf{n} v\,ds.$$

By combining this with

$$\|v\|^2_{H^1(\Omega)} = \sum_{\omega\in\mathcal{T}}\|v_\omega\|^2_{H^1(\omega)} \lesssim \sum_{\omega\in\mathcal{T}} 2^{|\omega|}\|v_\omega\|^2_{L_2(\partial\omega_N)} = \sum_{\omega\in\mathcal{T}} 2^{-|\omega|}\|\vec g_2\cdot\mathbf{n}|_\omega\|^2_{L_2(\partial\omega_N)},$$

and $g(v) = \int_{\Gamma_N}\vec g_2\cdot\mathbf{n} v\,dx$, we arrive at

$$\sqrt{\sum_{\omega\in\mathcal{T}} 2^{-|\omega|}\|g_2|_\omega\cdot\mathbf{n}\|^2_{L_2(\partial\omega_N)}} \lesssim \frac{g(v)}{\|v\|_{H^1(\Omega)}} \le \|g\|_{\mathcal{Y}'},$$

in the case that $\{w\in H^1(\Omega)\colon w|_{\Gamma_D}=0\}\subset\mathcal{V}_1$.

Otherwise, when $\mathcal{V}_1 = H^1(\Omega)/\mathbb{R}$, we take $\bar v = v - \frac{\int_\Omega v\,dx}{\mathrm{meas}(\Omega)}\mathbb{1}\in\mathcal{V}$. Then $\|\bar v\|_{H^1(\Omega)}\lesssim \|v\|_{H^1(\Omega)}$, $g(v)=g(\bar v)$ by assumption, and so $\frac{g(v)}{\|v\|_{H^1(\Omega)}} \lesssim \frac{g(\bar v)}{\|\bar v\|_{H^1(\Omega)}} \le \|g\|_{\mathcal{Y}'}$. □

An easy version of the proof of Proposition 2.9.1 shows the following result, which has been used to bound $\|\mathbf{r}_{11}-\tilde{\mathbf{r}}_{11}\|$ in the proof of Theorem 2.5.14.

Proposition 2.9.3. *For a tiling $\mathcal{T} \subset \mathcal{O}_\Omega$, and $g \in P_m(\mathcal{T})$, it holds that*

$$\big\|\langle \Psi^{\mathscr{U}}, g\rangle_{L_2(\Omega)}\big|_{\vee_{\mathscr{U}}\setminus\Lambda^{\mathscr{U}}(\mathcal{T},k)}\big\| \lesssim 2^{-k}\|g\|_{\mathscr{U}'}$$

(*uniform in $\mathcal{T}$ and g*).

The statements from the following proposition have been used to bound the terms $\|\mathbf{r}_{11} - \tilde{\mathbf{r}}_{11}\|$ (first statement) and $\|\mathbf{r}_2 - \tilde{\mathbf{r}}_2\|$ (both statements) in the proof of Theorem 2.5.14.

Proposition 2.9.4. *For a tiling $\mathcal{T} \subset \mathcal{O}_\Omega$, $g \in P_m(\mathcal{T})$, $1 \le q \le n$, and $\vec{g} \in P_m(\mathcal{T})^n$, it holds that*

$$\big\|\langle \Psi^{\mathscr{T}_q}, g\rangle_{L_2(\Omega)}\big|_{\vee_{\mathscr{T}_q}\setminus\Lambda^{\mathscr{T}_q}(\mathcal{T},k)}\big\| \lesssim 2^{-k/2}\|g\|_{L_2(\Omega)}$$
$$\big\|\langle \nabla\Psi^{\mathscr{U}}, \vec{g}\rangle_{L_2(\Omega)^n}\big|_{\vee_{\mathscr{U}}\setminus\Lambda^{\mathscr{U}}(\mathcal{T},k)}\big\| \lesssim 2^{-k/2}\|\vec{g}\|_{L_2(\Omega)^n}$$

(*uniform in $\mathcal{T}$, g, and $\vec{g}$*).

Proof. Since by assumption (w_2), for $\lambda \in \vee_{\mathscr{T}_q} \setminus \Lambda^{\mathscr{T}_q}(\mathcal{T},k)$, $\operatorname{supp}\psi_\lambda^{\mathscr{T}_q}$ has non-empty intersection with a uniformly bounded number of $\omega \in \mathcal{T}$, we have

$$\sum_{\lambda\in\vee_{\mathscr{T}_q}\setminus\Lambda^{\mathscr{T}_q}(\mathcal{T},k)} \big|\sum_{\omega\in\mathcal{T}}\langle\psi_\lambda^{\mathscr{T}_q}, g\rangle_{L_2(\omega)}\big|^2 \lesssim \sum_{\omega\in\mathcal{T}}\sum_{\lambda\in\vee_{\mathscr{T}_q}\setminus\Lambda^{\mathscr{T}_q}(\mathcal{T},k)} |\langle\psi_\lambda^{\mathscr{T}_q}, g\rangle_{L_2(\omega)}|^2. \tag{2.9.3}$$

Given $\omega \in \mathcal{T}$ and $\ell \in \mathbb{N}_0$, we set

$$\Lambda^{(1)}_{\omega,\ell} = \{\lambda \in \vee_{\mathscr{T}_q} : |\lambda| = \ell,\ \operatorname{supp}\psi_\lambda^{\mathscr{T}_q} \subset \omega,\ \psi_\lambda^{\mathscr{T}_q} \text{ has a vanishing moment}\}$$
$$\Lambda^{(2)}_{\omega,\ell} = \{\lambda \in \vee_{\mathscr{T}_q} \setminus \Lambda^{(1)}_{\omega,\ell} : |\lambda| = \ell,\ \operatorname{meas}(\operatorname{supp}\psi_\lambda^{\mathscr{T}_q} \cap \omega) > 0\}.$$

For $\lambda \in \Lambda^{(2)}_{\omega,\ell}$, we estimate

$$|\langle\psi_\lambda^{\mathscr{T}_q}, g\rangle_{L_2(\omega)}| \le \|\psi_\lambda^{\mathscr{T}_q}\|_{L_1(\Omega)}\|g\|_{L_\infty(\omega)} \lesssim 2^{-\ell n/2}2^{|\omega|n/2}\|g\|_{L_2(\omega)}. \tag{2.9.4}$$

Using that $\#\Lambda^{(2)}_{\omega,\ell} \lesssim 2^{(\ell-|\omega|)(n-1)}$ (cf. (w_4)) , we infer that

$$\sum_{\ell>|\omega|+k}\sum_{\lambda\in\Lambda^{(2)}_{\omega,\ell}} |\langle\psi_\lambda^{\mathscr{T}_q}, g\rangle_{L_2(\omega)}|^2 \lesssim 2^{-k}\|g\|^2_{L_2(\omega)}. \tag{2.9.5}$$

Using that for $\lambda \in \Lambda^{(1)}_{\omega,\ell}$, $\psi_\lambda^{\mathscr{T}_q}$ has a vanishing moment, we find that

$$|\langle\psi_\lambda^{\mathscr{T}_q}, g\rangle_{L_2(\omega)}| \lesssim \|\psi_\lambda^{\mathscr{T}_q}\|_{L_1(\Omega)}2^{-\ell}\|g\|_{W^1_\infty(\omega)} \lesssim 2^{-\ell n/2}2^{-\ell}2^{|\omega|}2^{|\omega|n/2}\|g\|_{L_2(\omega)}. \tag{2.9.6}$$

From $\#\Lambda^{(1)}_{\omega,\ell} \lesssim 2^{(\ell-|\omega|)n}$, we obtain

$$\sum_{\ell>|\omega|+k}\sum_{\lambda\in\Lambda^{(1)}_{\omega,\ell}} |\langle\psi_\lambda^{\mathscr{T}_q}, g\rangle_{L_2(\omega)}|^2 \lesssim 4^{-k}\|g\|^2_{L_2(\omega)}. \tag{2.9.7}$$

The proof of the first inequality follows from (2.9.3), (2.9.5), and (2.9.7).

To prove the second inequality, for any $\lambda \in \vee_{\mathscr{U}}$, similar to (2.9.4) we have

$$|\langle \nabla \psi_\lambda^{\mathscr{U}}, \vec{g}\rangle_{L_2(\omega)^n}| \le \|\psi_\lambda^{\mathscr{U}}\|_{W_1^1(\Omega)} \|\vec{g}\|_{L_\infty(\omega)^n} \lesssim 2^{-|\lambda|n/2} 2^{|\omega|n/2} \|\vec{g}\|_{L_2(\omega)^n}.$$

When $\psi_\lambda^{\mathscr{U}}$ has a vanishing moment and $\operatorname{supp} \psi_\lambda^{\mathscr{U}} \subset \omega$, we have

$$\begin{aligned} |\langle \nabla \psi_\lambda^{\mathscr{U}}, \vec{g}\rangle_{L_2(\omega)^n}| &= |\langle \psi_\lambda^{\mathscr{U}}, \operatorname{div} \vec{g}\rangle_{L_2(\omega)}| \lesssim \|\psi_\lambda^{\mathscr{U}}\|_{L_1(\Omega)} 2^{-|\lambda|} \|\vec{g}\|_{W_\infty^2(\omega)^n} \\ &\lesssim 2^{-|\lambda|n/2} 4^{-|\lambda|} 4^{|\omega|} 2^{|\omega|/2} \|\vec{g}\|_{L_2(\omega)^n} \end{aligned}$$

replacing (2.9.6). From these two estimates, the second inequality follows similarly to the first one.

□

The following proposition has been used to bound the term $\|\mathbf{r}_3^{(\frac{1}{2})} - \tilde{\mathbf{r}}_3^{(\frac{1}{2})}\|$ in the proof of Theorem 2.5.14.

Proposition 2.9.5. *For a boundary tiling $\mathcal{T}_{\Gamma_D} \subset \mathcal{O}_{\Gamma_D}$, and $g \in \mathcal{P}_m(\mathcal{T}_{\Gamma_D}) \cap C(\Gamma_D)$,*

$$\|\langle \Psi^{\mathscr{V}_2}, g\rangle_{L_2(\Gamma_D)}|_{\vee_{\mathscr{V}_2} \setminus \Lambda^{\mathscr{V}_2}(\mathcal{T}_{\Gamma_D}, k)}\| \lesssim 2^{-k/2} \|g\|_{H^{\frac{1}{2}}(\Gamma_D)}.$$

Proof. Using (2.5.8), similar to (2.9.1)

$$\begin{aligned} \sum_{\lambda \in \vee_{\mathscr{V}_2} \setminus \Lambda^{\mathscr{V}_2}(\mathcal{T}_{\Gamma_D}, k)} \Big| \int_{\Gamma_D} g \psi_\lambda^{\mathscr{V}_2} \, ds \Big|^2 &\le \sum_{\lambda \in \vee_{\mathscr{V}_2} \setminus \Lambda^{\mathscr{V}_2}(\mathcal{T}_{\Gamma_D}, k)} \sum_{\omega \in \mathcal{T}_{\Gamma_D}} 2^{-|\lambda|} \|g\|^2_{H^1(\omega \cap \operatorname{supp} \psi_\lambda^{\mathscr{V}_2})} \\ &\lesssim 2^{-k} \sum_{\omega \in \mathcal{T}_{\Gamma_D}} 2^{-|\omega|} \|g\|^2_{H^1(\omega)} \lesssim 2^{-k} \|g\|^2_{H^{\frac{1}{2}}(\Gamma_D)}, \end{aligned}$$

by an application of an inverse inequality. □

The following proposition has been used to bound the term $\|\mathbf{r}_3 - \tilde{\mathbf{r}}_3\|$ in the proof of Theorem 2.5.14.

Proposition 2.9.6. *For a boundary tiling $\mathcal{T}_{\Gamma_D} \subset \mathcal{O}_{\Gamma_D}$, and $g \in \mathcal{P}_m(\mathcal{T}_{\Gamma_D})$,*

$$\|\langle \Psi^{\mathscr{U}}, g\rangle_{L_2(\Gamma_D)}|_{\vee_{\mathscr{U}} \setminus \Lambda^{\mathscr{U}}(\mathcal{T}_{\Gamma_D}, k)}\| \lesssim 2^{-k/2} \|g\|_{H^{-\frac{1}{2}}(\Gamma_D)}.$$

Proof. In the proof of Proposition 2.9.1, we saw that $\|\Psi_\lambda^{\mathscr{V}_1}\|_{L_2(\Gamma_N)} \lesssim 2^{-|\lambda|/2}$. The same arguments show that $\|\Psi_\lambda^{\mathscr{U}}\|_{L_2(\Gamma_D)} \lesssim 2^{-|\lambda|/2}$. Consequently, similar to (2.9.1),

$$\begin{aligned} \sum_{\lambda \in \vee_{\mathscr{U}} \setminus \Lambda^{\mathscr{U}}(\mathcal{T}_{\Gamma_D}, k)} \Big| \int_{\Gamma_D} g \psi_\lambda^{\mathscr{U}} \, ds \Big|^2 &\le \sum_{\lambda \in \vee_{\mathscr{U}} \setminus \Lambda^{\mathscr{U}}(\mathcal{T}_{\Gamma_D}, k)} \sum_{\omega \in \mathcal{T}_{\Gamma_D}} 2^{-|\lambda|} \|g\|^2_{L_2(\omega \cap \operatorname{supp} \psi_\lambda^{\mathscr{U}})} \\ &\lesssim 2^{-k} \sum_{\omega \in \mathcal{T}_{\Gamma_D}} 2^{-|\omega|} \|g\|^2_{L_2(\omega)} \lesssim 2^{-k} \|g\|^2_{H^{-\frac{1}{2}}(\Gamma_D)}, \end{aligned}$$

where the last inequality follows from analogous arguments as were applied in the first paragraph of the proof of Lemma 2.9.2. □

3. A quadratic finite element wavelet Riesz basis

In this chapter, continuous piecewise quadratic finite element wavelets are constructed on general polygons in $\mathbb{R}^2$. The wavelets are stable in H^s for $|s| < \frac{3}{2}$ and have two vanishing moments. Each wavelet is a linear combination of 11 or 13 nodal basis functions. Numerically computed condition numbers for $s \in \{-1, 0, 1\}$ are provided for the unit square.

3.1 Introduction

It is well-known that, properly scaled, an infinite countable collection of wavelets can generate Riesz bases for a scale of Sobolev spaces. Such wavelets can be applied for the numerical solution of PDEs and singular integral equations. For suitable wavelets, the (infinite) stiffness matrix of such an operator equation is boundedly invertible, so that the residual of an approximation is equivalent to its error, meaning that this residual can be used as an a posteriori error estimator for driving an adaptive algorithm. Using that wavelets have vanishing moments, even for singular integral equations the matrix is close to being sparse, and therefore its application can be efficiently approximated. Furthermore, in any case for elliptic equations, any principal submatrix is uniformly well-conditioned allowing for an efficient iterative solution of the arising Galerkin systems.

As with most wavelet applications, it is important that the wavelets have local supports. Other than with classical wavelet applications, as data compression and signal analysis, for solving operator equations the corresponding dual wavelets do not enter the computations, and their support sizes are irrelevant. This induces a lot of freedom in the construction of suitable wavelet bases. For more information on the application of wavelets for the (adaptive) solution of operator equations, we refer to [Dah97, Ste09, Urb09] and the references cited there.

Traditionally, wavelets are constructed on the line or on the interval $[0, 1]$. Then the application of a tensor product construction yields wavelets on $\mathbb{R}^n$ or $[0, 1]^n$. For equipping more general domains in $\mathbb{R}^n$, or their boundaries, with wavelet bases domain decomposition techniques have been developed (e.g. [CTU99, DS99a, DS99b]).

Another approach to treat non-product domains is to construct wavelets in finite element spaces w.r.t. a nested sequence of meshes. This approach, to which also this chapter is devoted, inherits the full flexibility from the finite element method concerning the shape of the domain.

To realize the Riesz basis property, one can rely on the theory of biorthogonal space decompositions ([Dah96]). It starts with two multiresolution analyses $(V_j)_{j\geq 0}$, $(\tilde{V}_j)_{j\geq 0}$ on the given domain that both satisfy Jackson and Bernstein estimates, and for which V_j and $\tilde{V}_j$ are relatively close in the sense that they satisfy inf-sup conditions, uniformly in j. Then for each j, one constructs the wavelets on 'level' j as a uniformly L_2 well-conditioned basis for the biorthogonal complement $V_j \cap \tilde{V}_{j-1}^{\perp_{L_2}}$ of V_{j-1} in V_j. The union over j of such wavelets form, properly scaled, a Riesz basis for the Sobolev space with smoothness index s for s in an interval $(s_{\min}, s_{\max}) \ni 0$ determined by the aforementioned Jackson and Bernstein estimates. With finite element wavelets, both multiresolution analyses are sequences of finite element spaces w.r.t. a common sequence of meshes. Linear finite elements of this type, with $\tilde{V}_j = V_j$, were constructed in [KO95, Ste98a, FQ00, HM00], and higher order ones, also with $\tilde{V}_j \neq V_j$, can be found in [DS99c, NS09]. With the exception of [Ste98a], the constructions in these references were restricted to two space dimensions.

An alternative possibility is to relax 'true' biorthogonality w.r.t. L_2 to an approximate biorthogonality, for instance biorthogonality w.r.t. to a level-dependent, approximate L_2-scalar product. Usually the resulting wavelet bases have smaller supports than those that span truly L_2-biorthogonal complements, but on the other hand typically their $s_{\min}$ is larger and sometimes positive. Linear finite element wavelets of this type can be found in [VW96, Ste98b, LO96], and quadratic ones in [Liu06].

In Chapter 4, we apply an adaptive wavelet method for solving time-dependent parabolic PDEs in a simultaneous space-time variational formulation. One of the arising spaces that has to be equipped with a wavelet Riesz basis is the intersection of the Bochner spaces $L_2((0,T);H_0^1(\Omega)) \cap H^1((0,T);H^{-1}(\Omega))$, where $\Omega \subset \mathbb{R}^2$ denotes the spatial domain. We will equip this space with a basis of tensor products of temporal wavelets and spatial wavelets. In order to obtain a Riesz basis for the aforementioned intersection space, the collection of temporal wavelets has to be, properly scaled, a Riesz basis for $L_2(0,T)$ and for $H^1(0,T)$, which collections of wavelets are amply available, whereas the collection Ψ of spatial wavelets has to be, properly scaled, a Riesz basis for $H_0^1(\Omega)$ and for its dual $H^{-1}(\Omega)$. Moreover, in order to obtain a sufficient near-sparsity of the resulting stiffness matrix, we need these wavelets from Ψ to have 2 vanishing moments, or more precisely, cancellation properties of order 2. Finally, since we write the PDE as a first order system least squares problem, and a suitable wavelet basis for the flux variable is at least of order 2, we need the wavelets from Ψ to be of order 3, i.e. piecewise quadratics.

Piecewise quadratics wavelets in finite element spaces were constructed in [DS99c, Liu06, NS09]. Those in [Liu06] have small supports, but are not stable in H^s for $s \leq 0$ and do not have vanishing moments. The quadratic wavelets in the other two references satisfy our needs, and have even 3 vanishing moments ($\tilde{V}_j = V_j$). Unfortunately

the condition numbers of those from [DS99c] turned out to be very large. For that reason, in [NS09] we applied the available freedom in the general construction from [DS99c] to arrive at wavelets that are much better conditioned. The price to be paid was an increased support size. The continuous piecewise quadratic finite element wavelets from [NS09] are linear combinations of 87 nodal basis functions. In the current work, we change $\tilde{V}_j$ into the space of continuous piecewise linears w.r.t. a dyadically refined mesh, and use the fact that the latter space can be equipped with basis functions of smaller supports to construct piecewise quadratic wavelets that satisfy all our needs, have condition numbers similar to those from [NS09], and are given as linear combinations of 11 or 13 nodal basis functions (these numbers depend on the valence of the vertices in the initial mesh, and the current numbers apply when this valence is 6). The same construction principle can be applied in three dimensions, but it will require calculations on a reference tetrahedron that so far we have not performed.

This chapter is organized as follows: In Sect. 3.2 we review the general theory on biorthogonal space decompositions, and in Sect. 3.3 we apply the general principles of the element-by-element construction of finite element wavelets to construct continuous piecewise quadratic wavelets in two space dimensions, with small supports and 2 vanishing moments. We provide numerically computed condition numbers in H^1, L_2 and H^{-1}-norms.

We will use the following notations. By $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For normed linear spaces $\mathscr{A}$ and $\mathscr{B}$, for convenience in this chapter always over $\mathbb{R}$, $\mathcal{L}(\mathscr{A}, \mathscr{B})$ will denote the space of bounded linear mappings $\mathscr{A} \to \mathscr{B}$ endowed with the operator norm $\|\cdot\|_{\mathcal{L}(\mathscr{A},\mathscr{B})}$.

For a countable set $\vee$, the norm and scalar product on $\ell_2(\vee)$ will be denoted as $\|\cdot\|$ and $\langle\,,\,\rangle$, respectively. For real square matrices A and B of the same size, we write $A \geq B$ when for all real vectors x, $\langle Ax, x\rangle \geq \langle Bx, x\rangle$.

A countable collection of functions Σ will be formally viewed as a column vector. Then for a sequence of scalars $\mathbf{c} = (c_\sigma)_{\sigma\in\Sigma}$, we set $\mathbf{c}^\top\Sigma := \sum_{\sigma\in\Sigma} c_\sigma\sigma$. For countable collections of functions Σ and Φ in a Hilbert space $\mathscr{H}$, we define the (formal) matrix $\langle\Sigma, \Phi\rangle_{\mathscr{H}} := [\langle\sigma, \phi\rangle_{\mathscr{H}}]_{\sigma\in\Sigma,\phi\in\Phi}$.

We use the symbol $\mathbb{N}_0$ to denote $\{0, 1, \cdots\}$.

3.2 Theory on biorthogonal wavelet bases

Let $\mathscr{V}, \tilde{\mathscr{V}}, \mathscr{H}$ be separable Hilbert spaces with $\mathscr{V}, \tilde{\mathscr{V}} \hookrightarrow \mathscr{H}$ with dense embedding. Identifying $\mathscr{H}$ with its dual, we obtain the Gelfand triples $\mathscr{V} \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{V}'$ and $\tilde{\mathscr{V}} \hookrightarrow \mathscr{H} \hookrightarrow \tilde{\mathscr{V}}'$ with dense embeddings. For $s \in [-1, 1]$, we set the interpolation spaces $\mathscr{V}^s := [\mathscr{V}', \mathscr{V}]_{\frac{1}{2}s+\frac{1}{2}}$ and $\tilde{\mathscr{V}}^s := [\tilde{\mathscr{V}}', \tilde{\mathscr{V}}]_{\frac{1}{2}s+\frac{1}{2}}$. The following theorem is a special case of an even more general result proven in [Dah96].

Theorem 3.2.1 (Biorthogonal space decompositions, [DS99c])**.** *Consider two* multiresolution analyses

$$V_0 \subset V_1 \subset \cdots \subset \mathscr{H}, \text{ with } \operatorname{clos}_{\mathscr{H}}(\cup_{j\geq 0} V_j) = \mathscr{H},$$
$$\tilde{V}_0 \subset \tilde{V}_1 \subset \cdots \subset \mathscr{H}, \text{ with } \operatorname{clos}_{\mathscr{H}}(\cup_{j\geq 0} \tilde{V}_j) = \mathscr{H}.$$

Suppose that for $j \geq 0$ there exist uniformly bounded biorthogonal projectors $Q_j \in \mathcal{L}(\mathscr{H}, \mathscr{H})$ such that

$$\operatorname{ran} Q_j = V_j, \quad \operatorname{ran}(I - Q_j) = \tilde{V}_j^{\perp_{\mathscr{H}}}, \tag{3.2.1}$$

and that, for some ρ, $\tilde{\rho} > 1$,

$$\begin{aligned} \inf_{v_j \in V_j} \|v - v_j\|_{\mathscr{H}} &\lesssim \rho^{-j} \|v\|_{\mathscr{V}} \quad (v \in \mathscr{V}), \\ \inf_{\tilde{v}_j \in \tilde{V}_j} \|\tilde{v} - \tilde{v}_j\|_{\mathscr{H}} &\lesssim \tilde{\rho}^{-j} \|\tilde{v}\|_{\tilde{\mathscr{V}}} \quad (\tilde{v} \in \tilde{\mathscr{V}}), \end{aligned} \tag{3.2.2}$$

and

$$\|v_j\|_{\mathscr{V}} \lesssim \rho^j \|v_j\|_{\mathscr{H}} \quad (v_j \in V_j), \quad \|\tilde{v}_j\|_{\tilde{\mathscr{V}}} \lesssim \tilde{\rho}^j \|\tilde{v}_j\|_{\mathscr{H}} \quad (\tilde{v} \in \tilde{V}_j). \tag{3.2.3}$$

Then, with $Q_{-1} := 0$, for every $s \in (-1, 1)$ it holds that

$$\|v\|_{\mathscr{V}^s}^2 \eqsim \sum_{j=0}^{\infty} \rho^{2js} \|(Q_j - Q_{j-1})v\|_{\mathscr{H}}^2 \quad (v \in \mathscr{V}^s).$$

Remark 3.2.2 (e.g. [Ste03]). Existence of the biorthogonal projector Q_j as in (3.2.1) is equivalent to each of the following conditions:

(1) $\beta_j := \inf_{0\neq v_j \in V_j} \sup_{0\neq \tilde{v}_j \in \tilde{V}_j} \frac{\langle v_j, \tilde{v}_j\rangle_{\mathscr{H}}}{\|v_j\|_{\mathscr{H}} \|\tilde{v}_j\|_{\mathscr{H}}} = \inf_{0\neq \tilde{v}_j \in \tilde{V}_j} \sup_{0\neq v_j \in V_j} \frac{\langle v_j, \tilde{v}_j\rangle_{\mathscr{H}}}{\|v_j\|_{\mathscr{H}} \|\tilde{v}_j\|_{\mathscr{H}}} > 0,$

(2) for any Riesz basis Φ_j for V_j there exists a (unique) $\mathscr{H}$-dual Riesz basis $\tilde{\Phi}_j$ for $\tilde{V}_j$,

(3) there exist some Riesz bases Φ_j and $\tilde{\Phi}_j$ for V_j and $\tilde{V}_j$, respectively, such that $\langle \Phi_j, \tilde{\Phi}_j\rangle_{\mathscr{H}}$ is bounded invertible.

It holds that $\|Q_j\|_{\mathcal{L}(\mathscr{H},\mathscr{H})}^{-1} = \beta_j \geq \frac{\|\langle \Phi_j, \tilde{\Phi}_j\rangle_{\mathscr{H}}^{-1}\|^{-1}}{\sqrt{\|\langle \Phi_j, \Phi_j\rangle_{\mathscr{H}}\| \|\langle \tilde{\Phi}_j, \tilde{\Phi}_j\rangle_{\mathscr{H}}\|}}$.

Corollary 3.2.3. *In the situation of Thm. 3.2.1, for $j \geq 0$ let $\Psi_j = \{\psi_{j,x} \colon x \in J_j\}$ be a* uniform *Riesz basis for $\operatorname{ran}(Q_j - Q_{j-1}) = V_j \cap \tilde{V}_{j-1}^{\perp_{\mathscr{H}}}$ ($\tilde{V}_{-1} := \{0\}$), i.e., with $\kappa_{\mathscr{H}}(\Psi_j) := \|\langle \Psi_j, \Psi_j\rangle_{\mathscr{H}}\| \|\langle \Psi_j, \Psi_j\rangle_{\mathscr{H}}^{-1}\|$ it holds that $\sup_j \kappa_{\mathscr{H}}(\Psi_j) < \infty$. Then for $s \in (-1, 1)$,*

$$\bigcup_{j=0}^{\infty} \rho^{-js} \Psi_j \text{ is a Riesz basis for } \mathscr{V}^s.$$

In particular, with κ_s denoting the quotient of the supremum and infimum over $v \in \mathscr{V}_s$ of $\|v\|^2_{\mathscr{V}^s} / \sum_{j=0}^{\infty} \rho^{2js} \|(Q_j - Q_{j-1})v\|^2_{\mathscr{H}}$, it holds that

$$\kappa_{\mathscr{V}^s}\Big(\bigcup_{j=0}^{\infty} \rho^{-js}\Psi_j\Big) \leq \kappa_s \times \sup_{j\geq 0} \kappa_{\mathscr{H}}(\Psi_j).$$

Recalling that $\operatorname{ran}(Q_0 - Q_{-1}) = V_0$, the remaining challenge is the construction of Ψ_j for $j \geq 1$, i.e. Ψ_{j+1} for $j \geq 0$:

Proposition 3.2.4 ([NS09]). *For $j \geq 0$, let $\Theta_j \cup \Sigma_{j+1}$ and $\tilde{\Phi}_j$ be uniform Riesz bases for V_{j+1} and $\tilde{V}_j$, respectively, such that $\langle \Theta_j, \tilde{\Phi}_j \rangle_{\mathscr{H}} = \mathrm{Id}$. Then*

$$\Psi_{j+1} = \Xi_{j+1} - \langle \Xi_{j+1}, \tilde{\Phi}_j \rangle_{\mathscr{H}} \Theta_j \tag{3.2.4}$$

is a uniform Riesz basis for $V_{j+1} \cap \tilde{V}_j^{\perp_{\mathscr{H}}} = \operatorname{ran}(Q_{j+1} - Q_j)$.

With

$$\delta_j := \inf_{0\neq \hat{v}_j \in \operatorname{span}\Theta_j} \sup_{0\neq \tilde{v}_j \in \tilde{V}_j} \frac{\langle \hat{v}_j, \tilde{v}_j \rangle_{\mathscr{H}}}{\|\hat{v}_j\|_{\mathscr{H}} \|\tilde{v}_j\|_{\mathscr{H}}},$$

$$\varepsilon_j := \sup_{0\neq \hat{v}_j \in \operatorname{span}\Theta_j} \sup_{0\neq w_{j+1} \in \operatorname{span}\Xi_{j+1}} \frac{\langle \hat{v}_j, w_{j+1} \rangle_{\mathscr{H}}}{\|\hat{v}_j\|_{\mathscr{H}} \|w_{j+1}\|_{\mathscr{H}}},$$

it holds that

$$\kappa_{\mathscr{H}}(\Psi_{j+1}) \leq \frac{(1+\delta_j^{-1})}{\sqrt{1-\varepsilon_j}} \kappa_{\mathscr{H}}(\Xi_{j+1}).$$

To complete our overview of the theory of biorthogonal space decompositions and wavelets, in the next two remarks we discuss dual wavelets. The first remark will become relevant for the numerical approximation of the $H^{-1}(\Omega)$-condition numbers of the (primal) wavelets that we will construct. The second remark will shed light on the difference between the general setting of biorthogonal space decompositions that we consider, and the more restricted framework for the construction of locally supported wavelets, whose duals are locally supported too, that is usually considered in one dimension.

Remark 3.2.5 (dual wavelets). With $(\,)^*$ denoting the adjoint w.r.t. $\langle\,,\,\rangle_{\mathscr{H}}$, under the conditions of Thm 3.2.1 equivalently it holds that for every $s \in (-1, 1)$,

$$\|\tilde{v}\|^2_{\tilde{\mathscr{V}}^s} \eqsim \sum_{j=0}^{\infty} \rho^{2js} \|(Q_j^* - Q_{j-1}^*)\tilde{v}\|^2_{\mathscr{H}} \quad (\tilde{v} \in \tilde{\mathscr{V}}^s),$$

so that for $\tilde{\Psi}_j$ being *any* uniform Riesz basis for $\operatorname{ran}(Q_j^* - Q_{j-1}^*) = \tilde{V}_{j+1} \cap V_j^{\perp_{\mathscr{H}}}$,

$$\bigcup_{j=0}^{\infty} \rho^{-js}\tilde{\Psi}_j \text{ is a Riesz basis for } \tilde{\mathscr{V}}^s.$$

One verifies that the pair $(\operatorname{ran}(Q_j^*-Q_{j-1}^*),\operatorname{ran}(Q_j-Q_{j-1}))$ satisfies (1) with infsup constant $\gamma_j := \|Q_j-Q_{j-1}\|^{-1}_{\mathcal{L}(\mathscr{H},\mathscr{H})} \gtrsim 1$. Consequently, as stated in Remark 3.2.2, given a Riesz basis Ψ_j for $\operatorname{ran}(Q_j-Q_{j-1})$, there exists a unique dual or biorthogonal Riesz basis $\tilde\Psi_j$ for $\operatorname{ran}(Q_j^*-Q_{j-1}^*)$, i.e. with $\langle\Psi_j,\tilde\Psi_j\rangle_{\mathscr{H}}=\mathrm{Id}$. From

$$\gamma_j\|\mathbf{c}_j^\top\tilde\Psi_j\|_{\mathscr{H}} \le \sup_{0\neq v_j\in\operatorname{ran}(Q_j-Q_{j-1})}\frac{\langle\mathbf{c}_j^\top\tilde\Psi_j,v_j\rangle_{\mathscr{H}}}{\|v_j\|_{\mathscr{H}}} \le \|\mathbf{c}_j^\top\tilde\Psi_j\|_{\mathscr{H}},$$

and

$$\sup_{0\neq v_j\in\operatorname{ran}(Q_j-Q_{j-1})}\frac{\langle\mathbf{c}_j^\top\tilde\Psi_j,v_j\rangle_{\mathscr{H}}}{\|v_j\|_{\mathscr{H}}} = \sup_{0\neq\mathbf{d}_j}\frac{\langle\mathbf{c}_j^\top\tilde\Psi_j,\mathbf{d}_j^\top\Psi_j\rangle_{\mathscr{H}}}{\|v_j\|_{\mathscr{H}}} = \sup_{0\neq\mathbf{d}_j}\frac{\langle\mathbf{c}_j,\mathbf{d}_j\rangle}{\|\mathbf{d}_j\|}\frac{\|\mathbf{d}_j\|}{\|\mathbf{d}_j^\top\Psi_j\|_{\mathscr{H}}}$$

one infers that $\kappa_{\mathscr{H}}(\tilde\Psi_j)\le\gamma_j^{-1}\kappa_{\mathscr{H}}(\Psi_j)$. Consequently, when the Ψ_j are uniform Riesz bases for $\operatorname{ran}(Q_j-Q_{j-1})$, then their duals are uniform Riesz bases for $\operatorname{ran}(Q_j^*-Q_{j-1}^*)$, and for $s\in(-1,1)$, $\bigcup_{j=0}^\infty\rho^{-js}\Psi_j$ and $\bigcup_{j=0}^\infty\rho^{js}\tilde\Psi_j$ are $\mathscr{H}$-biorthogonal Riesz bases for $\mathscr{V}^s$ and $\tilde{\mathscr{V}}^{-s}$, respectively.

Moreover, if $\tilde{\mathscr{V}}=\mathscr{V}$, then from

$$\|\mathbf{c}^\top\mathbf{D}\Psi\|_{\mathscr{V}^s} = \sup_{0\neq\mathbf{d}\in\ell_2}\frac{\langle\mathbf{c}^\top\mathbf{D}\Psi,\mathbf{d}^\top\mathbf{D}^{-1}\tilde\Psi\rangle_{\mathscr{H}}}{\|\mathbf{d}^\top\mathbf{D}^{-1}\tilde\Psi\|_{\tilde{\mathscr{V}}^{-s}}} = \sup_{0\neq\mathbf{d}\in\ell_2}\frac{\langle\mathbf{c},\mathbf{d}\rangle}{\|\mathbf{d}\|}\frac{\|\mathbf{d}\|}{\|\mathbf{d}^\top\mathbf{D}^{-1}\tilde\Psi\|_{\tilde{\mathscr{V}}^{-s}}},$$

where $\mathbf{D}$ is an invertible diagonal matrix, $\Psi=\cup_j\Psi_j$, and $\tilde\Psi=\cup_j\tilde\Psi_j$, and the analogous result with interchanged roles of $(\Psi,\mathbf{D})$ and $(\tilde\Psi,\mathbf{D}^{-1})$, one infers that $\kappa_{\mathscr{V}^s}(\mathbf{D}\Psi_j)=\kappa_{\tilde{\mathscr{V}}^{-s}}(\mathbf{D}^{-1}\tilde\Psi)$, and so in particular that $\kappa_{\mathscr{V}^s}(\bigcup_{j=0}^\infty\rho^{-js}\Psi_j)=\kappa_{\tilde{\mathscr{V}}^{-s}}(\bigcup_{j=0}^\infty\rho^{js}\tilde\Psi_j)$.

Remark 3.2.6 (dual wavelets cont'd). An explicit expression for $\tilde\Psi_j$ can be obtained in the following special case: For $j\ge0$, let Φ_j and $\tilde\Phi_j$ be $\mathscr{H}$-biorthogonal Riesz bases for V_j and $\tilde V_j$, respectively. Let $\mathbf{M}_j=[\mathbf{M}_{j,0}\,\mathbf{M}_{j,1}]$ be the basis transformation from $\Phi_j\cup\Psi_{j+1}$ to Φ_{j+1}, i.e., $[\Phi_j^\top\,\Psi_{j+1}^\top]=\Phi_{j+1}^\top\mathbf{M}_j$. Analogously, let $[\tilde\Phi_j^\top\,\tilde\Psi_{j+1}^\top]=\tilde\Phi_{j+1}^\top\tilde{\mathbf{M}}_j$. Biorthogonality shows that $\mathbf{M}_{j,0}=\langle\tilde\Phi_{j+1},\Phi_j\rangle_{\mathscr{H}}$, $\mathbf{M}_{j,1}=\langle\tilde\Phi_{j+1},\Psi_{j+1}\rangle_{\mathscr{H}}$, and analogous relations at the dual side, as well as $\tilde{\mathbf{M}}_j=\mathbf{M}_j^{-\top}$.

Now let Ψ_j be constructed as in Prop. 3.2.4 for the case that $\Theta_j=\Phi_j$ and thus $\operatorname{span}\Theta_j=V_j$. Then (3.2.4) reads as $\mathbf{M}_{j,1}=(\mathrm{Id}-\mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^\top)\mathbf{R}_{j,1}$, where $\mathbf{R}_{j,1}=\langle\tilde\Phi_{j+1},\Xi_{j+1}\rangle_{\mathscr{H}}$. We conclude that

$$\mathbf{M}_j=[\mathbf{M}_{j,0}\;\mathbf{R}_{j,1}]\begin{bmatrix}\mathrm{Id} & -\tilde{\mathbf{M}}_{j,0}^\top\mathbf{R}_{j,1}\\ 0 & \mathrm{Id}\end{bmatrix},\ \text{i.e. }\tilde{\mathbf{M}}_j=\begin{bmatrix}\mathrm{Id} & 0\\ \mathbf{R}_{j,1}^\top\tilde{\mathbf{M}}_{j,0} & \mathrm{Id}\end{bmatrix}[\mathbf{M}_{j,0}\;\mathbf{R}_{j,1}]^{-\top},$$

meaning that dual wavelets become explicitly available in terms of $\tilde\Phi_{j+1}$ when additionally Ξ_{j+1} is chosen such that the basis transformation $[\mathbf{M}_{j,0}\;\mathbf{R}_{j,1}]^{-1}$ from Φ_{j+1} to the two-level basis $\Phi_j\cup\Xi_{j+1}$ is explicitly available.

Classical 'stationary' wavelet constructions on the line provide explicitly given (primal and) dual wavelets. Explicit knowledge of dual wavelets is crucial for applications

as data compression and data analysis. For applications as preconditioning and the (adaptive) solving of operator equations, however, dual wavelets do not enter the computation, and wavelet constructions on general, non-rectangular domains usually do not provide them (an exception is [Ste03]). Allowing $\operatorname{span}\Theta_j \neq V_j$, thus giving up explicit knowledge of dual wavelets, gives an enormous additional freedom in the construction of Ψ_{j+1} in Prop. 3.2.4, which we will also exploit in the current work.

3.3 Construction of quadratic Lagrange finite element wavelets

3.3.1 Multi-resolution analyses

Given a conforming triangulation $\mathcal{T}_0$ of a polygon $\Omega \subset \mathbb{R}^2$, let $(\mathcal{T}_j)_{j\geq 0}$ be the sequence of triangulations where $\mathcal{T}_{j+1}$ is created from $\mathcal{T}_j$ by subdividing each triangle $T \in \mathcal{T}_j$ into four sub-triangles by connecting the midpoints of the edges of T (*red-refinement*).

For $\Gamma \subset \partial\Omega$ being a union of closed edges of triangles $T \in \mathcal{T}_0$, we define V_j ($\tilde{V}_j$) as the space of continuous piecewise quadratics (linears) w.r.t. $\mathcal{T}_j$ ($\mathcal{T}_{j+1}$) that vanish at Γ. We let $\mathcal{N}(\mathcal{T}_j)$ denote the set of vertices that are not on Γ of $T \in \mathcal{T}_j$.

We take $\mathscr{H} := L_2(\Omega)$, and for some $t \in [1, \frac{3}{2})$, define $\mathscr{V} = \tilde{\mathscr{V}} := H^1_{0,\Gamma}(\Omega) \cap H^t(\Omega)$. With these definitions, the Jackson and Bernstein estimates (3.2.2)–(3.2.3) are satisfied with $\rho = 2^t$. After having equipped V_j and $\tilde{V}_j$ with uniform Riesz bases, later in Sect. 3.3.3, with the aid of Remark 3.2.2 we will verify also the remaining condition (3.2.1) of Theorem 3.2.1. For $|s| \leq 2$, we define

$$\mathscr{Z}^s := \big[(H^1_{0,\Gamma}(\Omega)\cap H^2(\Omega))', H^1_{0,\Gamma}(\Omega)\cap H^2(\Omega)\big]_{\frac{s}{4}+\frac{1}{2}} \simeq \begin{cases} H^1_{0,\Gamma}(\Omega)\cap H^s(\Omega) & s\in[1,2],\\ H^s_{0,\Gamma}(\Omega) & s\in(\frac{1}{2},1),\\ [L_2(\Omega), H^1_{0,\Gamma}(\Omega)]_{\frac{1}{2}} & s=\frac{1}{2},\\ H^s(\Omega) & s\in[0,\frac{1}{2}),\\ (\mathscr{Z}^{-s})' & s\in[-2,0). \end{cases}$$

Thanks to the *reiteration theorem* we have that $\mathscr{V}^s \simeq \mathscr{Z}^{st}$. Since $t \in [1, \frac{3}{2})$ was arbitrary, from Corollary 3.2.3 we conclude that if Ψ_j is a uniform $L_2(\Omega)$-Riesz basis for $V_j \cap \tilde{V}_{j-1}^{\perp_{L_2(\Omega)}}$, then for $|s| < \frac{3}{2}$,

$$\bigcup_{j=0}^{\infty} 2^{-js}\Psi_j \text{ is a Riesz basis for } \mathscr{Z}^s.$$

From the approximation properties of $(V_j)_j$, we infer that for $j \geq 0$, $\psi_{j+1,x} \in \Psi_{j+1}$, $p \in [1,\infty]$ and $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned} |\langle u, \psi_{j+1,x}\rangle_{L_2(\Omega)}| &\leq \|\psi_{j+1,x}\|_{L_p(\Omega)} \inf_{v_j\in V_j} \|u - v_j\|_{L_q(\operatorname{supp}\psi_{j,x})} \\ &\lesssim \|\psi_{j+1,x}\|_{L_p(\Omega)} (2^{-j})^2 |u|_{W^2_q(\text{conv hull}(\operatorname{supp}\psi_{j,x}))} \quad (u \in W^2_q(\Omega)\cap H^1_{0,\Gamma}(\Omega)), \end{aligned}$$

that is, the wavelets will have *cancellation properties of order* 2. In particular, when conv hull$(\operatorname{supp}\psi_{j+1,x})\cap\Gamma=\emptyset$, $\psi_{j+1,x}$ has 2 *vanishing moments*.

3.3.2 Local-to-global basis construction

In this subsection it will be shown how to reduce the construction of the various collections of functions on Ω, that are needed for the construction of the biorthogonal wavelet basis corresponding to the primal and dual multi-resolution analyses $(V_j)_j$ and $(\tilde{V}_j)_j$, to the construction of corresponding collections of 'local' functions on the reference triangle

$$\boldsymbol{T}=\{\lambda\in\mathbb{R}^3\colon \sum_{i=1}^{3}\lambda_i=1,\lambda_i\geq 0\}.$$

For $1\leq i\leq 3$, let $\boldsymbol{T}_i=\{\lambda\in\boldsymbol{T}\colon \lambda_i\leq\frac{1}{2}\}$, and let $\boldsymbol{T}_4=\overline{\boldsymbol{T}\setminus\cup_{i=1}^{3}\boldsymbol{T}_i}$ (*red-refinement*).

For any closed triangle T, let $\lambda_T(x)\in\boldsymbol{T}$ denote the barycentric coordinates of $x\in T$ with respect to the set of vertices of T ordered in some way.

We consider finite collections of functions $\boldsymbol{\Sigma}=\{\boldsymbol{\sigma}_\lambda:\lambda\in\boldsymbol{I}_{\boldsymbol{\Sigma}}\}$ on $\boldsymbol{T}$, where $\boldsymbol{I}_{\boldsymbol{\Sigma}}$ is a finite set of points in $\boldsymbol{T}$, that satisfy

- $(\mathcal{V})$ $\boldsymbol{\sigma}_\lambda$ vanishes on any edge or vertex that does not include λ,
- $(\mathcal{S})$ $\pi(\boldsymbol{I}_{\boldsymbol{\Sigma}})=\boldsymbol{I}_{\boldsymbol{\Sigma}}$ and $\boldsymbol{\sigma}_\lambda=\boldsymbol{\sigma}_{\pi(\lambda)}\circ\pi$ for any permutation $\pi:\mathbb{R}^3\to\mathbb{R}^3$,
- $(\mathcal{I})$ $\boldsymbol{\Sigma}$ is a linearly independent collection of continuous functions.

Such collections of local functions can be used to assemble collections of global functions in a way known from finite element methods: For $j\geq 0$ and with

$$I_{\Sigma_j}:=\{x\in\bar{\Omega}\setminus\Gamma:\lambda_T(x)\in\boldsymbol{I}_{\boldsymbol{\Sigma}}\text{ for some }T\in\mathcal{T}_j\},\tag{3.3.1}$$

we define the collection $\Sigma_j=\{\sigma_{j,x}\colon x\in I_{\Sigma_j}\}$ of functions on Ω by

$$\sigma_{j,x}(y)=\begin{cases}\mu(x;\mathcal{T}_j)\boldsymbol{\sigma}_{\lambda_T(x)}(\lambda_T(y)) & \text{if } x,y\in T\in\mathcal{T}_j\\ 0 & \text{otherwise}\end{cases}\tag{3.3.2}$$

with scaling factor $\mu(x;\mathcal{T}_j):=\big(\sum_{\{T\in\mathcal{T}_j\colon T\ni x\}}\operatorname{vol}(T)\big)^{-\frac{1}{2}}$. Note that the assumptions $(\mathcal{V})$, $(\mathcal{S})$ and $(\mathcal{I})$ show that Σ_j are independent collections of well-defined, continuous functions on Ω.

Lemma 3.3.1. *Let $\boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Sigma}}$ be two collections of 'local' functions on $\boldsymbol{T}$ both satisfying $(\mathcal{V})$, $(\mathcal{S})$ and $(\mathcal{I})$. Let Σ_j and $\tilde{\Sigma}_j$ denote the corresponding collections of 'global' functions on Ω. Then:*

(i) $\|\langle\Sigma_j,\Sigma_j\rangle_{L_2(\Omega)}\|\leq\big\|\frac{\langle\boldsymbol{\Sigma},\boldsymbol{\Sigma}\rangle_{L_2(\boldsymbol{T})}}{\operatorname{vol}(\boldsymbol{T})}\big\|,\quad\|\langle\Sigma_j,\Sigma_j\rangle^{-1}_{L_2(\Omega)}\|\leq\Big\|\Big(\frac{\langle\boldsymbol{\Sigma},\boldsymbol{\Sigma}\rangle_{L_2(\boldsymbol{T})}}{\operatorname{vol}(\boldsymbol{T})}\Big)^{-1}\Big\|.$

(ii) If $\boldsymbol{I}_{\boldsymbol{\Sigma}}=\boldsymbol{I}_{\tilde{\boldsymbol{\Sigma}}}$, and $\mu\mathrm{Id}\leq\frac{\langle\boldsymbol{\Sigma},\tilde{\boldsymbol{\Sigma}}\rangle_{L_2(\boldsymbol{T})}}{\operatorname{vol}(\boldsymbol{T})}\leq M\mathrm{Id}$ then $\mu\mathrm{Id}\leq\langle\Sigma_j,\tilde{\Sigma}_j\rangle_{L_2(\Omega)}\leq M\mathrm{Id}$.

(iii) $$\sup_{\substack{0\neq v_j\in\operatorname{span}\Sigma_j\\ 0\neq\tilde v_j\in\operatorname{span}\tilde\Sigma_j}}\frac{\langle v_j,\tilde v_j\rangle_{L_2(\Omega)}}{\|v_j\|_{L_2(\Omega)}\|\tilde v_j\|_{L_2(\Omega)}}\le\sup_{\substack{0\neq v\in\operatorname{span}\boldsymbol\Sigma\\ 0\neq\tilde v\in\operatorname{span}\tilde{\boldsymbol\Sigma}}}\frac{\langle v,\tilde v\rangle_{L_2(\boldsymbol\Sigma)}}{\|v\|_{L_2(\boldsymbol\Sigma)}\|\tilde v\|_{L_2(\boldsymbol\Sigma)}}.$$

Proof. We note that for $\tilde{\mathbf c}_j\in\ell_2(I_{\tilde\Sigma_j})$, $\mathbf c_j\in\ell_2(I_{\Sigma_j})$, it holds that

$$\begin{aligned}
&\big\langle\langle\Sigma_j,\tilde\Sigma_j\rangle_{L_2(\Omega)}\tilde{\mathbf c}_j,\mathbf c_j\big\rangle=\langle\mathbf c_j^\top\Sigma_j,\tilde{\mathbf c}_j^\top\tilde\Sigma_j\rangle_{L_2(\Omega)}\\
&=\sum_{T\in\mathcal T_j}\frac{\operatorname{vol}(T)}{\operatorname{vol}(\boldsymbol T)}\Big\langle\sum_{x\in I_{\Sigma_j}\cap T}c_{j,x}\mu(x;\mathcal T_j)\boldsymbol\sigma_{\lambda_T(x)},\sum_{y\in I_{\tilde\Sigma_j}\cap T}\tilde c_{j,y}\mu(y;\mathcal T_j)\tilde{\boldsymbol\sigma}_{\lambda_T(y)}\Big\rangle_{L_2(\boldsymbol T)}\\
&=\sum_{T\in\mathcal T_j}\Big\langle\frac{\langle\boldsymbol\Sigma,\tilde{\boldsymbol\Sigma}\rangle_{L_2(\boldsymbol T)}}{\operatorname{vol}(\boldsymbol T)}\tilde{\mathbf c}_{\mathcal T_j,T},\mathbf c_{\mathcal T_j,T}\Big\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathbf c}_{\mathcal T_j,T}&:=\Big(\tilde c_{j,\lambda_T^{-1}(\lambda)}\mu(\lambda_T^{-1}(\lambda);\mathcal T_j)\sqrt{\operatorname{vol}(T)}\Big)_{\lambda\in\boldsymbol I_{\tilde{\boldsymbol\Sigma}}},\\
\mathbf c_{\mathcal T_j,T}&:=\Big(c_{j,\lambda_T^{-1}(\lambda)}\mu(\lambda_T^{-1}(\lambda);\mathcal T_j)\sqrt{\operatorname{vol}(T)}\Big)_{\lambda\in\boldsymbol I_{\boldsymbol\Sigma}}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{T\in\mathcal T_j}\|\tilde{\mathbf c}_{\mathcal T_j,T}\|^2&=\sum_{T\in\mathcal T_j}\sum_{y\in I_{\tilde\Sigma_j}\cap T}|\tilde c_{j,y}|^2\mu(y;\mathcal T_j)^2\operatorname{vol}(T)\\
&=\sum_{y\in I_{\tilde\Sigma_j}}|\tilde c_{j,y}|^2\mu(y;\mathcal T_j)^2\sum_{\{T\in\mathcal T_j\,:\,T\ni y\}}\operatorname{vol}(T)=\|\tilde{\mathbf c}_j\|^2,
\end{aligned}$$

and similarly $\sum_{T\in\mathcal T_j}\|\mathbf c_{\mathcal T_j,T}\|^2=\|\mathbf c_j\|^2$.

These relations show (ii) and (i), for the latter using that for $A=A^\top>0$, $\|A\|=\sup_{0\neq x}\frac{\langle Ax,x\rangle}{\|x\|^2}$, $\|A^{-1}\|^{-1}=\inf_{0\neq x}\frac{\langle Ax,x\rangle}{\|x\|^2}$.

Denoting the value of the supremum at the right hand side of (iii) as Q, we have

$$\begin{aligned}
\langle v_j,\tilde v_j\rangle_{L_2(\Omega)}&=\sum_{T\in\mathcal T_j}\frac{\operatorname{vol}(T)}{\operatorname{vol}(\boldsymbol T)}\langle v_j\circ\lambda_T^{-1},\tilde v_j\circ\lambda_T^{-1}\rangle_{L_2(\boldsymbol T)}\\
&\le Q\sum_{T\in\mathcal T_j}\frac{\operatorname{vol}(T)}{\operatorname{vol}(\boldsymbol T)}\|v_j\circ\lambda_T^{-1}\|_{L_2(\boldsymbol T)}\|\tilde v_j\circ\lambda_T^{-1}\|_{L_2(\boldsymbol T)}\\
&\le Q\Big[\sum_{T\in\mathcal T_j}\frac{\operatorname{vol}(T)}{\operatorname{vol}(\boldsymbol T)}\|v_j\circ\lambda_T^{-1}\|^2_{L_2(\boldsymbol T)}\Big]^{\frac12}\Big[\sum_{T\in\mathcal T_j}\frac{\operatorname{vol}(T)}{\operatorname{vol}(\boldsymbol T)}\|\tilde v_j\circ\lambda_T^{-1}\|^2_{L_2(\boldsymbol T)}\Big]^{\frac12}\\
&=Q\|v_j\|_{L_2(\Omega)}\|\tilde v_j\|_{L_2(\Omega)},
\end{aligned}$$

which completes the proof. □

3.3.3 Verification of the uniform inf-sup conditions for $(V_j, \tilde{V}_j)_{j\geq 0}$

As a first application of the local-to-global basis construction discussed in the previous subsection, we will verify the existence of the uniform bounded biorthogonal projectors from (3.2.1).

Setting $\boldsymbol{I}_0 := \mathbb{N}_0^3 \cap \boldsymbol{T}$, and for $i \geq 1$, $\boldsymbol{I}_i := (2^{-i}\mathbb{N}_0^3 \cap \boldsymbol{T}) \setminus \boldsymbol{I}_{i-1}$, we let $\boldsymbol{N} := \{n_\lambda \colon \lambda \in \boldsymbol{I}_0 \cup \boldsymbol{I}_1\}$ and $\tilde{\boldsymbol{N}} := \{\tilde{n}_\lambda \colon \lambda \in \boldsymbol{I}_0 \cup \boldsymbol{I}_1\}$ denote the standard nodal bases of $P_2(\boldsymbol{T})$ and $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_1(\boldsymbol{T}_i)$, respectively.

The with $\boldsymbol{N}$ and $\tilde{\boldsymbol{N}}$ corresponding 'global' collections N_j and $\tilde{N}_j$, defined by (3.3.1)–(3.3.2), span the spaces V_j and $\tilde{V}_j$, respectively. The index sets of these collections satisfy $I_{N_j} = I_{\tilde{N}_j} = \mathcal{N}(\mathcal{T}_{j+1})$. Lemma 3.3.1(i) shows that N_j and $\tilde{N}_j$ are uniform $L_2(\Omega)$-Riesz bases for their spans. With $\boldsymbol{I}_0 \cup \boldsymbol{I}_1$ (and $\boldsymbol{I}_2$) numbered as indicated in Figure 3.1, a direct computation shows that

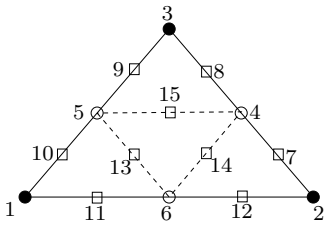


Figure 3.1: Numbering of $\boldsymbol{I}_0 \cup \boldsymbol{I}_1 \cup \boldsymbol{I}_2$.

$$\frac{\langle \boldsymbol{N}, \tilde{\boldsymbol{N}} \rangle_{L_2(\boldsymbol{T})}}{\operatorname{vol}(\boldsymbol{T})} = \frac{1}{480} \begin{bmatrix} 16 & -3 & -3 & -10 & 0 & 0 \\ -3 & 16 & -3 & 0 & -10 & 0 \\ -3 & -3 & 16 & 0 & 0 & -10 \\ 2 & 14 & 14 & 70 & 30 & 30 \\ 14 & 2 & 14 & 30 & 70 & 30 \\ 14 & 14 & 2 & 30 & 30 & 70 \end{bmatrix}.$$

With $\lambda := \lambda_{\min}\big(\frac{1}{2\operatorname{vol}(\boldsymbol{T})}(\langle \boldsymbol{N}, \tilde{\boldsymbol{N}} \rangle_{L_2(\boldsymbol{T})} + \langle \boldsymbol{N}, \tilde{\boldsymbol{N}} \rangle_{L_2(\boldsymbol{T})}^\top)\big) \approx 0.0191$, it holds that $\frac{\langle \boldsymbol{N}, \tilde{\boldsymbol{N}} \rangle_{L_2(\boldsymbol{T})}}{\operatorname{vol}(\boldsymbol{T})} \geq \lambda \mathrm{Id}$ and thus, by Lemma 3.3.1(ii), $\langle N_j, \tilde{N}_j \rangle_{L_2(\Omega)} \geq \lambda \mathrm{Id}$, which shows that $\sup_j \|\langle N_j, \tilde{N}_j \rangle_{L_2(\Omega)}^{-1}\| \leq \lambda^{-1}$. Now Remark 3.2.2 shows that the inf-sup condition (1), and thus equivalently (3.2.1), are indeed satisfied.

3.3.4 Local collections $\boldsymbol{\Theta}$, $\boldsymbol{\Xi}$, and $\tilde{\boldsymbol{\Phi}}$ underlying the construction of Ψ_{j+1}

In order to construct Ψ_{j+1} for $j \geq 0$ by means of equation (3.2.4), we need to specify the collections Θ_j, Ξ_{j+1} and $\tilde{\Phi}_j$ of functions on Ω. We will construct them from

collections $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_\lambda \colon \lambda \in \boldsymbol{I}_0 \cup \boldsymbol{I}_1\}$, $\boldsymbol{\Xi} = \{\boldsymbol{\xi}_\lambda \colon \lambda \in \boldsymbol{I}_2\}$, and $\tilde{\boldsymbol{\Phi}} = \{\tilde{\boldsymbol{\phi}}_\lambda \colon \lambda \in \boldsymbol{I}_0 \cup \boldsymbol{I}_1\}$, respectively, of functions on $\boldsymbol{T}$ that satisfy $(\mathcal{V})$, $(\mathcal{S})$, and $(\mathcal{J})$, and for which $\boldsymbol{\Theta} \cup \boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Phi}}$ are bases for $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_2(\boldsymbol{T}_i)$ and $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_1(\boldsymbol{T}_i)$, respectively, and $\langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})} = \operatorname{vol}(\boldsymbol{T})\mathrm{Id}$. Then by an application of Lemma 3.3.1, we know that, as required, $\Theta_j \cup \Sigma_{j+1}$ and $\tilde{\Phi}_j$ are uniform Riesz bases for V_{j+1} and $\tilde{V}_j$, respectively, and $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)} = \mathrm{Id}$. The index sets of these collections satisfy $I_{\Theta_j} = I_{\tilde{\Phi}_j} = \mathcal{N}(\mathcal{T}_{j+1})$, and $I_{\Sigma_{j+1}} = \mathcal{N}(\mathcal{T}_{j+2}) \setminus \mathcal{N}(\mathcal{T}_{j+1})$.

We will specify $\boldsymbol{\Theta} \cup \boldsymbol{\Sigma}$ and $\tilde{\boldsymbol{\Phi}}$ in terms of the usual nodal bases $\boldsymbol{N}_f$ and $\tilde{\boldsymbol{N}}$ for $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_2(\boldsymbol{T}_i)$ and $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_1(\boldsymbol{T}_i)$, respectively. For completeness, with $\boldsymbol{N}_f$ we mean the collection of functions in $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_2(\boldsymbol{T}_i)$ that have value 1 in one of the points of $\boldsymbol{I}_0 \cup \boldsymbol{I}_1 \cup \boldsymbol{I}_2$ and vanish at the others. Aiming at wavelets that have relatively small supports, we exploit the available freedom in the construction to obtain $\boldsymbol{\theta}_\lambda$ and $\boldsymbol{\xi}_\lambda$ with small supports and a matrix $\langle \boldsymbol{\Xi}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})}$ that is sparsely populated.

We define $\tilde{\boldsymbol{\Phi}}$ as indicated in Figure 3.2, and $\boldsymbol{\Theta}$ as indicated in Figure 3.3. Both these collections satisfy $(\mathcal{V})$, $(\mathcal{S})$, and $(\mathcal{J})$, and $\tilde{\boldsymbol{\Phi}}$ is a basis for $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_1(\boldsymbol{T}_i)$. A direct computation shows that $\langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}} \rangle_{L_2(\boldsymbol{T})} = \operatorname{vol}(\boldsymbol{T})\mathrm{Id}$.

We define $\boldsymbol{\Xi}$ as indicated in Figure 3.4. It satisfies $(\mathcal{V})$, $(\mathcal{S})$, and $(\mathcal{J})$, and since

$$\begin{vmatrix}
72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1560}{81} & 0 & 0 & \frac{-12}{25} & \frac{-12}{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1560}{81} & 0 & 0 & 0 & \frac{-12}{25} & \frac{-12}{25} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1560}{81} & 0 & 0 & 0 & 0 & \frac{-12}{25} & \frac{-12}{25} & 0 & 0 & 0 \\
0 & 6 & 0 & \frac{530}{81} & 0 & 0 & 1 & \frac{-2}{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & \frac{530}{81} & 0 & 0 & \frac{-2}{25} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & \frac{530}{81} & 0 & 0 & 0 & 1 & \frac{-2}{25} & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & \frac{530}{81} & 0 & 0 & 0 & \frac{-2}{25} & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & \frac{530}{81} & 0 & 0 & 0 & 0 & 1 & \frac{-2}{25} & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & \frac{530}{81} & 0 & 0 & 0 & 0 & \frac{-2}{25} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{50}{81} & \frac{50}{81} & 0 & 0 & \frac{1}{25} & \frac{-8}{25} & \frac{-8}{25} & \frac{1}{25} & \frac{-5}{4} & 1 & 1 \\
0 & 0 & 0 & \frac{50}{81} & 0 & \frac{50}{81} & \frac{-8}{25} & \frac{1}{25} & 0 & 0 & \frac{1}{25} & \frac{-8}{25} & 1 & \frac{-5}{4} & 1 \\
0 & 0 & 0 & \frac{50}{81} & \frac{50}{81} & 0 & \frac{1}{25} & \frac{-8}{25} & \frac{-8}{25} & \frac{1}{25} & 0 & 0 & 1 & 1 & \frac{-5}{4}
\end{vmatrix} \neq 0,$$

we conclude that $\boldsymbol{\Theta} \cup \boldsymbol{\Xi}$ is a basis for $C(\boldsymbol{T}) \cap \prod_{i=1}^4 P_2(\boldsymbol{T}_i)$.

It holds that

$$\langle \boldsymbol{\xi}_{(\frac{3}{4},\frac{1}{4},0)}, \tilde{\boldsymbol{\phi}}_\lambda \rangle_{L_2(\boldsymbol{T})} = \operatorname{vol}(\boldsymbol{T}) \times \begin{cases} \frac{3}{100} & \lambda = (1,0,0), \\ 0 & \lambda \in (\boldsymbol{I}_0 \cup \boldsymbol{I}_1) \setminus \{(1,0,0)\}, \end{cases}$$

$$\langle \boldsymbol{\xi}_{(\frac{1}{4},\frac{1}{4},\frac{1}{2})}, \tilde{\boldsymbol{\phi}}_\lambda \rangle_{L_2(\boldsymbol{T})} = \operatorname{vol}(\boldsymbol{T}) \times \begin{cases} \frac{-1}{48} & \lambda = (0,0,1), \\ \frac{27}{240} & \lambda = (\frac{1}{2},\frac{1}{2},0), \\ 0 & \lambda \in (\boldsymbol{I}_0 \cup \boldsymbol{I}_1) \setminus \{(0,0,1),(\frac{1}{2},\frac{1}{2},0)\}, \end{cases}$$

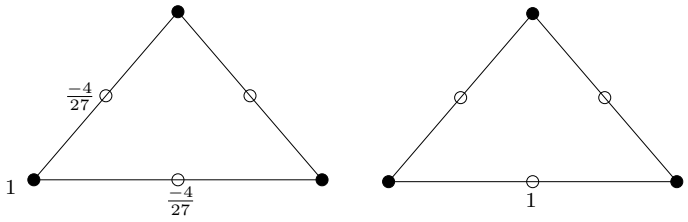


Figure 3.2: $\tilde{\boldsymbol{\phi}}_{(1,0,0)}$ (left) and $\tilde{\boldsymbol{\phi}}_{(\frac{1}{2},\frac{1}{2},0)}$ (right) in terms of $\tilde{\boldsymbol{N}}$. The other $\tilde{\boldsymbol{\phi}}_\lambda$ are obtained by permuting the barycentric coordinates.

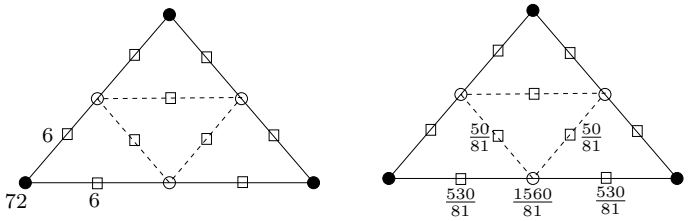


Figure 3.3: $\boldsymbol{\theta}_{(1,0,0)}$ (left) and $\boldsymbol{\theta}_{(\frac{1}{2},\frac{1}{2},0)}$ (right) in terms of $\boldsymbol{N}_f$. The other $\boldsymbol{\theta}_\lambda$ are obtained by permuting the barycentric coordinates.

where the other values of $\langle \boldsymbol{\xi}_\mu, \tilde{\boldsymbol{\phi}}_\lambda \rangle_{L_2(\boldsymbol{T})}$ for $(\mu, \lambda) \in \boldsymbol{I}_2 \times (\boldsymbol{I}_1 \cup \boldsymbol{I}_0)$ are obtained by permuting the barycentric coordinates.

3.3.5 Definition of the Ψ_j

We take $\Psi_0 = N_0$, being a Riesz basis for V_0. In the previous subsection, from corresponding local collections we have constructed uniform Riesz bases $\Theta_j \cup \Sigma_{j+1}$ and $\tilde{\Phi}_j$ for V_{j+1} and $\tilde{V}_j$, respectively, such that $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Omega)} = \mathrm{Id}$. For $j \geq 0$, the collection of wavelets Ψ_{j+1} is now given by the explicit formula $\Psi_{j+1} = \Xi_{j+1} - \langle \Xi_{j+1}, \tilde{\Phi}_j \rangle_{L_2(\Omega)} \Theta_j$.

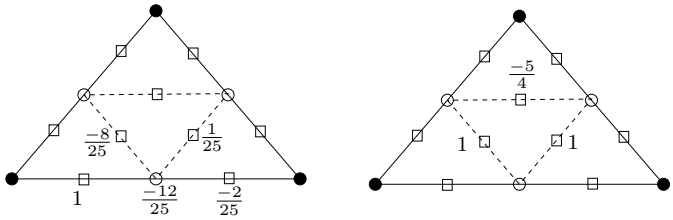


Figure 3.4: $\boldsymbol{\xi}_{(\frac{3}{4},\frac{1}{4},0)}$ (left) and $\boldsymbol{\xi}_{(\frac{1}{4},\frac{1}{4},\frac{1}{2})}$ (right) in terms of $\boldsymbol{N}_f$. The other $\boldsymbol{\xi}_\lambda$'s (five and two) are obtained by permuting the barycentric coordinates.

These wavelets will depend on the topology of $\mathcal{T}_0$ via the the local-to-global basis construction (3.3.1)–(3.3.2) that we applied for the definition of Θ_j, Σ_{j+1}, and $\tilde{\Phi}_j$, as well via the inner product $\langle \Xi_{j+1}, \tilde{\Phi}_j \rangle_{L_2(\Omega)}$.

Despite of these dependencies on $\mathcal{T}_0$, as well as that on Γ, we can distinguish between two types of wavelets: A wavelet $\psi_{j+1,x}$ of the first type stems from $\boldsymbol{\xi}_\lambda$ of type $\boldsymbol{\xi}_{(\frac{3}{4},\frac{1}{4},0)}$ (see Figure 3.4), so that $x \in \mathcal{N}(\mathcal{T}_{j+2}) \setminus \mathcal{N}(\mathcal{T}_{j+1})$ is on an edge of a $T \in \mathcal{T}_j$. It equals $\xi_{j+1,x}$ minus a multiple of *one* (or, near the Dirichlet boundary, possibly zero) $\theta_{j,y}$ with $y \in \mathcal{N}(\mathcal{T}_j)$ (left picture in Figure 3.2). A wavelet $\psi_{j+1,x}$ of the second type stems from $\boldsymbol{\xi}_\lambda$ of type $\boldsymbol{\xi}_{(\frac{1}{4},\frac{1}{4},\frac{3}{4})}$, so that $x \in \mathcal{N}(\mathcal{T}_{j+2}) \setminus \mathcal{N}(\mathcal{T}_{j+1})$ is interior to a $T \in \mathcal{T}_j$. It equals $\xi_{j+1,x}$ minus a multiple of *two* (or, near the Dirichlet boundary, possibly one or zero) $\theta_{j,y}$'s, where one $y \in \mathcal{N}(\mathcal{T}_j)$ and the other is in $\mathcal{N}(\mathcal{T}_{j+1}) \setminus \mathcal{N}(\mathcal{T}_j)$ (right picture in Figure 3.2). The supports of both types of wavelets (away from the Dirichlet boundary) are illustrated in Figure 3.5.

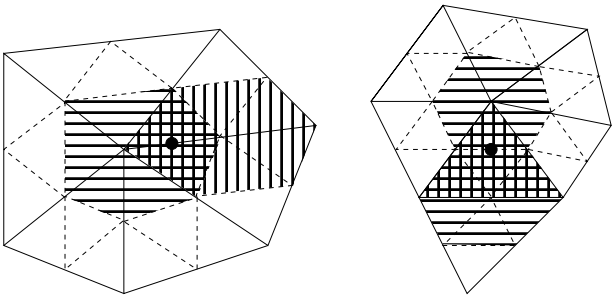


Figure 3.5: Wavelets $\psi_{j+1,x}$ for $x \in \mathcal{N}(\mathcal{T}_{j+2}) \setminus \mathcal{N}(\mathcal{T}_{j+1})$ on an edge of a $T \in \mathcal{T}_j$ (left) or interior to a $T \in \mathcal{T}_j$ (right). Indicated are x (•), the support of $\xi_{j+1,x}$ (vertical shading), and that of the $\theta_{j,y}$'s (horizontal shading). The support of the wavelet is the union of these supports.
The wavelets are linear combinations of 11 (left) or 13 (right) quadratic nodal basis functions (i.e. functions from N_{j+1}) (the numbers 11 and 13 apply when the valence of each interior vertex in $\mathcal{T}_j$ is 6).

3.3.6 Condition numbers

For $\Omega = (0,1)^2$, $\mathcal{T}_0 = \big\{\{(x,y)\colon 0 \le x \le y \le 1\}, \{(x,y)\colon 0 \le y \le x \le 1\}\big\}$, and $\Gamma = \partial\Omega$, we have computed $\kappa_{L_2(\Omega)}(\bigcup_{j=0}^J \Psi_j)$ and $\kappa_{H_0^1(\Omega)}(\bigcup_{j=0}^J \Psi_j)$, with $H_0^1(\Omega)$ equipped with $|\cdot|_{H^1(\Omega)}$, and with the wavelets normalized in the corresponding norm. That is, we have computed the condition numbers of the (normalized) mass and stiffness matrices. The results are given in Tables 3.1 and 3.2.

Remark 3.3.2. The principles behind the finite element wavelet construction applied in this work were introduced in [Ste98a, DS99c]. The realizations of finite element

Table 3.1: L_2-condition numbers of the L_2-normalized wavelets up to level J.

J	0	1	2	3	4	5	6	7	8
$\kappa_{L_2((0,1)^2)}$	1	4.8	7.3	8.3	8.9	9.2	9.7	9.8	9.9

Table 3.2: H_0^1-condition numbers of the H_0^1-normalized wavelets up to level J.

J	0	1	2	3	4	5	6	7	8
$\kappa_{H_0^1((0,1)^2)}$	1	27	41	54	63	70	76	81	85

wavelets for given primal and dual orders from [DS99c] were revisited in [NS09] in order to obtain smaller condition numbers. The two-dimensional piecewise quadratic wavelets from [NS09] have 3 vanishing moments (since $\tilde{V}_j = V_j$). Their L_2- and H_0^1-condition numbers on the square on level $J = 6$ were 12 and 60, respectively, for the H_0^1-case somewhat improving upon the condition numbers from Table 3.2. On the other hand, for a regular mesh where all vertices have valence 6, each quadratic wavelet away from the boundary in [NS09] is a linear combination of not less than 87 nodal basis functions, which high number was the motivation for the current work.

Two-dimensional quadratic finite element wavelets where each wavelet is a linear combination of only 4 or 6 nodal basis functions were introduced in [Liu06]. These wavelets are based on a splitting of V_{j+1} into V_j and an orthogonal complement w.r.t. a 'discrete' level-dependent scalar product on V_{j+1}. It was shown that the wavelets generate a Riesz basis for H^s for $s \in (0.3974, \frac{3}{2})$. These wavelets, however, have no vanishing moment, and, consequently, they cannot be expected to generate a Riesz basis for H^s for $s \leq 0$.

Finally, since for our application in solving parabolic PDEs it is needed that the wavelets form a Riesz basis in $H^{-1}(\Omega)$, based on $\kappa_{H^{-1}(\Omega)}(\mathbf{D}\Psi) = \kappa_{H_0^1(\Omega)}(\mathbf{D}^{-1}\tilde{\Psi})$ (see Remark 3.2.5), where $\tilde{\Psi}$ is the dual wavelet basis and $\mathbf{D} \eqsim \text{blockdiag}[2^{-j}]_{j\geq 0} \eqsim \text{diag}[|\tilde{\psi}|_{H^1(\Omega)}]_{\tilde{\psi}\in\tilde{\Psi}}$, additionally we have computed $\kappa_{H_0^1(\Omega)}(\bigcup_{j=0}^{J}\tilde{\Psi}_j)$ with these dual wavelets being normalized w.r.t. $|\cdot|_{H^1(\Omega)}$. The results are given in Table 3.3. Since, as explained in Remark 3.2.6, these computations involve the applications of inverses of mass matrices and that of two-level transforms at the primal side, which inverses

Table 3.3: H^1-condition numbers of the H^1-normalized *dual* wavelets up to level J.

J	0	1	2	3	4	5	6
$\kappa_{H_0^1((0,1)^2)}$	1	6.5	14	22	28	32	36

in our case are densely populated, we computed these condition numbers only up to level 6.

4. An optimal adaptive tensor product wavelet solver of a space-time FOSLS formulation of parabolic evolution problems

In this chapter we construct a well-posed first order system least squares (FOSLS) simultaneously space-time formulation of parabolic PDEs. Using an adaptive wavelet solver, this problem is solved with the best possible rate in linear complexity. Thanks to the use of a basis that consists of tensor products of wavelets in space and time, this rate is equal to that when solving the corresponding stationary problem. Our findings are illustrated by numerical results.

4.1 Introduction

After pioneering earlier work in [BJ89, BJ90], in recent years one witnesses a renewed and growing interest in simultaneously space-time solvers for evolutionary PDEs (e.g. [And14, MST14, Ste15, LMN16, ECD16, DFW16]). Instead of reducing the problem to a coupled system of ODEs, as with the method of lines, or to a sequence of stationary PDEs, as with time marching schemes like Rothe's method, one aims to solve the problem as a whole. The motivations to do so are to obtain a reduction in the computational complexity or storage requirements by exploiting the product structure of the space-time cylinder for tensor product approximation (see [GO07]); to construct meshes that are optimally adapted to singularities that are local in space and time (improving upon the rather limited possibilities of local time stepping); and, finally, to use the additional time dimension to enhance the possibilities for a massive paralellization (e.g. [GN16]).

When aiming at a convergent or rather optimally convergent adaptive solution method, a major obstruction is that a space-time variational formulation does not lead to a bilinear form that is symmetric or coercive. In [SS09, GK11, KSU16], considering parabolic PDEs, this problem was tackled as follows: By equipping the (Bochner) spaces w.r.t. which the bilinear form is well-posed with (wavelet) Riesz bases, the vari-

ational problem has an equivalent formulation as a well-posed bi-infinite matrix-vector formulation. By now forming normal equations one arrives at a well-posed symmetric positive bi-infinite matrix-vector problem whose solution can be approximated at the best possible nonlinear approximation rate using the adaptive wavelet scheme developed in [CDD01, GHS07].

For both the evaluation of the a posteriori error estimator, and for solving the arising Galerkin systems, a key ingredient of these schemes is, for the current finite approximate solution vector, the approximate evaluation of the residual of the infinite system of equations within a tolerance that is equal to a fixed multiple of the current error, and at a cost such that the overall scheme has optimal computational complexity. In the original scheme this was performed by approximating separately both the right-hand side vector and the matrix-vector multiplication each within half this tolerance. For the second task an ingenious scheme was constructed, the so-called **apply**-routine, that approximates each column of the matrix with an accuracy dependent on the modulus of the corresponding entry in the input vector. To have a proper decay of the matrix entries away from the diagonal, it requires that the wavelets have sufficiently many vanishing moments dependent on their order. Even for linear operators the apply routine is non-linear, and unfortunately it turned out to be quantitatively quite demanding.

In Chapter 2 we developed an alternative for the **apply** routine. It is based on the observation that if the wavelets are sufficiently smooth, then each entry of the residual vector equals the residual of the equation in mild form integrated against a wavelet. Now it can be shown that if the current approximation is from the span of the wavelets up to some level, then a sufficiently accurate residual vector approximation is obtained by ignoring all entries of this vector that correspond to wavelets whose levels exceed that level by a fixed constant. This observation also applies to non-uniform, adaptive settings as long as the approximate solutions are sought in spans of wavelets whose indices form trees. For this scheme, the number of vanishing moments of the wavelets does not have to grow with their order, and often one vanishing moment suffices. To compute the remaining entries of the residual vector in linear complexity, i.e., those that cannot be ignored, the current approximate solution is first written in a locally finite single-scale basis, then its residual is integrated against single-scale basis functions, after which the transposed multi-to-single scale transformation is applied. Without difficulty the scheme applies to semi-linear PDEs as well.

The key idea behind this alternative residual evaluation scheme is not to split the residual functional into two terms and to approximate them separately, but to represent both terms in one common dictionary before integrating their difference against a wavelet basis. In this way one benefits from the fact that the norm of this functional decreases proportionally to the error in the approximation, meaning that one is allowed to make a fixed relative error in the second step. The approach requires that the application of the PDO to any wavelet lands in L_2, for operators of 2nd order meaning that the (piecewise polynomial) wavelets have to be globally C^1. To avoid this unpleasant condition, in Chapter 2 we showed that any well-posed semi-linear 2nd order PDE can be written as a well-posed First Order System Least Squares (FOSLS)

problem to which the whole machinery can be applied. In this case, it suffices to have bases of globally continuous (piecewise polynomial) wavelets which are available on arbitrary polytopes.

In the current chapter we apply the approach introduced in Chapter 2 to parabolic evolutionary PDEs written in a well-posed simultaneously space-time variational formulation. We transform this problem to a well-posed FOSLS formulation, and equip the arising Bochner spaces with bases that are tensor products of temporal and spatial wavelet bases. The advantage of tensor product approximation is that it allows to solve the full time evolution at an order of computational complexity that is equal as when solving the corresponding stationary problem.

This advantage, however, does not come for free. As a counterpart for the tree approximations with non-tensor product wavelet approximation for stationary problems studied in Chapter 2, here we will allow only approximate solutions from spans of tensor-product wavelets whose index pairs form multi-trees (more particular, double trees). With this restriction, we will be able to show that a sufficiently accurate residual vector approximation is obtained as follows: Drop each entry of this residual vector that corresponds to a tensor-wavelet for which one or both its levels exceed by more than a fixed constant the corresponding level of any tensor-wavelet in the multi-tree with which it has an overlapping support.

To compute the remaining entries of the residual vector in linear complexity, we will rely on a generalisation from [KS14] of an algorithm originally developed in [BZ96] in the (non-adaptive) sparse-grid setting. Note that now we cannot rely on a residual evaluation by writing the current approximation in a locally finite single-scale representation, since the complexity of such a representation generally will not be of the order of complexity of the tensor-wavelet representation. The unavailability of such a locally finite single-scale representation has the consequence that we are not able to handle non-linear PDEs.

In a non-FOSLS setting, we studied adaptive tensor product wavelet methods for solving simultaneously space-time variational formulations of parabolic PDEs already in [SS09, CS11]. In those works, we considered unconstrained approximation, i.e. no multi-tree constraints, and relied on the **apply**-routine for the approximate residual evaluation. In [SS09] quantitative aspects were not considered, and in order to get a reasonably fast implementation in [CS11] we considered custom designed wavelets that for a PDO with constant coefficients on rectangular spatial domains yield a stiffness matrix that is truly sparse, and therefore can be applied exactly in optimal complexity.

This chapter is organized as follows: In Sect. 4.2 we give a well-posed FOSLS formulation of a parabolic PDE. One of the residuals in this system will be measured in a dual norm. To circumvent its evaluation, we replace this norm by an equivalent sequence norm of this residual integrated against a wavelet basis.

In Sect. 4.3 we recall some relevant facts about the adaptive wavelet Galerkin method (**awgm**) for solving general well-posed operator equations.

In Sect. 4.4 we apply the **awgm** to the FOSLS formulation of the parabolic PDE.

We construct tensor product bases for the arising Bochner spaces, specify the multi-tree constraint on the index sets of the bases, and investigate the best possible approximation rate that can be achieved. Most of our efforts will be devoted to the construction of an approximate residual evaluation such that the overall scheme converges with the best possible rate in linear complexity. A number of technical decay estimates will be postponed to the appendix.

In Sect. 4.5, we test our **awgm** on the heat equation on a two-dimensional L-shaped spatial domain with homogenous Dirichlet boundary conditions. In two examples where the initial condition satisfies the usual condition to vanish at the lateral boundary, we observe the best possible converge rate. In an example where the initial condition violates this lowest order compatibility condition, and thus the exact solution is discontinuous along a two-dimensional manifold, a reduced rate is observed. We envisage that the 'full' rate will be restored by replacing the isotropic spatial wavelets by (piecewise) tensor product wavelets.

A short conclusion and a brief discussion of issues that are open to further investigations are presented in Sect. 4.6.

In this work, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For normed linear spaces $\mathscr{A}$ and $\mathscr{B}$, $\mathcal{L}(\mathscr{A},\mathscr{B})$ will denote the space of bounded linear mappings $\mathscr{A} \to \mathscr{B}$ endowed with the operator norm $\|\cdot\|_{\mathcal{L}(\mathscr{A},\mathscr{B})}$. The subset of invertible operators in $\mathcal{L}(\mathscr{A},\mathscr{B})$ with inverses in $\mathcal{L}(\mathscr{B},\mathscr{A})$ will be denoted as $\mathcal{L}\mathrm{is}(\mathscr{A},\mathscr{B})$.

For countable sets $\vee_1, \vee_2$, the norms on $\ell_2(\vee_i)$ and on $\mathcal{L}(\ell_2(\vee_1),\ell_2(\vee_2))$ will often simply denoted as $\|\cdot\|$.

4.2 Well-posed FOSLS formulation of a parabolic PDE

For a bounded domain $\Omega \subset \mathbb{R}^n$, $\mathrm{I} := (0,T)$ for some $T > 0$, and $A = A^\top \in L_\infty(\mathrm{I}\times\Omega)^{n\times n}$ with $\xi^\top A(\cdot)\xi \eqsim \|\xi\|^2$ ($\xi \in \mathbb{R}^n$, a.e. on $\mathrm{I}\times\Omega$), we consider the semi-linear parabolic time evolution problem

$$\left\{\begin{array}{rl} \frac{\partial u}{\partial t} - \nabla_x \cdot A\nabla_x u + N(u) = g & \text{on } \mathrm{I}\times\Omega, \\ u = 0 & \text{on } \mathrm{I}\times\partial\Omega, \\ u(0,\cdot) = h & \text{on } \Omega, \end{array}\right. \tag{4.2.1}$$

Multiplying this equation by smooth test functions v of time and space that vanish at $[0,T]\times\partial\Omega$, integrating both sides over time and space, and, applying integration by parts in space, we arrive at the variational problem of finding

$$u \in \boxed{\mathscr{U} := L_2(\mathrm{I}; H^1_0(\Omega)) \cap H^1(\mathrm{I}; H^{-1}(\Omega)),}$$

such that for all

$$v \in \boxed{\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2 := L_2(\mathrm{I}; H^1_0(\Omega)) \times L_2(\Omega),}$$

it holds that

$$(Gu)(v) := \int_{\mathrm{I}}\int_{\Omega} \big(\frac{\partial u}{\partial t} + N(u) - g\big)v_1 + A\nabla_x u \cdot \nabla_x v_1 \, dx\, dt + \int_{\Omega} (u(0,\cdot) - h)v_2 \, dx = 0.$$

We assume that

- a solution u exists,
- G, i.e., N, is two times continuously Fréchet differentiable in a neighborhood of u,
- $DG(u) \in \mathcal{L}(\mathscr{U}, \mathscr{V}')$ is a homeomorphism with its range,

i.e., that conditions (i)-(iii) from Sect. 2.2 are satisfied.

Remark 4.2.1 (linear case). In the case that for some $\vec{b} \in L_\infty(\mathrm{I}\times\Omega)^n$ and $c \in L_\infty(\mathrm{I}\times\Omega)$,

$$N(u) = Nu := \vec{b} \cdot \nabla_x u + cu, \tag{4.2.2}$$

these conditions are known to be satisfied even with $DG(u) \in \mathcal{L}\mathrm{is}(\mathscr{U}, \mathscr{V}')$, see e.g. [DL92, Ch.XVIII], [SS09]. In this linear case, G is affine and so $DG(u)(v) = G(v) - G(0)$, and the solution u is unique.

For the verification of the conditions in semi-linear cases, we refer to e.g. [Tem97].

Using the general framework outlined in Sect. 2.2, we write our second order PDE as a *first order system least squares* (FOSLS) problem: With

$$\boxed{\vec{\mathscr{T}} := L_2(\mathrm{I}; L_2(\Omega)^n)},$$

it holds that

$$\vec{p} \mapsto \big((v_1, v_2) \mapsto \int_{\mathrm{I}}\int_{\Omega} \vec{p} \cdot \nabla_x v_1 \, dx\big) \in \mathcal{L}(\vec{\mathscr{T}}, \mathscr{V}'), \qquad u \mapsto A\nabla_x u \in \mathcal{L}(\mathscr{U}, \vec{\mathscr{T}}).$$

Consequently, for any solution u of $G(u) = 0$, it holds that $(u, \vec{p}) := (u, A\nabla_x u) \in \mathscr{U} \times \vec{\mathscr{T}}$ is a zero of the least squares functional

$$Q(u, \vec{p}) := \tfrac{1}{2}\Big(\|H_1(u, \vec{p})\|^2_{\mathscr{V}_1'} + \|H_2(u, \vec{p})\|^2_{L_2(\Omega)} + \|H_3(u, \vec{p})\|^2_{\vec{\mathscr{T}}}\Big), \tag{4.2.3}$$

where

$$\begin{aligned} H_1(u, \vec{p}) &:= v_1 \mapsto \int_{\mathrm{I}}\int_{\Omega} \big(\frac{\partial u}{\partial t} + N(u) - g\big)v_1 + \vec{p} \cdot \nabla_x v_1 \, dx\, dt, \\ H_2(u, \vec{p}) &:= u(0,\cdot) - h, \quad H_3(u, \vec{p}) := \vec{p} - A\nabla_x u, \end{aligned}$$

and so a solution of $DQ(u, \vec{p}) = 0$. This latter operator equation is well-posed in the sense that

- $DQ : \mathscr{U} \times \vec{\mathscr{F}} \supset \mathrm{dom}(DQ) \to (\mathscr{U} \times \vec{\mathscr{F}})'$ is continuously Fréchet differentiable in a neighborhood of any solution $(u, \vec{p})$,
- $0 < D^2Q(u, \vec{p}) = D^2Q(u, \vec{p})' \in \mathcal{L}\mathrm{is}(\mathscr{U} \times \vec{\mathscr{F}}, (\mathscr{U} \times \vec{\mathscr{F}})')$, so that any solution is $(u, \vec{p})$ locally unique,
- for $(u, \vec{p})$ being a zero of Q, it holds that

$$Q(w, \vec{q}) \eqsim \|u - w\|^2_{\mathscr{U}} + \|\vec{p} - \vec{q}\|^2_{\vec{\mathscr{F}}} \tag{4.2.4}$$

  in a neighborhood of $(u, \vec{p})$.

Remark 4.2.2 (linear case cont'd). If N is linear, and thus G is affine, then DQ is affine, and therefore $D^2Q(u, p)(v, \vec{q}) = DQ(v, \vec{q}) - DQ(0, \vec{0})$; in particular, D^2Q is constant. The equation $DQ(u, \vec{p}) = 0$ is uniquely solvable, and therefore its solution $(u, \vec{p})$ is the zero of Q and u is the solution of $G(u) = 0$. The relation (4.2.4) holds globally.

Remark 4.2.3. In [MS01, MS02] a least-squares functional similar to (4.2.3) has been studied with, essentially, $\|\cdot\|^2_{\mathscr{V}_1'}$ being replaced by $\|\cdot\|^2_{L_2(\mathrm{I};L_2(\Omega)^n)}$. As a consequence, a norm equivalence as in (4.2.4) could not be demonstrated, but on the other hand the evaluation of the dual norm is avoided.

To deal with the dual norm in the definition of Q, we equip $\mathscr{V}_1$ with a *Riesz basis*

$$\Psi^{\mathscr{V}_1} = \{\psi^{\mathscr{V}_1}_\lambda : \lambda \in \vee_{\mathscr{V}_1}\},$$

meaning that the *analysis operator*

$$\mathcal{F}_{\mathscr{V}_1} : g \mapsto g(\Psi^{\mathscr{V}_1}) := [g(\psi^{\mathscr{V}_1}_\lambda)]_{\lambda \in \vee_{\mathscr{V}_1}} \in \mathcal{L}\mathrm{is}(\mathscr{V}_1', \ell_2(\vee_{\mathscr{V}_1})),$$

and so its adjoint, known as the *synthesis operator*,

$$\mathcal{F}'_{\mathscr{V}_1} : \mathbf{v} \mapsto \mathbf{v}^\top \Psi^{\mathscr{V}_1} := \sum_{\lambda \in \vee_{\mathscr{V}_1}} v_\lambda \psi^{\mathscr{V}_1}_\lambda \in \mathcal{L}\mathrm{is}(\ell_2(\vee_{\mathscr{V}_1}), \mathscr{V}_1).$$

In the definition of the least squares functional Q, we now *replace* the standard dual norm on $\mathscr{V}_1'$ by the *equivalent* norm $\|\mathcal{F}_{\mathscr{V}_1} \cdot\|_{\ell_2(\vee_{\mathscr{V}_1})}$, which yields that

$$\begin{aligned} DQ(u, \vec{p})(w, \vec{q}) =& \big\langle DH_1(u, \vec{p})(w, \vec{q})(\Psi^{\mathscr{V}_1}), H_1(u, \vec{p})(\Psi^{\mathscr{V}_1})\big\rangle_{\ell_2(\vee_{\mathscr{V}_1})} + \\ & \langle DH_2(u, \vec{p})(w, \vec{q}), H_2(u, \vec{p})\rangle_{L_2(\Omega)} + \langle DH_3(u, \vec{p})(w, \vec{q}), H_3(u, \vec{p})\rangle_{\vec{\mathscr{F}}}. \end{aligned}$$

To solve this operator equation $DQ(u, \vec{p}) = 0$ we are going to apply the adaptive wavelet Galerkin method.

4.3 The adaptive wavelet Galerkin method (**awgm**)

In this section, we summarize findings about the **awgm** from Chapter 2 and [Ste14, CS15]. Let

(I) $F : \mathscr{H} \supset \mathrm{dom}(F) \to \mathscr{H}'$, with $\mathscr{H}$ being a separable Hilbert space;

(II) $F(z) = 0$;

(III) F be continuously differentiable in a neighborhood of z;

(IV) $0 < DF(z) = DF(z)' \in \mathcal{L}\mathrm{is}(\mathscr{H}, \mathscr{H}')$.

In our applications, the triple $(F, \mathscr{H}, z)$ will read as $(DQ, \mathscr{U} \times \vec{\mathscr{T}}, (u, \vec{p}))$, so that we already know that (I)-(IV) are valid.

Let $\Psi = \{\psi_\lambda : \lambda \in \vee\}$ be a *Riesz basis* for $\mathscr{H}$, with analysis operator $\mathcal{F} : g \mapsto g(\Psi) \in \mathcal{L}\mathrm{is}(\mathscr{H}', \ell_2(\vee))$, and so synthesis operator $\mathcal{F}' : \mathbf{v} \mapsto \mathbf{v}^\top \Psi \in \mathcal{L}\mathrm{is}(\ell_2(\vee), \mathscr{H})$. For any $\Lambda \subset \vee$, we set

$$\ell_2(\Lambda) := \{\mathbf{v} \in \ell_2(\vee) \colon \operatorname{supp} \mathbf{v} \subset \Lambda\}.$$

For satisfying the forthcoming Condition 4.3.4 that concerns the computational cost, it will be relevant that Ψ is a basis of *wavelet* type.

Writing $z = \mathcal{F}'\mathbf{z}$, and with

$$\mathbf{F} := \mathcal{F}F\mathcal{F}' \colon \ell_2(\vee) \to \ell_2(\vee),$$

an *equivalent* formulation of $F(z) = 0$ is given by

$$\mathbf{F}(\mathbf{z}) = 0.$$

We are going to approximate $\mathbf{z}$, and so z, by a sequence of Galerkin approximations from the spans of increasingly larger sets of wavelets, which sets are created by an adaptive process. Given $\Lambda \subset \vee$, the Galerkin approximation $\mathbf{z}_\Lambda$, or equivalently, $z_\Lambda := \mathbf{z}_\Lambda^\top \Psi$, are the solutions of $\langle \mathbf{F}(\mathbf{z}_\Lambda), \mathbf{v}_\Lambda \rangle_{\ell_2(\vee)} = 0$ $(\mathbf{v}_\Lambda \in \ell_2(\Lambda))$, i.e., $\mathbf{F}(\mathbf{z}_\Lambda)|_\Lambda = 0$, and $F(z_\Lambda)(v_\Lambda) = 0$ $(v_\Lambda \in \mathrm{span}\{\psi_\lambda : \lambda \in \Lambda\})$, respectively.

In order to be able to construct efficient algorithms, it will be needed to consider only sets Λ from a certain subset of all finite subsets of $\vee$. This collection of so-called *admissible* Λ will specified later. For the moment, it suffices to know that the union of any two admissible sets is again admissible.

To provide a benchmark to evaluate our adaptive algorithm, for $s > 0$, we define the nonlinear *approximation class*

$$\mathcal{A}^s := \Big\{\mathbf{z} \in \ell_2(\vee) \colon \|\mathbf{z}\|_{\mathcal{A}^s} := \sup_{\varepsilon > 0} \varepsilon \times \min\big\{(\#\Lambda)^s \colon \Lambda \text{ is admissible}, \inf_{\tilde{\mathbf{z}} \in \ell_2(\Lambda)} \|\mathbf{z} - \tilde{\mathbf{z}}\| \le \varepsilon\big\} < \infty\Big\}. \tag{4.3.1}$$

A vector $\mathbf{z}$ is in $\mathcal{A}^s$ if and only if there exists a sequence of admissible $(\Lambda_i)_i$, with $\lim_{i\to\infty}\#\Lambda_i=\infty$, such that $\sup_i \inf_{\mathbf{z}_i\in\ell_2(\Lambda_i)}(\#\Lambda_i)^s\|\mathbf{z}-\mathbf{z}_i\|<\infty$.

The adaptive wavelet Galerkin method (**awgm**) defined below produces a sequence of increasingly more accurate Galerkin approximations $\mathbf{z}_\Lambda$ to $\mathbf{z}$. The, generally, infinite residual $\mathbf{F}(\mathbf{z}_\Lambda)$ is used as an a posteriori error estimator.

This a posteriori error estimator guides an appropriate enlargement of the current set Λ using a bulk chasing strategy, so that the sequence of approximations converge with the best possible rate to $\mathbf{z}$. To arrive at an implementable method, that is even of optimal computational complexity, both the Galerkin solution and its residual are allowed to be computed inexactly within sufficiently small relative tolerances.

Algorithm 4.3.1 (awgm).

% Let $0<\mu_0\le\mu_1<1$, $\delta,\gamma>0$ be constants, $\Lambda_0\subset\vee$ be admissible,
% and $\mathbf{z}_{\Lambda_0}\in\ell_2(\Lambda_0)$. Let $\mathbf{Z}$ be a neighborhood of $\mathbf{z}\in\ell_2(\vee)$.

`for` $i=0,1,\ldots$ `do`

(R) $\zeta:=\frac{2\delta}{1+\delta}\|\tilde{\mathbf{r}}_{i-1}\|$. *% (Read $\|\tilde{\mathbf{r}}_{-1}\|$ as some scalar $\eqsim\|\mathbf{z}\|$.)*
`do` $\zeta:=\zeta/2$*; Compute $\tilde{\mathbf{r}}_i\in\ell_2(\vee)$ such that $\|\tilde{\mathbf{r}}_i-\mathbf{F}(\mathbf{z}_{\Lambda_i})\|\le\zeta$.*
`until` $\zeta\le\frac{\delta}{1+\delta}\|\tilde{\mathbf{r}}_i\|$.

(B) *Determine an admissible $\Lambda_{i+1}\supset\Lambda_i$ with $\|\tilde{\mathbf{r}}_i|_{\Lambda_{i+1}}\|\ge\mu_0\|\tilde{\mathbf{r}}_i\|$ such that $\#(\Lambda_{i+1}\setminus\Lambda_i)\lesssim\#(\tilde{\Lambda}\setminus\Lambda_i)$ for any admissible $\tilde{\Lambda}\supset\Lambda_i$ with $\|\tilde{\mathbf{r}}_i|_{\tilde{\Lambda}}\|\ge\mu_1\|\tilde{\mathbf{r}}_i\|$.*

(G) *Compute $\mathbf{z}_{\Lambda_{i+1}}\in\ell_2(\Lambda_{i+1})\cap\mathbf{Z}$ with $\|\mathbf{F}(\mathbf{z}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}\|\le\gamma\|\tilde{\mathbf{r}}_i\|$.*

`endfor`

In step (R), by means of a loop in which an absolute tolerance is decreased, the true *residual* $\mathbf{F}(\mathbf{z}_{\Lambda_i})$ is approximated within a relative tolerance δ. In step (B), *bulk chasing* is performed on the approximate residual. The idea is to find a smallest admissible $\Lambda_{i+1}\supset\Lambda_i$ with $\|\tilde{\mathbf{r}}_i|_{\Lambda_{i+1}}\|\ge\mu_0\|\tilde{\mathbf{r}}_i\|$. For reasons of computational efficiency, the condition of having a truly smallest Λ_{i+1} has been relaxed. Finally, in step (G), a sufficiently accurate approximation of the *Galerkin* solution w.r.t. the new set Λ_{i+1} is determined.

Convergence of the adaptive wavelet Galerkin method, with the *best possible rate*, is stated in the following theorem.

Theorem 4.3.2 ([Ste14, Thm. 3.9])**.** *Let μ_1,γ,δ, $\inf_{\mathbf{v}_{\Lambda_0}\in\ell_2(\Lambda_0)}\|\mathbf{z}-\mathbf{v}_{\Lambda_0}\|$, $\|\mathbf{F}(\mathbf{z}_{\Lambda_0})|_{\Lambda_0}\|$, and the neighborhood $\mathbf{Z}$ of the solution $\mathbf{z}$ all be sufficiently small. Then, for some $\alpha=\alpha[\mu_0]<1$, the sequence $(\mathbf{z}_{\Lambda_i})_i$ produced by* **awgm** *satisfies*

$$\|\mathbf{z}-\mathbf{z}_{\Lambda_i}\|\lesssim\alpha^i\|\mathbf{z}-\mathbf{z}_{\Lambda_0}\|.$$

If, for whatever $s>0$, $\mathbf{z}\in\mathcal{A}^s$, then $\#(\Lambda_{i+1}\setminus\Lambda_0)\lesssim\|\mathbf{z}-\mathbf{z}_{\Lambda_i}\|^{-1/s}$.

The *computation* of the approximate Galerkin solution $\mathbf{z}_{\Lambda_{i+1}}$ can be implemented by performing the simple fixed point iteration

$$\mathbf{z}^{(j+1)}_{\Lambda_{i+1}}=\mathbf{z}^{(j)}_{\Lambda_{i+1}}-\omega\mathbf{F}(\mathbf{z}^{(j)}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}. \tag{4.3.2}$$

Taking $\omega > 0$ to be a sufficiently small constant and starting with $\mathbf{z}^{(0)}_{\Lambda_{i+1}} = \mathbf{z}_{\Lambda_i}$, a fixed number of iterations suffices to meet the condition $\|\mathbf{F}(\mathbf{z}^{(j+1)}_{\Lambda_{i+1}})|_{\Lambda_{i+1}}\| \leq \gamma\|\tilde{\mathbf{r}}_i\|$. This holds also true when each of the $\mathbf{F}()|_{\Lambda_{i+1}}$ evaluations is performed within an absolute tolerance that is a sufficiently small fixed multiple of $\|\tilde{\mathbf{r}}_i\|$.

Optimal *computational* complexity of the **awgm** –meaning that the work to obtain an approximation within a given tolerance $\varepsilon > 0$ can be bounded on some constant multiple of the bound on its support length from Theorem 4.3.2,– is guaranteed under the following two conditions concerning the cost of the "bulk chasing" process, and that of the approximate residual evaluation, respectively.

Condition 4.3.3. The determination of the next index set Λ_{i+1} in Algorithm 4.3.1 is performed in $\mathcal{O}(\#\operatorname{supp}\tilde{\mathbf{r}}_i + \#\Lambda_i)$ operations.

Condition 4.3.4 (Cost condition)**.** For a sufficiently small, fixed $\varsigma > 0$, there exists a neighborhood $\mathbf{Z}$ of the solution $\mathbf{z}$ of $\mathbf{F}(\mathbf{z}) = 0$, such that for all admissible $\Lambda \subset \vee$, $\tilde{\mathbf{z}} \in \ell_2(\Lambda) \cap \mathbf{Z}$, and $\varepsilon > 0$, there exists an $\tilde{\mathbf{r}} \in \ell_2(\vee)$ with

$$\|\mathbf{F}(\tilde{\mathbf{z}}) - \tilde{\mathbf{r}}\| \leq \varsigma\|\mathbf{z} - \tilde{\mathbf{z}}\| + \varepsilon,$$

that one can compute in $\mathcal{O}(\varepsilon^{-1/s} + \#\Lambda)$ operations. Here $s > 0$ is such that $\mathbf{z} \in \mathcal{A}^s$.

Under both conditions, the **awgm** has optimal computational complexity:

Theorem 4.3.5. *In the setting of Theorem 4.3.2, and under Conditions 4.3.3 and 4.3.4, not only $\#\mathbf{z}_{\Lambda_i}$, but also the number of arithmetic operations required by* **awgm** *for the computation of $\mathbf{z}_{\Lambda_i}$ is $\mathcal{O}(\|\mathbf{z} - \mathbf{z}_{\Lambda_i}\|^{-1/s})$.*

In the setting of $F = DQ$, and Q being the FOSLS-functional associated to our parabolic time evolution problem, we will be able to verify Condition 4.3.4 only when DQ is affine, i.e., when our PDO is *linear*, i.e., $N(u) = Nu$ as given in (4.2.2). In this case where

$$0 < D\mathbf{F}(z)^\top = D\mathbf{F}(z) \equiv D\mathbf{F}\colon \tilde{\mathbf{z}} \mapsto \mathbf{F}(\tilde{\mathbf{z}}) - \mathbf{F}(\mathbf{0}) \in \mathcal{L}(\ell_2(\vee), \ell_2(\vee)),$$

first of all the conditions in Theorem 4.3.2 of $\inf_{\mathbf{v}_{\Lambda_0} \in \ell_2(\Lambda_0)} \|\mathbf{z} - \mathbf{v}_{\Lambda_0}\|$, $\|\mathbf{F}(\mathbf{z}_{\Lambda_0})|_{\Lambda_0}\|$, and the neighborhood $\mathbf{Z}$ being sufficiently small can be dropped.

Secondly, for any $\Lambda \subset \vee$ and $\varepsilon > 0$, our approximate residual $\tilde{\mathbf{r}}$ as meant in Condition 4.3.4 will be of the form $\mathbf{A}_{\Lambda,\varepsilon}\tilde{\mathbf{z}} + \mathbf{b}_\varepsilon$, where $\mathbf{A}_{\Lambda,\varepsilon} = \mathbf{A}^\top_{\Lambda,\varepsilon} \in \mathcal{L}(\ell_2(\Lambda), \ell_2(\Lambda))$. The construction of $\tilde{\mathbf{r}}$ will show that, in addition to $\|\mathbf{F}(\tilde{\mathbf{z}}) - (\mathbf{A}_{\Lambda,\varepsilon}\tilde{\mathbf{z}} + \mathbf{b}_\varepsilon)\| \leq \varsigma\|\mathbf{F}(\tilde{\mathbf{z}})\| + \varepsilon$ for all $\tilde{\mathbf{z}} \in \ell_2(\Lambda)$, and thus $\|\mathbf{F}(\mathbf{0}) - \mathbf{b}_\varepsilon\| \leq \varepsilon$, it holds that $\|D\mathbf{F} - \mathbf{A}_{\Lambda,\varepsilon}\|_{\mathcal{L}(\ell_2(\Lambda),\ell_2(\vee))} \lesssim \zeta\|D\mathbf{F}\|_{\mathcal{L}(\ell_2(\vee),\ell_2(\vee))}$. Consequently, by taking ζ sufficiently small, this $\mathbf{A}_{\Lambda,\varepsilon}$ can be used for solving the arising Galerkin problems by a Krylov iteration assuming that the initial residual is computed sufficiently accurate (see [GHS07, Thm. 2.5] for details). This method is much more efficient than the simple fixed point iteration applicable in the general nonlinear case.

4.4 Application to the FOSLS formulation

4.4.1 Expression for the residual

As announced before, we apply the **awgm** to solving $DQ(u,\vec{p})=0$ where, in order to be able to satisfy the cost condition Condition 4.3.4, we take $N(u)=Nu$ as given in (4.2.2). Besides the Riesz basis $\Psi^{\mathscr{V}_1}$ for $\mathscr{V}_1$ introduced earlier, let $\Psi^{\mathscr{U}}:=\{\psi^{\mathscr{U}}_\lambda:\lambda\in\vee_{\mathscr{U}}\}$ and $\Psi^{\vec{\mathscr{F}}}:=\{\psi^{\vec{\mathscr{F}}}_\lambda:\lambda\in\vee_{\vec{\mathscr{F}}}\}$ be Riesz bases for $\mathscr{U}$ and $\vec{\mathscr{F}}$, respectively. Then

$$(\Psi^{\mathscr{U}},\vec{0}_{\vec{\mathscr{F}}})\cup(0_{\mathscr{U}},\Psi^{\vec{\mathscr{F}}}):=\{(\psi^{\mathscr{U}}_\lambda,\vec{0}_{\vec{\mathscr{F}}})\colon\lambda\in\vee_{\mathscr{U}}\}\cup\{(0_{\mathscr{U}},\psi^{\vec{\mathscr{F}}}_\lambda)\colon\lambda\in\vee_{\vec{\mathscr{F}}}\}$$

is a Riesz basis for $\mathscr{U}\times\vec{\mathscr{F}}$, with analysis operator $\mathcal{F}\in\mathcal{L}\mathrm{is}((\mathscr{U}\times\vec{\mathscr{F}})',\ell_2(\vee_{\mathscr{U}\times\vec{\mathscr{F}}}))$, with $\vee_{\mathscr{U}\times\vec{\mathscr{F}}}:=\vee_{\mathscr{U}}\cup\vee_{\vec{\mathscr{F}}}$. Since with $\mathscr{T}:=L_2(\mathrm{I},L_2(\Omega))$, it holds that $\vec{\mathscr{F}}=\mathscr{T}^n$, we select $\Psi^{\vec{\mathscr{F}}}$ of the form $\{\psi^{\mathscr{T}}_\lambda\vec{e}_i\colon(\lambda,i)\in\vee_{\vec{\mathscr{F}}}:=\vee_{\mathscr{T}}\times\{1,\dots,n\}\}$ with $\Psi^{\mathscr{T}}:=\{\psi^{\mathscr{T}}_\lambda:\lambda\in\vee_{\mathscr{T}}\}$ being a Riesz basis for $\mathscr{T}$. We write $(u,\vec{p})=\mathcal{F}'[\mathbf{u}^\top\,\mathbf{p}^\top]^\top$.

For the application of the **awgm**, for each $[\mathbf{w}^\top\,\mathbf{q}^\top]^\top$ supported on an admissible (and thus finite) subset of $\vee_{\mathscr{U}\times\vec{\mathscr{F}}}$ we have to construct a computationally efficient approximation for the residual $D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top)$, where $D\mathbf{Q}:=\mathcal{F}DQ\mathcal{F}'$. For that goal we impose the condition that $\Psi^{\mathscr{T}}\subset L_2(\mathrm{I};H^1(\Omega))$, so that

$$\Psi^{\vec{\mathscr{F}}}\subset L_2(\mathrm{I};H(\mathrm{div};\Omega)). \tag{4.4.1}$$

Then with $(w,\vec{q}):=(\mathbf{w}^\top\Psi^{\mathscr{U}},\mathbf{q}^\top\Psi^{\vec{\mathscr{F}}})$, we obtain that

$$\begin{aligned}&D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top)=\\&\begin{bmatrix}\langle(\frac{\partial}{\partial t}+N)\Psi^{\mathscr{U}},\Psi^{\mathscr{V}_1}\rangle_{L_2(\mathrm{I}\times\Omega)}\\ \langle\Psi^{\vec{\mathscr{F}}},\nabla_x\Psi^{\mathscr{V}_1}\rangle_{L_2(\mathrm{I}\times\Omega)^n}\end{bmatrix}\big\langle\Psi^{\mathscr{V}_1},\frac{\partial w}{\partial t}+Nw-\nabla_x\cdot\vec{q}-g\big\rangle_{L_2(\mathrm{I}\times\Omega)}\\&+\begin{bmatrix}\langle\Psi^{\mathscr{U}}(0,\cdot),w(0,\cdot)-h\rangle_{L_2(\Omega)}\\0\end{bmatrix}+\begin{bmatrix}\langle-A\nabla_x\Psi^{\mathscr{U}},\vec{q}-A\nabla_x w\rangle_{L_2(\mathrm{I}\times\Omega)^n}\\\langle\Psi^{\vec{\mathscr{F}}},\vec{q}-A\nabla_x w\rangle_{L_2(\mathrm{I}\times\Omega)^n}\end{bmatrix},\end{aligned} \tag{4.4.2}$$

where we applied (4.4.1), and the zero boundary conditions satisfied by $\Sigma^{\mathscr{V}_1}$, to write $\langle\nabla_x\Psi^{\mathscr{V}_1},\vec{q}\rangle_{L_2(\mathrm{I}\times\Omega)^n}$ as $\langle\Psi^{\mathscr{V}_1},-\nabla_x\cdot\vec{q}\rangle_{L_2(\mathrm{I}\times\Omega)}$. Note that the residual consists of three terms, each of them being essentially one of the three terms of the least squares functional (4.2.3) in strong form integrated against a wavelet basis.

In view of (4.2.4), for the current application of **awgm** the cost condition can be reformulated as follows

Condition 4.3.4*. *For a sufficiently small, fixed $\varsigma>0$, for all admissible $\Lambda\subset\vee_{\mathscr{U}\times\vec{\mathscr{F}}}$, $[\mathbf{w}^\top\,\mathbf{q}^\top]^\top\in\ell_2(\Lambda)$, and $\varepsilon>0$, there exists an $\tilde{\mathbf{r}}\in\ell_2(\vee_{\mathscr{U}\times\vec{\mathscr{F}}})$ with*

$$\begin{aligned}&\|D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top)-\tilde{\mathbf{r}}\|\le\\&\quad\varsigma\Big(\|\frac{\partial w}{\partial t}+Nw-\nabla_x\cdot\vec{q}-g\|_{\mathscr{V}_1'}+\|w(0,\cdot)-h\|_{L_2(\Omega)}+\|\vec{q}-A\nabla_x w\|_{\vec{\mathscr{F}}}\Big)+\varepsilon,\end{aligned} \tag{4.4.3}$$

that one can compute in $\mathcal{O}(\varepsilon^{-1/s}+\#\Lambda)$ *operations, where* $s>0$ *is such that* $[\mathbf{u}^\top\ \mathbf{p}^\top]^\top \in \mathcal{A}^s$.

4.4.2 Tensor product bases

In view of the definitions of $\mathscr{U}$, $\mathscr{V}_1$, and $\mathscr{T}$ as being Bochner spaces, their bases will consist of tensor products of functions of collections of temporal and spatial functions, apart from a normalisation in case of $\mathscr{U}$. For $*\in\{\mathscr{U},\mathscr{V}_1,\mathscr{T}\}$, let

$$\Theta^* = \{\theta^*_\lambda\colon \lambda\in\triangleleft_*\}$$

be collections of *'temporal'* wavelets on I, such that

$$\Theta^{\mathscr{V}_1},\ \Theta^{\mathscr{T}} \text{ are Riesz bases for } L_2(\mathrm{I}).$$

We assume that $\Theta^{\mathscr{U}}\subset H^1(\mathrm{I})$, and that

$$\frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}},\ \frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{H^1(\mathrm{I})}} \text{ are Riesz bases for } L_2(\mathrm{I}), H^1(\mathrm{I}),$$

respectively. Here with $\Theta^{\mathscr{U}}/\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}$, and similarly for other normalisations or collections, we mean the collection $\{\theta^{\mathscr{U}}_\lambda/\|\theta^{\mathscr{U}}_\lambda\|_{L_2(\mathrm{I})}\colon \lambda\in\triangleleft_{\mathscr{U}}\}$.

For $*\in\{\mathscr{U},\mathscr{V}_1,\mathscr{T}\}$, let

$$\Sigma^* = \{\sigma^*_\mu\colon \mu\in\diamondsuit_*\}$$

be collections of *'spatial'* wavelets on Ω, such that,

$$\Sigma^{\mathscr{V}_1},\ \Sigma^{\mathscr{T}} \text{ are Riesz bases for } H^1_0(\Omega),\ L_2(\Omega),$$

respectively, $\Sigma^{\mathscr{U}}\subset H^1_0(\Omega)$, and

$$\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^{-1}(\Omega)}},\ \frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^1(\Omega)}} \text{ are Riesz bases for } H^{-1}(\Omega),\ H^1_0(\Omega), \tag{4.4.4}$$

respectively. An interpolation argument shows that, consequently, $\Sigma^{\mathscr{U}}/\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}$ is a Riesz basis for $L_2(\Omega)$.

Under the above assumptions, we have that

$$\boxed{\Psi^{\mathscr{V}_1}:=\Theta^{\mathscr{V}_1}\otimes\Sigma^{\mathscr{V}_1},\quad \Psi^{\mathscr{T}}:=\Theta^{\mathscr{T}}\otimes\Sigma^{\mathscr{T}},\quad \Psi^{\mathscr{U}}:=\frac{\Theta^{\mathscr{U}}\otimes\Sigma^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\otimes\Sigma^{\mathscr{U}}\|_{\mathscr{U}}}} \tag{4.4.5}$$

are Riesz bases for $\mathscr{V}_1$, $\mathscr{T}$, and $\mathscr{U}$ with index sets $\vee_*=\triangleleft_*\times\diamondsuit_*$ for $*$ being $\mathscr{V}_1$, $\mathscr{T}$, or $\mathscr{U}$, respectively. For the last statement we refer to [GO95].

4.4.3 Piecewise polynomial spatial and temporal wavelets

For $* \in \{\mathscr{U}, \mathscr{T}, \mathscr{V}_1\}$, we collect a number of (standard) assumptions on the *spatial* wavelet collections $\Sigma^* = \{\sigma^*_\lambda : \lambda \in \diamondsuit_*\}$ on Ω. To each $\lambda \in \diamondsuit_*$, we associate a value $|\lambda| \in \mathbb{N}_0$, which is called the *level* of λ. We will assume that the elements of Σ^* are *locally supported, piecewise polynomial* of some degree m, w.r.t. *dyadically nested partitions* in the following sense:

(s_1) There exists a collection $\mathcal{O}_\Omega := \{\omega : \omega \in \mathcal{O}_\Omega\}$ of closed polytopes, such that, with $|\omega| \in \mathbb{N}_0$ being the *level* of ω, $\mathrm{meas}(\omega \cap \omega') = 0$ when $|\omega| = |\omega'|$ and $\omega \neq \omega'$; for any $\ell \in \mathbb{N}_0$, $\bar{\Omega} = \cup_{|\omega|=\ell}\omega$; $\mathrm{diam}\,\omega \eqsim 2^{-|\omega|}$; and ω is the union of ω' for some ω' with $|\omega'| = |\omega| + 1$. We call ω the *parent* of its *children* ω'. Moreover, we assume that the $\omega \in \mathcal{O}_\Omega$ are uniformly *shape regular*, in the sense that they satisfy a uniform Lipschitz condition.

(s_2) $\mathrm{supp}\,\sigma^*_\lambda$ is contained in a connected union of a uniformly bounded number of ω's with $|\omega| = |\lambda|$, and restricted to each of these ω's is σ^*_λ a polynomial of degree m.

(s_3) Each ω is intersected by the supports of a uniformly bounded number of σ^*_λ's with $|\lambda| = |\omega|$.

(s_4) Σ^* has the *cancellation property of order 1* meaning that

$$|\int_\Omega \sigma^*_\mu v\,dx| \lesssim 2^{-|\mu|}\|\sigma^*_\mu\|_{L_1(\Omega)}|v|_{W^1_\infty(\mathrm{supp}\,\sigma^*_\mu)} \quad (\sigma \in \diamondsuit_*,\ v \in W^1_\infty(\Omega) \cap H^1_0(\Omega)).$$

Generally, the polynomial degree m will be different for the different bases, but otherwise fixed. The collection $\mathcal{O}_\Omega$ is shared among all bases.

In addition to (s_1)–(s_4), we assume that $\Sigma^{\mathscr{U}}$ has the *cancellation properties of order 2*:

($s_4^{\mathscr{U}}$) $|\int_\Omega \sigma^{\mathscr{U}}_\mu v\,dx| \lesssim 4^{-|\mu|}\|\sigma^{\mathscr{U}}_\mu\|_{L_1(\Omega)}|v|_{W^2_\infty(\mathrm{supp}\,\sigma^{\mathscr{U}}_\mu)}$ $(\sigma \in \diamondsuit_{\mathscr{U}},\ v \in W^2_\infty(\Omega)\cap H^1_0(\Omega))$.

Wavelets of in principle arbitrary order that satisfy all these assumptions can be found in e.g. [DS99c, NS09].

Remark 4.4.1. In both (s_4) and ($s_4^{\mathscr{U}}$), $\mathrm{supp}\,\sigma^*_\mu$ could be read as a neighborhood of this support of diameter $2^{-|\mu|}$, which requires an only minor adaptation of some proofs.

Definition 4.4.2 (tiling)**.** A collection $\mathcal{T} \subset \mathcal{O}_\Omega$ such that $\overline{\Omega} = \cup_{\omega\in\mathcal{T}}\omega$, and for $\omega_1 \neq \omega_2 \in \mathcal{T}$, $\mathrm{meas}(\omega_1 \cap \omega_2) = 0$ will be called a *tiling*. With $\mathcal{P}_m(\mathcal{T})$, we denote the space of piecewise polynomials of degree m w.r.t. $\mathcal{T}$. The smallest common refinement of tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ is denoted as $\mathcal{T}_1 \oplus \mathcal{T}_2$.

To be able to find, in linear complexity, a representation of a function, given as linear combination of wavelets, as a piecewise polynomial w.r.t. a tiling we will impose a tree constraint on the underlying set of wavelet indices:

Definition 4.4.3 (trees)**.** To each $\lambda \in \diamondsuit_*$ with $|\lambda| > 0$, we associate one $\mu \in \diamondsuit_*$ with $|\mu| = |\lambda| - 1$ and $\operatorname{meas}(\operatorname{supp}\sigma_\lambda^* \cap \operatorname{supp}\sigma_\mu^*) > 0$. We call μ the *parent* of λ, and so λ a *child* of μ.

To each $\lambda \in \diamondsuit_*$, we associate some neighbourhood $\mathcal{S}(\sigma_\lambda^*)$ of $\operatorname{supp}\sigma_\lambda^*$, with diameter $\lesssim 2^{-|\lambda|}$, such that $\mathcal{S}(\sigma_\lambda^*) \subset \mathcal{S}(\sigma_\mu^*)$ when λ is a child of μ.

We call a finite $\Lambda \subset \diamondsuit_*$ a *tree*, if it contains all $\lambda \in \diamondsuit_*$ with $|\lambda| = 0$, as well as the parent of any $\lambda \in \Lambda$ with $|\lambda| > 0$.

Remark 4.4.4. Note that we have parent-child relations on the set $\mathcal{O}_\Omega$ of polytopes as well as on the index sets $\diamondsuit_*$ (and similarly later on the index sets $\triangleleft_*$). We trust that no confusion will arise.

For some collections of wavelets, as the Haar or more generally, Alpert wavelets ([Alp93]), it suffices to take $\mathcal{S}(\sigma_\lambda^*) := \operatorname{supp}\sigma_\lambda^*$ in order to satisfy the nestedness assumption made in Definition 4.4.3. The next result shows that, thanks to (s_1)-(s_2), a suitable neighbourhood $\mathcal{S}(\sigma_\lambda^*)$ always exist.

Lemma 4.4.5. *With* $C := \sup_{\lambda\in\diamondsuit_*} 2^{|\lambda|} \operatorname{diam}\operatorname{supp}\sigma_\lambda^*$*, a valid choice of* $\mathcal{S}(\sigma_\lambda^*)$ *is given by* $\{x \in \Omega\colon \operatorname{dist}(x, \operatorname{supp}\sigma_\lambda^*) \le C2^{-|\lambda|}\}$.

A proof of the following proposition, as well as an algorithm to apply the multi-to-single-scale transformation that is mentioned, is given in [Ste14, §4.3].

Proposition 4.4.6 (tree-to-tiling)**.** *Given a tree* $\Lambda \subset \diamondsuit_*$*, there exists a tiling* $\mathcal{T}(\Lambda) \subset \mathcal{O}_\Omega$ *with* $\#\mathcal{T}(\Lambda) \lesssim \#\Lambda$ *such that* $\operatorname{span}\{\sigma_\lambda^*\colon \lambda \in \Lambda\} \subset \mathcal{P}_m(\mathcal{T}(\Lambda))$*. Moreover, equipping* $\mathcal{P}_m(\mathcal{T}(\Lambda))$ *with a basis of functions, each of which supported in* ω *for one* $\omega \in \mathcal{T}(\Lambda)$*, the representation of this embedding, known as the multi- to single-scale transform, can be applied in* $\mathcal{O}(\#\Lambda)$ *operations.*

Conversely, given a tiling, we define an element-tree and, given an integer k, a wavelet-tree:

Definition 4.4.7 (tiling-to-tree)**.** Given a tiling $\mathcal{T} \subset \mathcal{O}_\Omega$, let $t(\mathcal{T}) \subset \mathcal{O}_\Omega$ be its enlargement by adding all ancestors of all $\omega \in \mathcal{T}$. Given a $k \in \mathbb{N}_0$, we set the *k-neighborhood of $\mathcal{T}$ in $\diamondsuit_*$* by

$$\diamondsuit_*(\mathcal{T}, k) := \big\{\lambda \in \diamondsuit_*\colon \operatorname{meas}\big(\mathcal{S}(\sigma_\lambda^*) \cap \bigcup_{\{\omega\in t(\mathcal{T})\colon |\omega|=\max(|\lambda|-k,0)\}} \omega\big) > 0\big\}.$$

Proposition 4.4.8. *The set* $\diamondsuit_*(\mathcal{T}, k)$ *is a tree, and* $\#\diamondsuit_*(\mathcal{T}, k) \lesssim \#\mathcal{T}$ *(dependent on* $k \in \mathbb{N}_0$*).*

Remark 4.4.9. The idea behind the definitions related to tilings is the following: With the application of the **awgm** to FOSLS formulations of PDEs where the wavelet bases are of non-tensor product form as studied in Chapter 2, the residual consists of terms of the form $\langle\Sigma^*, g\rangle_{L_2(\Omega)}$, where, for some tiling $\mathcal{T}$, $g \in \mathcal{P}_m(\mathcal{T})$ because it is from the span of a set of wavelets with indices from a tree. Now estimates of the form

$$\lim_{k\to\infty} \sup_{0\neq g\in\mathcal{P}_m(\mathcal{T})} \frac{\|\langle\Sigma^*, g\rangle_{L_2(\Omega)}|_{\diamondsuit_*\setminus\diamondsuit_*(\mathcal{T},k)}\|}{\|g\|_{*'}} = 0$$

were shown, meaning that in order to approximate $\langle \Sigma^*, g\rangle_{L_2(\Omega)}$ within some given relative error it is sufficient to compute this vector on a k-neighborhood of $\mathcal{T}$ in $\diamondsuit_*$, where k is a suitable constant. Furthermore, with the aid of multi- to locally single-scale transformations, $\langle \Sigma^*, g\rangle_{L_2(\Omega)}|_{\diamondsuit_*(\mathcal{T},k)}$ can be exactly evaluated in $\lesssim \#\diamondsuit_*(\mathcal{T},k) \lesssim \#\mathcal{T}$ operations.

Such results, together with analogous ones for temporal wavelets, will be the basis for the residual approximation in the current setting of the application of tensor product wavelets, where will restrict to approximations from spans of sets of wavelets with indices that from multi-trees.

Moving to the *temporal* wavelet collections, for $* \in \{\mathscr{U}, \mathscr{T}, \mathscr{V}_1\}$ we assume that $\Theta^* = \{\theta^*_\lambda \colon \lambda \in \triangleleft_*\}$ satisfies conditions (t_1)–(t_3) analogous to (s_1)–(s_3) with $\mathcal{O}_\Omega$ reading as $\mathcal{O}_\mathrm{I} = \{[i2^{-\ell}T, (i+1)2^{-\ell}T] \colon \ell \in \mathbb{N}_0,\, i = 0, \dots, 2^\ell - 1\}$, and the level $|\omega|$ of $\omega = [i2^{-\ell}T, (i+1)2^{-\ell}T]$ being defined as ℓ. In addition, we assume that Θ^* has the *cancellation property of order 1*:

(t_4) $|\int_\mathrm{I} \theta^*_\mu v\, dx| \lesssim 2^{-|\mu|}\|\theta^*_\mu\|_{L_1(\mathrm{I})}|v|_{W^1_\infty(\operatorname{supp}\theta^*_\mu)}$ $(\sigma \in \triangleleft_*,\, v \in W^1_\infty(\mathrm{I}))$.

Remark 4.4.10. Compared to (s_4), note that (t_4) is imposed for all $v \in W^1_\infty(\mathrm{I})$ instead of for only $v \in W^1_\infty(\mathrm{I}) \cap H^1_0(\mathrm{I})$.

To each $\lambda \in \triangleleft_*$ with $|\lambda| > 0$, we associate one $\mu \in \triangleleft_*$ with $|\mu| = |\lambda| - 1$ and $\operatorname{meas}(\operatorname{supp}\theta^*_\lambda \cap \operatorname{supp}\theta^*_\mu) > 0$. We call μ the *parent* of λ, and so λ a *child* of μ. To each $\lambda \in \triangleleft_*$, we associate some neighbourhood $\mathcal{S}(\theta^*_\lambda)$ of $\operatorname{supp}\theta^*_\lambda$, with diameter $\lesssim 2^{-|\lambda|}$, such that $\mathcal{S}(\theta^*_\lambda) \subset \mathcal{S}(\theta^*_\mu)$ when λ is a child of μ. We call a finite $\Lambda \subset \triangleleft_*$ a *tree*, if it contains all $\lambda \in \triangleleft_*$ with $|\lambda| = 0$, as well as the parent of any $\lambda \in \Lambda$ with $|\lambda| > 0$.

Finally in this subsection, we add one more assumption on our PDE: We assume that its coefficients

$$A,\ \vec{b},\ \text{and } c \text{ (cf. (4.2.2)) are piecewise polynomial w.r.t. the coarsest possible tiling } \{\omega_\mathrm{I} \times \omega_\Omega \colon (\omega_\mathrm{I}, \omega_\Omega) \in \mathcal{O}_\mathrm{I} \times \mathcal{O}_\Omega,\ |\omega_\mathrm{I}| = |\omega_\Omega| = 0\} \text{ of } \bar{I} \times \bar{\Omega}. \tag{4.4.6}$$

4.4.4 Alpert wavelets

Recall the least squares functional Q from (4.2.3). It consists of three 'residuals' $\frac{\partial w}{\partial t} + Nw - \nabla_x \cdot \vec{q} - g$, $w(0,\cdot) - h$, and $\vec{q} - A\nabla_x w$ (not to be confused with the residual $D\mathbf{Q}([\mathbf{w}^\top\, \mathbf{q}^\top]^\top)$), whose norms are minimized. A main ingredient of our approximate evaluation of $D\mathbf{Q}([\mathbf{w}^\top\, \mathbf{q}^\top]^\top)$ will consist of representing all terms in each of the three 'residuals' in a common dictionary. If w and $\vec{q}$ would be from spans of sets of non-tensor product wavelets whose index sets form trees, then such a dictionary can consist of the piecewise polynomials of some degree w.r.t. a tiling whose cardinality is of the order of the sum of the cardinalities of both trees. This is the setting considered in Chapter 2 for the solution of stationary PDEs. In the current setting of tensor product approximation, such a 'single-scale' representation of optimal cardinality does not exist unless we put conditions on the wavelet index sets that are so restrictive that the advantages of tensor product approximation concerning favourable approximation rates are lost.

Instead, focussing to the first and third 'residual', we employ a representation in terms of tensor products of temporal and spatial Alpert wavelets. Unfortunately, this procedure does not apply to nonlinear terms being the reason for our restriction to $N(u) = Nu$ from (4.2.2).

Definition 4.4.11 (Alpert wavelets [Alp93])**.** We let $\Theta^a = \{\theta^a_\lambda \colon \lambda \in \lhd_a\}$ denote an orthonormal basis for $L_2(\mathrm{I})$ such that $\operatorname{supp}\theta^a_\lambda = \omega$ for some $\omega \subset \mathcal{O}_\mathrm{I}$ with $|\omega| = \max(|\lambda| - 1, 0)$, and $\operatorname{span}\{\theta^a_\lambda \colon |\lambda| \le \ell\} = \mathcal{P}_m(\{\omega \in \mathcal{O}_\mathrm{I} \colon |\omega| = \ell\})$.

Similarly, we let $\Sigma^a = \{\sigma^a_\mu \colon \mu \in \diamondsuit_a\}$ denote an orthonormal basis for $L_2(\Omega)$ such that $\operatorname{supp}\sigma^a_\mu = \omega$ for some $\omega \subset \mathcal{O}_\Omega$ with $|\omega| = \max(|\mu| - 1, 0)$, and $\operatorname{span}\{\sigma^a_\mu \colon |\lambda| \le \ell\} = \mathcal{P}_m(\{\omega \in \mathcal{O}_\Omega \colon |\omega| = \ell\})$.

We set $\Psi^a := \Theta^a \otimes \Sigma^a$.

4.4.5 Multi-tree approximation

We need a definition of admissible subsets of the index set of our basis for $\mathscr{U} \times \vec{\mathscr{T}}$ that on the one hand is sufficiently restrictive to allow for the evaluation of the approximate residuals in linear complexity, and on the other hand yields the favourable approximation rates known from unconstrained tensor product approximation. For that goal we consider multi-trees as a substitute for the concept of a tree in the non-tensor product case.

Definition 4.4.12 (multi-trees)**.** For $* \in \{\mathscr{U}, \mathscr{V}_1, \mathscr{T}, a\}$, $\Lambda \subset \vee_* := \lhd_* \times \diamondsuit_*$ is called a *multi-tree* when for any $(\lambda, \mu) \in \Lambda$, $\Lambda_2(\lambda) := \{\gamma \colon (\lambda, \gamma) \in \Lambda\}$ and $\Lambda_1(\mu) := \{\gamma \colon (\gamma, \mu) \in \Lambda\}$ are trees in $\diamondsuit_*$ and $\lhd_*$, respectively. We set $\Lambda_2 := \cup_{\lambda \in \lhd_*} \Lambda_2(\lambda)$, $\Lambda_1 := \cup_{\mu \in \diamondsuit_*} \Lambda_1(\mu)$.

If $\Lambda \subset \vee_*$ is multi-tree, then Λ_1 and Λ_2, being unions of trees, are trees in $\lhd_*$ and $\diamondsuit_*$, respectively.

Simple examples of multi-trees, suited for approximation of functions without local singularities, are sets $\{(\lambda, \mu) \in \vee_* \colon (|\lambda|, |\mu|) \in S\}$ for finite $\emptyset \ne S \subset \mathbb{N}_0^2$ with the property that if $(i, j) \in S$ then $\{(\max(i-1, 0), j), (i, \max(j-1, 0))\} \in S$. Examples of such multi-trees are index sets corresponding to 'full' or 'sparse-grids', see [BG04].

Concerning the efficient computation of residuals we recall the following result from [KS14] that builds on earlier work from [BZ96] dealing with sparse-grids: Let a be a bilinear form such that for $u(t, x) = u_1(t)u_2(x)$ and $v(t, x) = v_1(t)v_2(x)$, it holds that $a(u, v) = a_1(u_1, v_1)a_2(u_2, v_2)$, where the a_i are local, i.e., $a_i(u_i, v_i) = 0$ when $|\operatorname{supp} u_i \cap \operatorname{supp} v_i| = 0$, and such that for $\omega \in \mathcal{O}_\Omega$ or $\omega \in \mathcal{O}_\mathrm{I}$ and $p, q \in \mathcal{P}_m(\omega)$ the evaluation of $a_i(p, q)$ can be performed in $\mathcal{O}(1)$ operations. Then for $*, \circ \in \{\mathscr{U}, \mathscr{V}_1, \mathscr{T}, a\}$, multi-trees $\Lambda_* \subset \vee_*$, $\Lambda_\circ \subset \vee_\circ$, and $\mathbf{w} \in \ell_2(\Lambda_*)$, the matrix-vector product

$$\big(a(\Psi^\circ, \Psi^*)\mathbf{w}\big)|_{\Lambda_\circ} \tag{4.4.7}$$

can be evaluated in $\mathcal{O}(\max(\#\Lambda^*, \#\Lambda^\circ))$ operations.

Although the definition in [KS14] of a tree and therefore that of a multi-tree are slightly different from the current definitions, the results from [KS14] carry over to the current setting without much difficulty. For details we refer to Chapter 5, Sect. 5.3.

Recalling that the solution of our operator equation $DQ(u,\vec{p}) = 0$ lives in $\mathscr{U} \times \vec{\mathscr{T}}$, being the Cartesian product of $\mathscr{U}$ and the n-fold Cartesian product of $\mathscr{T}$, and which has been equipped with Riesz basis $(\Psi^{\mathscr{U}}, \vec{0}_{\vec{\mathscr{T}}}) \cup (0_{\mathscr{U}}, \Psi^{\vec{\mathscr{T}}})$ with index set $\vee_{\mathscr{U}\times\vec{\mathscr{T}}} = \vee_{\mathscr{U}} \cup \vee_{\vec{\mathscr{T}}} = \vee_{\mathscr{U}} \cup (\vee_{\mathscr{T}}, 1) \cup \cdots \cup (\vee_{\mathscr{T}}, n)$, the following definition is natural.

Definition 4.4.13 (admissible index sets)**.** A set $\Lambda \subset \vee_{\mathscr{U}\times\vec{\mathscr{T}}}$ is called *admissible* when $\Lambda_{\mathscr{U}} := \Lambda \cap \vee_{\mathscr{U}}$ and, for all $1 \le i \le n$, $\Lambda_{\vec{\mathscr{T}}_i} := \{\lambda\colon (\lambda, i) \in \Lambda \cap (\vee_{\mathscr{T}}, i)\} \subset \vee_{\mathscr{T}}$ are multi-trees. We set $\Lambda_{\vec{\mathscr{T}}} := \Lambda \cap \vee_{\vec{\mathscr{T}}}$.

Definition 4.4.14. For $*, \circ \in \{\mathscr{U}, \mathscr{V}_1, \mathscr{T}, a\}$, a multi-tree $\Lambda \subset \vee_*$, and $k \in \mathbb{N}_0$, we define its *k-neighborhood in $\vee_\circ$* by

$$\begin{aligned}\vee_\circ(\Lambda,k) := \{(\lambda',\mu') \in \vee_\circ\colon \exists (\lambda,\mu) \in \Lambda \text{ with } |\lambda| = \max(|\lambda'|-k,0),\\ |\mu| = \max(|\mu'|-k,0), \text{and } \operatorname{meas}\big(\mathcal{S}(\theta^\circ_{\lambda'}) \times \mathcal{S}(\sigma^\circ_{\mu'}) \cap \mathcal{S}(\theta^*_\lambda) \times \mathcal{S}(\sigma^*_\mu)\big) > 0\}.\end{aligned}$$

We set $\vee_{\vec{\mathscr{T}}}(\Lambda,k) := \cup_{i=1}^n (\vee_{\mathscr{T}}(\Lambda,k), i)$.

Proposition 4.4.15. *The k-neighborhood $\vee_\circ(\Lambda,k)$, defined in Definition 4.4.14, is a multi-tree, and $\#\vee_\circ(\Lambda,k) \lesssim \#\Lambda$ (dependent on k).*

Proof. Thanks to Λ being a multi-tree, the definition of $\vee_\circ(\Lambda,k)$ would not change if the conditions $|\lambda| = \max(|\lambda'|-k,0)$, $|\mu| = \max(|\mu'|-k,0)$ read as $|\lambda| \ge |\lambda'|-k$, $|\mu| \ge |\mu'|-k$. One infers that $\vee_\circ(\Lambda,k)$ is a multi-tree. The statement $\#\vee_\circ(\Lambda,k) \lesssim \#\Lambda$ (dependent on k) follows from the locality of the wavelets. □

4.4.6 Best possible rate

Let the bases $\Theta^{\mathscr{U}}$, $\Sigma^{\mathscr{U}}$, $\Theta^{\mathscr{T}}$, and $\Sigma^{\mathscr{T}}$ be of orders $d_{\mathscr{U}_t} > 1$, $d_{\mathscr{U}_x} > 1$, $d_{\mathscr{T}_t} > 0$, and $d_{\mathscr{T}_x} > 0$, respectively. Recalling the definition of the approximation class $\mathcal{A}^s$, the largest value $s_{\max}$ of s for which $[\mathbf{u}^\top\, \mathbf{p}^\top]^\top \in \mathcal{A}^s$ can be expected, and for sufficiently smooth $(u,\vec{p}) \in \mathscr{U} \times \vec{\mathscr{T}}$ actually $[\mathbf{u}^\top\, \mathbf{p}^\top]^\top \in \mathcal{A}^s$, is given by

$$s_{\max} = \min\Big(d_{\mathscr{U}_t} - 1, \frac{d_{\mathscr{U}_x}-1}{n}, d_{\mathscr{T}_t}, \frac{d_{\mathscr{T}_x}}{n}\Big),$$

see [SS09, Sect. 7]. This holds true assuming the minimum is not attained for simultaneously $d_{\mathscr{T}_t}$ and $\frac{d_{\mathscr{T}_x}}{n}$, in which case this maximal rate is attained up to a log-factor.

Note that when limited by the orders of the spatial wavelets, the value $s_{\max}$ for the approximation rate of the time-dependent problem is *equal* to the approximate rate for the corresponding stationary problem, being the major advantage of tensor product approximation.

The value of $s_{\max}$ has been derived using 'sparse-grid type' multi-trees assuming sufficient smoothness of the solution. Using multi-trees that are adapted to local singularities of the solution, it can be expected that this rate can be attained to a much larger class of functions. Results on unconstrained tensor-product approximation can be found in [Nit06, SU09]. Based on results on non-tensor product tree approximation

([CDDD01]), we anticipate that the multi-tree constraint on the index sets makes the approximation classes only 'slightly' smaller. See also [DS10] for results on multi-tree tensor-product approximation of the solution of elliptic PDEs.

4.4.7 Constructing the approximate residual

Given a fixed $\varsigma > 0$, for all admissible $\Lambda \subset \vee_{\mathscr{U}\times\vec{\mathscr{T}}}$, $[\mathbf{w}^\top\,\mathbf{q}^\top]^\top \in \ell_2(\Lambda)$, and $\varepsilon > 0$, we construct an approximation $\tilde{\mathbf{r}} \in \ell_2(\vee_{\mathscr{U}\times\vec{\mathscr{T}}})$ to $D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top)$ that satisfies the error bound (4.4.3) from the cost condition Condition 4.3.4*.

The first statement of the following lemma shows that with $(w,\vec{q}) := (\mathbf{w}^\top\Psi^{\mathscr{U}}, \mathbf{q}^\top\Psi^{\vec{\mathscr{T}}})$ it holds that $\frac{\partial w}{\partial t} + Nw - \nabla_x\cdot\vec{q} \in \operatorname{span}\Psi^a\big|_{\vee_a(\Lambda,0)}$. The second and third statements of this lemma will imply a similar statement for the third 'residual', being $\vec{q} - A\nabla_x w$.

Lemma 4.4.16. *Let $\Lambda \subset \vee_{\mathscr{U}\times\vec{\mathscr{T}}}$ be admissible. Then*

$$\vee_a(\Lambda,0) := \vee_a(\Lambda_{\mathscr{U}},0) \cup \cup_{i=1}^n \vee_a(\Lambda_{\vec{\mathscr{T}}_i},0)$$

is a multi-tree, $\#\vee_a(\Lambda,0) \lesssim \#\Lambda$ and

$$\begin{aligned}
\operatorname{span}(\frac{\partial}{\partial t} + N)\Psi^{\mathscr{U}}\big|_{\Lambda_{\mathscr{U}}} + \operatorname{span}\nabla_x\cdot\Psi^{\vec{\mathscr{T}}}\big|_{\Lambda_{\vec{\mathscr{T}}}} &\subset \operatorname{span}\Psi^a\big|_{\vee_a(\Lambda,0)},\\
\operatorname{span}A\nabla_x\Psi^{\mathscr{U}}\big|_{\Lambda_{\mathscr{U}}} + \operatorname{span}\Psi^{\vec{\mathscr{T}}}\big|_{\Lambda_{\vec{\mathscr{T}}}} &\subset \prod_{i=1}^n \operatorname{span}\Psi^a\big|_{\vee_a(\Lambda,0)}\mathbf{e}_i,\\
\operatorname{span}A^\top A\nabla_x\Psi^{\mathscr{U}}\big|_{\Lambda_{\mathscr{U}}} + \operatorname{span}A^\top\Psi^{\vec{\mathscr{T}}}\big|_{\Lambda_{\vec{\mathscr{T}}}} &\subset \prod_{i=1}^n \operatorname{span}\Psi^a\big|_{\vee_a(\Lambda,0)}\mathbf{e}_i.
\end{aligned}$$

Proof. The first statement follows by the L_2-orthogonality of the Alpert wavelets and the fact that, using assumption (4.4.6), the element of the collections $(\frac{\partial}{\partial t}+N)\Psi^{\mathscr{U}}\big|_{\Lambda_{\mathscr{U}}}$ and $\operatorname{span}\nabla_x\cdot\Psi^{\vec{\mathscr{T}}}\big|_{\Lambda_{\vec{\mathscr{T}}}}$ are piecewise polynomials. The proofs of the second and third statements are similar. □

The term $\langle\Psi^{\mathscr{U}}(0,\cdot), w(0,\cdot)-h\rangle_{L_2(\Omega)}$ in $D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top)$, resulting from the second 'residual' $w(0,\cdot)-h$, reads as $\mathbf{E}\langle\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}}, (\mathbf{E}^\top\mathbf{w})^\top\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}} - h\rangle_{L_2(\Omega)}$, with the $\vee_{\mathscr{U}}\times\diamond_{\mathscr{U}}$-matrix $\mathbf{E}$ defined by

$$\mathbf{E}_{(\lambda,\mu),\mu'} = \begin{cases} \frac{\theta^{\mathscr{U}}_\lambda(0)\|\sigma^{\mathscr{U}}_\mu\|_{L_2(\Omega)}}{\|\theta^{\mathscr{U}}_\lambda\otimes\sigma^{\mathscr{U}}_\mu\|_{\mathscr{U}}} & \text{if } (\lambda,\mu)\in\vee_{\mathscr{U}},\ \mu'=\mu,\\ 0 & \text{if } \mu\neq\mu'\in\diamond_{\mathscr{U}}.\end{cases}$$

Its transpose $\mathbf{E}^\top$ represents the trace mapping at $t=0$ w.r.t. to the bases $\frac{\Theta_{\mathscr{U}}\times\Sigma_{\mathscr{U}}}{\|\Theta_{\mathscr{U}}\times\Sigma_{\mathscr{U}}\|_{\mathscr{U}}}$ and $\frac{\Sigma_{\mathscr{U}}}{\|\Sigma_{\mathscr{U}}\|_{L_2(\mathscr{U})}}$ for $\mathscr{U}$ and $L_2(\Omega)$, respectively.

Since $\mathbf{w}$ is finitely supported, $\mathbf{E}^\top\mathbf{w}$ can be computed exactly in optimal complexity. The function $w(0,\cdot) = (\mathbf{E}^\top\mathbf{w})^\top\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}}$ is piecewise polynomial w.r.t. some tiling $\mathcal{T}$.

Given an $\varepsilon > 0$, h will be approximated within tolerance ε by a piecewise polynomial h_ε w.r.t. some tiling $\mathcal{T}(\varepsilon)$, so that $w(0,\cdot) - h_\varepsilon$ is piecewise polynomial w.r.t. the tiling $\mathcal{T} \oplus \mathcal{T}(\varepsilon)$. Now $\langle \frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}}, w(0,\cdot) - h\rangle_{L_2(\Omega)} \in \ell_2(\diamond_{\mathscr{U}})$ will be approximated by restricting it to a k-neighborhood of $\mathcal{T} \oplus \mathcal{T}(\varepsilon)$ (cf. Def. 4.4.7). The remaining issue how to approximate the application of $\mathbf{E}$ is dealt with in the following lemma.

Lemma 4.4.17. *For $k \in \mathbb{N}_0$ define $\mathbf{E}_k$ by*

$$(\mathbf{E}_k)_{(\lambda,\mu),\mu'} = \begin{cases} (\mathbf{E}_{(\lambda,\mu),\mu'} & \text{if } \big||\lambda| - 2|\mu|\big| \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{E}, \mathbf{E}_k \in \mathcal{L}(\ell_2(\diamond_{\mathscr{U}}), \ell_2(\vee_{\mathscr{U}}))$, $\|\mathbf{E} - \mathbf{E}_k\| \lesssim 2^{-k/2}$, and $\#\operatorname{supp}\mathbf{E}_k\mathbf{v} \lesssim \#\operatorname{supp}\mathbf{v}$ (dependent on k).

Proof. By the inequalities

$$\|\theta_\lambda^{\mathscr{U}}\|_{H^1(\mathrm{I})} \lesssim 2^{|\lambda|}\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})} \lesssim \|\theta_\lambda^{\mathscr{U}}\|_{H^1(\mathrm{I})}, \tag{4.4.8}$$

$$\|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)} \lesssim 2^{-|\mu|}\|\sigma_\mu^{\mathscr{U}}\|_{H^1(\Omega)}, \quad \|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)} \lesssim 2^{|\mu|}\|\sigma_\mu^{\mathscr{U}}\|_{H^{-1}(\Omega)}, \tag{4.4.9}$$

which will be demonstrated below, $|\theta_\lambda^{\mathscr{U}}(0)|^2 \lesssim \|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})}\|\theta_\lambda^{\mathscr{U}}\|_{H^1(\mathrm{I})}$ by the trace inequality, and, finally, the definition of $\mathscr{U}$, we obtain

$$\begin{aligned}\Big|\frac{\theta_\lambda^{\mathscr{U}}(0)\|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)}}{\|\theta_\lambda^{\mathscr{U}} \otimes \sigma_\mu^{\mathscr{U}}\|_{\mathscr{U}}}\Big|^2 &\lesssim \Big(\frac{\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})}}{\|\theta_\lambda^{\mathscr{U}}\|_{H^1(\mathrm{I})}}\frac{\|\sigma_\mu^{\mathscr{U}}\|^2_{H^1(\Omega)}}{\|\sigma_\mu^{\mathscr{U}}\|^2_{L_2(\Omega)}} + \frac{\|\theta_\lambda^{\mathscr{U}}\|_{H^1(\mathrm{I})}}{\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})}}\frac{\|\sigma_\mu^{\mathscr{U}}\|^2_{H^{-1}(\Omega)}}{\|\sigma_\mu^{\mathscr{U}}\|^2_{L_2(\Omega)}}\Big)^{-1} \\ &\lesssim \big(2^{-|\lambda|}4^{|\mu|} + 2^{|\lambda|}4^{-|\mu|}\big)^{-1} \le 2^{-|\ell|},\end{aligned}$$

when $\ell := |\lambda| - 2|\mu| \in \mathbb{Z}$. From this result, one easily infers the statements of the lemma.

From (s_4), we have

$$\begin{aligned}\|\sigma_\mu^{\mathscr{U}}\|^2_{L_2(\Omega)} &\lesssim 2^{-|\mu|}\|\sigma_\mu^{\mathscr{U}}\|_{L_1(\Omega)}|\sigma_\mu^{\mathscr{U}}|_{W^1_\infty(\Omega)} \\ &\lesssim 2^{-|\mu|}2^{-|\mu|n/2}\|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)}2^{|\mu|n/2}\|\sigma_\mu^{\mathscr{U}}\|_{H^1(\Omega)},\end{aligned}$$

which shows first inequality in (4.4.9). The second one is a consequence of $\|\sigma_\mu^{\mathscr{U}}\|^2_{L_2(\Omega)} \le \|\sigma_\mu^{\mathscr{U}}\|_{H^1(\Omega)}\|\sigma_\mu^{\mathscr{U}}\|_{H^{-1}(\Omega)} \lesssim 2^{|\mu|}\|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)}\|\sigma_\mu^{\mathscr{U}}\|_{H^{-1}(\Omega)}$.

The first inequality in (4.4.8) is the inverse inequality for polynomials. Thanks to (t_4), the second one in a consequence of $\|\theta_\lambda^{\mathscr{U}}\|^2_{L_2(\mathrm{I})} \lesssim 2^{-|\lambda|}\|\theta_\lambda^{\mathscr{U}}\|_{L_1(\mathrm{I})}|\theta_\lambda^{\mathscr{U}}|_{W^1_\infty(\mathrm{I})} \eqsim 2^{-|\lambda|}\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})}|\theta_\lambda^{\mathscr{U}}|_{H^1(\mathrm{I})}$. Note that here (t_4) has been used also for $v \notin H^1_0(\mathrm{I})$, cf. Remark 4.4.10. □

We are ready to specify our approximate evaluation of $D\mathbf{Q}([\mathbf{w}^\top\ \mathbf{q}^\top]^\top)$:

Algorithm 4.4.18 (approximate residual evaluation)**.**

Input: data $g \in L_2(\mathrm{I}; H^{-1}(\Omega))$, $h \in L_2(\Omega)$, an admissible $\Lambda \subset \vee_{\mathscr{U}\times\mathscr{F}}$, $[\mathbf{w}^\top\ \mathbf{q}^\top]^\top \in \ell_2(\Lambda)$, $\varepsilon > 0$, and $k_2, k_3, k_4, k_5, k_6 \in \mathbb{N}_0$.

Output: $\tilde{\mathbf{r}} := \tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3 \approx D\mathbf{Q}([\mathbf{w}^\top\ \mathbf{q}^\top]^\top)$ defined as follows:

(*S*1) *Find a multi-tree* $\vee_a(\varepsilon) \subset \vee_a$ *such that*

$$\inf_{g_\varepsilon \in \operatorname{span} \Psi^a\big|_{\vee_a(\varepsilon)}} \|g - g_\varepsilon\|_{L_2(\mathrm{I};H^{-1}(\Omega))} \leq \varepsilon.$$

Find a tiling $\mathcal{T}(\varepsilon) \in \mathcal{O}_\Omega$ *such that*

$$\inf_{h_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\varepsilon))} \|h - h_\varepsilon\|_{L_2(\Omega)} \leq \varepsilon.$$

(*S*2) *With* $\vee_a(\Lambda,\varepsilon) := \vee_a(\Lambda,0) \cup \vee_a(\varepsilon)$ *and* $(w,\vec{q}) := [\mathbf{w}^\top\, \mathbf{q}^\top]\Psi$, *approximate*

$$\mathbf{r}_{\frac{1}{2}} := \langle \Psi^{\mathscr{V}_1}, \frac{\partial w}{\partial t} + Nw - \nabla_x \cdot \vec{q} - g\rangle_{L_2(\mathrm{I}\times\Omega)}$$

by $\tilde{\mathbf{r}}_{\frac{1}{2}} := \langle \Psi^{\mathscr{V}_1}, \frac{\partial w}{\partial t} + Nw - \nabla_x \cdot \vec{q} - g_\varepsilon\rangle_{L_2(\mathrm{I}\times\Omega)}|_{\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k_2)}$.

(*S*3) *With* $\tilde{r}_{\frac{1}{2}} := \tilde{\mathbf{r}}_{\frac{1}{2}}^\top \Psi^{\mathscr{V}_1}$, *approximate*

$$\mathbf{r}_1 := \begin{bmatrix} \langle (\frac{\partial}{\partial t} + N)\Psi^{\mathscr{U}}, \tilde{r}_{\frac{1}{2}}\rangle_{L_2(\mathrm{I}\times\Omega)} \\ \langle \Psi^{\vec{\mathscr{T}}}, \nabla_x \tilde{r}_{\frac{1}{2}}\rangle_{L_2(\mathrm{I}\times\Omega)^n} \end{bmatrix}$$

by $\tilde{\mathbf{r}}_1 := \mathbf{r}_1|_{\vee_{\mathscr{U}}(\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k_2),k_3)\cup\vee_{\vec{\mathscr{T}}}(\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k_2),k_3)}$.

(*S*4) *With* $\Lambda_\downarrow := \cup_{\{\lambda\in(\Lambda\cap\vee_{\mathscr{U}})_1\,:\,\theta_\lambda^{\mathscr{U}}(0)\neq 0\}}(\Lambda\cap\vee_{\mathscr{U}})_2(\lambda) \subset \diamond_{\mathscr{U}}$, *set* $\mathcal{T}(\Lambda_\downarrow,\varepsilon) := \mathcal{T}(\varepsilon) \oplus \mathcal{T}(\Lambda_\downarrow)$, *and approximate*

$$\mathbf{r}_{\frac{3}{2}} := \Big\langle \frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}}, w(0,\cdot) - h\Big\rangle_{L_2(\Omega)}$$

by $\tilde{\mathbf{r}}_{\frac{3}{2}} := \Big\langle \frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}}, w(0,\cdot) - h_\varepsilon\Big\rangle_{L_2(\Omega)}|_{\diamond_{\mathscr{U}}(\mathcal{T}(\Lambda_\downarrow,\varepsilon),k_4)}$.

(*S*5) *Approximate* $\mathbf{r}_2 := \begin{bmatrix} \mathbf{E}\tilde{\mathbf{r}}_{\frac{3}{2}} \\ 0 \end{bmatrix}$ *by* $\tilde{\mathbf{r}}_2 := \begin{bmatrix} \mathbf{E}_{k_5}\tilde{\mathbf{r}}_{\frac{3}{2}} \\ 0 \end{bmatrix}$.

(*S*6) *Approximate*

$$\mathbf{r}_3 := \begin{bmatrix} \langle \nabla_x \Psi^{\mathscr{U}}, A^\top(A\nabla_x w - \vec{q})\rangle_{L_2(\mathrm{I}\times\Omega)^n} \\ \langle \Psi^{\vec{\mathscr{T}}}, \vec{q} - A\nabla_x w\rangle_{L_2(\mathrm{I}\times\Omega)^n} \end{bmatrix}$$

by $\tilde{\mathbf{r}}_3 = \tilde{\mathbf{r}}_3(k_6) := \mathbf{r}_3|_{\vee_{\mathscr{U}}(\vee_a(\Lambda,0),k_6)\cup\vee_{\vec{\mathscr{T}}}(\vee_a(\Lambda,0),k_6)}$.

Theorem 4.4.19. *For* $g \in L_2(\mathrm{I};H^{-1}(\Omega))$, $h \in L_2(\Omega)$, *let* $s > 0$ *be such that the solution of* $D\mathbf{Q}([\mathbf{u}^\top\, \mathbf{p}^\top]^\top) = 0$ *satisfies* $[\mathbf{u}^\top\, \mathbf{p}^\top]^\top \in \mathcal{A}^s$. *Let* $\vee_a(\varepsilon)$ *and* $\mathcal{T}(\varepsilon)$ *from* (*S*1) *satisfy* $\max(\#\vee_a(\varepsilon), \#\mathcal{T}(\varepsilon)) \lesssim \varepsilon^{-1/s}$. *Then given an admissible* $\Lambda \subset \vee_{\mathscr{U}\times\vec{\mathscr{T}}}$, $[\mathbf{w}^\top\, \mathbf{q}^\top]^\top \in \ell_2(\Lambda)$, *and* $\varepsilon > 0$, $\tilde{\mathbf{r}}$ *produced by Algorithm 4.4.18 satisfies*

$$\begin{aligned}\|D\mathbf{Q}([\mathbf{w}^\top\, \mathbf{q}^\top]^\top) - \tilde{\mathbf{r}}\| \lesssim 2^{-k/2}\big[\|\frac{\partial w}{\partial t} + Nw - \nabla_x\cdot\vec{q} - g\|_{L_2(\mathrm{I};H^{-1}(\Omega))} + \\ \|w(0,\cdot) - h\|_{L_2(\Omega)} + \|\vec{q} - A\nabla_x w\|_{L_2(\mathrm{I};L_2(\Omega)^n)}\big] + \varepsilon,\end{aligned} \tag{4.4.10}$$

where $k := \min(k_2, k_3, k_4, k_5, k_6)$, and its computation requires $\mathcal{O}(\#\Lambda + \varepsilon^{-1/s})$ operations. So by taking k a sufficiently large constant, Assumption 4.3.4 is satisfied.*

Remark 4.4.20. Collections $\vee_a(\varepsilon)$ and $\mathcal{T}(\varepsilon)$ as in the statement of the theorem exist, and so the condition about them concerns their actual construction.

Indeed, when $[\mathbf{u}^\top\,\mathbf{p}^\top]^\top \in \mathcal{A}^s$, then given an $\varepsilon > 0$ there exists an admissible $\Lambda_\varepsilon \subset \vee_{\mathscr{U}\times\mathscr{F}}$ with $\#\Lambda_\varepsilon \lesssim \varepsilon^{-1/s}$ and a $[\mathbf{u}_\varepsilon^\top\,\mathbf{p}_\varepsilon^\top]^\top \in \ell_2(\Lambda_\varepsilon)$ with, for a constant $C > 0$ determined below, $\|[\mathbf{u}^\top\,\mathbf{p}^\top]^\top - [\mathbf{u}_\varepsilon^\top\,\mathbf{p}_\varepsilon^\top]^\top\| \leq \frac{\varepsilon}{C}$, and thus with $(u,\vec{p}) := (\mathbf{u}^\top\Psi^{\mathscr{U}}, \mathbf{p}^\top\Psi^{\vec{\mathscr{F}}})$ and $(u_\varepsilon,\vec{p}_\varepsilon) := (\mathbf{u}_\varepsilon^\top\Psi^{\mathscr{U}}, \mathbf{p}_\varepsilon^\top\Psi^{\vec{\mathscr{F}}})$, $\|u-u_\varepsilon\|_{\mathscr{U}} + \|\vec{p}-\vec{p}_\varepsilon\|_{\vec{\mathscr{F}}} \lesssim \varepsilon$. Using that $g = \frac{\partial}{\partial t}u + Nu - \nabla_x\cdot\vec{p}$, we infer that, by selecting a proper C, with $g_\varepsilon := \frac{\partial}{\partial t}u_\varepsilon + Nu_\varepsilon - \nabla_x\cdot\vec{p}_\varepsilon$, it holds that $\|g-g_\varepsilon\|_{L_2(\mathrm{I};H^{-1}(\Omega))} \leq \varepsilon$. Since as shown in Lemma 4.4.16, $\operatorname{span}(\frac{\partial}{\partial t}+N)\Psi^{\mathscr{U}}\big|_{(\Lambda_\varepsilon)_{\mathscr{U}}} + \operatorname{span}\nabla_x\cdot\Psi^{\vec{\mathscr{F}}}\big|_{(\Lambda_\varepsilon)_{\vec{\mathscr{F}}}} \subset \operatorname{span}\Psi^a\big|_{\vee_a(\Lambda_\varepsilon,0)}$ and $\#\vee_a(\Lambda_\varepsilon,0) \lesssim \#\Lambda_\varepsilon \lesssim \varepsilon^{-1/s}$, we conclude that a suitable $\vee_a(\varepsilon)$ exists.

Similarly, since $h = u(0,\cdot)$, with $(\Lambda_\varepsilon)_\downarrow$ defined similarly as in $(S4)$, by taking $h_\varepsilon := (\mathbf{u}_\varepsilon|_{(\Lambda_\varepsilon)_\downarrow})^\top\Sigma^{\mathscr{U}}$ and possibly adjusting C, it holds that $\|h-h_\varepsilon\|_{L_2(\Omega)} \leq \varepsilon$. The collection $(\Lambda_\varepsilon)_\downarrow$ is a tree in $\diamondsuit_{\mathscr{U}}$, and so $h_\varepsilon \in \mathcal{P}_m(\mathcal{T}(\varepsilon))$ with, thanks to Proposition 4.4.6, $\#\mathcal{T}(\varepsilon) \lesssim \#(\Lambda_\varepsilon)_\downarrow \lesssim \#\Lambda_\varepsilon \lesssim \varepsilon^{-1/s}$.

Proof of Theorem 4.4.19. The expression for $D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top)$ given in (4.4.2), and the definitions of $\mathbf{r}_i$ and $\tilde{\mathbf{r}}_i$, for $i \in \{\frac{1}{2},1,\frac{3}{2},2,3\}$, show that

$$D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top) - \tilde{\mathbf{r}} = \mathbf{r}_1 - \tilde{\mathbf{r}}_1 + \begin{bmatrix} \langle\frac{\partial}{\partial t}\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\rangle_{L_2(\Omega)} \\ \langle\Psi^{\vec{\mathscr{F}}}, \nabla_x\Psi^{\mathscr{V}_1}\rangle_{L_2(\Omega)^n}\end{bmatrix}(\mathbf{r}_{\frac{1}{2}} - \tilde{\mathbf{r}}_{\frac{1}{2}}) + \mathbf{r}_2 - \tilde{\mathbf{r}}_2 + \mathbf{E}(\mathbf{r}_{\frac{3}{2}} - \tilde{\mathbf{r}}_{\frac{3}{2}}) + \mathbf{r}_3 - \tilde{\mathbf{r}}_3.$$

The boundedness of $\mathbf{E}$ (cf. Lemma 4.4.17) and, by the Riesz bases properties of $\Psi^{\mathscr{U}}$, $\Psi^{\mathscr{V}_1}$ and $\Psi^{\vec{\mathscr{F}}}$, that of $\langle\Psi^{\vec{\mathscr{F}}}, \nabla_x\Psi^{\mathscr{V}_1}\rangle_{L_2(\mathrm{I}\times\Omega)^n}$ and $\langle\frac{\partial}{\partial t}\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\rangle_{L_2(\mathrm{I}\times\Omega)}$, show that

$$\|D\mathbf{Q}([\mathbf{w}^\top\,\mathbf{q}^\top]^\top) - (\tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \tilde{\mathbf{r}}_3)\| \lesssim \sum_{i\in\{\frac{1}{2},1,\frac{3}{2},2,3\}} \|\mathbf{r}_i - \tilde{\mathbf{r}}_i\|.$$

Below, we will show that all five terms at the right-hand side are bounded by a multiple of the right-hand side of (4.4.10).

For $g_\varepsilon \in \operatorname{span}\Psi^a\big|_{\vee_a(\varepsilon)}$, we write

$$\mathbf{r}_{\frac{1}{2}} - \tilde{\mathbf{r}}_{\frac{1}{2}} = \langle\Psi^{\mathscr{V}_1}, \frac{\partial w}{\partial t} + Nw - \nabla_x\cdot\vec{q} - g_\varepsilon\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{\vee_{\mathscr{V}_1}\setminus\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k)} + \langle\Psi^{\mathscr{V}_1}, g_\varepsilon - g\rangle_{L_2(\mathrm{I}\times\Omega)}.$$

From the first statement of Lemma 4.4.16, we know that $\frac{\partial}{\partial t}w + Nw - \nabla_x\cdot\vec{q} - g_\varepsilon \in \operatorname{span}\Psi^a|_{\vee_a(\Lambda,\varepsilon)}$. An application of the forthcoming Theorem 4.7.3 shows that consequently the norm of the first term is $\lesssim 2^{-k/2}\|\frac{\partial}{\partial t}w + Nw - \nabla_x\cdot\vec{q} - g_\varepsilon\|_{L_2(\mathrm{I};H^{-1}(\Omega))}$. From $\Psi^{\mathscr{V}_1}$ being a Riesz basis for $L_2(\mathrm{I};H_0^1(\Omega))$, it follows that the norm of the second term is $\lesssim \|g-g_\varepsilon\|_{L_2(\mathrm{I};H^{-1}(\Omega))}$. From $(S1)$, we infer that

$$\|\mathbf{r}_{\frac{1}{2}} - \tilde{\mathbf{r}}_{\frac{1}{2}}\| \lesssim 2^{-k/2}\|\frac{\partial w}{\partial t} + Nw - \nabla_x\cdot\vec{q} - g\|_{L_2(\mathrm{I};H^{-1}(\Omega))} + \varepsilon.$$

Applications of the forthcoming Corollaries 4.7.7 and 4.7.9 show that $\|\mathbf{r}_1-\tilde{\mathbf{r}}_1\|\lesssim 2^{-k/2}\|\mathbf{r}_{\frac12}\|$, whereas $\|\mathbf{r}_{\frac12}\|\leq\|\mathbf{r}_{\frac12}-\tilde{\mathbf{r}}_{\frac12}\|+\|\tilde{\mathbf{r}}_{\frac12}\|$ and $\|\tilde{\mathbf{r}}_{\frac12}\|\lesssim\|\frac{\partial w}{\partial t}+Nw-\nabla_x\cdot\vec{q}-g\|_{L_2(\mathrm{I};H^{-1}(\Omega))}$.

For $h_\varepsilon\in\mathcal{P}_m(\mathcal{T}(\varepsilon))$, we write

$$\mathbf{r}_{\frac32}-\tilde{\mathbf{r}}_{\frac32}=\Big\langle\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}},w(0,\cdot)-h_\varepsilon\Big\rangle_{L_2(\Omega)}\Big|_{\diamond_{\mathscr{U}}\setminus\diamond_{\mathscr{U}}(\mathcal{T}(\Lambda_\downarrow,\varepsilon),k)}+\Big\langle\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}},h_\varepsilon-h\Big\rangle_{L_2(\Omega)}.$$

Note that $w(0,\cdot)\in\operatorname{span}\Sigma^{\mathscr{U}}\big|_{\Lambda_\downarrow}$, and that $\Lambda_\downarrow$ is a tree in $\diamond_{\mathscr{U}}$. Using that $\Sigma^{\mathscr{U}}$ satisfies (s_1)-(s_4), analogously to Proposition 2.9.4 one shows that that the norm of the first term is $\lesssim 2^{-k/2}\|w(0,\cdot)-h_\varepsilon\|_{L_2(\Omega)}$. From $\Sigma^{\mathscr{U}}/\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}$ being a Riesz basis it follows that the norm of the second term is $\lesssim\|h-h_\varepsilon\|_{L_2(\Omega)}$. From $(S1)$, we infer that $\|\mathbf{r}_{\frac32}-\tilde{\mathbf{r}}_{\frac32}\|\lesssim 2^{-k/2}\|w(0,\cdot)-h\|_{L_2(\Omega)}+\varepsilon$.

An application of Lemma 4.4.17 shows that $\|\mathbf{r}_2-\tilde{\mathbf{r}}_2\|\lesssim 2^{-k/2}\|\tilde{\mathbf{r}}_{\frac32}\|$, whereas $\|\tilde{\mathbf{r}}_{\frac32}\|\lesssim\|w(0,\cdot)-h\|_{L_2(\Omega)}$.

From the second and third statements of Lemma 4.4.16, we know that $\vec{q}-A\nabla_x w$, $A^\top(A\nabla_x w-\vec{q})\in\prod_{i=1}^n\operatorname{span}\Psi^a|_{\vee_a(\Lambda,0)}\mathbf{e}_i$. Now an application of the forthcoming Corollary 4.7.11 shows that $\|\mathbf{r}_3-\tilde{\mathbf{r}}_3\|\lesssim 2^{-k/2}\|\vec{q}-A\nabla_x w\|_{L_2(\mathrm{I}\times\Omega)^n}$, which completes the proof of (4.4.10).

The computation of $\tilde{\mathbf{r}}_{\frac12}$, $\tilde{\mathbf{r}}_1$, $\tilde{\mathbf{r}}_{\frac32}$, $\tilde{\mathbf{r}}_2$ or $\tilde{\mathbf{r}}_3$ requires a number of operations that is of the order

$$\begin{aligned}
&\#\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k),\\
\#\vee_{\mathscr{U}}(\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k),k)+\#\vee_{\vec{\mathscr{F}}}(\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k),k),\\
&\#\diamond_{\mathscr{U}}(\mathcal{T}(\Lambda_\downarrow,\varepsilon),k)+\#\Lambda,\\
&k\#\diamond_{\mathscr{U}}(\mathcal{T}(\Lambda_\downarrow,\varepsilon),k),\\
\#\vee_{\mathscr{U}}(\vee_a(\Lambda,0),k)+\#\vee_{\vec{\mathscr{F}}}(\vee_a(\Lambda,0),k),
\end{aligned}$$

respectively. Each of these numbers can be bounded by a multiple (dependent on k) of $\#\Lambda+\varepsilon^{-1/s}$ which proves the statement about the total complexity of computing $\tilde{\mathbf{r}}$.

The statement about the cost of computing $\tilde{\mathbf{r}}_{\frac12}$ follows by expressing $\frac{\partial}{\partial t}w+Nw-\nabla_x\cdot\vec{q}-g_\varepsilon$ in terms of $\Psi^a|_{\vee_a(\Lambda,\varepsilon)}$, and then by applying the statement about the cost of the evaluation of (4.4.7). A similar argument applies for the computation of $\tilde{\mathbf{r}}_1$ when $\langle N\Psi^{\mathscr{U}},\tilde{r}_{\frac12}\rangle_{L_2(\mathrm{I}\times\Omega)}$ is written as $\langle\Psi^{\mathscr{U}},\Psi^a\rangle_{L_2(\mathrm{I}\times\Omega)}\mathbf{d}$ for some $\mathbf{d}\in\vee_a(\vee_{\mathscr{V}_1}(\vee_a(\Lambda,\varepsilon),k),0)$, as well as for the computation of $\tilde{\mathbf{r}}_3$. The evaluation of $\tilde{\mathbf{r}}_{\frac32}$ requires first the computation of $\mathbf{E}^\top\mathbf{w}$, which takes $\mathcal{O}(\#\Lambda)$ operations, then h_ε needs to be subtracted taking $\mathcal{O}(\#\Lambda_\downarrow+\#\mathcal{T}(\varepsilon))$ operations, and finally $\Big\langle\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{L_2(\Omega)}},w(0,\cdot)-h_\varepsilon\Big\rangle_{L_2(\Omega)}|_{\diamond_{\mathscr{U}}(\mathcal{T}(\Lambda_\downarrow,\varepsilon),k)}$ needs to be evaluated which takes $\mathcal{O}(\#\diamond_{\mathscr{U}}(\mathcal{T}(\Lambda_\downarrow,\varepsilon),k))$ operations (cf. Remark 4.4.9). The statement about the cost of the evaluation of $\tilde{\mathbf{r}}_2$ follows directly from the definition of $\mathbf{E}_k$. □

Finally, recall that in addition to the cost condition Condition 4.3.4* that has been verified in Theorem 4.4.19, another condition, Condition 4.3.3, is required to conclude by Theorem 4.3.5 that the **awgm** is optimal. This condition requires the determination of an admissible $\tilde{\Lambda} \supset \Lambda$ with essentially quasi-minimal $\#(\tilde{\Lambda} \setminus \Lambda)$ such that $\|\tilde{\mathbf{r}}|_{\tilde{\Lambda}}\| \geq \mu_0 \|\tilde{\mathbf{r}}\|$. Unfortunately in our setting of multi-tree approximation, we are not aware of an algorithm that is guaranteed to yield such a $\tilde{\Lambda}$. In our numerical experiments, as a first step we constructed some set $\hat{\Lambda} \supset \Lambda$ with quasi-minimal $\#(\hat{\Lambda} \setminus \Lambda)$ such that $\|\tilde{\mathbf{r}}|_{\hat{\Lambda}}\| \geq \mu_0 \|\tilde{\mathbf{r}}\|$ by applying a bucket sort procedure on the entries of $\tilde{\mathbf{r}}$. Secondly, we enlarged $\hat{\Lambda}$ to an admissible set. In experiments we observed that $\#(\tilde{\Lambda} \setminus \hat{\Lambda})$ is at most a small multiple of $\#(\hat{\Lambda} \setminus \Lambda)$ which means that Condition 4.3.3 is satisfied, but we do not have a proof of this.

4.5 Numerical results

We consider the *heat equation*, i.e. $A = \mathrm{Id}$ and $N = 0$, on the L-shaped domain $\Omega = (0,1) \setminus [\frac{1}{2}, 1)^2$ and $\mathrm{I} = (0,1)$. For our convenience we take $g = 1$, and consider three different initial conditions $h = 0$, $h = 1$, and $h(x,y) = 50x(x-1)(x-\frac{1}{2})y(y-1)(y-\frac{1}{2})$.

We select the spatial wavelet collections Σ^* for $* \in \{\mathscr{U}, \mathscr{V}_1, \mathscr{T}\}$ s.t. σ^*_λ for $|\lambda| = \ell$ is piecewise polynomial w.r.t. the triangulation of Ω indicated in Figure 4.1, which specifies the collection $\mathcal{O}_\Omega$. We take $\Sigma^{\mathscr{V}_1}$ to be the continuous piecewise linear three-

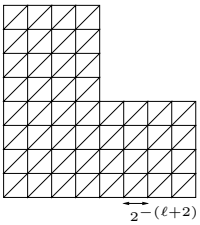


Figure 4.1: Partition of Ω on level $\ell \in \mathbb{N}_0$.

point wavelet basis from [Ste98b], satisfying homogenous boundary conditions, and normalized such that it is a Riesz basis for $H^1_0(\Omega)$, and for $\Sigma^{\mathscr{T}}$ we select this three-point wavelet basis, now without boundary conditions, and normalized such that it is a Riesz basis for $L_2(\Omega)$ (for a brief presentation see Sect. 5.2). For $\Sigma^{\mathscr{U}}$ we take a continuous piecewise quadratic wavelet basis that was constructed in Chapter 3, and that when normalized in $H^1(\Omega)$ or in $H^{-1}(\Omega)$ is a Riesz basis for $H^1_0(\Omega)$ or $H^{-1}(\Omega)$, respectively.

For $\Theta^{\mathscr{V}_1}$ and $\Theta^{\mathscr{T}}$ we take a discontinuous $L_2(\mathrm{I})$-orthonormal piecewise linear wavelet basis, and for $\Theta^{\mathscr{U}}$ the continuous piecewise linear three-point wavelet basis (without boundary conditions).

These collections satisfy all conditions (s_1)-(s_4), $(s_4^{\mathscr{U}})$, and (t_1)-(t_4), and using them we build the tensor product wavelet bases $\Psi^{\mathscr{V}_1}$, $\Psi^{\mathscr{T}}$ and $\Psi^{\mathscr{U}}$ as in (4.4.5).

It holds that $d_{\mathscr{U}_t} = d_{\mathscr{T}_t} = d_{\mathscr{T}_x} = 2$ and $d_{\mathscr{U}_x} = 3$, meaning that the best possible rate $s_{\max} = 1$.

Recall that $D\mathbf{Q}$ is affine, so that its (symmetric positive definite) linear part is given by $D^2\mathbf{Q}\colon [\mathbf{w}^\top\, \mathbf{q}^\top]^\top \mapsto D\mathbf{Q}[\mathbf{w}^\top\, \mathbf{q}^\top]^\top - D\mathbf{Q}[0\,0]^\top$. We have approximated the condition number of this system matrix by restricting it to the square block corresponding to all wavelets with indices $\lambda \in \vee_{\mathscr{U}}$, $(\mu, i) \in \vee_{\mathscr{T}} \times \{1, 2\}$ with $|\lambda|, |\mu|$ less than some integer. Even these Galerkin matrices cannot be evaluated exactly because they are of the form $\mathbf{A}_1^\top \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3$ where the rows of $\mathbf{A}_1$ run over the infinite index set $\vee_{\mathscr{V}_1}$. Similarly to Step $(S2)$ in Algorithm 4.4.18 we made an approximation by omitting all rows corresponding to indices $\gamma \in \vee_{\mathscr{V}_1}$ for which $|\gamma|$ exceeds any value of $|\lambda|$ or $|\mu|$ by more than a constant k that was taken sufficiently large so that it hardly affected the computed condition numbers. These numbers, illustrated in Figure 4.2 indicate that the condition number of the infinite system matrix can be expected to be of the order of 700.

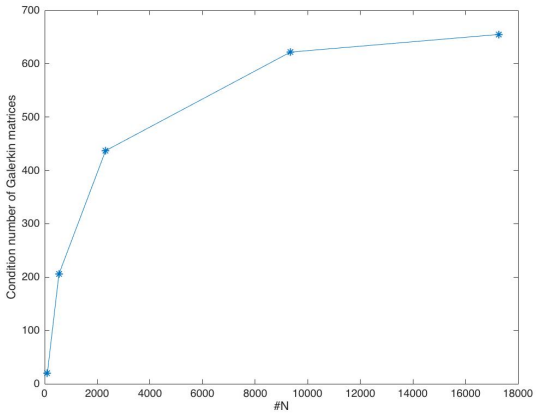


Figure 4.2: Approximate condition numbers of the Galerkin matrices vs. their dimension.

In Sect. 2.6 for an analogous FOSLS formulation of the corresponding stationary operator, i.e. the Laplace operator, and with the same spatial wavelets (and thus without temporal wavelets), we found a condition number of the order of 550.

We applied the **awgm** given in Algorithm 4.3.1. In step (R) of this algorithm, instead of performing a loop we simply computed the approximated residual by one application of Algorithm 4.4.18. In step (B) we applied bulk chasing as explained in the last paragraph of the previous Section 4.4 with parameter $\mu = 0.5$. The Galerkin matrices in step (G) were approximated using Algorithm 4.4.18, and approximately solved with parameter $\gamma = 0.2$ using CG iteration as explained in the last paragraph of Section 4.3.

The parameters k_2, k_3 and k_6 in Algorithm 4.4.18 were chosen to be equal to 1.

Furthermore, (k_4, k_5) was taken to be equal to $(1,3)$, $(4,7)$ or $(1,5)$ for $h = 0$, $h = 1$, and $h(x,y) = 50x(x-1)(x-\frac{1}{2})y(y-1)(y-\frac{1}{2})$, respectively. Since g, and in all three cases, h are global polynomials, ε in $(S1)$ equals 0, where $\vee_a(0)$ corresponds to the Cartesian product of the indices corresponding to the temporal and spatial 'scaling functions', and $\mathcal{T}(0)$ is the initial triangulation of Ω. For details about the implementation we refer to Chapter 5.

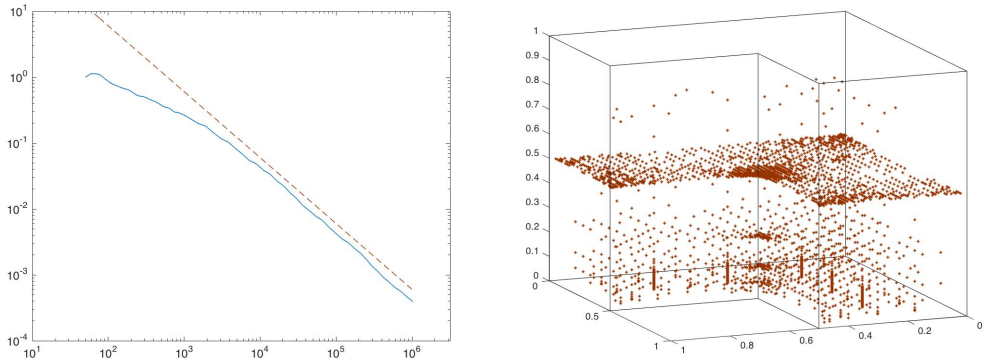


Figure 4.3: Norm residual vs. number of wavelets (dashed line has slope -1), and centers supports of the selected 4500 wavelets for $h = 0$ and $g = 1$.

In the left pictures in Figures 4.3-4.5, for right-hand side $g = 1$, and initial conditions $h = 0$, $h = 1$, and $h(x,y) = 50x(x-1)(x-\frac{1}{2})y(y-1)(y-\frac{1}{2})$, respectively, the ℓ_2-norm of the (approximate) residual is given vs. the number of wavelets from the basis for $\mathscr{U} \times \vec{\mathscr{T}} = L_2(\mathrm{I}; H^1_0(\Omega)) \cap H^1(\mathrm{I}; H^{-1}(\Omega)) \times L_2(\mathrm{I}; L_2(\Omega)^2)$.

For $h = 0$ and $h(x,y) = 50x(x-1)(x-\frac{1}{2})y(y-1)(y-\frac{1}{2})$, one observes that the **awgm** converges with the best possible rate $s = 1$. Moreover, thanks to the tensor product approximation not only the rate but also in an absolute sense the results are rather close to the results we found in Figure 2.4 for the corresponding stationary problem with errors in $(u, \nabla_x u)$ measured in $H^1_0(\Omega) \times L_2(\Omega)^2$. This means that one obtains the additional time dimension nearly for free.

This norm is equivalent to the $\mathscr{U} \times \vec{\mathscr{T}}$-norm in the error of the approximation to $(u, \vec{p}) = (u, \nabla_x u)$. In the right pictures one finds the centers of the supports of the tensor product wavelets that were selected by the adaptive method.

For $h = 1$, the observed rate indicates that the exact solution is only in $\mathcal{A}^{\frac{1}{2}}$. This reduction in the best approximation rate can be understood as follows: With this initial condition, the solution u is discontinuous at the full intersection of the lateral boundary and the bottom of the space-time space cylinder, inducing strong refinements near this intersection, as illustrated in the right picture of Figure 4.4. Although the solution is smooth in the direction tangential to this intersection, since the spatial

wavelets are isotropic the method cannot benefit from this smoothness causing the reduced best approximation rate.

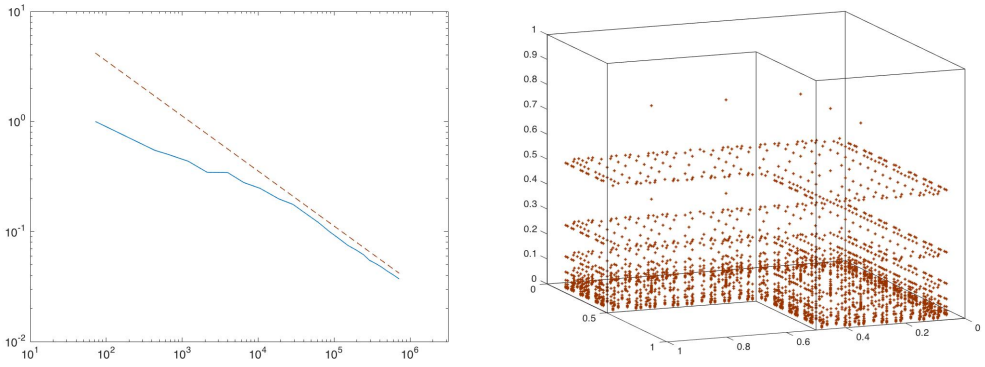


Figure 4.4: Norm residual vs. number of wavelets (dashed line has slope $-\frac{1}{2}$), and centers supports of the selected 4131 wavelets for $h = 1$ and $g = 1$.

We expect that if we would have applied a spatial piecewise tensor product wavelet basis as constructed in [CDFS13], this reduction would not have occurred, and also for this problem we would have obtained rates as if we would solve a one-dimensional problem. We have however chosen for the current isotropic spatial wavelets because of their (relatively) easy construction, and because they apply to any polygon. Moreover, parabolic problems are usually studied assuming that the data satisfies the lowest order compatibility condition of a vanishing initial condition at the homogenous Dirichlet boundary.

Note that the initial condition $h(x, y) = 50x(x - 1)(x - \frac{1}{2})y(y - 1)(y - \frac{1}{2})$ does vanish at $\partial\Omega$, but does not satisfy the next compatibility condition of $-\Delta h = g(= 1)$ at $\partial\Omega$. Nevertheless, it seems to give rise to the best possible rate allowed by the polynomial orders that were applied.

4.6 Conclusion

In this chapter an optimal adaptive wavelet solver has been developed for solving a simultaneously space-time FOSLS formulation of parabolic PDEs. Thanks to the use of tensor products of wavelets in space and time the whole time evolution can be solved at a complexity of solving the corresponding stationary problem, which has been illustrated by numerical results.

A theoretical issue that has not yet been satisfactorily solved is that of bulk chasing under a multi-tree constraint (cf. last paragraph of Sect. 4.4). It may require a

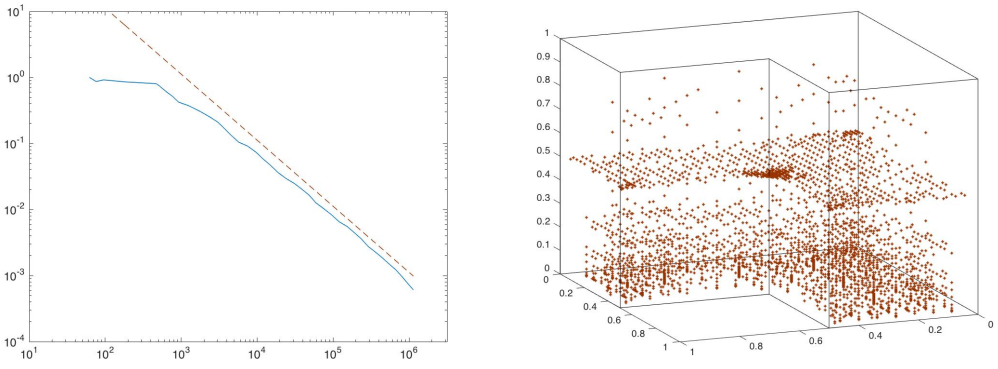


Figure 4.5: Norm residual vs. number of wavelets (dashed line has slope -1), and centers supports of the selected 4345 wavelets for $h(x,y) = 50x(x-1)(x-\frac{1}{2})y(y-1)(y-\frac{1}{2})$ and $g = 1$.

generalisation to multi-trees of the tree approximation routines given in [BD04].

Other than in Chapter 2 dealing with stationary PDEs and non-tensor product approximation, in order to construct an approximate residual evaluation of linear complexity we had to restrict ourselves to linear PDOs. It would be interesting to circumvent this restriction.

In [SS17], we constructed well-posed simultaneously space-time saddle-point formulations of instationary Stokes and Navier-Stokes equations. Using the approach from Chapter 2, these formulations can be recast as well-posed FOSLS so that, modulo the treatment of the nonlinear term in NSE, the adaptive scheme from the current work applies.

Adaptive finite element (afem) schemes usually have better quantitative properties than adaptive wavelet schemes. To the best of our knowledge, for simultaneously space-time variational formulations, currently there are no afem schemes available that are proven to converge, let alone to be optimal or to give rates as for stationary problems. Our FOSLS formulation might give an opening towards such results because it gives a well-posed symmetric positive definite problem.

4.7 Appendix: Decay estimates

In this appendix we prove the technical results Theorem 4.7.3, Corollaries 4.7.7, 4.7.9, and 4.7.11 that were used in the proof of Theorem 4.4.19.

The following lemma is an application of Schur's lemma that is often used to bound the spectral norm of a matrix whose the row and column indices run over index sets

of multi-level bases.

Lemma 4.7.1. *For index sets J, J', let $|\cdot|\colon J \cup J' \to \mathbb{N}_0$, and let $M := [m_{\lambda',\lambda}]_{(\lambda',\lambda)\in J'\times J}$ be such that for some $\xi \geq 0$, $\varrho > 0$,*

$$\#\{\lambda'\colon m_{\lambda',\lambda} \neq 0,\, |\lambda'| = |\lambda| + k\} \lesssim 2^{\xi k} \quad (\lambda \in J,\, k \in \mathbb{N}_0),$$
$$\#\{\lambda\colon m_{\lambda',\lambda} \neq 0,\, |\lambda'| = |\lambda| + k\} \lesssim 1 \quad (\lambda' \in J',\, k \in \mathbb{N}_0),$$

and

$$|m_{\lambda',\lambda}| \lesssim 2^{(|\lambda|-|\lambda'|)(\varrho+\frac{\xi}{2})} \quad (|\lambda'| \geq |\lambda|).$$

Then

$$\||M\big|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}|\| \lesssim 2^{-\varrho k},$$

where $(M\big|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}})_{\lambda',\lambda} := \begin{cases} m_{\lambda',\lambda} & \text{when } |\lambda'| > |\lambda| + k \\ 0 & \text{otherwise} \end{cases}$, *and $\|\cdot\|$ denotes the matrix spectral norm, i.e., here the norm on $\mathcal{L}(\ell_2(J), \ell_2(J'))$. The absolute value refers to taking entry-wise absolute value. (Similar notations will be used at other occasions.)*

Proof. With $I_{\ell',\ell} := [|m_{\lambda',\lambda}|]_{\{(\lambda',\lambda)\colon |\lambda'|=\ell',|\lambda|=\ell\}}$, we have

$$\||M\big|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}|\|^2 \lesssim \max_{\ell'} \sum_{\ell<\ell'-k} \|I_{\ell',\ell}\| \times \max_{\ell} \sum_{\ell'>\ell+k} \|I_{\ell',\ell}\|,$$

where ℓ, ℓ' run over $\mathbb{N}_0$.

The number of non-zero entries in each column or row of $I_{\ell',\ell}$ is $\lesssim 2^{\xi(\ell'-\ell)}$ or $\lesssim 1$, respectively. Using $\|\cdot\|^2 \leq \|\cdot\|_1 \|\cdot\|_\infty$, we infer that $\|I_{\ell',\ell}\|^2 \lesssim 2^{\xi(\ell'-\ell)} \cdot 2^{(\frac{\xi}{2}+\varrho)(\ell-\ell')} \cdot 1 \cdot 2^{(\frac{\xi}{2}+\varrho)(\ell-\ell')} = 4^{\varrho(\ell-\ell')}$. □

The next lemma concerns near-sparsity of a generalized mass matrix corresponding to two temporal wavelet bases.

Lemma 4.7.2. *For $k \in \mathbb{N}_0$, $\Theta^*, \Theta^\circ \in \{\Theta^{\mathscr{V}_1}, \Theta^{\mathscr{T}}, \Theta^a, \Theta^{\mathscr{U}}/\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}\}$ we have*

$$\||\langle \Theta^*, \Theta^\circ\rangle_{L_2(\mathrm{I})}\big|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}|\| \lesssim 2^{-k/2}.$$

Proof. Using that Θ^* satisfies (t_1)–(t_4), being the counterparts of (s_1)-(s_4) for the spatial wavelets, we split the matrix into $B^r + B^s$, where B^r contains all its entries $\langle \theta^*_{\lambda'}, \theta^\circ_\lambda\rangle_{L_2(\mathrm{I})}$ for which $\operatorname{supp}\theta^*_{\lambda'}$ is contained in ω for some $\omega \in \mathcal{O}_\mathrm{I}$ with $|\omega| = |\lambda|$ (the 'regular' entries), and where B^s contains the remaining ('singular') entries.

The number of non-zero entries with $|\lambda'| = \ell'$ and $|\lambda| = \ell$ in each column or row of B^r is $\lesssim 2^{\ell'-\ell}$ or $\lesssim 1$, respectively. Thanks to (t_4), for each of these entries we have $|\langle \theta^*_{\lambda'}, \theta^\circ_\lambda\rangle_{L_2(\mathrm{I})}| \leq \|\theta^*_{\lambda'}\|_{L_1(\mathrm{I})} 2^{-\ell'} |\theta^\circ_\lambda|_{W^1_\infty(\operatorname{supp}\theta^*_{\lambda'})} \lesssim 2^{3(\ell-\ell')/2}$. An application of Lemma 4.7.1 with $\xi = \varrho = 1$ shows that $\||B^r|\| \lesssim 2^{-k}$.

The number of non-zero entries with $|\lambda'| = \ell'$ and $|\lambda| = \ell$ in each column or row of B^s is $\lesssim 1$. For each of these entries, we have $|\langle\theta^*_{\lambda'}, \theta^\circ_\lambda\rangle_{L_2(\mathrm{I})}| \le \|\theta^*_{\lambda'}\|_{L_1(\mathrm{I})}\|\theta^\circ_\lambda\|_{L_\infty(\mathrm{I})} \lesssim 2^{(\ell-\ell')/2}$. An application of Lemma 4.7.1 with $\xi = 0$, $\varrho = \frac{1}{2}$ shows that $\|\!|B^s|\!\| \lesssim 2^{-k/2}$. □

The following theorem provides the main ingredient for bounding $\|\mathbf{r}_{\frac{1}{2}} - \tilde{\mathbf{r}}_{\frac{1}{2}}\|$.

Theorem 4.7.3. *Let $\Lambda^a \subset \vee_a$ be a multi-tree, and $r \in \operatorname{span}\Psi^a\big|_{\Lambda^a}$ For $k \in \mathbb{N}_0$, it holds that*

$$\|\langle\Psi^{\mathscr{V}_1}, r\rangle_{L_2(\mathrm{I}\times\Omega)}|_{\vee_{\mathscr{V}_1}\setminus\vee_{\mathscr{V}_1}(\Lambda^a,k)}\| \lesssim 2^{-k/2}\|r\|_{L_2(\mathrm{I};H^{-1}(\Omega))}. \tag{4.7.1}$$

Proof. We write $r = \sum_{(\lambda,\mu)\in\Lambda^a} r_{\lambda\mu}\theta^a_\lambda \otimes \sigma^a_\mu$, and let

$$\delta_\lambda(\mu') := \begin{cases} 0 & |\mu'| \le \max\{|\mu|\colon \mu \in \Lambda^a_2(\lambda),\ \mathrm{meas}(\mathcal{S}(\sigma^{\mathscr{V}_1}_{\mu'}) \cap \operatorname{supp}\sigma^a_\mu) > 0\} + k \\ 1 & \text{elsewhere} \end{cases},$$

Writing $\Lambda^{\mathscr{V}_1} := \vee_{\mathscr{V}_1}(\Lambda^a, k)$, from $\frac{1}{2}(a+b)^2 \le (a^2+b^2)$, we have that

$$\begin{aligned}
&\frac{1}{2}\|\langle\Psi^{\mathscr{V}_1}, r\rangle_{L_2(\mathrm{I}\times\Omega)}|_{\vee_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}}\|^2 \le \\
&\sum_{\mu'\in\diamondsuit_{\mathscr{V}_1}}\sum_{\lambda'\in\triangleleft_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}_1(\mu')}\Big|\sum_{|\lambda'|>|\lambda|+k}\langle\theta^{\mathscr{V}_1}_{\lambda'}, \theta^a_\lambda\rangle_{L_2(\mathrm{I})}\langle\sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu\in\Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}\Big|^2 + \\
&\sum_{\mu'\in\diamondsuit_{\mathscr{V}_1}}\sum_{\lambda'\in\triangleleft_{\mathscr{V}_1}\setminus\Lambda^{\mathscr{V}_1}_1(\mu')}\Big|\sum_{|\lambda|\ge|\lambda'|-k}\langle\theta^{\mathscr{V}_1}_{\lambda'}, \theta^a_\lambda\rangle_{L_2(\mathrm{I})}\delta_\lambda(\mu')\langle\sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu\in\Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}\Big|^2
\end{aligned} \tag{4.7.2}$$

Here we could insert the factor $\delta_\lambda(\mu')$ in the second sum because of the following reason: Let $(\lambda', \mu') \in \vee_{\mathscr{V}_1} \setminus \Lambda^{\mathscr{V}_1}$ and $\lambda \in \Lambda^a_1$ with $|\lambda| \ge |\lambda'| - k$. If $\mathrm{meas}(\mathcal{S}(\theta^{\mathscr{V}_1}_{\lambda'}) \cap \operatorname{supp}\theta^a_\lambda) = 0$, then the value of $\delta_\lambda(\mu')$ is irrelevant. If $\mathrm{meas}(\mathcal{S}(\theta^{\mathscr{V}_1}_{\lambda'}) \cap \operatorname{supp}\theta^a_\lambda) > 0$, then the definition of $\Lambda^{\mathscr{V}_1} = \vee_{\mathscr{V}_1}(\Lambda^a, k)$ shows that $|\mu'| > |\mu| + k$ for all $\mu \in \Lambda^a_2(\lambda)$ with $\mathrm{meas}(\mathcal{S}(\sigma^{\mathscr{V}_1}_{\mu'}) \cap \operatorname{supp}\sigma^a_\mu) > 0$, meaning that $\delta_\lambda(\mu') = 1$.

Using Lemma 4.7.2 for $(\Theta^*, \Theta^\circ) = (\Theta^{\mathscr{V}_1}, \Theta^a)$, the first sum can be bounded on a multiple of

$$\begin{aligned}
&\sum_{\mu'\in\diamondsuit_{\mathscr{V}_1}} 2^{-k}\sum_{\lambda\in\triangleleft_a}\big|\langle\sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu\in\Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}\big|^2 \\
&= 2^{-k}\sum_{\lambda\in\triangleleft_a}\sum_{\mu'\in\diamondsuit_{\mathscr{V}_1}}\big|\langle\sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu\in\Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}\big|^2 \eqsim 2^{-k}\sum_{\lambda\in\triangleleft_a}\|\sum_{\mu\in\Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\|^2_{H^{-1}(\Omega)} \\
&\eqsim 2^{-k}\|\sum_{\lambda\in\triangleleft_a}\theta^a_\lambda \otimes \sum_{\mu\in\Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\|^2_{L_2(\mathrm{I};H^{-1}(\Omega))} = 2^{-k}\|r\|^2_{L_2(\mathrm{I};H^{-1}(\Omega))},
\end{aligned}$$

where we used that $\Sigma^{\mathscr{V}_1}$ is a Riesz basis for $H^1_0(\Omega)$, and that Θ^a is a Riesz basis for $L_2(\mathrm{I})$.

To bound the second sum, recall that for $\mu \in \diamondsuit_a$, it holds that $\operatorname{supp}\sigma^a_\mu = \omega_\mu$ for some $\omega_\mu \in \mathcal{O}_\Omega$ with $|\omega_\mu| = \max(|\mu|-1, 0)$. Define the tiling $\mathcal{T}(\lambda) \in \mathcal{O}_\Omega$ as the union, over the leaves μ of the tree $\Lambda^a_2(\lambda)$, of the children of ω_μ when $|\mu| > 0$, or of ω_μ itself when $|\mu| = 0$. Then $\operatorname{span}\{\sigma^a_\mu \colon \mu \in \Lambda^a_2(\lambda)\} = \mathcal{P}_m(\mathcal{T}(\lambda))$, and $\{\mu' \in \diamondsuit_{\mathscr{V}_1} \colon \delta_\lambda(\mu') = 1\} = \diamondsuit_{\mathscr{V}_1} \setminus \diamondsuit_{\mathscr{V}_1}(\mathcal{T}(\lambda), k)$, cf. Definition 4.4.7.

Since Θ^a and $\Theta^{\mathscr{V}_1}$ are Riesz bases for $L_2(\mathrm{I})$, and so $\langle \Theta^{\mathscr{V}_1}, \Theta^a\rangle_{L_2(\mathrm{I})} \in \mathcal{L}(\ell_2(\lhd_a), \ell_2(\lhd_{\mathscr{V}_1}))$, invoking Proposition 2.9.1 using that $\Sigma^{\mathscr{V}_1}$ satisfies (s_1)–(s_4), the second sum can be bounded on a multiple of

$$\begin{aligned}
&\sum_{\mu' \in \diamondsuit_{\mathscr{V}_1}} \sum_{\lambda \in \lhd_a} \big|\delta_\lambda(\mu')\langle \sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu \in \Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}\big|^2 \\
&= \sum_{\lambda \in \lhd_a} \sum_{\mu' \in \diamondsuit_{\mathscr{V}_1}} \big|\delta_\lambda(\mu')\langle \sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu \in \Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}\big|^2 \lesssim \sum_{\lambda \in \lhd_a} 4^{-k} \| \sum_{\mu \in \Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\|^2_{H^{-1}(\Omega)} \\
&\eqsim 4^{-k} \| \sum_{\lambda \in \lhd_a} \theta^a_\lambda \otimes \sum_{\mu \in \Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\|^2_{L_2(\mathrm{I};H^{-1}(\Omega))} = 4^{-k}\|r\|^2_{L_2(\mathrm{I};H^{-1}(\Omega))},
\end{aligned}$$

where we used that Θ^a is a Riesz basis for $L_2(\mathrm{I})$. □

If Θ^a would have been a Riesz basis for $H^{-1}(\Omega)$, then in the proof of Theorem 4.7.3 it would have been natural to write $\langle \sigma^{\mathscr{V}_1}_{\mu'}, \sum_{\mu \in \Lambda^a_2(\lambda)} r_{\lambda\mu}\sigma^a_\mu\rangle_{L_2(\Omega)}$ as $\langle \Sigma^{\mathscr{V}_1}, \Sigma^a\rangle_{L_2(\Omega)}[r_{\lambda\mu}]_{\mu \in \Lambda^a_2(\lambda)}$. In this case the approach of the insertion of the factor $\delta_\lambda(\mu')$ would have given the bound

$$\begin{aligned}
\tfrac{1}{2}\sqrt{2}\,\big\|\langle \Psi^{\mathscr{V}_1}, \Psi^a\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{\vee_{\mathscr{V}_1} \setminus \Lambda^{\mathscr{V}_1}(\Lambda^a,k)\times\Lambda^a}\big\| \leq \\
\big\|\langle \Theta^{\mathscr{V}_1}, \Theta^a\rangle_{L_2(\mathrm{I})}|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}\big\|\big\|\langle \Sigma^{\mathscr{V}_1}, \Sigma^a\rangle_{L_2(\Omega)}\big\| + \\
\big\|\langle \Theta^{\mathscr{V}_1}, \Theta^a\rangle_{L_2(\mathrm{I})}\big\|\big\|\langle \Sigma^{\mathscr{V}_1}, \Sigma^a\rangle_{L_2(\Omega)}\big|_{\{(\mu',\mu)\colon |\mu'|>|\mu|+k\}}\big\|.
\end{aligned}$$

Although in the current setting where $\langle \Sigma^{\mathscr{V}_1}, \Sigma^a\rangle_{L_2(\Omega)} \notin \mathcal{L}(\ell_2(\diamondsuit_a), \ell_2(\diamondsuit_{\mathscr{V}_1}))$, this estimate makes not much sense, for other collections this result, formulated in the next proposition, is going to be useful.

Proposition 4.7.4. *For $*, \circ \in \{\mathscr{U}, \mathscr{V}_1, \mathscr{T}, a\}$, let $M_\lhd := [m^\lhd_{\lambda',\lambda}]_{(\lambda',\lambda)} \in \mathcal{L}(\ell_2(\lhd_*), \ell_2(\lhd_\circ))$, $M_\diamondsuit := [m^\diamondsuit_{\mu',\mu}]_{(\mu',\mu)} \in \mathcal{L}(\ell_2(\diamondsuit_*), \ell_2(\diamondsuit_\circ))$, where $m^\lhd_{\lambda',\lambda} = 0$ when $\operatorname{meas}(\mathcal{S}(\theta^\circ_{\lambda'}) \cap \operatorname{supp}\theta^*_\lambda) = 0$, and $m^\diamondsuit_{\mu',\mu} = 0$ when $\operatorname{meas}(\mathcal{S}(\sigma^\circ_{\mu'}) \cap \operatorname{supp}\sigma^*_\mu) = 0$. Then for a multi-tree $\Lambda^* \subset \vee_*$, and $k \in \mathbb{N}_0$, it holds that*

$$\begin{aligned}
\tfrac{1}{2}\sqrt{2}\,\|M_\lhd \otimes M_\diamondsuit|_{(\vee_\circ \setminus \Lambda^\circ(\Lambda^*,k))\times\Lambda^*}\| \leq \\
\|M_\lhd|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}\|\|M_\diamondsuit\| + \|M_\lhd\|\|M_\diamondsuit|_{\{(\mu',\mu)\colon |\mu'|>|\mu|+k\}}\|.
\end{aligned}$$

The remaining of this appendix will consist of various applications of Proposition 4.7.4 for which in several lemmas we estimate norms of type

$$\|M_\lhd|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}\| \quad \text{or} \quad \|M_\lhd|_{\{(\lambda',\lambda)\colon |\lambda'|>|\lambda|+k\}}\|.$$

The next lemma deals with the first task.

Lemma 4.7.5. *For $k \in \mathbb{N}_0$, it holds that*

$$\Big|\Big|\Big|\Big\langle \frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^{-1}(\Omega)}}, \Sigma^{\mathscr{V}_1}\Big\rangle_{L_2(\Omega)}\Big|_{\{(\mu',\mu):\,|\mu'|>|\mu|+k\}}\Big|\Big|\Big| \lesssim 2^{-k/2},$$
$$\Big|\Big|\Big|\Big\langle \frac{\frac{\partial}{\partial x_i}\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^{1}(\Omega)}}, \Sigma^{a}\Big\rangle_{L_2(\Omega)}\Big|_{\{(\mu',\mu):\,|\mu'|>|\mu|+k\}}\Big|\Big|\Big| \lesssim 2^{-k/2}.$$

Proof. For proving the first inequality, we split the matrix into $B^r + B^s$, where B^r contains all its entries $\Big\langle \frac{\sigma^{\mathscr{U}}_{\mu'}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^{-1}(\Omega)}}, \sigma^{\mathscr{V}_1}_{\mu}\Big\rangle_{L_2(\Omega)}$ for which $\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'}$ is contained in ω for some $\omega \in \mathcal{O}_\Omega$ with $|\omega| = |\mu|$ (the 'regular' entries), and where B^s contains the remaining ('singular') entries.

Thanks to $(s_4^{\mathscr{U}})$, for the regular entries we can estimate

$$\begin{aligned}|\langle \sigma^{\mathscr{U}}_{\mu'}, \sigma^{\mathscr{V}_1}_{\mu}\rangle_{L_2(\Omega)}| &\lesssim \|\sigma^{\mathscr{U}}_{\mu'}\|_{L_1(\Omega)} 4^{-|\mu'|}|\sigma^{\mathscr{V}_1}_{\mu}|_{W^2_\infty(\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'})}\\ &\lesssim 4^{-|\mu'|}2^{-|\mu'|\frac{n}{2}}\|\sigma^{\mathscr{U}}_{\mu'}\|_{L_2(\Omega)} 2^{|\mu|}2^{|\mu|\frac{n}{2}}\|\sigma^{\mathscr{V}_1}_{\mu}\|_{H^1(\Omega)}\\ &\eqsim 2^{(|\mu|-|\mu'|)(1+\frac{n}{2})}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^{-1}(\Omega)}\end{aligned}$$

where we used $\|\sigma^{\mathscr{V}_1}_{\mu}\|_{H^1(\Omega)} \eqsim 1$, and

$$\|\sigma^{\mathscr{U}}_{\mu'}\|^2_{L_2(\Omega)} \le \|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^{-1}(\Omega)} \lesssim 2^{|\mu'|}\|\sigma^{\mathscr{U}}_{\mu'}\|_{L_2(\Omega)}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^{-1}(\Omega)}.$$

An application of Lemma 4.7.1 with $\xi = n$ and $\varrho = 1$ shows that $|||B^r||| \lesssim 2^{-k}$.

Since the wavelets $\sigma^{\mathscr{V}_1}_{\mu}$ are piecewise polynomial functions in $H^1(\Omega)$, they are contained in $W^1_\infty(\Omega)$. Using (s_4), for the remaining singular entries we estimate

$$\begin{aligned}|\langle \sigma^{\mathscr{U}}_{\mu'}, \sigma^{\mathscr{V}_1}_{\mu}\rangle_{L_2(\Omega)}| &\lesssim \|\sigma^{\mathscr{U}}_{\mu'}\|_{L_1(\Omega)} 2^{-|\mu'|}|\sigma^{\mathscr{V}_1}_{\mu}|_{W^1_\infty(\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'})}\\ &\eqsim 2^{(|\mu|-|\mu'|)\frac{n}{2}}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^{-1}(\Omega)}\end{aligned}$$

again by $\|\sigma^{\mathscr{V}_1}_{\mu}\|_{H^1(\Omega)} \eqsim 1$, and $\|\sigma^{\mathscr{U}}_{\mu'}\|_{L_2(\Omega)} \lesssim 2^{|\mu'|}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^{-1}(\Omega)}$ (cf. (4.4.9)). An application of Lemma 4.7.1 with $\xi = n-1$ and $\varrho = 1/2$ shows that $|||B^s||| \lesssim 2^{-k/2}$.

Moving to the second inequality, we split the matrix into $B^r + B^s$, where B^r contains all its entries $\Big\langle \frac{\frac{\partial}{\partial x_i}\sigma^{\mathscr{U}}_{\mu'}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}}, \sigma^{a}_{\mu}\Big\rangle_{L_2(\Omega)}$ for which $\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'}$ is contained in $\omega \cap \Omega$ for some $\omega \in \mathcal{O}_\Omega$ with $|\omega| = |\mu|$ (the 'regular' entries), and where B^s contains the remaining ('singular') entries.

Thanks to (s_4), for the regular entries we can estimate

$$\begin{aligned}|\langle \frac{\partial}{\partial x_i}\sigma^{\mathscr{U}}_{\mu'}, \sigma^{a}_{\mu}\rangle_{L_2(\Omega)}| &= |\langle \sigma^{\mathscr{U}}_{\mu'}, \frac{\partial}{\partial x_i}\sigma^{a}_{\mu}\rangle_{L_2(\Omega)}| \lesssim \|\sigma^{\mathscr{U}}_{\mu'}\|_{L_1(\Omega)} 2^{-|\mu'|}|\sigma^{a}_{\mu}|_{W^2_\infty(\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'})}\\ &\lesssim 4^{-|\mu'|}2^{-|\mu'|\frac{n}{2}}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)} 4^{|\mu|}2^{|\mu|\frac{n}{2}}\|\sigma^{a}_{\mu}\|_{L_2(\Omega)} \eqsim 2^{(|\mu|-|\mu'|)(2+\frac{n}{2})}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)},\end{aligned}$$

where we used that $\|\sigma^{\mathscr{U}}_{\mu'}\|_{L_2(\Omega)} \lesssim 2^{-|\mu'|}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}$ ((4.4.9)). An application of Lemma 4.7.1 with $\xi = n$ and $\varrho = 2$ shows that $\||B^r|\| \lesssim 4^{-k}$.

For the remaining singular entries we estimate

$$\begin{aligned}|\langle \frac{\partial}{\partial x_i}\sigma^{\mathscr{U}}_{\mu'}, \sigma^a_\mu\rangle_{L_2(\Omega)}| &\lesssim \|\sigma^{\mathscr{U}}_{\mu'}\|_{W^1_1(\Omega)}\|\sigma^a_\mu\|_{L_\infty(\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'})}\\ &\lesssim 2^{(|\mu|-|\mu'|)\frac{n}{2}}\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}\|\sigma^a_\mu\|_{L_2(\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'})}.\end{aligned}$$

An application of Lemma 4.7.1 with $\xi = n-1$ and $\varrho = 1/2$ shows that $\||B^s|\| \lesssim 2^{-k/2}$. □

Lemma 4.7.6. *For $k \in \mathbb{N}_0$, it holds that*

$$\Big\|\Big\langle \frac{(\Theta^{\mathscr{U}})'}{\|(\Theta^{\mathscr{U}})'\|_{L_2(\mathrm{I})}}, \Theta^{\mathscr{V}_1}\Big\rangle_{L_2(\mathrm{I})}\Big|_{\{(\lambda',\lambda):\,|\lambda'|>|\lambda|+k\}}\Big\| \lesssim 2^{-k/2}.$$

Proof. We split the matrix into $B^r + B^s$, where B^r contains all ('regular') entries

$$\Big\langle \frac{(\theta^{\mathscr{U}}_{\lambda'})'}{\|(\theta^{\mathscr{U}}_{\lambda'})'\|_{L_2(\mathrm{I})}}, \theta^{\mathscr{V}_1}_\lambda\Big\rangle_{L_2(\mathrm{I})}$$

for which $\operatorname{supp}\theta^{\mathscr{U}}_{\lambda'}$ is contained in $\omega \cap I$ for some $\omega \in \mathcal{O}_\mathrm{I}$ with $|\omega| = |\lambda|$ (so that in particular $\theta^{\mathscr{U}}_{\lambda'}$ vanishes on ∂I), and where B^s contains the remaining ('singular') entries.

For the regular entries, we can estimate

$$\begin{aligned}&|\langle(\theta^{\mathscr{U}}_{\lambda'})', \theta^{\mathscr{V}_1}_\lambda\rangle_{L_2(\mathrm{I})}| = |\langle\theta^{\mathscr{U}}_{\lambda'}, (\theta^{\mathscr{V}_1}_\lambda)'\rangle_{L_2(\mathrm{I})}| \lesssim \|\theta^{\mathscr{U}}_{\lambda'}\|_{L_1(\mathrm{I})}2^{-|\lambda'|}\|\theta^{\mathscr{V}_1}_\lambda\|_{W^2_\infty(\operatorname{supp}\theta^{\mathscr{U}}_{\lambda'})}\\ &\lesssim 2^{-|\lambda'|/2}2^{-|\lambda'|}\|(\theta^{\mathscr{U}}_{\lambda'})'\|_{L_2(\mathrm{I})}2^{-|\lambda'|}4^{|\lambda|}2^{|\lambda|/2}\|\theta^{\mathscr{V}_1}_\lambda\|_{L_2(\mathrm{I})} \eqsim 2^{\frac{5}{2}(|\lambda|-|\lambda'|)}\|(\theta^{\mathscr{U}}_{\lambda'})'\|_{L_2(\mathrm{I})},\end{aligned}$$

where we used (t_4), Poincaré's inequality, an inverse inequality, and $\|\theta^{\mathscr{V}_1}_\lambda\|_{L_2(\mathrm{I})} \eqsim 1$. An application of Lemma 4.7.1 with $\xi = 1$ and $\varrho = 2$ shows that $\||B^r|\| \lesssim 4^{-k}$.

For the remaining singular entries, we estimate

$$|\langle(\theta^{\mathscr{U}}_{\lambda'})', \theta^{\mathscr{V}_1}_\lambda\rangle_{L_2(\mathrm{I})}| \le \|(\theta^{\mathscr{U}}_{\lambda'})'\|_{L_1(\mathrm{I})}\|\theta^{\mathscr{V}_1}_\lambda\|_{L_\infty(\mathrm{I})} \lesssim 2^{-\frac{1}{2}(|\lambda|-|\lambda'|)}\|(\theta^{\mathscr{U}}_{\lambda'})'\|_{L_2(\mathrm{I})}.$$

An application of Lemma 4.7.1 with $\xi = 0$ and $\varrho = \frac{1}{2}$ shows that $\||B^s|\| \lesssim 2^{-k/2}$. □

The following Corollary will be used to bound $\|(\mathbf{r}_1 - \tilde{\mathbf{r}}_1)|_{\vee_{\mathscr{U}}}\|$.

Corollary 4.7.7. *Let $\Lambda^{\mathscr{V}_1} \subset \vee_{\mathscr{V}_1}$ be a multi-tree. Then for $k \in \mathbb{N}_0$,*

$$\left.\begin{aligned}\|\big\langle \tfrac{\partial}{\partial t}\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{(\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k))\times\Lambda^{\mathscr{V}_1}}\|\\ \|\big\langle b_i\tfrac{\partial}{\partial x_i}\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{(\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k))\times\Lambda^{\mathscr{V}_1}}\|\\ \|\big\langle c\Psi^{\mathscr{U}}, \Psi^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{(\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k))\times\Lambda^{\mathscr{V}_1}}\|\end{aligned}\right\} \lesssim 2^{-k/2}.$$

Proof. (a). From $\|\theta^{\mathscr{U}}_{\lambda}\otimes\sigma^{\mathscr{U}}_{\mu}\|_{\mathscr{U}}\geq\|\theta^{\mathscr{U}}_{\lambda}\|_{H^1(\mathrm{I})}\|\sigma^{\mathscr{U}}_{\mu}\|_{H^{-1}(\Omega)}$, for the first inequality it is sufficient to prove that

$$\|\big\langle\frac{(\Theta^{\mathscr{U}})'}{\|\Theta^{\mathscr{U}}\|_{H^1(\mathrm{I})}}\otimes\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^{-1}(\Omega)}},\Theta^{\mathscr{V}_1}\otimes\Sigma^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{(\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k))\times\Lambda^{\mathscr{V}_1}}\|\lesssim 2^{-k/2}.$$

From $\Theta^{\mathscr{U}}/\|\Theta^{\mathscr{U}}\|_{H^1(\mathrm{I})}$, $\Theta^{\mathscr{V}_1}$, $\Sigma^{\mathscr{U}}/\|\Sigma^{\mathscr{U}}\|_{H^{-1}(\Omega)}$, and $\Sigma^{\mathscr{V}_1}$ being Riesz bases for $H^1(\mathrm{I})$, $L_2(\mathrm{I})$, $H^{-1}(\Omega)$, and $H^1_0(\Omega)$, we have

$$\Big\langle\frac{(\Theta^{\mathscr{U}})'}{\|\Theta^{\mathscr{U}}\|_{H^1(\mathrm{I})}},\Theta^{\mathscr{V}_1}\Big\rangle_{L_2(\mathrm{I})}\in\mathcal{L}(\ell_2(\lhd_{\mathscr{V}_1}),\ell_2(\lhd_{\mathscr{U}})),$$
$$\Big\langle\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^{-1}(\Omega)}},\Sigma^{\mathscr{V}_1}\Big\rangle_{L_2(\Omega)}\in\mathcal{L}(\ell_2(\diamond_{\mathscr{V}_1}),\ell_2(\diamond_{\mathscr{U}})).$$

The proof of the first inequality is completed by applications of Proposition 4.7.4 and Lemmata 4.7.5(first statement)–4.7.6.

(b). From $\operatorname{span}\frac{\partial}{\partial x_i}b_i\Psi^{\mathscr{V}_1}|_{\Lambda^{\mathscr{V}_1}}\subset\operatorname{span}\Psi^a|_{\vee_a(\Lambda^{\mathscr{V}_1},0)}$ (similar to Lemma 4.4.16), for $\mathbf{c}\in\ell_2(\Lambda^{\mathscr{V}_1})$ there exists a $\mathbf{d}\in\ell_2(\vee_a(\Lambda^{\mathscr{V}_1},0))$ such that

$$\begin{aligned}\big\langle b_i\frac{\partial}{\partial x_i}\Psi^{\mathscr{U}},\Psi^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I}\times\Omega)}\mathbf{c}&=\big\langle\Psi^{\mathscr{U}},-\mathbf{c}^\top\frac{\partial}{\partial x_i}(b_i\Psi^{\mathscr{V}_1})\big\rangle_{L_2(\mathrm{I}\times\Omega)}\\&=\big\langle\Psi^{\mathscr{U}},\mathbf{d}^\top\Psi^a\big\rangle_{L_2(\mathrm{I}\times\Omega)}=\big\langle\Psi^{\mathscr{U}},\Psi^a\big\rangle_{L_2(\mathrm{I}\times\Omega)}\mathbf{d},\end{aligned}$$

where

$$\|\mathbf{d}\|\eqsim\|\mathbf{d}^\top\Psi^a\|_{L_2(\mathrm{I}\times\Omega)}=\|\mathbf{c}^\top\frac{\partial}{\partial x_i}(b_i\Psi^{\mathscr{V}_1})\|_{L_2(\mathrm{I}\times\Omega)}\lesssim\|\mathbf{c}\|.$$

From $\|\theta^{\mathscr{U}}_{\lambda}\otimes\sigma^{\mathscr{U}}_{\mu}\|_{\mathscr{U}}\geq\|\theta^{\mathscr{U}}_{\lambda}\|_{L_2(\mathrm{I})}\|\sigma^{\mathscr{U}}_{\mu}\|_{H^1(\Omega)}$, and $\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k)=\vee_{\mathscr{U}}(\vee_a(\Lambda^{\mathscr{V}_1},0),k)$ it remains to be proven that

$$\|\big\langle\frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}}\otimes\frac{\frac{\partial}{\partial x_i}\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^1(\Omega)}},\Theta^a\otimes\Sigma^a\big\rangle_{L_2(\mathrm{I}\times\Omega)}\big|_{(\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\vee_a(\Lambda^{\mathscr{V}_1},0),k))\times\vee_a(\Lambda^{\mathscr{V}_1},0)}\|\lesssim 2^{-k/2}.$$

Indeed, this gives

$$\begin{aligned}&\|\Big(\big\langle b_i\frac{\partial}{\partial x_i}\Psi^{\mathscr{U}},\Psi^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I}\times\Omega)}\mathbf{c}\Big)\Big|_{\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k)}\|\\&=\|\Big(\big\langle\Psi^{\mathscr{U}},\Psi^a\big\rangle_{L_2(\mathrm{I}\times\Omega)}\mathbf{d}\Big)\Big|_{\vee_{\mathscr{U}}\setminus\vee_{\mathscr{U}}(\Lambda^{\mathscr{V}_1},k)}\|\lesssim 2^{-k/2}\|\mathbf{d}\|\lesssim 2^{-k/2}\|\mathbf{c}\|,\end{aligned}$$

showing the second inequality.

From $\Theta^{\mathscr{U}}/\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}$, Θ^a, $\Sigma^{\mathscr{U}}/\|\Sigma^{\mathscr{U}}\|_{H^1(\Omega)}$, and Σ^a being Riesz bases for $L_2(\mathrm{I})$, $L_2(\mathrm{I})$, $H^1_0(\Omega)$, and $L_2(\Omega)$, we have

$$\Big\langle\frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}},\Theta^a\Big\rangle_{L_2(\mathrm{I})}\in\mathcal{L}(\ell_2(\lhd_a),\ell_2(\lhd_{\mathscr{U}})),$$
$$\Big\langle\frac{\frac{\partial}{\partial x_i}\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^1(\Omega)}},\Sigma^a\Big\rangle_{L_2(\Omega)}\in\mathcal{L}(\ell_2(\diamond_{\mathscr{V}_a}),\ell_2(\diamond_{\mathscr{U}})).$$

The proof of the remaining inequality is completed by applications of Proposition 4.7.4, Lemma 4.7.2 for $(\Theta^*,\Theta^\circ) = (\frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}},\Theta^a)$, and the second statement from Lemma 4.7.5.

(c). A subset of the arguments that showed the second inequality gives third one. □

Lemma 4.7.8. *For $k\in\mathbb{N}_0$ and $1\le i\le n$, it holds that*

$$\Big\|\Big|\Big\langle \Sigma^{\mathscr{T}},\partial_i\Sigma^{\mathscr{V}_1}\Big\rangle_{L_2(\Omega)^n}\big|_{\{(\mu',\mu)\colon |\mu'|>|\mu|+k\}}\Big|\Big\|\lesssim 2^{-k/2},$$

Proof. We split the matrix into B^r+B^s, where B^r contains all its entries $\Big\langle \sigma^{\mathscr{T}}_{\mu'},\partial_i\sigma^{\mathscr{V}_1}_{\mu}\Big\rangle_{L_2(\Omega)^n}$ for which $\operatorname{supp}\sigma^{\mathscr{T}}_{\mu'}$ is contained in $\omega\cap\Omega$ for some $\omega\in\mathcal{O}_\Omega$ with $|\omega|=|\mu|$ (the 'regular' entries), and where B^s contains the remaining ('singular') entries.

For the regular entries using (s_4) we can estimate

$$\begin{aligned}|\langle\sigma^{\mathscr{T}}_{\mu'},\partial_i\sigma^{\mathscr{V}_1}_{\mu}\rangle_{L_2(\Omega)^n}| &\lesssim \|\sigma^{\mathscr{T}}_{\mu'}\|_{L_1(\Omega)^n}2^{-|\mu'|}|\sigma^{\mathscr{V}_1}_{\mu}|_{W^2_\infty(\operatorname{supp}\sigma^{\mathscr{T}}_{\mu'})}\\ &\eqsim 2^{(|\mu|-|\mu'|)(1+\frac{n}{2})}\end{aligned}$$

An application of Lemma 4.7.1 with $\xi=n$ and $\varrho=1$ shows that $\||B^r|\|\lesssim 2^{-k}$.

Since the wavelets $\sigma^{\mathscr{V}_1}_{\mu}$ are piecewise polynomial, and functions in $H^1(\Omega)$, they are contained in $W^1_\infty(\Omega)$. For the remaining singular entries we estimate

$$|\langle\sigma^{\mathscr{T}}_{\mu'},\partial_i\sigma^{\mathscr{V}_1}_{\mu}\rangle_{L_2(\Omega)^n}|\lesssim\|\sigma^{\mathscr{T}}_{\mu'}\|_{L_1(\Omega)^n}|\sigma^{\mathscr{V}_1}_{\mu}|_{W^1_\infty(\operatorname{supp}\sigma^{\mathscr{T}}_{\mu'})}\lesssim 2^{(|\mu|-|\mu'|)\frac{n}{2}}.$$

An application of Lemma 4.7.1 with $\xi=n-1$ and $\varrho=1/2$ shows that $\||B^s|\|\lesssim 2^{-k/2}$. □

The following Corollary will be used to bound $\|(\mathbf{r}_1-\tilde{\mathbf{r}}_1)|_{\vee_{\mathscr{T}}}\|$.

Corollary 4.7.9. *Let $\Lambda^{\mathscr{V}_1}\subset\vee_{\mathscr{V}_1}$ be a multi-tree. Then for $k\in\mathbb{N}_0$,*

$$\Big\|\Big\langle\Psi^{\vec{\mathscr{T}}},\nabla_x\Psi^{\mathscr{V}_1}\Big\rangle_{L_2(\mathrm{I}\times\Omega)^n}\Big|_{(\vee_{\mathscr{T}}\setminus\vee_{\mathscr{T}}(\Lambda^{\mathscr{V}_1},k))\times\Lambda^{\mathscr{V}_1}}\Big\|\lesssim 2^{-k/2}.$$

Proof. Using Lemma 4.7.2 for $(\Theta^*,\Theta^\circ)=(\Theta^{\mathscr{T}},\Theta^{\mathscr{V}_1})$, Lemma 4.7.8, $\big\langle\Theta^{\mathscr{T}},\Theta^{\mathscr{V}_1}\big\rangle_{L_2(\mathrm{I})}\in\mathcal{L}(\lhd_{\mathscr{V}_1},\lhd_{\mathscr{T}})$, and $\Big\langle\Sigma^{\mathscr{T}},\partial_i\Sigma^{\mathscr{V}_1}\Big\rangle_{L_2(\Omega)^n}\in\mathcal{L}(\diamond_{\mathscr{V}_1},\diamond_{\mathscr{T}})$, the proof follows from Proposition 4.7.4. □

Lemma 4.7.10. *For $k\in\mathbb{N}_0$, $1\le i\le n$, it holds that*

$$\begin{aligned}\Big\|\Big|\Big\langle\frac{\nabla\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^1(\Omega)}},\Sigma^a\mathbf{e}_i\Big\rangle_{L_2(\Omega)^n}\big|_{\{(\mu',\mu)\colon|\mu'|>|\mu|+k\}}\Big|\Big\|&\lesssim 2^{-k/2},\\ \Big\|\langle\Sigma^{\mathscr{T}},\Sigma^a\rangle_{L_2(\Omega)^n}\big|_{\{(\mu',\mu)\colon|\mu'|>|\mu|+k\}}\Big\|&\lesssim 2^{-k/2}.\end{aligned}$$

Proof. For proving the first inequality, we split the matrix into $B^r + B^s$, where B^r contains all its entries $\Big\langle \frac{\nabla \sigma^{\mathscr{U}}_{\mu'}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}}, \sigma^a_\mu \mathbf{e}_i \Big\rangle_{L_2(\Omega)^n}$ for which $\operatorname{supp} \sigma^{\mathscr{U}}_{\mu'}$ is contained in $\omega \cap \Omega$ for some $\omega \in \mathcal{O}_\Omega$ with $|\omega| = |\mu|$ (the 'regular' entries), and where B^s contains the remaining ('singular') entries.

For the regular entries, using (s_4) and the first inequality in (4.4.9), we estimate

$$\begin{aligned}
&\Big|\Big\langle \frac{\partial_i \sigma^{\mathscr{U}}_{\mu'}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}}, \sigma^a_\mu \Big\rangle_{L_2(\Omega)^n}\Big| = \Big|\Big\langle \frac{\sigma^{\mathscr{U}}_{\mu'}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}}, \partial_i \sigma^a_\mu \Big\rangle_{L_2(\Omega)^n}\Big| \\
&\lesssim 2^{-|\mu'|} \frac{\|\sigma^{\mathscr{U}}_{\mu'}\|_{L_1(\Omega)}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}} |\sigma^a_\mu|_{W^2(\operatorname{supp}\sigma^{\mathscr{U}}_{\mu'})} \lesssim 2^{-|\mu'|} 2^{-|\mu'|n/2} 2^{-|\mu'|} 2^{|\mu|n/2} 2^{2|\mu|} \\
&= 2^{(2+n/2)(|\mu'|-|\mu|)}.
\end{aligned}$$

An application of Lemma 4.7.1 with $\xi = n$ and $\varrho = 2$ shows that $\|B^r\| \lesssim 4^{-k}$.

For the singular entries, we estimate

$$\Big|\Big\langle \frac{\partial_i \sigma^{\mathscr{U}}_{\mu'}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}}, \sigma^a_\mu \Big\rangle_{L_2(\Omega)^n}\Big| \lesssim \frac{\|\partial_i \sigma^{\mathscr{U}}_{\mu'}\|_{L_1(\Omega)}}{\|\sigma^{\mathscr{U}}_{\mu'}\|_{H^1(\Omega)}} \|\sigma^a_\mu\|_{L_\infty(\Omega)} \lesssim 2^{(|\mu'|-|\mu|)n/2}.$$

An application of Lemma 4.7.1 with $\xi = n-1$ and $\varrho = 1/2$ shows that $\||B^s|\| \lesssim 2^{-k/2}$.

The proof of the second inequality proceeds along the by now well-known steps. Using assumption (s_4) on $\Sigma^{\mathscr{T}}$ one shows that $\||B^r|\| \lesssim 2^{-k}$, whereas $\||B^s|\| \lesssim 2^{-k/2}$. □

The following Corollary will be used to bound $\|\mathbf{r}_3 - \tilde{\mathbf{r}}_3\|$.

Corollary 4.7.11. *Let $\Lambda^a \subset \vee_a$ be a multi-tree. Then for $k \in \mathbb{N}_0$,*

$$\Big\| \big\langle \nabla_x \Psi^{\mathscr{U}}, \Psi^a \mathbf{e}_i \big\rangle_{L_2(\mathrm{I}\times\Omega)^n}\big|_{(\vee_{\mathscr{U}} \setminus \vee_{\mathscr{U}}(\Lambda^a,k))\times\Lambda^a} \Big\| \lesssim 2^{-k/2}.$$

$$\Big\| \big\langle \Psi^{\vec{\mathscr{T}}}, \Psi^a \mathbf{e}_i \big\rangle_{L_2(\mathrm{I}\times\Omega)^n}\big|_{(\vee_{\vec{\mathscr{T}}} \setminus \vee_{\vec{\mathscr{T}}}(\Lambda^a,k))\times\Lambda^a} \Big\| \lesssim 2^{-k/2}.$$

Proof. From $\|\theta^{\mathscr{U}}_\lambda \otimes \sigma^{\mathscr{U}}_\mu\|_{\mathscr{U}} \geq \|\theta^{\mathscr{U}}_\lambda\|_{L_2(\mathrm{I})} \|\sigma^{\mathscr{U}}_\mu\|_{H^1(\Omega)}$, in order to prove the first result it suffices to show that

$$\Big\| \Big\langle \frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}} \otimes \frac{\nabla \Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^1(\Omega)}}, \Theta^a \otimes \Sigma^a \mathbf{e}_i \Big\rangle_{L_2(\mathrm{I}\times\Omega)^n}\Big|_{\vee_{\mathscr{U}} \setminus \vee_{\mathscr{U}}(\Lambda^a,k)\times\Lambda^a} \Big\| \lesssim 2^{-k/2}.$$

This and the second result follow from applications of Proposition 4.7.4, Lemma 4.7.2, and Lemma 4.7.10. □

5. On the awgm implementation

In this chapter we aim to demonstrate the developments towards an implementation of the Adaptive Wavelet Galerkin Method (**awgm**) scheme described in Sect. 2.3 and Sect. 4.3, and applied to the elliptic (stationary case) and parabolic (non-stationary or time dependent case) partial differential equations reformulated as first order system least squares (**FOSLS**). These formulations were presented in detail in Chapter 2 and Chapter 4. The developments consist of relevant definitions, the design and possible implementation of algorithms with the focus on obtaining optimal computational complexity.

5.1 Introduction

The **awgm** primarily builds on a sufficiently accurate finite approximation of the generally infinite residual. Furthermore, it employs a bulk chasing strategy to determine subsequent nested approximation spaces based on an a posteriori error indicator given by the residual. At last, it solves the arising Galerkin systems that correspond to those approximation spaces.

Optimality of the **awgm** is characterised by two features, best convergence rate and linear computational cost. The solution u of an operator equation $F(u) = 0$ belonging to the approximation class $\mathcal{A}^s$ (see Def. 2.3.1) means it can be approximated from the spans of subsets of the wavelet basis at a rate s. When the number of arithmetic operations is an absolute multiple of the cardinality of a wavelet subset Λ, then the computational time taken is a linear function of $\#\Lambda$.

A primary constraint on the support of approximations to u turns out to be the restriction to a tree structure. This makes the approximation class $\mathcal{A}^s$ only slightly smaller (cf. [CDDD01], see [CDD03b] for details), but from a computational point of view it forms the ground for a more efficient application of matrix-vector multiplications arising in the adaptive solver. Tree approximation even constitutes a necessity for realising optimal computational complexity in case of a non-linear operator. It allows for a transition from a multilevel to a single-scale representation of a function u in linear complexity. A similar approach has been taken earlier in [DSX00, CDD03b, XZ05, BU08, Vor09]. In consequence, knowing how to apply a multi- to single-scale transformation in $\mathcal{O}(\#\Lambda)$ operations, one can compute the ap-

plication of a stiffness matrix $a(\Psi, \Psi)$ w.r.t. a wavelet basis Ψ by first applying the multi- to single-scale transformation, then, with Φ denoting the single-scale basis, applying $a(\Phi, \Phi)$, whose individual entries one can compute in $\mathcal{O}(1)$ operations, and eventually returning to wavelet coordinates by applying the adjoint transformation.

The outline of this chapter is as follows: In Sect. 5.2 we give a brief overview of the wavelets used in our applications.

In Sect. 5.3 we give a series of necessary definitions. We discuss the definition a wavelet tree and that of a tiling to facilitate an efficient implementation of multi- to single-scale transformations. The creation of a tiling goes through the construction of an element tree, which is therefore defined subsequently. This section ends with definitions of extended wavelet (single or double) trees, and algorithms for producing them in linear complexity.

In Sect. 5.4, following a prior rigorous work in [KS14], we present three algorithms for fast-evaluation of system matrices w.r.t wavelet bases, applying either the full matrix, or its upper- or lower-triangular part, respectively. We adapt the first algorithm to overcome its limitation of being only applicable to bilinear forms. Proofs of correctness, and of a linear operation count of the algorithms are also included. A procedure to apply tensor-product operators completes this section.

In Sect. 5.5 we describe briefly the main data structures used in our implementations.

In Sect. 5.6 and 5.7, all the aforementioned parts find their use in the development of efficient implementations of the **awgm** applied to an elliptic semi-linear, and a parabolic evolution partial differential equation, both formulated as first order system least squares problems. Those implementations have produced the numerical results found in Sect. 2.6 and Sect. 4.5.

5.2 Wavelets bases: A brief presentation of collections in use

In our wavelet approximation methods we consider wavelet collections, commonly denoted as $\Psi = \{\psi_\lambda : \lambda \in \vee\}$, in one and two dimensions, which are piecewise polynomial functions of a given polynomial degree. When appropriately scaled, these collections form Riesz bases for a range of Sobolev spaces.

5.2.1 Haar wavelets

The collection of Haar wavelets will be considered on the domain $\Omega = (0,1)$. It consists of the union of the function $\psi_{0,0} \equiv 1$, and, for $\ell \in \mathbb{N}$ and $k = 0, \ldots, 2^{\ell-1} - 1$, the functions

$$\psi_{\ell,k} := 2^{\frac{\ell-1}{2}} \psi(2^{\ell-1}(\cdot) - k),$$

where $\psi \equiv 1$ on $[0, \frac{1}{2}]$ and $\psi \equiv -1$ on $(\frac{1}{2}, 1]$. We write a wavelet index $\lambda = (\ell, k)$ and let $|\lambda| = \ell$ denote its level. Consequently, a Haar wavelet at level $\ell > 0$ and with index k will be a non-zero piecewise constant function on the interval $2^{-(\ell-1)}[k, k+1]$. One may verify that the Haar wavelets form an *orthonormal discontinuous piecewise constant* wavelet basis for $L_2(\Omega)$.

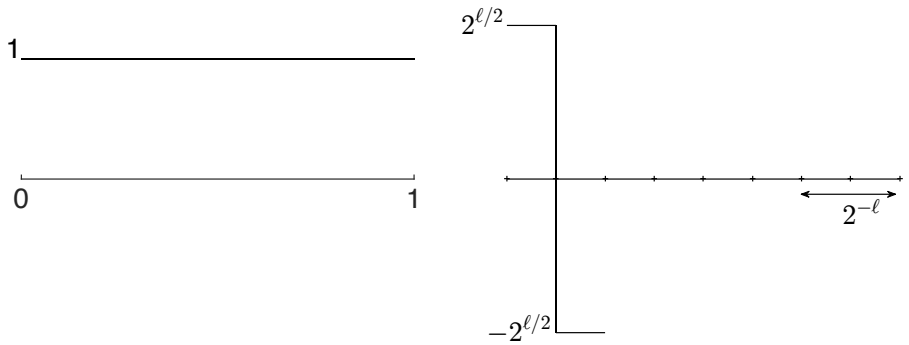


Figure 5.1: $L_2(\Omega)$-orthonormal Haar wavelets (left: scaling function on coarsest level, right: $2^{\ell-1}$ wavelets for $\ell = 1, \cdots$)

5.2.2 Orthonormal discontinuous piecewise linear wavelets

A collection of *orthonormal discontinuous piecewise linear* wavelets was constructed on $\Omega = (0,1)$, starting with $\psi_{0,0} \equiv 1$ and following a *Gram-Schmidt* process. Noting that $P_1(0,1)$ has dimension two, the second wavelet on level 0 is $\psi_{0,1}(t) = 2\sqrt{3}(t - \frac{1}{2})$ (as

seen in Figure 5.2, left). Similarly, a basis for the space of $\prod_{i=0}^{2^J-1} P_1(i2^{-J},(i+1)2^{-J})$ is given by $\{\psi_{0,0},\psi_{0,1}\}\cup\{\psi_{\ell,k} : 1\le\ell\le J, 0\le k\le 2^\ell-1\}$, where

$$\psi_{\ell,k}(t)=\begin{cases} 2^{\ell/2}(1-3(2^\ell t-k)), & t\in 2^{-\ell}[k,k+1]\\ 2^{\ell/2}(2+(2^\ell t-k)), & t\in 2^{-\ell}[k+1,k+2]\end{cases}$$

if k is even (Figure 5.2, right in black) and

$$\psi_{\ell,k}(t)=\begin{cases} \sqrt{3}2^{\ell/2}(1-2(2^\ell t-k)), & t\in 2^{-\ell}[k-1,k]\\ -\sqrt{3}2^{\ell/2}(1-2(2^\ell t-k)), & t\in 2^{-\ell}[k,k+1]\end{cases}$$

if k is odd (Figure 5.2, right in red).

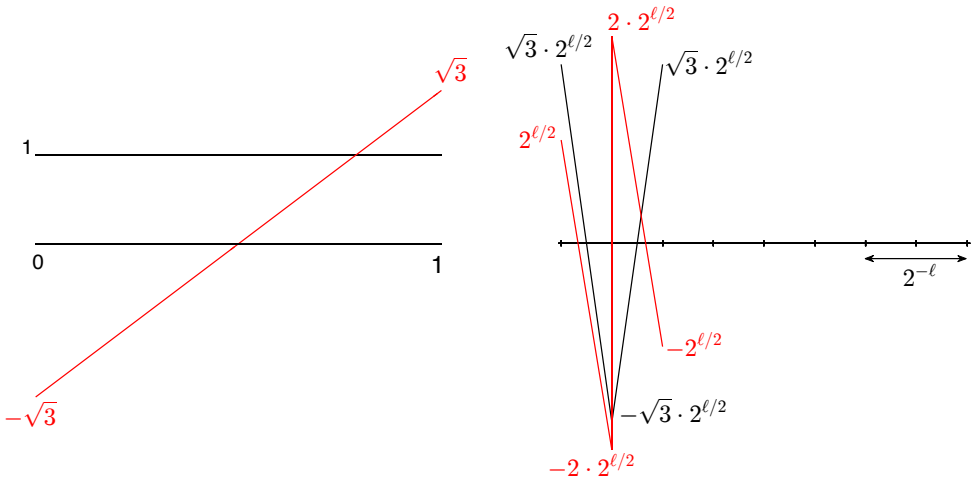


Figure 5.2: $L_2(\Omega)$-orthonormal discontinuous piecewise linear wavelets.

5.2.3 Continuous piecewise linear 3-point wavelets in 1D

5.2.3.1 The collection without boundary conditions

With $\Omega=[0,1]$ we consider a *continuous piecewise linear* 3*-point* wavelet collection without boundary conditions that for a set including all wavelets up to level J spans the space $H^1(\Omega)\cap\prod_{i=0}^{2^J-1} P_1(i2^{-J},(i+1)2^{-J})$. Properly scaled it provides a basis for $L_2(\Omega)$ as well as for $H^1(\Omega)$. For proofs and details of construction we refer to [Ste96].

We equip level 0 with scaling functions $\psi_{0,0}=1$ and $\psi_{0,1}(t)=2t-1$. Then for $\ell\in\mathbb{N}$ and $k=1,\dots,2^\ell-1$, with the exception of level 1 where $\psi_{1,1}(t)=4t-1$ if $t\in[0,\frac12]$ and $\psi_{1,1}(t)=-2t+2$ if $t\in[\frac12,1]$, the masks of $\psi_{\ell,k}$, w.r.t. linear nodal basis and with nodal points being in $\{0,2^{-\ell},2\cdot2^{-\ell},\cdots,1\}$, are given by $[-1\quad 1\quad -\frac12]$ if $k=1$, $[-\frac12\quad 1\quad -1]$ if $k=2^\ell-1$ and $[-\frac12\quad 1\quad -\frac12]$ otherwise.

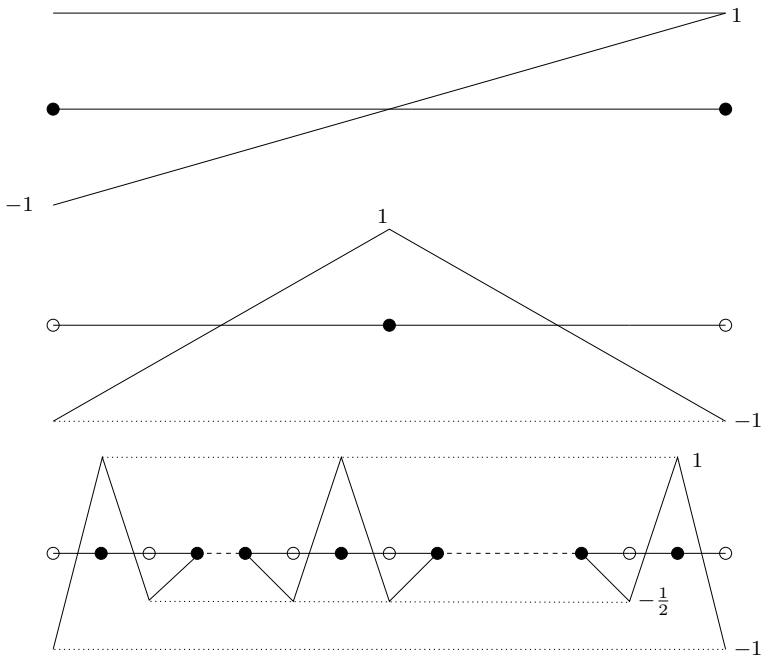


Figure 5.3: Continuous, piecewise linear wavelets without boundary conditions.

5.2.3.2 The collection with boundary conditions

Applying the same approach, one can construct a continuous piecewise linear 3-point wavelet collection vanishing at $t = 0$. A set including all wavelets up to level J spans the space $\left(\prod_{i=0}^{2^J-1} P_1(i2^{-J}, (i+1)2^{-J})\right) \cap H^1_{0,\{0\}}(\Omega)$. With appropriate scaling this collection gives wavelet bases for $L_2(\Omega)$ and $H^1_{0,\{0\}}(\Omega)$.

The coarsest level is equipped with $\psi_{0,0}(t) = t$, while level 1 consists of $\psi_{1,1}(t) = 2t$ if $t \in [0, \frac{1}{2}]$ and $\psi_{1,1}(t) = -2t+2$ if $t \in [\frac{1}{2}, 1]$. Then, with $\ell \in \mathbb{N}$, $\ell > 1$, $k = 1, \ldots, 2^\ell - 1$ and with nodal points in $\{0, 2^{-\ell}, 2 \cdot 2^{-\ell}, \cdots, 1\}$, the masks now read as $[1 \; -\frac{1}{2}]$ if $k = 1$, and $[-\frac{1}{2} \quad 1 \quad -1]$ if $k = 2^\ell - 1$, and $[-\frac{1}{2} \quad 1 \quad -\frac{1}{2}]$ otherwise.

5.2.4 Continuous piecewise linear 3-point wavelets in 2D

A collection of *continuous piecewise linear 3-point hierarchical wavelets* can be constructed on general meshes in two dimensions [Ste98b]. Scaled accordingly, in our applications, the collection with zero boundary conditions generates a basis for the space $H^1_0(\Omega)$, or a basis for $L_2(\Omega)$.

We describe briefly the construction from [Ste98b]. We consider a general polygon $\Omega \subset \mathbb{R}^2$, and an initial *triangulation* $\mathcal{T}_0$ of $\overline{\Omega}$, i.e. a conforming partition of $\overline{\Omega}$ into

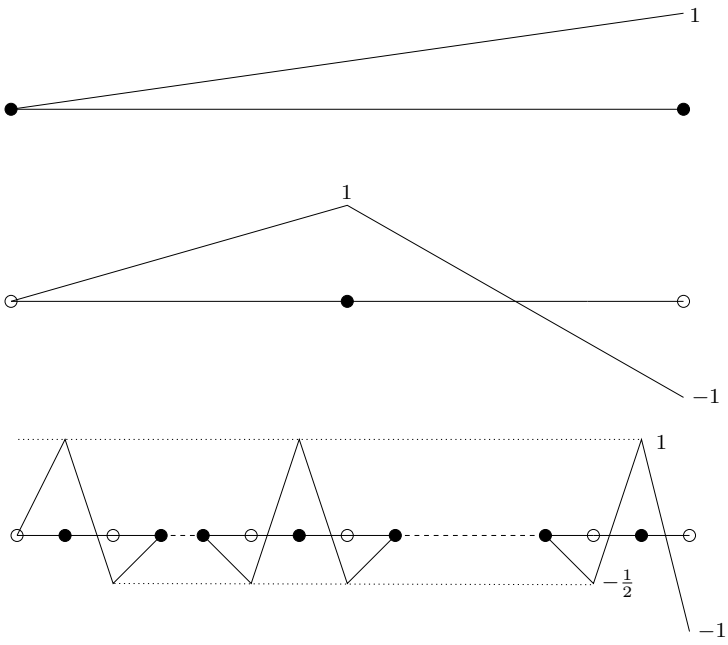


Figure 5.4: Continuous, piecewise linear wavelets, zero at the left boundary.

triangles. A sequence of nested finite element spaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset H$ is considered, with $\mathcal{V}_\ell$ corresponding to a partition $\mathcal{T}_\ell$ of $\overline{\Omega}$, at level ℓ, w.r.t to triangle splitting by connecting the mid-points of the edges of triangles $T \in \mathcal{T}_{\ell-1}$. All the vertices of triangles in $T \in \mathcal{T}_\ell$, also known as *nodes*, are collected in Ω_ℓ. If zero boundary conditions are imposed, we exclude the nodes lying on $\partial\Omega$.

We define a discrete scalar product on $\mathcal{V}_\ell$ with the use of the trapezoidal rule on a triangle by $(u,v)_{\mathcal{T}_\ell} = \sum_{T\in\mathcal{T}_\ell} \frac{\mathrm{vol}(T)}{3} \sum_{x\in T} u(x)\overline{v(x)}$. With $\mathcal{S}_0 = \mathcal{V}_0$ and for $\ell \geq 0$, a multiscale decomposition, i.e. a sequence of spaces $\mathcal{S}_\ell$ such that $\mathcal{V}_J = \sum_{\ell=0}^{J} \mathcal{S}_\ell$ and $\mathcal{S}_\ell \subset \mathcal{V}_\ell$, is defined by

$$\mathcal{S}_{\ell+1} = \mathcal{V}_{\ell+1} \ominus^{\perp_{(\cdot,\cdot)_{\mathcal{T}_{\ell+1}}}} \mathcal{V}_\ell.$$

Subsequently, the space $\mathcal{V}_\ell$ is equipped with the linear nodal basis $\{\phi_{\ell,x} : x \in \Omega_\ell\}$. We equip $\mathcal{S}_0$ with the same basis as $\mathcal{V}_0$, and, for $\ell \geq 1$, the space $\mathcal{S}_\ell$ with $\{\psi_{\ell,x} : x \in \Omega_\ell \setminus \Omega_{\ell-1}\}$ where

$$\psi_{\ell,x} = \phi_{\ell,x} - \sum_{y\in\Omega_{\ell-1}} \frac{(\phi_{\ell,x}, \phi_{\ell-1,y})_{\mathcal{T}_\ell}}{(\phi_{\ell,y}, \phi_{\ell-1,y})_{\mathcal{T}_\ell}} \phi_{\ell,y}.$$

It follows that each wavelet $\psi_{\ell,x}$ is a linear combination of at most three nodal basis function. In particular, away from the boundary, a wavelet at level ℓ is a linear combination of two nodal basis functions corresponding to vertices of a single edge of

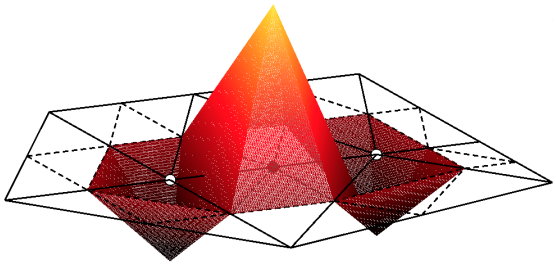


Figure 5.5: A continuous piecewise linear 3-point wavelet, away from the boundary, at level 1. This wavelet consists of a linear combination of three scaling functions, associated to three nodes lying on the edge of a triangle $T \in \mathcal{T}_0$ ($\mathcal{T}_0$ is drawn in black solid lines). The middle node is a vertex in Ω_1 and is indicated with a '$\bullet$', while the nodes lying in the coarser level are indicated with a blank point '$\circ$'.

a triangle $T \in \mathcal{T}_{\ell-1}$, together with the nodal basis function on the mid-point of this edge which is a node in Ω_ℓ (see Figure 5.5).

5.2.5 Continuous piecewise quadratic wavelets in 2D

In Chapter 3, we presented a collection of *continuous piecewise quadratic wavelets* constructed on general polygons $\Omega \subset \mathbb{R}^2$, with each wavelet being a linear combination of 11 or 13 nodal basis functions. In our applications, properly scaled they generate Riesz bases for $H^{-1}(\Omega)$, $L_2(\Omega)$, and $H_0^1(\Omega)$.

For details of construction and definition of the wavelets we refer to Sect. 3.3.5. In Figure 5.6 we illustrate the two main wavelet types whose supports were pictured in Figure 3.5 of the aforementioned section.

5.3 Definition and construction of trees

5.3.1 The wavelet tree

Admissible approximations of functions for the **awgm** are finite linear combinations of wavelets with the corresponding set of indices organised as a tree. The definition of a *wavelet tree* with respect to a wavelet collection Ψ is given below.

Definition 5.3.1. Let $\vee$ be an infinite set of indices so that $\Psi = \{\psi_\lambda : \lambda \in \vee\}$. To each $\lambda \in \vee$ with $|\lambda| > 0$, where $|\lambda|$ denotes the level of the wavelet, we associate one $\mu \in \vee$ with $|\mu| = |\lambda| - 1$ and $\operatorname{meas}(\operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_\mu) > 0$. We call μ the *parent* of λ, and so λ a *child* of μ.

Moreover, to each $\lambda \in \vee$, we associate some neighborhood $\mathcal{S}(\psi_\lambda)$ of $\operatorname{supp}\psi_\lambda$, with diameter $\lesssim 2^{-|\lambda|}$, such that $\mathcal{S}(\psi_\lambda) \subset \mathcal{S}(\psi_\mu)$ when λ is a child of μ.

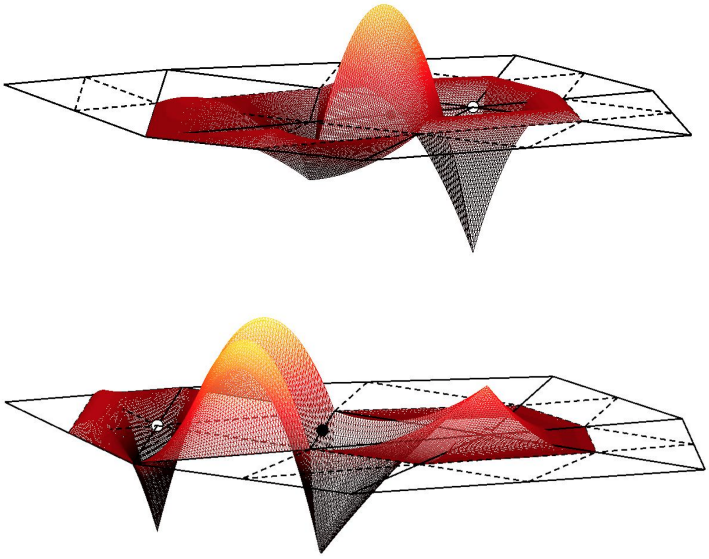


Figure 5.6: Continuous piecewise quadratic wavelets on level 1, consisting of a linear combination of 11 (top) and 13 (bottom) basis functions respectively (cf. Figure 3.5). In terms of the explicit formula given in Sect. 3.3.5, the wavelet on top is a linear combination of one function $\boldsymbol{\xi}$ (center indicated with a '$\bullet$') and one function $\boldsymbol{\theta}$ (center indicated with '$\circ$'), while the wavelet at the bottom consists of one $\boldsymbol{\xi}$ and two $\boldsymbol{\theta}$'s.

A finite $\Lambda \subset \vee$ is called a *wavelet tree*, if it contains all $\lambda \in \vee$ with $|\lambda| = 0$, as well as the parent of any $\lambda \in \Lambda$ with $|\lambda| > 0$. We denote by Λ_ℓ its subset with all $\lambda \in \Lambda$ with $|\lambda| = \ell$.

Note that when wavelets on the same level have overlapping supports, then for each wavelet generally there are more than one candidates to meet the requirements for being its parent.

In other works concerning tree approximation, the definition of a tree sometimes includes *multiple* parents assigned to a wavelet such that the support of a child is covered by the union of the supports of its parents ([KS14]), or a similar coverage is achieved by the formation of a *graded tree* (for this approach see [Ste14]). This *nestedness* of the supports is a necessary characteristic of the tree structure in view of the application of multi- to single-scale transformations in linear complexity. For reasons of efficiency and control of the computational costs we opt for *single* parents, and stretch the notion of nestedness with the use of the $\mathcal{S}$-neighborhood.

To capture the benefit from the use of the $\mathcal{S}$-neighborhood, we recall here Lemma 2.5.7 without repeating the proof.

Lemma 5.3.2. *Let $\overline{\Omega} = \Omega_0 \supseteq \Omega_1 \supseteq \cdots$ with Ω_ℓ being subdomains of Ω. Then*

$$\Lambda := \big\{\lambda \in \vee\colon \operatorname{meas}(\mathcal{S}(\psi_\lambda) \cap \Omega_{|\lambda|}) > 0\big\}$$

is a tree.

The most representative case to reveal the significance of the single parent property is lying on the adaptive mechanism. For instance, let $\Lambda \subset \vee$ be the occurring set of indices after the enlargement by a bulk chasing procedure, meaning the set carries the wavelet indices which corresponded to the largest in modulus components of the approximate residual. This set is often not a tree and must be extended to one by adding the missing parents. Hence, the density of the new approximation set is greatly influenced by the choice of the number of parents. For the goal of an optimal error reduction rate, it is desired to have a fine control of the density of these sets, in other words to maintain admissibility of Λ under (near-) minimal cardinality.

5.3.2 Tilings and the element tree

The wavelet collections in consideration are constructed in sequences of Lagrange finite-element spaces w.r.t. uniformly dyadically refined nested sequences of underlying partitions starting from an initial partition of a polytopal domain Ω. We assume there exists a collection of closed polytopes shared by all wavelets as in (w_1) given in Sect. 2.5.2 and that (w_2)-(w_4) are satisfied.

Definition 5.3.3. A collection $\mathcal{T} \subset \mathcal{O}_\Omega$ such that $\overline{\Omega} = \cup_{\omega\in\mathcal{T}}\omega$, and for $\omega_1 \neq \omega_2 \in \mathcal{T}$, $\operatorname{meas}(\omega_1 \cap \omega_2) = 0$ will be called a *tiling*. With $\mathcal{P}_m(\mathcal{T})$, we denote the space of piecewise polynomials of degree m w.r.t. $\mathcal{T}$. The smallest common refinement of tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ is denoted as $\mathcal{T}_1 \oplus \mathcal{T}_2$.

As a consequence of the assumption (w_2), wavelet supports are inherently decomposed in polytopes belonging to $\mathcal{O}_\Omega$. The consideration of tilings then reveals the contribution of the tree structure in transforming a function given by a linear combination of wavelets to a locally finite single scale representation while the amount of work needed scales linearly with the number of wavelets. Such transformations are included in the algorithms presented in Sect. 5.4. A proof of the following proposition is given in [Ste14, §4.3].

Proposition 5.3.4. *Given a tree $\Lambda \subset \vee$, there exists a tiling $\mathcal{T}(\Lambda) \subset \mathcal{O}_\Omega$ with $\#\mathcal{T}(\Lambda) \lesssim \#\Lambda$ such that $\operatorname{span}\{\psi_\lambda\colon \lambda \in \Lambda\} \subset \mathcal{P}_m(\mathcal{T}(\Lambda))$. Moreover, equipping $\mathcal{P}_m(\mathcal{T}(\Lambda))$ with a basis of functions, each of which supported in ω for one $\omega \in \mathcal{T}(\Lambda)$, the representation of this embedding, known as the multi- to single-scale transform, can be applied in $\mathcal{O}(\#\Lambda)$ operations.*

In order to create the tiling $\mathcal{T}(\Lambda)$, exploiting Lemma 5.3.2 we would like to construct *level-wise* sets of elements w.r.t to the subdomains Ω_ℓ. This leads us to define the concept of an *element tree*.

For simplicity we confine our discussion to two dimensions and take the aforementioned polytopes as being triangles. Starting with $\mathcal{T}_0 = \overline{\Omega}$, for $\ell \in \mathbb{N}_0$ we consider $\mathcal{T}_\ell$ to be partitions of Ω, with triangular elements, corresponding to uniform *red-refinement*, i.e. the element splitting by connecting the mid-points of the edges. By red-refinement each triangular element T has four children. We call ℓ the level of the partition $\mathcal{T}_\ell$. With $\mathfrak{T} := \{T \in \mathcal{T}_\ell : \ell \in \mathbb{N}_0\}$, for a set $\Sigma \subset \mathfrak{T}$ we define

$$\texttt{child}(\Sigma) := \{T \in \mathfrak{T} \colon T \text{ is a child of a } T' \in \Sigma\},$$

and when $\Sigma \subset \mathfrak{T} \setminus \mathcal{T}_0$,

$$\texttt{parent}(\Sigma) := \{T \in \mathfrak{T} \colon T \text{ is the parent of a } T' \in \Sigma\}.$$

Definition 5.3.5. A multilevel set of elements $\Sigma \subset \mathfrak{T}$ is an *element tree*, when

$$\texttt{parent}(\Sigma \setminus \mathcal{T}_0) \cup \texttt{child}(\texttt{parent}(\Sigma \setminus \mathcal{T}_0)) \cup \mathcal{T}_0 \subset \Sigma.$$

Note that in the preceding definition an element tree must include a union of three sets. The first set provides that the parent of each element is included in Σ. The second requires that all siblings of an element belong to Σ. The third inclusion means that Σ contains the initial partition $\mathcal{T}_0$, i.e. its *roots*.

After a prerequisite definition that links the wavelet levels with that of a partition, in the next proposition we sketch the construction of an element tree arising from a wavelet tree.

Definition 5.3.6. Given a wavelet tree $\Lambda \subset \vee$, the level of each wavelet coincides with the level of the coarsest uniform partition with respect to which it is a piecewise polynomial of the specified degree.

Lemma 5.3.7. *Given a wavelet tree $\Lambda \subset \vee$, valid $\mathcal{S}$-neighborhoods are given by*

$$\mathcal{S}(\psi_\lambda^*) := \{\texttt{child}(T), T \in \mathcal{T}_{|\lambda|-1} \colon |T \cap \hat{\mathcal{S}}(\psi_\lambda^*)| > 0\}. \tag{5.3.1}$$

where $\hat{\mathcal{S}}$ is a (preferably small) neighborhood of $\operatorname{supp}\psi_\lambda$ *with* $\operatorname{diam}(\hat{\mathcal{S}}(\psi_\lambda)) \lesssim 2^{-|\lambda|}$, *such that $\mathcal{S}(\psi_\lambda) \subset \mathcal{S}(\psi_{\texttt{parent}(\lambda)})$ holds $\forall \lambda \in \Lambda$. For $|\lambda| = 0$ one can take $\mathcal{S}(\psi_\lambda) := \overline{\Omega}$.*

Proposition 5.3.8. *Let a wavelet tree $\Lambda \subset \vee$. Then, with the $\mathcal{S}$-neighborhood defined as in Lemma 5.3.7, and with $\ell \in \mathbb{N}_0$, the set*

$$\Sigma(\Lambda) := \bigcup_{\ell \geq 0} \bigcup_{\lambda \in \Lambda_\ell} \{T \in \mathcal{T}_\ell \colon |T \cap \mathcal{S}(\psi_\lambda)| > 0\}$$

is an element tree with $\#\Sigma(\Lambda) \lesssim \#\Lambda$.

Proof. The set $\Sigma(\Lambda)$ is an element tree if it satisfies Definition 5.3.5. When $\ell = 0$ by definition $\mathcal{S}(\psi_\lambda) := \overline{\Omega}$ and thus $\mathcal{T}_0 \subset \Sigma(\Lambda)$. One easily deduces that $\texttt{parent}(\Sigma(\Lambda) \setminus \mathcal{T}_0)$, and $\texttt{child}(\texttt{parent}(\Sigma(\Lambda) \setminus \mathcal{T}_0))$ are included in $\Sigma(\Lambda)$ by definition of $\mathcal{S}$.

The support of each wavelet contains a uniformly bounded number of triangles and so does its $\mathcal{S}$-neighborhood. From this we conclude that $\#\Sigma(\Lambda) \lesssim \#\Lambda$. □

Remark 5.3.9. A reasonable approach to keep the area of $\hat{\mathcal{S}}(\psi_\lambda)$ close to its minimum is to choose $\texttt{parent}(\lambda)$ to be such that $\text{meas}(\hat{\mathcal{S}}(\psi_\lambda) \cap \hat{\mathcal{S}}(\psi_{\texttt{parent}(\lambda)}))$ is maximal. A direct verification for all the two dimensional wavelet collections presented in Sect. 5.2 shows that a suitable $\hat{\mathcal{S}}$ is given by

$$\hat{\mathcal{S}}(\psi_\lambda) := \operatorname{supp} \psi_\lambda.$$

Definition 5.3.10. Given an element tree Σ, we define the tiling $\mathcal{T}(\Sigma)$ as the collection of *leaves* of Σ, where the *leaves* are the elements without children in this set.

To construct the sets of leaves one has to remove those elements that have children in the set, and thus the set will eventually contain *all* the last generation of children from each tree branch. This means the covering property is kept, and that all pairs of remaining elements do not have a mutual overlap with positive measure. The opposite procedure is followed in the next definition.

Definition 5.3.11. Given a tiling $\mathcal{T}$, we define the element tree $\Sigma(\mathcal{T})$ as the collection that is generated by adding all ancestors to any $T \in \mathcal{T}$.

5.3.3 The extended wavelet tree.

A motivation for the proposed schemes on the approximate evaluation of the residual in wavelet coordinates was to provide an alternative for the standard way of approximation, that involves a separate approximation of the right hand side and that of the application of the operator using the **APPLY**-routine (see Sect. 2.1.1). If at a given iteration of the **awgm** the approximate solution is supported on a wavelet set Λ, it is shown that when evaluating the residual as a whole, it suffices at step (R) to approximate the residual on some neighborhood of Λ and still obtain the optimal error reduction. Such neighborhoods are given in the next definition for single trees, whereas a multi-dimensional version is described in the next section.

Definition 5.3.12. Given an element tree Σ and a $k \in \mathbb{N}$, we define the *extended wavelet tree* as

$$\Lambda^*(\Sigma, k) := \{\lambda \in \vee^* \colon |\mathcal{S}(\psi_\lambda) \cap T| > 0 \text{ for some } T \in \Sigma \text{ with } |T| = \max(|\lambda| - k, 0)\}$$

In order to *create* $\Lambda^*(\Sigma, k)$ level-by-level, we need the sets

$$\Sigma_\ell := \{T \in \Sigma \colon |T| = \ell\}. \tag{5.3.2}$$

An algorithm `growtree` that given an element tree produces the extended tree Λ is given below.

$[\Lambda^*] := \texttt{growtree}(\Sigma, k)$
% *Input: element tree* Σ, $k \in \mathbb{N}$.
% *Output: extended tree* Λ^*.

$\Lambda^* := \{\lambda \in \vee, |\lambda| = 0\}$ *% Initialise* Λ^* *with all wavelet indices on level* 0

```
for ℓ = 0 to max_{T∈Σ} |T|
    for T ∈ Σ_ℓ
        m = 1
        while m ≤ k
            add to Λ* all λ ∈ ∨* with |λ| = ℓ + m and |S(ψ_λ) ∩ T| > 0
            m = m + 1
        endwhile
    endfor
endfor
```

Theorem 5.3.13. *A call of* `growtree` *yields the output as specified, at the cost of* $\mathcal{O}(\#\Sigma)$ *operations.*

Proof. As a consequence on locality of wavelets in $\vee^*$ (cf. (w_2)-(w_3)), each element at a certain level can only intersect a uniformly bounded number of wavelets on the finer level, thus, $\#\Lambda \lesssim \#\Sigma$. After running through all elements of the element tree, and with use of Lemma 5.3.2, one easily verifies that Λ^* is produced as specified. We conclude that the total operations cost is $\mathcal{O}(\#\Sigma)$. □

5.3.4 The double-tree and its k^{th}-neighborhood.

Let us now define a multi-dimensional version of the tree concept. With Ψ now reading as

$$\Psi := \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} := \otimes_i \psi^{(i)}_{\lambda_i} : \boldsymbol{\lambda} \in \vee := \prod_i \vee_i\}.$$

we denote a collection of tensor product wavelets where $\boldsymbol{\lambda}$ is now a *multi-index*. In view of our forthcoming applications, w.l.o.g. hereafter we will consider $i \in \{1, 2\}$ and hence a two-dimensional multi-tree, called *double-tree*.

To prepare for the definition of a double-tree, we define $\neg i$ to be the value in $\{1, 2\}$ unequal to i. Furthermore, if $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ is a pair of indices, the projector onto the i^{th} argument is $P_i(\lambda_1, \lambda_2) := \lambda_i$.

Definition 5.3.14. For $\boldsymbol{\Lambda} \subset \vee = \vee_1 \times \vee_2$ we call $\boldsymbol{\Lambda}$ a *double-tree* when for $i \in \{1, 2\}$ and any $\mu \in P_{\neg i}\boldsymbol{\Lambda}$, the fiber

$$\boldsymbol{\Lambda}_{i,\mu} := P_i(P_{\neg i}|_{\boldsymbol{\Lambda}})^{-1}\{\mu\}$$

is a tree (in $\vee_i$). That is, $\boldsymbol{\Lambda}$ is a double-tree when 'frozen' in any of its coordinates, at any value of that coordinate, it is a tree in the remaining coordinate.

From $\boldsymbol{\Lambda} = \cup_{\mu \in P_{\neg i}\boldsymbol{\Lambda}}(P_{\neg i}|_{\boldsymbol{\Lambda}})^{-1}\{\mu\}$, we have $P_i\boldsymbol{\Lambda} = \cup_{\mu \in P_{\neg i}\boldsymbol{\Lambda}}\boldsymbol{\Lambda}_{i,\mu}$, which, being a union of trees, is a tree itself. Moreover, we set $\boldsymbol{\Lambda}_i := P_i\boldsymbol{\Lambda}$.

In Definition 5.3.12 we presented the k^{th}-neighborhood of a (single) tree, also called the extended tree. Analogously, we will construct the k^{th}-neighborhood of a double-tree applying unidirectional operations following the same principles. Let us first give a definition for the extended double-tree.

Definition 5.3.15. For a double-tree $\mathbf{\Lambda} \subset \vee$, and $k \in \mathbb{N}_0$, we define its *k-neighborhood in* $\bar{\nabla} = \bar{\vee}_1 \times \bar{\vee}_2$ by

$$\bar{\mathbf{\Lambda}} = \bar{\nabla}(\mathbf{\Lambda}, k) := \{(\lambda_1', \lambda_2') \in \bar{\nabla} \colon \exists (\lambda_1, \lambda_2) \in \mathbf{\Lambda} \text{ with } |\lambda_1| = \max(|\lambda_1'| - k, 0),$$
$$|\lambda_2| = \max(|\lambda_2'| - k, 0), \text{and } \operatorname{meas}\left(\mathcal{S}(\psi^{(1)}_{\lambda_1'}) \times \mathcal{S}(\psi^{(2)}_{\lambda_2'}) \cap \mathcal{S}(\psi^{(1)}_{\lambda_1}) \times \mathcal{S}(\psi^{(2)}_{\lambda_2}\right) > 0\}.$$

Remark 5.3.16. The $\mathcal{S}$-neighborhood appearing in the preceding definition is equal to the one given in (5.3.1).

Proposition 5.3.17. *The k^{th}-neighborhood $\bar{\mathbf{\Lambda}}$, defined in Definition 5.3.15, is a double-tree, and $\#\bar{\mathbf{\Lambda}} \lesssim \#\mathbf{\Lambda}$ (dependent on k).*

5.3.5 A routine `GROWDBLTREE` for extending a double-tree.

We present a model algorithm `growDbltree` that given a double-tree $\mathbf{\Lambda} \subset \vee = \vee_1 \times \vee_2$ creates its k^{th} neighborhood in $\bar{\nabla} = \bar{\vee}_1 \times \bar{\vee}_2$. Considering that the algorithm constructs the new double-tree using unidirectional operations, for simplicity, and w.l.o.g. we choose to start the extension by the first direction, i.e., in $\bar{\vee}_1$.

$[\bar{\mathbf{\Lambda}}] :=$ `growDbltree`$(\mathbf{\Lambda}, k)$
% *Input: double-tree* $\mathbf{\Lambda}$, $k \in \mathbb{N}$.
% *Output:* k^{th}*-neighborhood* $\bar{\mathbf{\Lambda}}$ *in* $\bar{\nabla}$.

$\bar{\mathbf{\Lambda}} := \{(\lambda_1', \lambda_2') \in \bar{\nabla}, |\lambda_1'| = 0, |\lambda_2'| = 0\}$
`for` $\lambda_1' \in \bar{\mathbf{\Lambda}}_1(\Sigma(P_1\mathbf{\Lambda}), k)$ % *unidirectionally extended tree in the* 1^{st}*-direction*
    `if` $|\lambda_1'| = 0$ `then` $m = 0$ `else` $m = 1$ `endif`
    `while` $m \le k$ *and* $|\lambda_1'| \ge m$
        `Cover`$(\lambda_1', m) := \{\lambda_1 \in P_1\mathbf{\Lambda}, |\lambda_1| = |\lambda_1'| - m, |\mathcal{S}(\psi^{(1)}_{\lambda_1'}) \cap \mathcal{S}(\psi^{(1)}_{\lambda_1})| > 0\}$
        `for` $\mu \in$ `Cover`(λ_1', m)
        $\bar{\mathbf{\Lambda}}_{2,\lambda_1'} := \bar{\mathbf{\Lambda}}_2(\Sigma(\mathbf{\Lambda}_{2,\mu}), k)$ % *unidirectional extention in the* 2^{nd}*-direction*
        $\bar{\mathbf{\Lambda}} := \bar{\mathbf{\Lambda}} \cup (\{\lambda_1'\} \times \bar{\mathbf{\Lambda}}_{2,\lambda_1'})$
        `endfor`
    $m = m + 1$
    `endwhile`
`endfor`

Theorem 5.3.18. *A call of* `growDbltree` *yields the output as specified, at the cost of* $\mathcal{O}(\#\mathbf{\Lambda})$ *operations.*

Proof. At first, the routine creates an element tree by means of $\Sigma(P_1\mathbf{\Lambda})$. By Prop. 5.3.8 and locality of wavelets in $\bar{\nabla}$ we infer $\#\Sigma(P_1\mathbf{\Lambda})) \lesssim \#P_1\mathbf{\Lambda}$ and $\bar{\mathbf{\Lambda}}_1(\Sigma(P_1\mathbf{\Lambda}), k) \lesssim \#P_1\mathbf{\Lambda}$ respectively. Moreover, following Lemma 5.3.2, the set $P_1\bar{\mathbf{\Lambda}} = \bar{\mathbf{\Lambda}}_1(\Sigma(P_1\mathbf{\Lambda}), k)$ is a tree. Thereafter, the extension of the double-tree $\mathbf{\Lambda}$ in the 1^{st} direction w.r.t. $\bar{\vee}_1$ is created in $\mathcal{O}(\#P_1\mathbf{\Lambda})$ operations.

For all $\lambda_1' \in P_1\bar{\mathbf{\Lambda}}$, we create the sets $\texttt{Cover}(\lambda_1', m)$ as defined. Assuming we can find these sets in $\mathcal{O}(1)$ operations, we freeze the double-tree $\mathbf{\Lambda}$ in each $\mu \in \texttt{Cover}(\lambda_1', m)$ and grow the remaining tree in the other direction according to Definition 5.3.12. Again with the use of Lemma 5.3.2 the sets added to $\bar{\mathbf{\Lambda}}$ are trees. After repetition for all λ_1''s the desired extended tree $\bar{\mathbf{\Lambda}}$ is a double-tree.

In addition, $\forall \lambda_1' \in P_1\bar{\mathbf{\Lambda}}$, with $|\lambda_1'| > 0$, we have

$$\#\bigcup_{m=1}^{\max\{|\lambda_1'|,k\}} \bigcup_{\mu\in\texttt{Cover}(\lambda_1',m)} \Sigma(\mathbf{\Lambda}_{2,\mu}) \lesssim \#\bigcup_{m=1}^{\max\{|\lambda_1'|,k\}} \bigcup_{\mu\in\texttt{Cover}(\lambda_1',m)} \mathbf{\Lambda}_{2,\mu}$$

and

$$\#\bar{\mathbf{\Lambda}}_{2,\lambda_1'} \lesssim \#\bigcup_{m=1}^{\max\{|\lambda_1'|,k\}} \bigcup_{\mu\in\texttt{Cover}(\lambda_1',m)} \Sigma(\mathbf{\Lambda}_{2,\mu}).$$

If $|\lambda_1'| = 0$ then the following holds

$$\#\bar{\mathbf{\Lambda}}_{2,\lambda_1'} \lesssim \#\bigcup_{\mu\in\texttt{Cover}(\lambda_1',0)} \Sigma(\mathbf{\Lambda}_{2,\mu}).$$

Then, the last three relations imply,

$$\sum_{\lambda_1'\in P_1\bar{\mathbf{\Lambda}}} \#\bar{\mathbf{\Lambda}}_{2,\lambda_1'} \lesssim \#\bigcup_{\mu\in\texttt{Cover}(\lambda_1',0)} \mathbf{\Lambda}_{2,\mu} + \sum_{\substack{\lambda_1'\in P_1\bar{\mathbf{\Lambda}}\\ |\lambda_1'|\neq 0}} \#\bigcup_{m=1}^{\max\{|\lambda_1'|,k\}} \bigcup_{\mu\in\texttt{Cover}(\lambda_1',m)} \mathbf{\Lambda}_{2,\mu}.$$

Note that the sets $\bar{\mathbf{\Lambda}}_{2,\lambda_1'}$ and $\texttt{Cover}(\lambda_1', m)$ are dependent on k. Thus, considering each $\lambda_1' \in P_1\bar{\mathbf{\Lambda}}$ was genererated by a uniformly bounded number of wavelets in $P_1\mathbf{\Lambda}$ with at most k levels difference, we infer that $\#\bar{\mathbf{\Lambda}} \lesssim \#\mathbf{\Lambda}$ dependent on k. Since $\#P_1\mathbf{\Lambda}$ is a bounded multiple of $\#\mathbf{\Lambda}$, we conclude that the total operations cost is $\mathcal{O}(\#\mathbf{\Lambda})$. □

Remark 5.3.19. Depending on the wavelet collections composing the tensor-product wave- lets, a careful selection of the starting direction to build $\bar{\mathbf{\Lambda}}$ could result in smaller computational cost overall.

5.4 The application of system matrices

The overall quantitative performance of the **awgm** relies heavily on the efficient evaluation of matrix-vector multiplications which carry the main weight of calculations for both the approximation of the residual and the solution of Galerkin systems. In this section we will present modified versions of algorithms for the application of system matrices w.r.t. (tensor product) wavelet bases developed in [KS14]. These will form the cornerstone for the design of the forthcoming implementations (Sect. 5.6.6) in optimal computational complexity. Three auxiliary algorithms for applying a lower

and an upper triangular matrix as well as their corresponding sum are presented first. Finally, we present a procedure to apply matrices corresponding to tensor product operators.

5.4.1 A routine EVAL corresponding to a mapping between scalar-valued functions.

The evaluation of products between stiffness matrices w.r.t. wavelet bases and coefficient vectors in linear complexity requires special attention. Since the matrix entries are inner products between basis functions the cost of a direct calculation per entry could be intolerably large in view of the multilevel structure of the basis. Pairs of wavelets that result in non-zero matrix entries have to be identified, and these entries need to be approximated using numerical quadrature. In addition, the desired complexity may not be achieved for general index sets.

On the other hand, under the tree constraint one can achieve a transition between a multi- and a single-scale representation by applying multi- to single-scale transformations and their adjoints respectively in linear complexity (see Prop. 5.3.4). In the algorithm presented here, these transformations are incorporated to facilitate a transition to a single-scale representation simultaneously with building the matrix-vector product for which only the calculation of products between scaling functions on the same level is needed. Afterwards, the transpose transformation is applied to return to the wavelet representation. The locality of the bilinear forms and that of the wavelet supports lead to sparse single-scale and transformation matrices. Moreover, due to the fact that scalar products between scaling functions can be computed in $\mathcal{O}(1)$ operations, the operation count is proportional to the vector's size.

Consider wavelet collections $\breve{\Psi}$, Ψ of orders m_1 and m_2 and $m := \max\{m_1, m_2\}$, with corresponding sets of indices in $\breve{\vee}$ and $\vee$ respectively. The sets $\breve{\Psi}_\ell$, Ψ_ℓ are the restrictions of $\breve{\Psi}$ and Ψ to their elements at level ℓ. Furthermore, $\breve{\vee}_{\ell\uparrow}$ and $\vee_{\ell\uparrow}$ denote the subsets of $\breve{\vee}$ and $\vee$ of indices with levels being greater or equal to ℓ.

For $T \in \mathfrak{T}$, let Φ_T denote the *nodal* basis of $\mathcal{P}_d(T)$ with $d := m - 1$, and let $\Phi := (\Phi_T)_{T\in\mathfrak{T}}$, $\Phi_\ell := (\Phi_T)_{\{T\in\mathfrak{T}:\,|T|=\ell\}}$. The dimension of $P_d(T)$ is given by

$$n = n(d) := \frac{1}{2}(d+1)(d+2) \tag{5.4.1}$$

Viewing sets of functions as column vectors, there exist matrices $\mathfrak{p}_{\ell+1}$, $\mathfrak{q}_\ell$, such that $\Phi_\ell^\top = \Phi_{\ell+1}^\top \mathfrak{p}_{\ell+1}$, $\Psi_\ell^\top = \Phi_\ell^\top \mathfrak{q}_\ell$ so that $\mathbf{d}_\ell^\top \Phi_\ell = (\mathfrak{p}_{\ell+1}\mathbf{d}_\ell)^\top \Phi_{\ell+1}$ and $\mathbf{c}_\ell^\top \Psi_\ell = (\mathfrak{q}_\ell \mathbf{c}_\ell)^\top \Phi_\ell$. Similarly, there exist analogous matrices $\breve{\mathfrak{q}}_\ell$ related to $\breve{\Psi}$. The matrices $\mathfrak{q}_\ell$, $\breve{\mathfrak{q}}_\ell$ (but not the $\mathfrak{p}_\ell$) depend on the type of wavelet.

Below we give a modified version of the algorithm `evalA`, found in [KS14], such that it can handle possible non-linear forms arising in our applications. With $(v, u) \in \operatorname{span}\breve{\Psi} \times \operatorname{span}\Psi$, we consider $a(v, u) = a(v, u|_{\operatorname{supp} v})$ to be *local*, i.e. $a(v, u) = 0$ whenever $|\operatorname{supp} u \cap \operatorname{supp} v| = 0$, and linear in the first argument. The idea behind the modification is to avoid splitting $a(v, u)$ as $a(v, u_1) + a(v, u_2)$ for some $u_1 + u_2 = u$ with $|\operatorname{supp} u_1 \cap \operatorname{supp} u_2| > 0$. To avoid confusion we note that in applications the first

argument is the test function, and the second argument is the trial function. A model algorithm is as follows.

$[\vec{e},\, \vec{f}] := \texttt{eval}(a)(\ell, \breve{\Pi}, \breve{\Lambda}, \Pi, \Lambda, \vec{d}, \vec{c})$
% *Input:* $\ell \in \mathbb{N}$, $\breve{\Pi}, \Pi \subset \mathcal{T}_{\ell-1}$, ℓ*-trees* $\breve{\Lambda} \subset \breve{\vee}_{\ell\uparrow}$ *and* $\Lambda \subset \vee_{\ell\uparrow}$,
% $\vec{d} = (d_T)_{T\in\Pi} \subset \mathbb{R}^n$ (n *as in* (5.4.1)), $\vec{c} = (c_\lambda)_{\lambda\in\Lambda} \subset \mathbb{R}$.
% *Output: With* $u := \vec{d}^\top \Phi|_\Pi + \vec{c}^\top \Psi|_\Lambda$,
% $\vec{e} = a(\Phi|_{\breve{\Pi}}, u)$, $\vec{f} = a(\breve{\Psi}|_{\breve{\Lambda}}, u)$.

```
if Π̆ ≠ ∅ then
```
$\breve{\Pi}_B := \Sigma_\ell(\Lambda) \cap \breve{\Pi}$, $\breve{\Pi}_A := \breve{\Pi} \setminus \breve{\Pi}_B$
$\Pi_B := \big(\Sigma_\ell(\breve{\Lambda}) \cup \breve{\Pi}_B\big) \cap \Pi$, $\Pi_A := \Pi \setminus \Pi_B$
$\underline{\breve{\Pi}} := \texttt{child}(\breve{\Pi}_B) \cup \Sigma_\ell(\breve{\Lambda})$
$\underline{\Pi} := \texttt{child}(\Pi_B) \cup \Sigma_\ell(\Lambda)$

$\underline{\vec{d}} = \vec{d}|_{\underline{\Pi}} := \mathfrak{p}_\ell \vec{d}|_{\Pi_B} + \mathfrak{q}_\ell \vec{c}|_{\Lambda_\ell}$

$[\underline{\vec{e}},\, \underline{\vec{f}}] := \texttt{eval}(a)(\ell+1, \underline{\breve{\Pi}}, \breve{\Lambda}_{\ell+1\uparrow}, \underline{\Pi}, \Lambda_{\ell+1\uparrow}, \underline{\vec{d}}, \vec{c}|_{\Lambda_{\ell+1\uparrow}})$

$$\vec{e} = \begin{bmatrix} \vec{e}|_{\breve{\Pi}_A} \\ \vec{e}|_{\breve{\Pi}_B} \end{bmatrix} := \begin{bmatrix} a(\Phi|_{\breve{\Pi}_A}, \vec{d}^\top \Phi|_\Pi) \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}})|_{\breve{\Pi}_B} \end{bmatrix}$$
$$\vec{f} = \begin{bmatrix} \vec{f}|_{\breve{\Lambda}_\ell} \\ \vec{f}|_{\breve{\Lambda}_{\ell+1\uparrow}} \end{bmatrix} := \begin{bmatrix} (\breve{\mathfrak{q}}_\ell^\top \underline{\vec{e}})|_{\breve{\Lambda}_\ell} \\ \underline{\vec{f}} \end{bmatrix}$$
```
endif
```

Remark 5.4.1. If, for some trees Λ, $\breve{\Lambda}$, and $\vec{c} \in \ell_2(\Lambda)$, one seeks $a(\breve{\Psi}|_{\breve{\Lambda}}, \vec{c}^\top \Psi|_\Lambda)$, then with

$$\vec{d} := \mathfrak{q}_0 \mathbf{c}|_{\Lambda_0}, \quad [\vec{e},\, \vec{f}] := \texttt{eval}(a)(1, \mathcal{T}_0, \breve{\Lambda}_{1\uparrow}, \mathcal{T}_0, \Lambda_{1\uparrow}, \vec{d}, \vec{c}|_{\Lambda_{1\uparrow}}),$$

(and $u := \vec{d}^\top \Phi|_\Pi + \vec{c}|_{\Lambda_{1\uparrow}}^\top \Psi|_{\Lambda_{1\uparrow}}$), it holds that

$$a(\breve{\Psi}|_{\breve{\Lambda}}, \vec{c}^\top \Psi|_\Lambda)\Big(= a(\breve{\Psi}|_{\breve{\Lambda}}, u) = \begin{bmatrix} a(\breve{\Psi}|_{\breve{\Lambda}_0}, u) \\ a(\breve{\Psi}|_{\breve{\Lambda}_{1,\uparrow}}, u) \end{bmatrix} \Big) = \begin{bmatrix} (\breve{\mathfrak{q}}_0^\top \vec{e})|_{\breve{\Lambda}_0} \\ \vec{f} \end{bmatrix},$$

where we used a being linear in its first argument.

Theorem 5.4.2. *A call of* `eval` *yields the output as specified, at the cost of* $\mathcal{O}(\#\breve{\Pi} + \#\breve{\Lambda} + \#\Pi + \#\Lambda)$ *operations.*

Proof. By its definition, $\#\underline{\breve{\Pi}} \lesssim \#\Sigma_\ell(\breve{\Lambda}) + \#\breve{\Pi}_B \lesssim \#\breve{\Lambda}_\ell + \#\Lambda_\ell$. So after sufficiently many steps, the current set $\breve{\Pi}$ will be empty. For use later, we note that $\#\underline{\Pi} \lesssim \#\Lambda_\ell + \#\Pi_B \lesssim \#\Lambda_\ell + \#\breve{\Lambda}_\ell + \#\breve{\Pi}_B \lesssim \#\Lambda_\ell + \#\breve{\Lambda}_\ell$.

For $\breve{\Pi} = \emptyset$, the call produces nothing, which is correct because $\breve{\Pi} = \emptyset$ implies $\breve{\Lambda} = \emptyset$.

Now let $\breve{\Pi} \neq \emptyset$. From $\breve{\Pi}_A \cap \mathtt{parent}(\Sigma_\ell(\Lambda)) = \emptyset$, Λ being a tree, $\Sigma(\Lambda)$ an element tree, and a being local, one has

$$\vec{e}|_{\breve{\Pi}_A} = a(\Phi|_{\breve{\Pi}_A}, u) = a(\Phi|_{\breve{\Pi}_A}, \vec{d}^\top \Phi|_\Pi).$$

The definition of $\underline{\vec{d}}$ together with the property that $\underline{\Pi} \supset \mathtt{child}(\Pi_B) \cup \Sigma_\ell(\Lambda)$ show that

$$\underline{u} := \underline{\vec{d}}^\top \Phi|_{\underline{\Pi}} + \vec{c}|^\top_{\Lambda_{\ell+1\uparrow}} \Psi|_{\Lambda_{\ell+1\uparrow}} = (\vec{d}|_{\Pi_B})^\top \Phi|_{\Pi_B} + \vec{c}^\top \Psi|_\Lambda = u - (\vec{d}|_{\Pi_A})^\top \Phi|_{\Pi_A}.$$

By the property of $\breve{\Lambda}$ and Λ being trees and $\Sigma(\breve{\Lambda})$, $\Sigma(\Lambda)$ being element trees, we have $\underline{\breve{\Pi}} \supset \mathtt{parent}(\Sigma_{\ell+1}(\breve{\Lambda}))$ and $\underline{\Pi} \supset \mathtt{parent}(\Sigma_{\ell+1}(\Lambda))$ needed for the recursive call. By induction, the recursive call yields $\underline{\vec{e}} = a(\Phi|_{\underline{\breve{\Pi}}}, \underline{u})$ and $\underline{\vec{f}} = a(\breve{\Psi}|_{\breve{\Lambda}_{\ell+1\uparrow}}, \underline{u}) = a(\breve{\Psi}|_{\breve{\Lambda}_{\ell+1\uparrow}}, u)$ where the latter equality follows from $\Pi_A \cap \mathtt{parent}(\Sigma_\ell(\breve{\Lambda})) = \emptyset$, a being local, and $\breve{\Lambda}$ being a tree and $\Sigma(\breve{\Lambda})$ an element tree.

From $\mathtt{child}(\breve{\Pi}_B) \subset \underline{\breve{\Pi}}$, we have

$$\Phi|_{\breve{\Pi}_B} = (\mathfrak{p}_\ell^\top \Phi|_{\underline{\breve{\Pi}}})|_{\breve{\Pi}_B}.$$

By $\Pi_A \cap \breve{\Pi}_B = \emptyset$, a being local, and linear in its first argument, we conclude that

$$\vec{e}|_{\breve{\Pi}_B} = a(\Phi|_{\breve{\Pi}_B}, u) = a(\Phi|_{\breve{\Pi}_B}, \underline{u}) = \big(\mathfrak{p}_\ell^\top \underline{\vec{e}}\big)|_{\breve{\Pi}_B}.$$

From $\Sigma_\ell(\breve{\Lambda}) \subset \underline{\breve{\Pi}}$, a being local, and linear in its first argument, we have

$$\breve{\Psi}|_{\breve{\Lambda}_\ell} = (\breve{\mathfrak{q}}_\ell^\top \Phi|_{\underline{\breve{\Pi}}})|_{\breve{\Lambda}_\ell}.$$

By $\Pi_A \cap \mathtt{parent}(\Sigma_\ell(\breve{\Lambda})) = \emptyset$, we conclude that

$$\vec{f}|_{\breve{\Lambda}_\ell} = a(\breve{\Psi}|_{\breve{\Lambda}_\ell}, u) = a(\breve{\Psi}|_{\breve{\Lambda}_\ell}, \underline{u}) = \big(\breve{\mathfrak{q}}_\ell^\top \underline{\vec{e}}\big)|_{\breve{\Lambda}_\ell}.$$

From the assumptions on the collections Φ, $\breve{\Psi}$, and Ψ, and their consequences on the sparsity of the matrices $\mathfrak{p}_\ell$, $\mathfrak{q}_\ell$, and $\breve{\mathfrak{q}}_\ell$, one easily infers that the total cost of the evaluations of the statements in `eval` is $\mathcal{O}(\#\breve{\Pi} + \#\breve{\Lambda}_\ell + \#\Pi + \#\Lambda_\ell)$ plus the cost of the recursive call. Using $\#\underline{\breve{\Pi}} + \#\underline{\Pi} \lesssim \#\breve{\Lambda}_\ell + \#\Lambda_\ell$ and induction, we conclude the second statement of the theorem. □

Example 5.4.3. We give an example to demonstrate the action of the modified routine. The Figure 5.7 visualises the process of marking the supports of elements involved in the evaluation of the matrix-vector product $a(\Psi_{\breve{\Lambda}}, \vec{c}^\top \Psi_\Lambda)$ with $a(\cdot,\cdot) := \langle \cdot, \cdot \rangle_{L_2(\Omega)}$ and $n \in \mathbb{Z}^+$, $\Omega = [0,1]$, Ψ being the one dimensional Haar wavelets (see Sect. 5.2.1) serving as a basis for $L_2(\Omega)$, and $\breve{\Lambda}$ and Λ two given sets of wavelet indices. The Haar wavelets are preferred for convenience of presentation because of their non-overlapping supports.

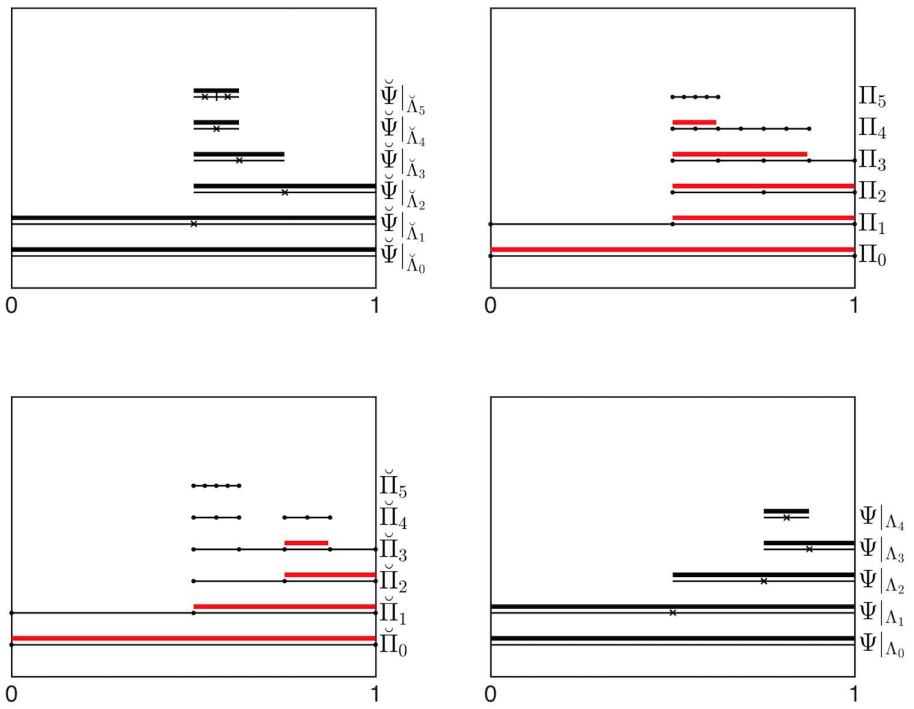


Figure 5.7: A visualisation of supports of wavelets in $\breve{\Psi}|_{\breve{\Lambda}}$ and $\Psi|_{\Lambda}$, and the sets Π_ℓ, $\breve{\Pi}_\ell$, for $\ell = 0, \cdots, 5$, as produced by the application of `eval` in the case explained in Example 5.4.3.

In the upper left and bottom right picture of Figure 5.7, the supports of wavelets with indices in $\breve{\Lambda}$ and Λ, respectively, are represented with black-and-white stripes, and arranged in layers corresponding to different levels of the two trees. The center of each wavelet support is pointed with an '$\times$'.

In the upper right and bottom left picture, with single black lines we mark the intervals included in Π_ℓ and $\breve{\Pi}_\ell$, respectively, i.e. the sets Π and $\breve{\Pi}$ in the arguments of the ℓ^{th} recursive call of `eval`. Moreover, with a point '$\bullet$' we draw the two ends of each interval. Finally, we indicate in red line shading the intervals included in $(\Pi_B)_\ell \subset \Pi_\ell$ or $(\breve{\Pi}_B)_\ell \subset \breve{\Pi}_\ell$.

A sequence of recursive calls of `eval` produces a single-scale representation of $\vec{c}^{\top}\Psi_\Lambda$ w.r.t. $\mathcal{P}_d(\mathcal{T}(\Lambda))$, by constructing appropriately the sets Π_ℓ, and applying accordingly the single- to multi-scale transformations $\mathfrak{q}_\ell$ and the prolongation operators $\mathfrak{p}_\ell$. During these calls, `eval` also creates $\breve{\Pi}_\ell$, an essential step for the efficient evaluation of the L_2-products between scaling functions. Note that $\breve{\Pi}_6 = \emptyset$, thus, the last recursive

call is for $\ell = 6$. In order to compute $\langle \breve{\Psi}|_{\breve{\Lambda}_5}, \vec{c}^\top \Psi|_\Lambda \rangle_{L_2(\Omega)}$, we only have to compute $\langle \Phi|_{(\breve{\Pi}_A)_5}, (\vec{d}|_{\Phi|_{\Pi_5}})^\top \Phi|_{\Pi_5} \rangle_{L_2(\Omega)}$ for scaling functions supported on $(\breve{\Pi}_A)_5$, i.e. intervals in Π_5 with no red shading (thus, the set $\breve{\Pi}_5$ itself), and subsequently apply the single- to multi-scale transformation $\breve{\mathfrak{q}}_5^\top$ to the last result. Counting backwards in ℓ, for $\ell = 4$ we have $(\breve{\Pi}_A)_4 = \breve{\Pi}_4$, therefore we calculate $\langle \breve{\Psi}|_{\breve{\Lambda}_4}, \vec{c}^\top \Psi|_\Lambda \rangle_{L_2(\Omega)}$ similarly. Then for $\ell = 3$, we have $\langle \breve{\Psi}|_{\breve{\Lambda}_3}, \vec{c}^\top \Psi|_\Lambda \rangle_{L_2(\Omega)} = \breve{\mathfrak{q}}_3^\top (\langle \Phi|_{(\breve{\Pi}_A)_3}, (\vec{d}|_{\Phi|_{\Pi_3}})^\top \Phi|_{\Pi_3} \rangle_{L_2(\Omega)} + (\breve{\mathfrak{p}}_3^\top \langle \Phi|_{\breve{\Pi}_4}, (\vec{d}|_{\Phi|_{\Pi_4}})^\top \Phi|_{\Pi_4} \rangle_{L_2(\Omega)})|_{\Phi|_{(\breve{\Pi}_B)_3}})$, hence, there is no need to compute $\langle \Phi|_{(\breve{\Pi}_B)_3}, \vec{d}^\top \Phi|_{\Pi_3} \rangle_{L_2(\Omega)}$. The calculation of $\langle \breve{\Psi}|_{\breve{\Lambda}}, \vec{c}^\top \Psi|_\Lambda \rangle_{L_2(\Omega)}$ is completed by repeating the last step analogously for $\ell = 2, 1, 0$.

5.4.2 A routine EVALUPP corresponding to a mapping between scalar-valued functions.

In the same setting, where $(v, u) \in \operatorname{span} \breve{\Psi} \times \operatorname{span} \Psi$, we consider $a(v, u) = a(v, u|_{\operatorname{supp} v})$ to be *local* and *bilinear*. To prepare for the application of tensor product operators, we consider a splitting of the matrix $A = [a(\breve{\psi}_\lambda, \psi_\mu)]_{(\lambda,\mu)\in\breve{\vee}\times\vee}$ into an upper and lower triangular part. We set $U = [a(\breve{\psi}_\lambda, \psi_\mu)]_{|\lambda|\le|\mu|}$, $L = [a(\breve{\psi}_\lambda, \psi_\mu)]_{|\lambda|>|\mu|}$ such that $A = L + U$.

The next algorithm `evalupp` performs the application of U to a vector, with the entries of both the input and output vector corresponding to a tree structured index set, driven by the same principles underlying the routine `eval`.

$[\vec{e},\, \vec{f}] := \texttt{evalupp}(a)(\ell, \breve{\Pi}, \breve{\Lambda}, \Pi, \Lambda, \vec{d}, \vec{c})$
% *Input:* $\ell \in \mathbb{N}$, $\breve{\Pi}, \Pi \subset \mathcal{T}_{\ell-1}$, *$\ell$-trees* $\breve{\Lambda} \subset \breve{\vee}_{\ell\uparrow}$ *and* $\Lambda \subset \vee_{\ell\uparrow}$,
% $\vec{d} = (d_T)_{T\in\Pi} \subset \mathbb{R}^n$ (*n as in* (5.4.1)), $\vec{c} = (c_\lambda)_{\lambda\in\Lambda} \subset \mathbb{R}$.
% *Output: With* $u := \vec{d}^\top \Phi|_\Pi + \vec{c}^\top \Psi|_\Lambda$,
% $\vec{e} = a(\Phi|_{\breve{\Pi}}, u)$, $\vec{f} = U|_{\breve{\Lambda}\times\Lambda}\vec{c}$.

`if` $\breve{\Pi} \neq \emptyset$ `then`
  $\breve{\Pi}_B := \Sigma_\ell(\Lambda) \cap \breve{\Pi}$, $\breve{\Pi}_A := \breve{\Pi} \setminus \breve{\Pi}_B$
  $\underline{\breve{\Pi}} := \texttt{child}(\breve{\Pi}_B) \cup \Sigma_\ell(\breve{\Lambda})$
  $\underline{\Pi} := \Sigma_\ell(\Lambda)$

  $\underline{\vec{d}} = \underline{\vec{d}}|_{\underline{\Pi}} := \mathfrak{q}_\ell \vec{c}|_{\Lambda_\ell}$

  $[\underline{\vec{e}},\, \underline{\vec{f}}] := \texttt{evalupp}(a)(\ell+1, \underline{\breve{\Pi}}, \breve{\Lambda}_{\ell+1\uparrow}, \underline{\Pi}, \Lambda_{\ell+1\uparrow}, \underline{\vec{d}}, \vec{c}|_{\Lambda_{\ell+1\uparrow}})$

  $$\vec{e} = \begin{bmatrix} \vec{e}|_{\breve{\Pi}_A} \\ \vec{e}|_{\breve{\Pi}_B} \end{bmatrix} := \begin{bmatrix} a(\Phi|_{\breve{\Pi}_A}, \vec{d}^\top \Phi|_\Pi) \\ a(\Phi|_{\breve{\Pi}_B}, \vec{d}^\top \Phi|_\Pi) + (\mathfrak{p}_\ell^\top \underline{\vec{e}})|_{\breve{\Pi}_B} \end{bmatrix}$$

$$\vec{f} = \begin{bmatrix} \vec{f}|_{\breve{\Lambda}_\ell} \\ \vec{f}|_{\breve{\Lambda}_{\ell+1\uparrow}} \end{bmatrix} := \begin{bmatrix} (\breve{\mathfrak{q}}_\ell^\top \underline{\vec{e}})|_{\breve{\Lambda}_\ell} \\ \underline{\vec{f}} \end{bmatrix}$$

`endif`

Remark 5.4.4. If, for some trees Λ, $\breve{\Lambda}$, and $\vec{c} \in \ell_2(\Lambda)$, one seeks $U|_{\breve{\Lambda}\times\Lambda}\vec{c}$, then with

$$\vec{d} := \mathfrak{q}_0\mathbf{c}|_{\Lambda_0}, \quad [\vec{e}, \vec{f}] := \texttt{evalupp}(a)(1, \mathcal{T}_0, \breve{\Lambda}_{1\uparrow}, \mathcal{T}_0, \Lambda_{1\uparrow}, \vec{d}, \vec{c}|_{\Lambda_{1\uparrow}}),$$

(and $u := \vec{d}^\top \Phi|_\Pi + \vec{c}|_{\Lambda_{1\uparrow}}^\top \Psi|_{\Lambda_{1\uparrow}}$), it holds that

$$U|_{\breve{\Lambda}\times\Lambda}\vec{c} \Big(= \begin{bmatrix} a(\breve{\Psi}|_{\breve{\Lambda}_0}, u) \\ U|_{\breve{\Lambda}_{1\uparrow}\times\Lambda_{1\uparrow}}\vec{c}|_{\Lambda_{1\uparrow}} \end{bmatrix} \Big) = \begin{bmatrix} (\breve{\mathfrak{q}}_0^\top \vec{e})|_{\breve{\Lambda}_0} \\ \vec{f} \end{bmatrix}.$$

Theorem 5.4.5. *A call of* `evalupp` *yields the output as specified, at the cost of* $\mathcal{O}(\#\breve{\Pi} + \#\breve{\Lambda} + \#\Pi + \#\Lambda)$ *operations.*

Proof. By its definition, $\#\underline{\breve{\Pi}} \lesssim \#\Sigma_\ell(\breve{\Lambda}) + \#\breve{\Pi}_B \lesssim \#\breve{\Lambda}_\ell + \#\Lambda_\ell$. So after sufficiently many steps, the current set $\breve{\Pi}$ will be empty. For use later, we note that $\#\underline{\Pi} \lesssim \#\Lambda_\ell + \#\Pi_B \lesssim \#\Lambda_\ell + \#\breve{\Lambda}_\ell + \#\breve{\Pi}_B \lesssim \#\Lambda_\ell + \#\breve{\Lambda}_\ell$.

For $\breve{\Pi} = \emptyset$, the call produces nothing, which is correct because $\breve{\Pi} = \emptyset$ implies $\breve{\Lambda} = \emptyset$.

Now let $\breve{\Pi} \neq \emptyset$. From $\breve{\Pi}_A \cap \texttt{parent}(\Sigma_\ell(\Lambda)) = \emptyset$, Λ being a tree, $\Sigma(\Lambda)$ being an element tree, and a being local, one has

$$\vec{e}|_{\breve{\Pi}_A} = a(\Phi|_{\breve{\Pi}_A}, u) = a(\Phi|_{\breve{\Pi}_A}, \vec{d}^\top \Phi|_\Pi).$$

The definition of $\underline{\vec{d}}$ together with the property that $\underline{\Pi} \supset \Sigma_\ell(\Lambda)$ show that

$$\underline{u} := \underline{\vec{d}}^\top \Phi|_{\underline{\Pi}} + \vec{c}|_{\Lambda_{\ell+1\uparrow}}^\top \Psi|_{\Lambda_{\ell+1\uparrow}} = \vec{c}^\top \Psi|_\Lambda = u - \vec{d}^\top \Phi|_\Pi.$$

By the property of $\breve{\Lambda}$ and Λ being trees and $\Sigma(\breve{\Lambda})$, $\Sigma(\Lambda)$ being element trees, we have $\underline{\breve{\Pi}} \supset \texttt{parent}(\Sigma_{\ell+1}(\breve{\Lambda}))$ and $\underline{\Pi} \supset \texttt{parent}(\Sigma_{\ell+1}(\Lambda))$ needed for the recursive call. By induction, this call yields $\underline{\vec{e}} = a(\Phi|_{\underline{\breve{\Pi}}}, \underline{u})$ and $\underline{\vec{f}} = U|_{\breve{\Lambda}_{\ell+1\uparrow}\times\Lambda_{\ell+1\uparrow}}\vec{c}_{\Lambda_{\ell+1\uparrow}} = (U|_{\breve{\Lambda}\times\Lambda}\vec{c})|_{\breve{\Lambda}_{\ell+1\uparrow}} = \vec{f}|_{\breve{\Lambda}_{\ell+1\uparrow}}$.

From $\texttt{child}(\breve{\Pi}_B) \subset \underline{\breve{\Pi}}$, we have

$$\Phi|_{\breve{\Pi}_B} = (\mathfrak{p}_\ell^\top \Phi|_{\underline{\breve{\Pi}}})|_{\breve{\Pi}_B}.$$

We conclude that

$$\vec{e}|_{\breve{\Pi}_B} = a(\Phi|_{\breve{\Pi}_B}, u) = a(\Phi|_{\breve{\Pi}_B}, \underline{u}) + a(\Phi|_{\breve{\Pi}_B}, \vec{d}^\top \Phi|_\Pi) = \big(\mathfrak{p}_\ell^\top \underline{\vec{e}}\big)|_{\breve{\Pi}_B} + a(\Phi|_{\breve{\Pi}_B}, \vec{d}^\top \Phi|_\Pi).$$

From $\Sigma_\ell(\breve{\Lambda}) \subset \underline{\breve{\Pi}}$, we have

$$\breve{\Psi}|_{\breve{\Lambda}_\ell} = (\breve{\mathfrak{q}}_\ell^\top \Phi|_{\underline{\breve{\Pi}}})|_{\breve{\Lambda}_\ell}.$$

We conclude that

$$\vec{f}|_{\breve{\Lambda}_\ell} = a(\breve{\Psi}|_{\breve{\Lambda}_\ell}, \vec{c}^{\top}\Psi|_{\Lambda}) = a(\breve{\Psi}|_{\breve{\Lambda}_\ell}, \underline{u}) = (\breve{\mathfrak{q}}_\ell^{\top}\vec{\underline{e}})|_{\breve{\Lambda}_\ell}.$$

From the assumptions on the collections Φ, $\breve{\Psi}$, and Ψ, and their consequences on the sparsity of the matrices $\mathfrak{p}_\ell$, $\mathfrak{q}_\ell$, and $\breve{\mathfrak{q}}_\ell$, one easily infers that the total cost of the evaluations of the statements in `evalupp` is $\mathcal{O}(\#\breve{\underline{\Pi}} + \#\breve{\Lambda}_\ell + \#\Pi + \#\Lambda_\ell)$ plus the cost of the recursive call. Using $\#\breve{\underline{\Pi}} + \#\underline{\Pi} \lesssim \#\breve{\Lambda}_\ell + \#\Lambda_\ell$ and induction, we conclude the second statement of the theorem. □

5.4.3 A routine `EVALLOW` corresponding to a mapping between scalar-valued functions.

What remains to be reported is a routine for the application of the lower triangular part $L = [a(\breve{\psi}_\lambda, \psi_\mu)]_{|\lambda|>|\mu|}$ as defined in the previous section.

The algorithm `evallow` reads as follows.

$\vec{f} :=$ `evallow`$(a)(\ell, \breve{\Lambda}, \Pi, \Lambda, \vec{d}, \vec{c})$
% *Input:* $\ell \in \mathbb{N}$, $\Pi \subset \mathcal{T}_{\ell-1}$, ℓ*-trees* $\breve{\Lambda} \subset \breve{\vee}_{\ell\uparrow}$ *and* $\Lambda \subset \vee_{\ell\uparrow}$,
% $\vec{d} = (d_T)_{T\in\Pi} \subset \mathbb{R}^n$ (n *as in* (5.4.1)), $\vec{c} = (c_\lambda)_{\lambda\in\Lambda} \subset \mathbb{R}$.
% *Output:* $\vec{f} = a(\breve{\Psi}|_{\breve{\Lambda}}, \Phi|_{\Pi})\vec{d} + L|_{\breve{\Lambda}\times\Lambda}\vec{c}$.

`if` $\breve{\Lambda} \neq \emptyset$ `then`
  $\Pi_B := \Sigma_\ell(\breve{\Lambda}) \cap \Pi$
  $\underline{\Pi} :=$ `child`$(\Pi_B) \cup \Sigma_\ell(\Lambda)$

  $\vec{\underline{e}} := a(\Phi|_{\texttt{child}(\Pi_B)}, \Phi|_{\texttt{child}(\Pi_B)})\mathfrak{p}_\ell\vec{d}|_{\Pi_B}$
  $\vec{\underline{d}} := \mathfrak{p}_\ell\vec{d}|_{\Pi_B} + \mathfrak{q}_\ell\vec{c}|_{\Lambda_\ell}$
  $\vec{f} = \begin{bmatrix} \vec{f}|_{\breve{\Lambda}_\ell} \\ \vec{f}|_{\breve{\Lambda}_{\ell+1\uparrow}} \end{bmatrix} := \begin{bmatrix} (\breve{\mathfrak{q}}_\ell^{\top}\vec{\underline{e}})|_{\breve{\Lambda}_\ell} \\ \texttt{evallow}(a)(\ell+1, \breve{\Lambda}_{\ell+1\uparrow}, \underline{\Pi}, \Lambda_{\ell+1\uparrow}, \vec{\underline{d}}, \vec{c}|_{\Lambda_{\ell+1\uparrow}}) \end{bmatrix}$
`endif`

Remark 5.4.6. If, for some trees Λ, $\breve{\Lambda}$, and $\vec{c} \in \ell_2(\Lambda)$, one seeks $L|_{\breve{\Lambda}\times\Lambda}\vec{c}$, then with

$$\vec{d} := \mathfrak{q}_0\mathbf{c}|_{\Lambda_0}, \quad L|_{\breve{\Lambda}\times\Lambda}\vec{c} = L|_{\breve{\Lambda}_{1\uparrow}\times\Lambda}\vec{c} = \texttt{evallow}(a)(1, \breve{\Lambda}_{1\uparrow}, \mathcal{T}_0, \Lambda_{1\uparrow}, \vec{d}, \vec{c}|_{\Lambda_{1\uparrow}}).$$

Theorem 5.4.7. *A call of* `evallow` *yields the output as specified, at the cost of* $\mathcal{O}(\#\breve{\Lambda} + \#\Pi + \#\Lambda)$ *operations.*

Proof. For use later, we note that $\#\underline{\Pi} \lesssim \#\Lambda_\ell + \#\Pi_B \lesssim \#\Lambda_\ell + \#\breve{\Lambda}_\ell + \#\breve{\Pi}_B \lesssim \#\Lambda_\ell + \#\breve{\Lambda}_\ell$.

For $\breve{\Lambda} = \emptyset$, the call produces nothing.

Let $\breve{\Lambda} \neq \emptyset$ and $\Pi \neq \emptyset$. The recursive call is allowed by $\underline{\Pi} \supset \Sigma_\ell(\Lambda) \supset \mathtt{parent}(\Sigma_{\ell+1}(\Lambda))$, by Λ being a tree and $\Sigma(\Lambda)$ being an element tree, and produces

$$\begin{aligned}\vec{f}|_{\breve{\Lambda}_\ell} = a(\breve{\Psi}_{\breve{\Lambda}_\ell}, \Phi|_\Pi)\vec{d} = a(\breve{\Psi}_{\breve{\Lambda}_\ell}, \Phi|_\Pi)\vec{d}|_{\Pi_B} &= (\breve{\mathfrak{q}}_\ell^\top a(\Phi|_{\mathtt{child}(\Pi_B)}, \Phi_{\mathtt{child}(\Pi_B)})\mathfrak{p}_\ell \vec{d}|_{\Pi_B})|_{\breve{\Lambda}_\ell} \\ &= (\breve{\mathfrak{q}}_\ell^\top \underline{\vec{e}})|_{\breve{\Lambda}_\ell}\end{aligned}$$

Using that $\breve{\Lambda}$ is a tree,

$$\begin{aligned} f|_{\breve{\Lambda}_{\ell+1\uparrow}} &= a(\breve{\Psi}|_{\breve{\Lambda}_{\ell+1\uparrow}}, \Phi|_\Pi)\vec{d} + L|_{\breve{\Lambda}_{\ell+1\uparrow}\times\Lambda_\ell}\vec{c}|_{\Lambda_\ell} + L|_{\breve{\Lambda}_{\ell+1\uparrow}\times\Lambda_{\ell+1\uparrow}}\vec{c}|_{\Lambda_{\ell+1\uparrow}} \\ &= a(\breve{\Psi}|_{\breve{\Lambda}_{\ell+1\uparrow}}, \Phi|_\Pi)\vec{d}|_{\Pi_B} + a(\breve{\Psi}|_{\breve{\Lambda}_{\ell+1\uparrow}}, \Psi|_{\Lambda_\ell})\vec{c}|_{\Lambda_\ell} + L|_{\breve{\Lambda}_{\ell+1\uparrow}\times\Lambda_{\ell+1\uparrow}}\vec{c}|_{\Lambda_{\ell+1\uparrow}} \\ &= a(\breve{\Psi}|_{\breve{\Lambda}_{\ell+1\uparrow}}, \Phi|_{\underline{\Pi}})\underline{\vec{d}} + L|_{\breve{\Lambda}_{\ell+1\uparrow}\times\Lambda_{\ell+1\uparrow}}\vec{c}|_{\Lambda_{\ell+1\uparrow}} \\ &= \mathtt{evallow}(a)(\ell+1, \breve{\Lambda}_{\ell+1\uparrow}, \underline{\Pi}, \Lambda_{\ell+1\uparrow}, \underline{\vec{d}}, \vec{c}|_{\Lambda_{\ell+1\uparrow}}) \end{aligned}$$

by induction.

By the assumptions on the collections Φ, $\breve{\Psi}$, and Ψ, and their consequences on the sparsity of the matrices $\mathfrak{p}_\ell$, $\mathfrak{q}_\ell$, and $\breve{\mathfrak{q}}_\ell$, one easily infers that the total cost of the evaluations of the statements in `evallow` is $\mathcal{O}(\#\breve{\Lambda}_\ell + \#\Pi + \#\Lambda_\ell)$ plus the cost of the recursive call. Using $\#\underline{\Pi} \lesssim \#\breve{\Lambda}_\ell + \#\Lambda_\ell$ and induction, we conclude the second statement of the theorem. □

5.4.4 The application of tensor product operators

We now consider tensor product wavelet collections

$$\breve{\Psi} := \{\breve{\boldsymbol{\psi}}_{\boldsymbol{\lambda}} := \otimes_i \breve{\psi}^{(i)}_{\lambda_i} : \boldsymbol{\lambda} \in \breve{\vee} := \prod_i \breve{\vee}_i\}, \quad \Psi := \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} := \otimes_i \psi^{(i)}_{\lambda_i} : \boldsymbol{\lambda} \in \vee := \prod_i \vee_i\}.$$

Moreover, on $\operatorname{span}\breve{\Psi} \times \operatorname{span}\Psi$ we consider the *bilinear* form,

$$\boldsymbol{a}(\otimes_i v_i, \otimes_i u_i) = \prod_i a_i(v_i, u_i)$$

with $a_i(\cdot,\cdot)$ being *local*, *bilinear* forms so that the bi-infinite wavelet matrix representation of the associated tensor product operator is,

$$\boldsymbol{a}(\breve{\Psi}, \Psi) = \otimes_i A_i.$$

where $A_i = a_i(\breve{\psi}_i, \psi_i)$ and $\breve{\psi}_i = \{\breve{\psi}^{(i)}_\lambda : \lambda \in \breve{\vee}_i\}$, $\psi_i = \{\psi^{(i)}_\lambda : \lambda \in \vee_i\}$. A splitting of A_i into $L_i + U_i$ is considered, where $U_i := [(A_i)_{\lambda,\mu}]_{|\lambda|\leq|\mu|}$, $L_i := [(A_i)_{\lambda,\mu}]_{|\lambda|>|\mu|}$, and will reveal its use in Corollary 5.4.9.

For a subset $\triangleleft$ of a double index set $\diamondsuit$, let $I^\diamondsuit_\triangleleft$ denote the extension operator with zeros of a vector supported on $\triangleleft$ to one on $\diamondsuit$, and with $R^\diamondsuit_\triangleleft$ its (formal) adjoint, being the restriction operator of a vector supported on $\diamondsuit$ to one on $\triangleleft$. Since the set $\diamondsuit$ will

always be clear from the context, we will refer to these operators simply by $I_\triangleleft$ and $R_\triangleleft$.

Then, focusing on the case $i = 2$, and for double-trees $\breve{\mathbf{\Lambda}}, \mathbf{\Lambda} \subset \diamond$ the challenge lies in the application of the following matrix in optimal complexity

$$\boldsymbol{a}(\breve{\Psi}|_{\breve{\mathbf{\Lambda}}}, \Psi|_{\mathbf{\Lambda}}) = R_{\breve{\mathbf{\Lambda}}}(A_1 \otimes A_2) I_{\mathbf{\Lambda}}.$$

If $\breve{\mathbf{\Lambda}}, \mathbf{\Lambda}$ correspond to *full grid* index sets, meaning they consist of all wavelets up to some level $\breve{\ell}, \ell$ respectively, then it is easy to conclude that the product can be achieved in $\mathcal{O}(\#\breve{\mathbf{\Lambda}} + \#\mathbf{\Lambda})$ operations [KS14].

Let us consider the case where $\breve{\mathbf{\Lambda}} = \mathbf{\Lambda} = \{\lambda \in \diamond : \max_i |\lambda_i| \leq \ell\}$, i.e., $\breve{\mathbf{\Lambda}}, \mathbf{\Lambda}$ being *sparse grid* index sets. Splitting the application of the tensor product operator into *unidirectional operations* as

$$R_{\mathbf{\Lambda}}(A_1 \otimes \mathrm{Id}) I_{\tilde{\mathbf{\Lambda}}} R_{\tilde{\mathbf{\Lambda}}} (\mathrm{Id} \otimes A_2) I_{\mathbf{\Lambda}} \tag{5.4.2}$$

one realises that the intermediate set $\tilde{\mathbf{\Lambda}}$ can not be smaller than the full grid index set, meaning that $\#\tilde{\mathbf{\Lambda}} \not\lesssim \#\mathbf{\Lambda}$. Hence, the operator given by (5.4.2) can not be applied in optimal complexity.

Next we present an algorithm introduced in [KS14] that for double-trees $\breve{\mathbf{\Lambda}}, \mathbf{\Lambda}$, so in particular for sparse grid index sets, performs the matrix-vector application with matrix $\boldsymbol{a}(\breve{\Psi}|_{\breve{\mathbf{\Lambda}}}, \Psi|_{\mathbf{\Lambda}})$ in linear complexity. Following an idea introduced in [BZ96], the key is to split A_1 into a lower and an upper triangular matrix, and, for one of the two resulting terms, to reverse the order of the tensor product operations.

In [KS14], the optimal evaluation of $\boldsymbol{a}(\breve{\Psi}|_{\breve{\mathbf{\Lambda}}}, \Psi|_{\mathbf{\Lambda}})$ was generalised to the case that $\breve{\mathbf{\Lambda}}, \mathbf{\Lambda}$ are index sets with a multi-tree structure. The following theorem describes the fast evaluation scheme restricted to double-trees, therefore for completeness, we include a proof with a sequence of conclusions.

Theorem 5.4.8. *Let $\breve{\mathbf{\Lambda}} \subset \breve{\vee}_1 \times \breve{\vee}_2$, $\mathbf{\Lambda} \subset \vee_1 \times \vee_2$ be finite double-trees. Then*

$$\mathbf{\Sigma} := \bigcup_{\lambda \in P_1 \mathbf{\Lambda}} \{\lambda\} \times \big\{\breve{\mathbf{\Lambda}}_{2,\mu} : \mu \in P_1 \breve{\mathbf{\Lambda}},\ |\mu| = |\lambda| + 1,\ |\mathcal{S}(\breve{\psi}_\mu^{(1)}) \cap \mathcal{S}(\psi_\lambda^{(1)})| > 0\big\},$$

$$\mathbf{\Theta} := \bigcup_{\lambda \in P_2 \mathbf{\Lambda}} \{\mu \in P_1 \breve{\mathbf{\Lambda}} : \exists \gamma \in \mathbf{\Lambda}_{1,\lambda} \ s.t.\ |\gamma| = |\mu|,\ |\mathcal{S}(\breve{\psi}_\mu^{(1)}) \cap \mathcal{S}(\psi_\gamma^{(1)})| > 0\} \times \{\lambda\},$$

are double-trees with $\#\mathbf{\Sigma} \lesssim \#\breve{\mathbf{\Lambda}}$ and $\#\mathbf{\Theta} \lesssim \#\mathbf{\Lambda}$, and

$$\begin{aligned} R_{\breve{\mathbf{\Lambda}}}(A_1 \otimes A_2) I_{\mathbf{\Lambda}} = {} & R_{\breve{\mathbf{\Lambda}}}(L_1 \otimes \mathrm{Id}) I_{\mathbf{\Sigma}} R_{\mathbf{\Sigma}} (\mathrm{Id} \otimes A_2) I_{\mathbf{\Lambda}} + \\ & R_{\breve{\mathbf{\Lambda}}}(\mathrm{Id} \otimes A_2) I_{\mathbf{\Theta}} R_{\mathbf{\Theta}} (U_1 \otimes \mathrm{Id}) I_{\mathbf{\Lambda}}. \end{aligned} \tag{5.4.3}$$

Before we give the proof of this theorem we discuss its implications. The application of $R_{\breve{\mathbf{\Lambda}}}(L_1 \otimes \mathrm{Id}) I_{\mathbf{\Sigma}}$ boils down to the application of $R_{\breve{\mathbf{\Lambda}}_{1,\mu}} L_1 I_{\mathbf{\Sigma}_{1,\mu}}$ for all $\mu \in P_2 \mathbf{\Sigma}$. Each of these applications can be performed in $\mathcal{O}(\#\breve{\mathbf{\Lambda}}_{1,\mu} + \#\mathbf{\Sigma}_{1,\mu})$ operations by means of a call of `evallow`(a_1). Since $\cup_{\mu \in P_2 \mathbf{\Sigma}} \#\mathbf{\Sigma}_{1,\mu} = \#\mathbf{\Sigma}$ and $\cup_{\mu \in P_2 \mathbf{\Sigma}} \#\breve{\mathbf{\Lambda}}_{1,\mu} \leq \#\breve{\mathbf{\Lambda}}$, we

conclude that the application of $R_{\breve{\mathbf{\Lambda}}}(L_1 \otimes \mathrm{Id})I_{\mathbf{\Sigma}}$ can be performed in $\mathcal{O}(\#\breve{\mathbf{\Lambda}} + \#\mathbf{\Sigma})$ operations.

Similarly, the applications of $R_{\mathbf{\Sigma}}(\mathrm{Id}\otimes A_2)I_{\mathbf{\Lambda}}$, $R_{\breve{\mathbf{\Lambda}}}(\mathrm{Id}\otimes A_2)I_{\mathbf{\Theta}}$, and $R_{\mathbf{\Theta}}(U_1\otimes\mathrm{Id})I_{\mathbf{\Lambda}}$ by means of calls of $\texttt{eval}(a_2)$, $\texttt{eval}(a_2)$, and $\texttt{evalupp}(a_1)$, respectively, can be performed in $\mathcal{O}(\#\mathbf{\Sigma} + \#\mathbf{\Lambda})$, $\mathcal{O}(\#\breve{\mathbf{\Lambda}} + \#\mathbf{\Theta})$, and $\mathcal{O}(\#\mathbf{\Theta} + \#\mathbf{\Lambda})$ operations.

From $\#\mathbf{\Sigma} \lesssim \#\breve{\mathbf{\Lambda}}$ and $\#\mathbf{\Theta} \lesssim \#\mathbf{\Lambda}$, and the expression of $R_{\breve{\mathbf{\Lambda}}}(A_1 \otimes A_2)I_{\mathbf{\Lambda}}$ given by the right hand side of (5.4.3), we conclude the following:

Corollary 5.4.9. *Let $\breve{\mathbf{\Lambda}} \subset \breve{\vee}_1 \times \breve{\vee}_2$, $\mathbf{\Lambda} \subset \vee_1 \times \vee_2$ be finite double-trees, then $R_{\breve{\mathbf{\Lambda}}}(A_1 \otimes A_2)I_{\mathbf{\Lambda}}$ can be applied in $\mathcal{O}(\#\breve{\mathbf{\Lambda}} + \#\mathbf{\Lambda})$ operations.*

Proof of Thm. 5.4.8. We write

$$\begin{aligned} R_{\breve{\mathbf{\Lambda}}}(A_1 \otimes A_2)I_{\mathbf{\Lambda}} =& R_{\breve{\mathbf{\Lambda}}}((L_1 + U_1) \otimes A_2)I_{\mathbf{\Lambda}} \\ =& R_{\breve{\mathbf{\Lambda}}}(L_1 \otimes \mathrm{Id})(\mathrm{Id} \otimes A_2)I_{\mathbf{\Lambda}} + \qquad (5.4.4) \\ & R_{\breve{\mathbf{\Lambda}}}(\mathrm{Id} \otimes A_2)(U_1 \otimes \mathrm{Id})I_{\mathbf{\Lambda}}. \qquad (5.4.5) \end{aligned}$$

Considering (5.4.4), the range of $(\mathrm{Id} \otimes A_2)I_{\mathbf{\Lambda}}$ consists of vectors whose entries with first index outside $P_1\mathbf{\Lambda}$ are zero. In view of the subsequent application of $L_1 \otimes I$, furthermore only those indices $(\lambda, \gamma) \in P_1\mathbf{\Lambda} \times \breve{\vee}_2$ of these vectors might be relevant for which $\exists(\mu, \gamma) \in \breve{\mathbf{\Lambda}}$ with $|\mu| > |\lambda|$ and $|\mathcal{S}(\breve{\psi}_\mu^{(1)}) \cap \mathcal{S}(\psi_\lambda^{(1)})| > 0$. Note that $|\mathcal{S}(\breve{\psi}_\mu^{(1)}) \cap \mathcal{S}(\psi_\lambda^{(1)})| = 0$ implies $|\operatorname{supp} \breve{\psi}_\mu^{(1)} \cap \operatorname{supp} \psi_\lambda^{(1)}| = 0$, and so $a_1(\breve{\psi}_\mu^{(1)}, \psi_\lambda^{(1)}) = 0$. If for given (λ, γ) such a pair (μ, γ) exists for $|\mu| > |\lambda|$, then such a pair exists for $|\mu| = |\lambda| + 1$ as well, because $\breve{\mathbf{\Lambda}}_{1,\gamma}$ is a tree, and $\mathcal{S}(\breve{\psi}_{\mu'}^{(1)}) \supset \mathcal{S}(\breve{\psi}_\mu^{(1)})$ for any ancestor μ' of μ. In other words, the condition $|\mu| > |\lambda|$ can be read as $|\mu| = |\lambda| + 1$. The set of (λ, γ) that we just described is given by the set $\mathbf{\Sigma}$, and so we infer that

$$R_{\breve{\mathbf{\Lambda}}}(L_1 \otimes \mathrm{Id})(\mathrm{Id} \otimes A_2)I_{\mathbf{\Lambda}} = R_{\breve{\mathbf{\Lambda}}}(L_1 \otimes \mathrm{Id})I_{\mathbf{\Sigma}}R_{\mathbf{\Sigma}}(\mathrm{Id} \otimes A_2)I_{\mathbf{\Lambda}}.$$

Now let $(\lambda, \gamma) \in \mathbf{\Sigma}$. Using that $P_1\mathbf{\Lambda}$ is a tree, and $\mathcal{S}(\psi_\lambda^{(1)}) \subset \mathcal{S}(\psi_{\lambda'}^{(1)})$ for any ancestor λ' of λ, we infer that $(\lambda', \gamma) \in \mathbf{\Sigma}$. Using that for any $\mu \in P_1\breve{\mathbf{\Lambda}}$, $\breve{\mathbf{\Lambda}}_{2,\mu}$ is a tree, we infer that for any ancestor γ' of γ, $(\lambda, \gamma') \in \mathbf{\Sigma}$, so that $\mathbf{\Sigma}$ is a double-tree.

For any $\mu \in \breve{\vee}_1$, the number of $\lambda \in \vee_1$ with $|\mu| = |\lambda|+1$ and $|\mathcal{S}(\breve{\psi}_\mu^{(1)}) \cap \mathcal{S}(\psi_\lambda^{(1)})| > 0$ is uniformly bounded, from which we infer that $\#\mathbf{\Sigma} \lesssim \sum_{\mu \in P_1\breve{\mathbf{\Lambda}}} \#\breve{\mathbf{\Lambda}}_{2,\mu} = \#\breve{\mathbf{\Lambda}}$.

Considering (5.4.5), the range of $(U_1 \otimes \mathrm{Id})I_{\mathbf{\Lambda}}$ consists of vectors that can only have non-zero entries for indices $(\mu, \lambda) \in \breve{\vee}_1 \times P_2\mathbf{\Lambda}$ for which there exists a $\gamma \in \mathbf{\Lambda}_{1,\lambda}$ with $|\gamma| \geq |\mu|$ and $|\mathcal{S}(\breve{\psi}_\mu^{(1)}) \cap \mathcal{S}(\psi_\gamma^{(1)})| > 0$. Since $\mathbf{\Lambda}_{1,\lambda}$ is a tree, and $\mathcal{S}(\psi_{\gamma'}^{(1)}) \supset \mathcal{S}(\psi_\gamma^{(1)})$ for any ancestor γ' of γ, equivalently $|\gamma| \geq |\mu|$ can be read as $|\gamma| = |\mu|$. Furthermore, in view of the subsequent application of $R_{\breve{\mathbf{\Lambda}}}(\mathrm{Id} \otimes A_2)$, it suffices to consider those indices (μ, λ) with $\mu \in P_1\breve{\mathbf{\Lambda}}$. The set of (μ, λ) that we just described is given by the set $\mathbf{\Theta}$, and so we infer that

$$R_{\breve{\mathbf{\Lambda}}}(\mathrm{Id} \otimes A_2)(U_1 \otimes \mathrm{Id})I_{\mathbf{\Lambda}} = R_{\breve{\mathbf{\Lambda}}}(\mathrm{Id} \otimes A_2)I_{\mathbf{\Theta}}R_{\mathbf{\Theta}}(U_1 \otimes \mathrm{Id})I_{\mathbf{\Lambda}}.$$

Now let $(\mu,\lambda)\in\boldsymbol{\Theta}$. If λ' is an ancestor of λ, then from $\boldsymbol{\Lambda}_{2,\gamma}$ being a tree for any $\gamma\in P_1\boldsymbol{\Lambda}$, we have $(\mu,\lambda')\in\boldsymbol{\Theta}$. If μ' is an ancestor of μ, then from $P_1\breve{\boldsymbol{\Lambda}}$ being a tree, and $\mathcal{S}(\breve{\psi}^{(1)}_{\mu'})\supset\mathcal{S}(\breve{\psi}^{(1)}_{\mu})$, we infer that $(\mu',\lambda)\in\boldsymbol{\Theta}$, and thus that $\boldsymbol{\Theta}$ is a double-tree.

For any $\gamma\in\vee_1$, the number of $\mu\in\breve{\vee}_1$ with $|\mu|=|\gamma|$ and $|\mathcal{S}(\breve{\psi}^{(1)}_{\mu})\cap\mathcal{S}(\psi^{(1)}_{\gamma})|>0$ is uniformly bounded, from which we infer that $\#\boldsymbol{\Theta}\lesssim\sum_{\lambda\in P_2\boldsymbol{\Lambda}}\#\boldsymbol{\Lambda}_{1,\lambda}=\#\boldsymbol{\Lambda}$. □

5.5 Data structures

In order to achieve optimal computational complexity, it is required to choose appropriate data structures to host the data involved in calculations. Basic actions as lookups, addition and deletion of data entries is often expected to be performed in $\mathcal{O}(1)$ operations. This can be achieved by the so-called *hashtables*. To distribute and control efficiently data information in a hashtable, special functions need to be found in order to map data to the desired position. This is achieved by a suitable indexing and a *hash function*.

In other circumstances, a level-wise ordering of the wavelets is needed to enable probing through a list within a given level, or through consecutive ones, in $\mathcal{O}(\#N)$ operations, with N being the number of cells in the list. A level-wise ordered structure is obtained by an array of *linked lists*. In addition, a similar structure may be used to accommodate double-trees.

5.5.1 Hashtables and hash functions

Thinking, briefly, of finding a solution $u\in X$ of a well-posed linear operator equation $A:X\to Y'$ and with right hand side $f\in Y'$, when X,Y are equipped with Riesz wavelet bases the equation $Au=f$ can be transformed in a well conditioned discrete ℓ_2-problem. In this case the solution is identified by an ℓ_2-vector with wavelet coefficients.

A way to store vectors produced by wavelet discretisations when one needs to add, delete and search for an entry in $\mathcal{O}(1)$ operations is using a *hashtable*. A hashtable is made by allocating a finite array of memory slots and using a *hash function* $h:\mathbb{N}_0\to\mathbb{N}_0$. The latter maps a natural number called the *key*, such as a wavelet index, to an index corresponding to an array slot. When more than one entries hash to the same position in the array, these could still be added and reached by creating a chain (also called a linked list, see next Sect. 5.5.2) under the hashed position. This is called a *collision*, a case whose occurrence we would like to control as much as possible in our effort to distribute the keys evenly in the table. In consequence, a desired hash function is the one that minimises collisions so that the complexity of the main operations is maintained of $\mathcal{O}(1)$, i.e., the number of collisions under a slot is bounded by a reasonably small constant.

Given an array $(1,\dots,P-1)$ where P is a prime number, a simple hash function that maps any $\lambda\in\mathbb{N}_0$ to an array slot is

$$\lambda\mapsto\lambda \bmod P. \tag{5.5.1}$$

Assuming a consecutive numbering of the adaptively generated wavelets starting from 0, if the size of the table is large enough, that is P is equal to or greater than the total number of wavelets $\#N$ then (5.5.1) is an injective mapping and one achieves a perfect hashing. If $\#N > P$, the selection of P being prime provides a better distribution of the keys in the array *buckets*. In fact, for $\#N = nP$ with $n \in \mathbb{N}$ and an array of $nP-1$ buckets, it is easily deduced that the number of collisions in each bucket would be exactly n.

In practice though, we use a level-by-level numbering of the infinite wavelet collection and use these indices as the keys to be hashed to the table. Although far from the ideal case we described before, numerical experiments did not reveal an intolerable performance and taking into account its simplicity and low computational cost the function in (5.5.1) was the hash function of choice. In the case of tensor-product wavelet approximation though, where one needs to hash pairs of indices, a more sophisticated hash function is used which we describe in Sect. 5.5.4.

5.5.2 Level-wise linked lists

When a level-wise ordering is required, there is no need for a hashing technique. The range of the allocated array now indicates levels of wavelets or partitions etc. As a consequence, an entry finds its place by *direct hashing*, meaning a simple correspondence to a level. The cells of a linked list then are pair-wise connected with pointers pointing to the memory address of the next one. The pointer of the last cell in each list, namely the *tail*, is fused by pointing to NULL.

The main instance in our applications when arrays of linked-lists find their use is for storing information about element trees, such as successive triangulations. In this case, we wish to scroll through a complete set of elements and therefore looking for an $\mathcal{O}(\#elements)$ complexity. Unlike sequential search, insertion of a new entry in a single level's linked list is done by replacing the head of the list and shifting the chain downwards which counts to $\mathcal{O}(1)$ time. Deleting individual entries is not preferred, thus, deleting linked lists as a whole realises $\mathcal{O}(\#elements)$ computational time.

5.5.3 The double tree structure

The application of tensor product operators raises the need for storing double trees. In particular, fixing a wavelet index, in either direction, we would like to recover the tree in the other one. Furthermore, for any fixed index μ we wish to store trees of the form

$$\mathbf{\Lambda}_{i,\mu} := P_i(P_{\neg i}|_{\mathbf{\Lambda}})^{-1}\{\mu\}$$

in the spirit of Definition 5.3.14.

For $i \in \{1,2\}$ we assign a consecutive natural number to each element of $P_i\mathbf{\Lambda}$ during the construction of the set and so create an index collection $K = \{0,\dots,\#P_i\mathbf{\Lambda}-1\}$. Then $\forall \mu \in P_{\neg i}\mathbf{\Lambda}$, the sets $P_i(P_{\neg i}|_{\mathbf{\Lambda}})^{-1}\{\mu\}$ are stored in an array of linked lists with exact length being $\#P_{\neg i}\mathbf{\Lambda}$. The double-tree structure is depicted in Figure 5.8 when $N = \#P_{\neg i}\mathbf{\Lambda}$.

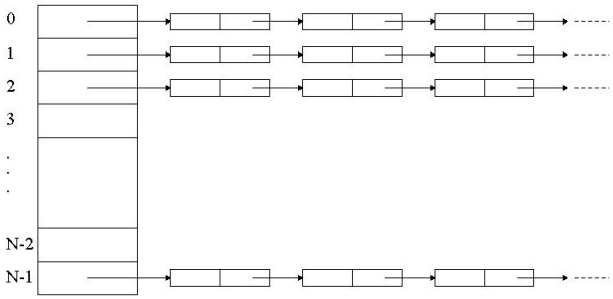


Figure 5.8: (i) For $N = P$ it corresponds to the hashtable with chaining. (ii) In case the array counts for levels of a tree, with $N = \max\{\ell, \ell \text{ denoting levels}\} + 1$ it reads as an array of linked-lists (with direct hashing). (iii) The figure shows the double-tree structure when $N = \#P_{\neg i}\mathbf{\Lambda}$.

5.5.4 On numbering tensor-product wavelets

Ideally, in order to minimise collisions, before hashing, the wavelets are numbered in the order they are generated. This ordering is however unknown due to the adaptive wavelet generation, so let us assume an ordering as if the solution would be infinitely smooth.

In the tensor case, any wavelet is the tensor product of a temporal wavelet and a spatial one. Let us number both the temporal wavelets and the spatial wavelets level-by-level (starting from the coarsest one), and on each level in an arbitrarily chosen order (a natural one would be lexicographical).

Now a tensor product wavelet is indexed by a pair $(\lambda_1, \lambda_2) \in \mathbb{N}_0^2$, where λ_1 is the number of the temporal wavelet, and λ_2 is the number of the spatial wavelet.

For smooth functions, wavelets with indices (λ_1, λ_2) will be generated before (λ_1', λ_2') whenever $|\vec{\lambda}|_1 < |\vec{\lambda}'|_1$. In view of this property, what would be a suitable numbering, i.e., a mapping $h_2 : \mathbb{N}_0^2 \to \mathbb{N}_0$, is one that satisfies $h_2(\vec{\lambda}) < h_2(\vec{\lambda}')$ whenever $|\vec{\lambda}|_1 < |\vec{\lambda}'|_1$.

Instead of 2-tuples, let us consider n-tuples. Define

$$N(n, i) := \#\{\lambda \in \mathbb{N}_0^n \colon |\vec{\lambda}|_1 = i\}.$$

Then

$$\sum_{j=0}^{i} N(n, j) = \sum_{j=0}^{i} N(n, i-j) = N(n+1, i)$$

and also

$$N(n, i) = \binom{i+n-1}{n-1}.$$

With $h_1(\lambda) := \lambda$, for $n > 1$ we define

$$h_n(\vec{\lambda}) := \begin{cases} 0 & \vec{\lambda} = 0 \\ \binom{|\vec{\lambda}|_1 - 1 + n}{n} + h_{n-1}(\lambda_1, \ldots, \lambda_{n-1}) & \vec{\lambda} \neq 0 \end{cases}$$

Note for $n = 2$,

$$h_2(0,0) = 0,\ h_2(0,1) = 1,\ h_2(1,0) = 2,\ h_2(0,2) = 3,\ h_2(1,1) = 4,\ h_2(2,0) = 5.$$

To prevent overflow, i.e., the creation of numerical values exceeding the range of the hosting integer data class, we use that

$$\Big(\binom{\lambda_1 + \lambda_2 + 1}{2} + \lambda_1\Big) \mathrm{mod} P = \Big(\binom{(\lambda_1 + \lambda_2 + 1) \mathrm{mod}(2P)}{2} + \lambda_1\Big) \mathrm{mod} P = \Big(\binom{(\lambda_1 \mathrm{mod}(2P) + \lambda_2 \mathrm{mod}(2P) + 1)}{2} + \lambda_1 \mathrm{mod}(2P)\Big) \mathrm{mod} P$$

meaning that we never generate the number $(\lambda_1 + \lambda_2 + 1)(\lambda_1 + \lambda_2)$, or even not λ_1 or λ_2.

5.6 An implementation of the elliptic case

The definitions, data structures and algorithms presented until this point are employed to compose an adaptive wavelet solver, based on the **awgm** algorithm in Sect. 2.3, and applied to the stationary PDE reformulated as a first order least squares problem described in Sect. 2.5.1. This solver produced the numerical results shown in Sect. 2.6. Here we give an overview of the main components and demonstrate custom routines for an efficient implementation of the newly developed residual evaluation scheme.

5.6.1 The elliptic semi-linear problem

Recall the semi-linear elliptic PDE (2.6.1) with Dirichlet boundary conditions on the L-shaped domain $\Omega = (0,1)^2 \setminus [\frac{1}{2}, 1)^2$,

$$\begin{cases} -\Delta u + u^3 = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.6.1}$$

where, for simplicity, $f = 1$.

5.6.2 Spaces configuration

After the reformulation of (5.6.1) in Sect. 2.5.1 as a first order system least squares problem, a sequence of choices are made to equip the corresponding function spaces with Riesz wavelet bases.

The space $\mathscr{U} = H_0^1(\Omega)$ is equipped with the continuous piecewise quadratic wavelets, constructed in Chapter 3 and visualised in Sect. 5.2.5, appropriately scaled for being a basis for $H_0^1(\Omega)$. Then, $\mathscr{V} = \mathscr{V}_1 = H_0^1(\Omega)$ is equipped with continuous piecewise linear three-point hierarchical wavelets with zero boundary conditions, as found in Sect. 5.2.4, scaled accordingly for being a basis for $H_0^1(\Omega)$. Finally, both components of $\mathscr{T} = L_2(\Omega) \times L_2(\Omega)$ are equipped with continuous piecewise linear three-point hierarchical wavelets *without* boundary conditions, with a suitable scaling for being a basis for $L_2(\Omega)$.

5.6.3 Approximate Residual Scheme

At step (R) the **awgm** produces a finite residual approximation (cf. Sect. 2.3). Elaborating this, starting with an approximate solution the residual is approximated on a sufficiently larger approximation space which then provides an *a posteriori error indicator*. The procedure is described below.

For given trees

$$\Lambda^{\mathscr{U}} \subset \vee^{\mathscr{U}},\ \Lambda^{\mathscr{T}_1} \subset \vee^{\mathscr{T}_1},\ \Lambda^{\mathscr{T}_2} \subset \vee^{\mathscr{T}_2},$$

and approximations

$$\tilde{\mathbf{u}} \in \ell_2(\Lambda^{\mathscr{U}}),\ \tilde{\theta}_{\mathbf{1}} \in \ell_2(\Lambda^{\mathscr{T}_1}),\ \tilde{\theta}_{\mathbf{2}} \in \ell_2(\Lambda^{\mathscr{T}_2}).$$

one has to compute a sufficiently accurate approximation to

$$D\mathbf{Q}([\tilde{\mathbf{u}}^\top,\tilde{\boldsymbol{\theta}}^\top]^\top) = \begin{bmatrix} \langle \partial_1\Psi^{\mathscr{U}}, \partial_1\tilde{u} - \tilde{\theta}_1\rangle_{L_2(\Omega)} + \langle \partial_2\Psi^{\mathscr{U}}, \partial_2\tilde{u} - \tilde{\theta}_2\rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}_1}, \tilde{\theta}_1 - \partial_1\tilde{u}\rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}_2}, \tilde{\theta}_2 - \partial_2\tilde{u}\rangle_{L_2(\Omega)} \end{bmatrix}$$

$$\begin{bmatrix} \langle \Psi^{\mathscr{U}}, DN(\tilde{u})\Psi^{\mathscr{V}}\rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}_1}, \partial_1\Psi^{\mathscr{V}}\rangle_{L_2(\Omega)} \\ \langle \Psi^{\mathscr{T}_2}, \partial_2\Psi^{\mathscr{V}}\rangle_{L_2(\Omega)} \end{bmatrix} \left\langle \Psi^{\mathscr{V}_1}, N(\tilde{u}) - f - \operatorname{div}\vec{\tilde{\theta}} \right\rangle_{L_2(\Omega)}$$

with the nonlinear term being $N(\tilde{u}) = \tilde{u}^3$ and thus $DN(\tilde{u}) = 3\tilde{u}^2$.

The accuracy of this approximation depends on the size of the final and intermediate output sets on which each of the three main parts of $D\mathbf{Q}$ is evaluated. For the purpose of these evaluations, we set the following enlarged wavelet trees as

$$\bar{\Lambda}^* := \Lambda^*(\Sigma(\Lambda^{\mathscr{U}}) \oplus \Sigma(\Lambda^{\mathscr{T}_1}) \oplus \Sigma(\Lambda^{\mathscr{T}_2}), k),$$

with $^* \in \{\mathscr{U}, \mathscr{V}, \mathscr{T}_1, \mathscr{T}_2\}$, as well as

$$\bar{\bar{\Lambda}}^* := \Lambda^*(\Sigma(\bar{\Lambda}^{\mathscr{V}}), k),$$

with $^* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$. Recall that such sets have been defined in (5.3.12) as well as a sequence of steps to construct them is discussed in Sect. 5.3.3.

As implied in the previous paragraph, we compute the approximate residual by splitting it to three steps. First, in the routine called eval**1** we perform the computation of

$$\begin{bmatrix} \Big[\langle \partial_1 \Psi^{\mathscr{U}}, \partial_1 \tilde{u} - \tilde{\theta}_1 \rangle_{L_2(\Omega)} + \langle \partial_2 \Psi^{\mathscr{U}}, \partial_2 \tilde{u} - \tilde{\theta}_2 \rangle_{L_2(\Omega)} \Big] \Big|_{\bar{\Lambda}^{\mathscr{U}}} \\ \langle \Psi^{\mathscr{T}_1}, \tilde{\theta}_1 - \partial_1 \tilde{u} \rangle_{L_2(\Omega)} |_{\bar{\Lambda}^{\mathscr{T}_1}} \\ \langle \Psi^{\mathscr{T}_2}, \tilde{\theta}_2 - \partial_2 \tilde{u} \rangle_{L_2(\Omega)} |_{\bar{\Lambda}^{\mathscr{T}_2}} \end{bmatrix}, \tag{5.6.2}$$

followed by

$$\tilde{\mathbf{v}} := \big\langle \Psi^{\mathscr{V}_1}, N(\tilde{u}) - f - \operatorname{div} \vec{\tilde{\theta}} \big\rangle_{L_2(\Omega)} |_{\bar{\Lambda}^{\mathscr{V}}} \tag{5.6.3}$$

in eval**2**, and then, with $\tilde{v} := \tilde{\mathbf{v}}^\top \Psi^{\mathscr{V}}$, eval**3** gives

$$\begin{bmatrix} \langle \Psi^{\mathscr{U}}, DN(\tilde{u})\tilde{v} \rangle_{L_2(\Omega)} |_{\bar{\Lambda}^{\mathscr{U}}} \\ \langle \Psi^{\mathscr{T}_1}, \partial_1 \tilde{v} \rangle_{L_2(\Omega)} |_{\bar{\Lambda}^{\mathscr{T}_1}} \\ \langle \Psi^{\mathscr{T}_2}, \partial_2 \tilde{v} \rangle_{L_2(\Omega)} |_{\bar{\Lambda}^{\mathscr{T}_2}} \end{bmatrix}. \tag{5.6.4}$$

The sum $\tilde{\mathbf{r}}$ of the vectors produced in (5.6.2) and (5.6.4), which is supported on the union of trees $\bar{\bar{\Lambda}}^{\mathscr{U}}$, $\bar{\bar{\Lambda}}^{\mathscr{T}_1}$, and $\bar{\bar{\Lambda}}^{\mathscr{T}_2}$, is now the sufficiently accurate approximation of the residual. The routines eval**1**, eval**2**, eval**3** are given in Sect. 5.6.6.

5.6.4 Bulk chasing

As indicated in the second step (B) of the **awgm** algorithm in Sect. 2.3, we determine an admissible set of wavelets whose span gives the next approximation space. After having computed the approximate residual $\tilde{\mathbf{r}}$, its entries are used to guide the refinement of the previous approximation space. This procedure is known as *bulk chasing*.

In particular, one orders the entries of $\tilde{\mathbf{r}}$ by their modulus or, as we do in practice, apply an approximate bucket sort (details in [Ste09]). Subsequently, we collect the largest in modulus in a set Σ such that, for some fixed parameter $\theta \in (0,1)$, $\|\tilde{\mathbf{r}}|_\Sigma\| \geq \theta \|\tilde{\mathbf{r}}\|$ holds. The latter condition is called the *bulk criterion*. Writing $\Sigma = \Sigma^{\mathscr{U}} \cup \Sigma^{\mathscr{T}_1} \cup \Sigma^{\mathscr{T}_2}$, each of the three components are generally no trees, and should be expanded to trees that contain $\Lambda^{\mathscr{U}}$, $\Lambda^{\mathscr{T}_1}$, or $\Lambda^{\mathscr{T}_2}$, respectively.

An algorithm `TreeIt` that completes sets to trees after bulk chasing is considered below for Σ^*, where $* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$:

$[\Lambda^*] := \texttt{TreeIt}(\Sigma^*, \Lambda^*)$
% *Input: tree* Λ^**, unconstrained set* Σ^* *with* $\Sigma^* \cap \Lambda^* = \emptyset$
% *Output: smallest tree* Λ^* *that includes both* Σ^* *and the original tree* Λ^*

```
while Σ* ≠ ∅ do
      extract λ ∈ Σ*
      if λ ∉ Λ* then
         Λ* := λ ∪ Λ*
         if |λ| > 0 then
```

```
            Σ* := parent(λ) ∪ Σ*
          endif
        endif
        Σ* := Σ* \ λ
endwhile
```

Remark 5.6.1. Note that this routine *overwrites* the original Λ^* by a tree, being the next Λ^*, which contains $\Sigma^* \cup \Lambda^*$.

5.6.5 Galerkin solve

The approximation of the solution to the arising Galerkin systems, in the third step (G) of the **awgm**, is achieved by an (inexact) fixed point iteration as given in (2.3.2). The, generally nonlinear, operator evaluations on subspaces are given by replacing the sets $\bar{\Lambda}^{\mathscr{U}}$, $\bar{\Lambda}^{\mathscr{T}_1}$, $\bar{\Lambda}^{\mathscr{T}_2}$ in (5.6.2), and $\bar{\bar{\Lambda}}^{\mathscr{U}}$, $\bar{\bar{\Lambda}}^{\mathscr{T}_1}$, $\bar{\bar{\Lambda}}^{\mathscr{T}_2}$ in (5.6.4), by $\Lambda^{\mathscr{U}}$, $\Lambda^{\mathscr{T}_1}$, $\Lambda^{\mathscr{T}_2}$ in both instances. A near optimal value of the damping parameter ω for the fixed point iteration was determined experimentally by a bisection strategy.

5.6.6 The routines eval**1**, eval**2**, and eval**3**

Based on the principles of `eval` presented in Sect. 5.4.1, we designed three algorithms each of them computing one of the aforementioned pieces of the approximate residual. They incorporate multiple simultaneous multi- to single-scale wavelet transformations and their adjoints to perform several evaluations of L_2-products, as dictated by each of the three parts of $D\mathbf{Q}$ respectively. All the algorithms are recursive in their coarsest level and they are shown numerically to have optimal computational complexity.

The first algorithm `eval1`, computing the quantity given in (5.6.2), is exhibited below.

$[\vec{e}^{\mathscr{U}}, \vec{f}^{\mathscr{U}}, \vec{e}^{\mathscr{T}_1}, \vec{f}^{\mathscr{T}_1}, \vec{e}^{\mathscr{T}_2}, \vec{f}^{\mathscr{T}_2}] :=$`eval1`$(\ell, \breve{\Pi}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{U}}, \breve{\Pi}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_1}, \breve{\Pi}^{\mathscr{T}_2}, \breve{\Lambda}^{\mathscr{T}_2}, \Pi^{\mathscr{U}}, \Lambda^{\mathscr{U}},$
$\Pi^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_1}, \Pi^{\mathscr{T}_2}, \Lambda^{\mathscr{T}_2}, \vec{d}^{\mathscr{U}}, \vec{c}^{\mathscr{U}}, \vec{d}^{\mathscr{T}_1}, \vec{c}^{\mathscr{T}_1}, \vec{d}^{\mathscr{T}_2}, \vec{c}^{\mathscr{T}_2})$
% *Input:* $\ell \in \mathbb{N}$, $\breve{\Pi}^{\mathscr{U}}, \breve{\Pi}^{\mathscr{T}_1}, \breve{\Pi}^{\mathscr{T}_2}, \Pi^{\mathscr{U}}, \Pi^{\mathscr{T}_1}, \Pi^{\mathscr{T}_2} \subset \mathcal{T}_{\ell-1}$,
% ℓ-*trees* $\breve{\Lambda}^{\mathscr{U}}, \Lambda^{\mathscr{U}} \subset \vee^{\mathscr{U}}_{\ell\uparrow}$, $\breve{\Lambda}^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_2}, \Lambda^{\mathscr{T}_2} \subset \vee^{\mathscr{T}}_{\ell\uparrow}$,
% % $\vec{d}^{\mathscr{U}} = (d^{\mathscr{U}}_\lambda)_{\lambda\in\Pi^{\mathscr{U}}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{U}} = (c^{\mathscr{U}}_\lambda)_{\lambda\in\Lambda^{\mathscr{U}}} \subset \mathbb{R}$.
% $\vec{d}^{\mathscr{T}_1} = (d^{\mathscr{T}_1}_\lambda)_{\lambda\in\Pi^{\mathscr{T}_1}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{T}_1} = (c^{\mathscr{T}_1}_\lambda)_{\lambda\in\Lambda^{\mathscr{T}_1}} \subset \mathbb{R}$.
% $\vec{d}^{\mathscr{T}_2} = (d^{\mathscr{T}_2}_\lambda)_{\lambda\in\Pi^{\mathscr{T}_2}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{T}_2} = (c^{\mathscr{T}_2}_\lambda)_{\lambda\in\Lambda^{\mathscr{T}_2}} \subset \mathbb{R}$.
% *Output: With* $\tilde{u} := (\vec{d}^{\mathscr{U}})^\top \Phi|_{\Pi^{\mathscr{U}}} + (\vec{c}^{\mathscr{U}})^\top \Psi^{\mathscr{U}}|_{\Lambda^{\mathscr{U}}}$,
% $\tilde{\theta}_1 := (\vec{d}^{\mathscr{T}_1})^\top \Phi|_{\Pi^{\mathscr{T}_1}} + (\vec{c}^{\mathscr{T}_1})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_1}}$,
% $\tilde{\theta}_2 := (\vec{d}^{\mathscr{T}_2})^\top \Phi|_{\Pi^{\mathscr{T}_2}} + (\vec{c}^{\mathscr{T}_2})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_2}}$,
% $\vec{e}^{\mathscr{U}} = \langle \nabla\Phi|_{\breve{\Pi}^{\mathscr{U}}}, \nabla\tilde{u}\rangle_{L_2(\Omega)^2} - \sum_{i=1}^2 \langle \partial_i \Phi|_{\breve{\Pi}^{\mathscr{U}}}, \tilde{\theta}_i\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{U}} = \langle \nabla\Psi^{\mathscr{U}}|_{\breve{\Lambda}^{\mathscr{U}}}, \nabla\tilde{u}\rangle_{L_2(\Omega)^2} - \sum_{i=1}^2 \langle \partial_i \Psi^{\mathscr{U}}|_{\breve{\Lambda}^{\mathscr{U}}}, \tilde{\theta}_i\rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_1} = \langle \Phi|_{\breve{\Pi}^{\mathscr{T}_1}}, \tilde{\theta}_1 - \partial_1\tilde{u}\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_1} = \langle \Psi^{\mathscr{T}}|_{\breve{\Lambda}^{\mathscr{T}_1}}, \tilde{\theta}_1 - \partial_1\tilde{u}\rangle_{L_2(\Omega)}$,

$\%\ \vec{e}^{\mathscr{T}_2} = \langle \Phi|_{\breve{\Pi}^{\mathscr{T}_2}}, \tilde{\theta}_2 - \partial_2 \tilde{u} \rangle_{L_2(\Omega)}$,
$\%\ \vec{f}^{\mathscr{T}_2} = \langle \Psi^{\mathscr{T}}|_{\breve{\Lambda}^{\mathscr{T}_2}}, \tilde{\theta}_2 - \partial_2 \tilde{u} \rangle_{L_2(\Omega)}$

`if` $\breve{\Pi}^{\mathscr{U}} \cup \breve{\Pi}^{\mathscr{T}_1} \cup \breve{\Pi}^{\mathscr{T}_2} \neq \emptyset$ `and` $\Pi^{\mathscr{U}} \cup \Pi^{\mathscr{T}_1} \cup \Pi^{\mathscr{T}_2} \neq \emptyset$ `then`

$\breve{G} := \Sigma_\ell(\Lambda^{\mathscr{U}}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_1}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_2})$,
$\breve{\Pi}_B^{\mathscr{U}} := \breve{G} \cap \breve{\Pi}^{\mathscr{U}}$, $\breve{\Pi}_A^{\mathscr{U}} := \breve{\Pi}^{\mathscr{U}} \setminus \breve{\Pi}_B^{\mathscr{U}}$
$\breve{\Pi}_B^{\mathscr{T}_1} := \breve{G} \cap \breve{\Pi}^{\mathscr{T}_1}$, $\breve{\Pi}_A^{\mathscr{U}} := \breve{\Pi}^{\mathscr{T}_1} \setminus \breve{\Pi}_B^{\mathscr{T}_1}$
$\breve{\Pi}_B^{\mathscr{T}_2} := \breve{G} \cap \breve{\Pi}^{\mathscr{T}_2}$, $\breve{\Pi}_A^{\mathscr{U}} := \breve{\Pi}^{\mathscr{T}_2} \setminus \breve{\Pi}_B^{\mathscr{T}_2}$

$G := \Sigma_\ell(\breve{\Lambda}^{\mathscr{U}}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_1}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_2}) \cup \breve{\Pi}_B^{\mathscr{U}} \cup \breve{\Pi}_B^{\mathscr{T}_1} \cup \breve{\Pi}_B^{\mathscr{T}_2}$,
$\Pi_B^{\mathscr{U}} := G \cap \Pi^{\mathscr{U}}$, $\Pi_A^{\mathscr{U}} := \Pi^{\mathscr{U}} \setminus \Pi_B^{\mathscr{U}}$
$\Pi_B^{\mathscr{T}_1} := G \cap \Pi^{\mathscr{T}_1}$, $\Pi_A^{\mathscr{U}} := \Pi^{\mathscr{T}_1} \setminus \Pi_B^{\mathscr{T}_1}$
$\Pi_B^{\mathscr{T}_2} := G \cap \Pi^{\mathscr{T}_2}$, $\Pi_A^{\mathscr{U}} := \Pi^{\mathscr{T}_2} \setminus \Pi_B^{\mathscr{T}_2}$

$\underline{\breve{\Pi}}^{\mathscr{U}} := \mathtt{child}(\breve{\Pi}_B^{\mathscr{U}}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{U}})$
$\underline{\breve{\Pi}}^{\mathscr{T}_1} := \mathtt{child}(\breve{\Pi}_B^{\mathscr{T}_1}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_1})$
$\underline{\breve{\Pi}}^{\mathscr{T}_2} := \mathtt{child}(\breve{\Pi}_B^{\mathscr{T}_2}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_2})$

$\underline{\Pi}^{\mathscr{U}} := \mathtt{child}(\Pi_B^{\mathscr{U}}) \cup \Sigma_\ell(\Lambda^{\mathscr{U}})$
$\underline{\Pi}^{\mathscr{T}_1} := \mathtt{child}(\Pi_B^{\mathscr{T}_1}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_1})$
$\underline{\Pi}^{\mathscr{T}_2} := \mathtt{child}(\Pi_B^{\mathscr{T}_2}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_2})$

$\underline{\vec{d}}^{\mathscr{U}} := \mathfrak{p}_\ell \vec{d}^{\mathscr{U}}|_{\Pi_B^{\mathscr{U}}} + \mathfrak{q}_\ell^{\mathscr{U}} \vec{c}^{\mathscr{U}}|_{\Lambda_\ell^{\mathscr{U}}}$
$\underline{\vec{d}}^{\mathscr{T}_1} := \mathfrak{p}_\ell \vec{d}^{\mathscr{T}_1}|_{\Pi_B^{\mathscr{T}_1}} + \mathfrak{q}_\ell^{\mathscr{T}} \vec{c}^{\mathscr{T}_1}|_{\Lambda_\ell^{\mathscr{T}_1}}$
$\underline{\vec{d}}^{\mathscr{T}_2} := \mathfrak{p}_\ell \vec{d}^{\mathscr{T}_2}|_{\Pi_B^{\mathscr{T}_2}} + \mathfrak{q}_\ell^{\mathscr{T}} \vec{c}^{\mathscr{T}_2}|_{\Lambda_\ell^{\mathscr{T}_2}}$

$[\underline{\vec{e}}^{\mathscr{U}}, \underline{\vec{f}}^{\mathscr{U}}, \underline{\vec{e}}^{\mathscr{T}_1}, \underline{\vec{f}}^{\mathscr{T}_1}, \underline{\vec{e}}^{\mathscr{T}_2}, \underline{\vec{f}}^{\mathscr{T}_2}] := \mathtt{eval1}(\ell+1, \underline{\breve{\Pi}}^{\mathscr{U}}, \breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{U}}, \underline{\breve{\Pi}}^{\mathscr{T}_1}, \breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{T}_1}, \underline{\breve{\Pi}}^{\mathscr{T}_2}, \breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{T}_2},$
$\underline{\Pi}^{\mathscr{U}}, \Lambda_{\ell+1\uparrow}^{\mathscr{U}}, \underline{\Pi}^{\mathscr{T}_1}, \Lambda_{\ell+1\uparrow}^{\mathscr{T}_1}, \underline{\Pi}^{\mathscr{T}_2}, \Lambda_{\ell+1\uparrow}^{\mathscr{T}_2}, \underline{\vec{d}}^{\mathscr{U}}, \vec{c}^{\mathscr{U}}|_{\Lambda_{\ell+1\uparrow}^{\mathscr{U}}}, \underline{\vec{d}}^{\mathscr{T}_1}, \vec{c}^{\mathscr{T}_1}|_{\Lambda_{\ell+1\uparrow}^{\mathscr{T}_1}}, \underline{\vec{d}}^{\mathscr{T}_2}, \vec{c}^{\mathscr{T}_2}|_{\Lambda_{\ell+1\uparrow}^{\mathscr{T}_2}})$

$$\vec{e}^{\mathscr{U}} = \begin{bmatrix} \vec{e}^{\mathscr{U}}|_{\breve{\Pi}_A^{\mathscr{U}}} \\ \vec{e}^{\mathscr{U}}|_{\breve{\Pi}_B^{\mathscr{U}}} \end{bmatrix} := \begin{bmatrix} \langle \nabla\Phi|_{\breve{\Pi}_A^{\mathscr{U}}}, \nabla\Phi|_{\Pi^{\mathscr{U}}} \rangle_{L_2(\Omega)^2} \vec{d}^{\mathscr{U}} - \sum_{i=1}^{2} \langle \partial_i \Phi|_{\breve{\Pi}_A^{\mathscr{U}}}, \Phi|_{\Pi^{\mathscr{T}_i}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{T}_i} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{U}})|_{\breve{\Pi}_B^{\mathscr{U}}} \end{bmatrix}$$
$$\vec{f}^{\mathscr{U}} = \begin{bmatrix} \vec{f}^{\mathscr{U}}|_{\breve{\Lambda}_\ell^{\mathscr{U}}} \\ \vec{f}^{\mathscr{U}}|_{\breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{U}}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}_\ell^{\mathscr{U}})^\top \underline{\vec{e}}^{\mathscr{U}})|_{\breve{\Lambda}_\ell^{\mathscr{U}}} \\ \underline{\vec{f}}^{\mathscr{U}} \end{bmatrix}$$
$$\vec{e}^{\mathscr{T}_1} = \begin{bmatrix} \vec{e}^{\mathscr{T}_1}|_{\breve{\Pi}_A^{\mathscr{T}_1}} \\ \vec{e}^{\mathscr{T}_1}|_{\breve{\Pi}_B^{\mathscr{T}_1}} \end{bmatrix} := \begin{bmatrix} \langle \Phi|_{\breve{\Pi}_A^{\mathscr{T}_1}}, \Phi|_{\Pi^{\mathscr{T}_1}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{T}_1} - \langle \Phi|_{\breve{\Pi}_A^{\mathscr{T}_1}}, \partial_1 \Phi|_{\Pi^{\mathscr{U}}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{U}} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{T}_1})|_{\breve{\Pi}_B^{\mathscr{T}_1}} \end{bmatrix}$$
$$\vec{f}^{\mathscr{T}_1} = \begin{bmatrix} \vec{f}^{\mathscr{T}_1}|_{\breve{\Lambda}_\ell^{\mathscr{T}_1}} \\ \vec{f}^{\mathscr{T}_1}|_{\breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{T}_1}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}_\ell^{\mathscr{T}})^\top \underline{\vec{e}}^{\mathscr{T}_1})|_{\breve{\Lambda}_\ell^{\mathscr{T}_1}} \\ \underline{\vec{f}}^{\mathscr{T}_1} \end{bmatrix}$$

$$\vec{e}^{\mathscr{T}_2} = \begin{bmatrix} \vec{e}^{\mathscr{T}_2}|_{\breve{\Pi}_A^{\mathscr{T}_2}} \\ \vec{e}^{\mathscr{T}_2}|_{\breve{\Pi}_B^{\mathscr{T}_2}} \end{bmatrix} := \begin{bmatrix} \langle \Phi|_{\breve{\Pi}_A^{\mathscr{T}_2}}, \Phi|_{\Pi^{\mathscr{T}_2}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{T}_2} - \langle \Phi|_{\breve{\Pi}_A^{\mathscr{T}_2}}, \partial_2 \Phi|_{\Pi^{\mathscr{U}}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{U}} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{T}_2})|_{\breve{\Pi}_B^{\mathscr{T}_2}} \end{bmatrix}$$

$$\vec{f}^{\mathscr{T}_2} = \begin{bmatrix} \vec{f}^{\mathscr{T}_2}|_{\breve{\Lambda}_\ell^{\mathscr{T}_2}} \\ \vec{f}^{\mathscr{T}_2}|_{\breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{T}_2}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}_\ell^{\mathscr{T}})^\top \underline{\vec{e}}^{\mathscr{T}_2})|_{\breve{\Lambda}_\ell^{\mathscr{T}_2}} \\ \underline{\vec{f}}^{\mathscr{T}_2} \end{bmatrix}$$

`endif`

The next algorithm **eval2** which calculates the part in (5.6.3) is as follows.

$[\vec{e},\, \vec{f}] :=$**eval2**$(\ell, \breve{\Pi}^{\mathscr{V}}, \breve{\Lambda}^{\mathscr{V}}, \Pi^{\mathscr{U}}, \Lambda^{\mathscr{U}}, \Pi^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_1}, \Pi^{\mathscr{T}_2}, \Lambda^{\mathscr{T}_2}, \vec{d}^{\mathscr{U}}, \vec{c}^{\mathscr{U}}, \vec{d}^{\mathscr{T}_1}, \vec{c}^{\mathscr{T}_1}, \vec{d}^{\mathscr{T}_2}, \vec{c}^{\mathscr{T}_2})$
% *Input:* $f \in L_2(\Omega)$, $\ell \in \mathbb{N}$, $\breve{\Pi}^{\mathscr{V}}, \Pi^{\mathscr{U}}, \Pi^{\mathscr{T}_1}, \Pi^{\mathscr{T}_2} \subset \mathcal{T}_{\ell-1}$,
% ℓ-*trees* $\breve{\Lambda}^{\mathscr{V}} \subset \vee_{\ell\uparrow}^{\mathscr{V}}, \Lambda^{\mathscr{U}} \subset \vee_{\ell\uparrow}^{\mathscr{U}}$, $\Lambda^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_2} \subset \vee_{\ell\uparrow}^{\mathscr{T}}$,
% $\vec{d}^{\mathscr{U}} = (d_\lambda^{\mathscr{U}})_{\lambda \in \Pi^{\mathscr{U}}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{U}} = (c_\lambda^{\mathscr{U}})_{\lambda \in \Lambda^{\mathscr{U}}} \subset \mathbb{R}$.
% $\vec{d}^{\mathscr{T}_1} = (d_\lambda^{\mathscr{T}_1})_{\lambda \in \Pi^{\mathscr{T}_1}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{T}_1} = (c_\lambda^{\mathscr{T}_1})_{\lambda \in \Lambda^{\mathscr{T}_1}} \subset \mathbb{R}$.
% $\vec{d}^{\mathscr{T}_2} = (d_\lambda^{\mathscr{T}_2})_{\lambda \in \Pi^{\mathscr{T}_2}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{T}_2} = (c_\lambda^{\mathscr{T}_2})_{\lambda \in \Lambda^{\mathscr{T}_2}} \subset \mathbb{R}$.
% *Output: With* $\tilde{u} := (\vec{d}^{\mathscr{U}})^\top \Phi|_{\Pi^{\mathscr{U}}} + (\vec{c}^{\mathscr{U}})^\top \Psi^{\mathscr{U}}|_{\Lambda^{\mathscr{U}}}$,
% $\tilde{\theta}_1 := (\vec{d}^{\mathscr{T}_1})^\top \Phi|_{\Pi^{\mathscr{T}_1}} + (\vec{c}^{\mathscr{T}_1})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_1}}$,
% $\tilde{\theta}_2 := (\vec{d}^{\mathscr{T}_2})^\top \Phi|_{\Pi^{\mathscr{T}_2}} + (\vec{c}^{\mathscr{T}_2})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_2}}$,
% $\vec{e}^{\mathscr{V}} = \langle \Phi|_{\breve{\Pi}^{\mathscr{V}}}, N(\tilde{u}) - f - \sum_{i=1}^2 \partial_i \tilde{\theta}_i \rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{V}} = \langle \Psi^{\mathscr{V}}|_{\breve{\Lambda}^{\mathscr{V}}}, N(\tilde{u}) - f - \sum_{i=1}^2 \partial_i \tilde{\theta}_i \rangle_{L_2(\Omega)}$.

`if` $\breve{\Pi}^{\mathscr{V}} \neq \emptyset$ `then`
 $\breve{G} := \Sigma_\ell(\Lambda^{\mathscr{U}}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_1}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_2})$,
 $\breve{\Pi}_B^{\mathscr{V}} := \breve{\Pi}^{\mathscr{V}} \cap \breve{G}$, $\breve{\Pi}_A^{\mathscr{V}} := \breve{\Pi}^{\mathscr{V}} \setminus \breve{\Pi}_B^{\mathscr{V}}$

 $G := \Sigma_\ell(\breve{\Lambda}^{\mathscr{V}}) \cup \breve{\Pi}_B^{\mathscr{V}}$,
 $\Pi_B^{\mathscr{U}} := \Pi^{\mathscr{U}} \cap G$, $\Pi_A^{\mathscr{U}} := \Pi^{\mathscr{U}} \setminus \Pi_B^{\mathscr{U}}$
 $\Pi_B^{\mathscr{T}_1} := \Pi^{\mathscr{T}_1} \cap G$, $\Pi_A^{\mathscr{T}_1} := \Pi^{\mathscr{T}_1} \setminus \Pi_B^{\mathscr{T}_1}$
 $\Pi_B^{\mathscr{T}_2} := \Pi^{\mathscr{T}_2} \cap G$, $\Pi_A^{\mathscr{T}_2} := \Pi^{\mathscr{T}_2} \setminus \Pi_B^{\mathscr{T}_2}$

 $\underline{\breve{\Pi}}^{\mathscr{V}} := \mathtt{child}(\breve{\Pi}_B^{\mathscr{V}}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{V}})$

 $\underline{\Pi}^{\mathscr{U}} := \mathtt{child}(\Pi_B^{\mathscr{U}}) \cup \Sigma_\ell(\Lambda^{\mathscr{U}})$
 $\underline{\Pi}^{\mathscr{T}_1} := \mathtt{child}(\Pi_B^{\mathscr{T}_1}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_1})$
 $\underline{\Pi}^{\mathscr{T}_2} := \mathtt{child}(\Pi_B^{\mathscr{T}_2}) \cup \Sigma_\ell(\Lambda^{\mathscr{T}_2})$

 $\underline{\vec{d}}^{\mathscr{U}} := \big(\mathfrak{p}_\ell \vec{d}^{\mathscr{U}}|_{\Pi_B^{\mathscr{U}}} + \mathfrak{q}_\ell^{\mathscr{U}} \vec{c}^{\mathscr{U}}|_{\Lambda_\ell^{\mathscr{U}}}\big)|_{\underline{\Pi}^{\mathscr{U}}}$
 $\underline{\vec{d}}^{\mathscr{T}_1} := \big(\mathfrak{p}_\ell \vec{d}^{\mathscr{T}_1}|_{\Pi_B^{\mathscr{T}_1}} + \mathfrak{q}_\ell^{\mathscr{T}} \vec{c}^{\mathscr{T}_1}|_{\Lambda_\ell^{\mathscr{T}_1}}\big)|_{\underline{\Pi}^{\mathscr{T}_1}}$
 $\underline{\vec{d}}^{\mathscr{T}_2} := \big(\mathfrak{p}_\ell \vec{d}^{\mathscr{T}_2}|_{\Pi_B^{\mathscr{T}_2}} + \mathfrak{q}_\ell^{\mathscr{T}} \vec{c}^{\mathscr{T}_2}|_{\Lambda_\ell^{\mathscr{T}_2}}\big)|_{\underline{\Pi}^{\mathscr{T}_2}}$

$$[\underline{\vec{e}}^{\mathscr{V}},\,\underline{\vec{f}}^{\mathscr{V}}] := \texttt{eval2}(\ell+1, \breve{\underline{\Pi}}^{\mathscr{V}}, \breve{\Lambda}^{\mathscr{V}}_{\ell+1\uparrow}, \underline{\Pi}^{\mathscr{U}}, \Lambda^{\mathscr{U}}_{\ell+1\uparrow}, \underline{\Pi}^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_1}_{\ell+1\uparrow}, \underline{\Pi}^{\mathscr{T}_2}, \Lambda^{\mathscr{T}_2}_{\ell+1\uparrow}, \underline{\vec{d}}^{\mathscr{U}}, \vec{c}^{\mathscr{U}}|_{\Lambda^{\mathscr{U}}_{\ell+1\uparrow}}, \underline{\vec{d}}^{\mathscr{T}_1}, \vec{c}^{\mathscr{T}_1}|_{\Lambda^{\mathscr{T}_1}_{\ell+1\uparrow}}, \underline{\vec{d}}^{\mathscr{T}_2}, \vec{c}^{\mathscr{T}_2}|_{\Lambda^{\mathscr{T}_2}_{\ell+1\uparrow}})$$

$$\vec{e}^{\mathscr{V}} = \begin{bmatrix} \vec{e}^{\mathscr{V}}|_{\breve{\Pi}^{\mathscr{V}}_A} \\ \vec{e}^{\mathscr{V}}|_{\breve{\Pi}^{\mathscr{V}}_B} \end{bmatrix} := \begin{bmatrix} \left[\left\langle \Phi_T, N((d^{\mathscr{U}}_T)^\top \Phi_T) - f|_T - \sum_{i=1}^2 (d^{\mathscr{T}_i}_T)^\top \partial_i \Phi_T \right\rangle_{L_2(T)}\right]_{T\in\breve{\Pi}^{\mathscr{V}}_A} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{V}})|_{\breve{\Pi}^{\mathscr{V}}_B} \end{bmatrix}$$

$$\vec{f}^{\mathscr{V}} = \begin{bmatrix} \vec{f}^{\mathscr{V}}|_{\breve{\Lambda}^{\mathscr{V}}_\ell} \\ \vec{f}^{\mathscr{V}}|_{\breve{\Lambda}^{\mathscr{V}}_{\ell+1\uparrow}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}^{\mathscr{V}}_\ell)^\top \underline{\vec{e}}^{\mathscr{V}})|_{\breve{\Lambda}^{\mathscr{V}}_\ell} \\ \underline{\vec{f}}^{\mathscr{V}} \end{bmatrix}$$

`endif`

The series of algorithms is completed by **eval3** that computes the third part of the residual as given in (5.6.4).

$$[\vec{e}^{\mathscr{U}},\,\vec{f}^{\mathscr{U}}, \vec{e}^{\mathscr{T}_1}, \vec{f}^{\mathscr{T}_1}, \vec{e}^{\mathscr{T}_2},\,\vec{f}^{\mathscr{T}_2}] := \texttt{eval3}(\ell, \breve{\Pi}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{U}}, \breve{\Pi}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_1}, \breve{\Pi}^{\mathscr{T}_2}, \breve{\Lambda}^{\mathscr{T}_2}, \Pi^{\mathscr{U}}, \Lambda^{\mathscr{U}}, \Pi^{\mathscr{V}}, \Lambda^{\mathscr{V}}, \vec{d}^{\mathscr{U}}, \vec{c}^{\mathscr{U}}, \vec{d}^{\mathscr{V}}, \vec{c}^{\mathscr{V}})$$

% *Input:* $\ell \in \mathbb{N}$, $\breve{\Pi}^{\mathscr{U}}, \breve{\Pi}^{\mathscr{T}_1}, \breve{\Pi}^{\mathscr{T}_2}, \Pi^{\mathscr{U}}, \Pi^{\mathscr{V}} \subset \mathcal{T}_{\ell-1}$,
% ℓ-trees $\breve{\Lambda}^{\mathscr{U}}, \Lambda^{\mathscr{U}} \subset \vee^{\mathscr{U}}_{\ell\uparrow}$, $\breve{\Lambda}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_2} \subset \vee^{\mathscr{T}}_{\ell\uparrow}$, $\Lambda^{\mathscr{V}} \subset \vee^{\mathscr{V}}_{\ell\uparrow}$
% $\vec{d}^{\mathscr{U}} = (d^{\mathscr{U}}_\lambda)_{\lambda\in\Pi^{\mathscr{U}}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{U}} = (c^{\mathscr{U}}_\lambda)_{\lambda\in\Lambda^{\mathscr{U}}} \subset \mathbb{R}$.
% $\vec{d}^{\mathscr{V}} = (d^{\mathscr{V}}_\lambda)_{\lambda\in\Pi^{\mathscr{V}}} \subset \mathbb{R}^6$, $\vec{c}^{\mathscr{V}} = (c^{\mathscr{V}}_\lambda)_{\lambda\in\Lambda^{\mathscr{V}}} \subset \mathbb{R}$.
% *Output: With* $\tilde{u} := (\vec{d}^{\mathscr{U}})^\top \Phi|_{\Pi^{\mathscr{U}}} + (\vec{c}^{\mathscr{U}})^\top \Psi^{\mathscr{U}}|_{\Lambda^{\mathscr{U}}}$,
% $\tilde{v} := (\vec{d}^{\mathscr{V}})^\top \Phi|_{\Pi^{\mathscr{V}}} + (\vec{c}^{\mathscr{V}})^\top \Psi^{\mathscr{V}}|_{\Lambda^{\mathscr{V}}}$,
% $\vec{e}^{\mathscr{U}} = \langle \Phi|_{\breve{\Pi}^{\mathscr{U}}}, DN(\tilde{u})\tilde{v}\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{U}} = \langle \Psi^{\mathscr{U}}|_{\breve{\Lambda}^{\mathscr{U}}}, DN(\tilde{u})\tilde{v}\rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_1} = \langle \Phi|_{\breve{\Pi}^{\mathscr{T}_1}}, \partial_1\tilde{v}\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_1} = \langle \Psi^{\mathscr{T}}|_{\breve{\Lambda}^{\mathscr{T}_1}}, \partial_1\tilde{v}\rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_2} = \langle \Phi|_{\breve{\Pi}^{\mathscr{T}_2}}, \partial_2\tilde{v}\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_2} = \langle \Psi^{\mathscr{T}}|_{\breve{\Lambda}^{\mathscr{T}_2}}, \partial_2\tilde{v}\rangle_{L_2(\Omega)}$

`if` $\breve{\Pi}^{\mathscr{U}} \cup \breve{\Pi}^{\mathscr{T}_1} \cup \breve{\Pi}^{\mathscr{T}_2} \neq \emptyset$ `and` $\Pi^{\mathscr{U}} \cup \Pi^{\mathscr{V}} \neq \emptyset$ `then`

$\breve{G} := \Sigma_\ell(\Lambda^{\mathscr{U}}) \cup \Sigma_\ell(\Lambda^{\mathscr{V}})$,
$\breve{\Pi}^{\mathscr{U}}_B := \breve{\Pi}^{\mathscr{U}} \cap \breve{G}$, $\breve{\Pi}^{\mathscr{U}}_A := \breve{\Pi}^{\mathscr{U}} \setminus \breve{\Pi}^{\mathscr{U}}_B$
$\breve{\Pi}^{\mathscr{T}_1}_B := \breve{\Pi}^{\mathscr{T}_1} \cap \breve{G}$, $\breve{\Pi}^{\mathscr{T}_1}_A := \breve{\Pi}^{\mathscr{T}_1} \setminus \breve{\Pi}^{\mathscr{T}_1}_B$
$\breve{\Pi}^{\mathscr{T}_2}_B := \breve{\Pi}^{\mathscr{T}_2} \cap \breve{G}$, $\breve{\Pi}^{\mathscr{T}_2}_A := \breve{\Pi}^{\mathscr{T}_2} \setminus \breve{\Pi}^{\mathscr{T}_2}_B$

$G := \Sigma_\ell(\breve{\Lambda}^{\mathscr{U}}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_1}) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_2}) \cup \breve{\Pi}^{\mathscr{U}}_B \cup \breve{\Pi}^{\mathscr{T}_1}_B \cup \breve{\Pi}^{\mathscr{T}_2}_B$,
$\Pi^{\mathscr{U}}_B := \Pi^{\mathscr{U}} \cap G$, $\Pi^{\mathscr{U}}_A := \Pi^{\mathscr{U}} \setminus \Pi^{\mathscr{U}}_B$
$\Pi^{\mathscr{V}}_B := \Pi^{\mathscr{V}} \cap G$, $\Pi^{\mathscr{V}}_A := \Pi^{\mathscr{V}} \setminus \Pi^{\mathscr{V}}_B$

$\underline{\breve{\Pi}}^{\mathscr{U}} := \texttt{child}(\breve{\Pi}^{\mathscr{U}}_B) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{U}})$
$\underline{\breve{\Pi}}^{\mathscr{T}_1} := \texttt{child}(\breve{\Pi}^{\mathscr{T}_1}_B) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_1})$
$\underline{\breve{\Pi}}^{\mathscr{T}_2} := \texttt{child}(\breve{\Pi}^{\mathscr{T}_2}_B) \cup \Sigma_\ell(\breve{\Lambda}^{\mathscr{T}_2})$

$$\underline{\Pi}^{\mathscr{U}} := \texttt{child}(\Pi_B^{\mathscr{U}}) \cup \Sigma_\ell(\Lambda^{\mathscr{U}})$$
$$\underline{\Pi}^{\mathscr{V}} := \texttt{child}(\Pi_B^{\mathscr{V}}) \cup \Sigma_\ell(\Lambda^{\mathscr{V}})$$

$$\underline{\vec{d}}^{\mathscr{U}} := \big(\mathfrak{p}_\ell \vec{d}^{\mathscr{U}}|_{\Pi_B^{\mathscr{U}}} + \mathfrak{q}_\ell^{\mathscr{U}} \vec{c}^{\mathscr{U}}|_{\Lambda_\ell^{\mathscr{U}}}\big)|_{\underline{\Pi}^{\mathscr{U}}}$$
$$\underline{\vec{d}}^{\mathscr{V}} := \big(\mathfrak{p}_\ell \vec{d}^{\mathscr{V}}|_{\Pi_B^{\mathscr{V}}} + \mathfrak{q}_\ell^{\mathscr{V}} \vec{c}^{\mathscr{V}}|_{\Lambda_\ell^{\mathscr{V}}}\big)|_{\underline{\Pi}^{\mathscr{V}}}$$

$$[\underline{\vec{e}}^{\mathscr{U}}, \underline{\vec{f}}^{\mathscr{U}}, \underline{\vec{e}}^{\mathscr{T}_1}, \underline{\vec{f}}^{\mathscr{T}_1}, \underline{\vec{e}}^{\mathscr{T}_2}, \underline{\vec{f}}^{\mathscr{T}_2}] := \texttt{eval3}(\ell+1, \breve{\underline{\Pi}}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{U}}_{\ell+1\uparrow}, \breve{\underline{\Pi}}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_1}_{\ell+1\uparrow}, \breve{\underline{\Pi}}^{\mathscr{T}_2}, \breve{\Lambda}^{\mathscr{T}_2}_{\ell+1\uparrow},$$
$$\underline{\Pi}^{\mathscr{U}}, \Lambda^{\mathscr{U}}_{\ell+1\uparrow}, \underline{\Pi}^{\mathscr{V}}, \Lambda^{\mathscr{V}}_{\ell+1\uparrow}, \underline{\vec{d}}^{\mathscr{U}}, \vec{c}^{\mathscr{U}}|_{\Lambda^{\mathscr{U}}_{\ell+1\uparrow}}, \underline{\vec{d}}^{\mathscr{V}}, \vec{c}^{\mathscr{V}}|_{\Lambda^{\mathscr{V}}_{\ell+1\uparrow}})$$

$$\vec{e}^{\mathscr{U}} = \begin{bmatrix} \vec{e}^{\mathscr{U}}|_{\breve{\Pi}_A^{\mathscr{U}}} \\ \vec{e}^{\mathscr{U}}|_{\breve{\Pi}_B^{\mathscr{U}}} \end{bmatrix} := \begin{bmatrix} \Big[\big\langle \Phi_T, DN((d_T^{\mathscr{U}})^\top \Phi_T)(d_T^{\mathscr{V}})^\top \Phi_T \big\rangle_{L_2(T)}\Big]_{T\in\breve{\Pi}_A^{\mathscr{U}}} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{U}})|_{\breve{\Pi}_B^{\mathscr{U}}} \end{bmatrix}$$
$$\vec{f}^{\mathscr{U}} = \begin{bmatrix} \vec{f}^{\mathscr{U}}|_{\breve{\Lambda}_\ell^{\mathscr{U}}} \\ \vec{f}^{\mathscr{U}}|_{\breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{U}}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}_\ell^{\mathscr{U}})^\top \underline{\vec{e}}^{\mathscr{U}})|_{\breve{\Lambda}_\ell^{\mathscr{U}}} \\ \underline{\vec{f}}^{\mathscr{U}} \end{bmatrix}$$
$$\vec{e}^{\mathscr{T}_1} = \begin{bmatrix} \vec{e}^{\mathscr{T}_1}|_{\breve{\Pi}_A^{\mathscr{T}_1}} \\ \vec{e}^{\mathscr{T}_1}|_{\breve{\Pi}_B^{\mathscr{T}_1}} \end{bmatrix} := \begin{bmatrix} \langle \Phi|_{\breve{\Pi}_A^{\mathscr{T}_1}}, \partial_1 \Phi|_{\Pi^{\mathscr{V}}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{V}} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{T}_1})|_{\breve{\Pi}_B^{\mathscr{T}_1}} \end{bmatrix}$$
$$\vec{f}^{\mathscr{T}_1} = \begin{bmatrix} \vec{f}^{\mathscr{T}_1}|_{\breve{\Lambda}_\ell^{\mathscr{T}_1}} \\ \vec{f}^{\mathscr{T}_1}|_{\breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{T}_1}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}_\ell^{\mathscr{T}})^\top \underline{\vec{e}}^{\mathscr{T}_1})|_{\breve{\Lambda}_\ell^{\mathscr{T}_1}} \\ \underline{\vec{f}}^{\mathscr{T}_1} \end{bmatrix}$$
$$\vec{e}^{\mathscr{T}_2} = \begin{bmatrix} \vec{e}^{\mathscr{T}_2}|_{\breve{\Pi}_A^{\mathscr{T}_2}} \\ \vec{e}^{\mathscr{T}_2}|_{\breve{\Pi}_B^{\mathscr{T}_2}} \end{bmatrix} := \begin{bmatrix} \langle \Phi|_{\breve{\Pi}_A^{\mathscr{T}_2}}, \partial_2 \Phi|_{\Pi^{\mathscr{V}}} \rangle_{L_2(\Omega)} \vec{d}^{\mathscr{V}} \\ (\mathfrak{p}_\ell^\top \underline{\vec{e}}^{\mathscr{T}_2})|_{\breve{\Pi}_B^{\mathscr{T}_2}} \end{bmatrix}$$
$$\vec{f}^{\mathscr{T}_2} = \begin{bmatrix} \vec{f}^{\mathscr{T}_2}|_{\breve{\Lambda}_\ell^{\mathscr{T}_2}} \\ \vec{f}^{\mathscr{T}_2}|_{\breve{\Lambda}_{\ell+1\uparrow}^{\mathscr{T}_2}} \end{bmatrix} := \begin{bmatrix} (\mathfrak{q}_\ell^{\mathscr{T}})^\top \underline{\vec{e}}^{\mathscr{T}_2})|_{\breve{\Lambda}_\ell^{\mathscr{T}_2}} \\ \underline{\vec{f}}^{\mathscr{T}_2} \end{bmatrix}$$

`endif`

5.6.7 A possible implementation of eval**1**, eval**2**, and eval**3**

At this point we propose a possible implementation by using pseudocode to interpret the algorithms eval**1**, eval**2**, and eval**3** in a structured body of commands oriented, but not bounded, to an implementation in the C programming language. We first give some information that apply to all the algorithms implemented.

To begin with, we consider an initial triangulation of the L-shaped domain Ω and store it in a *linked list* of data cells (see Figure 5.8) at level 0. Gradually, any new triangulation is added in a linked list in subsequent slots. The level of any triangulation agrees with the level of a wavelet with respect to which the latter are piecewise polynomials of their specified degree (cf. Remark 5.3.6). For instance, a triangulation on level ℓ, contains triangles of mesh size $h = 2^{-\ell-1}$ and relates to the support of quadratic wavelets on the same level. On the other hand, the coarsest level

of piecewise linear wavelets is level 1 since no linear scaling functions appear on the initial triangulation in view of the Dirichlet boundary conditions.

Data cells in triangulations contain information about the position of triangles in Ω. A stable way to store the *position* of a triangle T is the following: we keep record of the vertices $\vec{w}_1, \vec{w}_2, \vec{w}_3$ of its ancestor in the initial triangulation by storing a pointer `ancestor` pointing to a place where these positions are stored. For each vertex $\vec{v}_i$ of T we find the barycentric coordinates $(\sigma_1^{(i)}, \sigma_2^{(i)}, \sigma_3^{(i)})$ of $\vec{v}_i$ w.r.t. $(\vec{w}_1, \vec{w}_2, \vec{w}_3)$, meaning that $\vec{v}_i = \sum_{j=1}^3 \sigma_j^{(i)} \vec{w}_j$. These barycentric coordinates are of the form $\vec{m}^{(i)}h$, being in our case $\vec{m}^{(i)} 2^{-\ell-1}$, where ℓ is the level of T, and $\vec{m}^{(i)} \in \mathbb{N}_0^3$. The vector of $\vec{m}^{(i)}$'s is then placed in a record, erasing any potential rounding error. Additionally, if you split a triangle into 4, their barycentric coordinates w.r.t. $(\vec{w}_1, \vec{w}_2, \vec{w}_3)$ read as

$$
\begin{aligned}
(2\vec{m}^{(1)}, \vec{m}^{(1)} + \vec{m}^{(2)}, \vec{m}^{(1)} + \vec{m}^{(3)}) 2^{-\ell-1} \\
(\vec{m}^{(1)} + \vec{m}^{(2)}, 2\vec{m}^{(2)}, \vec{m}^{(2)} + \vec{m}^{(3)}) 2^{-\ell-1} \\
(\vec{m}^{(1)} + \vec{m}^{(2)}, \vec{m}^{(2)} + \vec{m}^{(3)}, 2\vec{m}^{(3)}) 2^{-\ell-1} \\
(\vec{m}^{(1)} + \vec{m}^{(2)}, \vec{m}^{(2)} + \vec{m}^{(3)}, \vec{m}^{(1)} + \vec{m}^{(3)}) 2^{-\ell-1}.
\end{aligned}
$$

Triangulations are made of Lagrange triangular elements of at most order two and they form the support for the single-scale representation of any relevant function. For a more convenient implementation we take *discontinuous* scaling functions. For $T \in \mathfrak{T}$, Φ_T will denote the *nodal* basis of $P_2(T)$, $\Phi := (\Phi_T)_{T\in\mathfrak{T}}$, and $\Phi_\ell := (\Phi_T)_{\{T\in\mathfrak{T}:\ |T|=\ell\}}$. Coefficients corresponding to scaling functions in such a representation are stored in vectors at each triangle T, with the length of these vectors being equal to the dimension of $P_2(T)$, given by (5.4.1), and counts to 6.

Wavelets on the lowest level $\ell = 0$ are usually called scaling functions. To avoid confusion with the aforementioned discontinuous scaling functions, from now on we abandon this terminology and call also the wavelets on the lowest level 'wavelets'.

At last, as described in Sect. 5.5.1, all coefficient vectors w.r.t. wavelet bases are stored in hash-tables using the hash function defined in (5.5.1).

5.6.7.1 An implementation of `eval1`

Recall that the expression of the approximate residual includes several instances of L_2-products. Any object which corresponds to the left argument of the product is denoted with a '$\breve{}$' on top of its notation and any right one is left plain.

To accommodate transitions to single scale representation of the involved functions at both sides, we create sets of triangles associated to the left or right arguments under the notations $\breve{\Pi}$ or Π, respectively. We store $\Pi \cup \breve{\Pi}$ as a set U of triangles, implemented as a linked list.

Remark 5.6.2. It would be easier to implement two separate collections Π and $\breve{\Pi}$. In view of the definition of Π_B however, which involves the intersection $\Pi \cap \breve{\Pi}_B$, we need the possibility to check whether a triangle is in a set, in $\mathcal{O}(1)$ operations. To be able

to do this without having to store Π and $\breve{\Pi}$ in hash tables, we benefit from the use of boolean *flags* indicating membership of a triangle in different sets.

Then a triangle T is a record with data:

- booleans $\texttt{in}\breve{\Pi}^{\mathscr{U}}$, $\texttt{in}\breve{\Pi}^{\mathscr{T}_1}$, $\texttt{in}\breve{\Pi}^{\mathscr{T}_2}$; $\texttt{in}\Pi^{\mathscr{U}}$, $\texttt{in}\Pi^{\mathscr{T}_1}$, $\texttt{in}\Pi^{\mathscr{T}_2}$; $\texttt{in}\breve{\Pi}_B^{\mathscr{U}}$, $\texttt{in}\breve{\Pi}_B^{\mathscr{T}_1}$, $\texttt{in}\breve{\Pi}_B^{\mathscr{T}_2}$; $\texttt{in}\Pi_B^{\mathscr{U}}$, $\texttt{in}\Pi_B^{\mathscr{T}_1}$, $\texttt{in}\Pi_B^{\mathscr{T}_2}$
- the position of the triangle
- $\vec{d}^{\mathscr{U}}, \vec{d}^{\mathscr{T}_1}, \vec{d}^{\mathscr{T}_2}, \vec{e}^{\mathscr{U}}, \vec{e}^{\mathscr{T}_1}, \vec{e}^{\mathscr{T}_2} \in \mathbb{R}^6$ [1].
- set of 4 pointers `children`

At initialisation of any triangle, we set the booleans as `false`, and the vectors as zero (or do the latter only when the vectors are generated). Moreover, we initialise in the hash tables $\vec{f}^{\mathscr{U}}$, $\vec{f}^{\mathscr{T}_1}$, $\vec{f}^{\mathscr{T}_2}$ as $\vec{0}$.

Then, a call of `eval1` performs as follows.

eval**1**$(\ell, U, \breve{\Lambda}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_2}, \Lambda^{\mathscr{U}}, \Lambda^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_2})$
% Input: $\ell \in \mathbb{N}$, $U \subset \mathcal{T}_{\ell-1}$ is a set of triangles (records).
% For $T \in U$, if $T.\texttt{in}\Pi^*$=`true`, then $T \in \Pi^*$, and $T.\vec{d}^* = d_T^* \in \mathbb{R}^6$,
% if $T.\texttt{in}\breve{\Pi}^*$=`true`, then $T \in \breve{\Pi}^*$ ($* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$).
% $\breve{\Lambda}^*, \Lambda^* \subset \vee^*$ are trees stored as hash-tables but only their parts in $\vee^*_{\ell\uparrow}$ are touched.
% $\Lambda^* \cap \vee^*_{\ell\uparrow}$ contains $\vec{c}^*$.
% At termination: With $\tilde{u} := (\vec{d}^{\mathscr{U}})^\top \Phi|_{\Pi^{\mathscr{U}}} + (\vec{c}^{\mathscr{U}})^\top \Psi^{\mathscr{U}}|_{\Lambda^{\mathscr{U}} \cap \vee^{\mathscr{U}}_{\ell\uparrow}}$,
% $\tilde{\theta}_1 := (\vec{d}^{\mathscr{T}_1})^\top \Phi|_{\Pi^{\mathscr{T}_1}} + (\vec{c}^{\mathscr{T}_1})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_1} \cap \vee^{\mathscr{T}}_{\ell\uparrow}}$,
% $\tilde{\theta}_2 := (\vec{d}^{\mathscr{T}_2})^\top \Phi|_{\Pi^{\mathscr{T}_2}} + (\vec{c}^{\mathscr{T}_2})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_2} \cap \vee^{\mathscr{T}}_{\ell\uparrow}}$,
% $\vec{e}^{\mathscr{U}} := \langle \nabla\Phi, \nabla\tilde{u}\rangle_{L_2(\Omega)^2} - \sum_{i=1}^2 \langle \partial_i \Phi, \tilde{\theta}_i\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{U}} := \langle \nabla\Psi^{\mathscr{U}}, \nabla\tilde{u}\rangle_{L_2(\Omega)^2} - \sum_{i=1}^2 \langle \partial_i \Psi^{\mathscr{U}}, \tilde{\theta}_i\rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_1} := \langle \Phi, \tilde{\theta}_1 - \partial_1\tilde{u}\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_1} := \langle \Psi^{\mathscr{T}}, \tilde{\theta}_1 - \partial_1\tilde{u}\rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_2} := \langle \Phi, \tilde{\theta}_2 - \partial_2\tilde{u}\rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_2} := \langle \Psi^{\mathscr{T}}, \tilde{\theta}_2 - \partial_2\tilde{u}\rangle_{L_2(\Omega)}$,
% for $T.\texttt{in}\breve{\Pi}^*$=`true` it will hold that $T.\vec{e}^* = \vec{e}^*|_T$,
% and $\breve{\Lambda}^* \cap \vee^*_{\ell\uparrow}$ will contain $\vec{f}^*|_{\breve{\Lambda}^* \cap \vee^*_{\ell\uparrow}}$.
Run over the $T \in U$ until you have found at least one T with $T.\texttt{in}\breve{\Pi}^*$=true for some $* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, and one T' (possibly $T = T'$) with $T'.\texttt{in}\Pi^*$=true for some $* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$. If you don't succeed, then `stop` because apparently $\Pi^{\mathscr{U}} \cup \Pi^{\mathscr{T}_1} \cup \Pi^{\mathscr{T}_2}$ or $\breve{\Pi}^{\mathscr{U}} \cup \breve{\Pi}^{\mathscr{T}_1} \cup \breve{\Pi}^{\mathscr{T}_2}$ is empty.

`for` $T \in U$ `do`
  `for` $* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$ `do`
    `for` $\lambda \in \Lambda^*_\ell$ with $|T \cap \mathcal{S}(\psi^*_\lambda)| > 0$ `do` % i.e. $T \in$ `parent`$(\Sigma_\ell(\Lambda^*))$

[1] or include only pointers to such vectors, and create them only when $T.\texttt{in}\Pi^{\mathscr{U}}$, $T.\texttt{in}\Pi^{\mathscr{T}_1}$, or $T.\texttt{in}\Pi^{\mathscr{T}_2}$ is `true`, or when $T.\texttt{in}\breve{\Pi}^{\mathscr{U}}$, $T.\texttt{in}\breve{\Pi}^{\mathscr{T}_1}$, or $T.\texttt{in}\breve{\Pi}^{\mathscr{T}_2}$ respectively is `true`

```
        for T' ∈ child(T) do
            if T' ∉ {T.children}
            then create T' ∈ U
                 {T.children}:={T.children} ∪ &T'
            endif
            T'.inΠ*:=true
            if |T' ∩ S(ψ*_λ)| > 0        % i.e. T' ∈ Σ_ℓ(Λ*)
            then T'.d⃗* := T'.d⃗* + (q*_ℓ c⃗*_λ)|_T'
            endif
        endfor
        boolean beenset:=false
        for + ∈ {U, T1, T2} do
            if T.inΠ̆+
            then T.inΠ̆+_B:=true, beenset:=true
            endif
        endfor
        if beenset then        % T ∈ Π̆^U_B ∪ Π̆^T1_B ∪ Π̆^T2_B
           for + ∈ {U, T1, T2} do
               if T.inΠ+
               then T.inΠ+_B:=true
               endif
           endfor
        endif
    endfor
endfor

% ∀∗ ∈ {U, T1, T2}, Π̆*_B ∩ T has been set,
% U* ∩ child(T) = Σ_ℓ(Λ*) ∩ child(T),
% Π*_B = Π* ∩ (Π̆^U ∪ Π̆^T1 ∪ Π̆^T2) ∩ T,
% and d⃗* |_child(T)∩Σ_ℓ(Λ*) = (q*_ℓ c⃗*|_Λ*_ℓ)|_child(T)∩Σ_ℓ(Λ*).

for ∗ ∈ {U, T1, T2} do
    for λ ∈ Λ̆*_ℓ with |T ∩ S(ψ̆*_λ)| > 0 do        % i.e. T ∈ parent(Σ_ℓ(Λ̆*))
        for T' ∈ child(T) do
            if T' ∉ {T.children}
            then create T' ∈ U
                 {T.children}:={T.children} ∪ &T'
            endif
            T'.inΠ̆*:=true
        endfor
        for + ∈ {U, T1, T2} do
            if T.inΠ+
            then T.inΠ+_B:=true
```

```
            endif
        endfor
    endfor
endfor
```

% $\forall * \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, $\Pi^*_B \cap T$ has been set,
% and $\underline{\breve{\Pi}}^* \cap \mathtt{child}(T) = \Sigma_\ell(\breve{\Lambda}^*) \cap \mathtt{child}(T)$.

```
for * ∈ {𝒰, 𝒯1, 𝒯2} do
    if T.in𝚷̆*_B
    then for T' ∈ child(T) do
            if T' ∉ {T.children}
            then create T' ∈ U
                 {T.children}:={T.children} ∪ &T'
            endif
            T'.in𝚷̆*:=true
         endfor
    endif
endfor
```

% $\forall * \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, $\underline{\breve{\Pi}}^* \cap \mathtt{child}(T)$ has been set

```
for * ∈ {𝒰, 𝒯1, 𝒯2} do
    if T.in𝚷*_B
    then for T' ∈ child(T) do
            if T' ∉ {T.children}
            then create T' ∈ U
                 {T.children}:={T.children} ∪ &T'
            endif
            T'.in𝚷*:=true
            T'.d⃗* := T'.d⃗* + (𝔭ℓT.d⃗*)|T'
         endfor
    endif
endfor
```

% $\forall * \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, $\underline{\Pi}^* \cap \mathtt{children}(T)$ and $\underline{\vec{d}^*}|_{\mathtt{child}(T)}$ have been set.

`endfor` % end of the loop over $T \in U$

`eval1`$(\ell+1, \underline{U}, \breve{\Lambda}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_2}, \Lambda^{\mathscr{U}}, \Lambda^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_2})$

```
for T ∈ U with T.in𝚷̆^𝒰 = true do
    if T.in𝚷̆^𝒰_B
```

```
    then T.ē^𝒰 := Σ_{T'∈{T.children}} 𝔭_ℓ^⊤ T'.ē^𝒰
    else if T.in Π^𝒰
         then T.ē^𝒰 := ⟨∇Φ_T, ∇Φ_T⟩_{L2(T)²} T.d̄^𝒰
         endif
         if T.in Π^{𝒯1}
         then T.ē^𝒰 := T.ē^𝒰 − ⟨∂1Φ_T, Φ_T⟩_{L2(T)} T.d̄^{𝒯1}
         endif
         if T.in Π^{𝒯2}
         then T.ē^𝒰 := T.ē^𝒰 − ⟨∂2Φ_T, Φ_T⟩_{L2(T)} T.d̄^{𝒯2}
         endif
    endif
    for λ ∈ Λ̆_ℓ^𝒰 with |T ∩ S(ψ_λ^𝒰)| > 0 do
        for T' ∈ {T.children} with T'.in Π̆^𝒰 =true do
            f_λ^𝒰 := f_λ^𝒰 + (𝔮_ℓ^𝒰)^⊤ T'.ē^𝒰
        endfor
    endfor
endfor

for T ∈ U with T.in Π̆^{𝒯1} = true do
    if T.in Π̆_B^{𝒯1}
    then T.ē^{𝒯1} := Σ_{T'∈{T.children}} 𝔭_ℓ^⊤ T'.ē^{𝒯1}
    else if T.in Π^{𝒯1}
         then T.ē^{𝒯1} := ⟨Φ_T, Φ_T⟩_{L2(T)} T.d̄^{𝒯1}
         endif
         if T.in Π^𝒰
         then T.ē^{𝒯1} := T.ē^{𝒯1} − ⟨Φ_T, ∂1Φ_T⟩_{L2(T)} T.d̄^𝒰
         endif
    endif
    for λ ∈ Λ̆_ℓ^{𝒯1} with |T ∩ S(ψ_λ^𝒯)| > 0 do
        for T' ∈ {T.children} with T'.in Π̆^{𝒯1} =true do
            f_λ^{𝒯1} := f_λ^{𝒯1} + (𝔮_ℓ^𝒯)^⊤ T'.ē^{𝒯1}
        endfor
    endfor
endfor

for T ∈ U with T.in Π̆^{𝒯2} = true do
    if T.in Π̆_B^{𝒯2}
    then T.ē^{𝒯2} := Σ_{T'∈{T.children}} 𝔭_ℓ^⊤ T'.ē^{𝒯2}
    else if T.in Π^{𝒯2}
         then T.ē^{𝒯2} := ⟨Φ_T, Φ_T⟩_{L2(T)} T.d̄^{𝒯2}
         endif
         if T.in Π^𝒰
```

```
            then T.ē^{𝒯₂} := T.ē^{𝒯₂} − ⟨Φ_T, ∂₂Φ_T⟩_{L₂(T)} T.d⃗^{𝒰}
            endif
        endif
        for λ ∈ Λ̆_ℓ^{𝒯₂} with |T ∩ S(ψ_λ^{𝒯})| > 0 do
            for T′ ∈ {T.children} with T′.inΠ̆^{𝒯₂}=true do
                f_λ^{𝒯₂} := f_λ^{𝒯₂} + (q_ℓ^{𝒯})^⊤ T′.ē^{𝒯₂}
            endfor
        endfor
    endfor
```

After the execution, the set $\underline{U}$ can be deleted.

5.6.7.2 An implementation of eval2

The next implementation to consider is that of `eval2` where the main patterns of our practice will be identical. Again, a triangle set U contains $\breve{\Pi} \cup \Pi$ and it is used in a similar fashion as in `eval1`.

The contents of a triangle T are adapted to the current needs and the record consists of:

- booleans $\texttt{in}\breve{\Pi}$; $\texttt{in}\Pi^{\mathscr{U}}$, $\texttt{in}\Pi^{\mathscr{T}_1}$, $\texttt{in}\Pi^{\mathscr{T}_2}$; $\texttt{in}\breve{\Pi}_B$; $\texttt{in}\Pi_B^{\mathscr{U}}$, $\texttt{in}\Pi_B^{\mathscr{T}_1}$, $\texttt{in}\Pi_B^{\mathscr{T}_2}$
- the position of the triangle
- $\vec{d}^{\mathscr{U}}, \vec{d}^{\mathscr{T}_1}, \vec{d}^{\mathscr{T}_2}, \vec{e}^{\mathscr{V}} \in \mathbb{R}^6$ [2]
- set of 4 pointers `children`

At initialisation of any triangle, we set the booleans as `false`, and the vectors as zero (or the latter is done only when the vectors are generated). In addition we initialise in the hash tables $\vec{f}^{\mathscr{V}}$ as $\vec{0}$.

A call of `eval2` is executed in the following manner.

eval**2**$(\ell, U, \breve{\Lambda}^{\mathscr{V}}, \Lambda^{\mathscr{U}}, \Lambda^{\mathscr{T}_1}, \Lambda^{\mathscr{T}_2})$
% Input: $\ell \in \mathbb{N}$, $U \subset \mathcal{T}_{\ell-1}$ is a set of triangles (records).
% For $T \in U$, if $T.\texttt{in}\Pi^*$=`true`, then $T \in \Pi^*$, and $T.\vec{d}^* = d_T^* \in \mathbb{R}^6$ $(* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\})$,
% if $T.\texttt{in}\breve{\Pi}^{\mathscr{V}}$=`true`, then $T \in \breve{\Pi}^{\mathscr{V}}$
% $\breve{\Lambda}^{\mathscr{V}} \subset \vee^{\mathscr{V}}, \Lambda^* \subset \vee^*$ are trees stored as hash-tables
% but only their parts in $\vee^{\mathscr{V}}_{\ell\uparrow}$, $\vee^*_{\ell\uparrow}$ are touched.
% $\Lambda^* \cap \vee^*_{\ell\uparrow}$ contains $\vec{c}^*$.
% At termination: With $\tilde{u} := (\vec{d}^{\mathscr{U}})^\top \Phi|_{\Pi^{\mathscr{U}}} + (\vec{c}^{\mathscr{U}})^\top \Psi^{\mathscr{U}}|_{\Lambda^{\mathscr{U}} \cap \vee^{\mathscr{U}}_{\ell\uparrow}}$,
% $\tilde{\theta}_1 := (\vec{d}^{\mathscr{T}_1})^\top \Phi|_{\Pi^{\mathscr{T}_1}} + (\vec{c}^{\mathscr{T}_1})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_1} \cap \vee^{\mathscr{T}}_{\ell\uparrow}}$,
% $\tilde{\theta}_2 := (\vec{d}^{\mathscr{T}_2})^\top \Phi|_{\Pi^{\mathscr{T}_2}} + (\vec{c}^{\mathscr{T}_2})^\top \Psi^{\mathscr{T}}|_{\Lambda^{\mathscr{T}_2} \cap \vee^{\mathscr{T}}_{\ell\uparrow}}$,
% $\bar{e}^{\mathscr{V}} := \langle \Phi, N(\tilde{u}) - f - \sum_{i=1}^2 \partial_i \tilde{\theta}_i \rangle_{L_2(\Omega)}$,

[2] or include only pointers to such vectors, and create then only when $T.\texttt{in}\Pi^{\mathscr{U}}$, $T.\texttt{in}\Pi^{\mathscr{T}_1}$, or $T.\texttt{in}\Pi^{\mathscr{T}_2}$, respectively, is `true`, or, when $T.\texttt{in}\breve{\Pi}^{\mathscr{V}}$ is `true`

% $\vec{f}^{\breve{\mathscr{V}}} := \langle \Psi^{\breve{\mathscr{V}}}, N(\tilde{u}) - f - \sum_{i=1}^{2} \partial_i \tilde{\theta}_i \rangle_{L_2(\Omega)}$.
% for $T.\mathtt{in}\breve{\Pi}^{\breve{\mathscr{V}}}$=`true` it will hold that $T.\vec{e}^{\breve{\mathscr{V}}} = \vec{e}^{\breve{\mathscr{V}}}|_T$,
% and $\breve{\Lambda}^{\breve{\mathscr{V}}} \cap \vee^{\breve{\mathscr{V}}}_{\ell\uparrow}$ will contain $\vec{f}^{\breve{\mathscr{V}}}|_{\breve{\Lambda}^{\breve{\mathscr{V}}} \cap \vee^{\breve{\mathscr{V}}}_{\ell\uparrow}}$.

Run over the $T \in U$ until you have found at least one T with $T.\mathtt{in}\breve{\Pi}^{\breve{\mathscr{V}}}$=true.
If you don't succeed, then `stop` because apparently $\breve{\Pi}^{\breve{\mathscr{V}}}$ is empty.

```
for T ∈ U do
    for ∗ ∈ {𝒰, 𝒯1, 𝒯2} do
        for λ ∈ Λ*_ℓ with |T ∩ 𝒮(ψ*_λ)| > 0 do        % i.e. T ∈ parent(Σ_ℓ(Λ*))
            for T' ∈ child(T) do
                if T' ∉ {T.children}
                then create T' ∈ U
                     {T.children}:={T.children} ∪ &T'
                endif
                T'.inΠ*:=true
                 if |T' ∩ 𝒮(ψ*_λ)| > 0        % i.e. T' ∈ Σ_ℓ(Λ*)
                 then T'.d* := T'.d* + (q*_ℓ c*_λ)|_T'
                 endif
            endfor
            if T.in Π̆^𝒱
            then T.in Π̆^𝒱_B:=true
                 for + ∈ {𝒰, 𝒯1, 𝒯2} do
                     if T.inΠ^+
                     then T.inΠ^+_B:=true
                     endif
                 endfor
            endif
        endfor
    endfor

    % Π̆^𝒱_B ∩ T has been set; ∀∗ ∈ {𝒰, 𝒯1, 𝒯2},
    % Π*_∗ ∩ child(T) = Σ_ℓ(Λ*) ∩ child(T),
    % Π*_B = Π* ∩ Π̆^𝒱 ∩ T, and d*|_{child(T)∩Σ_ℓ(Λ*)} = (q*_ℓ c*|_{Λ*_ℓ})|_{child(T)∩Σ_ℓ(Λ*)}.

    for λ ∈ Λ̆^𝒱_ℓ with |T ∩ 𝒮(ψ̆^𝒱_λ)| > 0 do        % i.e. T ∈ parent(Σ_ℓ(Λ̆^𝒱))
        for T' ∈ child(T) do        % i.e. T' ∈ Σ_ℓ(Λ̆^𝒱)
            if T' ∉ {T.children}
            then create T' ∈ U
                 {T.children}:={T.children} ∪ &T'
            endif
            T'.in Π̆^𝒱:=true
        endfor
```

```
        for ∗ ∈ {𝒰, 𝒯1, 𝒯2} do
            if T.in∏^∗
            then T.in∏^∗_B:=true
            endif
        endfor
    endfor

    % ∀∗ ∈ {𝒰, 𝒯1, 𝒯2}, ∏^∗_B ∩ T has been set,
    % and ∐^𝒱 ∩ child(T) = Σ_ℓ(Λ̆^𝒱) ∩ child(T).

    if T.in∐^𝒱_B
    then for T′ ∈ child(T) do
             if T′ ∉ {T.children}
             then create T′ ∈ U
                  {T.children}:={T.children} ∪ &T′
             endif
             T′.in∏̆^𝒱:=true
         endfor
    endif
    % ∐^𝒱 ∩ child(T) has been set

    for ∗ ∈ {𝒰, 𝒯1, 𝒯2} do
        if T.in∏^∗_B
        then for T′ ∈ child(T) do
                 if T′ ∉ {T.children}
                 then create T′ ∈ U
                      {T.children}:={T.children} ∪ &T′
                 endif
                 T′.in∏^∗:=true
                 T′.d⃗^∗ := T′.d⃗^∗ + (𝔭_ℓT.d⃗^∗)|_T′
             endfor
        endif
    endfor

    % ∀∗ ∈ {𝒰, 𝒯1, 𝒯2}, ∏^∗ ∩ child(T) and d⃗^∗|_child(T) have been set.

endfor % end of the loop over T ∈ U

eval2(ℓ + 1, U, Λ̆^𝒱, Λ^𝒰, Λ^𝒯1, Λ^𝒯2)

for T ∈ U with T.in∐̆^𝒱 = true do
    if T.in∐̆^𝒱_B
```

```
    then T.ē^𝒱 := Σ_{T'∈{T.children}} 𝔭_ℓ^⊤ T'.ē^𝒱
    else T.ē^𝒱 := −⟨Φ_T, f⟩_{L2(T)}        % this f is the rhs (so not f̄^𝒱)
         if T.inΠ^𝒰
         then T.ē^𝒱 := T.ē^𝒱 + ⟨Φ_T, N((T.d̄^𝒰)^⊤ Φ_T)⟩_{L2(T)}
         endif
         if T.inΠ^{𝒯1}
         then T.ē^𝒱 := T.ē^𝒱 − ⟨Φ_T, ∂1Φ_T⟩_{L2(T)} T.d̄^{𝒯1}
         endif
         if T.inΠ^{𝒯2}
         then T.ē^𝒱 := T.ē^𝒱 − ⟨Φ_T, ∂2Φ_T⟩_{L2(T)} T.d̄^{𝒯2}
         endif
    endif
    for λ ∈ Λ̆_ℓ^𝒱 with |T ∩ S(ψ_λ^𝒱)| > 0 do
        for T' ∈ {T.children} with T'.inΠ̆^𝒱 =true do
            f_λ^𝒱 := f_λ^𝒱 + (𝔮_ℓ^𝒱)^⊤ T'.ē^𝒱
        endfor
    endfor
endfor
```

Similarly, $\underline{U}$ is of no use at this end and can be released from memory.

Remark 5.6.3. After having performed the multi- to single-scale transformations to the term $\left\langle \Psi^{\mathscr{V}_1}, N(\tilde{u}) - f - \operatorname{div} \vec{\theta} \right\rangle_{L_2(\Omega)}$, one has to compute

$$\langle \Phi_T, N((T.\vec{d}^{\mathscr{U}})^\top \Phi_T) - f + T.\vec{d}^{\mathscr{T}_1} \partial_1 \Phi_T + T.\vec{d}^{\mathscr{T}_2} \partial_2 \Phi_T \rangle_{L_2(T)}.$$

This product could either be calculated term-by-term or as a whole. After testing both approaches with no sign of numerical instability, we chose the first option for reasons of speed. The products $\langle \Phi_T, \partial_i \Phi_T \rangle_{L_2(\Omega)}$ are 6×6 matrices which are simply given as an input. The term $\langle \Phi_T, N((T.\vec{d}^{\mathscr{U}})^\top \Phi_T) \rangle_{L_2(T)}$ is viewed as the integral of a polynomial of degree 8 which we calculate exactly using a 16-point rule found in [ZCL09]. Regarding the term $\langle \Phi_T, f \rangle_{L_2(T)}$, when $f = 1$, it is precomputed and inserted compactly in a coefficient vector. To test our code, different polynomial right hand sides were used which, however, did not exceed degree 6. Thus, the same integration rule was used to calculate the resulting integrals.

5.6.7.3 An implementation of `eval3`

The last routine receives as an input the result of `eval2`, namely the vector $\vec{f}^{\mathscr{V}}$ and the strategy that is applied is similar. In this case, a triangle T is a record with the following information:

- booleans $\texttt{in}\breve{\Pi}^{\mathscr{U}}$, $\texttt{in}\breve{\Pi}^{\mathscr{T}_1}$, $\texttt{in}\breve{\Pi}^{\mathscr{T}_2}$; $\texttt{in}\Pi^{\mathscr{U}}$, $\texttt{in}\Pi^{\mathscr{V}}$; $\texttt{in}\breve{\Pi}_B^{\mathscr{U}}$, $\texttt{in}\breve{\Pi}_B^{\mathscr{T}_1}$, $\texttt{in}\breve{\Pi}_B^{\mathscr{T}_2}$; $\texttt{in}\Pi_B^{\mathscr{U}}$, $\texttt{in}\Pi_B^{\mathscr{V}}$,

- the position of the triangle
- $\vec{d}^{\mathscr{U}}$, $\vec{d}^{\mathscr{V}}$, $\vec{e}^{\mathscr{U}}$, $\vec{e}^{\mathscr{T}_1}$, $\vec{e}^{\mathscr{T}_2} \in \mathbb{R}^6$ [3]
- set of 4 pointers `children`

At initialisation of any triangle, we set the booleans as `false`, and the vectors as zero (or the latter is done only when the vectors are generated). In the hash tables the coefficient vectors $\vec{f}^{\mathscr{U}}$, $\vec{f}^{\mathscr{T}_1}$, $\vec{f}^{\mathscr{T}_2}$ are initialised as $\vec{0}$.

The body of `eval3` is organised and executed according to the pseudocode given below.

eval**3**$(\ell, U, \breve{\Lambda}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_2}, \Lambda^{\mathscr{U}}, \Lambda^{\mathscr{V}})$
% Input: $\ell \in \mathbb{N}$, $U \subset \mathcal{T}_{\ell-1}$ is a set of triangles (records).
% For $T \in U$, if $T.\mathtt{in}\Pi^{+}$=`true`, then $T \in \Pi^{+}$, and $T.\vec{d}^{+} = \vec{d}^{+}_{T} \in \mathbb{R}^6$ $(+ \in \{\mathscr{U}, \mathscr{V}\})$,
% if $T.\mathtt{in}\breve{\Pi}^{*}$=`true`, then $T \in \breve{\Pi}^{*}$ $(* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\})$.
% $\breve{\Lambda}^{*}, \Lambda^{+} \subset \vee^{+}$ are trees stored as hash-tables
% but only their parts in $\vee^{*}_{\ell\uparrow}$, $\vee^{+}_{\ell\uparrow}$ are touched.
% $\Lambda^{+} \cap \vee^{+}_{\ell\uparrow}$ contains $\vec{c}^{+}$.
% At termination: With $\tilde{u} := (\vec{d}^{\mathscr{U}})^{\top}\Phi|_{\Pi^{\mathscr{U}}} + (\vec{c}^{\mathscr{U}})^{\top}\Psi^{\mathscr{U}}|_{\Lambda^{\mathscr{U}} \cap \vee^{\mathscr{U}}_{\ell\uparrow}}$,
% $\tilde{v}_1 := (\vec{d}^{\mathscr{V}})^{\top}\Phi|_{\Pi^{\mathscr{V}}} + (\vec{c}^{\mathscr{V}})^{\top}\Psi^{\mathscr{V}}|_{\Lambda^{\mathscr{V}} \cap \vee^{\mathscr{V}}_{\ell\uparrow}}$,
% $\vec{e}^{\mathscr{U}} := \langle \Phi, DN(\tilde{u})\tilde{v} \rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{U}} := \langle \Psi^{\mathscr{U}}, DN(\tilde{u})\tilde{v} \rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_1} := \langle \Phi, \partial_1\tilde{v} \rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_1} := \langle \Psi^{\mathscr{T}}, \partial_1\tilde{v} \rangle_{L_2(\Omega)}$,
% $\vec{e}^{\mathscr{T}_2} := \langle \Phi, \partial_2\tilde{v} \rangle_{L_2(\Omega)}$,
% $\vec{f}^{\mathscr{T}_2} := \langle \Psi^{\mathscr{T}}, \partial_2\tilde{v} \rangle_{L_2(\Omega)}$
% for $T.\mathtt{in}\breve{\Pi}^{*}$=`true` it will hold that $T.\vec{e}^{*} = \vec{e}^{*}|_T$,
% and $\breve{\Lambda}^{*} \cap \vee^{*}_{\ell\uparrow}$ will contain $\vec{f}^{*}|_{\breve{\Lambda}^{*} \cap \vee^{*}_{\ell\uparrow}}$.

Run over the $T \in U$ until you have found at least one T with $T.\mathtt{in}\breve{\Pi}^{*}$=true for some $* \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, and one T' (possibly $T = T'$) with $T'.\mathtt{in}\Pi^{+}$=true for some $+ \in \{\mathscr{U}, \mathscr{V}\}$. If you don't succeed, then `stop` because apparently $\Pi^{\mathscr{U}} \cup \Pi^{\mathscr{V}}$ or $\breve{\Pi}^{\mathscr{U}} \cup \breve{\Pi}^{\mathscr{T}_1} \cup \breve{\Pi}^{\mathscr{T}_2}$ is empty.

```
for T ∈ U do
    for + ∈ {𝒰, 𝒱} do
        for λ ∈ Λ_ℓ^+ with |T ∩ S(ψ_λ^+)| > 0 do      % i.e. T ∈ parent(Σ_ℓ(Λ^+))
            for T' ∈ child(T) do
                if T' ∉ {T.children}
                then create T' ∈ U
                        {T.children}:={T.children} ∪ &T'
```

[3] or include only pointers to such vectors, and create then only when $T.\mathtt{in}\Pi^{\mathscr{U}}$ or $T.\mathtt{in}\Pi^{\mathscr{V}}$ is `true`, or, when $T.\mathtt{in}\breve{\Pi}^{\mathscr{U}}$, $T.\mathtt{in}\breve{\Pi}^{\mathscr{T}_1}$, or $T.\mathtt{in}\breve{\Pi}^{\mathscr{T}_2}$, respectively, is `true`

```
            endif
            T'.in\Pi^+:=true
            if |T' \cap \mathcal{S}(\psi_\lambda^+)| > 0        % i.e. T' \in \Sigma_\ell(\Lambda^+)
            then T'.\vec{d}^+ := T'.\vec{d}^+ + (\mathfrak{q}_\ell^+ \vec{c}_\lambda^+)|_{T'}
            endif
        endfor
        boolean beenset:=false
        for * \in {\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2} do
            if T.in\breve{\Pi}^*
            then T.in\breve{\Pi}^*_B:=true, beenset:=true
            endif
        endfor
        if beenset then        % T \in \breve{\Pi}^{\mathscr{U}}_B \cup \breve{\Pi}^{\mathscr{T}_1}_B \cup \breve{\Pi}^{\mathscr{T}_2}_B
           for - \in {\mathscr{U}, \mathscr{V}} do
               if T.in\Pi^-
               then T.in\Pi^-_B:=true
               endif
           endfor
        endif
    endfor
endfor

% \forall * \in {\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2}, \forall + \in {\mathscr{U}, \mathscr{V}}, \breve{\Pi}^*_B \cap T has been set,
% \underline{\Pi}^+ \cap child(T) = \Sigma_\ell(\Lambda^+) \cap child(T),
% \Pi^+_B = \Pi^+ \cap (\breve{\Pi}^{\mathscr{U}} \cup \breve{\Pi}^{\mathscr{T}_1} \cup \breve{\Pi}^{\mathscr{T}_2}) \cap T,
% and \underline{\vec{d}}^+|_{child(T)\cap\Sigma_\ell(\Lambda^+)} = (\mathfrak{q}_\ell^+ \vec{c}^+|_{\Lambda_\ell^+})|_{child(T)\cap\Sigma_\ell(\Lambda^+)}.

for * \in {\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2} do
    for \lambda \in \breve{\Lambda}^*_\ell with |T \cap \mathcal{S}(\breve{\psi}^*_\lambda)| > 0 do        % i.e. T \in parent(\Sigma_\ell(\breve{\Lambda}^*))
        for T' \in child(T) do
            if T' \notin {T.children}
            then create T' \in \underline{U}
                 {T.children}:={T.children} \cup &T'
            endif
            T'.in\breve{\Pi}^*:=true
        endfor
        for + \in {\mathscr{U}, \mathscr{V}} do
            if T.in\Pi^+
            then T.in\Pi^+_B:=true
            endif
        endfor
    endfor
endfor
```

% $\forall + \in \{\mathscr{U}, \mathscr{V}\}$, $\Pi_B^+ \cap T$ has been set,
% and $\forall * \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, $\breve{\underline{\Pi}}^* \cap \mathtt{child}(T) = \Sigma_\ell(\breve{\Lambda}^*) \cap \mathtt{child}(T)$.

```
for ∗ ∈ {𝒰, 𝒯1, 𝒯2} do
    if T.in Π̆*_B
    then for T′ ∈ child(T) do
              if T′ ∉ {T.children}
              then create T′ ∈ U
                   {T.children}:={T.children} ∪ &T′
              endif
              T′.in Π̆*:=true
         endfor
    endif
endfor
```

% $\forall * \in \{\mathscr{U}, \mathscr{T}_1, \mathscr{T}_2\}$, $\breve{\underline{\Pi}}^* \cap \mathtt{child}(T)$ has been set

```
for + ∈ {𝒰, 𝒱} do
    if T.in Π+_B
    then for T′ ∈ child(T) do
              if T′ ∉ {T.children}
              then create T′ ∈ U
                   {T.children}:={T.children} ∪ &T′
              endif
              T′.in Π+:=true
              T′.d⃗+ := T′.d⃗+ + (𝔭ℓ T.d⃗+)|T′
         endfor
    endif
endfor
```

% $\forall + \in \{\mathscr{U}, \mathscr{V}\}$, $\underline{\Pi}^+ \cap \mathtt{child}(T)$ and $\underline{\vec{d}}^+|_{\mathtt{child}(T)}$ have been set.

`endfor` % end of the loop over $T \in U$

eval**3**$(\ell + 1, \underline{U}, \breve{\Lambda}^{\mathscr{U}}, \breve{\Lambda}^{\mathscr{T}_1}, \breve{\Lambda}^{\mathscr{T}_2}, \Lambda^{\mathscr{U}}, \Lambda^{\mathscr{V}})$

```
for T ∈ U with T.in Π̆^𝒰 = true do
    if T.in Π̆^𝒰_B
    then T.e⃗^𝒰 := Σ_{T′∈{T.children}} 𝔭ℓ^⊤ T′.e⃗^𝒰
    else if T.in Π^𝒰 and T.in Π^𝒱
         then T.e⃗^𝒰 := ⟨Φ_T, DN((T.d⃗^𝒰)^⊤ Φ_T)(T.d⃗^𝒱)^⊤ Φ_T⟩_{L2(T)}
```

```
        endif
    endif
    for λ ∈ Λ̆_ℓ^𝒰 with |T ∩ S(ψ_λ^𝒰)| > 0 do
        for T' ∈ {T.children} with T'.inΠ̆^𝒰 =true do
            f_λ^𝒰 := f_λ^𝒰 + (q_ℓ^𝒰)^⊤ T'.ē^𝒰
        endfor
    endfor
endfor

for T ∈ U with T.inΠ̆^{𝒯1} = true do
    if T.inΠ̆_B^{𝒯1}
    then T.ē^{𝒯1} := Σ_{T'∈{T.children}} p_ℓ^⊤ T'.ē^{𝒯1}
    else if T.inΠ^𝒱
         then T.ē^{𝒯1} := ⟨Φ_T, ∂1 Φ_T⟩_{L2(T)} T.d̄^𝒱
         endif
    endif
    for λ ∈ Λ̆_ℓ^{𝒯1} with |T ∩ S(ψ_λ^𝒯)| > 0 do
        for T' ∈ {T.children} with T'.inΠ̆^{𝒯1} =true do
            f_λ^{𝒯1} := f_λ^{𝒯1} + (q_ℓ^𝒯)^⊤ T'.ē^{𝒯1}
        endfor
    endfor
endfor

for T ∈ U with T.inΠ̆^{𝒯2} = true do
    if T.inΠ̆_B^{𝒯2}
    then T.ē^{𝒯2} := Σ_{T'∈{T.children}} p_ℓ^⊤ T'.ē^{𝒯2}
    else if T.inΠ^𝒱
         then T.ē^{𝒯2} := ⟨Φ_T, ∂2 Φ_T⟩_{L2(T)} T.d̄^𝒱
         endif
    endif
    for λ ∈ Λ̆_ℓ^{𝒯2} with |T ∩ S(ψ_λ^𝒯)| > 0 do
        for T' ∈ {T.children} with T'.inΠ̆^{𝒯2} =true do
            f_λ^{𝒯2} := f_λ^{𝒯2} + (q_ℓ^𝒯)^⊤ T'.ē^{𝒯2}
        endfor
    endfor
endfor
```

As previously, the set $\underline{U}$ can now be deleted.

Remark 5.6.4. The calculation of $\langle \Phi_T, DN((T.\bar{d}^{\mathscr{U}})^\top \Phi_T)(T.\bar{d}^{\mathscr{V}})^\top \Phi_T \rangle_{L_2(T)}$ is done using the 16-point, 8^{th} degree rule reported in Remark 5.6.3.

5.7 An implementation of the parabolic case

In Sect. 5.3 we gave the definitions of a wavelet and an element tree, and then the definition of a multi-dimensional version of a tree, in particular a double tree. Subsequently, the algorithm `growDbltree` that constructs the k-neighborhood of a double tree was given. Furthermore, in Sect. 5.4.4 we presented a fast-evaluation procedure for applying tensor product operators in linear complexity. In Sect. 5.5.3 we described a special data structure to store a double tree and in Sect. 5.5.4 we gave an efficient way to number tensor product wavelet indices and to apply a hash function. Here, we employ all the aforementioned parts to apply the **awgm** to a parabolic evolution problem. The numerical results produced by this implementation were given in Sect. 4.5.

5.7.1 First order least squares problem

The partial differential equation considered here is the *heat equation* given below, i.e. the equation (4.2.1) with $A = \mathrm{Id}$ and $N = 0$, on the L-shaped domain $\Omega = (0,1) \setminus [\frac{1}{2},1)^2$ and $\mathrm{I} = (0,1)$

$$\begin{cases} \frac{\partial u}{\partial t} - \triangle u = g & \mathrm{I} \times \Omega \\ u = 0 & \mathrm{I} \times \partial\Omega \\ u(0,\cdot) = h & \Omega \end{cases}$$

For our convenience we take $g = 1$, and we will consider three different initial conditions, namely, $h = 0$, $h = 1$ and $h(x,y) = 50x(x-1)(x-\frac{1}{2})y(y-1)(y-\frac{1}{2})$.

Introducing the new variable $\vec{p} = \nabla u$, a first order formulation reads as

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div} \vec{p} = g & \mathrm{I} \times \Omega \\ \vec{p} - \nabla u = 0 & \mathrm{I} \times \Omega \\ u = 0 & \mathrm{I} \times \partial\Omega \\ u(0,\cdot) = h & \Omega \end{cases}$$

Then with $\mathscr{U} := L_2(\mathrm{I}; H_0^1(\Omega)) \cap H^1(\mathrm{I}; H^{-1}(\Omega))$, $\mathscr{T}_1 = \mathscr{T}_2 = L_2(\mathrm{I}; L_2(\Omega))$, and $\mathscr{V} := L_2(\mathrm{I}; H_0^1(\Omega))$ one seeks $(u, p_1, p_2) \in \mathscr{U} \times \mathscr{T}_1 \times \mathscr{T}_2$ that minimizes the least squares functional

$$Q(u,p_1,p_2) := \tfrac{1}{2}\Big(\big\| v \mapsto \int_{\mathrm{I}} \int_{\Omega} (\frac{\partial u}{\partial t} - g)v + \sum_{i=1,2} p_i \partial_i v \, dx \, dt \big\|_{\mathscr{V}'}^2 + \sum_{i=1,2} \|p_i - \partial_i u\|_{\mathscr{T}_i}^2 + \|u(0,\cdot) - h\|_{L_2(\Omega)}^2 \Big),$$

by solving $DQ(u,p_1,p_2) = 0$. The details of the current formulation were given in Sect. 4.2. A representation of DQ in wavelet coordinates follows in the next section.

5.7.2 Representation in wavelet coordinates

Letting $\Psi^{\mathscr{U}}$, $\Psi^{\mathscr{T}_i}$ be Riesz bases for $\mathscr{U}$ and $\mathscr{T}_i$, we write the solution to $DQ(u,p_1,p_2)=0$ as $u=\mathbf{u}^\top\Psi^{\mathscr{U}}$, $p_i=\mathbf{p}_i^\top\Psi^{\mathscr{T}_i}$. With $\Psi^{\mathscr{V}}$ being a Riesz basis for $\mathscr{V}$, we replace $\|\cdot\|_{\mathscr{V}'}$ by the equivalent ℓ_2-norm of the sequence of applications of the functional to all elements from $\Psi^{\mathscr{V}}$.

Then, taking $\Psi^{\mathscr{T}_i}\subset L_2(\mathrm{I};H^1(\Omega))$, $DQ(u,p_1,p_2)=0$ if and only if

$$\boxed{\begin{aligned}&D\mathbf{Q}([\mathbf{u}^\top,\mathbf{p}_1^\top,\mathbf{p}_2^\top]^\top)=\\&\begin{bmatrix}\langle\frac{\partial\Psi^{\mathscr{U}}}{\partial t},\Psi^{\mathscr{V}}\rangle_{L_2(\mathrm{I}\times\Omega)}\\ \langle\Psi^{\mathscr{T}_1},\partial_1\Psi^{\mathscr{V}}\rangle_{L_2(\mathrm{I}\times\Omega)}\\ \langle\Psi^{\mathscr{T}_2},\partial_2\Psi^{\mathscr{V}}\rangle_{L_2(\mathrm{I}\times\Omega)}\end{bmatrix}\langle\Psi^{\mathscr{V}},\frac{\partial u}{\partial t}-g-\sum_{i=1,2}\partial_i p_i\rangle_{L_2(\mathrm{I}\times\Omega)}\\+&\begin{bmatrix}\langle\Psi^{\mathscr{U}}(0,\cdot),u(0,\cdot)-h\rangle_{L_2(\Omega)}\\0\\0\end{bmatrix}+\begin{bmatrix}\langle-\nabla\Psi^{\mathscr{U}},\vec{p}-\nabla u\rangle_{L_2(\mathrm{I}\times\Omega)^n}\\ \langle\Psi^{\mathscr{T}_1},p_1-\partial_1 u\rangle_{L_2(\mathrm{I}\times\Omega)}\\ \langle\Psi^{\mathscr{T}_2},p_2-\partial_2 u\rangle_{L_2(\mathrm{I}\times\Omega)}\end{bmatrix}=0\end{aligned}} \tag{5.7.1}$$

We will solve equation (5.7.1) with the adaptive wavelet scheme given in Algorithm 4.3.1.

5.7.3 Tensor bases

The Riesz bases for the Bochner spaces on which we formulated the **FOSLS** functional, consist of tensor product wavelets. Below we describe the composition of these tensor product bases.

For $*\in\{\mathscr{U},\mathscr{T}_1,\mathscr{T}_2,\mathscr{V}\}$, let

$$\Theta^*=\{\theta^*_\lambda\colon\lambda\in\triangleleft_*\}$$

be collections of *'temporal'* wavelets on I, such that

$$\Theta^{\mathscr{T}_1},\ \Theta^{\mathscr{T}_2},\ \Theta^{\mathscr{V}}\text{ are Riesz bases for }L_2(\mathrm{I}).$$

We assume that $\Theta^{\mathscr{U}}\subset H^1_0(\mathrm{I})$, and that

$$\frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}},\ \frac{\Theta^{\mathscr{U}}}{\|\Theta^{\mathscr{U}}\|_{H^1_0(\mathrm{I})}}\text{ are Riesz bases for }L_2(\mathrm{I}),H^1_0(\mathrm{I}),$$

respectively. Here with $\Theta^{\mathscr{U}}/\|\Theta^{\mathscr{U}}\|_{L_2(\mathrm{I})}$, and similarly for other normalisations or collections, we mean the collection $\{\theta^{\mathscr{U}}_\lambda/\|\theta^{\mathscr{U}}_\lambda\|_{L_2(\mathrm{I})}\colon\lambda\in\triangleleft_{\mathscr{U}}\}$.

Similarly, with $*\in\{\mathscr{U},\mathscr{T}_1,\mathscr{T}_2,\mathscr{V}\}$, let

$$\Sigma^*=\{\sigma^*_\mu\colon\mu\in\diamondsuit_*\}$$

be collections of *'spatial'* wavelets on Ω, such that,

$$\Sigma^{\mathscr{V}}\text{ is a Riesz basis for }H^1_0(\Omega),$$

$$\Sigma^{\mathscr{T}_1},\ \Sigma^{\mathscr{T}_2} \subset H^1(\Omega) \text{ are Riesz bases for } L_2(\Omega),$$

and with $\Sigma^{\mathscr{U}} \subset H_0^1(\Omega)$,

$$\frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H^{-1}(\Omega)}},\ \frac{\Sigma^{\mathscr{U}}}{\|\Sigma^{\mathscr{U}}\|_{H_0^1(\Omega)}} \text{ are Riesz bases for } H^{-1}(\Omega),\ H_0^1(\Omega), \tag{5.7.2}$$

respectively.

Under above assumptions, we have that

$$\Psi^{\mathscr{V}} := \Theta^{\mathscr{V}} \otimes \Sigma^{\mathscr{V}}, \quad \Psi^{\mathscr{T}_i} := \Theta^{\mathscr{T}_i} \otimes \Sigma^{\mathscr{T}_i}, \quad \Psi^{\mathscr{U}} := \frac{\Theta^{\mathscr{U}} \otimes \Sigma^{\mathscr{U}}}{\|\Theta^{\mathscr{U}} \otimes \Sigma^{\mathscr{U}}\|_{\mathscr{U}}}$$

are Riesz bases for $\mathscr{V}$, $\mathscr{T}_i$, and $\mathscr{U}$, respectively. In addition, we have

$$\begin{aligned}\|\theta_\lambda^{\mathscr{U}} \otimes \sigma_\mu^{\mathscr{U}}\|_{\mathscr{U}} &= \sqrt{\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})}^2 \|\nabla\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)^n}^2 + \|(\theta_\lambda^{\mathscr{U}})'\|_{L_2(\mathrm{I})}^2 \|\sigma_\mu^{\mathscr{U}}\|_{H^{-1}(\Omega)}^2} \\ &\eqsim \sqrt{\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})}^2 \|\nabla\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)^n}^2 + \|(\theta_\lambda^{\mathscr{U}})'\|_{L_2(\mathrm{I})}^2 \frac{\|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)}^4}{\|\nabla\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)^n}^2}} =: \mathbf{D}_{\lambda,\mu}\end{aligned} \tag{5.7.3}$$

by an application of the relation $\|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)}^2 \eqsim \|\sigma_\mu^{\mathscr{U}}\|_{H^{-1}(\Omega)} \|\nabla\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)^n}$ (stated in the proof of Lemma 4.4.17).

In order to obtain a Riesz basis for $\mathscr{U}$, we can use the latter computable normalization in (5.7.3) instead of $\|\theta_\lambda^{\mathscr{U}} \otimes \sigma_\mu^{\mathscr{U}}\|_{\mathscr{U}}$. Moreover, $\mathbf{D}_{\lambda,\mu}$ can be replaced by a *uniform* equivalent quantity. For instance, if $\Theta^{\mathscr{U}}$, $\Psi^{\mathscr{U}}$ are scaled s.t. $\|\theta_\lambda^{\mathscr{U}}\|_{L_2(\mathrm{I})} \eqsim 1 \eqsim \|\nabla\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)^n}$, then one can take

$$\mathbf{D}_{\lambda,\mu} := \sqrt{1 + \|(\theta_\lambda^{\mathscr{U}})'\|_{L_2(\mathrm{I})}^2 \|\sigma_\mu^{\mathscr{U}}\|_{L_2(\Omega)}^4} \tag{5.7.4}$$

Thus, with $\mathbf{D}$ as in (5.7.3), or in (5.7.4) under specified scaling for $\Theta^{\mathscr{U}}$ and $\Sigma^{\mathscr{U}}$, we have that

$$\Psi^{\mathscr{U}} = \{\mathbf{D}_{\lambda,\mu}^{-1} \theta_\lambda^{\mathscr{U}} \otimes \sigma_\mu^{\mathscr{U}} : (\lambda,\mu) \in \lhd_{\mathscr{U}} \times \diamond_{\mathscr{U}}\}$$

is a Riesz basis for $\mathscr{U}$.

As described in Sect. 4.5, we performed numerical experiments to approximate the condition numbers of square blocks of the matrix $\mathrm{D}\mathbf{Q}$ w.r.t. to subsequent wavelet sets for approximations of u and p_i produced by the application of the **awgm** on (5.7.1). We tested both scalings in (5.7.3) and (5.7.4) and found nearly indistinguishable results. Therefore, in favour of a more convenient implementation and computational performance we preferred the scaling given in (5.7.4).

5.7.4 Wavelet bases selection

At this point, we specify the wavelet bases involved in the representation of DQ, in the spatial and the temporal dimension respectively. The tensor product wavelet bases are then composed as described in the previous section.

Below we record the wavelet bases in use in the spatial dimension:

- For $\Sigma^{\mathscr{U}}$ we take the continuous piecewise *quadratic* wavelet basis, zero at $\partial\Omega$. Properly scaled, it gives a Riesz basis for both $H^1_0(\Omega)$ *and* $H^{-1}(\Omega)$ as required.
- For $\Sigma^{\mathscr{T}_i}$ we take the continuous piecewise linear three-point wavelet basis, *without* boundary conditions, (approximately) normalized in $\|\cdot\|_{L_2(\Omega)}$.
- For $\Sigma^{\mathscr{V}}$ we take the continuous piecewise linear three-point wavelet basis, zero at $\partial\Omega$, (approximately) normalized in $\|\nabla\cdot\|_{L_2(\Omega)^n}$.

The wavelet bases in the temporal dimension are as follows:

- For $\Theta^{\mathscr{U}}$ we take continuous piecewise linear wavelets without boundary conditions (see Figure 5.3), (approximately) normalized in $L_2(\mathrm{I})$.
- For both $\Theta^{\mathscr{V}}$ *and* $\Theta^{\mathscr{T}_i}$ we use $L_2(\mathrm{I})$-orthonormal discontinuous piecewise linear wavelets (see Figure 5.2).

5.7.5 Approximate Residual evaluation

Given double-trees $\mathbf{\Lambda}_{\mathscr{U}} \subset \vee_{\mathscr{U}}$, $\mathbf{\Lambda}_{\mathscr{T}_i} \subset \vee_{\mathscr{T}_i}$, and approximations $\mathbf{w} \in \ell_2(\mathbf{\Lambda}_{\mathscr{U}})$, $\mathbf{q}_i \in \ell_2(\mathbf{\Lambda}_{\mathscr{T}_i})$, the approximate evaluation of $D\mathbf{Q}([\mathbf{w}^\top, \mathbf{q}_1^\top, \mathbf{q}_2^\top]))$ at step (R) of the **awgm**, is built in the following steps, where $k \in \mathbb{N}_0$ is a sufficiently large constant. With $w := \mathbf{w}^\top \Psi^{\mathscr{U}}$ and $q_i := \mathbf{q}_i^\top \Psi^{\mathscr{T}_i}$

($t1$) We approximate

$$\mathbf{r}_{\frac{1}{2}} := \langle \Psi^{\mathscr{V}}, \frac{\partial w}{\partial t} - g - \sum_{i=1,2} \partial_i q_i \rangle_{L_2(\mathrm{I}\times\Omega)}$$

by $\tilde{\mathbf{r}}_{\frac{1}{2}} := \mathbf{r}_{\frac{1}{2}}|_{\underline{\mathbf{\Lambda}}^{\mathscr{V}}}$, where $\underline{\mathbf{\Lambda}}^{\mathscr{V}} := \mathbf{\Lambda}^{\mathscr{V}}(\mathbf{\Lambda}_{\mathscr{U}}, k) \cup \mathbf{\Lambda}^{\mathscr{V}}(\mathbf{\Lambda}_{\mathscr{T}_1}, k) \cup \mathbf{\Lambda}^{\mathscr{V}}(\mathbf{\Lambda}_{\mathscr{T}_2}, k)$.

($t2$) With $\tilde{r}_{\frac{1}{2}} := \tilde{\mathbf{r}}_{\frac{1}{2}}^\top \Psi^{\mathscr{V}}$, we approximate

$$\mathbf{r}_1 := \begin{bmatrix} \langle \frac{\partial \Psi^{\mathscr{U}}}{\partial t}, \tilde{r}_{\frac{1}{2}} \rangle_{L_2(\mathrm{I}\times\Omega)} \\ \langle \Psi^{\mathscr{T}_1}, \partial_1 \tilde{r}_{\frac{1}{2}} \rangle_{L_2(\mathrm{I}\times\Omega)} \\ \langle \Psi^{\mathscr{T}_2}, \partial_2 \tilde{r}_{\frac{1}{2}} \rangle_{L_2(\mathrm{I}\times\Omega)} \end{bmatrix}$$

by $\tilde{\mathbf{r}}_1 = \tilde{\mathbf{r}}_1(k) := \mathbf{r}_1|_{\mathbf{\Lambda}^{\mathscr{U}}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k) \cup \mathbf{\Lambda}^{\mathscr{T}_1}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k) \cup \mathbf{\Lambda}^{\mathscr{T}_2}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}$.

($t3$) We approximate

$$\mathbf{r}_3 := \begin{bmatrix} \langle -\nabla \Psi^{\mathscr{U}}, \vec{q} - \nabla w \rangle_{L_2(\mathrm{I}\times\Omega)^n} \\ \langle \Psi^{\mathscr{T}_1}, q_1 - \partial_1 w \rangle_{L_2(\mathrm{I}\times\Omega)} \\ \langle \Psi^{\mathscr{T}_2}, q_2 - \partial_2 w \rangle_{L_2(\mathrm{I}\times\Omega)} \end{bmatrix}$$

by $\tilde{\mathbf{r}}_3 = \tilde{\mathbf{r}}_3(k) := \mathbf{r}_3|_{\bar{\mathbf{\Lambda}}}$, where $\bar{\mathbf{\Lambda}}$ is defined as

$$\bar{\mathbf{\Lambda}} := \big(\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}_{\mathscr{U}}, k) \cup \mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}_{\mathscr{T}_1}, k) \cup \mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}_{\mathscr{T}_2}, k)\big) \cup \big(\mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}_{\mathscr{U}}, k) \cup \mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}_{\mathscr{T}_1}, k)\big) \cup \big(\mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}_{\mathscr{U}}, k) \cup \mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}_{\mathscr{T}_2}, k)\big).$$

($t4$) With $\mathbf{\Lambda}_\downarrow := \cup_{\{\lambda\in(\mathbf{\Lambda}\cap\vee_{\mathscr{U}})_1:\,\theta^{\mathscr{U}}_\lambda(0)\neq 0\}}(\mathbf{\Lambda}\cap\vee_{\mathscr{U}})_2(\lambda)$, we approximate

$$\mathbf{r}_{\frac{3}{2}} := \Big\langle \Sigma^{\mathscr{U}}, w(0,\cdot)-h\Big\rangle_{L_2(\Omega)}$$

by $\tilde{\mathbf{r}}_{\frac{3}{2}} = \tilde{\mathbf{r}}_{\frac{3}{2}}(k) := \mathbf{r}_{\frac{3}{2}}\big|_{\Lambda^{\Sigma^{\mathscr{U}}}(\mathcal{T}(\mathbf{\Lambda}_\downarrow),k)}$.

($t5$) Finally we approximate $\mathbf{r}_2 := \mathbf{E}\tilde{\mathbf{r}}_{\frac{3}{2}}$ by $\tilde{\mathbf{r}}_2 := \mathbf{E}_k\tilde{\mathbf{r}}_{\frac{3}{2}}$, where $\mathbf{E}$ and $\mathbf{E}_k$ (for $k\in\mathbb{N}_0$) are defined on $\ell_2(\diamond_{\mathscr{U}})$ by

$$(\mathbf{E}\mathbf{v})_{\lambda,\mu} := \mathbf{D}^{-1}_{\lambda,\mu}\theta^{\mathscr{U}}_\lambda(0)\mathbf{v}_\mu,\quad ((\lambda,\mu)\in\vee_{\mathscr{U}}),$$

$$\text{and } (\mathbf{E}_k\mathbf{v})_{\lambda,\mu} = \begin{cases} (\mathbf{E}\mathbf{v})_{\lambda,\mu} & \text{when } \big||\lambda|-2|\mu|\big|\le k,\\ 0 & \text{otherwise.}\end{cases}$$

after which the whole approximation is $\tilde{\mathbf{r}}_1+\tilde{\mathbf{r}}_2+\tilde{\mathbf{r}}_3$.

Note that, expanding each expression in steps ($t1$)-($t4$) w.r.t. the tensor product structure of the wavelet bases, we *compute* accordingly

$$\begin{aligned}\tilde{\mathbf{r}}_{\frac{1}{2}} =& R_{\underline{\mathbf{\Lambda}}^{\mathscr{V}}}\Big(-\langle\Psi^{\mathscr{V}},g\rangle_{L_2(\mathrm{I}\times\Omega)} + \langle\Theta^{\mathscr{V}},\frac{\partial}{\partial t}\Theta^{\mathscr{U}}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{V}},\Sigma^{\mathscr{U}}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{U}}}\mathbf{D}^{-1}\mathbf{w}\Big) -\\ & R_{\underline{\mathbf{\Lambda}}^{\mathscr{V}}}\Big(\sum_{i=1,2}\langle\Theta^{\mathscr{V}},\Theta^{\mathscr{T}_i}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{V}},\partial_i\Sigma^{\mathscr{T}_i}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{T}_i}}\mathbf{q}_i\Big),\end{aligned}$$

followed by

$$\tilde{\mathbf{r}}_1 = \begin{bmatrix}\mathbf{D}^{-1}R_{\mathbf{\Lambda}^{\mathscr{U}}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}\langle\frac{d\Theta^{\mathscr{U}}}{dt},\Theta^{\mathscr{V}}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{U}},\Sigma^{\mathscr{V}}\rangle_{L_2(\Omega)}I_{\underline{\mathbf{\Lambda}}^{\mathscr{V}}}\tilde{\mathbf{r}}_{\frac{1}{2}}\\ R_{\mathbf{\Lambda}^{\mathscr{T}_1}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}\langle\Theta^{\mathscr{T}_1},\Theta^{\mathscr{V}}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{T}_1},\partial_1\Sigma^{\mathscr{V}}\rangle_{L_2(\Omega)}I_{\underline{\mathbf{\Lambda}}^{\mathscr{V}}}\tilde{\mathbf{r}}_{\frac{1}{2}}\\ R_{\mathbf{\Lambda}^{\mathscr{T}_2}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}\langle\Theta^{\mathscr{T}_2},\Theta^{\mathscr{V}}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{T}_2},\partial_2\Sigma^{\mathscr{V}}\rangle_{L_2(\Omega)}I_{\underline{\mathbf{\Lambda}}^{\mathscr{V}}}\tilde{\mathbf{r}}_{\frac{1}{2}}\end{bmatrix},$$

subsequently at ($t3$)

$$\tilde{\mathbf{r}}_3 = \begin{bmatrix}-\mathbf{D}^{-1}R_{\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}_{\mathscr{U}},k)\cup\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}_{\mathscr{T}_1},k)\cup\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}_{\mathscr{T}_2},k)}\Big(\langle\Theta^{\mathscr{U}},\Theta^{\mathscr{T}_1}\rangle_{L_2(\mathrm{I})}\otimes\langle\partial_1\Sigma^{\mathscr{U}},\Sigma^{\mathscr{T}_1}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{T}_1}}\mathbf{q}_1\\ +\langle\Theta^{\mathscr{U}},\Theta^{\mathscr{T}_2}\rangle_{L_2(\mathrm{I})}\otimes\langle\partial_2\Sigma^{\mathscr{U}},\Sigma^{\mathscr{T}_2}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{T}_2}}\mathbf{q}_2\\ -\langle\Theta^{\mathscr{U}},\Theta^{\mathscr{U}}\rangle_{L_2(\Omega)^n}\otimes\langle\nabla\Sigma^{\mathscr{U}},\nabla\Sigma^{\mathscr{U}}\rangle_{L_2(\Omega)^n}I_{\mathbf{\Lambda}_{\mathscr{U}}}\mathbf{D}^{-1}\mathbf{w}\Big)\\ R_{\mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}_{\mathscr{U}},k)\cup\mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}_{\mathscr{T}_1},k)}\Big(\langle\Theta^{\mathscr{T}_1},\Theta^{\mathscr{T}_1}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{T}_1},\Sigma^{\mathscr{T}_1}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{T}_1}}\mathbf{q}_1\\ -\langle\Theta^{\mathscr{T}_1},\Theta^{\mathscr{U}}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{T}_1},\partial_1\Sigma^{\mathscr{U}}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{U}}}\mathbf{D}^{-1}\mathbf{w}\Big)\\ R_{\mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}_{\mathscr{U}},k)\cup\mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}_{\mathscr{T}_2},k)}\Big(\langle\Theta^{\mathscr{T}_2},\Theta^{\mathscr{T}_2}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{T}_2},\Sigma^{\mathscr{T}_2}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{T}_2}}\mathbf{q}_2\\ -\langle\Theta^{\mathscr{T}_2},\Theta^{\mathscr{U}}\rangle_{L_2(\mathrm{I})}\otimes\langle\Sigma^{\mathscr{T}_2},\partial_2\Sigma^{\mathscr{U}}\rangle_{L_2(\Omega)}I_{\mathbf{\Lambda}_{\mathscr{U}}}\mathbf{D}^{-1}\mathbf{w}\Big)\end{bmatrix},$$

and then in the intermediate step ($t4$), we have

$$\tilde{\mathbf{r}}_{\frac{3}{2}} = R_{\Lambda^{\Sigma^{\mathscr{U}}}(\mathcal{T}(\mathbf{\Lambda}_\downarrow),k)}\langle\Sigma^{\mathscr{U}},(I_{\mathbf{\Lambda}_\downarrow}\mathbf{z})^\top\Sigma^{\mathscr{U}}-h\rangle_{L_2(\Omega)}$$

where $z_\mu = \sum_{\lambda\in\lhd_{\mathscr{U}}}\mathbf{D}^{-1}_{\lambda,\mu}w_{\lambda\mu}\theta^{\mathscr{U}}_\lambda(0)$. The output of the latter is the input for the computation of $\tilde{\mathbf{r}}_2$ described on the last step ($t5$).

Remark 5.7.1. For the computation of $\tilde{\mathbf{r}}_1$, numerical experiments showed that it is sufficient to approximate this quantity on smaller sets, and thus one could replace

$$R_{\mathbf{\Lambda}^{\mathscr{U}}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}, \quad R_{\mathbf{\Lambda}^{\mathscr{T}_1}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}, \quad R_{\mathbf{\Lambda}^{\mathscr{T}_2}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)},$$

by

$$R_{\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}^{\mathscr{U}},k)\cup\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}^{\mathscr{T}_1},k)\cup\mathbf{\Lambda}^{\mathscr{U}}(\mathbf{\Lambda}^{\mathscr{T}_2},k)}, \quad R_{\mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}^{\mathscr{U}},k)\cup\mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}^{\mathscr{T}_1},k)\cup\mathbf{\Lambda}^{\mathscr{T}_1}(\mathbf{\Lambda}^{\mathscr{T}_2},k)},$$

$$R_{\mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}^{\mathscr{U}},k)\cup\mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}^{\mathscr{T}_1},k)\cup\mathbf{\Lambda}^{\mathscr{T}_2}(\mathbf{\Lambda}^{\mathscr{T}_2},k)},$$

respectively.

Each of the tensor products of matrices included in the previous steps is evaluated according to the procedure described in Theorem 5.4.8. The intermediate sets $\mathbf{\Theta}$ and $\mathbf{\Sigma}$ involved in these evaluations are determined efficiently exploiting the properties of the double tree data structure (see Sect. 5.5.3). Moreover, the enlarged double trees specified as the input and output sets of the different parts of the residual are determined by appropriate applications of `growDbltree`.

Note that with the current selection of the bases for $\Theta^{\mathscr{V}}$ and $\Theta^{\mathscr{T}_i}$, $\langle\Theta^{\mathscr{V}},\Theta^{\mathscr{T}_i}\rangle_{L_2(\mathrm{I})}$ in $\tilde{\mathbf{r}}_{\frac{1}{2}}$, $\langle\Theta^{\mathscr{T}_i},\Theta^{\mathscr{V}}\rangle_{L_2(\mathrm{I})}$ in $\tilde{\mathbf{r}}_1$, and $\langle\Theta^{\mathscr{T}_i},\Theta^{\mathscr{T}_i}\rangle_{L_2(\mathrm{I})}$ in $\tilde{\mathbf{r}}_3$ are identity matrices, which allows for a much more efficient evaluation of the corresponding tensor product operations. In this case, in (5.4.3), $L_1 = 0$, so the first term on the right-hand side vanishes. The second term is just $R_{\mathbf{\Lambda}^{\mathscr{V}}}(I\otimes A_2)I_{\mathbf{\Lambda}^{\mathscr{U}}}$. Hence, in addition there is no need to determine the intermediate double trees $\mathbf{\Theta}$ (not to be confused with the bases Θ^*) and $\mathbf{\Sigma}$.

5.7.6 Bulk chasing

As in the elliptic case, at step (B) of the **awgm** we perform bulk chasing on the entries of $\tilde{\mathbf{r}}$. The principles of this procedure do not differ from those applied on the elliptic case and thus we refer to Sect. 5.6.4 for details. However, for the parabolic problem we can not prove that the new admissible set $\mathbf{\Lambda}_{i+1}$ is optimal w.r.t. to the condition stated in the algorithm. Nevertheless, numerical experiments validated the use of the same procedure (for details, see the last paragraph of Sect. 4.4).

5.7.7 Galerkin solve

At step (G), we approximate the Galerkin matrices using steps $(t1)$-$(t5)$ computing

- $\mathbf{r}_1|_{\mathbf{\Lambda}_{\mathscr{U}}\cup\mathbf{\Lambda}_{\mathscr{T}_1}\cup\mathbf{\Lambda}_{\mathscr{T}_2}}$ instead of $\mathbf{r}_1|_{\mathbf{\Lambda}^{\mathscr{U}}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)\cup\mathbf{\Lambda}^{\mathscr{T}_1}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)\cup\mathbf{\Lambda}^{\mathscr{T}_2}(\underline{\mathbf{\Lambda}}^{\mathscr{V}},k)}$
- $\mathbf{r}_3|_{\mathbf{\Lambda}_{\mathscr{U}}\cup\mathbf{\Lambda}_{\mathscr{T}_1}\cup\mathbf{\Lambda}_{\mathscr{T}_2}}$ instead of $\mathbf{r}_3|_{\bar{\mathbf{\Lambda}}}$, with $\bar{\mathbf{\Lambda}}$ as defined in $(t3)$.
- $\mathbf{r}_{\frac{3}{2}}|_{\mathbf{\Lambda}_\downarrow}$ instead of $\mathbf{r}_{\frac{3}{2}}|_{\Lambda^{\Sigma^{\mathscr{U}}}(\mathcal{T}(\mathbf{\Lambda}_\downarrow),k)}$, and consequently $\mathbf{r}_2|_{\mathbf{\Lambda}_\downarrow}$

in order to retrieve an square system of equations. Subsequently, we solve the resulting Galerkin systems by a *conjugate gradient* (CG) routine.

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Summary

Optimal Adaptive Wavelet Methods for solving First Order System Least Squares.

The efficient numerical approximation of the solution of a partial differential equation (PDE) requires the use of an approximation space with a local resolution that is adjusted to the local smoothness of the solution. Since a priori this solution is unknown, such a space has to be created adaptively by means of a loop in which increasingly more accurate approximations to the solution are built. Two instances of such adaptive solution methods are adaptive finite element methods (**afem**) and adaptive wavelet-Galerkin methods (**awgm**). It is fair to say that the latter methods are more cumbersome to implement, but on the other hand their use is not restricted to essentially elliptic PDEs.

Any well-posed operator equation on a Hilbert space equipped with a Riesz basis has an equivalent formulation as an bi-infinite matrix-vector equation. Given a finite $\Lambda \subset \mathbb{N}$, and having computed a (quasi-) best approximation to the solution by a vector supported on Λ, the norm of the residual vector of this approximation is proportional to the norm of its error. By adding those indices of the residual vector that correspond to its largest entries to the set Λ an extended set is created, and the loop can be repeated.

Generally the aforementioned residual vector has infinite support, and therefore has to be approximated. This residual vector is the difference of the infinite load vector and the bi-infinite stiffness matrix applied to the current approximation vector. The standard approach is to approximate both terms separately, the second one using the nonlinear approximate matrix-vector multiplication routine, known as the **apply**-routine. In contrast to their difference, being the residual vector, the aforementioned terms do not tend to zero when the iteration proceeds. Therefore, in order to approximate the residual within some fixed relative tolerance, they have to be approximated within a relative tolerance that gets increasingly smaller.

In order to improve the quantitative properties of the adaptive wavelet method, Chapter 2 of this thesis is devoted to an approach to approximate the residual vector without splitting it. In order to do so, it is needed to be able to represent both parts of the residual in a common dictionary. In the standard setting of a PDE of 2nd order and continuous piecewise polynomial wavelets, the operator applied to a wavelet is a distribution, and the splitting cannot be avoided. Therefore, we consider

a reformulation of such a PDE as a first order system by introducing the flux as a separate unknown. We show that any well-posed semi-linear PDE of 2nd order allows a reformulation as a well-posed first order system least squares (FOSLS) problem. For such system we design an alternative, more efficient approximate residual evaluation scheme that, in case of a linear PDE, depends linearly on the current approximation.

In Chapter 4 we apply this machinery to parabolic evolutionary PDEs in a simultaneously space-time variational formulation. In the recent years, one witnesses a growing interest in such formulations as an alternative for the usual time marching schemes that are inherently sequential, and that are not suited to efficiently approximating singularities that are local in both space and time. The arising Hilbert spaces in this setting are (intersections of) Bochner spaces that are most naturally equipped with Riesz bases that are tensor products of temporal and spatial wavelet bases. These tensor product wavelet bases require a modified, more complicated routine for the approximate residual evaluation, since one cannot afford a transformation to a locally single representation. On the other hand, they give the major advantage of an effective dimension reduction known from sparse-grid methods in a non-adaptive setting. We will be able to solve the whole time evolution at a complexity equal to that of solving one stationary problem.

For running the **awgm**, wavelet bases are needed for the generally non-square domains on which the equations are posed. Several constructions of continuous piecewise linear wavelets on general polygonal or polyhedral domains are known. Because of our FOSLS formulation, we need in addition a wavelet Riesz basis consisting of piecewise quadratics. Such bases are constructed in Chapter 3.

Finally, in Chapter 5 details are provided on several aspects of the implementation of the various routines that we have developed.

Samenvatting

Optimale Adaptieve Wavelet Methoden voor het vinden van een kleinste kwadraten oplossing van eerste orde stelsels.

Het efficiënt numeriek benaderen van de oplossing van een partiële differentiaal vergelijking (PDV) vereist een benaderingsruimte met een lokale resolutie welke aangepast is aan de lokale gladheid van de oplossing. Aangezien deze oplossing a priori onbekend is, moet een dergelijke ruimte adaptief gemaakt worden met behulp van een iteratie waarin benaderingen worden gecreëerd met toenemende nauwkeurigheid. Voorbeelden van dergelijke adaptieve oplossingsmethoden zijn de adaptieve eindige elementen methode en de adaptieve wavelet-Galerkin methode. De tweede methode is ingewikkelder te implementeren, maar heeft aan de andere kant het voordeel dat deze tevens toegepast kan worden op niet-elliptische PDV's.

Iedere goed-gestelde operatorvergelijking op een Hilbert ruimte welke uitgerust is met een Riesz basis, heeft een equivalente formulering als een oneindig matrix-vector probleem. Voor een gegeven eindige $\Lambda \subset \mathbb{N}$, en een (quasi-) beste benadering van de oplossing door een vector met drager in Λ, is de norm van de residu vector equivalent aan de norm van de fout. Door het toevoegen aan Λ van die indices van de residu vector welke corresponderen met de grootste elementen, verkrijgt men een uitbreiding van Λ waarmee de iteratie herhaald kan worden.

In het algemeen heeft voornoemde residu vector een oneindige drager, en moet deze daarom benaderd worden. Deze vector is het verschil van de oneindige rechterlid vector en de vector welke het resultaat is van de toepassing van de oneindige stijfheidsmatrix op de huidige benaderingsvector. De gebruikelijke aanpak is om beide termen afzonderlijk te benaderen, de tweede m.b.v. de niet-lineaire benaderende matrix-vector vermenigvuldigingsroutine, welke bekend staat als de **apply**-routine. In tegenstelling tot hun verschil, nl. de residu vector, convergeren beide voornoemde termen niet naar nul bij een klimmend aantal iteraties. Als gevolg hiervan moeten ze beide met een steeds kleinere relatieve tolerantie benaderd worden om ervoor te zorgen dat het residu met een vaste relatieve tolerantie benaderd wordt.

Om de kwantitatieve eigenschappen van de adaptieve wavelet methode te verbeteren, is hoofdstuk 2 van dit proefschrift gewijd aan een aanpak om de residu vector te benaderen zonder deze te splitsen. Om dit te kunnen doen is het nodig om beide delen van het residu te representeren in een gemeenschappelijk systeem. Voor het

standaard geval van een PDV van tweede orde en continue stuksgewijze polynomiale wavelets, resulteert de toepassing van de operator op een wavelet in een distributie, in welk geval de splitsing niet vermeden kan worden. Om deze reden zullen we een dergelijke PDV schrijven als een stelsel van PDV's van eerste orde door de gradiënt van de oplossing te introduceren als een aparte onbekende. We tonen aan dat iedere goed-gestelde semi-lineaire PDV van tweede orde geschreven kan worden als een goed gesteld kleinste kwadraten probleem voor een stelsel van PDV's van eerste orde. Voor een dergelijk systeem ontwikkelen we een alternatieve, efficiëntere residu benadering welke, voor een lineaire PDV, lineair afhangt van de huidige benaderende oplossing.

In hoofdstuk 4 passen we deze methodologie toe op parabolische tijdsafhankelijke PDV's in een tegelijkertijd ruimte-tijd variationele formulering. Gedurende de laatste jaren is er een toenemende belangstelling voor dergelijke formuleringen als een alternatief voor de gebruikelijke tijdsevolutie schema's, welke inherent sequentieel zijn, en niet geschikt voor het benaderen van singulariteiten die lokaal zijn in tijd en plaats. De relevante Hilbert ruimten voor deze problemen zijn (doorsneden van) Bochner ruimten welke canoniek uitgerust worden met Riesz bases, welke tensor producten zijn van wavelet bases in tijd en ruimte. Deze tensor product wavelet bases vereisen een gecompliceerdere routine voor de benadering van het residu, aangezien in dit geval een transformatie naar een lokale enkele schaal basis te kostbaar is. Aan de andere kant geven deze bases het belangrijke voordeel van een effectieve dimensie reductie, welke bekend is van de niet-adaptieve 'sparse-grid' methoden. We zullen in staat zijn het hele tijdsevolutie probleem op te lossen met een aantal operaties dat van dezelfde orde is als het aantal operaties waarmee het corresponderende stationaire probleem opgelost kan worden.

Voor het toepassen van de adaptieve wavelet methode zijn wavelet bases nodig op de in het algemeen niet-vierkante domein waarop de PDV geformuleerd is. Verschillende constructies zijn beschikbaar van stuksgewijze lineaire wavelets op algemene polygonen en polyeders. Vanwege onze formulering van de PDV als een eerste orde stelsel hebben we daarnaast ook een wavelet basis nodig bestaande uit stuksgewijze kwadratische functies. Dergelijke bases worden geconstrueerd in hoofdstuk 3.

Tenslotte worden in hoofdstuk 5 details gepresenteerd over aspecten van de implementatie van de verschillende routines welke we ontwikkeld hebben.

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