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The Composite Iteration Algorithm for Finding Efficient and Financially Fair Risk-Sharing Rules

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Abstract

We consider the problem of finding an efficient and fair ex-ante rule for division of an uncertain monetary outcome among a finite number of von Neumann-Morgenstern agents. Efficiency is understood here, as usual, in the sense of Pareto efficiency subject to the feasibility constraint. Fairness is defined as financial fairness with respect to a predetermined pricing functional. We show that efficient and financially fair allocation rules are in one-to-one correspondence with positive eigenvectors of a nonlinear homogeneous and monotone mapping associated to the risk sharing problem. We establish relevant properties of this mapping. On the basis of this, we obtain a proof of existence and uniqueness of solutions via nonlinear Perron-Frobenius theory, as well as a proof of global convergence of the natural iterative algorithm. We argue that this algorithm is computationally attractive, and discuss its rate of convergence.

Keywords: risk sharing, fair division, Perron-Frobenius theory, eigenvector computation, collectives.

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1 Introduction

This paper is concerned with the design of risk sharing systems. For an example of the type of situation we have in mind, consider a collective pension fund of the type existing for instance in the Netherlands. The claim to future benefits that participants receive in return for their contributions is a contingent claim, since benefits depend on the funding status at the time of payment, and the funding status in turn depends on realized investment returns as well as on prevailing interest rates. In the design of a system of this nature, it would seem reasonable to include considerations relating to *preferences* (different degrees of risk aversion among participants) as well as considerations relating to *financial fairness* (balance between the value of agents' contributions on the one hand, and the value of the contingent claims they receive in return on the other hand). The aspect of value brings prices into play. Since the agents in the risk sharing systems we have in mind constitute only a small part of the entire economy, prices will be taken as exogenously given.

The model that we use as a basis for risk sharing design is a two-period model (time points 0 and 1) with a finite number of von Neumann-Morgenstern agents. We allow for a continuum of possible time-1 states of nature. We assume the availability of a valuation operator that is of sufficiently wide scope to determine the value of any contingent claim that might be defined as a result of risk sharing. The inputs to the design problem are (i) agents' preferences, specified by utility functions and objective probabilities, (ii) their claim values (in monetary units),[1] and (iii) the aggregate endowment (i.e., shared risk—for instance, the uncertain outcome of joint investment). The objective of the design is to find a Pareto efficient allocation of the aggregate endowment such that all agents' allotments are within their budget sets as determined by their claim values and by the given valuation operator. We refer to such allocations as being *Pareto efficient and financially fair* (PEFF).

This form of the risk sharing problem has been formulated in the literature already several decades ago (Gale, 1977; Gale and Sobel, 1979; Bühlmann and Jewell, 1978, 1979; Balasko, 1979). Results on existence and uniqueness of PEFF solutions are available in the cited papers under various assumptions. The purpose of the present paper is to propose an effective and easily implemented computational algorithm, which may stimulate a more widespread use of the PEFF solution concept. We suggest an iterative method that is

[1] The term "claim value" refers to the time-0 value of an agent's share. For instance, if a project with time-0 value 100 is jointly owned by two agents A and B who hold 60% and 40% of the ownership rights respectively, then the claim value of agent A is 60 and the claim value of agent B is 40. These claim values can be achieved in many ways; for instance, agent A's claim might be 50% of the outcome of the project up to a certain threshold plus 100% of the amount by which the project outcome exceeds the threshold.

built up from simple steps. We provide a proof of convergence of the iteration, and we demonstrate that the asymptotic rate of convergence is linear. The analysis is cast in the framework of nonlinear Perron-Frobenius theory.

The model used in this paper can be looked at from the point of view of optimal risk sharing, but it also relates to the theory of fixed-price equilibria, and to the theory of fair division. A discussion of these relationships can be given as follows.

Research on optimal risk sharing has a long history. The origins of the theory of reciprocal reinsurance treaties are traced back by Seal (1969) to de Finetti (1942). Borch (1962) obtained a parametrization of the collection of all Pareto optimal solutions to a risk sharing problem, when the preferences of agents can be described by expected utility. The value-based notion of fairness that is used in this paper was proposed by Gale (1977) in the context of distribution of a random harvest in proportion to ownership rights. The applicability of Gale's ideas to risk sharing was noted by Bühlmann and Jewell (1978, 1979), who generalized the problem formulation by allowing the weights of future states that are used in the fairness condition to be different from probabilities as perceived by the agents. For the allocation problem as formulated by Gale (1977), the uniqueness of Pareto optimal and fair allocations was shown by Gale and Sobel (1979) under the assumption a finite number of possible future states, and by Gale and Sobel (1982) in the continuous case, under somewhat restrictive conditions on utility functions. The proof of uniqueness in these papers is based on the construction of a "social welfare function", which is such that it reaches its optimum on the set of financially fair allocations at a Pareto efficient point. Bühlmann and Jewell (1979) note that essentially the same technique can be applied as well to their formulation of the problem. Sobel (1981) gives a proof of uniqueness that avoids the introduction of the social welfare function, in order to accommodate a generalization in which agents use private valuation functionals.

In recent years, formulations of the risk sharing problem in which the preferences of agents are specified by risk measures (monetary valuation functionals) have attracted considerable interest; see for instance Chateauneuf et al. (2000); Barrieu and El Karoui (2005); Acciaio (2007); Jouini et al. (2008); Filipovic and Svindland (2008); Kiesel and Rüschendorf (2008). When all agents use a translation invariant risk measure (as in Artzner et al. (1999)) for evaluation, Pareto optimal solutions can only be unique up to addition of deterministic side payments which sum to zero. In such a case, the existence of Pareto optimal solutions automatically implies the existence of solutions that are both Pareto optimal and financially fair, and the question of uniqueness comes down to uniqueness of Pareto optimal solutions up to "rebalancing the cash". Uniqueness results of this type were given by Filipovic and Svindland (2008) and Kiesel and Rüschendorf (2008) under a condition of strict convexity.

A model similar to those proposed by Gale (1977) and by Bühlmann and Jewell (1979), but using more general preference specifications, was developed contemporaneously and independently by Balasko (1979). Balasko was motivated by developments in general equilibrium theory, in particular fixed-price equilibria as studied by Drèze (1975) and Benassy (1975). He used methods of differential topology to show existence of Pareto efficient and financially fair allocations. Keiding (1981) gives an existence result under a very general preference specification and mentions that, at this level of generality, uniqueness cannot be guaranteed.

In more recent work, Herings and Polemarchakis (2002, 2005) have studied fixed-price equilibria from the point of view of Pareto-improving interventions. The interventions discussed by these authors are based on price regulation. It may be viewed as an advantage of such interventions that they operate anonymously, by means of market variables. In this paper, intervention takes place directly through allocation, but is subject to the constraint of respecting agents' claim values.

Fair division problems have been studied extensively in social choice theory; see for instance Brams and Taylor (1996) and Brandt et al. (2016). A typical setting, as used for instance by Brams and Taylor (1996), equips agents with a linear valuation operator, which generally is different for different agents. This operator serves to define the ranking of alternatives by agents, and at the same time it also supports a notion of fairness. Fairness can be expressed as proportionality: all agents receive a share that, according to their own valuation, is at least equal to $1/n$-th of the total, where n is the number of agents. More generally, the fractions $1/n$ may be replaced by "entitlements" that are not necessarily equal to each other. These fairness constraints are expressed through inequalities, and consequently they usually do not determine a unique solution. As a stronger notion, envy-freeness has been used extensively (no agent should prefer another agent's allotment to his or her own).

One way in which the setting of the present paper is different from the framework commonly used in fair division theory is that a distinction is made between *utility* value on the one hand, and *financial* value on the other hand. Moreover, financial value is taken to be agent-independent. The notion of "claim value" used in this paper is similar to the notion of "entitlement" (applied to financial value). However, while entitlements are used to formulate *inequality* constraints, claim values are used to specify *equality* constraints. Indeed, due to the agent-independent nature of financial value, the sum of the claim values of the agents is fixed, which makes it impossible to raise one agent's claim value without reducing the claim values of others. The separation between utility value and financial value makes it possible to define the notion of efficiency in terms of utility value, so that a distinction between efficient and inefficient solutions can still be made, even though claim values are fixed. This is analogous to the classical single-agent problem of portfolio optimization, in which the role of the claim value is played by the budget constraint, and the agent aims to maximize utility subject to the given budget.

Both in general equilibrium theory and in the literature on fair division, much attention has been paid to computational methods. In comparison, algorithms for finding PEFF solutions have not been explored as extensively. One possible approach is to employ the transformation to single-objective optimization problems from Gale (1977) that was used in subsequent papers to prove existence and uniqueness of solutions. Since the optimization problems resulting from this transformation are convex, any method for solving convex optimization problems subject to equality constraints (see for instance Boyd and Vandenberghe (2004, Ch. 10)) can qualify as a method for obtaining PEFF solutions. However, it would be of interest to make more use of the particular structure of PEFF problems. An early discussion of specialized methods is given by Bühlmann and Jewell (1979). They discuss in particular the case of two agents, for which a line search suffices, and the case of exponential utility.

The iterative algorithm for finding PEFF solutions that is proposed in this paper can be related to a matrix algorithm known as the "iterative proportional fitting procedure" (IPFP). This algorithm finds, among all matrices with given positive row and column sums, the one that is closest, in the sense of Kullback-Leibler divergence, to a given nonnegative matrix. The procedure was proposed as a matrix fitting method by Kruithof (1937) and independently by Deming and Stephan (1940), but the optimization problem solved by it was only identified decades later by Ireland and Kullback (1968). Convergence of IPFP was proved by Csiszár (1975) in the discrete case and by Rüschendorf (1995) in the continuous case.

The iterative proportional fitting procedure can be viewed as an implementation of the method of successive projections due to Bregman (1967). This method is generally applicable to convex optimization problems with equality constraints. In particular, it may be applied to the optimization problems that result from Gale's transformation of PEFF problems. Making use of the Borch parametrization (see Section 3.2), one then arrives at the same procedure as the one that is studied in this paper, and that is motivated below directly from the PEFF problem.

In the framework of successive projections, one would be led to convergence analysis in the style of Csiszár (1975), based on generalized versions of the Pythagorean theorem. While convergence of Bregman projections has been discussed extensively in the literature (see for instance Censor and Lent (1981); Bauschke and Borwein (1996)), the authors are unaware of a result along these lines that would apply directly to the situation considered in the present paper. Below we use an alternative perspective, which relates PEFF solutions to positive eigenvectors of a nonlinear mapping. The analogous approach in the context of IPFP has been pioneered by Menon (1967) and Brualdi et al. (1966).

If the approximation problem solved by IPFP would be translated to the PEFF context by applying Gale's transformation backwards, it would lead to a problem in which the number of future states is finite, the probabilities of future states are agent-dependent (i.e., subjective probabilities), and agents' utility functions are logarithmic. In this paper it is assumed that all agents assign the same probabilities to future states (since this is a standard assumption in a large part of the literature, and it simplifies notation), but extension to the case of subjective probabilities would be straightforward; cf. for instance Wilson (1968). IPFP would then become a special case of the algorithm considered here.

In the present paper, we work within the classical framework of expected utility. We demonstrate that the problem of finding a Pareto efficient and financially fair allocation can be written as the problem of finding a positive eigenvector of a homogeneous nonlinear mapping from the nonnegative cone into itself. This leads naturally to the use of nonlinear Perron-Frobenius theory. For a general introduction to this subject, see for instance the book by Lemmens and Nussbaum (2012). We show that the homogeneous mapping associated to the fair allocation problem enjoys a number of properties that are useful within the Perron-Frobenius theory, such as continuity, monotonicity, and a more exotic property called nonsectionality.

The formulation as an eigenvector problem suggests an iterative solution method, analogous to the "power method" in the linear case (Wilkinson, 1965). We prove that convergence takes place from any given initial point within the positive cone. The iteration is built up from mappings that are easy to compute, so that it offers an attractive alternative to other methods which call for solution of large nonlinear equation systems.

This paper takes a "social planner" point of view. We do not model a negotiation process between the agents, as for instance in Boonen (2016). State prices, which are used to determine financial fairness, are assumed to be given. The risk to be shared is taken to be given as well, as in the paper by Borch (1962). The reader may refer to Pazdera et al. (2016) for the construction of a suitable homogeneous mapping in the context of risk sharing situations as in Wilson (1968), where the risk itself is subject to a decision by the collective.

The paper is organized as follows. Notation and assumptions are covered in the section following this introduction. Section 3 discusses the problem formulation, and Section 4 presents a brief review of requisite mathematical material. The main results of the paper are in Section 5. A special case is discussed in Section 6, followed by a discussion of the rate of convergence in Section 7. Finally, Section 8 concludes.

2 Notation and assumptions

The model that we use in this paper can be understood as a two-period exchange economy under uncertainty. There is a single good and a continuum of future states of nature. State prices are supposed to be given and are described by means of a pricing measure Q. There is a finite number n of agents. The agents' preferences across distributions of future consumption are of the von Neumann-Morgenstern type: agent i ranks distributions on the basis of expected utility of future consumption, where expectation is taken under an objective probability measure P, and utility is measured by a utility function $u_i(x)$. Subject to the given pricing measure Q, the budget set of agent i is determined by a number v_i which quantifies the ownership rights of agent i, and which can be interpreted as the value (under the given pricing measure Q) of the agent's initial endowment. The aggregate endowment is denoted by X. The relationship

$$\sum_{i=1}^{n} v_i = E^Q[X] \tag{2.1}$$

holds, where the symbol E^Q denotes expectation under Q. This relation states that the sum of the claim values of the agents is equal to the time-0 value of the aggregate endowment.

The aggregate endowment X is also referred to as the total risk that is to be shared among the agents. Our sign convention is that positive values of X indicate gains and negative values indicate losses, so that the term "risk" is to be understood as "uncertain outcome" without necessarily a negative connotation.

In mathematical terms, risks are modeled as bounded random variables on a measurable space $(\Omega, \mathcal{F})$. The agents' utility functions are taken to be defined on an intervals of the form (b_i, ∞) with $b_i \in [-\infty, \infty)$, and will always be assumed to satisfy the following conditions.

Assumption 2.1 For each $i = 1, \ldots, n$, the function $u_i : (b_i, \infty) \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, and strictly concave. Moreover, the following Inada conditions are satisfied:

$$\lim_{x \downarrow b_i} u_i'(x) = \infty, \qquad \lim_{x \to \infty} u_i'(x) = 0. \tag{2.2}$$

As a result of this assumption, all marginal utilities u_i' are continuous and strictly decreasing functions whose range covers the positive real axis. The inverse marginal utility of agent i will be denoted by I_i. In other words, I_i is the function from $(0,\infty)$ to (b_i,∞) that is defined implicitly by

$$u_i'(I_i(z)) = z \qquad (z > 0). \tag{2.3}$$

The inverse marginal utility is a continuous and strictly decreasing function that has the interval (b_i,∞) as its image. We write

$$b := \sum_{i=1}^{n} b_i, \qquad D := (b,\infty) \tag{2.4}$$

with the convention that $b = -\infty$ if there is an index i such that $b_i = -\infty$. The space of continuous functions from D to $\mathbb{R}$ will be denoted by $C(D,\mathbb{R})$.

If all lower bounds b_i are finite and if the total risk X is such that $P(X \leq \sum_{i=1}^{n} b_i) > 0$, then it is not possible to allocate the risk in such a way that the expected utility of each agent is finite. To make the problem feasible, we need to impose that $P(X \in D) = 1$. We shall in fact work under the stronger assumption that the risk X is bounded away from the critical level.

Assumption 2.2 The total risk X takes values in a compact set $A \subset (b,\infty)$.

For vectors α, $\beta \in \mathbb{R}^n$, the notation $\alpha < \beta$ ($\alpha \leq \beta$) indicates that $\alpha_i < \beta_i$ ($\alpha_i \leq \beta_i$) for all i, whereas $\alpha \lneq \beta$ means $\alpha \leq \beta$ and $\alpha \neq \beta$. Similar notation will be used for real-valued functions: in particular, for functions f, $g \in C(D,\mathbb{R})$ we write $f < g$ when $f(x) < g(x)$ for all $x \in D$. A mapping f from one ordered space into another will be said to be *monotone* if $x \leq y$ implies $f(x) \leq f(y)$, *strictly monotone* if it is monotone and $x < y$ implies $f(x) < f(y)$, and *strongly monotone* if $x \lneq y$ implies $f(x) < f(y)$.

The nonnegative cone $\{\alpha \in \mathbb{R}^n \mid \alpha \geq 0\}$ is denoted by $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$ indicates the positive cone $\{\alpha \in \mathbb{R}^n \mid \alpha > 0\}$. When α is a given vector in $\mathbb{R}^n$ and $S = \{i_1,\ldots,i_k\}$ is a nonempty subset of the index set $\{1,\ldots,n\}$, we write $\alpha_S := (\alpha_{i_1},\ldots,\alpha_{i_k})$. If $(\alpha^k)_{k=1,2,\ldots}$ is a sequence of vectors in $\mathbb{R}^n$, the notation $\alpha^k \to \infty$ means that $\alpha_i^k \to \infty$ for all $i = 1,\ldots,n$.

The valuation functional that is used in the financial fairness condition is obtained from a probability measure Q defined on $(\Omega,\mathcal{F})$. In the two-period model that we consider, discounting can be dispensed with. The time-0 value of a random payoff X at time 1 will therefore simply be represented by the expectation of X with respect to the measure Q.

We do not require the valuation measure Q to be absolutely continuous with respect to the probability measure P used by agents to compute expected utility, nor do we require that the measure P should be absolutely continuous with respect to Q. The development of this paper still applies even when the measure Q is concentrated on a single outcome of the total risk X. Such a situation may be realistic; it occurs when a group, in neglect of the stochasticity of X, has only made a decision in advance about how to divide a particular outcome. The principles of Pareto efficiency and financial fairness are then sufficient, given the agents' utility functions, to arrive at a well-defined allocation even if a different outcome is realized.

3 The allocation problem

3.1 Definitions

A *risk-sharing rule* is a collection $(y_1, \dots, y_n)$ of functions in $C(D, \mathbb{R})$ satisfying the feasibility condition

$$\sum_{i=1}^{n} y_i(x) = x \qquad (x \in D). \tag{3.1}$$

By requiring the equality to hold for all x in the domain D, which is determined by the preferences of the agents, we avoid dependence on the range of values taken by a specific risk X. It will be seen below that this extension of the problem setting does not affect either existence or uniqueness of solutions.

The risk of agent i after allocation is $Y_i := y_i(X)$, and the corresponding utility for agent i is $E^P[u_i(Y_i)]$. Risk sharing can be thought of as a particular form of allocation, so that we also sometimes use the term "allocation rule" or simply "allocation" instead of "risk-sharing rule". The functions $y_i(\,\cdot\,)$ are called *allocation functions*.

We will be looking for risk-sharing rules that are *Pareto efficient* as well as *financially fair*. The definition of Pareto efficiency is standard.

Definition 3.1 A risk-sharing rule $(y_1, \dots, y_n)$ is *Pareto efficient* (or *Pareto optimal*) if there does not exist a risk-sharing rule $(\tilde{y}_1, \dots, \tilde{y}_n)$, with associated allocated risks $(\tilde{Y}_1, \dots, \tilde{Y}_n)$, such that $(E^P[u_1(\tilde{Y}_1]), \dots, E^P[u_n(\tilde{Y}_n)]) \gneq (E^P[u_1(Y_1)], \dots, E^P[u_n(Y_n)])$.

To state the definition of financial fairness, we assume that for each agent a number $v_i > b_i$ is specified, which represents the claim value of agent i. These numbers may also be referred to as *ownership rights*. The claim value specifies only the time-0 value of the allotment to be received by agent i, not the allotment itself.

Definition 3.2 A risk-sharing rule $(y_1, \ldots, y_n)$ for the given total risk X is *financially fair* if, for each agent, the value of the allocated share of each agent is equal to that agents' claim value, i.e.,

$$E^Q[y_i(X)] = v_i \qquad (i = 1, \ldots, n). \tag{3.2}$$

Because the allocation functions are continuous, and because of Assumption (2.2), the random variables $Y_i = y_i(X)$ are bounded, so that their expectations under Q are indeed well defined. Feasibility of the requirement (3.2) taking into account the market clearing property (3.1) is guaranteed by the relation (2.1).

In this paper we are interested in allocation functions that combine financial fairness with Pareto efficiency. It should be noted that the notion of Pareto efficiency that we use here is subject to feasibility (3.1), but not to financial fairness. In other words, we want to find feasible allocations that are financially fair, and that are Pareto efficient even among feasible allocations that violate financial fairness.

3.2 Borch's parametrization

To convert the allocation problem into a set of equations, we use the parametrization of Pareto efficient risk-sharing rules that was devised by Borch in 1962.

Theorem 3.3 (Borch, 1962) *A risk-sharing rule $(y_1, \ldots, y_n)$ is Pareto efficient for any given total risk taking values in the domain D if and only if there exist a continuous function $J : D \to \mathbb{R}_{++}$ and positive constants $\alpha_1, \ldots, \alpha_n$ such that*

$$\alpha_i u_i'(y_i(x)) = J(x) \tag{3.3}$$

for all $x \in D$ and for all $i = 1, \ldots, n$.

Details of the proof can be found in DuMouchel (1968); Gerber and Pafumi (1998); Barrieu and Scandolo (2008). The quantity $J(x)$ can be interpreted as a Lagrange multiplier associated to the feasibility constraint (3.1). We can now state the central problem considered in this paper as follows.

Problem 3.4 Assume given: a finite number of agents, with utility functions u_i satisfying Assumption 2.1; a risk X satisfying Assumption 2.2; a valuation measure Q; and agents' claim values v_i satisfying (2.1). Find a collection of allocation functions $(y_1, \ldots, y_n)$ such that the following conditions are satisfied:

- feasibility, i.e. $\sum_{i=1}^{n} y_i(x) = x$ for all $x \in D$;

- Pareto efficiency, i.e. there exist positive constants $\alpha_1, \ldots, \alpha_n$ and a continuous function $J : D \to \mathbb{R}_{++}$ such that (3.3) holds for all $x \in D$ and for all i;
- financial fairness, i.e. $E^Q[y_i(X)] = v_i$ for all i.

The Borch condition (3.3) can be rewritten as follows in terms of the inverse marginal utilities (cf. (2.3)):

$$y_i(x) = I_i\big(J(x)/\alpha_i\big). \tag{3.4}$$

Since the functions y_i must satisfy the feasibility condition, the following condition has to be satisfied for all $x \in D$:

$$\sum_{i=1}^{n} I_i\big(J(x)/\alpha_i\big) = x. \tag{3.5}$$

For given $(\alpha_1, \ldots, \alpha_n)$ and given x, the above equation determines $J(x)$ uniquely, since the function $z \mapsto \sum_{i=1}^n I_i(z/\alpha_i)$ is strictly decreasing. We may therefore consider the function J to be defined by the relation (3.5); to emphasize this point of view, we will sometimes write $J(x;\alpha)$ instead of $J(x)$. Conversely, if (3.5) is satisfied for a set of positive numbers $\alpha_1, \ldots, \alpha_n$, then the functions $y_1, \ldots, y_n$ in (3.4) determine a Pareto efficient risk-sharing rule.

In this way, Borch's theorem provides a parametrization of Pareto efficient risk-sharing rules in terms of the utility weights $\alpha_1, \ldots, \alpha_n$. The effective number of parameters is in fact $n-1$ rather than n, since the allocation rule that is generated by a positive vector $(\alpha_1, \ldots, \alpha_n)$ does not change if all numbers α_i are multiplied by the same positive constant. Indeed, in this case the corresponding function J is multiplied by the same constant, so that the ratios $J(x)/\alpha_i$ remain the same.

Remark 3.5 Given a vector $\alpha \in \mathbb{R}^n_{++}$, let $(y_1, \ldots, y_n) = (y_1(\,\cdot\,;\alpha), \ldots, y_n(\,\cdot\,;\alpha))$ denote the Pareto efficient risk-sharing rule defined through (3.4) and (3.5). The "weighted group utility" $u(\,\cdot\,;\alpha)$ corresponding to given weights $\alpha = (\alpha_1, \ldots, \alpha_n)$ is defined by

$$u(x;\alpha) = \sum_{i=1}^{n} \alpha_i u_i(y_i(x)) \qquad (x \in D). \tag{3.6}$$

Under the assumption that the utility functions u_i are twice continuously differentiable, the inverse marginal utilities are continuously differentiable; it follows that the function J, being the inverse of the mapping $z \mapsto \sum_{i=1}^n I_i(z/\alpha_i)$, is differentiable as well. Consequently, the allocation functions y_i defined by (3.4) are likewise differentiable. We can then write (cf. for instance Xia (2004))

$$u'(x;\alpha) = \sum_{i=1}^{n} \alpha_i u_i'(y_i(x))y_i'(x) = J(x;\alpha)\sum_{i=1}^{n} y_i'(x) = J(x;\alpha) \tag{3.7}$$

where the second equality follows from Borch's condition (3.3) and the third uses the feasibility condition (3.1). The function $J(\,\cdot\,;\alpha)$ can thus be interpreted as the marginal group utility that corresponds to a given set of weights α.

3.3 Computational approaches

Consider now the problem of numerically solving the equation system consisting of the feasibility condition (3.1), the financial fairness condition (3.2), and the efficiency condition (3.3). The unknowns in these equations consist of the utility weights α_i, the marginal utility (or multipliers) $J(x)$, and the allocation functions y_i. The equation system suggests at least three broad computational approaches.

First of all, using the fact that the marginal utility J can be thought of as being defined by the utility weights, as discussed above, the system (3.1–3.2–3.3) can be rewritten as a system of n nonlinear equations in n unknowns $\alpha_1, \ldots, \alpha_n$:

$$E^Q\big[I_i(J(X;\alpha_1,\ldots,\alpha_n))/\alpha_i\big] = v_i \qquad (i = 1,\ldots,n). \tag{3.8}$$

Subsequently, a nonlinear equation solver may be applied. This approach, which uses utility weights as the reduced set of unknowns, is analogous to Negishi's method in the theory of Arrow-Debreu equilibrium (Negishi, 1960).

Alternatively, one can express the utility weights in terms of the marginal utility by making use of the financial fairness conditions. Indeed, as will be discussed in more detail below, the equations (3.2) and (3.4) determine the weights α_i when the function J is given. Writing $\alpha = \alpha(J)$ to indicate this dependence, we can find Pareto efficient and financially fair allocations by solving the equation

$$\sum_{i=1}^{n} I_i\big(J(x)/\alpha_i(J)\big) = x \qquad (x \in D) \tag{3.9}$$

for the unknown function J. This approach is analogous to the standard method of finding Arrow-Debreu equilibria by making use of the excess demand function; see for instance Kehoe (1991). Here, the excess demand is given by the difference of the left-hand side and the right-hand side in the equation above.

The weights that solve the equation system (3.8) can alternatively be characterized as fixed points of the composite mapping that is formed by applying the mapping $\alpha \mapsto J$ of the first method, followed by the mapping $J \mapsto \alpha$ of the second method. This leads to a third computational approach. The analogous technique in the case of Arrow-Debreu equilibrium appears in a paper by Dana (2001, p. 170) under the name "price-weight, weight-price approach". It is natural to attempt to find fixed points by iteration of the composite mapping. While the first two methods that we discussed produce nonlinear equation systems that can be challenging to solve, the composite iteration method relies on repeated use of mappings that are easy to compute. As already mentioned in the Introduction, the iterative method can be constructed as an application of Bregman's successive projections method via reformulation of the PEFF problem as an optimization problem; however, the motivation as given above seems more direct.

In the present paper we focus on the "price-weight, weight-price" approach; we call it here the composite iteration algorithm. We aim to establish relevant properties of the composite iteration mapping, which allow to prove convergence of the algorithm. In particular, we prove (Thm. 5.9 below) that the composite iteration mapping can be uniquely extended to a continuous, homogeneous,[2] and monotone mapping from the nonnegative cone to itself. Moreover, it will be shown that the fixed-point problem for the composite mapping can be reformulated as the problem of finding a positive eigenvector of the mapping. These facts lead towards the use of nonlinear Perron-Frobenius theory. Application of a theorem of Oshime (1983) (see Thm. 4.2 below) allows us to conclude existence and uniqueness of solutions as well as convergence of the iterative algorithm. Before proceeding to the main results, we first review mathematical preliminaries.

4 Preliminaries

Order-preserving (monotone) nonlinear maps can be viewed as generalizations of positive matrices. It turns out that much of the Perron-Frobenius theory concerning eigenvalues and eigenvectors of such matrices can be extended to the nonlinear case. An extensive discussion of nonlinear Perron-Frobenius theory is provided by Lemmens and Nussbaum (2012).

A particular class of interest is the class of monotone mappings that are *homogeneous* in the sense that $\varphi(\lambda x) = \lambda \varphi(x)$ for all positive λ. For continuous homogeneous mappings from the nonnegative cone into itself, the existence of nonnegative eigenvectors follows

[2] In this paper we always use the term "homogeneous" in the sense of "homogeneous of degree 1".

from Brouwer's fixed-point theorem. However, for the application to Pareto efficient and financially fair allocations, we need an eigenvector with entries that are strictly positive. Conditions for existence and uniqueness of such eigenvectors form an important topic in nonlinear Perron-Frobenius theory; see for instance (Lemmens and Nussbaum, 2012, Ch. 6). Here we will use a result of Oshime (1983) that guarantees the existence of a unique positive eigenvector. For ease of reference, this result is stated below. First the definition is given of a notion that can be thought of as a nonlinear variant of the irreducibility condition that is well known in linear Perron-Frobenius theory.

Definition 4.1 A mapping φ of $\mathbb{R}^n_+$ into itself is *nonsectional* if, for every decomposition of the index set $\{1, \ldots, n\}$ into two complementary nonempty subsets R and S, there exists $s \in S$ such that

(i) $(\varphi(x))_s > (\varphi(y))_s$ for all x, $y \in \mathbb{R}^n_+$ such that $x_R > y_R$ and $x_S = y_S > 0$;

(ii) $(\varphi(x^k))_s \to \infty$ for all sequences $(x^k)_{k=1,2,\ldots}$ in $\mathbb{R}^n_+$ such that $x^k_R \to \infty$ while x^k_S is fixed and positive.

Theorem 4.2 (Oshime, 1983, Thm. 8, Remark 2) *If a mapping φ from $\mathbb{R}^n_+$ into itself is continuous, monotone, homogeneous, and nonsectional, then the mapping φ has a positive eigenvector, which is unique up to scalar multiplication, with a positive associated eigenvalue. In other words, there exist a constant $\eta^* > 0$ and a vector $x^* \in \mathbb{R}^n_{++}$ such that $\varphi(x^*) = \eta^* x^*$, and if $\eta > 0$ and $x \in \mathbb{R}^n_{++}$ are such that $\varphi(x) = \eta x$, then x is a scalar multiple of x^*.*

The eigenvalue associated to the positive eigenvector in the above theorem is in fact the *maximal* eigenvalue of the mapping φ (Oshime, 1983, Thm. 3). Iteration is a standard method to find the eigenvector associated to the maximal eigenvalue. In the linear case, this technique is known as the *power method* (see for instance Wilkinson (1965, p. 570)). Due to the homogeneity of the problem, it is possible to reduce the iteration to the unit simplex. In relation to a given homogeneous mapping φ from the positive cone into itself, we can define a normalized mapping ψ from the open unit simplex $\{(x_1, \ldots, x_n) \in \mathbb{R}^n_{++} \mid \sum_{i=1}^n x_i = 1\}$ into itself by

$$\psi(x) = \frac{\varphi(x)}{\|\varphi(x)\|_1} \tag{4.1}$$

where $\|v\|_1 = \sum_{i=1}^n |v_i|$ is the 1-norm of $v \in \mathbb{R}^n$. Positive eigenvectors of the mapping φ correspond to fixed points of the mapping ψ.

To prove convergence of the iterative algorithm, it is natural to use a suitable contraction mapping theorem. First of all, an appropriate metric needs to be defined. A standard metric used in nonlinear Perron-Frobenius theory is the Hilbert metric, which is defined as follows.

Definition 4.3 The *Hilbert metric* assigns to a pair (x, y) with $x,\ y \in \mathbb{R}^n_{++}$ the distance $d(x, y)$ given by

$$d(x,y) = \log \frac{\max_i(x_i/y_i)}{\min_i(x_i/y_i)}\,.$$

Points on the same ray are equivalent with respect to the Hilbert metric, since

$$d(ax, by) = d(x, y) \quad \text{for all } a,\ b > 0 \tag{4.2}$$

On the positive cone, the Hilbert metric is therefore only a pseudometric. Alternatively, it can be viewed as a true metric on the space of positive rays, or on the open unit simplex (Lemmens and Nussbaum, 2012, Prop. 2.1.1).

The following lemma is a standard fact (see for instance Lemmens and Nussbaum (2012, Ch. 2)); for the reader's convenience, we provide a proof of the version that we need here. Recall that a mapping φ from a metric space into itself is said to be *contractive* if $d(\varphi(x), \varphi(y)) < d(x, y)$ for all x, y such that $d(x, y) > 0$.

Lemma 4.4 *If $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is homogeneous and strongly monotone (i.e., $x \lneq y$ implies $\varphi(x) < \varphi(y)$), then φ is contractive with respect to the Hilbert metric.*

Proof. Take $x,\ y \in \mathbb{R}^n_{++}$ with $d(x, y) > 0$. Define $M := \max_i(x_i/y_i)$, $m := \min_i(x_i/y_i)$. We then have $my \lneq x \lneq My$, and by homogeneity and strong monotonicity of φ we obtain $m\varphi(y) < \varphi(x) < M\varphi(y)$. Therefore,

$$\min_i \frac{\varphi(x)_i}{\varphi(y)_i} > m, \qquad \max_i \frac{\varphi(x)_i}{\varphi(y)_i} < M$$

and hence $d(\varphi(x), \varphi(y)) < \log(M/m) = d(x, y)$. □

As a consequence of the property (4.2), the mapping ψ that is obtained from φ by normalization to the unit simplex is contractive if φ is. The lemma above only establishes that φ is contractive, not that it is a contraction mapping; in other words, the lemma does not provide a positive number δ such that $d(\varphi(x), \varphi(y)) \le (1 - \delta)d(x, y)$ for all x and y. Therefore, we are not in a position to apply the Banach contraction mapping theorem. Instead we will use the following theorem due to Nadler, which in our application guarantees convergence as a result of the assumption that the number of agents is finite.

Theorem 4.5 (Nadler, 1972, Thm. 1) *When* $(\mathcal{X}, d)$ *is a locally compact and connected metric space, and* $f : \mathcal{X} \to \mathcal{X}$ *is a contractive mapping with fixed point* $x^* \in \mathcal{X}$*, then for every* $x \in \mathcal{X}$ *the sequence of iterates* $(f^{(k)}(x))_{k=1,2,\dots})$ *converges to the point* x^**.*

Alternatively, one might use an argument based on the fact that the open unit simplex with the Hilbert metric is a geodesic space (there is a geodesic path from any given point to any other given point); cf. Lemmens and Nussbaum (2012, Prop. 3.2.3, Thm. 6.5.1).

5 Main results

In this section, we prove the convergence of the composite iteration method. Our method of proof is based on application of nonlinear Perron-Frobenius theory, which calls for the verification of a number of properties of the iteration mapping. This will be done in a series of lemmas below. We also demonstrate that the approach via nonlinear Perron-Frobenius theory leads to a proof of existence and uniqueness of Pareto efficient and financially fair solutions, independent from the approach via reformulation as an optimization problem (Gale and Sobel, 1979; Bühlmann and Jewell, 1979).

First we need to introduce some notation. Recall that the domain D is defined as (b, ∞), where $b = \sum_{i=1}^n b_i$ and the bounds b_i are the left limits of the domains of the utility functions of the individual agents. Within the space $C(D, \mathbb{R}_+)$ of continuous functions from D to $[0, \infty)$, equipped with the topology of pointwise convergence, we define the cone of strictly decreasing functions

$$\mathcal{L} = \{f \in C(D, \mathbb{R}_+) \mid f(y) < f(x) \text{ for all } x, y \in D \text{ s. t. } y > x\} \cup \{0\}.$$

The inclusion of the zero function within this set is natural when the functions in $\mathcal{L}$ are thought of as in terms of their graphs as subsets of the region $[b, \infty] \times [0, \infty]$ in the extended two-dimensional space. The function 0 can then be viewed as a representation of the multivalued mapping whose graph is $(\{b\} \times [0, \infty]) \cup ([b, \infty] \times \{0\})$.

5.1 Mapping from utility weights to marginal group utility

Agents whose utility functions are defined on all of the real line will need to be distinguished from agents who can tolerate only a limited loss. We therefore introduce the index set (possibly empty)

$$U = \{i \mid b_i = -\infty\}. \tag{5.1}$$

For $\alpha \in \mathbb{R}^n_+$ such that $\alpha_U > 0$, define

$$F(z, \alpha) = \sum_{i:\alpha_i > 0} I_i(z/\alpha_i) + \sum_{i:\alpha_i = 0} b_i. \tag{5.2}$$

This function is a continuous mapping from the product space $\mathbb{R}_{++} \times \{\alpha \in \mathbb{R}^n_+ \mid \alpha_U > 0\}$ to $\mathbb{R}$. For fixed α, the function $F(\,\cdot\,, \alpha) : (0, \infty) \to (b, \infty)$ is continuous and strictly decreasing, and satisfies

$$\lim_{z \to \infty} F(z, \alpha) = b, \qquad \lim_{z \downarrow 0} F(z, \alpha) = \infty. \tag{5.3}$$

Therefore there is a well-defined inverse function, which is denoted by $J(\,\cdot\,, \alpha)$. Since $F(\,\cdot\,, \alpha)$ is strictly decreasing and continuous, the same properties hold for $J(\,\cdot\,, \alpha)$. For $\alpha \in \mathbb{R}^n_+$, we can therefore define the function $\varphi_1(\alpha) \in \mathcal{L}$ by

$$(\varphi_1(\alpha))(x) = \begin{cases} J(x, \alpha) & \text{if } \alpha \neq 0 \text{ and } \alpha_U > 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

for $x \in D$. For $\alpha > 0$, the defining relationship for the mapping φ_1 may also be written in a more implicit but perhaps also more evocative form as

$$\varphi_1 : \alpha \mapsto J, \qquad \sum_{i=1}^{n} I_i\big(J(x)/\alpha_i\big) = x \qquad (x \in D). \tag{5.5}$$

We now establish various properties of this mapping such as continuity and monotonicity.

Lemma 5.1 *The mapping φ_1 is homogeneous and monotone. If $\alpha^1 \in \mathbb{R}^n_+$ and $\alpha^2 \in \mathbb{R}^n_+$ are such that $\alpha^1_U > 0$ and $\alpha^1 \gneq \alpha^2$, then we have in fact $\varphi_1(\alpha^1) > \varphi_1(\alpha^2)$.*

Proof. The homogeneity is immediate from the definitions. Concerning the monotonicity, take α^1 and α^2 in $\mathbb{R}^n_+$ such that $\alpha^1 \geq \alpha^2$. First assume that $\alpha^2_U > 0$; then also $\alpha^1_U > 0$. Take $x \in D$, and let z_1 and z_2 be defined by $F(z_1, \alpha^1) = x$ and $F(z_2, \alpha^2) = x$. We then have $z_i = (\varphi_1(\alpha^i))(x)$ for $i = 1, 2$. Because the function $F(\,\cdot\,, \cdot\,)$ is strictly increasing in each of the components of its second argument and strictly decreasing in its first argument, the vector inequality $\alpha^1 \geq \alpha^2$ and the equality $F(z_1, \alpha^1) = F(z_2, \alpha^2)$ together imply that $z_1 \geq z_2$, with strict inequality as soon as α^1 and α^2 are not equal. If $\alpha^2_i = 0$ for some $i \in U$ while $\alpha^1_U > 0$, then the strict inequality $\varphi_1(\alpha^1) > \varphi_1(\alpha^2)$ trivially holds, since $\varphi_1(\alpha^1)$ takes positive values, while $\varphi_1(\alpha^2) = 0$ by definition. Finally, if there is $i \in U$ such that $\alpha^1_i = 0$, then $\varphi_1(\alpha^1) = \varphi_1(\alpha^2) = 0$. $\square$

To show the continuity of φ_1, we make use of the following lemma.

Lemma 5.2 *Let a topological space $\mathcal{Y}$ and a sequentially continuous mapping $f(\,\cdot\,,\,\cdot\,)$ from $\mathbb{R}_{++} \times \mathcal{Y}$ to $\mathbb{R}$ be given. Suppose that for every $y \in \mathcal{Y}$ there is exactly one $x \in \mathbb{R}_{++}$ such that $f(x, y) = 0$. Let $(y_k)_{k=1,2,\dots}$ be a sequence in $\mathcal{Y}$ that converges to $\bar{y} \in \mathcal{Y}$. Define x_k $(k = 1, 2, \dots)$ by the equations $f(x_k, y_k) = 0$, and let $\bar{x}$ be defined by $f(\bar{x}, \bar{y}) = 0$. If the collection $\{x_k \mid k \in \mathbb{N}\}$ is bounded, then $\lim_{k\to\infty} x_k = \bar{x}$.*

Proof. By the assumed boundedness of the collection $\{x_k \mid k \in \mathbb{N}\}$, it suffices to show that any accumulation point of this collection must coincide with $\bar{x}$. Let $\tilde{x}$ be an accumulation point, and let $(k_j)_{j=1,2,\dots}$ satisfy $\lim_{j\to\infty} x_{k_j} = \tilde{x}$. From the continuity of the mapping f, we have $f(\tilde{x}, \bar{y}) = \lim_{j\to\infty} f(x_{k_j}, y_{k_j}) = 0$. The assumed uniqueness of the solution of the equation $f(x, y) = 0$ for given y then implies the equality $\tilde{x} = \bar{x}$. □

Lemma 5.3 *The mapping $\varphi_1 : \mathbb{R}^n_+ \to \mathcal{L}$ is continuous.*

Proof. Let $(\alpha^k)_{k=1,2,\dots}$ be a sequence of vectors in $\mathbb{R}^n_+$ converging to a vector $\alpha \in \mathbb{R}^n_+$. Take $x \in D$; write $z_k := (\varphi_1(\alpha^k))(x)$ and $z := (\varphi_1(\alpha))(x)$. We need to show that the sequence $(z_k)_{k=1,2,\dots}$ converges to z.

First consider the case in which $\alpha_U > 0$. In this case we also have $\alpha^k_U > 0$ for all sufficiently large k. By definition, the numbers z_k and z are positive and satisfy $F(z_k, \alpha^k) = x$ and $F(z, \alpha) = x$. Suppose there would be a subsequence $(z_{k_j})_{j=1,2,\dots}$ that tends to infinity. For all i with $\alpha_i > 0$, the sequences $(\alpha_i^{k_j})_{j=1,2,\dots}$ tend to finite limits, namely α_i. Consequently, the quotients $z_{k_j}/\alpha_i^{k_j}$ tend to infinity, and therefore

$$x = \lim_{j\to\infty} F(z_{k_j}, \alpha^{k_j}) = b.$$

However, we have $x \in (b, \infty)$ so that $x > b$. From this contradiction it follows that the set $\{z_k \mid k \in \mathbb{N}\}$ is bounded, and it follows from Lemma 5.2 that $\lim_{k\to\infty} z_k = z$.

Now suppose that there is an index $\ell \in U$ such that $\alpha_\ell = 0$. By definition, we then have $z = 0$. To avoid trivialities, we may assume that $\alpha^k_U > 0$ for all k. The numbers $z_k > 0$ are then given as the solutions of $F(z_k, \alpha^k) = x$. Take $\varepsilon > 0$, and suppose there would be a subsequence $(z_{k_j})_{j=1,2,\dots}$ such that $z_{k_j} > \varepsilon$ for all j. The quotient $z_{k_j}/\alpha_\ell^{k_j}$ then tends to infinity because of the assumption that $\alpha_\ell = 0$, and the corresponding inverse marginal utility function $I_\ell(z)$ tends to $-\infty$ when its argument tends to infinity, due to the assumption that $\ell \in U$. Because $z_{k_j} > \varepsilon$ for all j and the sequences $(\alpha_i^{k_j})_{j=1,2,\dots}$ tend to finite limits, the inverse marginal utilities $I_i(z_{k_j}/\alpha_i^{k_j})$ $(i = 1, \ldots, n)$ are bounded from above. Therefore, we obtain

$$x = \lim_{j \to \infty} F(z_{k_j}, \alpha^{k_j}) = -\infty.$$

This is a contradiction. We therefore have $\lim_{k\to\infty} z_k = 0$, as was to be proved. □

The next lemma states a property of the mapping φ_1 that relates to nonsectionality.

Lemma 5.4 *Let $(\alpha^k)_{k=1,2,\ldots}$ be a sequence in $\mathbb{R}^n_+$ that has the following property: there exist complementary nonempty index sets R and S in $\{1, \ldots, n\}$ and a vector $\alpha_S \in \mathbb{R}^{|S|}_{++}$ such that $\alpha^k_R \to \infty$ as $k \to \infty$, while $\alpha^k_S = \alpha_S$ for all k. Then $(\varphi_1(\alpha^k))(x) \to \infty$ as $k \to \infty$ for all $x \in D$.*

Proof. Take $x \in D$. Since the entries with indices in S are assumed to be positive and those with indices in R tend to infinity, we can assume that all entries of α^k are positive. Then the numbers $z_k := \varphi_1(\alpha^k)(x)$ are defined implicitly by

$$\sum_{i \in R} I_i(z_k/\alpha_i^k) + \sum_{i \in S} I_i(z_k/\alpha_i) = x. \tag{5.6}$$

Suppose that $(z_k)_{k=1,2,\ldots}$ has a bounded subsequence $(z_{k_j})_{j=1,2,\ldots}$. The quotients $z_{k_j}/\alpha_i^{k_j}$ tend to zero for $i \in R$ so that the first term on the left-hand side in (5.6) tends to infinity. The quotients z_{k_j}/α_i for $i \in S$ remain bounded, so that the second term at the left-hand side is bounded from below. Therefore the left-hand side tends to infinity as $j \to \infty$, which leads to a contradiction. The statement in the lemma follows. □

5.2 Mapping from marginal group utility to utility weights

We now turn to the mapping φ_2. Recall that the numbers v_i $(i = 1, \ldots, n)$ represent the claim values of the agents, and that $v_i > b_i$ for all i. We have assumed that the total risk X is bounded; consequently, for any given nonzero function $J \in \mathcal{L}$, the random variable $J(X)$ is bounded as well. For each $i = 1, \ldots, n$, the mapping $\alpha_i \mapsto E^Q I_i(J(X)/\alpha_i)$ defines a strictly increasing function with

$$\lim_{\alpha_i \to \infty} E^Q I_i(J(X)/\alpha_i) = \infty, \qquad \lim_{\alpha_i \downarrow 0} E^Q I_i(J(X)/\alpha_i) = b_i.$$

By the assumed inequality $v_i > b_i$, the equation

$$E^Q I_i(J(X)/\alpha_i) = v_i \tag{5.7}$$

therefore has a unique solution $\alpha_i > 0$. The mapping from collective marginal utility J to utility weights α can be extended to a mapping defined on all of $\mathcal{L}$ by

$$(\varphi_2(J))_i = \begin{cases} \alpha_i \text{ satisfying (5.7)} & \text{if } J \neq 0 \\ 0 & \text{if } J = 0 \end{cases} \tag{5.8}$$

for $i = 1, \ldots, n$.

Lemma 5.5 *The mapping φ_2 is homogeneous and strictly monotone (i.e. $\varphi_2(J_1) \geq \varphi_2(J_2)$ when $J_1 \geq J_2$ and $\varphi_2(J_1) > \varphi_2(J_2)$ when $J_1 > J_2$).*

Proof. The homogeneity is immediate from the definition. The strict monotonicity follows from the fact that all inverse marginal utilities I_i are strictly decreasing; in case $J_2 = 0$, the strict monotonicity is immediate from the definition. □

Lemma 5.6 *The mapping φ_2 is sequentially continuous.*

Proof. Let $(J_k)_{k=1,2,\ldots}$ be a sequence in $\mathcal{L}$, converging pointwise to $J \in \mathcal{L}$, and fix $i \in \{1, \ldots, n\}$. Write $\alpha_i^k := (\varphi_2(J_k))_i$ and $\alpha_i := (\varphi_2(J))_i$. We want to show that $(\alpha_i^k)_{k=1,2,\ldots}$ converges to α_i.

First assume that the limit function J is nonzero; we can then assume that all elements of the sequence J_k are nonzero as well. Note that the collection of random variables $J_k(X)$ is bounded above by $\sup_k J_k(\inf X)$ and below by $\inf_k J_k(\sup X)$. Therefore, if there would exist a subsequence $(\alpha_i^{k_j})_{j=1,2,\ldots}$ converging to infinity, we would have

$$v_i = \lim_{j\to\infty} E^Q I_i(J_{k_j}(X)/\alpha_i^{k_j}) = \infty. \tag{5.9}$$

This would contradict the assumptions. Consequently, the collection $\{\alpha_i^k \mid k \in \mathbb{N}\}$ is bounded. Consider the function $G : \mathbb{R}_{++} \times \mathcal{L} \to \mathbb{R}$ defined by

$$G(\alpha_i, J) = E^Q I_i(J(X)/\alpha_i).$$

It follows from the bounded convergence theorem that this function is sequentially continuous. The relation $\lim \alpha_i^k = \alpha_i$ follows from Lemma 5.2.

Consider now the case in which $J = 0$. In this case, we have by definition $\alpha_i = 0$. Take $\varepsilon > 0$ and suppose that there would exist a subsequence $(\alpha_i^{k_j})_{j=1,2,\ldots}$ such that $\alpha_i^{k_j} > \varepsilon$ for all $j = 1, 2, \ldots$. The convergence of $(J_k)_{k=1,2,\ldots}$ to $J = 0$ would then imply the same conclusion as in (5.9). Consequently, we have $\lim_{k\to\infty} \alpha_i^k = 0$, as was to be shown. □

The final lemma establishes a property that will be used in a nonsectionality argument.

Lemma 5.7 *Let $(J_k)_{k=1,2,\dots}$ be a sequence in $\mathcal{L}$ such that $J_k(x) \to \infty$ for all $x \in D$ as $k \to \infty$. Then $(\varphi_2(J_k))_i \to \infty$ for all $i = 1, \dots, n$.*

Proof. Choose $i \in \{1, \dots, n\}$. Assume that the i-th entry of $\alpha^k := \varphi_2(J_k)$ does not tend to infinity. Then there exist a finite number M and a subsequence $(\alpha_i^{k_j})_{j=1,2,\dots}$ such that $\alpha_i^{k_j} < M$ for all j. We would then have

$$v_i = \lim_{j\to\infty} E^Q I_i(J^{k_j}(X)/\alpha_i^{k_j}) = b_i. \tag{5.10}$$

This is a contradiction, since it has been assumed that $v_i > b_i$. Therefore the statement of the lemma follows. □

5.3 The complete iteration mapping

With the mappings $\varphi_1 : \mathbb{R}_+^n \to \mathcal{L}$ and $\varphi_2 : \mathcal{L} \to \mathbb{R}_+^n$ in hand, one can define a mapping φ from $\mathbb{R}_+^n$ into itself in the obvious way by

$$\varphi(\alpha) = \varphi_2(\varphi_1(\alpha)). \tag{5.11}$$

It follows from the development above and the Borch parametrization (3.4) that Pareto efficient and financially fair solutions of the risk sharing problem are in one-to-one correspondence with vectors $\alpha \in \mathbb{R}_{++}^n$ such that $\varphi(\alpha) = \alpha$. The proposition below implies that it is in fact sufficient to look for positive eigenvectors of the mapping φ. A similar argument was used by Menon (1967) in an analysis of the IPFP.

Proposition 5.8 *The mapping φ can only have 1 as an eigenvalue corresponding to a positive eigenvector. In other words, if $\alpha \in \mathbb{R}_{++}^n$ is such that $\varphi(\alpha) = \lambda\alpha$, then $\lambda = 1$.*

Proof. Let $\alpha > 0$ be such that $\varphi(\alpha) = \lambda\alpha$. Since φ maps the positive cone into itself, the eigenvalue λ must be positive. Define $J = \varphi_1(\alpha)$; then $\varphi_2(J) = \lambda\alpha$. Note that $J(x) > 0$ for all $x \in D$. By definition, we have

$$\sum_{i=1}^n I_i(J(x)/\alpha_i) = x \qquad (x \in D)$$

$$E^Q I_i(J(X)/(\lambda\alpha_i)) = v_i \qquad (i = 1, \dots, n).$$

Therefore,

$$\sum_{i=1}^n E^Q I_i(J(X)/(\lambda\alpha_i)) = \sum_{i=1}^n v_i = E^Q X = \sum_{i=1}^n E^Q I_i(J(X)/\alpha_i).$$

The claim follows by noting that the function $\lambda \mapsto I_i(J(x)/(\lambda\alpha_i))$, for fixed x and fixed i, is strictly increasing in λ. □

Theorem 5.9 *The mapping φ defined by (5.4–5.8–5.11) has a unique continuous extension to a mapping from the nonnegative cone to itself. This extension is homogeneous, monotone, and nonsectional. On the positive cone, the mapping φ is strongly monotone.*

Proof. The continuity follows from Lemmas 5.3 and 5.6. Monotonicity and homogeneity follow from Lemmas 5.1 and 5.5. These lemmas also imply strong monotonicity on the positive cone. Consider now two nonempty complementary subsets R and S of the index set $\{1,\ldots,n\}$ as in Def. 4.1. If α^1 and α^2 are such that $\alpha^2_S > 0$, $\alpha^1_S = \alpha^2_S$, and $\alpha^1_R > \alpha^2_R$, then it follows from Lemma 5.1 that $\varphi_1(\alpha^1) > \varphi_1(\alpha^2)$. The strict inequality is preserved by the mapping φ_2 according to Lemma 5.5, so that item (i) in Def. 4.1 is satisfied. The condition in item (ii) is fulfilled due to Lemma 5.4 and Lemma 5.7. □

By Oshime's theorem (Thm. 4.2), we can now conclude the following.

Corollary 5.10 *Problem 3.4 has a unique solution. The unique allocation rule that is Pareto efficient and financially fair is given by*

$$y_i(x) = I_i(J(x)/\alpha_i) \tag{5.12}$$

for $i = 1,\ldots,n$ and $x \in D$, where I_i is the inverse marginal utility function of agent i, $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a positive eigenvector (unique up to multiplication by a positive scalar) of the mapping φ defined in (5.11), and J is given by $J = \varphi_1(\alpha)$ through the mapping φ_1 defined in (5.4).

The mapping φ induces a normalized mapping ψ from the open unit simplex into itself via (4.1). Under the assumptions of the above theorem, this mapping is contractive and has a fixed point. Using the fact that the open simplex in finite dimensions is a locally compact and connected metric space, we can therefore apply Thm. 4.5 to conclude the global convergence of the composite iteration algorithm.

Corollary 5.11 *Under Assumptions 2.1 and 2.2, the mapping ψ defined by (5.4–5.8–5.11) and (4.1) has the following property: for every α^0 in the open unit simplex $\{\alpha \in \mathbb{R}^n_{++} \mid \sum_{i=1}^n \alpha_i = 1\}$, the sequence of vectors $(\alpha^0, \alpha^1, \ldots)$ defined iteratively by*

$$\alpha^{i+1} = \psi(\alpha^i) \tag{5.13}$$

converges to the unique eigenvector in the open unit simplex of the mapping φ defined by (5.4–5.8–5.11).

Concerning implementation, it can be noted that the equations in (5.4) and (5.8) can be solved in parallel for different $x \in D$ and different i respectively, and that in each case the problem comes down to determining the root of a strictly monotone scalar function. The normalization is used above to simplify the proof of convergence. The fact that the eigenvalue associated to the positive eigenvector is equal to 1 suggests that normalization is not really needed. Computational experience indeed indicates that the composite iteration algorithm performs just as well, or perhaps even better, when normalization is not applied.

6 Equicautious HARA collectives

The class of utility functions with hyperbolic absolute risk aversion (HARA class) consists of the functions $u(\cdot)$ for which the corresponding coefficient of risk aversion $-u''(x)/u'(x)$ is of the form $1/(\sigma x + \tau)$, where σ and τ are constants. Special cases are the exponential utility functions ($\sigma = 0$) and the power utilities ($\tau = 0$). The coefficient σ has been called *cautiousness* by Wilson (1968); it measures how quickly the coefficient of risk aversion increases as wealth goes down. As noted by Wilson, collectives of agents who have identical cautiousness enjoy special properties. Such collectives have been called *equicautious* (Amershi and Stoeckenius, 1983). Examples are collectives of power utility agents who all have the same degree of relative risk aversion, and collectives consisting of exponential utility agents. The proposition below shows that, in the equicautious HARA case, the composite iteration mapping φ enjoys special properties too: the normalized iteration based on this mapping converges in one step.

Proposition 6.1 *In the case of an equicautious HARA collective, the mapping ψ defined by (5.4–5.8–5.11) and (4.1) satisfies $\psi(\psi(\alpha)) = \psi(\alpha)$ for all $\alpha \in \mathbb{R}^n_{++}$.*

Proof. Suppose that $u_1, \dots, u_n$ are utility functions of an equicautious HARA collective, and write $-u_i'(x)/u_i''(x) = \sigma x + \tau_i$. By definition of the functions I_i we have, for all $z > 0$, $u_i'(I_i(z)) = z$ and hence $u_i''(I_i(z))I_i'(z) = 1$, so that

$$-zI_i'(z) = -\frac{u_i'(I_i(z))}{u_i''(I_i(z))} = \sigma I_i(z) + \tau_i.$$

For any given weight vector $\alpha \in \mathbb{R}^n_{++}$, the function I defined by $I(z) = \sum_{i=1}^n I_i(z/\alpha_i)$ satisfies

$$-zI'(z) = -\sum_{i=1}^n (z/\alpha_i)I_i'(z/\alpha_i) = \sum_{i=1}^n \big(\sigma I_i(z/\alpha_i) + \tau_i\big) = \sigma I(z) + \sum_{i=1}^n \tau_i. \quad (6.1)$$

Write $\tau := \sum_{i=1}^{n} \tau_i$. Since J as defined in (5.4) is the inverse function of I, we have

$$-J(x)I'(J(x)) = \sigma x + \tau. \tag{6.2}$$

From the relation $I(J(x)) = x$ it follows that $I'(J(x))J'(x) = 1$; therefore (6.2) implies that $-J(x)/J'(x) = \sigma x + \tau$. This shows that the function J defined by (5.4) depends on the coefficients $\alpha_1, \ldots, \alpha_n$ only through a multiplicative factor. Consequently, the coefficients $\alpha_1, \ldots, \alpha_n$ that are determined from the function J via (5.7) represent a positive eigenvector of φ, so that convergence of the iteration (5.13) is achieved in one step. □

7 Rate of convergence

The asymptotic rate of convergence of the composite iteration algorithm is governed by the linearization of the iteration map around the fixed point of the iteration. As is well known (Wilkinson, 1965, Ch. 9), the power method for finding the eigenvector corresponding to the dominant eigenvalue of a matrix has a linear rate of convergence, and the speed of convergence is determined by the ratio of the absolute value of the second largest eigenvalue with respect to the absolute value of the largest eigenvalue. In our case the largest eigenvalue is equal to 1, and therefore the asymptotic convergence speed of the composite iteration algorithm is simply given by the size of the second largest eigenvalue of the Jacobian of this matrix at the fixed point of the iteration. The Jacobian matrix can be computed on the basis of the following proposition. First we introduce some notation: given a utility function u that is twice differentiable, strictly increasing, and strictly concave, the corresponding *coefficient of absolute risk tolerance* is the function T defined by

$$T(x) = -\frac{u'(x)}{u''(x)}\,. \tag{7.1}$$

This is the inverse of the usual Arrow-Pratt coefficient of absolute risk aversion.

Proposition 7.1 *Consider the mappings φ_1, φ_2, and the composite mapping φ as defined in (5.11) on the basis of a given group of agents with utility functions $u_i(\,\cdot\,)$ and claim values v_i. The agents' coefficients of absolute risk tolerance are denoted by $T_i(\,\cdot\,)$. Let $\alpha \in \mathbb{R}^n_{++}$ be such that $\varphi(\alpha) = \alpha$, and let $y_i(\,\cdot\,)$ denote the corresponding allocation functions as determined by (3.4). The linearization of the mapping φ_1 at the point α is given by*

$$D\varphi_1(\alpha) : \Delta\alpha \mapsto \Delta J, \quad \Delta J(x) = \frac{J(x)}{\sum_{i=1}^{n} T_i(y_i(x))} \sum_{i=1}^{n} T_i(y_i(x)) \frac{\Delta\alpha_i}{\alpha_i} \qquad (x \in D). \tag{7.2}$$

The linearization of the mapping φ_2 at $J = \varphi_1(\alpha)$ is given by

$$D\varphi_2(J) : \Delta J \mapsto \Delta\alpha, \quad \Delta\alpha_i = \frac{\alpha_i}{E^Q[T_i(y_i(X))]}\, E^Q\Big[T_i(y_i(X))\frac{\Delta J(X)}{J(X)}\Big] \qquad (i = 1, \dots, n). \tag{7.3}$$

The Jacobian at α of the composite mapping φ is given by

$$\big(D\varphi(\alpha)\big)_{ik} = \frac{\alpha_i/\alpha_k}{E^Q[T_i(y_i(X))]}\, E^Q\left[\frac{T_i(y_i(X))T_k(y_k(X))}{\sum_{i=1}^n T_i(y_i(X))}\right] \quad (i = 1, \dots, n;\ k = 1, \dots, n). \tag{7.4}$$

Proof. The linearization of the defining relationship (5.5) of the mapping φ_1 around a given point $\alpha \in \mathbb{R}^n_{++}$ is given by

$$\sum_{i=1}^n \frac{1}{\alpha_i}\, I_i'(J(x)/\alpha_i)\Delta J(x) - \sum_{i=1}^n \frac{J(x)}{\alpha_i^2}\, I_i'(J(x)/\alpha_i)\Delta\alpha_i = 0. \tag{7.5}$$

In terms of the allocation functions that are associated to the point α by means of Borch's condition for Pareto efficiency (3.3), we can write

$$I_i'(J(x)/\alpha_i) = I_i'(u_i'(y_i(x))) = \frac{1}{u_i''(y_i(x))} = \frac{\alpha_i}{J(x)}\,\frac{u_i'(y_i(x))}{u_i''(y_i(x))} = -\frac{\alpha_i}{J(x)}\, T_i(y_i(x)). \tag{7.6}$$

Together with (7.5), this leads to (7.2). The defining relationship (5.7) of the mapping φ_2 is linearized as follows:

$$E^Q\Big[\frac{1}{\alpha_i}\, I_i'(J(X)/\alpha_i)\Delta J(X)\Big] - E^Q\Big[\frac{J(X)}{\alpha_i^2}\, I_i'(J(X)/\alpha_i)\Delta\alpha_i\Big] = 0. \tag{7.7}$$

Together with (7.6), this leads to (7.3). Finally, the expression (7.4) is obtained by combining (7.2) and (7.3). □

The quantity $T_i(y_i(x))/\sum_{i=1}^n T_i(y_i(x))$ might be called the *tolerance share* of agent i at outcome x, within the allocation scheme defined by the functions y_i. Under the efficiency condition (3.3), this function can be given an alternative interpretation as follows. Since (3.3) implies the equality $J'(x) = \alpha_i u_i''(y_i(x))y_i'(x)$, we can write

$$T_i(y_i(x)) = -\frac{u_i'(y_i(x))}{u_i''(y_i(x))} = -\frac{J(x)}{J'(x)}\, y_i'(x).$$

Therefore we have

$$\sum_{i=1}^n T_i(y_i(x)) = -\frac{J(x)}{J'(x)} \tag{7.8}$$

and hence

$$\frac{T_i(y_i(x))}{\sum_{i=1}^n T_i(y_i(x))} = y_i'(x). \tag{7.9}$$

In other words, in Pareto efficient allocations the tolerance share of each agent is equal, at every outcome x, to the derivative of that agent's allocation function at the point x. Given the interpretation of $J(x)$ as a group marginal utility, the right hand side of (7.8) can be viewed as a group risk tolerance, which agrees with the natural interpretation of the left hand side.

The expression for the Jacobian can be simplified further by introducing a probability measure Q_i, which is associated to agent i under a given allocation scheme, as follows:

$$E^{Q_i}[Z] = \frac{E^Q[T_i(y_i(X))Z]}{E^Q[T_i(y_i(X))]} \qquad (Z \in L^\infty(\Omega, \mathcal{F}, Q)). \tag{7.10}$$

Using also (7.9), we can then write the elements of the Jacobian at the fixed point of the iteration mapping as

$$\big(D\varphi(\alpha)\big)_{ik} = \frac{\alpha_i}{\alpha_k} E^{Q_i}[y_k'(X)]. \tag{7.11}$$

While this is a short formula, an advantage of the expression (7.4) as given in the proposition above is that it does not require computing derivatives of the allocation functions.

Remark 7.2 In the case of equicautious HARA utilities, it is well known that efficient allocation rules must be linear (Amershi and Stoeckenius, 1983, Thm. 5). Suppose the allocation functions are given by $y_i(x) = a_i x + b_i$ where a_i and b_i ($i = 1, \ldots, n$) are constants. In this case, it follows from the expression (7.3) that the Jacobian of the iteration mapping at the fixed point is given by

$$\big(D\varphi(\alpha)\big)_{ik} = \frac{\alpha_i}{\alpha_k} a_k.$$

This implies that the Jacobian has rank 1, as expected from Section 6.

We conclude this section with a small numerical example in which we illustrate the convergence behavior of the composite iteration algorithm.

Example 7.3 Suppose a risk is to be divided between three agents who are referred to as "senior" (S), "mezzanine" (M), and "equity" (E). The agents use power utility $u_i(x) = x^{1-\gamma_i}/(1-\gamma_i)$, with different coefficients of relative risk aversion γ_i (10, 5, and 2). The agents have agreed on a pricing functional that gives positive weights to nine possible outcomes of the risk X. These outcomes are of the form $\exp z$, with $z = -2, -1.5, \ldots, 2$, and the corresponding weights (state prices) are proportional to $\exp(-\frac{1}{2}z^2)$. In other words, under the pricing measure, the risk X follows a discrete approximation to a lognormal

X	0.1353	0.2231	0.3679	0.6065	1.0000	1.6487	2.7183	4.4817	7.3891
q	0.0276	0.0663	0.1238	0.1802	0.2042	0.1802	0.1238	0.0663	0.0276
S	0.1138	0.1730	0.2554	0.3627	0.4888	0.6221	0.7554	0.8881	1.0228
M	0.0214	0.0495	0.1080	0.2178	0.3956	0.6408	0.9447	1.3060	1.7321
E	0.0001	0.0006	0.0045	0.0260	0.1155	0.3858	1.0182	2.2876	4.6342

Table 1: Pareto efficient and financially fair allocation of an approximately lognormal risk among three power utility agents labeled S, M, and E, with different coefficients of risk aversion, and with equal ownership rights. The row labeled X shows possible payoffs that are to be divided among the agents; the row labeled q shows valuation weights (measure Q).

distribution; numerical values are given in the second row of Table 1. There is no need to specify the probability measure P since the PEFF solution does not depend on it, due to the assumption that agents all use the same probabilities to compute expected utility. The three agents have equal ownership rights.

The composite iteration algorithm, initialized at the point $\alpha^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, without renormalization of iterates α^i, produces after four iterations a solution that satisfies the feasibility constraint (3.1) up to an error that is less than 0.5% (i.e. $\max \left| \left(\sum_{i=1}^{3} y_i(X) - X \right) \right| / E^Q[X] < 0.005$). The fairness constraints (3.2) are satisfied up to machine precision by the design of the algorithm. After three more iterations, the error in the feasibility constraint is less than 0.01%. The fixed point (corresponding to the scaling as given in the definition of the utility functions) is $\alpha = (0.03, 0.41, 0.56)$. The resulting allocation rule is shown in Table 1 for the outcomes of X that receive positive weights under Q. It can be verified that the claims held by the agents all have equal value when the value is computed by taking the weighted sum of the payoffs, with weights given in the row labeled q.

The size of the error as a function of the number of iterations is shown in Fig. 1. It is seen that the asymptotic regime sets in almost immediately. The slope of the line in the figure closely matches the magnitude of the second largest eigenvalue of the Jacobian, which can be found from (7.4) and which for the data as given equals 0.197. The convergence becomes only marginally slower when the grid size is increased to get a closer approximation to a lognormal variable under Q. Reducing the differences between the risk aversion coefficients of the agents tends to make the convergence faster, and the same holds when the differences between the ownership rights of the agents are made larger.

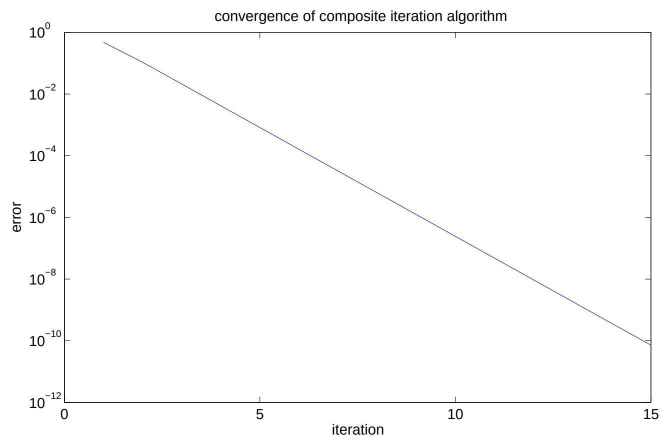

Figure 1: Error in the feasibility constraint as a function of the number of iterations

8 Conclusions and further research

In this paper we have studied the application of the composite iteration method to a fair division problem under a linear notion of fairness. The application features agents with concave and additively separable preferences. In this setting, the composite iteration map can be easily computed. We have established a number of relevant properties of the map, which allow to prove existence and uniqueness of solutions and global convergence of the corresponding iteration map.

We have assumed that the total risk X which is to be allocated among the agents is given. However, in many situations, the collective can decide to a certain extent how much risk it wants to take. Problems of collective investment decisions have been considered for instance by Wilson (1968) and Xia (2004). The notion of financial fairness would seem to be relevant in this context, but has not received much attention in the literature so far. A treatment of collective investment along the lines of the present paper has been given by Pazdera et al. (2016).

Multiperiod allocation problems have been considered for instance by Gale and Machado (1982); Barrieu and Scandolo (2008); Gollier (2008); Cui et al. (2011). Existence and uniqueness of Pareto efficient and financially fair allocation rules in the multiperiod context has been shown by Bao et al. (2017) using methods analogous to the ones in the present paper.

The composite iteration algorithm can be applied analogously (cf. Dana (2001)) in the case of Arrow-Debreu equilibrium. Among the known sufficient conditions for the composite iteration map to be strongly monotone in this case, the most important one is additive separability with low risk aversion; see Dana (2001) for details. The algorithm can be formulated for general preferences under suitable concavity assumptions, but simplifies notably in the case of additively separable preferences as considered in this paper.

It would be of interest to give an economic meaning to the magnitude of the second largest eigenvalue of the Jacobian of the composite iteration mapping at the fixed point. As noted in Remark 7.2, this number equals zero in the equicautious case. The question therefore arises to what extent the magnitude of the second largest eigenvalue could be viewed as a quantitative measure of nonequicautiousness, or (since equicautiousness makes it possible to aggregate preferences) as a quantitative measure of inaggregability of preferences.

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