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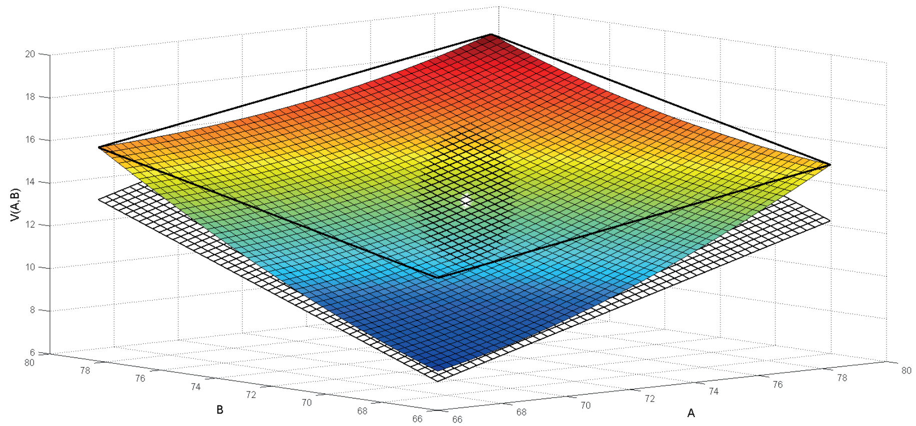

Essays on Derivatives Pricing

Marko Petrov

ESSAYS ON DERIVATIVES PRICING

ACADEMISCH PROEFSCHRIFT

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Dit proefschrift is tot stand gekomen in het kader van EDE-EM (European Doctorate in Economics – Erasmus Mundus), met als doel het behalen van een gezamenlijk doctoraat. Het proefschrift is voorbereid in de Faculteit Economie en Bedrijfskunde van de Universiteit van Amsterdam en in de Nova School of Business and Economics van de Nova Universiteit van Lissabon.

Esta tese foi escrita no âmbito de EDE-EM (European Doctorate in Economics – Erasmus Mundus), com o objetivo de obter o grau conjunto de doutorado. A tese foi preparada na Faculdade de Economia e Administração da Universidade de Amesterdão e na Faculdade de Economia da Universidade Nova de Lisboa.

This thesis has been written within the framework of EDE-EM (European Doctorate in Economics – Erasmus Mundus), with the purpose of obtaining a joint doctorate degree. The thesis was prepared in the Faculty of Economics and Business at the University of Amsterdam and in the Nova School of Business and Economics at the Nova University of Lisbon.

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Infinite thanks to my family - my sister, my parents and grandparents for their support. Their endless love has been always accompanying me, wherever I happened to be.

Finally, wordless thanks to my better half Nena for being an important constant in my life, through moments of happiness and through moments of distress.

I dedicate this thesis to all my friends.

…To those whose support I constantly had in many different ways, from my hometown Požarevac, from Belgrade and the rest of Serbia, and to those who are all around Europe and the world…

Посвећујем ову тезу свим својим пријатељима.

...Онима чију подршку сам константно осећао, на различите начине, из свог родног града Пожаревца, Београда и остатка Србије, и онима који су широм Европе и света...

Summary

This thesis consists of three separate articles.

In the first article we extend a fast algorithm to price European options on underlying assets which pay discrete dividends to the two-dimensional case. Firstly, by using convexity, we formulate upper and lower bounds for the price of a classical (univariate) European option written on a dividend-paying Black-Scholes asset in closed form, and show that those bounds converge to the true option price. The errors introduced by the method decrease with the square of the discretisation step used and scale with the gamma of the option. Secondly, the procedure is extended to obtain similar bounds for the price of a bivariate European call on the maximum of two underlying assets. Prices of other bivariate European options can then be found through put-call/min-max parity relations.

The second article concerns the derivation of analytical expressions for the future Expected Exposure in several Inflation Indexed Swaps under a stochastic model for inflation. These can be used to find a closed form solution for the Credit Value Adjustment (CVA). The CVA of a Zero-Coupon Inflation Indexed Swap is obtained analytically under this framework. For the Expected Exposure of a Year-on-Year Inflation Indexed Swap and for a portfolio of many Zero-Coupon Inflation Indexed Swap instruments, semi-analytical solutions are derived which are based on moment matching approximations. Extensive tests of the algorithms using Monte Carlo simulations show that the approximating formulae provide very fast and accurate methods to determine the CVA for different products.

In the third article we show that an equilibrium bid-ask spread for European derivatives may arise in dry markets for the underlying asset, even under symmetric information and absence of transaction costs. By dry markets we mean that the underlying asset may not be traded at all points in time, generating a particular form of market incompleteness. Using a partial equilibrium analysis in a one period model, we show two results. For monopolistic risk-neutral market makers, we fully characterise the bid-ask spread within the no-arbitrage bounds. For oligopolistic risk-neutral market makers, we prove that there is no pure symmetric Nash equilibrium of the game and that a bid-ask spread can only exist under a mixed strategy equilibrium.

Samenvatting

Dit proefschrift bestaat uit drie afzonderlijke delen.

In het eerste deel wordt een snel algoritme uitgebreid om de prijs te bepalen voor Europese opties op aandelen die een discreet dividend uitkeren naar twee dimensies. Door gebruik te maken van convexiteit formuleren we eerst boven- en ondergrenzen voor de prijs van een klassieke (univariate) Europese optie op een aandeel dat dividend betaalt onder Black-Scholes dynamics. We laten zien hoe we deze grenzen kunnen laten convergeren naar de werkelijke optieprijs. De fouten die door de methode worden geïntroduceerd nemen af met het kwadraat van de gebruikte discretisatiestap maal de gamma van de optie. Vervolgens wordt de procedure uitgebreid om soortgelijke boven- en ondergrenzen te verkrijgen voor de prijs van een bivariate Europese call op het maximum van twee onderliggende aandelen die dividend uitkeren. Prijzen voor andere bivariate Europese opties kunnen dan worden gevonden door middel van put-call-parity en min-max-parity relaties.

In het tweede deel worden analytische uitdrukkingen afgeleid voor de verwachte toekomstige blootstelling aan kredietrisico voor verschillende inflatieswaps, onder een stochastisch inflatiemodel. Deze kunnen worden gebruikt om een oplossing in gesloten vorm te vinden voor de zogenaamde Credit Value Adjustment (CVA). De CVA van een Zero-Coupon Inflation Indexed Swap kan onder onze aannamen in analytische vorm verkregen worden. Voor het toekomstige kredietrisico van een Year-on-Year Inflation Indexed Swap en voor een portfolio van diverse Zero-Coupon Inflation Indexed Swaps worden semi-analytische benaderingen afgeleid, die gebaseerd zijn op matching of moments technieken. Uitgebreide testen van de algoritmes met behulp van Monte Carlo simulatie laten zien dat met de benaderende formules zeer snelle en nauwkeurige benaderingen van de CVA bepaald kunnen worden voor deze verschillende producten.

In het derde deel laten we zien dat zich een evenwicht voor de bid-ask spread voor Europese derivaten voor kan doen in zogenaamde 'droge' markten voor het onderliggende aandeel, en dat dit zelfs kan gebeuren onder symmetrische informatie en in afwezigheid van transactiekosten. Met 'droge' markten bedoelen we markten waarin de onderliggende waarde niet kan worden verhandeld op alle momenten in de tijd, hetgeen leidt tot een specifieke vorm van markt incompleetheid. Met behulp van een analyse van het partiële evenwicht in een model met één tijdsperiode, leiden we een aantal verschillende resultaten af. Zo kunnen we voor monopolistische risico-neutrale marketmakers een volledige karakterisering geven van de bid-ask spread binnen de no-arbitrage grenzen. Voor oligopolistische risico-neutrale marketmakers bewijzen we dat er geen puur symmetrisch Nash-evenwicht bestaat, en dat een bid-ask spread alleen kan ontstaan bij een evenwicht dat gebaseerd is op gemengde strategieën.

Resumo

Esta tese consiste em três artigos distintos.

O primeiro artigo apresenta a extensão de um algoritmo rápido que valoriza opções europeias sobre ativos subjacentes que pagam dividendos discretos ao caso bi-dimensional. Em primeiro lugar e tendo em conta a convexidade de formula, é possível calcular limites superior e inferior para o preço de uma opção europeia clássica (univariada), escrito num ativo Black-Scholes de pagamento de dividendos, na forma fechada, e mostrar que esses limites convergem para o verdadeiro preço da opção. Os erros introduzidos pelo método descrito diminuem com o quadrado da janela de discretização utilizada e são proporcionais ao gama da opção. Em segundo lugar, o procedimento é estendido para obter limites semelhantes para o preço de uma opção *call* europeia bivariada no máximo de dois ativos subjacentes. Os preços de todas as outras opções europeias bivariadas podem ser encontrados através de relações de paridade *put-call/min-max*.

No segundo artigo, expressões analíticas para o futuro da exposição esperada em vários swaps indexados à inflação são derivadas. Esta derivação assenta num modelo estocástico para a inflação, o qual pode ser usado para encontrar uma solução de forma fechada para o *Credit Value Adjustment* (CVA). O CVA de um *Zero-Coupon Inflation Indexed Swap* é assim obtido analiticamente. Para a exposição esperada de um *Year-on-Year Inflation Indexed Swap* e para uma carteira de muitos instrumentos de *Zero-Coupon Inflation Indexed Swap*, soluções semi-analíticas são derivadas. Estas baseiam-se em aproximações dos momentos correspondentes. Testes extensivos dos algoritmos que usam simulações de Monte Carlo mostram que as fórmulas de aproximação fornecem métodos muito rápidos e precisos para determinar o CVA para os diferentes produtos.

O terceiro artigo mostra que um equilíbrio para um *bid-ask spread* de derivados europeus pode surgir no âmbito dos “mercados secos” para o ativo subjacente, mesmo na ausência de assimetria de informação e de custos de transação. Entende-se por “mercados secos” a incapacidade de comercializar o ativo subjacente em todos os pontos do tempo. Usando uma análise de equilíbrio parcial num modelo com um período, apresentamos os seguintes resultados: para intermediários monopolistas, neutros ao risco, caracterizamos completamente o *bid-ask spread* dentro dos limites de não arbitragem; para intermediários oligopolistas, neutros ao risco, provamos que não há equilíbrio de Nash simétrico puro do jogo e, um bid-ask spread só pode existir sob um equilíbrio de estratégia mista.

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Chapter 1

Introduction

This thesis deals with three separate problems in derivatives pricing. It considers three modifications of standard derivative pricing problems which make the analysis more challenging: the payment of discrete dividends, the introduction of credit risk, and the possibility that markets become less liquid. All these problems share the overall characteristic of considering derivatives under some deviations from the classical framework or different forms of market imperfections or incompleteness (illiquidity).

In the first problem, a modification of a classical model is generated by a discrete dividend on the underlying asset(s) which is paid at a predefined moment in time. This removes an essential feature of the standard option pricing model by Black and Scholes: the lognormal distribution of the underlying asset price at the derivative's maturity. The solution presented for the univariate case, initially proposed by Amaro de Matos *et al.* (2009), circumvents using the asset price distribution after the discrete dividend has been paid, by exploiting the convexity of the classical option price formula for stocks without dividends. This allows us to derive new analytical formulae for an upper and a lower bound for the value of the option on a stock paying a discrete dividend and to generalise results to the much harder bivariate case. The bounds are then shown to converge to the real price of the option, when we grant more computation time. The obtained algorithm turns out to be very fast and the pricing error can be controlled by the size of a discretisation step.

In the second problem, we deal with a specific form of market imperfection: default risk of the derivative's counterparty. The literature which emerged after the financial crisis of 2008 dictates that for such risks an additional charge must be added to the standard default-free price of derivatives, which is called the Credit Value Adjustment (CVA). In our work we are concerned with the CVA for inflation-linked derivatives. We provide novel (semi)-analytical formulae for the CVA of the two most common types, Zero-Coupon Inflation Indexed Swaps (ZCIIS) and Year-on-Year Inflation Indexed Swaps (YYIIS), and we define a closed form approximation for the CVA of a portfolio of inflation derivatives under netting agreements. As will be explained later, this is important since a CVA is typically calculated for a whole portfolio.

The third part of the work in this thesis considers a form of market incompleteness which is generated due to absence of trading possibilities for the underlying asset at some points in time; which we call market dryness. By considering a two period recombining binomial tree model in which trading is not permitted at the mid-time, the model becomes equivalent to a one period trinomial model. The standard optimisation problems in microeconomics for a representative agent will give us a demand and a supply curve for the derivative, which will depend on the choice of the agent's utility function. By introducing risk-neutral market makers who maximise their profits, we show that an equilibrium bid-ask spread for the derivative will be generated, even though our stylised model does not consider transaction costs or information asymmetries. As will be shown, for the case of a monopolistic market maker there is a result which fully characterises the equilibrium bid-ask spread within the no-arbitrage bounds of the derivative, whereas in the case of competition between market makers we prove that, under pure strategies, Nash equilibrium prices cannot exist. This is shown by identifying a more profitable strategy which players would choose to undertake in each of the possible cases. Furthermore, we can conclude that such a bid-ask equilibrium must exist under a mixed strategies game.

Chapter 2

Pricing of Bivariate Options on Stocks Paying Discrete Dividends

2.1 Introduction

Pricing of options on an asset that pays a discrete dividend has been a classical problem in quantitative finance since the early work of Black (1975), Merton (1976a, 1976b), Roll (1977), Geske (1979) and Whaley (1981). As Haug, Haug and Lewis (2003) correctly point out in their aptly named paper "Back to basics: a new approach to the discrete dividend problem", the inclusion of dividends in option pricing theory is a very important problem that has been overlooked throughout years of advances in the mathematical finance literature.

Merton (1973) was the first to relax the no-dividend assumption for the seminal Black and Scholes (1973) European option pricing formulae. He allowed for a deterministic dividend yield over the lifetime of the option or for a dividend specified as a fixed proportion of the stock price on the dividend date, and showed that options can then still be priced in a Black-Scholes-Merton economy with some minor modifications. However, in reality the majority of stocks on which options trade pay dividends that are specified as a fixed value. This implies that the stock price jumps downwards at dividend dates, with a jump size which equals this fixed value. Therefore, in realistic cases, we lose the essential property of the Black-Scholes model which makes explicit pricing formulae possible: the lognormal distribution of future asset prices.

As a consequence, the existing literature proposes numerous different approximations to such option prices. The approximation procedure that was first informally suggested by Black (1975), replaces the initial value of the stock S_0 in the Black-Scholes formula by its value minus the present value of dividends that are to be paid, i.e. the adjustment in the classical Black-Scholes formula would only adjust the current stock price: $S_0 \to S_0^* = S_0 - PV(D)$. This approximation is what is now known as the **Escrowed model** for options on a discrete dividend paying stock. The assumption is made that the asset price minus the present value of *all future dividends that are to be paid until the maturity of the option* follows a geometric Brownian motion. A number of other approximation models have been developed since, and a good overview is given in Frishling (2002) and in Vellekoop and Nieuwenhuis (2006).

The assumption that the asset price plus the forward value of *all past dividends that have been paid until today* follows a geometric Brownian motion is known as the **Forward model.**

In this alternative model, the adjustment to the classical Black-Scholes formula would only adjust the option's strike price: $K \to K^* = K - FV(D)$.

The **Modified escrowed model** is based on the slightly different assumption that the asset price minus the present value of *all dividends that are to be paid (at any time) in the future* follows a geometric Brownian motion. This model tries to remedy the obvious problem in the Escrowed model that it admits arbitrage: American calls expiring just before the dividend date can be more expensive than American calls expiring just after that date, since the model assumes different dynamics for the price processes at times before and after the dividend date. However, with the given modification, the option price will depend on dividends that will be paid after the maturity of the option, which seems unreasonable.

The combination of the Escrowed and Forward models introduced by Bos and Vandemark (2002) adjusts the time of dividends in the following way: a fraction of dividends is included in an adjusted current asset price and the rest in an adjusted strike.

Beneder and Vorst (2002) and Bos *et al.* (2003) extend the Escrowed model by adjusting the volatility to include the discrete dividends. This approach is based on the fact that the Escrowed model usually undervalues the options and an effort was made to account for this by adjusting the volatility parameter in the Black-Scholes formula using a weighted average of an adjusted and unadjusted variance of the stock returns, with weights that depend on the time of dividend payments.

Finally, the assumption that the asset price follows a geometric Brownian motion in between dividend dates is called the **Piecewise lognormal model.**

All the techniques mentioned above, except the Piecewise lognormal model, do not actually specify the asset price process underlying the models, but instead introduce simple *ad hoc* adjustments of the Black-Scholes formula for the classical European option. For the case of American options, the approaches mentioned above were used by Roll (1977), Geske (1979) and Whaley (1981), and this resulted in the Roll-Geske-Whaley model which has been considered for decades the best way to price American options on stocks which pay discrete dividends. The Roll-Geske-Whaley model is an Escrowed dividend model, but it uses a compound option approach to take into account the possibility of early exercise. However, it admits arbitrage opportunities, as pointed out in Haug, Haug and Lewis (2003).

As a result, in most later work the Piecewise lognormal model for the underlying asset price process is assumed. Although that approach seems to be preferred from a practical point of view, it is often too inefficient for accurate computations since, under this assumption, standard discrete Cox-Ross-Rubinstein (1979) tree schemes no longer recombine after dividend dates. A method to overcome this problem was suggested by Vellekoop and Nieuwenhuis (2006), and it can be used to price both European and American options with multiple dividends. It is based on interpolation steps within the tree. Interpolation methods had been suggested before by Wilmott *et al.* (1993) and Haug, Haug and Lewis (2003), but the first one suffers from possible negative risk-neutral probabilities, and the second one will not work for American options. For European options, the approximation method of Haug, Haug and Lewis (2003) replaces a multiple integration by a succession of single integrations over Black-Scholes-like approximating functions. It performs extremely well for this case and an advantage of the method is the fact that it is not limited to the Black-Scholes model for the underlying asset but can also treat more complex models, e.g. jump-diffusions or stochastic volatility models. Also, it introduces stock-price dependent dividend payments which can be a useful extension in practice.

In this thesis we take a different approach. We follow the one-dimensional approach by Amaro de Matos *et al.* (2009) and take the Piecewise Lognormal model for the underlying stock(s) dynamics and by using convexity of the classical option price formula for stocks without dividends, we derive analytical formulae for an upper and lower bound for the value of the option on stock(s) paying a discrete dividend. We then show convergence to the correct price for both bounds when the mesh size of an approximating grid goes to zero.

First, in the next section, we introduce the method for obtaining bounds for European options on a stock paying a single dividend, and analyse the approximation errors introduced by the method. Then, in Section 2.3, we extend the approach to bivariate options, i.e. options written on two underlying assets. Examples of papers concerning bivariate options on non-dividend paying stocks are Stulz (1982), Ouwehand and West (2006) and Margrabe (1978). However, to the best of our knowledge, there are no results in the literature to price such options on discrete dividend paying stocks, except in the master thesis by Kolb (2015), which is based on a similar idea, but does not provide any proofs. In Section 2.4 we present the results obtained by our algorithm and compare them with the values of the options obtained using Monte Carlo methods. Appendices can be found at the end of this thesis.

2.2 The One-dimensional Case

Following Amaro de Matos *et al.* (2009) we obtain upper and lower bounds for the price of a European option written on a stock S paying a discrete dividend of a fixed size $D \geq 0$ at a known time τ. We use the Piecewise lognormal model. The maturity of the option is T and its strike is K. The risk free interest rate is $r > 0$. We will price an option at time t, where $t < \tau < T$.

For all $t \in [0,T]\backslash\{\tau\}$, the underlying asset follows geometric Brownian motion dynamics

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.2.1}$$

with $\mu > r$ and $\sigma > 0$ known constants and W a standard Brownian motion on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}\right)$ which satisfies the usual conditions. The stock pays a known discrete dividend $D \geq 0$ at time τ, so $S_\tau = (S_{\tau-} - D)^+$. We will use the notation $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ as shorthand notation for $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$ which is the expectation under the unique martingale measure $\mathbb{Q}$, conditioned on the filtration $\mathcal{F}_t$ that is generated by the stochastic process (2.2.1). Here, $\mathbb{Q}$ is the unique risk-neutral probability measure that is equivalent to the original real-world probability measure $\mathbb{P}$, and under which the discounted underlying asset price process is a martingale.

We use $V^+(t,S)$ and $V^-(t,S)$ for the value of the option after and before the dividend date, at times $t \geq \tau$ and $t \leq \tau$ respectively, for stock prices $S \geq 0$. By a standard no-arbitrage argument the price of an option cannot have a discontinuity at τ, so the price of an option just after the dividend has been paid, $V^+(\tau, S_\tau)$, must be exactly the same as the price of the option just before the dividend payment, $V^-(\tau, S_{\tau-})$, i.e. we must have $V^-(\tau, S_{\tau-}) = V^+(\tau, S_\tau)$.

After the dividend date the option becomes an option on a non-dividend paying stock that follows geometric Brownian motion and thus its price after the dividend date is given by the classical Black-Scholes formula for the price of a European option, i.e. $V^+(\tau, S_\tau) = \mathrm{BS}\,(S_\tau,\, K,\, r,\, \sigma,\, T-\tau)$. By the no-arbitrage argument mentioned above this gives the price of the option just before the dividend date $V^-(\tau, S_\tau) = \mathrm{BS}\left((S_\tau - D)^+,\, K,\, r,\, \sigma,\, T-\tau\right)$. The price of the option at a time t before the dividend, for a stock price S_t, is then

$$V(t, S_t) = e^{-r(\tau-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V^-(\tau, S_{\tau-})\right], \quad t < \tau \tag{2.2.2}$$

with the function

$$V^-(t, S) = \mathrm{BS}\left((S-D)^+,\, K,\, r,\, \sigma,\, T-t\right). \tag{2.2.3}$$

Definition 2.1. Convexity in $\mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\alpha x^{(1)} + \beta x^{(2)}) \leq \alpha f(x^{(1)}) + \beta f(x^{(2)}),$$

for all $x^{(1)}, x^{(2)} \in \mathbb{R}^n$ and $\alpha, \beta \geq 0,\ \alpha + \beta = 1$.

As remarked by Amaro de Matos *et al.* (2009), Theorem 2.2, the value of the derivative at the dividend date is bounded from above:

$$V^-(\tau, S) \leq V^{-u}(\tau, S), \tag{2.2.4}$$

by convexity in $\mathbb{R}$. Here $V^{-u}(\tau, S)$, is a piecewise linear interpolation and extrapolation function between the points (S^i, V^i) that are defined for $i = 0, ..., M$ as

$$S^i = D + i\Delta S, \quad \Delta S = \frac{S^M - D}{M}, \tag{2.2.5}$$

$$V^i = V^-(\tau, S^i),$$

and $S^{M+1} = \infty$, $V^{M+1} = \infty$. The parameter M gives the number of intervals $[S^{i-1}, S^i)$ formed by discretisation points and the last point of the discretisation is S^M. Both M and S^M are input parameters for the method. We then set

$$V^{-u}(\tau, S) = \sum_{i=1}^{M}\left[\alpha^i(S - S^i) + V^i\right]1_{\{S\in[S^{i-1},S^i)\}} + \left[(S - S^M) + V^M\right]1_{\{S\geq S^M\}},$$

where the slopes of the piecewise linear function are ratios of differences, i.e.

$$\alpha^i = \frac{V^i - V^{i-1}}{S^i - S^{i-1}},\ i = 1, ..., M \tag{2.2.6}$$

and where in the last segment $[S^M, \infty)$ we have used the simple extrapolation of a linear function with slope 1. This must be an upper bound in this region, because the convex function $V^-(\tau, S)$ in (2.2.3) is such that its derivative in S is always below 1. Thus, the line with slope 1 starting from the point $\left(S^M, V^M\right)$ must lie above the function itself in the region $[S^M, \infty)$. By introducing more efficient notation all this can be merged together as

$$V^{-u}(\tau, S) = \sum_{i=1}^{M+1}\left[\alpha^i S + \xi^i\right]1_{\{S\in[S^{i-1},S^i)\}}, \tag{2.2.7}$$

with

$$\xi^i = V^i - \alpha^i S^i,\ i = 1, ..., M, \quad (2.2.8)$$
$$\alpha^{M+1} = 1,\ \ \xi^{M+1} = V^M - S^M.$$

Similarly, the value of the derivative at the dividend date is bounded from below:

$$V^-(\tau, S) \geq V^{-l}(\tau, S), \quad (2.2.9)$$

where $V^{-l}(\tau, S)$, is defined as the piecewise linear interpolation and extrapolation function on the mid-points of the segments defined above, i.e. on $(S^{i-\frac{1}{2}}, V^{i-\frac{1}{2}})$. Here $S^{i-\frac{1}{2}}$ and $V^{i-\frac{1}{2}}$ are defined by (2.2.5) while the slopes $\beta^{i-\frac{1}{2}}$ in $(S^{i-\frac{1}{2}}, V^{i-\frac{1}{2}})$ for this piecewise linear function are

$$\beta^{i-\frac{1}{2}} = \frac{\partial V^-}{\partial S}(\tau, S^{i-\frac{1}{2}}),\ i = 1, ..., M. \quad (2.2.10)$$

The partial derivative with respect to the underlying S for a European call option (known as an option's delta) is

$$\frac{\partial \mathrm{BS}\,(S,\ K,\ r,\ \sigma,\ T-t)}{\partial S} = N\left(\frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \quad (2.2.11)$$

for the Black-Scholes model, with N the standard Gaussian cumulative distribution function.

We define

$$V^{-l}(\tau, S) = \sum_{i=1}^{M} \left[\beta^{i-\frac{1}{2}}(S - S^{i-\frac{1}{2}}) + V^{i-\frac{1}{2}}\right] 1_{\{S\in[S^{i-1}, S^i)\}} + \left[S - D - Ke^{-r(T-\tau)}\right] 1_{\{S\geq S^M\}},$$

where the bound for the last segment is obtained from the fact that the Black-Scholes price of a European call option is bounded from below by[1]

$$\mathrm{BS}\,(S,\ K,\ r,\ \sigma,\ T-t) \geq S - Ke^{-r(T-t)}, \quad (2.2.12)$$

and thus, by (2.2.3),

$$V^-(t, S) \geq (S-D)^+ - Ke^{-r(T-t)} \geq S - D - Ke^{-r(T-t)}.$$

Again, by defining auxiliary variables $\chi^{i-\frac{1}{2}}$, we obtain the expression

$$V^{-l}(\tau, S) = \sum_{i=1}^{M+1} \left[\beta^{i-\frac{1}{2}} S + \chi^{i-\frac{1}{2}}\right] 1_{\{S\in[S^{i-1}, S^i)\}}, \quad (2.2.13)$$

with

$$\chi^{i-\frac{1}{2}} = V^{i-\frac{1}{2}} - \beta^{i-\frac{1}{2}} S^{i-\frac{1}{2}},\ i = 1, ...M, \quad (2.2.14)$$
$$\beta^{M+\frac{1}{2}} = 1,\ \ \chi^{M+\frac{1}{2}} = -D - Ke^{-r(T-\tau)}.$$

[1]See e.g. Hull (2006), "Options, futures and other derivatives", 8th edition, page 219.

So, from (2.2.2), (2.2.3), (2.2.4) and (2.2.7), we have for $t < \tau$:

$$\begin{aligned} V(t, S_t) &\leq e^{-r(\tau-t)} \sum_{i=1}^{M+1} \mathbb{E}_t^{\mathbb{Q}} \left[\left(\alpha^i S_{\tau-} + \xi^i \right) 1_{\{S_{\tau-} \in [S^{i-1}, S^i)\}} \right] \\ &= e^{-r(\tau-t)} \sum_{i=1}^{M+1} \left[\alpha^i f^i(t, S_t) + \xi^i g^i(t, S_t) \right], \end{aligned} \tag{2.2.15}$$

where the functions f^i and g^i are defined for $t < \tau$ as

$$f^i(t, S_t) = \mathbb{E}_t^{\mathbb{Q}} \left[S_{\tau-} 1_{\{S_{\tau-} \in [S^{i-1}, S^i]\}} \right] = S_t e^{r(\tau-t)} \left[N\left(d^i\right) - N\left(d^{i-1}\right) \right], \tag{2.2.16}$$

$$g^i(t, S_t) = \mathbb{E}_t^{\mathbb{Q}} \left[1_{\{S_{\tau-} \in [S^{i-1}, S^i]\}} \right] = N\left(d^i + \sigma\sqrt{\tau - t}\right) - N\left(d^{i-1} + \sigma\sqrt{\tau - t}\right), \tag{2.2.17}$$

with

$$d^i = \frac{\ln\left(\frac{S^i}{S_t}\right) - \left(r + \frac{1}{2}\sigma^2\right)(\tau - t)}{\sigma\sqrt{\tau - t}},$$

and N the standard normal cumulative distribution function.

Also, from (2.2.2), (2.2.3), (2.2.9) and (2.2.13) we have

$$\begin{aligned} V(t, S_t) &\geq e^{-r(\tau-t)} \sum_{i=1}^{M+1} \mathbb{E}_t^{\mathbb{Q}} \left[\left(\beta^{i-\frac{1}{2}} S_{\tau-} + \chi^{i-\frac{1}{2}} \right) 1_{\{S_{\tau-} \in [S^{i-1}, S^i)\}} \right] \\ &= e^{-r(\tau-t)} \sum_{i=1}^{M+1} \left[\beta^{i-\frac{1}{2}} f^i(t, S_t) + \chi^{i-\frac{1}{2}} g^i(t, S_t) \right], \end{aligned} \tag{2.2.18}$$

where the functions f^i and g^i are the same as above.

The result can now be summarised in the following proposition.

Proposition 2.1. *At any time $t < \tau$, the price of a European call option on the underlying asset S with dynamics as specified by* (2.2.1)*, has the following upper and lower bounds:*

$$V(t, S_t) \leq e^{-r(\tau-t)} \sum_{i=1}^{M+1} \left[\alpha^i f^i(t, S_t) + \xi^i g^i(t, S_t) \right],$$

$$V(t, S_t) \geq e^{-r(\tau-t)} \sum_{i=1}^{M+1} \left[\beta^{i-\frac{1}{2}} f^i(t, S_t) + \chi^{i-\frac{1}{2}} g^i(t, S_t) \right],$$

with coefficients α^i, ξ^i, $\beta^{i-\frac{1}{2}}$, $\chi^{i-\frac{1}{2}}$ and functions f^i and g^i as defined above.

The maximal error of the method, defined as the difference between the upper and lower bound, will be

$$V^{-u}(\tau, S) - V^{-l}(\tau, S) \leq \frac{1}{2} \left(\frac{\Delta S}{2} \right)^2 \max_{u>0} \left| \frac{\partial^2 V^-}{\partial S^2}(\tau, u) \right| + h\left(S^M\right), \tag{2.2.19}$$

for a function h which does not depend on ΔS, and which is such that $\lim_{S^M \to \infty} h\left(S^M\right) = 0$.

Proof. The first part of the Proposition follows from (2.2.15) and (2.2.18), by construction, i.e. by definitions of the upper and lower bounds $V^{-u}(\tau, S)$ and $V^{-l}(\tau, S)$ in (2.2.7) and (2.2.13).

The second part follows from a convergence analysis of our approximations. If we define the maximal error as the difference between the upper and lower bounds, we will have the following:

$$\begin{aligned} V^{-u}(\tau, S) - V^{-l}(\tau, S) \leq & \sum_{i=1}^{M} \max_{\widetilde{S} \in [S^{i-1}, S^i]} \left(\left(\alpha^i - \beta^{i-\frac{1}{2}} \right) \widetilde{S} + V^i - V^{i-\frac{1}{2}} - \alpha^i S^i + \beta^{i-\frac{1}{2}} S^{i-\frac{1}{2}} \right) 1_{\{S \in [S^{i-1}, S^i)\}} \\ & + \left(V^M - S^M + D + K e^{-r(T-\tau)} \right) 1_{\{S \geq S^M\}}. \end{aligned} \tag{2.2.20}$$

We have a linear function in each interval $[S^{i-1}, S^i]$ and since optimal values for a linear function always equal the value at one of the end points we have

$$\begin{aligned} V^{-u}(\tau, S) - V^{-l}(\tau, S) \leq & \sum_{i=1}^{M} \max \left\{ \begin{array}{c} V^{i-1} - V^{i-\frac{1}{2}} - \beta^{i-\frac{1}{2}} \left(S^{i-1} - S^{i-\frac{1}{2}} \right), \\ V^i - V^{i-\frac{1}{2}} - \beta^{i-\frac{1}{2}} \left(S^i - S^{i-\frac{1}{2}} \right) \end{array} \right\} 1_{\{S \in [S^{i-1}, S^i)\}} \\ & + \left(V^M - S^M + D + K e^{-r(T-\tau)} \right) 1_{\{S \geq S^M\}}. \end{aligned} \tag{2.2.21}$$

Recalling that $\beta^{i-\frac{1}{2}} = \frac{\partial V^-(\tau, S^{i-\frac{1}{2}})}{\partial S}$ we can recognise the arguments of the max function to be the first terms in a Taylor series expansion of the function $S \to V^-(\tau, S)$ around $S^{i-\frac{1}{2}}$, evaluated at $S = S^{i-1}$ and $S = S^i$. By considering Lagrange's form of the remainder R_1 of the Taylor's series expansion we have

$$V^{i-1} - V^{i-\frac{1}{2}} - \beta^{i-\frac{1}{2}} \left(S^{i-1} - S^{i-\frac{1}{2}} \right) = R_1(S^{i-1}) = \frac{1}{2} \frac{\partial^2 V^-(\tau, k_i)}{\partial S^2} \left(S^{i-1} - S^{i-\frac{1}{2}} \right)^2, \ k_i \in (S^{i-1}, S^{i-\frac{1}{2}});$$

$$V^i - V^{i-\frac{1}{2}} - \beta^{i-\frac{1}{2}} \left(S^i - S^{i-\frac{1}{2}} \right) = R_1(S^i) = \frac{1}{2} \frac{\partial^2 V^-(\tau, \widetilde{k}_i)}{\partial S^2} \left(S^i - S^{i-\frac{1}{2}} \right)^2, \ \widetilde{k}_i \in (S^{i-\frac{1}{2}}, S^i).$$

Due to the uniform discretisation (2.2.5), $S^i - S^{i-\frac{1}{2}} = S^{i-\frac{1}{2}} - S^i = \frac{\Delta S}{2}$, (2.2.21) becomes:

$$\begin{aligned} V^{-u}(\tau, S) - V^{-l}(\tau, S) \leq & \frac{1}{2} \left(\frac{\Delta S}{2} \right)^2 \sum_{i=1}^{M} \max \left\{ \frac{\partial^2 V^-(\tau, k_i)}{\partial S^2}, \frac{\partial^2 V^-(\tau, \widetilde{k}_i)}{\partial S^2} \right\} 1_{\{S \in [S^{i-1}, S^i)\}} \\ & + \left(V^-(\tau, S^M) - S^M + D + K e^{-r(T-\tau)} \right) 1_{\{S \geq S^M\}}. \end{aligned} \tag{2.2.22}$$

Replacing the maximum by $\max_{u>0} \left| \frac{\partial^2 V^-}{\partial S^2}(\tau, u) \right|$ we find the expression for the maximal error (2.2.19).

The last term $h\left(S^M\right) = V^-(\tau, S^M) - \left(S^M - D - K e^{-r(T-\tau)}\right) \to 0$ when $S^M \to \infty$, since the limit of the Black-Scholes price of the call option, given by (2.2.3), is

$$\lim_{S^M \to \infty} \left[\mathrm{BS}\left(S^M - D,\, K,\, r,\, \sigma,\, T - t\right) - \left(S^M - D - K e^{-r(T-t)}\right) \right] = 0,$$

i.e. the option's price reduces to its intrinsic value.[2] ■

[2]See e.g. Hull (2006), "Options, futures and other derivatives", 8th edition, page 315.

This result states that we are able to directly control the error introduced by the method through a suitable choice of the size of ΔS or, equivalently, the number of discretisation points M, and the value of S^M. It will scale with the second order sensitivity $\frac{\partial^2 V^-}{\partial S^2}(\tau, S)$, the option's gamma, which is readily available from the Black-Scholes formula for the function $V^-(\tau, S)$ in definition (2.2.3). The second term is a constant error term which stems from the choice of the truncation value S^M and it can be made negligible by choosing a large S^M.

In the last step of the proof above, equation (2.2.22), the uniform discretisation step size ΔS was used. There is no specific reason for taking a uniform mesh. The proof would also go through with a non-uniform mesh, i.e. similar result would hold in terms of the maximal mesh size step $\max(\Delta S)$. In that case the benefit would be the possibility of making a grid denser where curvature (gamma) is large, and thus improving the performance.

The result just presented shows that both bounds converge to the true option price (2.2.2) which, according to the standard option pricing theory, guarantees that the model is arbitrage free.

2.3 The Two-dimensional Case

We will extend the technique developed in the previous section for the option written on a single stock paying a discrete dividend, to a bivariate option with maturity T written on two correlated underlying stocks A and B. We therefore define for all $t \in [0, T] \backslash \{\tau\}$:

$$\begin{aligned} dA_t &= \mu_A A_t dt + \sigma_A A_t dW_t^A, \\ dB_t &= \mu_B B_t d + \sigma_B B_t dW_t^B, \\ d\left\langle W^A, W^B \right\rangle &= \rho dt, \end{aligned} \tag{2.3.1}$$

with, as before, $\mu_A > r$ and $\mu_B > r$ known rates of return and $\sigma_A > 0$ and $\sigma_B > 0$ known volatilities. The $\left(W^A, W^B\right)$ are correlated Brownian motions on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}\right)$. The stocks pay discrete dividends of known sizes D_A and D_B, respectively, at a known common time τ, so $A_\tau = (A_{\tau-} - D_A)^+$ and $B_\tau = (B_{\tau-} - D_B)^+$.

2.3.1 The Non-dividend Paying Case

Let us first describe some standard European options on two stocks with strike K and maturity T. Possible payoffs $V(T, A_T, B_T)$ of such options at the maturity time $T > t$ are

- a call on the maximum of the two assets: $(\max(A_T, B_T) - K)^+$,
- a call on the minimum of the two assets: $(\min(A_T, B_T) - K)^+$,
- a put on the maximum of the two assets: $(K - \max(A_T, B_T))^+$,
- a put on the minimum of the two assets: $(K - \min(A_T, B_T))^+$.

These options were priced by Stulz (1982) and Ouwehand and West (2006) for the case without dividends. Stulz (1982) obtains a bivariate density for the minimum of the two assets and derives a closed form solution for the price of a call on the minimum option by direct integration. The paper also states and proves put-call/min-max parity relations which allows us to use this result to price the other three types of options as well.

Ouwehand and West (2006) price both call-on-max and call-on-min options written on any number of underlyings (which makes them multivariate or so-called rainbow options), by using one of the assets as a numeraire. Their work shows how we can derive an analytical formula for the price of a call on the maximum option directly, which we are going to use in the rest of this chapter. A requirement to apply our method is that the option price is both convex and non-decreasing in the stock prices everywhere, and we will see that this is satisfied for that case.

We use the notation $N_2(\cdot, \cdot\,; \rho)$ for the bivariate standard normal cumulative distribution function with correlation coefficient ρ, i.e.

$$N_2(a, b;\, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{a} \int_{-\infty}^{b} e^{-\frac{x^2 - 2\rho x y + y^2}{2(1-\rho^2)}}\, dy\, dx. \tag{2.3.2}$$

and when the correlation parameter ρ is fixed, we also use the notation $N_2(a, b)$.

Lemma 2.1. *(Ouwehand and West, 2006) The bivariate European call on the maximum option with strike K and maturity T, i.e. the option with payoff $V(T, A, B) = (\max(A, B) - K)^+$, written on the (non-dividend paying) underlying assets A and B following geometric Brownian motion dynamics given by (2.3.1), at time $t < T$ has a price given by*

$$\begin{aligned} \mathrm{OW}\left(A_t,\, B_t,\; K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right) &= A_t N_2\left(\widetilde{c_A}, c_A; \rho_A\right) + B_t N_2\left(\widetilde{c_B}, c_B; \rho_B\right) \\ &\quad - K e^{-r(T-t)} \left(1 - N_2\left(-c_A + \sigma_A\sqrt{T-t}\,, -c_B + \sigma_B\sqrt{T-t}\,; \rho\right)\right), \end{aligned} \tag{2.3.3}$$

with

$$c_A = \frac{\ln\left(\frac{A_t}{K}\right) + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}}, \quad c_B = \frac{\ln\left(\frac{B_t}{K}\right) + \left(r + \frac{1}{2}\sigma_B^2\right)(T-t)}{\sigma_B\sqrt{T-t}},$$

and

$$\widetilde{c_A} = \frac{\ln\left(\frac{A_t}{B_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad \widetilde{c_B} = \frac{\ln\left(\frac{B_t}{A_t}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

where

$$\sigma = \sqrt{\sigma_A^2 - 2\rho\sigma_A\sigma_B + \sigma_B^2}, \quad \rho_A = \frac{\sigma_A - \rho\sigma_B}{\sigma}, \quad \rho_B = \frac{\sigma_B - \rho\sigma_A}{\sigma}.$$

An example of call-on-max prices as a function of underlying asset prices A and B (for the parameter values $K = 100$, $r = 3\%$, $\sigma_A = 0.2$, $\sigma_B = 0.3$, $\rho = 0.5$, $T = 1$, $t = 0.5$) is shown in Figure 2.1.

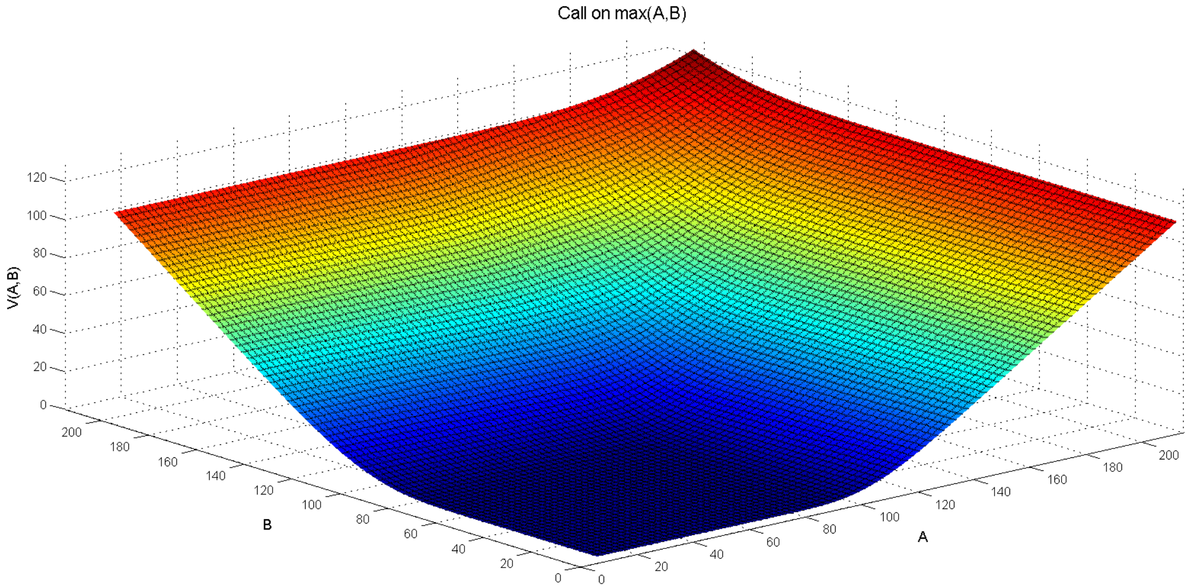

Figure 2.1: Price of a call on maximum option as a function of the two underlying assets.

Properties of the call on the maximum

As we can see, the price of the European call on the maximum option given by OW $(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t)$ in Lemma 2.1 is a convex and non-decreasing function of A_t and B_t. We now prove this property rigorously by showing it is inherited from the convexity and monotonicity of the payoff at maturity T.

Let r, σ_A, σ_B and ρ be given and also fix the strike K and time of maturity T. Given the dynamics of the underlying assets (2.3.1), according to standard no arbitrage pricing theory, the price OW $(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t) \equiv V(t, A_t, B_t)$ equals the following expectation:

$$V(t, A_t, B_t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V(T,\, A_T,\, B_T)\right],$$

with the payoff $V(T, A_T, B_T) = (\max(A_T, B_T) - K)^+ \equiv f(A_T, B_T)$. More generally, given a convex, non-decreasing and non-negative payoff function $f(A_T, B_T)$, the solution $V(t, A_t, B_t)$ is convex and non-decreasing in (A_t, B_t) as well, which we are going to show below.

Theorem 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$, be a non-negative and convex payoff function for an option with price function $V(t, A_t, B_t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[f(A_T,\, B_T)\right]$. Then, for any fixed $t < T$, the solution $V(t, A_t, B_t)$ is also convex, in (A_t, B_t).*

Furthermore, if f is non-decreasing in both its arguments, this also holds for $(A_t, B_t) \to V(t, A_t, B_t)$, for any fixed $t < T$.

Proof. From the convexity (Definition 2.1) of f, we have that its Hessian

$$\mathcal{H}_{f(x,y)} = \begin{bmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial y \partial x} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{bmatrix} \equiv \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix},$$

is positive semidefinite, i.e. all principal minors[3] of this matrix are non-negative:

$$f_{xx}(x,y) \geq 0, \quad f_{yy}(x,y) \geq 0, \quad \det\left(\mathcal{H}_{f(x,y)}\right) \geq 0. \tag{2.3.4}$$

The solutions of the SDEs for the geometric Brownian motion processes (2.3.1), at time T, under the risk neutral measure $\mathbb{Q}$, are

$$A_T = A_t e^{\left(r-\frac{1}{2}\sigma_A^2\right)(T-t)+\sigma_A\left(W_T^{\mathbb{Q},A}-W_t^{\mathbb{Q},A}\right)},$$

$$B_T = B_t e^{\left(r-\frac{1}{2}\sigma_B^2\right)(T-t)+\sigma_B\left(W_T^{\mathbb{Q},B}-W_t^{\mathbb{Q},B}\right)},$$

where the Brownian motion increments $W_T^{\mathbb{Q},A} - W_t^{\mathbb{Q},A}$ and $W_T^{\mathbb{Q},B} - W_t^{\mathbb{Q},B}$ are correlated normally distributed random variables with zero mean, variance $T-t$, and correlation ρ. This can thus be expressed as

$$\begin{aligned} A_T &= A_t R_A, \quad B_T = B_t R_B, \\ R_A &= e^{\left(r-\frac{1}{2}\sigma_A^2\right)(T-t)+\sigma_A\sqrt{T-t}\,Z_A}, \\ R_B &= e^{\left(r-\frac{1}{2}\sigma_B^2\right)(T-t)+\sigma_B\sqrt{T-t}\left(\rho Z_A+\sqrt{1-\rho^2}\,Z_B\right)}, \end{aligned} \tag{2.3.5}$$

where Z_A and Z_B are independent standard normal variables, and where the relative returns R_A and R_B do not depend on A_t and B_t. This means that our solution to the option pricing problem is now

$$V(t, A_t, B_t) = e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(A_t R_A,\, B_t R_B\right) \frac{e^{-\frac{1}{2}z_A^2-\frac{1}{2}z_B^2}}{2\pi}\, dz_A\, dz_B. \tag{2.3.6}$$

In order to show convexity of (2.3.6) in (A_t, B_t), we must show that its Hessian is positive semidefinite as well. The Hessian is

$$\begin{aligned} \mathcal{H}_{V(t,A_t,B_t)} &= \begin{bmatrix} \frac{\partial^2 V(t,A_t,B_t)}{\partial A_t^2} & \frac{\partial^2 V(t,A_t,B_t)}{\partial A_t \partial B_t} \\ \frac{\partial^2 V(t,A_t,B_t)}{\partial B_t \partial A_t} & \frac{\partial^2 V(t,A_t,B_t)}{\partial B_t^2} \end{bmatrix} \\ &= e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} \begin{bmatrix} R_A^2 f_{xx}\left(A_t R_A, B_t R_B\right) & R_A R_B f_{xy}\left(A_t R_A, B_t R_B\right) \\ R_B R_A f_{yx}\left(A_t R_A, B_t R_B\right) & R_B^2\, f_{yy}\left(A_t R_A, B_t R_B\right) \end{bmatrix} \frac{e^{-\frac{1}{2}z_A^2-\frac{1}{2}z_B^2}}{2\pi} dz_A dz_B, \end{aligned}$$

and hence, using (2.3.4), its determinant satisfies

$$\det\left(\mathcal{H}_{V(t,A_t,B_t)}\right) = e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} R_A^2 R_B^2 \det\left(\mathcal{H}_{f(A_t R_A,\, B_t R_B)}\right) \frac{e^{-\frac{1}{2}z_A^2-\frac{1}{2}z_B^2}}{2\pi}\, dz_A\, dz_B \geq 0,$$

as well as

$$\frac{\partial^2 V(t, A_t, B_t)}{\partial A_t^2} = e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} R_A^2\, f_{xx}\left(A_t R_A,\, B_t R_B\right) \frac{e^{-\frac{1}{2}z_A^2-\frac{1}{2}z_B^2}}{2\pi}\, dz_A\, dz_B \geq 0,$$

[3]The k-th principal minors are determinants formed from an n-by-n matrix by deleting any $n-k$ rows and the corresponding columns.

$$\frac{\partial^2 V(t, A_t, B_t)}{\partial B_t^2} = e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} R_B^2 \, f_{yy}\left(A_t R_A,\, B_t R_B\right) \frac{e^{-\frac{1}{2}z_A^2 - \frac{1}{2}z_B^2}}{2\pi} \, dz_A \, dz_B \geq 0,$$

which proves the convexity.[4]

The second part of the theorem follows if we apply the same procedure but instead consider the first order derivatives in (2.3.6) and the fact that $f(x, y)$ is non-decreasing: $f_x(x, y) \equiv \frac{\partial f(x,y)}{\partial x} \geq 0$ and $f_y(x, y) \equiv \frac{\partial f(x,y)}{\partial y} \geq 0$. Then we also have

$$\frac{\partial V(t, A_t, B_t)}{\partial A_t} = e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} R_A \, f_x\left(A_t R_A,\, B_t R_B\right) \frac{e^{-\frac{1}{2}z_A^2 - \frac{1}{2}z_B^2}}{2\pi} \, dz_A \, dz_B \geq 0,$$

$$\frac{\partial V(t, A_t, B_t)}{\partial B_t} = e^{-r(T-t)} \int_{\mathbb{R}} \int_{\mathbb{R}} R_B \, f_y\left(A_t R_A,\, B_t R_B\right) \frac{e^{-\frac{1}{2}z_A^2 - \frac{1}{2}z_B^2}}{2\pi} \, dz_A \, dz_B \geq 0,$$

since both R_A and R_B are positive functions, as given in (2.3.5). This proves the required monotonicity. ■

Corollary 2.1. *A bivariate European call on the maximum satisfies*

$$\mathrm{OW}\left(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right) \geq A_t - Ke^{-r(T-t)},$$

$$\mathrm{OW}\left(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right) \geq B_t - Ke^{-r(T-t)}.$$

Furthermore,

$$\lim_{A_t \to \infty} \frac{\mathrm{OW}\left(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{A_t - Ke^{-r(T-t)}} = 1, \tag{2.3.7}$$

for fixed B_t and other parameters, and

$$\lim_{B_t \to \infty} \frac{\mathrm{OW}\left(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{B_t - Ke^{-r(T-t)}} = 1, \tag{2.3.8}$$

for fixed A_t and other parameters.

Also

$$\frac{\partial \mathrm{OW}\left(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{\partial A} \leq 1, \quad \frac{\partial \mathrm{OW}\left(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{\partial B} \leq 1, \tag{2.3.9}$$

for any A, $B \geq 0$.

Proof. Consider buying the European call on the maximum option with strike K and maturity T, with price $\mathrm{OW}\left(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right) \equiv V_{max}^{Call-2D}(A_t,\, B_t)$, at time t, versus buying the portfolio Π that consists of the following:

- Any of the two underlying assets, say A_t,
- A borrowed amount $Ke^{-r(T-t)}$ in cash.

[4] The same approach can be adopted to show convexity in the one dimensional case, used in Section 2.2.

The cost of this portfolio would be $\Pi(t) = A_t - Ke^{-r(T-t)}$ and its value at time T will be exactly $\Pi(T) = A_T - K$. Note that it can also be negative, signifying a loss. For the call option the following outcomes are possible:

1. $A_T > B_T > K$ or $A_T > K > B_T \quad \Rightarrow \quad V_{max}^{Call-2D}(A_T,\, B_T) = A_T - K = \Pi(T)$,
2. $B_T > A_T > K$ or $B_T > K > A_T \quad \Rightarrow \quad V_{max}^{Call-2D}(A_T,\, B_T) = B_T - K > \Pi(T)$,
3. $K > A_T > B_T$ or $K > B_T > A_T \quad \Rightarrow \quad V_{max}^{Call-2D}(A_T,\, B_T) = 0 > \Pi(T)$.

Therefore, due to a no-arbitrage argument the value of the option at time t must be $V_{max}^{Call-2D}(A_t,\, B_t) \geq A_t - Ke^{-r(T-t)}$. The same conclusion follows if we choose to hold asset B_t instead.

For very large values of one of the underlying assets, while the other is kept fixed, the option value behaves like the classical (univariate) call option written only on that asset, and the option's price reduces to only its intrinsic value. This follows directly from (2.3.3), by noting that c_A, $\widetilde{c_A} \to \infty$ and $\widetilde{c_B} \to -\infty$ when $A_t \to \infty$. From the definition of the bivariate standard normal distribution (2.3.2) we conclude that factor after A_t becomes $N_2\left(\infty, \infty;\, \rho_A\right) = 1$ and the other two probability terms both become $N_2\left(-\infty, \cdot\,;\, \cdot\right) = 0$ which gives (2.3.7). The derivation is analogous for the case $B_t \to \infty$, giving (2.3.8).

Using L'Hôpital rule on (2.3.7) and (2.3.8) implies that the slope of the function in the A_t direction, when $A_t \to \infty$, for fixed B_t, and the slope in the B_t direction, when $B_t \to \infty$, for fixed A_t, equals 1. By convexity, shown in Theorem 2.1, those slopes are increasing functions. Hence, they must approach 1 from below, which completes the proof, i.e. both derivatives are always smaller than 1 for finite values of A_t and B_t.[5] ∎

However, on the diagonal, i.e. when $A_t = B_t$, we find different behaviour, which is described in the following corollary.

Corollary 2.2. *In the case of equal asset values $A_t = B_t$, the bivariate European call on the maximum option price takes the following form:*

$$\begin{aligned}\mathrm{OW}\left(A_t,\, A_t,\; K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right) = A_t &\left[N_2\left(\frac{1}{2}\sigma\sqrt{T-t}, c_A; \rho_A\right) + N_2\left(\frac{1}{2}\sigma\sqrt{T-t}, c_B; \rho_B\right)\right] \\ &- Ke^{-r(T-t)}\left(1 - N_2\left(-c_A + \sigma_A\sqrt{T-t}\,, -c_B + \sigma_B\sqrt{T-t}\,; \rho\right)\right). \qquad (2.3.10)\end{aligned}$$

Furthermore, in the limit $A_t = B_t \to \infty$, it degenerates to

$$\lim_{A_t \to \infty} \frac{\mathrm{OW}\left(A_t,\, A_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{2A_t N\left(\frac{1}{2}\sigma\sqrt{T-t}\right) - Ke^{-r(T-t)}} = 1.$$

Proof. From (2.3.3), we see that both $\widetilde{c_A}$ and $\widetilde{c_B}$ equal $\frac{1}{2}\sigma\sqrt{T-t}$ and (2.3.10) follows directly. For the second part, we have that c_A, $c_B \to \infty$ when A_t, $B_t \to \infty$, and thus the function becomes $2A_t N_2\left(\frac{1}{2}\sigma\sqrt{T-t}, \infty;\, \rho_A\right) - Ke^{-r(T-t)}\left(1 - N_2\left(-\infty, -\infty;\, \rho\right)\right)$. From

[5] The exact derivation of the first order derivatives $\frac{\partial \mathrm{OW}}{\partial A}$ and $\frac{\partial \mathrm{OW}}{\partial B}$ (i.e. the call-on-max option's deltas) and their plots (Figure A.2) are given in Appendix A.2.

definition (2.3.2) we can conclude that $N_2\left(-\infty\,,-\infty\,;\,\cdot\right)=0$ and $N_2\left(\frac{1}{2}\sigma\sqrt{T-t},\infty;\,\rho_A\right)=N\left(\frac{1}{2}\sigma\sqrt{T-t}\right)$, which completes the proof. ■

By L'Hôpital's rule, the slopes $\frac{\partial \mathrm{OW}}{\partial A}$ and $\frac{\partial \mathrm{OW}}{\partial B}$ of the function on the diagonal converge to the value of $2N\left(\frac{1}{2}\sigma\sqrt{T-t}\right)$ in the limit when $A_t=B_t\to\infty$.

The other bivariate option types

Prices for the other option types can be obtained from the following relationships. The price for a call-on-min with strike K is connected to the call-on-max via the max-min parity of Stulz (1982):

$$V^{Call}_{min}(t,\,A,\,B,\,K)=V^{Call}(t,\,A,\,K)+V^{Call}(t,\,B,\,K)-V^{Call}_{max}(t,\,A,\,B,\,K),\qquad(2.3.11)$$

which follows easily from the linear relationship at maturity. Here

$$V^{Call}_{max}(t,\,A,\,B,\,K)=\mathrm{OW}\left(A,\,B,\,K,\,r,\,\sigma_A,\,\sigma_B,\,\rho,\,T-t\right)$$

and

$$V^{Call}(t,\,A,\,K)=\mathrm{BS}\left(A,\,K,\,r,\,\sigma_A,\,T-t\right)$$

are the bivariate and univariate call options with the same strike and maturity.

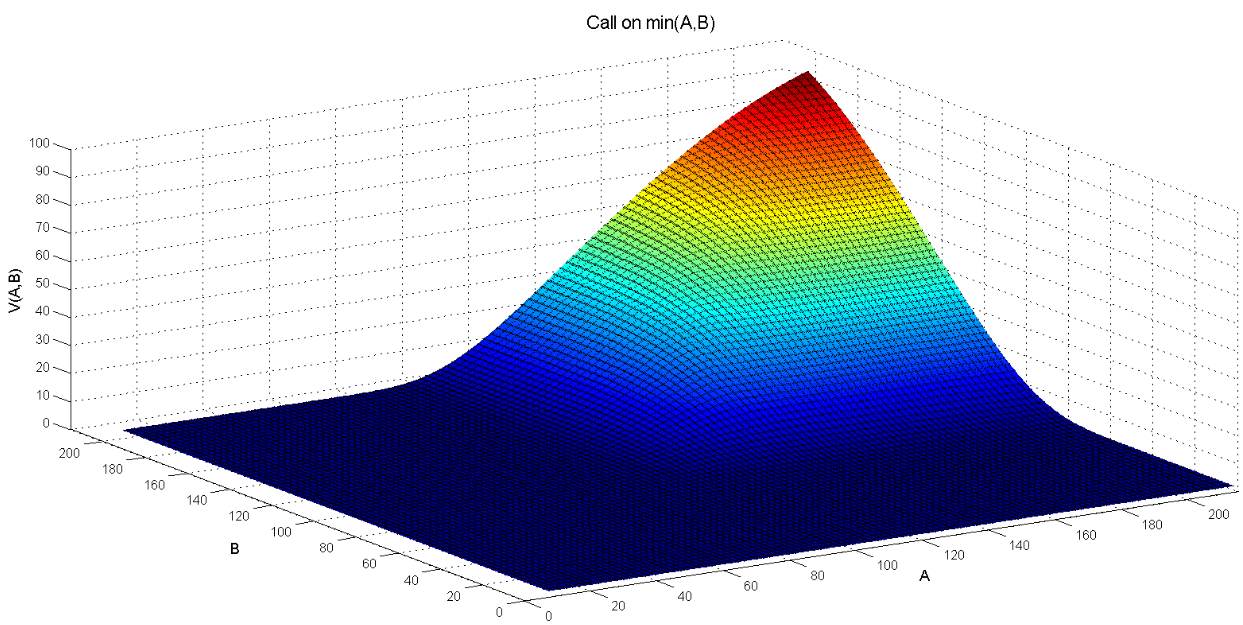

Figure 2.2: Price of a call on minimum option as a function of the two underlying assets.

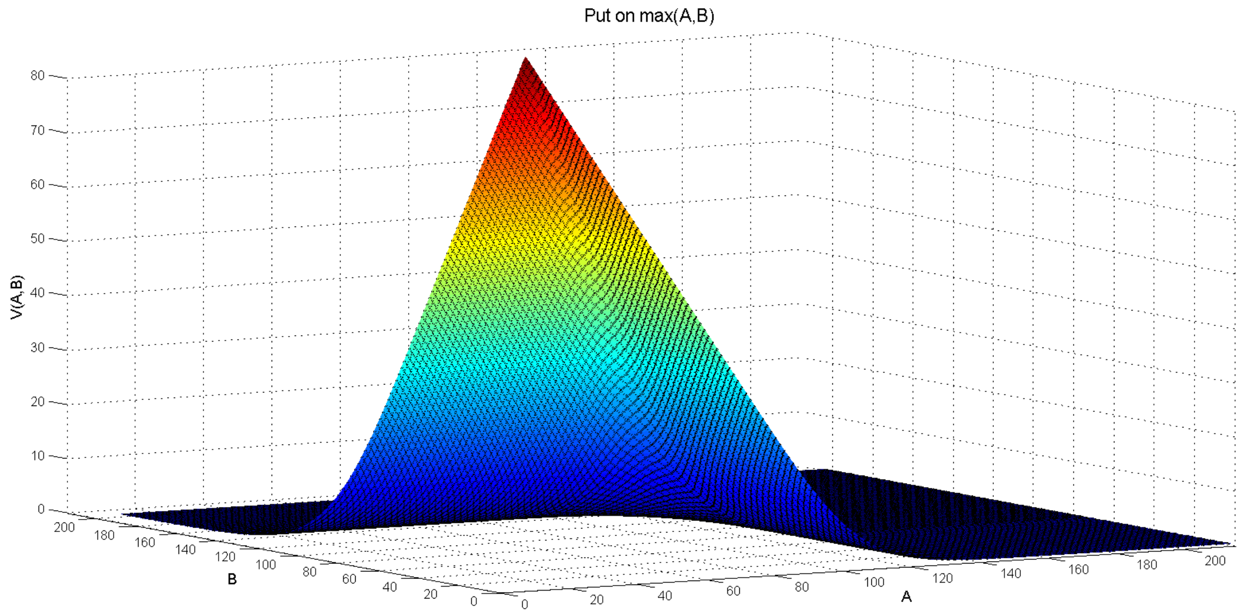

Figure 2.3: Price of a put on maximum option as a function of the two underlying assets.

The price for a put-on-max with strike K can be derived from the price for the call-on-max via put-call parity:

$$V_{max}^{Put}(t,\ A,\ B,\ K) = e^{-r(T-t)}K - V_{max}^{Call}(t,\ A,\ B,\ 0) + V_{max}^{Call}(t,\ A,\ B,\ K), \qquad (2.3.12)$$

and a similar relationship connects the prices of the put-on-min and the call-on-min:

$$V_{min}^{Put}(t,\ A,\ B,\ K) = e^{-r(T-t)}K - V_{min}^{Call}(t,\ A,\ B,\ 0) + V_{min}^{Call}(t,\ A,\ B,\ K). \qquad (2.3.13)$$

In the formulae above we use the call-on-max with zero strike $V_{max}^{Call}(t,\ A,\ B,\ 0)$, which can be found by taking the limit $K \to 0$ in (2.3.3), and call-on-min with zero strike $V_{min}^{Call}(t,\ A,\ B,\ 0)$, which can be determined from the price of the option to exchange one asset for another at maturity, i.e. the payoff $V_{Margrabe}^{Call}(T, A_T, B_T) = (A_T - B_T)^+$, priced by Margrabe (1978), in the following way:

$$V_{min}^{Call}(t,\ A,\ B,\ 0) = A - V_{Margrabe}^{Call}(t,\ A,\ B).$$

The prices of these other types of bivariate options, as functions of underlying asset prices A and B are illustrated in Figures 2.2, 2.3 and 2.4, respectively (for the same values of the parameters as specified above). As is obvious from the figures, and also from the payoff formulae at the beginning of this subsection, the price for a call on the minimum and for a put on the maximum are concave functions of the underlying stock prices, and therefore not suitable for direct application of our method. A put on the minimum option is convex but decreasing, which leaves us with a call on the maximum as the only candidate to apply our method, since it is both convex and increasing in its two arguments.

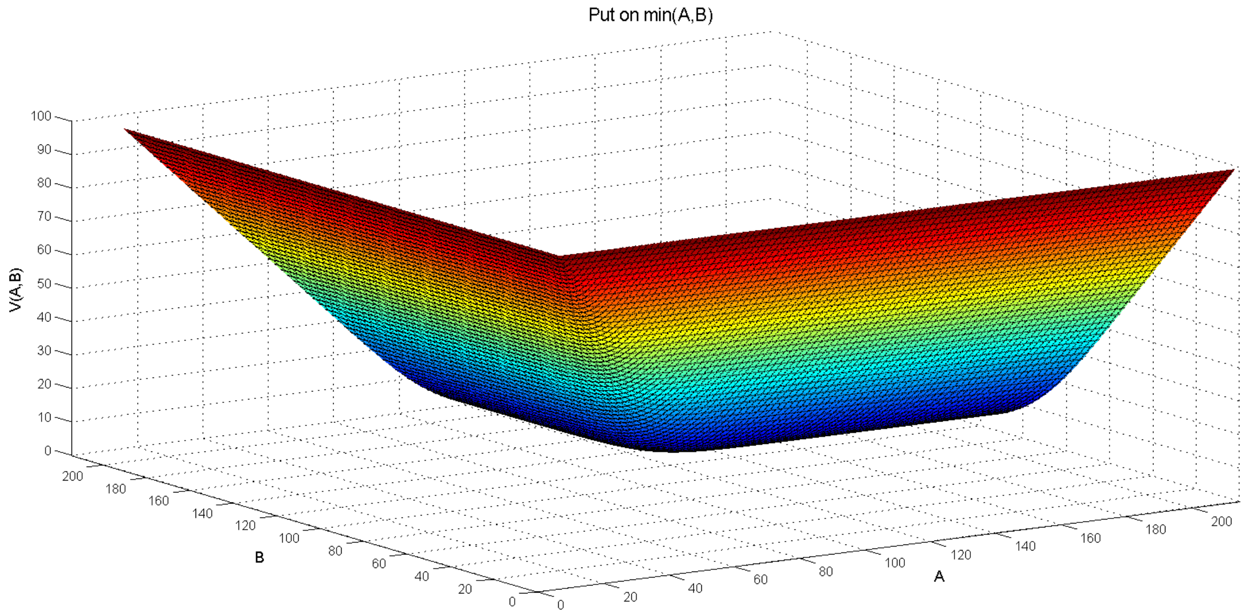

Figure 2.4: Price of a put on minimum option as a function of the two underlying assets.

2.3.2 The Dividend Paying Case

Since we now have an analytical and convex solution for the European call on the maximum of underlying assets that do not pay dividends, we can use it to price the same option but written on underlying assets that do pay discrete dividends. The upper and lower bounds needed to do this are given in the following proposition.

Proposition 2.2. *At time t, $(t < \tau < T)$, the price of a bivariate European call on the maximum, with strike K and maturity T, written on the underlying assets A and B as specified above by* (2.3.1)*, admits the following upper and lower bounds:*

$$V(t, A_t, B_t) \leq e^{-r(\tau-t)} \sum_{i=1}^{M_A+1} \sum_{j=1}^{M_B+1} \left[\alpha_A^{i,j} f_A^{i,j}(t, A_t, B_t) + \alpha_B^{i,j} f_B^{i,j}(t, A_t, B_t) + \xi^{i,j} g^{i,j}(t, A_t, B_t)\right],$$

$$V(t, A_t, B_t) \geq e^{-r(\tau-t)} \sum_{i=1}^{M_A+1} \sum_{j=1}^{M_B+1} \Big[\beta_A^{i-\frac{1}{2},j-\frac{1}{2}} f_A^{i,j}(t, A_t, B_t) + \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} f_B^{i,j}(t, A_t, B_t) + \chi^{i-\frac{1}{2},j-\frac{1}{2}} g^{i,j}(t, A_t, B_t)\Big],$$

with coefficients $\alpha_A^{i,j}$ and $\alpha_B^{i,j}$, $\xi^{i,j}$, $\beta_A^{i-\frac{1}{2},j-\frac{1}{2}}$ and $\beta_B^{i-\frac{1}{2},j-\frac{1}{2}}$, $\chi^{i-\frac{1}{2},j-\frac{1}{2}}$ defined below by (2.3.18), (2.3.20), (2.3.22) *and* (2.3.24)*, respectively, and functions $f_A^{i,j}$, $f_B^{i,j}$ and $g^{i,j}$ defined by* (2.3.27), (2.3.28) *and* (2.3.29)*. The discretisation grid is given by* (2.3.17) *and M_A, M_B, A^{M_A} and B^{M_B} are input parameters of the method.*

Proof. The price of this option at time $t < \tau < T$ is

$$V(t, A_t, B_t) = e^{-r(\tau-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V^{-}(\tau,\, A_{\tau-},\, B_{\tau-})\right], \tag{2.3.14}$$

with

$$V^{-}(t, A, B) = \mathrm{OW}\left((A - D_A)^{+},\, (B - D_B)^{+},\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T - t\right), \tag{2.3.15}$$

where $OW\,(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t)$ denotes the price of a non-dividend paying bivariate call on maximum option which is given by (2.3.3). The value of the derivative is bounded from above:

$$V^{-}(\tau, A, B) \le V^{-u}(\tau, A, B), \tag{2.3.16}$$

where $V^{-u}(\tau, A, B)$, is an upper bound, defined as a 2D piecewise linear interpolation and extrapolation. To specify $V^{-u}(\tau, A, B)$ we use points $(A^i, B^j, V^{i,j})$ which are defined as

$$A^i = D_A + i\Delta A, \quad \Delta A = \frac{A^{M_A} - D_A}{M_A}, \quad i = 0, ..., M_A; \tag{2.3.17}$$
$$B^j = D_B + j\Delta B, \quad \Delta B = \frac{B^{M_B} - D_B}{M_B}, \quad j = 0, ..., M_B;$$
$$V^{i,j} = V^{-}(\tau, A^i, B^j),$$

with $A^{M_A} - D_A = B^{M_B} - D_B$, and $A^{M_A+1} = B^{M_B+1} = \infty$. We ensure that $V^{-u}(\tau, A, B)$ is indeed an upper bound since in each discretisation region $[A^{i-1}, A^i] \times [B^{j-1}, B^j]$ it is defined as a plane through three points: $(A^i, B^j, V^{i,j})$, $(A^i, B^{j-1}, V^{i,j-1})$ and $(A^{i-1}, B^j, V^{i-1,j})$, an example of which is depicted in Figure 2.5.

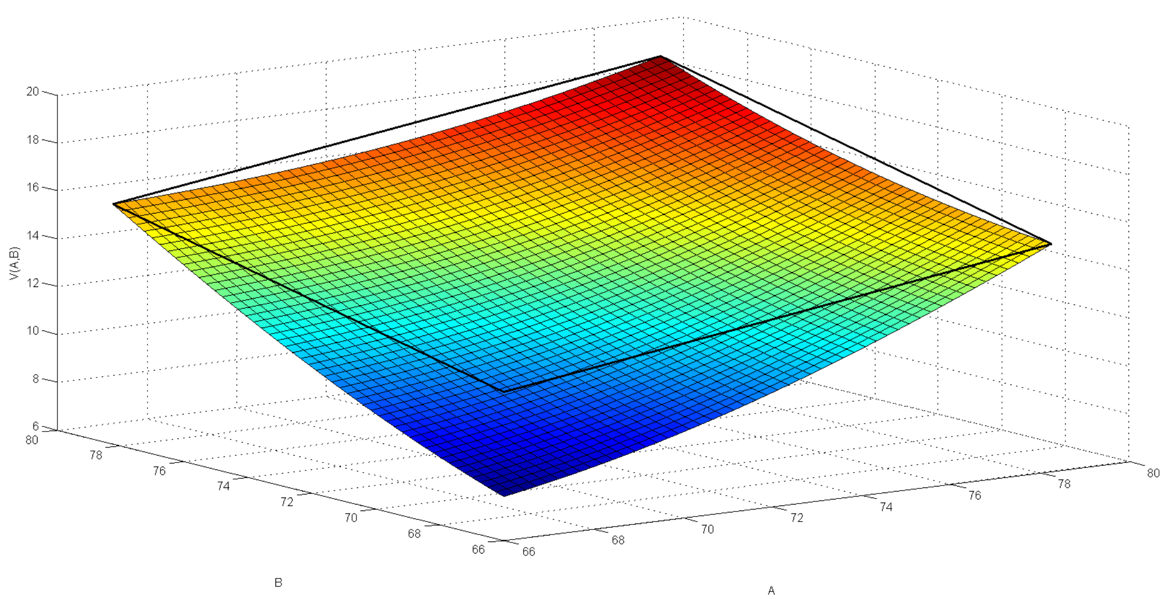

Figure 2.5: A plane through three upper points of the section of the surface $V^{-}(\tau, A, B)$.

Since Theorem 2.1 shows that the function $\mathrm{OW}\left(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-\tau\right)$ is both convex and increasing in A and B, and from equation (2.3.15) and the fact that $\left(A - D_A\right)^+$ and $\left(B - D_B\right)^+$are convex and non-decreasing functions we conclude that $V^-(\tau, A, B)$ must be convex and increasing in A and B as well, and thus this plane must always lie above the function surface itself. Thus, the slopes of the 2D piecewise linear function $V^{-u}(\tau, A, B)$, in both directions are defined as

$$\begin{aligned}
\alpha_A^{i,j} &= \frac{V^{i,j} - V^{i-1,j}}{A^i - A^{i-1}}, \qquad && i = 1, ..., M_A,\ j = 1, ..., M_B;\\
\alpha_A^{M_A+1,j} &= 1, && j = 1, ..., M_B;\\
\alpha_A^{i,M_B+1} &= \alpha_A^{i,M_B}, && i = 1, ..., M_A;
\end{aligned} \tag{2.3.18}$$

$$\begin{aligned}
\alpha_B^{i,j} &= \frac{V^{i,j} - V^{i,j-1}}{B^j - B^{j-1}}, \qquad && i = 1, ..., M_A,\ j = 1, ..., M_B;\\
\alpha_B^{i,M_B+1} &= 1, && i = 1, ..., M_A;\\
\alpha_B^{M_A+1,j} &= \alpha_B^{M_A,j}, && j = 1, ..., M_B;
\end{aligned}$$

$$\alpha_A^{M_A+1,M_B+1} = \alpha_B^{M_A+1,M_B+1} = 2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right).$$

Hence, by considering only segments inside the discretised space (i.e. for the summation indices up to M_A and M_B), we have

$$V_{inside}^{-u}(\tau, A, B) = \sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\left[\alpha_A^{i,j}\left(A - A^i\right) + \alpha_B^{i,j}\left(B - B^j\right) + V^{i,j}\right] 1_{\{A\in[A^{i-1},A^i)\}} 1_{\{B\in[B^{j-1},B^j)\}}.$$

In the space outside of the region $\left[D_A,\, A^{M_A}\right) \times \left[D_B,\, B^{M_B}\right)$, i.e. for indices $M_A + 1$ and $M_B + 1$, we take the following planes (implied by the definitions of alphas above).

In the regions $\left[A^{M_A},\, \infty\right) \times \left[D_B,\, B^{M_B}\right)$ and $\left[D_A,\, A^{M_A}\right) \times \left[B^{M_B},\, \infty\right)$ we take a simple extension of the two-dimensional piecewise linear function from the edges of the discretisation region, with corresponding slopes equal to 1. As before, in the one-dimensional case, this must be an upper bound in those regions, since, by Corollary 2.1, the derivatives of the option's value, for large A, with fixed B, and for large B, with fixed A, asymptotically approach the value of 1 in the corresponding direction. Therefore, the option's value surface always has gradients in both directions less than 1, and if we take linear extensions with gradient 1 in the appropriate direction, starting from points (A^{M_A}, B^j), $j = 1, ...M_B$, and (A^i, B^{M_B}), $i = 1, ...M_A$ on the surface, those extensions must lie above the function surface itself.

In the corner region $\left[A^{M_A},\, \infty\right) \times \left[B^{M_B},\, \infty\right)$, we take the plane with slopes in both directions equal to $2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right)$, the quantity which appears in Corollary 2.2, where $\sigma = \sqrt{\sigma_A^2 - 2\rho\sigma_A\sigma_B + \sigma_B^2}$, as defined before in (2.3.3). Since $\frac{1}{2}\sigma\sqrt{T-\tau} \geq 0$ and N is the standard normal cumulative distribution function, we have $0.5 \leq N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) \leq 1$, which means that those slopes are always at least 1. This plane, constructed such that is starts from the point (A^{M_A}, B^{M_B}) on the surface, must thus rise faster than the surface of the function $V^{-}(\tau, A, B)$ and is therefore always above it.[6] This is because the gradients of the function itself, the option's deltas, are never above 1, as proven in Corollary 2.1.

So finally, all this can be merged together in a single upper bound expression:

$$V^{-u}(\tau, A, B) = \sum_{i=1}^{M_A+1} \sum_{j=1}^{M_B+1} \left[\alpha_A^{i,j} A + \alpha_B^{i,j} B + \xi^{i,j}\right] 1_{\{A \in [A^{i-1}, A^i)\}} 1_{\{B \in [B^{j-1}, B^j)\}}, \tag{2.3.19}$$

with

$$\xi^{i,j} = V^{i,j} - \alpha_A^{i,j} A^i - \alpha_B^{i,j} B^j, \;\; i = 1, ..., M_A, \;\; j = 1, ..., M_B \tag{2.3.20}$$

and

$$\xi^{M_A+1,j} = V^{M_A,j} - A^{M_A} - \alpha_B^{M_A,j} B^j, \;\; j = 1, ..., M_B;$$
$$\xi^{i,M_B+1} = V^{i,M_B} - \alpha_A^{i,M_B} A^i - B^{M_B}, \;\; i = 1, ..., M_A;$$
$$\xi^{M_A+1,M_B+1} = V^{M_A,M_B} - 2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) A^{M_A} - 2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) B^{M_B}.$$

Similarly, from the convexity proven in Theorem 2.1, the value of the derivative can be bounded from below:

$$V^{-}(\tau, A, B) \geq V^{-l}(\tau, A, B), \tag{2.3.21}$$

where $V^{-l}(\tau, A, B)$, is defined as the piecewise linear interpolation and extrapolation in two dimensions on mid-points of the discretisation segments defined above, i.e. at points $(A^{i-\frac{1}{2}},\, B^{j-\frac{1}{2}},\, V^{i-\frac{1}{2},j-\frac{1}{2}})$. In each segment $[A^{i-1}, A^i] \times [B^{j-1}, B^j]$, the lower bound is a plane tangent to the surface $V^{-}(\tau, A, B)$ at the point $(A^{i-\frac{1}{2}}, B^{j-\frac{1}{2}}, V^{i-\frac{1}{2},j-\frac{1}{2}})$ and due to the convexity it will lie below the surface, as illustrated in Figure 2.6. This follows from the following known property of convex functions:[7]

Theorem 2.2. *Suppose that a convex function f, (as defined as above in Definition 2.1) is differentiable (i.e. its gradient ∇f exists at each point in the function's domain $\mathbb{R}^n$). Then, the inequality*

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

holds $\forall x, y \in \mathbb{R}^n$.

The function on the right hand side of the inequality is, of course, a first order Taylor approximation of f near x. The inequality thus states that for a convex function, the first order Taylor approximation in a point is in fact a global lower bound for the function. It follows that in two dimensions the tangent plane at any point x must always lie below the function itself.

[6] It rises even higher than the other "strip" surfaces in the edge regions just described above (for which the slopes are exactly 1).

[7] See e.g. "Complex optimization" by Boyd and Vandenberghe (2004), page 69, Chapter 3.1.3

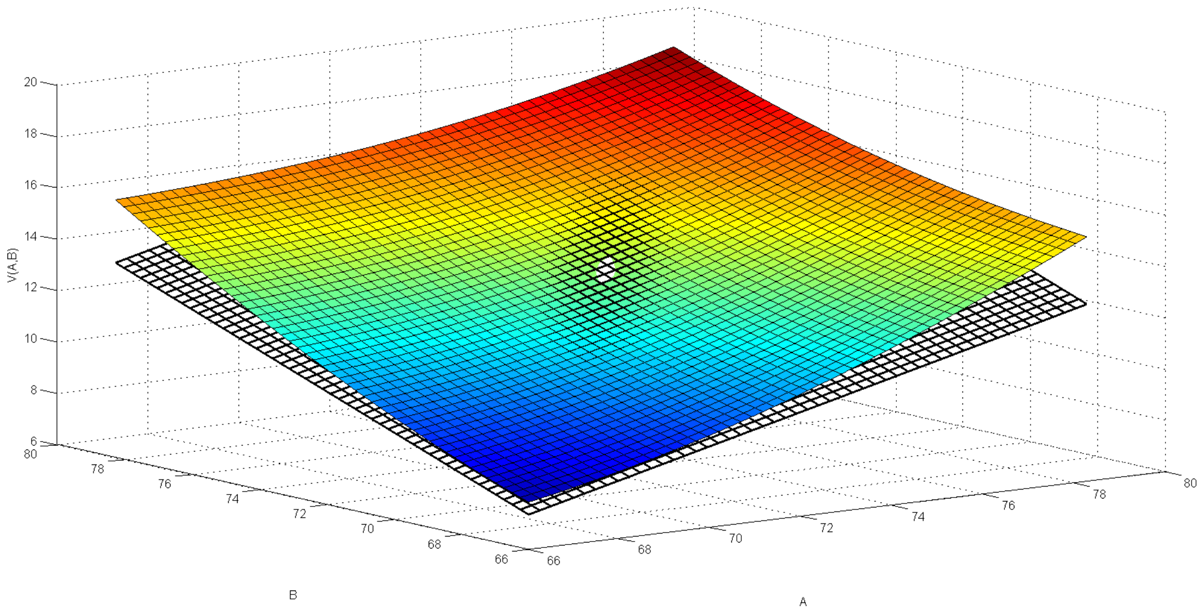

Figure 2.6: A plane through the middle point of the section of the surface $V^{-}(\tau, A, B)$.

The slopes (gradients) of this piecewise linear function $V^{-l}(\tau, A, B)$ at these points are

$$\begin{aligned}
\beta_A^{i-\frac{1}{2},j-\frac{1}{2}} &= \frac{\partial V^-}{\partial A}(\tau\,, A^{i-\frac{1}{2}},\, B^{j-\frac{1}{2}}), && i=1,...,M_A,\ j=1,...,M_B;\\
\beta_A^{M_A+\frac{1}{2},j-\frac{1}{2}} &= \beta_A^{M_A-\frac{1}{2},j-\frac{1}{2}}, && j=1,...,M_B+1;\\
\beta_A^{i-\frac{1}{2},M_B+\frac{1}{2}} &= \beta_A^{i-\frac{1}{2},M_B-\frac{1}{2}}, && i=1,...,M_A+1.
\end{aligned} \tag{2.3.22}$$

$$\begin{aligned}
\beta_B^{i-\frac{1}{2},j-\frac{1}{2}} &= \frac{\partial V^-}{\partial B}(\tau\,, A^{i-\frac{1}{2}},\, B^{j-\frac{1}{2}}), && i=1,...,M_A,\ j=1,...,M_B;\\
\beta_B^{i-\frac{1}{2},M_B+\frac{1}{2}} &= \beta_B^{i-\frac{1}{2},M_B-\frac{1}{2}}, && i=1,...,M_A+1;\\
\beta_B^{M_A+\frac{1}{2},j-\frac{1}{2}} &= \beta_B^{M_A-\frac{1}{2},j-\frac{1}{2}}, && j=1,...,M_B+1.
\end{aligned}$$

The partial derivatives $\frac{\partial V^-}{\partial A}$ and $\frac{\partial V^-}{\partial B}$ (i.e. the call-on-max option's deltas) for the Ouwehand-West formula (2.3.3) are given in Appendix A.2. For segments inside the discretised space (i.e. for the summation indices up to M_A and M_B), we have

$$V_{inside}^{-l}(\tau, A, B) = \sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\left[\beta_A^{i-\frac{1}{2},j-\frac{1}{2}}\left(A - A^{i-\frac{1}{2}}\right) + \beta_B^{i-\frac{1}{2},j-\frac{1}{2}}\left(B - B^{j-\frac{1}{2}}\right) + V^{i-\frac{1}{2},j-\frac{1}{2}}\right]$$
$$\cdot 1_{\{A\in[A^{i-1},A^i)\}}1_{\{B\in[B^{j-1},B^j)\}}.$$

For the values in the space outside $\left[D_A,\ A^{M_B}\right) \times \left[D_B,\ B^{M_B}\right)$, i.e. for indices M_A+1 and M_B+1, we take the extension of the planes that are tangent to V in points $(A^{M_A-\frac{1}{2}}, B^{j-\frac{1}{2}})$, $j=1,...M_B$ and $(A^{i-\frac{1}{2}}, B^{M_B-\frac{1}{2}})$, $i=1,...M_A$. That this indeed generates the lower bound follows directly from Theorem 2.2.

We can merge all this in a single expression:

$$V^{-l}(\tau,A,B) = \sum_{i=1}^{M_A+1}\sum_{j=1}^{M_B+1}\left[\beta_A^{i-\frac{1}{2},j-\frac{1}{2}}A + \beta_B^{i-\frac{1}{2},j-\frac{1}{2}}B + \chi^{i-\frac{1}{2},j-\frac{1}{2}}\right]1_{\{A\in[A^{i-1},A^i)\}}1_{\{B\in[B^{j-1},B^j)\}}, \tag{2.3.23}$$

with

$$\chi^{i-\frac{1}{2},j-\frac{1}{2}} = V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}}A^{i-\frac{1}{2}} - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}}B^{j-\frac{1}{2}},\ \ i=1,...,M_A,\ j=1,...,M_B$$

and

$$\begin{aligned}\chi^{M_A+\frac{1}{2},j-\frac{1}{2}} &= \chi^{M_A-\frac{1}{2},j-\frac{1}{2}},\ j=1,...,M_B+1;\\ \chi^{i-\frac{1}{2},M_B+\frac{1}{2}} &= \chi^{i-\frac{1}{2},M_B-\frac{1}{2}},\ i=1,...,M_A+1.\end{aligned} \tag{2.3.24}$$

So, from (2.3.14), (2.3.15), (2.3.16) and (2.3.19) we have

$$\begin{aligned}V(t,A_t,B_t) &\le e^{-r(\tau-t)}\sum_{i=1}^{M_A+1}\sum_{j=1}^{M_B+1}\mathbb{E}_t^{\mathbb{Q}}\left[\left(\alpha_A^{i,j}A_{\tau-} + \alpha_B^{i,j}B_{\tau-} + \xi^{i,j}\right)1_{\{A_{\tau-}\in[A^{i-1},A^i)\}}1_{\{B_{\tau-}\in[B^{j-1},B^j)\}}\right]\\ &= e^{-r(\tau-t)}\sum_{i=1}^{M_A+1}\sum_{j=1}^{M_B+1}\left[\alpha_A^{i,j}f_A^{i,j}(t,A_t,B_t) + \alpha_B^{i,j}f_B^{i,j}(t,A_t,B_t) + \xi^{i,j}g^{i,j}(t,A_t,B_t)\right],\end{aligned} \tag{2.3.25}$$

and from (2.3.14), (2.3.15), (2.3.21) and (2.3.23) we have

$$\begin{aligned}V(t,A_t,B_t) &\ge e^{-r(\tau-t)}\sum_{i=1}^{M_A+1}\sum_{j=1}^{M_B+1}\mathbb{E}_t^{\mathbb{Q}}\left[\left(\beta_A^{i-\frac{1}{2},j-\frac{1}{2}}A_{\tau-} + \beta_B^{i-\frac{1}{2},j-\frac{1}{2}}B_{\tau-} + \chi^{i-\frac{1}{2},j-\frac{1}{2}}\right)\right.\\ &\qquad\qquad \left.\cdot 1_{\{A_{\tau-}\in[A^{i-1},A^i)\}}1_{\{B_{\tau-}\in[B^{j-1},B^j)\}}\right]\\ &= e^{-r(\tau-t)}\sum_{i=1}^{M_A+1}\sum_{j=1}^{M_B+1}\left[\beta_A^{i-\frac{1}{2},j-\frac{1}{2}}f_A^{i,j}(t,A_t,B_t) + \beta_B^{i-\frac{1}{2},j-\frac{1}{2}}f_B^{i,j}(t,A_t,B_t)\right.\\ &\qquad\qquad \left.+\chi^{i-\frac{1}{2},j-\frac{1}{2}}g^{i,j}(t,A_t,B_t)\right],\end{aligned} \tag{2.3.26}$$

if the new functions $f_A^{i,j}$, $f_B^{i,j}$ and $g^{i,j}$ are defined[8] as

$$\begin{aligned} f_A^{i,j}(t,A_t,B_t) &= \mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}\in[A^{i-1},A^i)\}}1_{\{B_{\tau-}\in[B^{j-1},B^j)\}}\right] \\ &= A_t e^{r(\tau-t)}\left[N_2\left(d_A^i,\widetilde{d_B^j}\right)-N_2\left(d_A^{i-1},\widetilde{d_B^j}\right)-N_2\left(d_A^i,\widetilde{d_B^{j-1}}\right)+N_2\left(d_A^{i-1},\widetilde{d_B^{j-1}}\right)\right], \end{aligned} \tag{2.3.27}$$

$$\begin{aligned} f_B^{i,j}(t,A_t,B_t) &= \mathbb{E}_t^{\mathbb{Q}}\left[B_{\tau-}1_{\{A_{\tau-}\in[A^{i-1},A^i)\}}1_{\{B_{\tau-}\in[B^{j-1},B^j)\}}\right] \\ &= B_t e^{r(\tau-t)}\left[N_2\left(\widetilde{d_A^i},d_B^j\right)-N_2\left(\widetilde{d_A^{i-1}},d_B^j\right)-N_2\left(\widetilde{d_A^i},d_B^{j-1}\right)+N_2\left(\widetilde{d_A^{i-1}},d_B^{j-1}\right)\right], \end{aligned} \tag{2.3.28}$$

$$\begin{aligned} g^{i,j}(t,A_t,B_t) &= \mathbb{E}_t^{\mathbb{Q}}\left[1_{\{A_{\tau-}\in[A^{i-1},A^i)\}}1_{\{B_{\tau-}\in[B^{j-1},B^j)\}}\right] \\ &= N_2\left(d_A^i+\sigma_A\sqrt{\tau-t},d_B^j+\sigma_B\sqrt{\tau-t}\right)-N_2\left(d_A^{i-1}+\sigma_A\sqrt{\tau-t},d_B^j+\sigma_B\sqrt{\tau-t}\right) \\ &\quad -N_2\left(d_A^i+\sigma_A\sqrt{\tau-t},d_B^{j-1}+\sigma_B\sqrt{\tau-t}\right)+N_2\left(d_A^{i-1}+\sigma_A\sqrt{\tau-t},d_B^{j-1}+\sigma_B\sqrt{\tau-t}\right), \end{aligned} \tag{2.3.29}$$

where

$$d_A^i=\frac{\ln\left(\frac{A^i}{A_t}\right)-\left(r+\frac{1}{2}\sigma_A^2\right)(\tau-t)}{\sigma_A\sqrt{\tau-t}},\quad d_B^j=\frac{\ln\left(\frac{B^j}{B_t}\right)-\left(r+\frac{1}{2}\sigma_B^2\right)(\tau-t)}{\sigma_B\sqrt{\tau-t}},$$

and

$$\widetilde{d_A^i}=\frac{\ln\left(\frac{A^i}{A_t}\right)-\left(r-\frac{1}{2}\sigma_A^2+\rho\sigma_A\sigma_B\right)(\tau-t)}{\sigma_A\sqrt{\tau-t}},\quad \widetilde{d_B^j}=\frac{\ln\left(\frac{B^j}{B_t}\right)-\left(r-\frac{1}{2}\sigma_B^2+\rho\sigma_A\sigma_B\right)(\tau-t)}{\sigma_B\sqrt{\tau-t}}.$$

■

Proposition 2.3. *For the discretisation steps ΔA and ΔB, the difference between the bounds satisfies*[9]

$$\begin{aligned} \left|V^{-u}(\tau,A,B)-V^{-l}(\tau,A,B)\right| \leq & \frac{3}{2}\left(\frac{\max\{\Delta A,\Delta B\}}{2}\right)^2 \max_{u_A,u_B>0}\left\{\left|\frac{\partial^2 V^-}{\partial A^2}(\tau,u_A,u_B)\right|,\right. \\ & \left. 2\left|\frac{\partial^2 V^-}{\partial A\partial B}(\tau,u_A,u_B)\right|,\ \left|\frac{\partial^2 V^-}{\partial B^2}(\tau,u_A,u_B)\right|\right\} \\ & + h\left(\tau,A,B,A^{M_A},B^{M_B}\right), \end{aligned} \tag{2.3.30}$$

where $\lim_{A^{M_A}\to\infty,\,B^{M_B}\to\infty}\mathbb{E}_t^{\mathbb{Q}}\left[h\left(\tau,A_\tau,B_\tau,A^{M_A},B^{M_B}\right)\right]=0$, *for all* (A_τ,B_τ).

Therefore, the maximal error of the method to obtain the price $V(t,A_t,B_t)$ of a bivariate European call on maximum option, for $t<\tau$, becomes proportional to $(\max\{\Delta A,\Delta B\})^2$, *when A^{M_A}, $B^{M_B}\to\infty$.*

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[8]The detailed derivations are given in Appendix A.1.

[9]Second order derivatives such as $\frac{\partial^2 V^-}{\partial A^2}(\tau,A,B)$ may not exist for certain values of (A,B) since V^- may have discontinuous first order derivatives in the dividend value point. However, the lefthand and righthand second order derivatives always exist in these points and are bounded, so our derivations which are based on boundedness of these second order derivatives will remain valid.

Proof. If we define the absolute difference between the upper and lower bounds for $V^{-}(\tau, A, B)$, i.e. (2.3.19) minus (2.3.23), we will have the following :

$$\left|V^{-u}(\tau, A, B) - V^{-l}(\tau, A, B)\right| \leq \tag{2.3.31}$$

$$\sum_{i=1}^{M_A+1} \sum_{j=1}^{M_B+1} \max_{\substack{\widetilde{A} \in [A^{i-1}, A^i], \\ \widetilde{B} \in [B^{j-1}, B^j]}} \left| \left(\alpha_A^{i,j} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}}\right) \widetilde{A} + \left(\alpha_B^{i,j} - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}}\right) \widetilde{B} + \xi^{i,j} - \chi^{i-\frac{1}{2},j-\frac{1}{2}} \right| \begin{array}{l} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{array},$$

where

$$\xi^{i,j} - \chi^{i-\frac{1}{2},j-\frac{1}{2}} = V^{i,j} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \alpha_A^{i,j} A^i - \alpha_B^{i,j} B^j + \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} A^{i-\frac{1}{2}} + \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} B^{j-\frac{1}{2}}.$$

Since we consider a linear two-dimensional function in each region $[A^{i-1}, A^i] \times [B^{j-1}, B^j]$ the maxima always occur at the value in one of the corner points so, by recalling the definition of the alphas from (2.3.18), we obtain

$$\left|V^{-u}(\tau, A, B) - V^{-l}(\tau, A, B)\right| \leq \tag{2.3.32}$$

$$\sum_{i=1}^{M_A} \sum_{j=1}^{M_B} \max \left\{ \begin{array}{c} \left|V^{i,j} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^i - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^j - B^{j-\frac{1}{2}}\right)\right|, \\ \left|V^{i-1,j} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^{i-1} - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^j - B^{j-\frac{1}{2}}\right)\right|, \\ \left|V^{i,j-1} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^i - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^{j-1} - B^{j-\frac{1}{2}}\right)\right|, \\ \left|V^{i-1,j} + V^{i,j-1} - V^{i,j} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^{i-1} - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^{j-1} - B^{j-\frac{1}{2}}\right)\right| \end{array} \right\} \begin{array}{l} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{array}$$

$$+ \sum_{i=1}^{M_A} \max \left\{ \begin{array}{c} \left|V^{i,M_B} - V^{i-\frac{1}{2},M_B-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},M_B-\frac{1}{2}} \left(A^i - A^{i-\frac{1}{2}}\right) \right. \\ \left. -\beta_B^{i-\frac{1}{2},M_B-\frac{1}{2}} \left(B^{M_B} - B^{M_B-\frac{1}{2}}\right) + \left(1 - \beta_B^{i-\frac{1}{2},M_B-\frac{1}{2}}\right) (B - B^{M_B})\right|, \\ \left|V^{i-1,M_B} - V^{i-\frac{1}{2},M_B-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},M_B-\frac{1}{2}} \left(A^{i-1} - A^{i-\frac{1}{2}}\right) \right. \\ \left. -\beta_B^{i-\frac{1}{2},M_B-\frac{1}{2}} \left(B^{M_B} - B^{M_B-\frac{1}{2}}\right) + \left(1 - \beta_B^{i-\frac{1}{2},M_B-\frac{1}{2}}\right) (B - B^{M_B})\right| \end{array} \right\} \begin{array}{l} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \geq B^{M_B}\}} \end{array}$$

$$+ \sum_{j=1}^{M_B} \max \left\{ \begin{array}{c} \left|V^{M_A,j} - V^{M_A-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{M_A-\frac{1}{2},j-\frac{1}{2}} \left(A^{M_A} - A^{M_A-\frac{1}{2}}\right) \right. \\ \left. -\beta_B^{M_A-\frac{1}{2},j-\frac{1}{2}} \left(B^j - B^{j-\frac{1}{2}}\right) + \left(1 - \beta_A^{M_A-\frac{1}{2},j-\frac{1}{2}}\right) (A - A^{M_A})\right|, \\ \left|V^{M_A,j-1} - V^{M_A-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{M_A-\frac{1}{2},j-\frac{1}{2}} \left(A^{M_A} - A^{M_A-\frac{1}{2}}\right) \right. \\ \left. -\beta_B^{M_A-\frac{1}{2},j-\frac{1}{2}} \left(B^{j-1} - B^{j-\frac{1}{2}}\right) + \left(1 - \beta_A^{M_A-\frac{1}{2},j-\frac{1}{2}}\right) (A - A^{M_A})\right| \end{array} \right\} \begin{array}{l} 1_{\{A \geq A^{M_A}\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{array}$$

$$+ \left|V^{M_A,M_B} - V^{M_A-\frac{1}{2},M_B-\frac{1}{2}} - \beta_A^{M_A-\frac{1}{2},M_B-\frac{1}{2}} \left(A^{M_A} - A^{M_A-\frac{1}{2}}\right) - \beta_B^{M_A-\frac{1}{2},M_B-\frac{1}{2}} \left(B^{M_B} - B^{M_B-\frac{1}{2}}\right) \right.$$
$$\left. + \left(2N\left(\tfrac{1}{2}\sigma\sqrt{T-\tau}\right) - \beta_A^{M_A-\frac{1}{2},M_B-\frac{1}{2}}\right) (A - A^{M_A}) + \left(2N\left(\tfrac{1}{2}\sigma\sqrt{T-\tau}\right) - \beta_B^{M_A-\frac{1}{2},M_B-\frac{1}{2}}\right) (B - B^{M_B})\right| \begin{array}{l} 1_{\{A \geq A^{M_A}\}} \\ 1_{\{B \geq B^{M_B}\}} \end{array}.$$

Recalling that $\beta_A^{i-\frac{1}{2},j-\frac{1}{2}} = \frac{\partial V^-}{\partial A}(\tau, A^{i-\frac{1}{2}}, B^{j-\frac{1}{2}})$ and $\beta_B^{i-\frac{1}{2},j-\frac{1}{2}} = \frac{\partial V^-}{\partial B}(\tau, A^{i-\frac{1}{2}}, B^{j-\frac{1}{2}})$ we recognise terms in the two-dimensional Taylor series expansion of the functions $V^{i,j}$, $V^{i-1,j}$ and $V^{i,j-1}$ around $(A^{i-\frac{1}{2}}, B^{j-\frac{1}{2}})$.

By considering Lagrange's form of the remainder R_1 of the two-dimensional Taylor series expansion we have that there exist $k_A^i \in (A^{i-\frac{1}{2}}, A^i)$, $k_B^j \in (B^{j-\frac{1}{2}}, B^j)$, $\widetilde{k_A^i} \in (A^{i-1}, A^{i-\frac{1}{2}})$,

$\widetilde{k_B^j} \in (B^{j-\frac{1}{2}}, B^j)$ and $\widehat{k_A^i} \in (A^{i-\frac{1}{2}}, A^i)$, $\widehat{k_B^j} \in (B^{j-1}, B^{j-\frac{1}{2}})$ such that

$$\begin{aligned}
R_1(A^i, B^j) &= V^{i,j} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^i - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^j - B^{j-\frac{1}{2}}\right) \\
&= \frac{1}{2}\frac{\partial^2 V^-(\tau, k_A^i, k_B^j)}{\partial A^2} \left(A^i - A^{i-\frac{1}{2}}\right)^2 + \frac{\partial^2 V^-(\tau, k_A^i, k_B^j)}{\partial A \partial B} \left(A^i - A^{i-\frac{1}{2}}\right) \left(B^j - B^{j-\frac{1}{2}}\right) \\
&+ \frac{1}{2}\frac{\partial^2 V^-(\tau, k_A^i, k_B^j)}{\partial B^2} \left(B^j - B^{j-\frac{1}{2}}\right)^2,
\end{aligned}$$

$$\begin{aligned}
R_1(A^{i-1}, B^j) &= V^{i-1,j} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^{i-1} - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^j - B^{j-\frac{1}{2}}\right) \\
&= \frac{1}{2}\frac{\partial^2 V^-(\tau, \widetilde{k_A^i}, \widetilde{k_B^j})}{\partial A^2} \left(A^{i-\frac{1}{2}} - A^{i-1}\right)^2 + \frac{\partial^2 V^-(\tau, \widetilde{k_A^i}, \widetilde{k_B^j})}{\partial A \partial B} \left(A^{i-1} - A^{i-\frac{1}{2}}\right) \left(B^j - B^{j-\frac{1}{2}}\right) \\
&+ \frac{1}{2}\frac{\partial^2 V^-(\tau, \widetilde{k_A^i}, \widetilde{k_B^j})}{\partial B^2} \left(B^j - B^{j-\frac{1}{2}}\right)^2,
\end{aligned} \tag{2.3.33}$$

$$\begin{aligned}
R_1(A^i, B^{j-1}) &= V^{i,j-1} - V^{i-\frac{1}{2},j-\frac{1}{2}} - \beta_A^{i-\frac{1}{2},j-\frac{1}{2}} \left(A^i - A^{i-\frac{1}{2}}\right) - \beta_B^{i-\frac{1}{2},j-\frac{1}{2}} \left(B^{j-1} - B^{j-\frac{1}{2}}\right) \\
&= \frac{1}{2}\frac{\partial^2 V^-(\tau, \widehat{k_A^i}, \widehat{k_B^j})}{\partial A^2} \left(A^i - A^{i-\frac{1}{2}}\right)^2 + \frac{\partial^2 V^-(\tau, \widehat{k_A^i}, \widehat{k_B^j})}{\partial A \partial B} \left(A^i - A^{i-\frac{1}{2}}\right) \left(B^{j-1} - B^{j-\frac{1}{2}}\right) \\
&+ \frac{1}{2}\frac{\partial^2 V^-(\tau, \widehat{k_A^i}, \widehat{k_B^j})}{\partial B^2} \left(B^{j-\frac{1}{2}} - B^{j-1}\right)^2.
\end{aligned}$$

With those expressions, the maximal error (2.3.32) becomes:

$$\left|V^{-u}(\tau, A, B) - V^{-l}(\tau, A, B)\right| \leq \tag{2.3.34}$$

$$\begin{aligned}
&\sum_{i=1}^{M_A}\sum_{j=1}^{M_B} \max_{\substack{k_A^i, \widetilde{k_A^i}, \widehat{k_A^i} \in (A^{i-1}, A^i), \\ k_B^j, \widetilde{k_B^j}, \widehat{k_B^j} \in (B^{j-1}, B^j)}} \left\{ \begin{array}{c} \left|R_1(A^i, B^j)\right|, \\ \left|R_1(A^{i-1}, B^j)\right|, \\ \left|R_1(A^i, B^{j-1})\right|, \\ \left|R_1(A^{i-1}, B^j) + R_1(A^i, B^{j-1}) - R_1(A^i, B^j)\right| \end{array} \right\} \begin{array}{c} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{array} \\
&+ \sum_{i=1}^{M_A} \max_{k_A^i, \widetilde{k_A^i}, \widehat{k_A^i} \in (A^{i-1}, A^i)} \left\{ \begin{array}{c} \left|R_1(A^i, B^{M_B})\right| + \left(1 - \beta_B^{i-\frac{1}{2}, M_B - \frac{1}{2}}\right)\left(B - B^{M_B}\right), \\ \left|R_1(A^{i-1}, B^{M_B})\right| + \left(1 - \beta_B^{i-\frac{1}{2}, M_B - \frac{1}{2}}\right)\left(B - B^{M_B}\right) \end{array} \right\} \begin{array}{c} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \geq B^{M_B}\}} \end{array} \\
&+ \sum_{j=1}^{M_B} \max_{k_B^j, \widetilde{k_B^j}, \widehat{k_B^j} \in (B^{j-1}, B^j)} \left\{ \begin{array}{c} \left|R_1(A^{M_A}, B^j)\right| + \left(1 - \beta_A^{M_A-\frac{1}{2}, j - \frac{1}{2}}\right)\left(A - A^{M_A}\right), \\ \left|R_1(A^{M_A}, B^{j-1})\right| + \left(1 - \beta_A^{M_A-\frac{1}{2}, j - \frac{1}{2}}\right)\left(A - A^{M_A}\right) \end{array} \right\} \begin{array}{c} 1_{\{A \geq A^{M_A}\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{array} \\
&+ \Bigg[\left|R_1(A^{M_A}, B^{M_B})\right| + \left(2N\left(\tfrac{1}{2}\sigma\sqrt{T-\tau}\right) - \beta_A^{M_A-\frac{1}{2}, M_B-\frac{1}{2}}\right)\left(A - A^{M_A}\right) \\
&+ \left(2N\left(\tfrac{1}{2}\sigma\sqrt{T-\tau}\right) - \beta_B^{M_A-\frac{1}{2}, M_B-\frac{1}{2}}\right)\left(B - B^{M_B}\right)\Bigg] \begin{array}{c} 1_{\{A \geq A^{M_A}\}} \\ 1_{\{B \geq B^{M_B}\}} \end{array}.
\end{aligned}$$

For the uniform discretisation[10] (2.3.17), $A^i - A^{i-\frac{1}{2}} = A^{i-\frac{1}{2}} - A^{i-1} = \frac{\Delta A}{2}$ and $B^j - B^{j-\frac{1}{2}} = B^{j-\frac{1}{2}} - B^{j-1} = \frac{\Delta B}{2}$, so we can find the bound for all Taylor reminder terms (2.3.33), by using $|x + y \pm z| \le |x| + |y| + |z| \le 3\max\{|x|, |y|, |z|\}$, as

$$\left|R_1(A^i, B^j)\right| \le \frac{1}{2}\left(\frac{\max\{\Delta A, \Delta B\}}{2}\right)^2 \cdot 3\eta_i$$

where

$$\eta_i \equiv \max_{\substack{k_A^i \in (A^{i-1}, A^i), \\ k_B^i \in (B^{j-1}, B^j)}} \left\{\left|\frac{\partial^2 V^-(\tau, k_A^i, k_B^j)}{\partial A^2}\right|, \; 2\left|\frac{\partial^2 V^-(\tau, k_A^i, k_B^j)}{\partial A \partial B}\right|, \; \left|\frac{\partial^2 V^-(\tau, k_A^i, k_B^j)}{\partial B^2}\right|\right\}.$$

Thus, (2.3.34) becomes

$$\begin{aligned}
&\left|V^{-u}(\tau, A, B) - V^{-l}(\tau, A, B)\right| \le \qquad (2.3.35) \\
&\frac{3}{2}\left(\frac{\max\{\Delta A, \Delta B\}}{2}\right)^2 \sum_{i=1}^{M_A+1} \sum_{j=1}^{M_{B+1}} \max\ \eta_i \cdot 1_{\{A \in [A^{i-1}, A^i)\}} \cdot 1_{\{B \in [B^{j-1}, B^j)\}} \\
&+ \sum_{i=1}^{M_A}\left[\left(1 - \beta_B^{i-\frac{1}{2}, M_B - \frac{1}{2}}\right)(B - B^{M_B})\right] \begin{matrix} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \ge B^{M_B}\}} \end{matrix} + \sum_{j=1}^{M_B}\left[\left(1 - \beta_A^{M_A - \frac{1}{2}, j - \frac{1}{2}}\right)(A - A^{M_A})\right] \begin{matrix} 1_{\{A \ge A^{M_A}\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{matrix} \\
&+ \left[\left(2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) - \beta_A^{M_A - \frac{1}{2}, M_B - \frac{1}{2}}\right)\left(A - A^{M_A}\right) + \left(2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) - \beta_B^{M_A - \frac{1}{2}, M_B - \frac{1}{2}}\right)\left(B - B^{M_B}\right)\right] \begin{matrix} 1_{\{A \ge A^{M_A}\}} \\ 1_{\{B \ge B^{M_B}\}} \end{matrix}.
\end{aligned}$$

We thus conclude that an upper bound for the error is as given in (2.3.30), where

$$\begin{aligned}
h\left(\tau, A, B, A^{M_A}, B^{M_B}\right) = &\sum_{i=1}^{M_A}\left[\left(1 - \frac{\partial V^-}{\partial B}(\tau, A^{i-\frac{1}{2}}, B^{M_B - \frac{1}{2}})\right)\left(B - B^{M_B}\right)\right] \begin{matrix} 1_{\{A \in [A^{i-1}, A^i)\}} \\ 1_{\{B \ge B^{M_B}\}} \end{matrix} \\
&+ \sum_{j=1}^{M_B}\left[\left(1 - \frac{\partial V^-}{\partial A}(\tau, A^{M_A - \frac{1}{2}}, B^{j - \frac{1}{2}})\right)\left(A - A^{M_A}\right)\right] \begin{matrix} 1_{\{A \ge A^{M_A}\}} \\ 1_{\{B \in [B^{j-1}, B^j)\}} \end{matrix} \qquad (2.3.36) \\
&+ \left[\left(2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) - \frac{\partial V^-}{\partial A}(\tau, A^{M_A - \frac{1}{2}}, B^{M_B - \frac{1}{2}})\right)\left(A - A^{M_A}\right)\right. \\
&+ \left.\left(2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right) - \frac{\partial V^-}{\partial B}(\tau, A^{M_A - \frac{1}{2}}, B^{M_B - \frac{1}{2}})\right)\left(B - B^{M_B}\right)\right] \begin{matrix} 1_{\{A \ge A^{M_A}\}} \\ 1_{\{B \ge B^{M_B}\}} \end{matrix}
\end{aligned}$$

is the error term which stems from truncation, i.e. the choice of parameters A^{M_A} and B^{M_B}. In order to establish an error bound at pricing time $t < \tau$, we use (2.3.14):

$$\left|V^{-u}(t, A_t, B_t) - V^{-l}(t, A_t, B_t)\right| = e^{-r(\tau - t)}\mathbb{E}_t^{\mathbb{Q}}\left[\left|V^{-u}(\tau, A_\tau, B_\tau) - V^{-l}(\tau, A_\tau, B_\tau)\right|\right],$$

[10] Please note that also in this two-dimensional case there is no specific reason for taking a uniform grid. It can be established that the proof would go through with a non-uniform grid, i.e. similar result would also hold in terms of the maximal grid distances $\max(\Delta A)$ and $\max(\Delta B)$, at the cost of more cumbersome notation, since second derivatives are now matrices.

which means we take a time-t-conditional risk-neutral expectation of the entire expression (2.3.30), and thus also of the truncation error term (2.3.36). If we now also take the limit $\lim_{A^{M_A}\to\infty,\, B^{M_B}\to\infty} \mathbb{E}_t^{\mathbb{Q}}\left[h\left(\tau, A_\tau, B_\tau, A^{M_A}, B^{M_B}\right)\right]$, the individual terms which appear in the summations become zero, e.g.[11]

$$\lim_{A^{M_A}\to\infty}\left(1-\frac{\partial V^-}{\partial A}(\tau, A^{M_A},\, .)\right)\mathbb{E}_t^{\mathbb{Q}}\left[\left(A_\tau - A^{M_A}\right)^+\right]=0.$$

This is due to the fact that expectation $\mathbb{E}_t^{\mathbb{Q}}\left[\left(A_\tau - A^{M_A}\right)^+\right]$ is nothing else but the value of a call option at t, on the underlying asset A, with maturity τ, and strike A^{M_A}, and this converges to zero when the strike goes to infinity (irrespective of where the value of the other underlying asset B_τ happens to be). This reasoning can be applied to all four terms in the above expression (2.3.36) and thus the error term $\mathbb{E}_t^{\mathbb{Q}}\left[h\left(\tau, A_\tau, B_\tau, A^{M_A}, B^{M_B}\right)\right]$ goes to zero as A^{M_A}, $B^{M_B}\to\infty$.

Also, for the deltas of the Ouwehand-West option $V^-(\tau, A, B)$, from Corollary 2.2 we have $\lim_{A^{M_A}\to\infty}\frac{\partial V^-}{\partial A}(\tau, A^{M_A},\, .)=1$ and $\lim_{B^{M_B}\to\infty}\frac{\partial V^-}{\partial B}(\tau,\, .\, , B^{M_B})=1$, whereas from Corollary 2.2: $\lim_{A^{M_A}=B^{M_B}\to\infty}\frac{\partial V^-}{\partial A}(\tau, A^{M_A}, B^{M_B}) = \lim_{A^{M_A}=B^{M_B}\to\infty}\frac{\partial V^-}{\partial B}(\tau, A^{M_A}, B^{M_B}) = 2N\left(\frac{1}{2}\sigma\sqrt{T-\tau}\right)$.[12] As a matter of fact, the stronger result

$$\lim_{A^{M_A}\to\infty}\left(1-\frac{\partial V^-}{\partial A}(\tau, A^{M_A},\, .)\right)A^{M_A}=0$$

holds, which is shown in Appendix A.2.1.2.

Therefore, at time $t<\tau$, in the limit, the entire error of the method will be proportional to the square of the discretisation step, $(\max\{\Delta A, \Delta B\})^2$. ■

The other bivariate option types

Using the price of the call-on-max option written on the discrete dividend paying assets, we will now derive the prices for the other three option types. To do so, we will investigate which max-min/put-call parity formulae (2.3.11), (2.3.12), (2.3.13) of Stulz (1982) still hold in the dividend paying case. We first prove a max-min parity relationship.

Theorem 2.3. *At time t, ($t<\tau<T$), the price of a bivariate European call on the minimum option, with strike K and maturity T, written on the underlying assets A and B as specified above by* (2.3.1)*, is*

$$V_{min}^{Call-2D}(t,\, A,\, B,\, K) = V^{Call-1D}(t,\, A,\, K) + V^{Call-1D}(t,\, B,\, K) - V_{max}^{Call-2D}(t,\, A,\, B,\, K), \tag{2.3.37}$$

where $V_{max}^{Call-2D}(t,\, A,\, B,\, K)$ is the price of the bivariate call-on-max, as given by Proposition 2.2, and $V^{Call-1D}(t,\, A,\, K)$ and $V^{Call-1D}(t,\, B,\, K)$ are the prices of the call options (2.2.2)*, with underlying assets A and B, respectively, all with the same strike K and maturity T.*

[11] From the discretisation (2.3.17) and the way mid-points of the discretisation segments are defined, it is clear that when, for example, $A^{M_A}\to\infty$, then also $A^{M_A-\frac{1}{2}}\to\infty$.

[12] An expression for the Ouwehand-West delta is given in Appendix A.2 by (A.2.4), and an alternative derivation of its convergence in Appendix A.2.1.1.

Proof. Let us consider two different portfolios. Portfolio I consists of two bivariate options: a call on the minimum and a call on the maximum of the two assets, whereas Portfolio II consists of two univariate call options - one on each asset. All four options have the same strike K and maturity T. We will now take a look at the payoffs of those portfolios at maturity $T > \tau$, where τ is the time of the discrete dividend payment for both assets A and B. In Table 2.1 we show the payoff for each of the portfolios for every possible order of A_T, B_T and K. Without loss of generality we can assume that $A_T > B_T$. This is allowed because of the symmetry of the function $\max(A_T, B_T)$: if the opposite is the case we can just swap the asset names and the analysis that follows will not be affected. Now we can distinguish the following cases:

Payoffs	$A_T > B_T > K$	$A_T > K > B_T$	$K > A_T > B_T$
$V_{max}^{Call-2D}(T, A, B, K)$	$A_T - K$	$A_T - K$	0
$V_{min}^{Call-2D}(T, A, B, K)$	$B_T - K$	0	0
Portfolio I	$A_T + B_T - 2K$	$A_T - K$	0
$V^{Call-1D}(T, A, K)$	$A_T - K$	$A_T - K$	0
$V^{Call-1D}(T, B, K)$	$B_T - K$	0	0
Portfolio II	$A_T + B_T - 2K$	$A_T - K$	0

Table 2.1: The max-min parity proof.

Since both portfolios have the same payoff structure, due to absence of arbitrage arguments, their values today, at time t, must be the same:

$$V_{min}^{Call-2D}(t, A, B, K) + V_{max}^{Call-2D}(t, A, B, K) = V^{Call-1D}(t, A, K) + V^{Call-1D}(t, B, K),$$

and (2.3.37) follows. The fact that assets pay dividends at τ will not influence this analysis since, by absence of arbitrage, the value of any of the four options must not change across the dividend date. ■

Theorem 2.4. *At time t, $(t < \tau < T)$, the price of a bivariate European put on the maximum option, with strike K and maturity T, written on the underlying assets A and B as specified above by* (2.3.1)*, is*

$$V_{max}^{Put-2D}(t, A, B, K) = e^{-r(T-t)}K - V_{max}^{Call-2D}(t, A, B, 0) + V_{max}^{Call-2D}(t, A, B, K), \quad (2.3.38)$$

and the price of a bivariate European put on the minimum is

$$V_{min}^{Put-2D}(t, A, B, K) = e^{-r(T-t)}K - V_{min}^{Call-2D}(t, A, B, 0) + V_{min}^{Call-2D}(t, A, B, K). \quad (2.3.39)$$

Proof. The proof goes along the same lines as the proof for Theorem 2.3. We again consider two different portfolios. Portfolio I consists of two bivariate options: a put on the maximum with strike K and a call on the maximum with strike 0. The latter option's payoff is simply $(\max(A_T, B_T))^+$. Portfolio II consists of call on the maximum with strike K and $Ke^{-r(T-t)}$ in cash. In Table 2.2 we show the possible payoffs of those portfolios at maturity $T > \tau$. We can again, without loss of generality, assume that $A_T > B_T$ and we have the following cases:

Payoffs	$A_T > B_T > K > 0$	$A_T > K > B_T \geq 0$	$K > A_T > B_T \geq 0$	$K > A_T = B_T = 0$
$V_{max}^{Put-2D}(T, A, B, K)$	0	0	$K - A_T$	K
$V_{max}^{Call-2D}(T, A, B, 0)$	A_T	A_T	A_T	0
Portfolio I	A_T	A_T	K	K
$V_{max}^{Call-2D}(T, A, B, K)$	$A_T - K$	$A_T - K$	0	0
$Ke^{-r(T-t)}$	K	K	K	K
Portfolio II	A_T	A_T	K	K

Table 2.2: The call-put parity proof for max options.

We again have the same payoff for both portfolios and due to the absence of arbitrage arguments, their values today, at time t, must be the same:

$$V_{max}^{Put-2D}(t, A, B, K) + V_{max}^{Call-2D}(t, A, B, 0) = e^{-r(T-t)}K + V_{max}^{Call-2D}(t, A, B, K),$$

so (2.3.38) follows.

The reasoning for the second equality (2.3.13) goes as follows. Portfolio I consists of two bivariate options: a put on the minimum with strike K and a call on the minimum with strike 0. Portfolio II contains a call on the minimum with strike K and $Ke^{-r(T-t)}$ in cash. The payoffs at maturity are given in Table 2.3. We still assume, without loss of generality, that $A_T > B_T$ holds.

Payoffs	$A_T > B_T > K > 0$	$A_T > K > B_T > 0$	$A_T > K > B_T = 0$	$K > A_T > B_T > 0$	$K > A_T \geq B_T = 0$
$V_{min}^{Put-2D}(T, A, B, K)$	0	$K - B_T$	K	$K - B_T$	K
$V_{min}^{Call-2D}(T, A, B, 0)$	B_T	B_T	0	B_T	0
Portfolio I	B_T	K	K	K	K
$V_{min}^{Call-2D}(T, A, B, K)$	$B_T - K$	0	0	0	0
$Ke^{-r(T-t)}$	K	K	K	K	K
Portfolio II	B_T	K	K	K	K

Table 2.3: The call-put parity proof for min options.

Due to absence of arbitrage arguments, the portfolio values at time t must be the same

$$V_{min}^{Put-2D}(t, A, B, K) + V_{min}^{Call-2D}(t, A, B, 0) = e^{-r(T-t)}K + V_{min}^{Call-2D}(t, A, B, K),$$

so (2.3.39) follows. The payment of the dividends at τ will not affect the put-call parities, since the value of the components of the portfolios above can not exhibit any jumps across the dividend date. ■

2.4 Numerical Results

2.4.1 The One-dimensional Case

In this subsection we are going to show the performance of our method for a number of different options written on a single underlying stock paying a discrete dividend.

The results are obtained for the following values of the underlying asset parameters: $S_0 = 55$, $K = 60$, $r = 5\%$, $\sigma = 0.3$, $T = 1$, $\tau = 0.5$, $D = 5$ and $S^M = 250$ for the largest stock price on our numerical grid. The results for different values of the other method parameter M, i.e. the number of discretisation steps, are summarised in the Table 2.4.

M	V upper	V lower	Difference (ε)	CPU time (s)
10	4.675849	3.286415	1.389434	0.0080
20	4.041586	3.649472	0.392114	0.0082
40	3.846169	3.746931	0.099238	0.0085
80	3.796590	3.771706	0.024885	0.0086
160	3.784150	3.777924	0.006226	0.0090
320	3.781038	3.779481	0.001557	0.0098
640	3.780259	3.779870	0.000389	0.0112
1,280	3.780065	3.779967	0.000097	0.0135
2,560	3.780016	3.779992	0.000024	0.0192
5,120	3.780004	3.779998	$6.1 \cdot 10^{-6}$	0.0314
10,240	3.780000	3.779999	$1.5 \cdot 10^{-6}$	0.0459

Table 2.4: The call option values for different values of the parameter M.

Using a Monte Carlo result for the same option with 200,000,000 simulations we have obtained the value 3.7802 with the 95% confidence interval [3.7787, 3.7815] and a CPU time of 543 seconds. The results for different values for the parameter M (100, 200, ... 1000, 2000) are graphically presented in Figure 2.7. We see that our method indeed gives the correct value of the option and it is much more efficient than the Monte Carlo simulation method. Notice that already for 600 discretisation points our method produces price bounds that are closer than those of the Monte Carlo simulation 95% confidence interval for $2 \cdot 10^8$ simulations. For comparison, other Monte Carlo results are given in Table 2.5.

Sim. Num.	V MC	95% Confidence Interval	CPU time (s)
10^3	3.773922	[3.207120, 4.340724]	0.0036
10^4	3.649008	[3.476409, 3.821607]	0.0245
10^5	3.746018	[3.691415, 3.800621]	0.0306
10^6	3.782685	[3.765183, 3.800187]	0.1886
10^7	3.779123	[3.773523, 3.784624]	1.6334
10^8	3.781050	[3.779351, 3.782849]	17.5535

Table 2.5: The call option Monte Carlo values for different number of simulations.

The value of the same option obtained by the Black-Scholes discrete dividend approximation formula of the Escrowed model is 3.4998, which implies an error of around 7.4% with respect to the correct option value.

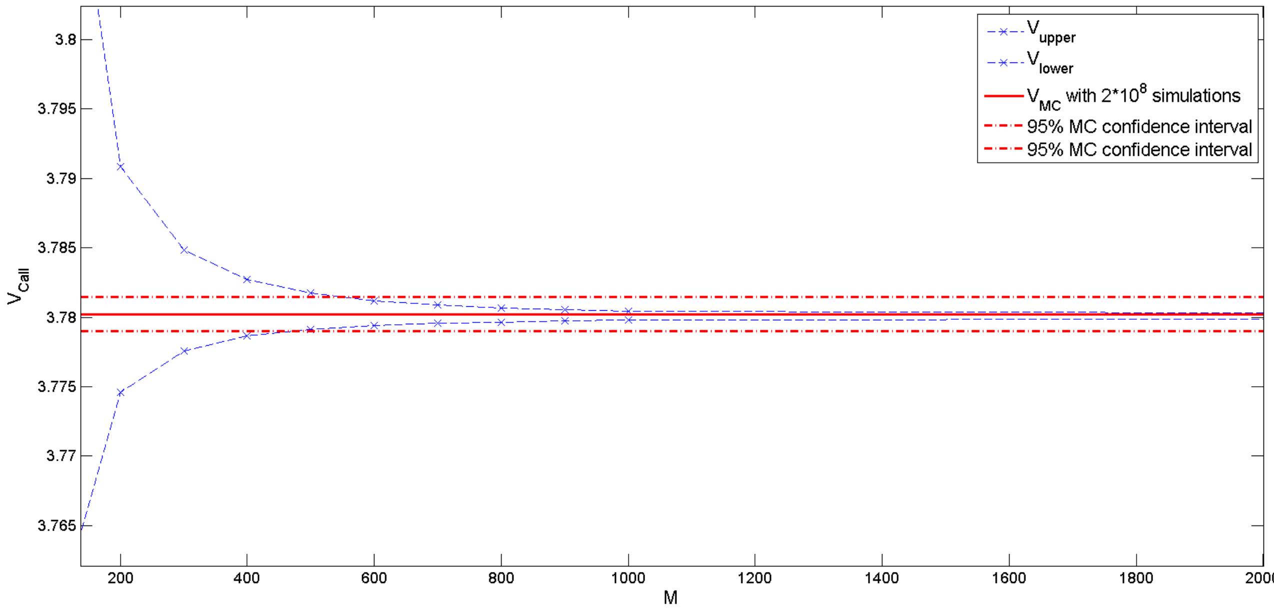

Figure 2.7: The bounds for the price of the option compared to Monte Carlo obtained result.

In order to better illustrate the differences in performance of our method and the Monte Carlo method we present the absolute error versus the CPU time for both methods on a logarithmic scale:

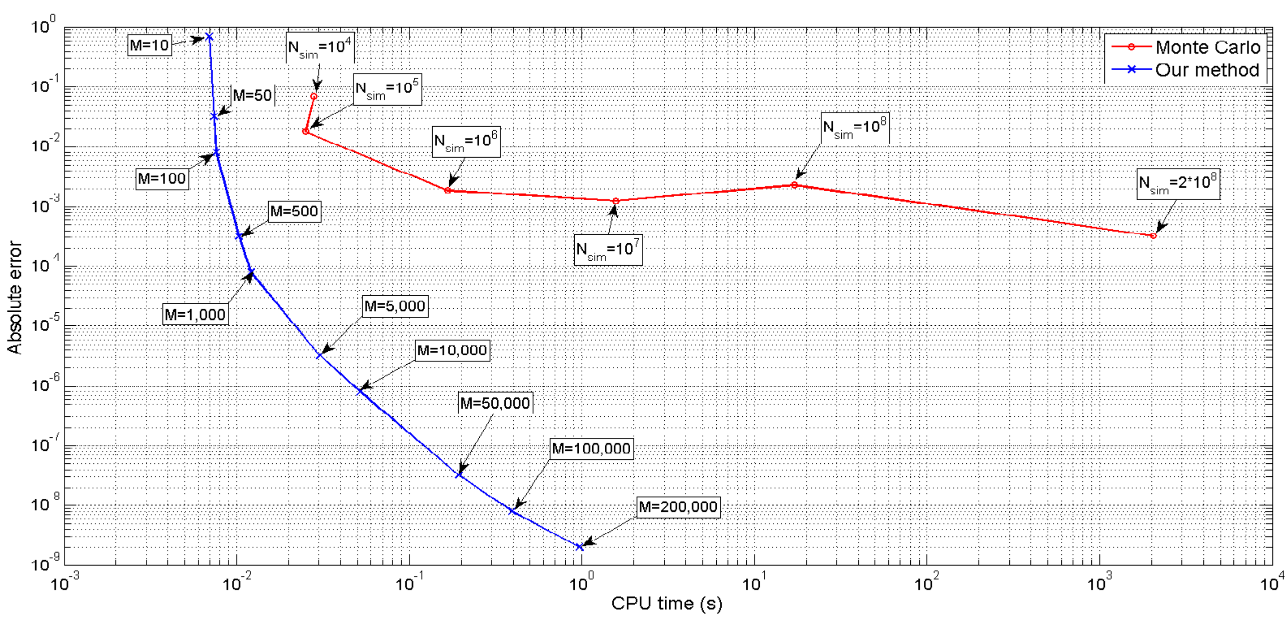

Figure 2.8: The absolute errors of both methods versus the CPU calculation time.

The following numerical test shows the convergence of the method. The error is defined as the maximal difference between the bounds for $V(t, S_t)$ as given by Proposition 2.1, in (2.2.19). We have used different values for the parameters and we have gradually increased the number of discretisation points M up to 10,000, thus making the discretisation step ΔS given by (2.2.5) progressively smaller. Figure 2.9 presents this error versus the parameter M in log-log scale, and as we can see the slope of the lines is exactly -2, as predicted by (2.2.19). In other words, the absolute value of the error of the method converges as ΔS^2.

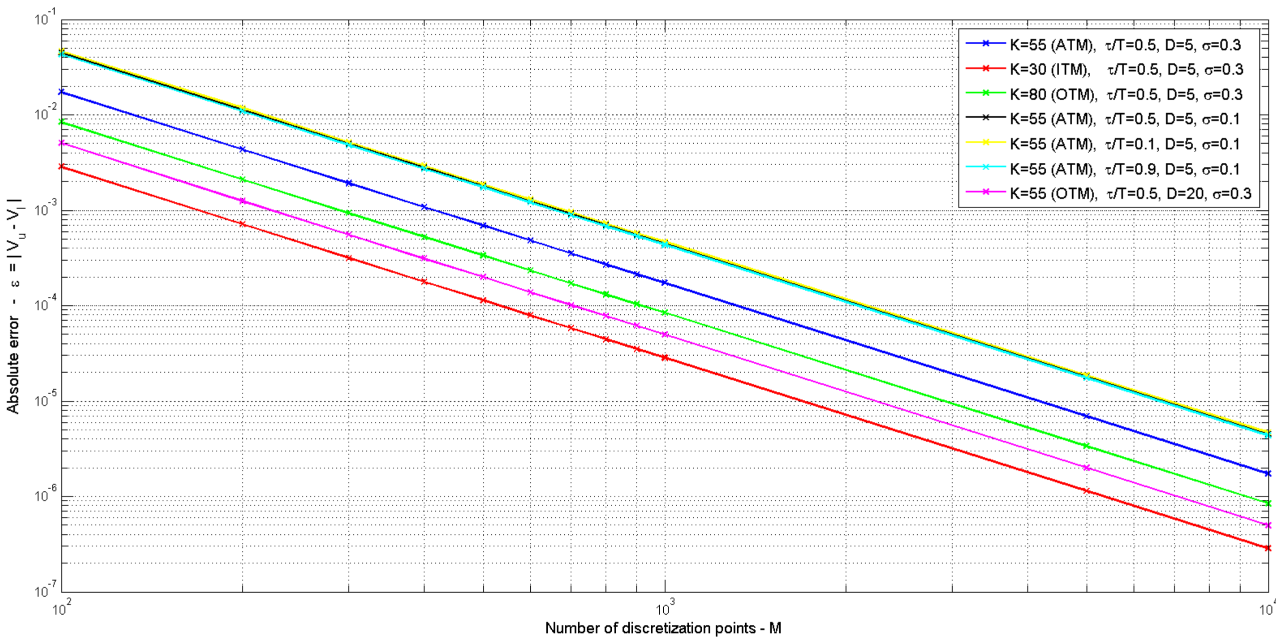

Figure 2.9: The difference between the bounds for the prices of different options versus the number of discretisation points.

The moneyness and maturity of the options were varied as well as the discrete dividend size and the payment time, but the results show that the method performs exceptionally well in all cases. We can see that different parameter values only lead to a parallel shift which means that the speed of convergence of the method is indeed as predicted in all cases, as long as the truncation value is sufficiently large.

However, if the truncation value S^M is not chosen large enough those lines will begin to curve as the second term $h\left(S^M\right)$ in (2.2.19) becomes non-negligible. As a rule of thumb we suggest at least $S^M \approx 10 S_0 \sigma T$ to avoid this undesired behaviour of the error convergence.

2.4.2 The Two-dimensional Case

We now turn to the analysis of the method's performance for a maximum option written on two underlying stocks which both pay discrete dividends.

The values of the underlying asset parameters are now $A_0 = 75$, $B_0 = 55$, $K = 60$, $r = 5\%$, $\sigma_A = 0.2$, $\sigma_B = 0.3$, $\rho = 0.7$, $T = 1$, $\tau = 0.5$, $D_A = 5$, $D_B = 5$ and the method parameters are $A^{M_A} = B^{M_B} = 250$. The other parameters of the method are set to be equal: $M_A = M_B$ and for their different values, results are summarised in the following Table 2.6.

$M_A = M_B$	V upper	V lower	Difference (ε)	CPU time (s)
10	17.400749	13.498299	3.902450	0.26
20	15.205101	14.322471	0.882629	0.48
40	14.686786	14.473807	0.212979	1.05
80	14.563401	14.510900	0.052501	2.17
160	14.533169	14.520109	0.013060	5.98
320	14.525666	14.522408	0.003259	16.69
640	14.523796	14.522982	0.000814	36.39
1,280	14.523329	14.523126	0.000203	161.52
2,560	14.523212	14.523161	0.000051	514.19
5,120	14.523183	14.523170	0.000013	1990.2
10,240	14.523176	14.523173	$3.2 \cdot 10^{-6}$	5243.7

Table 2.6: The call-on-max bivariate option values for different values of the parameters $M_A = M_B$.

The same option was also priced by a Monte Carlo method with 200,000,000 simulations and the result was 14.5232 with the 95% confidence interval [14.5213, 14.5250] and the CPU calculation time was 1938 seconds (32 minutes). This is shown in Figure 2.10 together with the bounds produced by our method, obtained for discretisation steps: 100, 200, ... 1000.

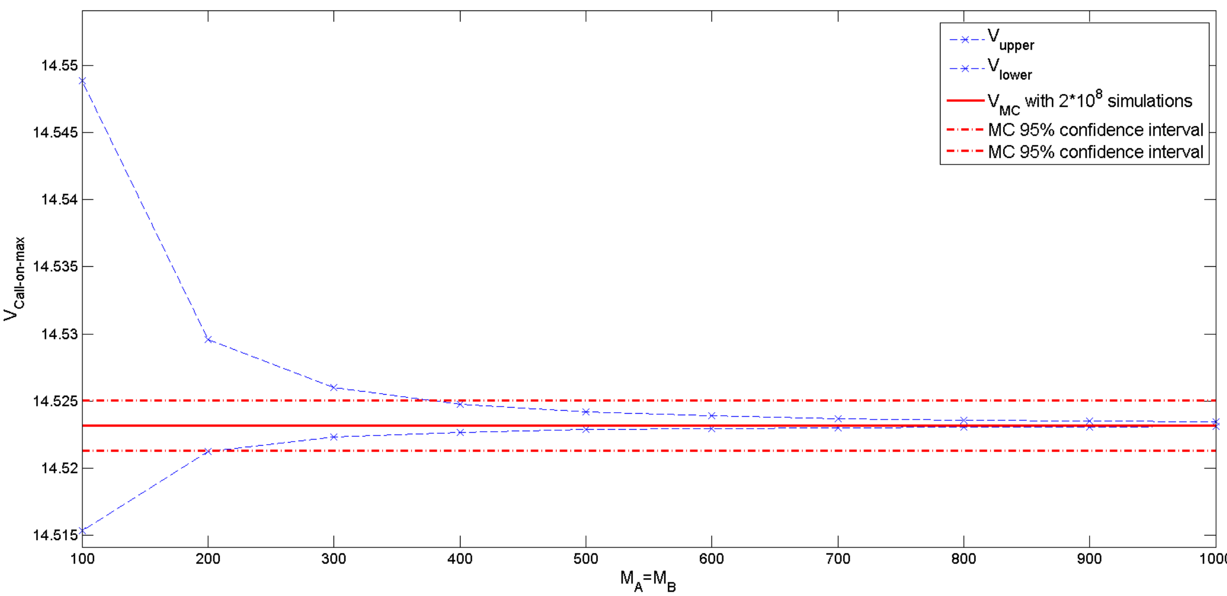

Figure 2.10: The bounds for the price of the bivariate option compared to the Monte Carlo result.

We see that the Monte Carlo result converges to the value obtained by our method but it is inferior in both the error introduced and in the calculation time spent. The result which our method reaches for 500 discretisation points in just 31 seconds can be obtained by the Monte Carlo method only when using $1.5 \cdot 10^8$ simulations in about 20 minutes of calculation time. The other examples of Monte Carlo results with their respective CPU times are given in Table 2.7.

Sim. Num.	V MC	95% Confidence Interval	CPU time (s)
10^3	14.621109	[13.755309, 15.487001]	0.0019
10^4	14.380231	[14.118832, 14.641530]	0.0050
10^5	14.500715	[14.416514, 14.584816]	0.0441
10^6	14.519798	[14.493299, 14.546400]	0.3005
10^7	14.524123	[14.515722, 14.532524]	3.3945
10^8	14.522501	[14.514102, 14.530900]	57.3142

Table 2.7: The call-on-max bivariate option Monte Carlo values for different number of simulations.

Again, for easier comparison of the performance of our method versus the Monte Carlo method, we show in Figure 2.11 the absolute error for both methods versus the CPU time.

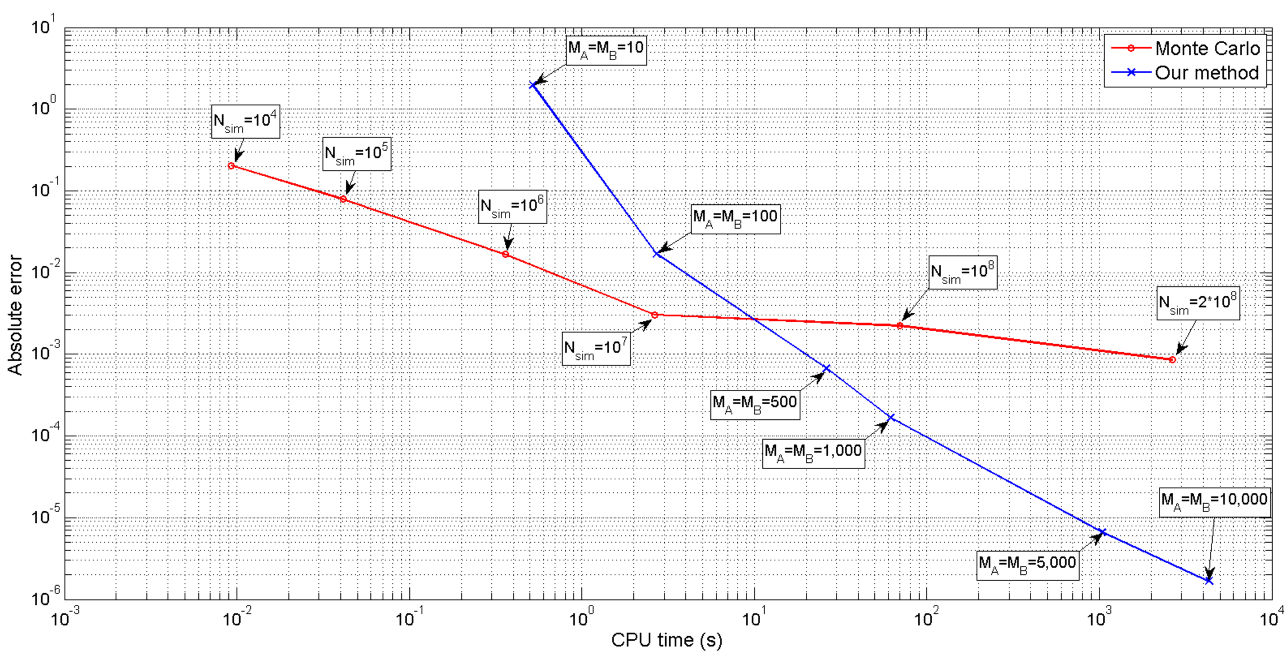

Figure 2.11: The absolute errors of both methods versus the CPU calculation time.

We have also performed a convergence analysis for the error of the method in the two-dimensional case. The error is defined as the absolute value of the maximal difference between the bounds for $V(t, A, B)$, given by Proposition 2.3. Figure 2.12 presents numerical results for this error, with an increasing number of discretisation points $M_A = M_B$ and with equal discretisation steps in both spatial dimensions, i.e. $\Delta A = \Delta B$.[13] In this case the slope is again exactly -2, as predicted by Proposition 2.3, equation (2.3.30).

[13]See discretisation formulae (2.3.17): we choose $M_A = M_B$, as well as $A^{M_A} - D_A = B^{M_B} - D_B$, which is already prescribed by the method.

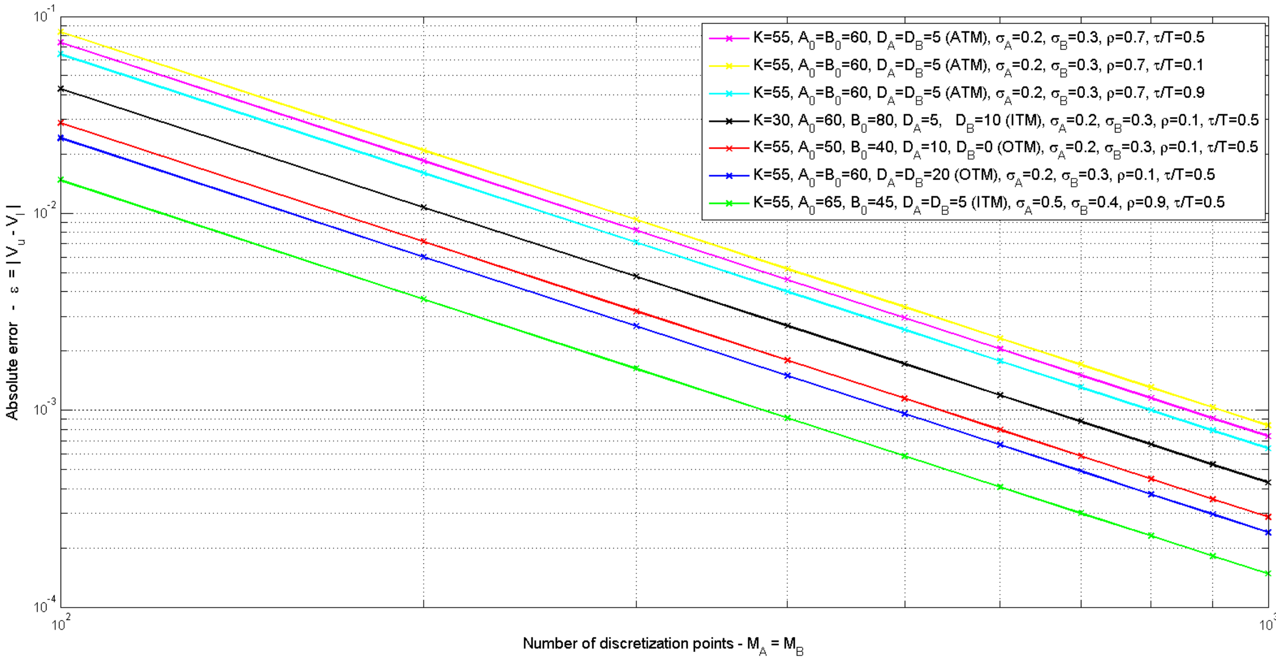

Figure 2.12: The difference between the bounds for the prices of different options versus the number of discretisation points.

The maturities and strike prices of options as well as the sizes of discrete dividends and their payment time were all varied. The results in Figure 2.12 show that also in the two-dimensional case the model performs extremely well for different parameter values.

2.5 Conclusion

We have presented a method for fast calculation of certain European option prices on stocks paying a discrete dividend. It is based on the fact that the option values are convex functions of the underlying asset prices at the dividend date with bounded derivatives, and hence piecewise linear upper and lower bounds can be constructed.

First, we have proven for the case with one underlying asset that the maximal error of the method scales with the gamma of the option, and that it is directly proportional to the square of the discretisation step used. This allows us to choose the input parameters in such a way that we achieve a desired precision.

Secondly, we have extended the methodology to the bivariate call on the maximum option when both underlying assets pay a known discrete dividend at a known time. We showed that the maximal error scales with $(\max\{\Delta A, \Delta B\})^2$, where ΔA and ΔB are the discretisation steps in each dimension. The values of other types of bivariate options can then be obtained by the max-min/put-call parities of Stulz (1982), which are shown to generalise to the dividend paying case.

The obvious limitation of our method is that it can be used only for cases in which there is only one known dividend payment date τ, at which all cash values of the dividends to be paid are known exactly. It cannot be used if the underlying stock pays two discrete dividends during the life of the option, or if the two underlying assets pay their respective dividends at two different times. This is due to the fact that after the option is priced for times between dividend payments, i.e. for t with $\tau_1 \le t < \tau_2$, the price $V^+(\tau_1, A_{\tau_1}, B_{\tau_1})$ will be given by Proposition 2.1 or 2.2, and the result must then be plugged again into an equation such as (2.2.2) or (2.3.14), to find values for $t < \tau_1$:

$$V(t, A_t, B_t) = e^{-r(\tau_1 - t)} \mathbb{E}_t^{\mathbb{Q}} \left[V^-(\tau_1, A_{\tau_1^-}, B_{\tau_1^-}) \right], \quad t < \tau_1,$$

for which we have not found an explicit bound.

However, in those cases it is possible to use $V^+(\tau_1, A_{\tau_1}, B_{\tau_1})$ obtained by our method as a control variate for the more accurate Monte Carlo simulation. Our method can also be extended to price an option written on more than two underlying assets (a classical basket option, which is indeed more commonly traded) as long as it has a convex payoff. But to obtain a closed form solution, all of the underlying assets that pay dividends during the lifetime of the option would need to pay them at the same time τ.

In practice, options exist on underlying stocks even before companies declare the expected dividend size. In those cases our model can actually be used "inversely", i.e. if we observe an option price in the market, possibly obtained by using a different (unknown) model, we can replicate it by our model and thus obtain an "implied" fixed dividend size D, which could be useful information for a trader to try and take advantage of the market, e.g. make an informed decision whether to buy/sell/hold the underlying stock. This inverse relation $V(t, S_t) \to D$ would not be in a simple explicit form since, through discretisation (2.2.5), D figures in many different parts of the expression given by Proposition 2.1, but it could certainly be obtained numerically through a couple of iterations, in a similar way the classical Black-Scholes formula is actually almost exclusively used to deduce implied volatility.

As pointed out by Haug, Haug and Lewis (2003), most of the known derivatives pricing methodologies for discrete dividend paying underlying assets deal with variations of the Black-Scholes model. Limitations of that model are very well known and many more advanced models now exist. If the underlying dynamics would be assumed to be that of any alternative model, this would lead to the same pricing formula (2.2.2), with different $V^-(t, S)$ at $t = \tau$. As long as the function $S \to V^+\left((S - D)^+, K, T - \tau\right)$ is convex, it would be possible to further continue with the construction of the numerical grid (2.2.5) and application of our method. The quantities $\alpha^i, \xi^i, \beta^{i-\frac{1}{2}}, \chi^{\beta^{i-\frac{1}{2}}}$ in the expressions for the bounds (2.2.15) and (2.2.18), at $t < \tau$, would remain the same, whereas the functions f^i and g^i would instead be expressed in terms of the specific probabilities $P_1(t, S, K, T)$ and $P_2(t, S, K, T)$:

$$f^i(t, S_t) = S_t e^{r(\tau - t)} \left[P_1(t, S_t, S^{i-1}, \tau) - P_1(t, S_t, S^i, \tau) \right], \tag{2.5.1}$$

$$g^i(t, S_t) = P_2(t, S_t, S^{i-1}, \tau) - P_2(t, S_t, S^i, \tau), \tag{2.5.2}$$

where $P_j(t, S, K, T) = \mathbb{Q}_j \left\{ \ln S_T \ge \ln K \mid S_t = S \right\}$, i.e. probabilities under the stock martingale measure, $j = 1$, and the T-forward martingale measure, $j = 2$ (which in the case of deterministic interest rates is actually just the standard risk-neutral measure), see e.g. Geman, El Karoui and Rochet (1995).

An appropriate alternative model could come from the class of exponential Lévy models, which offer analytically tractable positive jump processes, see e.g Cont and Tankov (2003). An exponential Lévy model is given by

$$S_T = S_t e^{(r+\omega)(T-t)+L_T-L_t},$$

where L is a Lévy process, r is the constant risk-free rate and ω is the compensator chosen to ensure that the discounted price process is a martingale. Since Lévy processes have characteristic functions available in closed form, this allows us to use Fourier transform methods for option pricing. There are a number of effective solutions available, e.g. using fast Fourier transform as in Carr and Madan (1999), or the COS method of Fang and Oosterlee (2008), which is based on Fourier cosine series expansions. The Bakshi and Madan (2000) approach, directly gives us the specific probabilities $P_j(t, S, K, T)$, $j = 1, 2$ in terms of Lévy's inversion formula.[14] This approach is applied to Merton's (1976b) jump-diffusion model in Minenna and Verzella (2008) and they generalise it to the entire class of exponential Lévy models (affine jump-diffusion models).[15] Madan, Carr and Chang (1998) consider a Lévy model with the Variance Gamma process and obtain a solution for $P_j(t, S, K, T)$, $j = 1, 2$ in terms of the modified Bessel function of the second kind. Other examples of exponential Lévy models are Normal Inverse Gaussian models, studied in Rydberg (1997), the KoBoL model of Boyarchenko and Levendorskii (2002)[16], the CGMY process of Carr *et al.* (2002) and the model of Kou (2002), which is another jump-diffusion model, where the distribution of jump sizes is taken to be an asymmetric exponential, as opposed to normally distributed jump sizes in Merton (1976b).

These approaches would allow fast and efficient pricing of the options on discrete dividend paying stocks presented in this thesis while giving the advantages of more advanced models for option pricing, such as inclusion of the volatility smile.

[14] Actually its variation: $P_j(t, S, K, T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\ln K}}{iu}\varphi_t^j(\ln S_T, u)\right] du$, known as Gil-Pelaez inversion formula, see e.g. Wendel (1961), for the characteristic function $\varphi_t^j(\ln S_T, u) = \mathbb{E}_t^{\mathbb{Q}_j}\left[e^{iu\ln S_T}\right]$, $j = 1, 2$.

[15] A stochastic volatility model, such as the one proposed by Heston (1993), also gives us an explicit solution obtained through Fourier transform methods. However, our method would not work under this particular model because the functions f^i and g^i will not be known in closed form since they will also depend on the volatility stochastic process at the dividend time.

[16] Initially introduced by Koponen (1995) under the name *truncated Lévy flight*, now also known as *the tempered stabe process,* see e.g Cont and Tankov (2003).

Chapter 3

Fast Calculation of Credit Value Adjustments for Inflation-linked Derivatives

3.1 Introduction

Counterparty risk has become increasingly important after the financial crisis of 2008, due to the new regulatory frameworks that have emerged after this crisis.[1] Such frameworks often require that a price for counterparty risk is charged on top of the standard default-free price of financial derivatives. This Credit Value Adjustment (CVA) is basically an addition to the price of a derivative to account for the fact that a counterparty is not default-free.

Most of the initial work on CVA calculations focused on interest rate derivatives since these are often traded over-the-counter and may thus carry a significant amount of counterparty risk. Results were first derived for standard interest rate swaps and these were then extended to portfolios of such swaps under netting agreements, see, for example, the work of Brigo and Masetti (2005). Today there is a myriad of counterparty models and methods to obtain the CVA for most asset classes, and these are used in banks all over the world. A good starting point for an overview of different methods is the Brigo, Morini and Pallavicini (2013) book.

In this thesis we focus on inflation derivatives, an asset class for which not much work seems to have been done on the CVA calculations. Pricing models for inflation derivatives are more complex than those for standard interest rate swaps since both real interest rates and (standard) nominal interest rates need to be included in the model. Well-known approaches in the scientific literature include Barone and Castagna (1997), who priced the very first inflation derivatives issued in US in January 1997 called TIPS (Treasury Inflation-Protected Securities) by extending the CIR short-rate model of Cox, Ingersoll, Ross (1985) for interest rate derivatives, followed by the doctoral thesis of Kazziha (1999)[2], Jarrow and Yildirim (2003), and market models of Belgrade, Benhamou and Koehler (2004) and Mercurio (2005).

[1]For a good overview, see e.g. Gregory (2010).

[2]This was the first work which imposes lognormal dynamics for the forward price of real zero-coupon bonds.

We will use the Jarrow-Yildirim (JY) model in this paper. In a similar way as in Barone and Castagna (1997), it exploits an analogy with derivatives on foreign currency, in the sense that the nominal rate is regarded as an interest rate on a "domestic" currency, the real rate as an interest rate on a "foreign" currency and the inflation index as the "exchange rate" between the currencies. However, the Jarrow-Yildirim model uses a different one-factor short-rate model developed by Hull and White (1990), which makes it an intuitive and natural extension of this classical model for interest rate derivatives. The dynamics for both the nominal and the real instantaneous short-rate processes have the same form as in the Hull-White model. The inflation index is assumed to follow geometric Brownian motion and the short rates and the rate of inflation can all be correlated. As in the Hull-White model, these modelling choices allow the derivation of explicit pricing formulae for the value of real and nominal bonds in terms of the current short rates and the current inflation rate. This makes calibration of such models to fixed income market data relatively easy. More advanced market models such as the Lognormal Forward-Libor Market Model in Mercurio (2005) or a similar approach of Belgrade, Benhamou and Koehler (2004) allow for more degrees of freedom in the calibration process, but at the price of considerably greater complexity.[3] For a good overview of these and other methods to model inflation derivatives, see the book by Brigo and Mercurio (2006).

The Zero-coupon Inflation Indexed Swaps (ZCIIS) that we want to price under counterparty risk also have payoffs for which the Jarrow-Yildirim model allows an exact closed form solution. For the related but more complicated payoffs of Year-on-Year Inflation Indexed Swaps (YYIIS) we will derive a semi-analytical approximation using moment matching. We will also show how a similar method can be applied to a whole portfolio of ZCIIS instruments if the portfolio consists entirely of payer swaps or receiver swaps.

3.2 Product and Model Specifications

We will now briefly discuss different types of inflation derivatives and pricing methods without counterparty risk adjustments.

3.2.1 Inflation-Linked Derivatives

The zero coupon inflation indexed swap (ZCIIS) exchanges two cash flows at the final time T: the fixed amount

$$N((1+K)^T - 1)$$

and the floating amount

$$N\left(\frac{I(T)}{I(0)} - 1\right),$$

[3] If we would have considered pricing of Inflation Indexed Caps or Floors, it would have been more natural to consider a volatility smile effect as it is done in other developed option markets. In that case the Jarrow-Yildirim model would no longer suffice and we would need to turn to different model classes such as those who allow stochastic volatilities, Mercurio and Moreni (2006, 2009), or jump-diffusion, Hinnerich (2008).

where $I(t)$ is the value of an underlying inflation index at time t. The time of maturity T will always be expressed in years and usually it is an integer number. The quantity N is the nominal value and K is the contract's fixed rate which is established at the beginning of the contract period. ZCIIS instruments are usually quoted in the market in terms of the par rate K, i.e. the rate at which the initial contract value equals zero. The payer ZCIIS swap[4] payoff at maturity in nominal terms is therefore

$$\widehat{\Pi}(T) = N\left(\frac{I(T)}{I(0)} - (1+K)^T\right)$$

and assuming that standard asset pricing methods for arbitrage-free and complete markets can be applied, the price of the instrument at time t can be written as

$$\text{ZCIIS}\left(t, T, I\left(0\right), N\right) = \mathbb{E}_t^{\mathbb{Q}_n}\left[D(t,T)\widehat{\Pi}(T)\right] = N\mathbb{E}_t^{\mathbb{Q}_n}\left[D(t,T)\left(\frac{I(T)}{I(0)} - (1+K)^T\right)\right], \tag{3.2.1}$$

where $D(t,T)$ is the nominal stochastic discount factor from time T to time t. The notation $\mathbb{E}_t^{\mathbb{Q}_n}[\cdot]$ is short for $\mathbb{E}^{\mathbb{Q}_n}[\cdot|\mathcal{F}_t]$ which is the expectation under the unique nominal martingale measure $\mathbb{Q}_n$, conditioned on the filtration $\mathcal{F}_t$ that is generated by the stochastic processes which drive the market and which will be specified later on. We will later make use of the unique real martingale measure $\mathbb{Q}_r$ as well but we simplify notation by writing $\mathbb{Q} \equiv \mathbb{Q}_n$, i.e. when no subscript is given, we refer to the nominal martingale measure.

The year-on-year inflation indexed swap (YYIIS) exchanges cash flows at given dates $T_i \in \{T_1, T_2, ..., T_M = M\}$ equal to the a priori fixed amount

$$N\varphi_i K$$

and the floating amount

$$N\varphi_i\left(\frac{I(T_i)}{I(T_{i-1})} - 1\right),$$

where φ_i is the year fraction between T_{i-1} and T_i. It usually has a value around 1, which implies an exchange of cashflows every year. Notice that the inflation index at the end of each period is expressed relative to the value of the inflation index at the beginning of that period and not relative to the inflation at the inception of the contract, $T_0 = 0$. The price at time t of the payer YYIIS instrument is thus given by

$$\text{YYIIS}\left(t, T_0, \{T_i, \varphi_i\}_{i=1,\ldots,M}, N\right) = N\sum_{i=1}^{M}\varphi_i\mathbb{E}_t^{\mathbb{Q}_n}\left[D(t,T_i)\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - K\right)\right]. \tag{3.2.2}$$

3.2.2 Credit Value Adjustment

The *CVA* for any derivative contract position is defined as[5]

$$\text{CVA} = (1-R)\,\mathbb{E}_0^{\mathbb{Q}}\left[1_{\{\tau\le T\}}D(0,\tau)E(\tau)\right]. \tag{3.2.3}$$

[4]The holder of the *payer ZCIIS* pays a fixed rate and receives floating rates, and the holder of the *receiver ZCIIS* receives a fixed rate and pays floating rates.

[5]See, for example, Brigo, Morini, Pallavicini (2013), Chapter 2.3, page 35.

The constant R is the recovery rate that is usually taken to have deterministic (which is not very realistic) value of around 40%.[6] The random variable τ is the stochastic first time of counterparty default and $1_{\{\tau \leq T\}}$ is the indicator function that takes value 1 if $\tau \leq T$ and zero otherwise. The notation $D(0, \tau)$ for the nominal stochastic discount factor from time τ until the initial time was introduced earlier.

The Exposure $E(t)$ at some future time $0 \leq t \leq T$ is defined as

$$E(t) = V(t)^+ \equiv \max\{0, V(t)\}, \qquad V(t) = \mathbb{E}_t^{\mathbb{Q}}[\Pi(t,T)], \tag{3.2.4}$$

with $\Pi(t,T)$ the discounted value of all payoffs between t and the maturity of the contract, T. For a vanilla swap this is the discounted sum of the random cash-flows on all different payment dates between the current time t and maturity. When derivatives generate a payoff only at maturity (which is the case for the ZCIIS), this quantity simply becomes $D(t,T)$, i.e. the discount factor from T to t, times the payoff at maturity.

Using an expectation which is conditional on the event that the random time of default τ happens before the maturity T we have:

$$\text{CVA} = (1-R)\,\mathbb{Q}\left(\tau \leq T\right)\mathbb{E}_0^{\mathbb{Q}}\left[D(0,\tau)(V(\tau))^+ | \tau \leq T\right].$$

The first factor $\mathbb{Q}(\tau \leq T)$ is the risk-neutral probability that the counterparty will default before the expiration of the contract T. The second factor is the expectation under the risk-neutral measure conditional on the event $\{\tau \leq T\}$, i.e. on the event that default occurs before the contract expires. It corresponds to a standard option pricing problem at the present time zero which involves a payoff at the later time τ.

To simplify the calculations, it is common in the financial industry to introduce what is known as a bucketing approximation.[7] We form a partition of $(0, T]$, i.e.

$$0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = T$$

and then approximate τ by t_i whenever τ is in $(t_{i-1}, t_i]$. This means that we are slightly postponing the default time, but the time intervals can be chosen arbitrarily small. In practice they are often chosen to be daily intervals, and we will follow that convention when generating our numerical results later in this chapter. The CVA is thus approximated by the bucketed CVA as:

$$\text{CVA} \approx \text{CVAB} = (1-R)\sum_{i=1}^{n}\mathbb{Q}\left(\tau \in (t_{i-1}, t_i]\right)\mathbb{E}_0^{\mathbb{Q}}\left[D(0,t_i)(V(t_i))^+ | \tau \in (t_{i-1}, t_i]\right].$$

We can make a further simplification by assuming independence between the default time and other stochastic factors driving the cash-flows paid out by the contract and the discount rates.[8]

[6] Ideally, recovery rates would be derived from market prices of recovery swaps, which unfortunately do not trade. Recoveries tend to show significant variations over time and across different industries and debt seniorities, Gregory (2012). Stochastic recovery is modelled in Li (2013) and a good overview of the topic and modelling possibilities is given by Altman *et al.* (2005).

[7] See Brigo, Morini, Pallavicini (2013), Chapter 2.3, page 36. Also see Pykhtin and Zhu (2007), last page, equation (17).

[8] Independence cannot be assumed if there would be a requirement to also include wrong way risk; see Pykhtin and Zhu (2007) for the definition and the book Brigo, Morini, Pallavicini (2013) for a detailed theoretical treatment and an overview of current industry practice to deal with this issue.

We then obtain what is known as the *independence based CVA with bucketing*:

$$\mathrm{CVA} \approx \mathrm{ICVAB} = (1-R)\sum_{i=1}^{n} \mathbb{Q}\left(\tau \in (t_i, t_{i-1}]\right) EE(t_i), \tag{3.2.5}$$

$$EE(t_i) = \mathbb{E}_0^{\mathbb{Q}}\left[D(0,t_i)(V(t_i))^+\right].$$

If the recovery rate is set at the industry standard value, we thus need to calculate the profile of the $EE(t_i)$, the Expected Exposure, and determine the risk-neutral probabilities of default of the counterparty in our bucketing intervals. These are usually obtained by interpolation from the probabilities of default that are implied by Credit Default Swap spreads. The calculation of the CVA is then reduced to obtaining the Expected Exposures at all time points t_i for our inflation-linked derivatives. To obtain these, we must now specify the market model for interest and inflation.

3.2.3 Jarrow-Yildirim Asset Price Dynamics

In Jarrow and Yildirim (2003) a model for real and nominal short rates r and n, and an inflation index I, is specified in the Heath-Jarrow-Morton framework, see Heath, Jarrow, Morton (1990, 1991, 1992). The dynamics of nominal and real forward rates f_n and f_r, and the dynamics for the inflation index $I(t)$, are given by

$$df_n\left(t,T\right) = \alpha_n\left(t,T\right)dt + \sigma_n\left(t,T\right)dW_n^{\mathbb{P}}(t), \tag{3.2.6}$$

$$df_r\left(t,T\right) = \alpha_r\left(t,T\right)dt + \sigma_r\left(t,T\right)dW_r^{\mathbb{P}}(t), \tag{3.2.7}$$

$$dI\left(t\right)/I\left(t\right) = \mu_I(t)dt + \sigma_I dW_I^{\mathbb{P}}(t), \tag{3.2.8}$$

with $(W_n^{\mathbb{P}}, W_r^{\mathbb{P}}, W_I^{\mathbb{P}})$ a three-dimensional Brownian motion with constant correlations $\rho_{nr},\ \rho_{nI}$ and ρ_{rI} under the original probability measure $\mathbb{P}$. This three-dimensional Brownian motion generates the filtration $\mathcal{F}_t = \sigma\left(\left\{\left(W_n^{\mathbb{P}}(s), W_r^{\mathbb{P}}(s), W_I^{\mathbb{P}}(s)\right),\ 0 \le s \le t\right\}\right)$. The short rates in this model follow from $r(t) = f_r(t,t)$ and $n(t) = f_n(t,t)$ which allows us to define the nominal and real money market account values

$$B_r(t) = \exp\left[\int_0^t r(u)du\right], \qquad B_n(t) = \exp\left[\int_0^t n(u)du\right],$$

while real and nominal zero-coupon bond prices at times $t \le T$, for maturity T, are found from

$$P_r(t,T) = \exp\left[-\int_t^T f_r(t,u)du\right], \qquad P_n(t,T) = \exp\left[-\int_t^T f_n(t,u)du\right].$$

Jarrow and Yildirim define Gaussian forward rates based on a specification of volatility term structures given by $\sigma_k\left(t,T\right) = \sigma_k e^{-a_k(T-t)},\quad k \in \{n,r\}$. Because of the HJM no-arbitrage condition, this choice determines the drifts of forward rates under the nominal martingale measure $\mathbb{Q}_n$, while the initial conditions $f_n(0,T)$ and $f_r(0,T)$ are also fixed if we want to match an existing initial term structure of interest rates. Jarrow and Yildirim impose that under this nominal martingale measure all nominally discounted tradables are martingales. This holds true in particular for the discounted prices of nominal zero-coupon bonds, inflation weighted real zero-coupon bonds (i.e. real zero-coupon bonds in nominal terms) and for the real bank account in nominal terms. As a result

$$\frac{P_n(t,T)}{B_n(t)},\ \frac{I(t)P_r(t,T)}{B_n(t)},\ \frac{I(t)B_r(t)}{B_n(t)} \tag{3.2.9}$$

must all be martingales under $\mathbb{Q}_n$. This implies that the dynamics of the nominal and real short interest rates and the inflation index are given by:

$$dn\left(t\right) = \left(\theta_n\left(t\right) - a_n n\left(t\right)\right)dt + \sigma_n dW_n\left(t\right), \tag{3.2.10}$$

$$dr\left(t\right) = \left(\theta_r\left(t\right) - a_r r\left(t\right) - \rho_{rI}\sigma_r\sigma_I\right)dt + \sigma_r dW_r\left(t\right), \tag{3.2.11}$$

$$dI\left(t\right) = \left(n\left(t\right) - r(t)\right)I\left(t\right)dt + \sigma_I I\left(t\right)dW_I\left(t\right), \tag{3.2.12}$$

with (W_n, W_r, W_I) a three-dimensional Brownian motion under $\mathbb{Q}_n$ with the same set of correlations $\rho_{nr},\ \rho_{nI}$ and ρ_{rI} as under the original probability measure $\mathbb{P}$.

If we write down the dynamics under the real martingale measure $\mathbb{Q}_r$ which is associated with the numeraire $B_r(t)$, instead of $\mathbb{Q}_n$ which is associated with the numeraire $B_n(t)$, the real short rate satisfies

$$dr\left(t\right) = \left(\theta_r\left(t\right) - a_r r\left(t\right)\right)dt + \sigma_r d\widetilde{W}_r\left(t\right), \tag{3.2.13}$$

with $\widetilde{W}_r$ a Brownian Motion under $\mathbb{Q}_r$.

Both short rates thus have the form of a Hull-White one-factor model under their respective martingale measures. The deterministic functions $\theta_n\left(t\right)$ and $\theta_r\left(t\right)$ in (3.2.10)-(3.2.11) can therefore be used to fit the initial term structures of nominal and real rates. This can be achieved by taking

$$\theta_k(T) = \frac{\partial f_k(0,T)}{\partial T} + a_k f_k(0,T) + \frac{\sigma_k^2}{2a_k}(1 - e^{-2a_k T}), \tag{3.2.14}$$

for $k \in \{n, r\}$, where the initial instantaneous forward rates

$$f_k\left(0,T\right) = -\frac{\partial}{\partial T}\ln P_k(0,T) \tag{3.2.15}$$

can be directly obtained from the initial real and nominal zero-coupon bond prices $P_k\left(0,T\right)$. The price for both the nominal and the real zero coupon bonds are then given by Hull and White (1990) formulae:

$$P_k(t,T) = \mathbb{E}_t^{\mathbb{Q}_k}\left[e^{-\int_t^T k(s)ds}\right] = A_k(t,T)e^{-B_{a_k}(t,T)k(t)}, \tag{3.2.16}$$

$$A_k(t,T) = \frac{P_k(0,T)}{P_k(0,t)}\exp\left\{B_{a_k}(t,T)f_k(0,t) - \frac{1}{2}B_{a_k}(t,T)^2 v_k(t)\right\}, \tag{3.2.17}$$

where the function B_a is defined as

$$B_a(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right), \tag{3.2.18}$$

which we will often use in the sequel. We define the function $v_k(t)$ in terms of the function B_a as

$$v_k(t) \equiv \frac{\sigma_k^2}{2a_k}(1 - e^{-2a_k t}) = \sigma_k^2 B_{2a_k}(0,t). \tag{3.2.19}$$

3.2.4 Second Order Moments

The SDEs in (3.2.10)-(3.2.12) have the following explicit solution (for $t \geq s$):

$$n(t) = n(s)\, e^{-a_n(t-s)} + \alpha_n(t) - \alpha_n(s)\, e^{-a_n(t-s)} + \sigma_n \int_s^t e^{-a_n(t-u)} dW_n(u), \tag{3.2.20}$$

$$r(t) = r(s)\, e^{-a_r(t-s)} + \alpha_r(t) - \alpha_r(s)\, e^{-a_r(t-s)} - R(s,t) + \sigma_r \int_s^t e^{-a_r(t-u)} dW_r(u) \tag{3.2.21}$$

$$I(t) = I(s) \exp\left[\int_s^t (n(u) - r(u))\, du - \frac{1}{2}\sigma_I{}^2 (t-s) + \sigma_I (W_I(t) - W_I(s))\right], \tag{3.2.22}$$

where the functions α_k, $k \in \{n, r\}$ are defined as

$$\alpha_k(t) \equiv f_k(0,t) + x_k(t), \tag{3.2.23}$$

with

$$x_k(t) \equiv \frac{\sigma_k^2}{2a_k}(1 - e^{-a_k t})^2 = \frac{1}{2}\sigma_k^2 B_{a_k}(0,t)^2, \tag{3.2.24}$$

and the real rate specific term (which stems from the change of measure from $\mathbb{Q}_r$ to $\mathbb{Q}_n$) is

$$R(s,t) = \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(1 - e^{-a_r(t-s)}\right). \tag{3.2.25}$$

The conditional moments of the short rate processes are thus given by

$$\mathbb{E}_t^{\mathbb{Q}^n}[n(T)] = \alpha_n(T) + (n(t) - \alpha_n(t))e^{-a_n(T-t)}, \tag{3.2.26}$$

$$\mathbb{E}_t^{\mathbb{Q}^n}[r(T)] = \alpha_r(T) + (r(t) - \alpha_r(t))e^{-a_r(T-t)} + R(t,T), \tag{3.2.27}$$

$$\mathrm{Var}_t^{\mathbb{Q}^n}[k(T)] = \sigma_k^2 \int_t^T e^{-2a_k(T-u)} du = \frac{\sigma_k^2}{2a_k}\left(1 - e^{-2a_k(T-t)}\right) = \sigma_k^2 B_{2a_k}(t,T). \tag{3.2.28}$$

Notice that $\alpha_k(0) = f_k(0,0) = k(0)$.[9] Therefore, the special case of conditioning on starting time $t = 0$ gives us

$$\mathbb{E}_0^{\mathbb{Q}^n}[n(t)] = \alpha_n(t), \tag{3.2.29}$$

$$\mathbb{E}_0^{\mathbb{Q}^n}[r(t)] = \alpha_r(t) + R(t), \tag{3.2.30}$$

$$\mathrm{Var}_0^{\mathbb{Q}^n}[k(t)] = \sigma_k^2 B_{2a_k}(0,t) = v_k(t), \tag{3.2.31}$$

where

$$R(t) \equiv R(0,t) = \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(1 - e^{-a_r t}\right) = \rho_{rI}\sigma_r\sigma_I B_{a_r}(0,t). \tag{3.2.32}$$

To characterise conditional moments of integrated short rate processes as well, we first define

$$\begin{aligned} V_k(t,T) &\equiv \sigma_k^2 \int_t^T B_{a_k}(u,T)^2 du = \sigma_k^2 \int_0^{T-t} B_{a_k}(0,u)^2 du \\ &= \frac{\sigma_k^2}{a_k^2}(T - t + 2(B_{a_k}(0,T) - B_{a_k}(0,t)) - (B_{2a_k}(0,T) - B_{2a_k}(0,t))), \end{aligned} \tag{3.2.33}$$

[9] The relationship between the instantaneous forward rate and the short rate is $f_k(t,t) = k(t)$.

where we have used the fact that $B_a(t,T) = B_a(0,T-t)$ for all $0 \le t \le T$. By integration of (3.2.26), and using (3.2.23) and (3.2.24), we find:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}^n}\left[\int_t^T n(s)ds\right] &= \int_t^T \alpha_n(s)ds + (n(t)-\alpha_n(t))\int_t^T e^{-a_n(t-s)}ds \qquad (3.2.34)\\ &= \int_t^T f_n(0,s)ds + \frac{1}{2}\sigma_n^2\int_t^T B_{a_n}(0,s)^2 ds + (n(t)-\alpha_n(t))\int_t^T e^{-a_n(t-s)}ds\\ &= \ln\frac{P_n(0,t)}{P_n(0,T)} + \tfrac{1}{2}\left(V_n(0,T)-V_n(0,t)\right) + (n(t)-\alpha_n(t))B_{a_n}(t,T),\end{aligned}$$

where in the last step we have used the definition of instantaneous forward rate (3.2.15), and also the second equality in (3.2.33). However,

$$\mathbb{E}_t^{\mathbb{Q}^n}\left[\int_t^T r(s)ds\right] = \ln\frac{P_r(0,t)}{P_r(0,T)} + \tfrac{1}{2}\left(V_r(0,T)-V_r(0,t)\right) + (r(t)-\alpha_r(t))B_{a_r}(t,T) + \overline{R}(t,T)$$

contains an extra term

$$\overline{R}(t,T) = \int_t^T R(t,s)ds = \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(T-t-\frac{1}{a_r}\left(1-e^{-a_r(T-t)}\right)\right), \qquad (3.2.35)$$

originating from (3.2.27). Variances of both integrals are derived from (3.2.20)-(3.2.21):

$$\begin{aligned}\mathrm{Var}_t^{\mathbb{Q}^n}\left(\int_t^T k(s)ds\right) &= \mathrm{Var}_t^{\mathbb{Q}^n}\left(\sigma_k\int_t^T\int_t^s e^{-a_k(s-u)}dW_k(u)\,ds\right) = \mathrm{Var}_t^{\mathbb{Q}^n}\left(\sigma_k\int_t^T\int_u^T e^{-a_k(s-u)}ds\,dW_k(u)\right)\\ &= \sigma_k^2\,\mathrm{Var}_t^{\mathbb{Q}^n}\left(\int_t^T B_{a_k}(u,T)dW_k(u)\right) = \sigma_k^2\int_t^T B_{a_k}(u,T)^2du \;=\; V_k(t,T), \qquad (3.2.36)\end{aligned}$$

where in the second step a change of the order of integration (a stochastic Fubini theorem), and in the fourth step Itô isometry was used, followed by definition (3.2.33). Again, the special case when considering the moments at the initial time gives us

$$\mathbb{E}_0^{\mathbb{Q}^n}\left[\int_0^t n(s)ds\right] = -\ln P_n(0,t) + \tfrac{1}{2}V_n(t), \qquad (3.2.37)$$

$$\mathbb{E}_0^{\mathbb{Q}^n}\left[\int_0^t r(s)ds\right] = -\ln P_r(0,t) + \tfrac{1}{2}V_r(t) + \overline{R}(t), \qquad (3.2.38)$$

and

$$\mathrm{Var}_0^{\mathbb{Q}^n}\left(\int_0^t k(s)ds\right) = V_k(t) \equiv V_k(0,t) = \frac{\sigma_k^2}{a_k^2}(t - 2B_{a_k}(0,t) - B_{2a_k}(0,t)), \qquad (3.2.39)$$

as well as

$$\overline{R}(t) \equiv \overline{R}(0,t) = \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t-\frac{1}{a_r}\left(1-e^{-a_r t}\right)\right) = \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t - B_{a_r}(0,t)\right). \qquad (3.2.40)$$

We will also need the following two useful identities.

The functions $V_{a_k,a_l}(t,T_1,T_2)$ and $U_{a_k}(s,t,T)$ are defined as

$$\begin{aligned}
V_{a_k,a_l}(t,T_1,T_2) &\equiv \int_0^t B_{a_k}(u,T_1)B_{a_l}(u,T_2)du \\
&= \frac{1}{a_k a_l}\Big\{ t + B_{a_k}(t,T_1) - B_{a_k}(0,T_1) + B_{a_l}(t,T_2) - B_{a_l}(0,T_2) \\
&\qquad - \Big[B_{a_k+a_l}(t, \tfrac{a_kT_1+a_lT_2}{a_k+a_l}) - B_{a_k+a_l}(0, \tfrac{a_kT_1+a_lT_2}{a_k+a_l})\Big]\Big\}, \qquad (3.2.41)\\
U_{a_k}(s,t,T) &\equiv \int_s^t B_{a_k}(u,T)du \;=\; \tfrac{1}{a_k}\,(\,t-s+B_{a_k}(t,T)-B_{a_k}(s,T)\,),
\end{aligned}$$

for $k,l \in \{n,r\}$, and where the function $B_{a_k}(t,T)$ is as given before by (3.2.18).

We remark that the functions v_k, x_k, R, $\overline{R}$, V_k, V_{a_k,a_l}, $\overline{V}_k$ and U_k can all easily be written in terms of the basic exponential functions $B_{a_k} \equiv B_k$, given by (3.2.18), in the equivalent notation which will be used throughout this part of the thesis. We summarise them all here:

$$\boxed{\begin{aligned}
B_k(t,T) &\equiv B_{a_k}(t,T) = \frac{1}{a_k}\left(1-e^{-a_k(T-t)}\right),\\
v_k(t) &\equiv \frac{\sigma_k^2}{2a_k}(1-e^{-2a_kt}) = \sigma_k^2 B_{2a_k}(0,t),\\
x_k(t) &\equiv \frac{\sigma_k^2}{2a_k}(1-e^{-a_kt})^2 = \frac{1}{2}\sigma_k^2 B_{a_k}(0,t)^2,\\
R(t) &\equiv \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(1-e^{-a_rt}\right) = \rho_{rI}\sigma_r\sigma_I B_{a_r}(0,t),\\
\overline{R}(t) &\equiv \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t-\frac{1}{a_r}\left(1-e^{-a_rt}\right)\right) = \frac{\rho_{rI}\sigma_r\sigma_I}{a_r}\left(t-B_{a_r}(0,t)\right),\\
V_k(t) &\equiv \frac{\sigma_k^2}{a_k^2}(t-2B_{a_k}(0,t)-B_{2a_k}(0,t)),\\
V_k(t,T) &\equiv \frac{\sigma_k^2}{a_k^2}(T-t+2(B_{a_k}(0,T)-B_{a_k}(0,t))-(B_{2a_k}(0,T)-B_{2a_k}(0,t))),\\
V_{k,l}(t,T_1,T_2) &\equiv V_{a_k,a_l}(t,T_1,T_2) = \frac{1}{a_ka_l}\,\{t+B_{a_k}(t,T_1)-B_{a_k}(0,T_1)+B_{a_l}(t,T_2)-B_{a_l}(0,T_2)\\
&\qquad - \Big[B_{a_k+a_l}(t,\tfrac{a_kT_1+a_lT_2}{a_k+a_l})-B_{a_k+a_l}(0,\tfrac{a_kT_1+a_lT_2}{a_k+a_l})\Big]\Big\},\\
\overline{V}_k(t,T) &\equiv \frac{\sigma_k^2}{a_k^2}\,(t+2(B_{a_k}(t,T)-B_{a_k}(0,T))-(B_{2a_k}(t,T)-B_{2a_k}(0,T))) = \sigma_k^2 V_{a_k,a_k}(t,T,T),\\
U_k(s,t,T) &\equiv U_{a_k}(s,t,T) = \tfrac{1}{a_k}\,(\,t-s+B_{a_k}(t,T)-B_{a_k}(s,T)\,).
\end{aligned}}$$

Now that we have derived the expressions for the second order moments and defined useful auxiliary functions, we can use them to price the Zero Coupon and Year-on-Year inflation derivatives and derive the Credit Value Adjustments.

3.2.5 The Margrabe Formula

We first formulate and prove a very useful Lemma that will be used repeatedly in the rest of this chapter. It is a version of the so-called Margrabe's formula, derived by William Margrabe to price an option to exchange one risky asset for another risky asset at maturity, see Margrabe (1978).

Lemma 3.1. *The expectation* $\mathbb{E}\left[\left(\omega\left(ae^X - be^Y\right)\right)^+\right]$, *where* (X,Y) *have a bivariate normal distribution and* a *and* b *are constants, has the closed form solution*

$$\omega a\mathbb{E}\left[e^X\right]\Phi\left(\omega\frac{\ln\frac{a\mathbb{E}\left[e^X\right]}{b\mathbb{E}\left[e^Y\right]}+\frac{1}{2}Var\left(X-Y\right)}{\sqrt{Var\left(X-Y\right)}}\right)-\omega b\mathbb{E}\left[e^Y\right]\Phi\left(\omega\frac{\ln\frac{a\mathbb{E}\left[e^X\right]}{b\mathbb{E}\left[e^Y\right]}-\frac{1}{2}Var\left(X-Y\right)}{\sqrt{Var\left(X-Y\right)}}\right), \tag{3.2.42}$$

where ω *is* 1 *or* -1 *(for a call and a put option respectively) and the function* Φ *is the standard Gaussian cumulative distribution function.*

In the special case where $\mathbb{E}\left[e^X\right]=\mathbb{E}\left[e^Y\right]=1$*:*

$$\mathbb{E}\left[\left(\omega\left(ae^X-be^Y\right)\right)^+\right]=\omega\left(a\Phi\left(\omega\frac{\ln\frac{a}{b}+\frac{1}{2}Var(X-Y)}{\sqrt{Var(X-Y)}}\right)-b\Phi\left(\omega\frac{\ln\frac{a}{b}-\frac{1}{2}Var(X-Y)}{\sqrt{Var(X-Y)}}\right)\right). \tag{3.2.43}$$

Proof. Without loss of generality we take $\omega=1$ since the case $\omega=-1$ then follows by switching the roles of a and b and of X and Y. We find by conditioning on $Z\equiv X-Y$,

$$\begin{aligned}\mathbb{E}\left[(ae^X-be^Y)^+\right]&=\mathbb{E}\left[\ \mathbb{E}\left[\left(ae^{X-Y}-b\right)1_{\{X-Y\geq\ln(b/a)\}}e^Y|X-Y\right]\ \right]\\&=\mathbb{E}\left[\ \left(ae^Z-b\right)1_{\{Z\geq\ln(b/a)\}}\mathbb{E}\left[e^Y|Z\right]\ \right].\end{aligned} \tag{3.2.44}$$

Since (Y,Z) are jointly Gaussian and since $\ln\mathbb{E}[e^A]=\mathbb{E}[A]+\frac{1}{2}\mathrm{Var}(A)$ for Gaussian variables A, we have that

$$\begin{aligned}\mathbb{E}\left[e^Y|Z\right]&=\exp\left(\mathbb{E}\left[Y|Z\right]+\tfrac{1}{2}\mathrm{Var}\left(Y|Z\right)\right)\\&=\exp\left(\qquad\mathbb{E}[Y]+\tfrac{\mathrm{Cov}(Y,Z)}{\mathrm{Var}(Z)}(Z-\mathbb{E}[Z])+\tfrac{1}{2}\left(\mathrm{Var}(Y)-\tfrac{\mathrm{Cov}(Y,Z)^2}{\mathrm{Var}(Z)}\right)\right) &(3.2.45)\\&=\exp\left(-Z+\mathbb{E}[X]+\tfrac{\mathrm{Cov}(X,Z)}{\mathrm{Var}(Z)}(Z-\mathbb{E}[Z])+\tfrac{1}{2}\left(\mathrm{Var}(X)-\tfrac{\mathrm{Cov}(X,Z)^2}{\mathrm{Var}(Z)}\right)\right). &(3.2.46)\end{aligned}$$

Substitution of (3.2.46) and (3.2.45) in the first and second term of (3.2.44) respectively gives

$$\begin{aligned}\mathbb{E}\left[(ae^X-be^Y)^+\right]=\ &a\mathbb{E}\left[\ 1_{\{Z\geq\ln(b/a)\}}\exp\left(\tfrac{\mathrm{Cov}(X,Z)}{\mathrm{Var}(Z)}(Z-\mathbb{E}[Z])\right)\ \right]\cdot\mathbb{E}[e^X]\ \exp\left(-\frac{\mathrm{Cov}(X,Z)^2}{2\mathrm{Var}(Z)}\right)\\&-b\mathbb{E}\left[\ 1_{\{Z\geq\ln(b/a)\}}\exp\left(\tfrac{\mathrm{Cov}(Y,Z)}{\mathrm{Var}(Z)}(Z-\mathbb{E}[Z])\right)\ \right]\cdot\mathbb{E}[e^Y]\ \exp\left(-\frac{\mathrm{Cov}(Y,Z)^2}{2\mathrm{Var}(Z)}\right).\end{aligned}$$

Using that for Gaussian Z we have

$$\mathbb{E}\left[e^{\gamma(Z-\mathbb{E}[Z])}1_{\{Z\geq c\}}\right]=\exp\left(\tfrac{1}{2}\gamma^2\mathrm{Var}(Z)\right)\ \Phi\left(\tfrac{\gamma\mathrm{Var}(Z)-c+\mathbb{E}[Z]}{\sqrt{\mathrm{Var}(Z)}}\right),$$

this gives that $\mathbb{E}\left[(ae^X - be^Y)^+\right]$ equals

$$
\begin{aligned}
& a\mathbb{E}[e^X]\Phi\left(\tfrac{\mathrm{Cov}(X,Z)+\ln(a/b)+\mathbb{E}[Z]}{\sqrt{\mathrm{Var}(Z)}}\right) - b\mathbb{E}[e^Y]\Phi\left(\tfrac{\mathrm{Cov}(Y,Z)+\ln(a/b)+\mathbb{E}[Z]}{\sqrt{\mathrm{Var}(Z)}}\right) \\
= \quad & a\mathbb{E}[e^X]\Phi\left(\tfrac{\mathrm{Cov}(X,X-Y)+\ln(a/b)+\ln\mathbb{E}[e^X]-\frac{1}{2}\mathrm{Cov}(X,X)-\ln\mathbb{E}[e^Y]+\frac{1}{2}\mathrm{Cov}(Y,Y)}{\sqrt{\mathrm{Var}(X-Y)}}\right) \\
& - b\mathbb{E}[e^Y]\Phi\left(\tfrac{\mathrm{Cov}(Y,X-Y)+\ln(a/b)+\ln\mathbb{E}[e^X]-\frac{1}{2}\mathrm{Cov}(X,X)-\ln\mathbb{E}[e^Y]+\frac{1}{2}\mathrm{Cov}(Y,Y)}{\sqrt{\mathrm{Var}(X-Y)}}\right),
\end{aligned}
$$

which gives the result after collecting terms.

The second part follows trivially if we substitute $\mathbb{E}\left[e^X\right] = \mathbb{E}\left[e^Y\right] = 1$. ■

Note that in the special case of $\mathbb{E}\left[e^X\right] = \mathbb{E}\left[e^Y\right] = 1$ the formula not only simplifies considerably but the end result (3.2.43) will only depend on the variance of $X - Y$. We will exploit this property repeatedly in our subsequent derivations.

3.3 The Zero Coupon Inflation Indexed Swap

For the Zero Coupon Inflation Indexed Swap, the pricing equation (3.2.1) reduces to

$$\mathrm{ZCIIS}\left(t, T, I\left(0\right), N\right) = N\mathbb{E}_t^{\mathbb{Q}_n}\left[e^{-\int_t^T n(s)ds}\left(\frac{I(T)}{I(0)} - (1+K)^T\right)\right]. \tag{3.3.1}$$

We use the fact that $P_r(t,T)I(t)$ is the nominal value of a tradeable asset, and thus a martingale under the nominal martingale measure after (nominal) discounting, to write

$$\mathbb{E}_t^{\mathbb{Q}_n}\left[e^{-\int_0^T n(s)ds} I(T)P_r(T,T)\right] = e^{-\int_0^t n(s)ds} I(t)P_r(t,T),$$

for the floating leg of the swap. For the fixed leg we have

$$\mathbb{E}_t^{\mathbb{Q}_n}\left[e^{-\int_t^T n(s)ds}(1+K)^T\right] = P_n\left(t,T\right)\left(1+K\right)^T. \tag{3.3.2}$$

The price of the payer ZCIIS swap at time t is thus

$$\mathrm{ZCIIS}\left(t, T, I\left(0\right), N\right) = N\left(\frac{I(t)}{I(0)}P_r\left(t,T\right) - \left(1+K\right)^T \cdot P_n\left(t,T\right)\right). \tag{3.3.3}$$

We see immediately from substituting $t = 0$ that the initial value is completely model independent since it depends only on the initial discount rates. Because of this model independence and since ZCIIS contracts are quoted in the market in terms of the fixed rate K which makes the initial contract value equal to zero, we can strip (bootstrap) real zero-coupon bond prices from those quotations for different maturities $K(T)$ to obtain the term structure of real interest rates, according to:

$$P_r\left(0,T\right) = P_n\left(0,T\right)\left(1+K(T)\right)^T. \tag{3.3.4}$$

ZCIIS CVA

We can now proceed to the CVA calculation for the ZCIIS instrument.

Proposition 3.1. *Under the Jarrow Yildirim model* (3.2.10) - (3.2.12)*, the Expected Exposure at the future time τ for the Zero-Coupon Inflation Indexed Swap contract* (3.3.3)*, with fixed rate K and maturity $T \geq \tau$, is given by*

$$EE(\tau) = \omega N \left(P_r(0,T)\, \Phi\left(\omega \frac{\ln \frac{P_r(0,T)}{\hat{K} P_n(0,T)} + \frac{\varsigma(\tau)^2}{2}}{\varsigma(\tau)}\right) - \hat{K} P_n(0,T)\, \Phi\left(\omega \frac{\ln \frac{P_r(0,T)}{\hat{K} P_n(0,T)} - \frac{\varsigma(\tau)^2}{2}}{\varsigma(\tau)}\right)\right), \tag{3.3.5}$$

where $\hat{K} \equiv (1+K)^T$, ω is 1 for a payer ZCIIS, and -1 for a receiver ZCIIS and Φ is the standard normal cumulative distribution function. The variance $\varsigma(\tau)^2$ is given by

$$\begin{aligned} \varsigma(\tau)^2 = {} & \overline{V}_n(\tau,T) + \overline{V}_r(\tau,T) + \sigma_I^2 \tau \\ & - 2\rho_{nr}\sigma_n\sigma_r V_{n,r}(\tau,T,T) + 2\rho_{nI}\sigma_n\sigma_I U_n(0,\tau,T) - 2\rho_{rI}\sigma_r\sigma_I U_r(0,\tau,T), \end{aligned} \tag{3.3.6}$$

with functions $\overline{V}_k(t,T)$, $V_{n,r}(t,T_1,T_2)$ and $U_k(s,t,T)$ defined above in (3.2.33) *and* (3.2.41).

Proof. In order to calculate $EE(\tau)$ we need to evaluate

$$\mathbb{E}_0^{\mathbb{Q}}\left[D(0,\tau)\left(\omega N \cdot \left(\frac{I(\tau)}{I(0)} P_r(\tau,T) - (1+K)^T \cdot P_n(\tau,T)\right)\right)^+\right],$$

with $D(0,\tau) = \exp\left(-\int_0^\tau n(s)\,ds\right)$. Since $n(\tau)$ and $r(\tau)$ are normally distributed and $P_n(\tau,T)$ and $P_r(\tau,T)$ are lognormally distributed, this can be interpreted as an option of Margrabe type which exchanges an inflation rate adjusted real zero-coupon bond for $\hat{K} = (1+K)^T$ nominal zero-coupon bonds. We can therefore use Lemma 3.1 to determine

$$\begin{aligned} EE(\tau) &= N \cdot \mathbb{E}_0^{\mathbb{Q}_n}\left[\left(\omega\left(D(0,\tau) \cdot \frac{I(\tau)}{I(0)} P_r(\tau,T) - \hat{K} \cdot D(0,\tau) \cdot P_n(\tau,T)\right)\right)^+\right] && (3.3.7) \\ &= N \cdot \mathbb{E}_0^{\mathbb{Q}_n}\left[\left(\omega\left(ae^X - be^Y\right)\right)^+\right], && (3.3.8) \end{aligned}$$

if we define

$$\begin{aligned} a &\equiv P_r(0,T), \qquad e^X \equiv D(0,\tau)\frac{I(\tau)}{I(0)}\frac{P_r(\tau,T)}{P_r(0,T)}, && (3.3.9) \\ b &\equiv \hat{K} P_n(0,T), \qquad e^Y \equiv D(0,\tau)\frac{P_n(\tau,T)}{P_n(0,T)}. \end{aligned}$$

For every discounted tradable asset price process S_τ in nominal terms we must have that $\mathbb{E}_0^{\mathbb{Q}_n}[S_\tau/S_0] = 1$. Therefore, from (3.2.9), we know that for our tradables, i.e. real and nominal zero-coupon bonds, the following must hold:

$$\mathbb{E}_0^{\mathbb{Q}_n}\left[D(0,\tau)\frac{I(\tau)}{I(0)}\frac{P_r(\tau,T)}{P_r(0,T)}\right] = 1, \qquad \mathbb{E}_0^{\mathbb{Q}_n}\left[D(0,\tau)\frac{P_n(\tau,T)}{P_n(0,T)}\right] = 1,$$

so $\mathbb{E}\left[e^X\right] = \mathbb{E}\left[e^Y\right] = 1$, and we can apply the second part of Lemma 3.1: equation (3.2.43). From that equation we see that we only need to determine $\mathrm{Var}(X - Y)$. This random variable $X - Y$ follows the normal distribution $\mathcal{N}\left(\mu(\tau), \varsigma(\tau)^2\right)$ with mean $\mu(\tau) = \mathbb{E}[X] - \mathbb{E}[Y]$ and variance $\varsigma(\tau)^2 = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X,Y)$. We notice that we must have $\mathbb{E}[X] = -\frac{1}{2}\mathrm{Var}(X)$ and $\mathbb{E}[Y] = -\frac{1}{2}\mathrm{Var}(Y)$, since for any Gaussian variable X we have that $\mathbb{E}\left[e^X\right] = e^{\mathbb{E}[X]+\frac{1}{2}\mathrm{Var}(X)}$. After including the inflation index dynamics from (3.2.22) we have

$$e^X = e^{-\int_0^\tau r(s)ds} e^{-\frac{1}{2}\sigma_I^2\tau+\sigma_I W_I(\tau)} \cdot \frac{P_r(\tau,T)}{P_r(0,T)}, \qquad e^Y = e^{-\int_0^\tau n(s)ds} \cdot \frac{P_n(\tau,T)}{P_n(0,T)}.$$

From the Hull-White zero-coupon bond formula (3.2.16), the dynamics of the short rate processes $n(\tau)$ and $r(\tau)$ from (3.2.20)-(3.2.21), and using the expression for the variance of the integral processes (3.2.36) we can decompose our normal variables X and Y into their means and their martingale increment parts:

$$\begin{aligned}
\mathbb{E}[X] &= -\frac{1}{2}\mathrm{Var}(X), \\
X - \mathbb{E}[X] &= -\sigma_r \int_0^\tau \int_0^s e^{-a_r(s-u)} dW_r(u) ds - B_r(\tau,T)\,\sigma_r \int_0^\tau e^{-a_r(\tau-u)} dW_r(u) + \sigma_I W_I(\tau); \\
\mathbb{E}[Y] &= -\frac{1}{2}\mathrm{Var}(Y), \qquad (3.3.10) \\
Y - \mathbb{E}[Y] &= -\sigma_n \int_0^\tau \int_0^s e^{-a_n(s-u)} dW_n(u) ds - B_n(\tau,T)\,\sigma_n \int_0^\tau e^{-a_n(\tau-u)} dW_n(u).
\end{aligned}$$

We will use the following identity to simplify the martingale increments:

$$\begin{aligned}
&\sigma_k \int_0^\tau \int_0^s e^{-a_k(s-u)} dW_k(u) ds + B_k(\tau,T)\,\sigma_k \int_0^\tau e^{-a_k(\tau-u)} dW_k(u) \\
&= \sigma_k \int_0^\tau \left(B_k(u,\tau) + B_k(\tau,T)\, e^{-a_k(\tau-u)}\right) dW_k(u) \\
&= \sigma_k \int_0^\tau \left(\frac{1-e^{-a_k(\tau-u)}}{a_k} + \frac{1-e^{-a_k(T-\tau)}}{a_k} e^{-a_k(\tau-u)}\right) dW_k(u) \\
&= \sigma_k \int_0^\tau \frac{1}{a_k}\left(1-e^{-a_k(T-u)}\right) dW_k(u) = \sigma_k \int_0^\tau B_k(u,T)\, dW_k(u). \qquad (3.3.11)
\end{aligned}$$

It is now clear that $\mathrm{Var}(X-Y)$ can be found by the clasical variance of the sum formula $\mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \sum_{j=1}^n \mathrm{Cov}\left(X_i, X_j\right)$ from

$$\begin{aligned}
\varsigma(\tau)^2 &= \mathrm{Var}(X-Y) = \mathrm{Var}\left(\sigma_n \int_0^\tau B_n(u,T)\, dW_n(u) - \sigma_r \int_0^\tau B_r(u,T)\, dW_r(u) + \sigma_I W_I(\tau)\right) \\
&= \sigma_n^2 \int_0^\tau B_n(u,T)^2 du + \sigma_r^2 \int_0^\tau B_r(u,T)^2 du + \sigma_I^2 \tau \qquad (3.3.12) \\
&- 2\rho_{nr}\sigma_n\sigma_r \int_0^\tau B_n(u,T) B_r(u,T) du + 2\rho_{nI}\sigma_n\sigma_I \int_0^\tau B_n(u,T) du - 2\rho_{rI}\sigma_r\sigma_I \int_0^\tau B_r(u,T) du.
\end{aligned}$$

Finally, we apply formula (3.2.43) to obtain (3.3.5), where the expectation in formula (3.3.7) is found using constants a and b from (3.3.9) and the variance from (3.3.12), which gives equation (3.3.6) of the proposition. ■

Once the analytical formula for $EE(\tau)$ is available, the CVA for ZCIIS instruments can be obtained from (3.2.5).

3.4 The Year-on-Year Inflation Indexed Swap

We now proceed with determining the CVA for the Year-on-Year Inflation Indexed Swap instrument. We first derive the price of this instrument without considering the possibility of default. We will consider individual terms i in the summation of equation (3.2.2). If the current time is $t \in [T_{i-1}, T_i)$, the pricing equation for such terms is completely analogous to the case of a ZCIIS valuation as in equation (3.3.3) and we find

$$\begin{aligned}\text{YYIIS}\,(t, T_{i-1}, T_i, \varphi_i, N) &= N\varphi_i \mathbb{E}_t^{\mathbb{Q}_n}\left[e^{-\int_t^{T_i} n(s)ds}\mathbb{E}_t^{\mathbb{Q}_n}\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - K\right)\right] \\ &= N\varphi_i\left(\frac{I(t)}{I(T_{i-1})}P_r\,(t, T_i) - (1+K)P_n\,(t, T_i)\right).\end{aligned} \tag{3.4.1}$$

For $t < T_{i-1}$ we cannot resort to the ZCIIS valuation since the value of $I(T_{i-1})$ is still unknown. We use that for Gaussian X and Y

$$\begin{aligned}\mathbb{E}\left[e^{X+Y-Z}\right] &= e^{\mathbb{E}[X]+\mathbb{E}[Y]-\mathbb{E}[Z]+\frac{1}{2}\mathrm{Var}(X+Y-Z)} = e^{\mathbb{E}[X]+\frac{1}{2}\mathrm{Var}(X)+\mathbb{E}[Y]+\frac{1}{2}\mathrm{Var}(Y)-\mathbb{E}[Z]-\frac{1}{2}\mathrm{Var}(Z)+\mathrm{Cov}(X-Z,\,Y-Z)} \\ &= e^{\mathrm{Cov}(X-Z,\,Y-Z)}\mathbb{E}[e^X]\mathbb{E}[e^Y]\,/\,\mathbb{E}[e^Z],\end{aligned}$$

and choose

$$X = -\int_t^{T_{i-1}} n(s)ds, \qquad Y = \ln\frac{I(T_i)}{I(t)} - \int_t^{T_i} n(s)ds, \qquad Z = \ln\frac{I(T_{i-1})}{I(t)} - \int_t^{T_{i-1}} n(s)ds,$$

so that the price for the derivative leg at time i in this case equals

$$\text{YYIIS}\,(t, T_{i-1}, T_i, \varphi_i, N) = N\varphi_i\left(P_n(t, T_{i-1})\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}e^{C(t, T_{i-1}, T_i)} - (1+K)P_n(t, T_i)\right),$$

with

$$\begin{aligned}C(t, T_{i-1}, T_i) &= \mathrm{Cov}_t^{\mathbb{Q}_n}\left(\,-\ln\tfrac{I(T_{i-1})}{I(t)},\ \ln\tfrac{I(T_i)}{I(T_{i-1})} - \textstyle\int_{T_{i-1}}^{T_i} n(s)ds\,\right) \\ &= \mathrm{Cov}_t^{\mathbb{Q}_n}\Big(-\sigma_n\textstyle\int_t^{T_{i-1}} B_n(u, T_{i-1})dW_n(u) + \sigma_r\int_t^{T_{i-1}} B_r(u, T_{i-1})dW_r(u) - \sigma_I\int_t^{T_{i-1}} dW_I(u), \\ &\qquad \sigma_r\textstyle\int_t^{T_i} B_r(u, T_i)dW_r(u) - \sigma_r\int_t^{T_{i-1}} B_r(u, T_{i-1})dW_r(u) + \sigma_I\int_{T_{i-1}}^{T_i} dW_I(u)\,\Big) \\ &= \textstyle\int_t^{T_{i-1}}\left[-\sigma_n\rho_{nr}B_n(u, T_{i-1}) + \sigma_r B_r(u, T_{i-1}) - \sigma_I\rho_{rI}\right]\left[\sigma_r B_r(u, T_{i-1}) - \sigma_r B_r(u, T_i)\right]du \\ &= -\sigma_r\textstyle\int_t^{T_{i-1}}\left[-\sigma_n\rho_{nr}B_n(u, T_{i-1}) + \sigma_r B_r(u, T_{i-1}) - \sigma_I\rho_{rI}\right]B_r(T_{i-1}, T_i)e^{-a_r(T_{i-1}-u)}du \\ &= \sigma_r B_r(T_{i-1}, T_i)\left[\sigma_n\rho_{nr}L_{a_r,a_n}(t, T_{i-1}) - \sigma_r L_{a_r,a_r}(t, T_{i-1}) + \sigma_I\rho_{rI}B_r(t, T_{i-1})\right],\end{aligned} \tag{3.4.2}$$

where

$$L_{a_k,a_l}(t, T) \equiv \int_t^T e^{-a_k(T-u)}B_{a_l}(u, T)du \;=\; \tfrac{1}{a_l}\left(\,B_{a_k}(t, T) - B_{a_k+a_l}(t, T)\,\right),$$

for $k, l \in \{n, r\}$, and the function $B_k(t,T) \equiv B_{a_k}(t,T)$ is as before given by (3.2.18). The first equality in the derivation above follows from the inflation index dynamics (3.2.22), by expressing stochastic parts of the corresponding $\int_t^T k(s)ds$ integrals in terms of $\int_t^T B_{a_k}(u,T)dW_k(u)$ as it was already done before when deriving (3.2.36) (in the penultimate step).

By summing the cashflows at all possible times, we obtain the following expression for the price of the payer YYIIS instrument at time t:

$$\begin{aligned} \text{YYIIS}\left(t,\, T_0,\, \{T_i,\varphi_i\}_{i=1,\dots M},\, N,\, K\right) &= N\varphi_l\left(\frac{I(t)}{I(T_{l-1})}P_r\left(t,T_l\right) - (1+K)\cdot P_n\left(t,T_l\right)\right) \qquad (3.4.3) \\ &+ N\sum_{i=l+1}^{M}\varphi_i\left(\frac{P_n\left(t,T_{i-1}\right)}{P_r\left(t,T_{i-1}\right)}P_r\left(t,T_i\right)\cdot e^{C(t,T_{i-1},T_i)} - (1+K)\cdot P_n\left(t,T_i\right)\right), \end{aligned}$$

where $l = \min\{i\,:\,T_i > t\}$ and $C(t,T_{i-1},T_i)$ is as defined in (3.4.2). Here it is clear that a valuation of a single YYIS cannot simply be done as a valuation of a portfolio of ZCIIS. Next section however deals with the separate case of a netted portfolio of ZCIIS's.

YYIIS CVA

For the CVA for an YYIIS instrument we use the independence and bucketing based CVA standard formula (3.2.5), which will take the following form:

$$\begin{aligned} CVA &\approx (1-R)\sum_{i=1}^{n}\mathbb{Q}\left\{\tau\in(t_i,t_{i-1}]\right\}\mathbb{E}_0^{\mathbb{Q}}D(0,t_i)\left[\left(\text{YYIIS}\left(t_i,T,\{T_j,\varphi_j\}_{\,j=1,\dots,M},N,K\right)\right)^+\right] \\ &= (1-R)\sum_{i=1}^{n}\mathbb{Q}\left\{\tau\in(t_i,t_{i-1}]\right\}EE(t_i), \qquad (3.4.4) \end{aligned}$$

where the bucketing of the default time is done over the partition

$$(0=t_0,t_1],(t_1,t_2],\dots,(t_{n-2},t_{n-1}],(t_{n-1},t_n=T_M],$$

with T_M the maturity of the YYIIS instrument. Note that l, the starting index of the summation in the expression (3.4.3) for YYIIS$(t_i,T,\{T_j,\varphi_j\}_{j=1,\dots,M},N,K)$, will now depend on t_i, so $l_i = \min\{j\,:\,T_j > t_i\}$.

The expression for the expected exposure at default corresponds to a t=0 option price for a payoff at τ: $EE(\tau) = \mathbb{E}_0^{\mathbb{Q}}\left[D(0,\tau)\left(\text{YYIIS}(\tau,T_0,\{T_j,\varphi_j\}_{j=1,\dots,M},N,K)\right)^+\right]$. By including our YYIIS valuation expression (3.4.3) we find the following expectation

$$\begin{aligned} EE(\tau) = \mathbb{E}_0^{\mathbb{Q}_n}\Bigg[N\cdot D(0,\tau)\Bigg(\omega\Bigg(\varphi_l\frac{I(\tau)}{I(T_{l-1})}P_r\left(\tau,T_l\right) + \sum_{j=l+1}^{M}\varphi_j\frac{P_n\left(\tau,T_{j-1}\right)}{P_r\left(\tau,T_{j-1}\right)}P_r\left(\tau,T_j\right)\cdot e^{C(\tau,T_{j-1},T_j)} \\ -(1+K)\cdot\sum_{j=l}^{M}\varphi_j P_n\left(\tau,T_j\right)\Bigg)\Bigg)^+\Bigg], \qquad (3.4.5) \end{aligned}$$

with $l = \min\{j\,:\,T_j > \tau\}$. The flag ω is either *+1* for a payer YYIIS, or *-1* for a receiver YYIIS.

The evaluation of this expression is the topic of the following proposition. We will show that a semi-analytical formula can be derived using *moment matching* approximations.[10] If we define the following quantities:

$$G \equiv e^{-\int_0^\tau n(s)ds}\left(\varphi_l \frac{I(\tau)}{I(T_{l-1})} P_r(\tau, T_l) + \sum_{j=l+1}^{M} \varphi_j \frac{P_n(\tau, T_{j-1})}{P_r(\tau, T_{j-1})} P_r(\tau, T_j) \cdot e^{C(\tau, T_{j-1}, T_j)}\right), \quad (3.4.6)$$

$$H \equiv e^{-\int_0^\tau n(s)ds} \cdot (1+K) \cdot \sum_{j=l}^{M} \varphi_j P_n(\tau, T_j),$$

we see that the expectation $EE(\tau)$ has the form $N \cdot \mathbb{E}_0^{\mathbb{Q}_n}\left[(\omega(G-H))^+\right]$. To apply the Margrabe formula (3.2.42), as we have done in the ZCIIS case, we require the quantities G and H to be lognormal. Unfortunately, they are not. Each of them is actually a sum of lognormal random variables for which no closed form probability distribution is available. However, we can argue that their distribution is not very far from lognormal and use auxiliary lognormal variables to approximate G and H, i.e.

$$G \approx ae^X, \quad H \approx be^Y,$$

with (X, Y) bivariate Gaussian variables.

This will allow us to evaluate our expectation

$$\mathbb{E}_0^{\mathbb{Q}_n}\left[(\omega(G-H))^+\right] \approx \mathbb{E}\left[\left(\omega\left(ae^X - be^Y\right)\right)^+\right]$$

in exactly the same manner as before by applying Margrabe formula - equation (3.2.42).

Proposition 3.2. *Under the Jarrow Yildirim model* (3.2.10) - (3.2.12)*, the moment matching approximation for the Expected Exposure at default τ for the Year on Year Inflation Indexed Swap* (3.4.3) *(payer when $\omega = 1$, receiver when $\omega = -1$ defined above) is*

$$EE(\tau) = \omega N \left(m_1(\tau)\Phi\left(\omega \frac{\ln \frac{m_1(\tau)}{b} + \frac{\varsigma(\tau)^2}{2}}{\varsigma(\tau)}\right) - b\Phi\left(\omega \frac{\ln \frac{m_1(\tau)}{b} - \frac{\varsigma(\tau)^2}{2}}{\varsigma(\tau)}\right)\right),$$

with

$$\varsigma(\tau)^2 = \ln \frac{\mathbb{E}[H^2]\,\mathbb{E}[G^2]}{\mathbb{E}[GH]^2}, \quad (3.4.7)$$

and

$$b = (1+K)\sum_{j=l}^{M} \varphi_j P_n(0, T_j),$$

[10] This kind of two-moment matching approximations to a lognormal distribution was first employed by Levy (1992) for a problem of pricing Asian options. Other approximations in the literature are Gentle's (1993) approximation by geometric average, Curran (1994) who uses conditioning on a geometric average price, Ju (1992) who uses Taylor series expansion approximations and Milevsky and Posner (1998a, 1998b) who match the two moments to a reciprocal gamma distribution instead.

where

$$m_1(\tau) \equiv \mathbb{E}[G] = e_l(\tau)e^{\frac{1}{2}\mathrm{Var}(Z_l)} + \sum_{j=l+1}^{m} c_j(\tau)e^{\frac{1}{2}\mathrm{Var}(X_j)},$$

$$\mathbb{E}\left[G^2\right] = e_l(\tau)^2 e^{2\mathrm{Var}(Z_l)} + 2\sum_{j=l+1}^{M} e_l(\tau)c_j(\tau)e^{\frac{1}{2}\mathrm{Var}(Z_l+X_j)} + \sum_{p=l+1}^{M}\sum_{q=l+1}^{M} c_p(\tau)c_q(\tau)e^{\frac{1}{2}\mathrm{Var}(X_p+X_q)},$$

$$\mathbb{E}\left[H^2\right] = (1+K)^2 \cdot \sum_{p=l}^{M}\sum_{q=l}^{M} d_p(\tau)d_q(\tau)e^{\frac{1}{2}\mathrm{Var}(Y_p+Y_q)},$$

$$\mathbb{E}[GH] = (1+K)\cdot\left\{\sum_{j=l}^{M} e_l(\tau)d_j(\tau)e^{\frac{1}{2}\mathrm{Var}(Z_l+Y_j)} + \sum_{p=l+1}^{M}\sum_{q=l}^{M} c_p(\tau)d_q(\tau)e^{\frac{1}{2}\mathrm{Var}(X_p+Y_q)}\right\},$$

with deterministic functions

$$\begin{aligned}
c_j(\tau) &= \varphi_j \frac{P_n(0,T_{j-1})}{P_r(0,T_{j-1})} P_r(0,T_j) e^{C(\tau,T_{j-1},T_j)} \cdot e^{-\frac{1}{2}V_n(\tau)-\frac{1}{2}B_n(\tau,T_{j-1})^2 v_n(\tau) - B_n(\tau,T_{j-1})x_n(\tau)} \\
&\qquad \cdot e^{-\frac{1}{2}\left(B_r(\tau,T_j)^2 - B_r(\tau,T_{j-1})^2\right)v_r(\tau) - (B_r(\tau,T_j)-B_r(\tau,T_{j-1}))(x_r(\tau)-R(\tau))}, \\
d_k(\tau) &= \varphi_k P_n(0,T_k) e^{-\frac{1}{2}V_n(\tau)-\frac{1}{2}B_n(\tau,T_k)^2 v_n(\tau) - B_n(\tau,T_k)x_n(\tau)}, \\
e_l(\tau) &= \varphi_l \frac{P_n(0,T_{l-1})}{P_r(0,T_{l-1})} P_r(0,T_l) e^{-\frac{1}{2}V_n(T_{l-1}) - \frac{1}{2}(V_r(\tau)-V_r(T_{l-1})) - \left(\overline{R}(\tau)-\overline{R}(T_{l-1})\right) - \frac{1}{2}\sigma_I^2(\tau - T_{l-1})} \\
&\quad \cdot e^{-\frac{1}{2}B_r(\tau,T_l)^2 v_r(\tau) - B_r(\tau,T_l)(x_r(\tau)+R(\tau))},
\end{aligned}$$

and where the functions $V_k(\tau)$, $v_k(\tau)$, $x_k(\tau)$, $R(\tau)$ and $\overline{R}(\tau)$ have been defined before. Variance-covariance terms that are needed in the moments expressions above can be expressed in terms of the functions $V_{k,l}(t,T_1,T_2)$ and $U_k(s,t,T)$ given in (3.2.41)*:*

$$\begin{aligned}
\mathrm{Cov}(X_i,X_j) &= \sigma_n^2 V_{n,n}(\tau,T_{i-1},T_{j-1}) + \sigma_r^2\left[V_{r,r}(\tau,T_{i-1},T_{j-1}) - V_{r,r}(\tau,T_{i-1},T_j)\right] \\
&\quad - \sigma_r^2\left[V_{r,r}(\tau,T_i,T_{j-1}) - V_{r,r}(\tau,T_i,T_j)\right] - \rho_{nr}\sigma_r\sigma_n\left[V_{n,r}(\tau,T_{i-1},T_{j-1})\right. \\
&\quad \left. - V_{n,r}(\tau,T_{i-1},T_j) + V_{n,r}(\tau,T_{j-1},T_{i-1}) - V_{n,r}(\tau,T_{j-1},T_i)\right], \\
\mathrm{Cov}(Y_i,Y_j) &= \sigma_n^2 V_{n,n}(\tau,T_i,T_j), \\
\mathrm{Cov}(Y_i,X_j) &= \sigma_n^2 V_{n,n}(\tau,T_i,T_{j-1}) - \rho_{nr}\sigma_n\sigma_r\left[V_{n,r}(\tau,T_i,T_{j-1}) - V_{n,r}(\tau,T_i,T_j)\right], \\
\mathrm{Cov}(Y_i,Z_k) &= \sigma_n^2 V_{n,n}(T_{k-1},T_i,T_{k-1}) - \rho_{nr}\sigma_r\sigma_n\left[V_{r,n}(T_{k-1},T_{k-1},T_i) - V_{r,n}(t,T_k,T_i)\right] \\
&\quad - \rho_{nI}\sigma_n\sigma_I U_n(T_{k-1},t,T_i), \\
\mathrm{Cov}(X_j,Z_k) &= \sigma_n^2 V_{n,n}(T_{k-1},T_{k-1},T_{j-1}) - \rho_{nI}\sigma_n\sigma_I U_n(T_{k-1},t,T_{j-1}) \\
&\quad - \rho_{nr}\sigma_r\sigma_n\left[V_{r,n}(T_{k-1},T_{k-1},T_{j-1}) - V_{r,n}(t,T_k,T_{j-1})\right] \\
&\quad - \rho_{nr}\sigma_r\sigma_n\left[V_{r,n}(T_{k-1},T_{j-1},T_{k-1}) - V_{r,n}(T_{k-1},T_j,T_{k-1})\right] \\
&\quad + \rho_{rI}\sigma_r\sigma_I\left[U_r(T_{k-1},t,T_{j-1}) - U_r(T_{k-1},t,T_j)\right] + \sigma_r^2\left[V_{r,r}(T_{k-1},T_{j-1},T_{k-1})\right. \\
&\quad \left. - V_{r,r}(T_{k-1},T_j,T_{k-1}) - V_{r,r}(t,T_{j-1},T_k) + V_{r,r}(t,T_j,T_k)\right], \\
\mathrm{Var}(Z_k) &= \sigma_n^2 V_{n,n}(T_{k-1},T_{k-1},T_{k-1}) - 2\rho_{nr}\sigma_r\sigma_n\left[V_{r,n}(T_{k-1},T_{k-1},T_{k-1})\right. \\
&\quad \left. - V_{r,n}(T_{k-1},T_k,T_{k-1})\right] + \sigma_I^2(t-T_{k-1}) - 2\rho_{rI}\sigma_r\sigma_I U_r(T_{k-1},t,T_k) \\
&\quad + \sigma_r^2 V_{r,r}(T_{k-1},T_{k-1},T_{k-1}) - 2\sigma_r^2 V_{r,r}(T_{k-1},T_{k-1},T_k) + \sigma_r^2 V_{r,r}(t,T_k,T_k).
\end{aligned} \tag{3.4.8}$$

Proof. Let
$$X \sim \mathcal{N}(\mu, \xi^2), \quad Y \sim \mathcal{N}(\nu, \eta^2), \quad \mathrm{Corr}(X, Y) = \rho.$$

It is then enough to match the first two moments of the (unknown) distribution of both our processes and their mixed moment, i.e. $\mathbb{E}[G]$, $\mathbb{E}[G^2]$, $\mathbb{E}[H]$, $\mathbb{E}[H^2]$ and $\mathbb{E}[GH]$ to the moments of the auxiliary lognormal random variables. The first order moments will give us $\mathbb{E}\left[ae^X\right] = \mathbb{E}[G]$ and $\mathbb{E}\left[be^Y\right] = \mathbb{E}[H]$ directly and the second order moments $\mathbb{E}[G^2]$, $\mathbb{E}[H^2]$ and $\mathbb{E}[GH]$ will provide us with three equations from which the variances ξ^2, η^2 and the covariance $\rho\xi\eta$ can be determined, and thus the key quantity $\mathrm{Var}(X - Y)$.

Looking at the definition of the quantities G and H in (3.4.6) we define

$$G = F_l^* + \sum_{j=l+1}^{M} G_j, \qquad H = (1 + K) \cdot \sum_{k=l}^{M} H_k \,;$$

$$F_l^* \equiv e^{-\int_0^\tau n(s)ds} \cdot \varphi_l \frac{I(\tau)}{I(T_{l-1})} P_r\left(\tau, T_l\right) = f_l e^{Z_l}, \tag{3.4.9}$$

$$G_j \equiv e^{-\int_0^\tau n(s)ds} \cdot \varphi_j \frac{P_n\left(\tau, T_{j-1}\right)}{P_r\left(\tau, T_{j-1}\right)} P_r\left(\tau, T_j\right) \cdot e^{C(\tau, T_{j-1}, T_j)} = g_j e^{X_j},$$

$$H_k \equiv e^{-\int_0^\tau n(s)ds} \cdot \varphi_k P_n\left(\tau, T_k\right) = h_k e^{Y_k},$$

where, using the Hull-White zero-coupon bond formula (3.2.16) and the dynamics of the inflation index (3.2.22)[11], we distinguish deterministic factors which involve the functions A_k defined in (3.2.17):

$$\begin{aligned} f_l &= \varphi_l A_r(\tau, T_l) e^{-\frac{1}{2}\sigma_I^2(\tau - T_{l-1})}, \\ g_j &= \varphi_j e^{C(\tau, T_{j-1}, T_j)} \frac{A_n(\tau, T_{j-1})}{A_r(\tau, T_{j-1})} A_r(\tau, T_j), \\ h_k &= \varphi_k A_n(\tau, T_k), \end{aligned} \tag{3.4.10}$$

and the stochastic processes in the exponents:

$$\begin{aligned} X_j &= -\int_0^\tau n(s)ds \; - \; B_n(\tau, T_{j-1})n(\tau) \; - \; (B_r(\tau, T_j) - B_r(\tau, T_{j-1}))r(\tau), \\ Y_k &= -\int_0^\tau n(s)ds \; - \; B_n(\tau, T_k)n(\tau), \\ Z_l &= -\int_0^\tau n(s)ds \; - \; B_r(\tau, T_l)r(\tau) \; + \int_{T_{l-1}}^\tau (n(s) - r(s))\,ds + \sigma_I\left(W_I(\tau) - W_I(T_{l-1})\right) \\ &= -\int_0^{T_{l-1}} (n(s) - r(s))\,ds \; + \; \sigma_I \int_{T_{l-1}}^\tau dW_I(u) \; - \; B_r(\tau, T_l)r(\tau) \; - \int_0^\tau r(s)ds. \end{aligned} \tag{3.4.11}$$

[11]with $s = T_{l-1}$ and $t = \tau$. From the definition of the index l after equation (3.4.5) it is clear that $T_{l-1} \leq \tau \leq T_l$.

From expressions (3.2.29)-(3.2.30) and (3.2.37)-(3.2.38), and using definitions (3.2.23), (3.2.32) and (3.2.39), (3.2.40) the expectations of those processes are

$$\mathbb{E}[X_j] = \ln P_n(0,\tau) - \frac{1}{2}V_n(\tau) - B_n(\tau,T_{j-1})\left(f_n(0,\tau)+x_n(\tau)\right)$$
$$-\left(B_r(\tau,T_j)-B_r(\tau,T_{j-1})\right)\left(f_r(0,\tau)+x_r(\tau)+R(\tau)\right),$$

$$\mathbb{E}[Y_k] = \ln P_n(0,\tau) - \frac{1}{2}V_n(\tau) - B_n(\tau,T_k)\left(f_n(0,\tau)+x_n(\tau)\right), \tag{3.4.12}$$

$$\mathbb{E}[Z_l] = \ln\left(\frac{P_n(0,T_{l-1})}{P_r(0,T_{l-1})}\right) - \frac{1}{2}[V_n(T_{l-1})-V_r(T_{l-1})] + \overline{R}(T_{l-1})$$
$$-B_r(\tau,T_l)\left(f_r(0,\tau)+x_r(\tau)+R(\tau)\right) + \ln P_r(0,\tau) - \frac{1}{2}V_r(\tau) - \overline{R}(\tau).$$

For moment matching, we will use the moment generating function, which for a normal random variable $S \sim \mathcal{N}(\mu,\sigma^2)$ and $m \in \mathbb{R}$ takes the form

$$\mathbb{E}\left[e^{mS}\right] = e^{\mu m + \frac{\sigma^2}{2}m^2}. \tag{3.4.13}$$

Moment matching technique for $\boldsymbol{H}$. Let us first start with the quantity H. The first two moments of H are

$$\mathbb{E}[H] = (1+K)\cdot\sum_{j=l}^{M} h_j\mathbb{E}\left[e^{Y_j(\tau)}\right] = (1+K)\cdot\sum_{j=l}^{M} h_j e^{\mathbb{E}[Y_j]+\frac{1}{2}\mathrm{Var}(Y_j)}, \tag{3.4.14}$$

$$\begin{aligned}\mathbb{E}\left[H^2\right] &= (1+K)^2\sum_{p=l}^{M}\sum_{q=l}^{M} h_p h_q\mathbb{E}\left[e^{Y_p(\tau)+Y_q(\tau)}\right] \\ &= (1+K)^2\sum_{p=l}^{M}\sum_{q=l}^{M} h_p h_q e^{\mathbb{E}[Y_p]+\mathbb{E}[Y_q]+\frac{1}{2}\mathrm{Var}(Y_p+Y_q)},\end{aligned} \tag{3.4.15}$$

that are to be matched with the first two moments of the auxiliary lognormal variable be^Y which, according to the moment generating function expression (3.4.13) are

$$\begin{aligned}\mathbb{E}\left[be^Y\right] &= be^{\nu+\frac{\eta^2}{2}}, \\ \mathbb{E}\left[(be^Y)^2\right] &= b^2e^{2\nu+2\eta^2}.\end{aligned} \tag{3.4.16}$$

Referring to the definition of h_j in (3.4.10), of $A_n(\tau, T)$ in (3.2.17) and of the mean $\mathbb{E}[Y_j]$ in (3.4.12), we define

$$\begin{aligned}
d_j(\tau) &\equiv h_j e^{\mathbb{E}[Y_j]} \qquad (3.4.17)\\
&= \varphi_j A_n(\tau, T_j) e^{\ln P_n(0,\tau) - \frac{1}{2} V_n(\tau) - B_n(\tau,T_j)(f_n(0,\tau)+x_n(\tau))}\\
&= \varphi_j \frac{P_n(0,T_j)}{P_n(0,\tau)} e^{B_n(\tau,T_j) f_n(0,\tau) - \frac{1}{2} B_n(\tau,T_j)^2 v_n(t)} P_n(0,\tau) e^{-\frac{1}{2} V_n(\tau) - B_n(\tau,T_j)(f_n(0,\tau)+x_n(\tau))}\\
&= \varphi_j P_n(0,T_j) e^{-\frac{1}{2} V_n(\tau) - \frac{1}{2} B_n(\tau,T_j)^2 v_n(\tau) - B_n(\tau,T_j) x_n(\tau)},
\end{aligned}$$

and using (3.4.11) we find the variances

$$\begin{aligned}
\mathrm{Var}\,(Y_j) &= \mathrm{Var}\left(-\int_0^\tau n(s)ds - B_n(\tau,T_j) n(\tau)\right)\\
&= \mathrm{Var}\left(\int_0^\tau n(s)ds\right) + B_n(\tau,T_j)^2 \mathrm{Var}\,(n(\tau)) + 2B_n(\tau,T_j)\mathrm{Cov}\left(\int_0^\tau n(s)ds,\, n(\tau)\right)\\
&= V_n(\tau) + B_n(\tau,T_j)^2 v_n(\tau) + 2B_n(\tau,T_j) x_n(\tau),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\,(Y_p + Y_q) &= Var\left(-2\int_0^\tau n(s)ds - (B_n(\tau,T_p) + B_n(\tau,T_q))\, n(\tau)\right)\\
&= 4V_n(\tau) + (B_n(\tau,T_p) + B_n(\tau,T_q))^2 v_n(\tau) + 4\,(B_n(\tau,T_p) + B_n(\tau,T_q))\, x_n(\tau),
\end{aligned}$$

where definitions of variances (3.2.31) and (3.2.39) are used, and the covariance is derived as follows, using dynamics of the rate process (3.2.20):

$$\begin{aligned}
\mathrm{Cov}\left(\int_0^\tau k\,(s)\,ds,\; k\,(\tau)\right) &= \mathrm{Cov}\left(\sigma_k \int_0^\tau \int_0^s e^{-a_k(s-u)} dW_k\,(u) ds,\; \sigma_k \int_0^\tau e^{-a_k(\tau-u)} dW_k\,(u)\right) =\\
&= \mathrm{Cov}\left(\sigma_k \int_0^\tau \left(\int_u^\tau e^{-a_k(s-u)} ds\right) dW_k\,(u),\; \sigma_k \int_0^\tau e^{-a_k(\tau-u)} dW_k\,(u)\right) =\\
&= \mathrm{Cov}\left(\frac{\sigma_k}{a_k} \int_0^\tau (1 - e^{-a_k(\tau-u)}) dW_k\,(u),\; \sigma_k \int_0^\tau e^{-a_k(\tau-u)} dW_k\,(u)\right) =\\
&= \frac{\sigma_k^2}{a_k} \int_0^\tau (1 - e^{-a_k(\tau-u)}) e^{-a_k(\tau-u)} du = \frac{\sigma_k^2}{2a_k^2}\left(1 - e^{-a_k\tau}\right)^2 = x_k(\tau),
\end{aligned}$$

so the end result turns out to be exactly the quantity $x_k(\tau)$ defined in (3.2.24).

The moment expression (3.4.14) becomes

$$\mathbb{E}[H] = (1+K) \cdot \sum_{j=l}^{M} d_j\,(\tau)\, e^{\frac{1}{2} Var(Y_j)} = (1+K) \sum_{j=l}^{M} \varphi_j P_n(0,T_j), \qquad (3.4.18)$$

as expected (since H represents a discounted tradable asset price), and (3.4.15) becomes

$$\mathbb{E}\left[H^2\right] = (1+K)^2 \sum_{p=l}^{M} \sum_{q=l}^{M} d_p(\tau)\, d_q(\tau)\, e^{\frac{1}{2}\mathrm{Var}(Y_p+Y_q)} \tag{3.4.19}$$

$$= (1+K)^2 \sum_{p=l}^{M} \sum_{q=l}^{M} \varphi_p \varphi_q P_n(0,T_p) P_n(0,T_q) e^{V_n(\tau)+B_n(\tau,T_p)B_n(\tau,T_q)v_n(\tau)+(B_n(\tau,T_p)+B_n(\tau,T_q))x_n(\tau)}.$$

The form of $\mathbb{E}[H]$ as given by (3.4.18) shows that $\mathbb{E}\left[e^Y\right] = 1$, so $\nu = -\frac{\eta^2}{2}$, leaving us with the system of two equations with two unknowns (b, η^2):

$$(1+K) \sum_{j=l}^{M} \varphi_j P_n(0,T_j) = b, \tag{3.4.20}$$

$$(1+K)^2 \sum_{p=l}^{M} \sum_{q=l}^{M} \varphi_p \varphi_q P_n(0,T_p) P_n(0,T_q) e^{V_n(\tau)+B_n(\tau,T_p)B_n(\tau,T_q)v_n(\tau)+(B_n(\tau,T_p)+B_n(\tau,T_q))x_n(\tau)} = b^2 e^{\eta^2},$$

which gives b and the solution for η:

$$\eta^2 = \ln \frac{\sum_{p=l}^{M} \sum_{q=l}^{M} \varphi_p \varphi_q P_n(0,T_p) P_n(0,T_q) e^{V_n(\tau)+B_n(\tau,T_p)B_n(\tau,T_q)v_n(\tau)+(B_n(\tau,T_p)+B_n(\tau,T_q))x_n(\tau)}}{\left(\sum_{j=l}^{M} \varphi_j P_n(0,T_j)\right)^2}. \tag{3.4.21}$$

Moment matching technique for G and for the mixed sum GH. By using equation (3.4.13) for G, as given by (3.4.9), its moments are

$$\mathbb{E}[G] = f_l \mathbb{E}\left[e^{Z_l}\right] + \sum_{j=l+1}^{M} g_j \mathbb{E}\left[e^{X_j}\right] = f_l e^{\mathbb{E}[Z_l]+\frac{1}{2}\mathrm{Var}(Z_l)} + \sum_{j=l+1}^{M} g_j e^{\mathbb{E}[X_j]+\frac{1}{2}\mathrm{Var}(X_j)},$$

$$\mathbb{E}\left[G^2\right] = f_l^2\, \mathbb{E}\left[e^{2Z_l}\right] + 2 \sum_{j=l+1}^{M} f_l g_j \mathbb{E}\left[e^{Z_l+X_j}\right] + \sum_{p=l+1}^{M} \sum_{q=l+1}^{M} g_p g_q \mathbb{E}\left[e^{X_p+X_q}\right] \tag{3.4.22}$$

$$= f_l^2\, e^{2\mathbb{E}[Z_l]+2\mathrm{Var}(Z_l)} + 2 \sum_{j=l+1}^{M} f_l g_j e^{\mathbb{E}[Z_l]+\mathbb{E}[X_j]+\frac{1}{2}\mathrm{Var}(Z_l+X_j)} + \sum_{p=l+1}^{M} \sum_{q=l+1}^{M} g_p g_q e^{\mathbb{E}[X_p]+\mathbb{E}[X_q]+\frac{1}{2}\mathrm{Var}(X_p+X_q)},$$

$$\mathbb{E}[GH] = (1+K) \cdot \left\{ \sum_{j=l}^{M} f_l h_j \mathbb{E}\left[e^{Z_l+Y_j}\right] + \sum_{p=l+1}^{M} \sum_{q=l}^{M} g_p h_q \mathbb{E}\left[e^{X_p+Y_q}\right] \right\}$$

$$= (1+K) \cdot \left\{ \sum_{j=l}^{M} f_l h_j e^{\mathbb{E}[Z_l]+\mathbb{E}[Y_j]+\frac{1}{2}\mathrm{Var}(Z_l+Y_j)} + \sum_{p=l+1}^{M} \sum_{q=l}^{M} g_p h_q e^{\mathbb{E}[X_p]+\mathbb{E}[Y_q]+\frac{1}{2}\mathrm{Var}(X_p+Y_q)} \right\}.$$

As above, from the definitions of g_j and f_l in (3.4.10), of $A_k(\tau, T)$ in (3.2.17) and the expectations $\mathbb{E}[X_j]$ and $\mathbb{E}[Z_l]$ in (3.4.12), we define new deterministic functions

$$\begin{aligned}
c_j(\tau) &\equiv g_j e^{\mathbb{E}[X_j]} \\
&= \varphi_j e^{C(\tau,T_{j-1},T_j)} A_n(\tau, T_{j-1}) \frac{A_r(\tau, T_j)}{A_r(\tau, T_{j-1})} e^{\ln P_n(0,\tau) - \frac{1}{2}V_n(\tau) - B_n(\tau,T_{j-1})(f_n(0,\tau)+x_n(\tau))} \\
&\quad \cdot e^{-(B_r(\tau,T_j)-B_r(\tau,T_{j-1}))(f_r(0,\tau)+x_r(\tau)+R(\tau))} \\
&= \varphi_j e^{C(\tau,T_{j-1},T_j)} \frac{P_n(0,T_{j-1})}{P_n(0,\tau)} e^{B_n(\tau,T_{j-1})f_n(0,\tau) - \frac{1}{2}B_n(\tau,T_{j-1})^2 v_n(\tau)} \\
&\quad \cdot \frac{P_r(0,T_j)}{P_r(0,T_{j-1})} e^{(B_r(\tau,T_j)-B_r(\tau,T_{j-1}))f_r(0,\tau) - \frac{1}{2}\left(B_r(\tau,T_j)^2 - B_r(\tau,T_{j-1})^2\right)v_r(\tau)} \\
&\quad \cdot P_n(0,\tau) e^{-\frac{1}{2}V_n(\tau) - B_n(\tau,T_{j-1})(f_n(0,\tau)+x_n(\tau))} \cdot e^{-(B_r(\tau,T_j)-B_r(\tau,T_{j-1}))(f_r(0,\tau)+x_r(\tau)+R(\tau))} \\
&= \varphi_j \frac{P_n(0,T_{j-1})}{P_r(0,T_{j-1})} P_r(0,T_j) e^{C(\tau,T_{j-1},T_j)} \cdot e^{-\frac{1}{2}V_n(\tau) - \frac{1}{2}B_n^2(\tau,T_{j-1})v_n(\tau) - B_n(\tau,T_{j-1})x_n(\tau)} \\
&\quad \cdot e^{-\frac{1}{2}\left(B_r(\tau,T_j)^2 - B_r(\tau,T_{j-1})^2\right)v_r(\tau) - (B_r(\tau,T_j)-B_r(\tau,T_{j-1}))(x_r(\tau)+R(\tau))},
\end{aligned} \tag{3.4.23}$$

$$\begin{aligned}
e_l(\tau) &\equiv f_l e^{\mathbb{E}[Z_l]} \\
&= \varphi_l A_r(\tau, T_l) e^{-\frac{1}{2}\sigma_I^2(\tau - T_{l-1})} e^{\ln\left(\frac{P_n(0,T_{l-1})}{P_r(0,T_{l-1})}\right) - \frac{1}{2}[V_n(T_{l-1}) - V_r(T_{l-1})] + \overline{R}(T_{l-1})} \\
&\quad \cdot e^{-B_r(\tau,T_l)(f_r(0,\tau)+x_r(\tau)+R(\tau)) + \ln P_r(0,\tau) - \frac{1}{2}V_r(\tau) - \overline{R}(\tau)} \\
&= \varphi_l \frac{P_r(0,T_l)}{P_r(0,\tau)} e^{B_r(\tau,T_l)f_r(0,\tau) - \frac{1}{2}B_r(\tau,T_l)^2 v_r(\tau)} \frac{P_n(0,T_{l-1})}{P_r(0,T_{l-1})} e^{-\frac{1}{2}[V_n(T_{l-1}) - V_r(T_{l-1})] + \overline{R}(T_{l-1})} \\
&\quad \cdot P_r(0,\tau) e^{-B_r(\tau,T_l)(f_r(0,\tau)+x_r(\tau)+R(\tau)) - \frac{1}{2}V_r(\tau) - \overline{R}(\tau)} \cdot e^{-\frac{1}{2}\sigma_I^2(\tau - T_{l-1})} \\
&= \varphi_l \frac{P_n(0,T_{l-1})}{P_r(0,T_{l-1})} P_r(0,T_l) e^{-\frac{1}{2}V_n(T_{l-1}) - \frac{1}{2}(V_r(\tau) - V_r(T_{l-1})) - \left(\overline{R}(\tau) - \overline{R}(T_{l-1})\right) - \frac{1}{2}\sigma_I^2(\tau - T_{l-1})} \\
&\quad \cdot e^{-\frac{1}{2}B_r(\tau,T_l)^2 v_r(\tau) - B_r(\tau,T_l)(x_r(\tau)+R(\tau))},
\end{aligned}$$

which finally give us the moment expressions

$$\mathbb{E}[G] = e_l(\tau) e^{\frac{1}{2}\mathrm{Var}(Z_l)} + \sum_{j=l+1}^{m} c_j(\tau) e^{\frac{1}{2}\mathrm{Var}(X_j)}, \tag{3.4.24}$$

$$\mathbb{E}\left[G^2\right] = e_l(\tau)^2 e^{2\mathrm{Var}(Z_l)} + 2\sum_{j=l+1}^{M} e_l(\tau)c_j(\tau)e^{\frac{1}{2}\mathrm{Var}(Z_l+X_j)} + \sum_{p=l+1}^{M}\sum_{q=l+1}^{M} c_p(\tau)c_q(\tau)e^{\frac{1}{2}\mathrm{Var}(X_p+X_q)},$$

$$\mathbb{E}[GH] = (1+K)\cdot\left\{\sum_{j=l}^{M} e_l(\tau)d_j(\tau)e^{\frac{1}{2}\mathrm{Var}(Z_l+Y_j)} + \sum_{p=l+1}^{M}\sum_{q=l}^{M} c_p(\tau)d_q(\tau)e^{\frac{1}{2}\mathrm{Var}(X_p+Y_q)}\right\}.$$

With this expression we have all the necessary terms to apply formula (3.2.42). The required variance is obtained from the following identity:

$$\frac{\mathbb{E}\left[H^2\right]\mathbb{E}\left[G^2\right]}{\mathbb{E}\left[GH\right]^2} = \frac{a^2e^{2\mathbb{E}[X]+2\mathrm{Var}(X)}b^2e^{2\mathbb{E}[Y]+2\mathrm{Var}(Y)}}{\left(abe^{\mathbb{E}[X+Y]+\frac{1}{2}\mathrm{Var}(X+Y)}\right)^2} = e^{\mathrm{Var}(X)+\mathrm{Var}(Y)-2\mathrm{Cov}(X,Y)} = e^{\mathrm{Var}(X-Y)},$$

and thus

$$\mathrm{Var}(X-Y) \equiv \varsigma^2 = \ln\frac{\mathbb{E}\left[H^2\right]\mathbb{E}\left[G^2\right]}{\mathbb{E}\left[GH\right]^2}.$$

This variance together with b from (3.4.21) and $m_1(\tau) \equiv E[G]$ from first line of (3.4.24), are all we need for the approximate semi-analytical solution for the expected exposure at default of the YYIIS instrument according to Lemma 3.1, and equation (3.2.42) takes the final form

$$EE(\tau) = \omega N\left\{m_1\Phi\left(\omega\frac{\ln\frac{m_1}{b}+\frac{\varsigma^2}{2}}{\varsigma}\right) - b\Phi\left(\omega\frac{\ln\frac{m_1}{b}-\frac{\varsigma^2}{2}}{\varsigma}\right)\right\}. \tag{3.4.25}$$

The remaining moments in (3.4.24) are obtained from definitions of our stochastic processes (3.4.11) and the key identity (3.3.11), by considering

$$X_j-\mathbb{E}\left[X_j\right] = -\sigma_n\int_0^\tau B_n(u,T_{j-1})dW_n\left(u\right)-\sigma_r\int_0^\tau\left(e^{-a_r(T_{j-1}-\tau)}-e^{-a_r(T_{j-1}-\tau)}\right)e^{-a_r(\tau-u)}dW_r\left(u\right)$$

$$= -\sigma_n\int_0^\tau B_n(u,T_{j-1})dW_n\left(u\right) - \sigma_r\int_0^\tau\left(B_r(u,T_{j-1})-B_r(u,T_j)\right)dW_r\left(u\right),$$

$$Y_k-\mathbb{E}\left[Y_k\right] = -\sigma_n\int_0^\tau B_n(u,T_k)dW_n\left(u\right), \tag{3.4.26}$$

$$Z_l-\mathbb{E}\left[Z_l\right] = -\sigma_n\int_0^{T_{l-1}} B_n(u,T_{l-1})dW_n\left(u\right) + \sigma_I\int_{T_{l-1}}^\tau dW_n\left(u\right)$$

$$+\sigma_r\int_0^{T_{l-1}} B_r(u,T_{l-1})dW_r\left(u\right) - \sigma_r\int_0^\tau B_r(u,T_l)dW_r\left(u\right).$$

By looking at all possible covariance combinations of the above expression and after some very tedious algebra, we finally obtain all variance-covariance terms that are given by (3.4.8) in Proposition 3.2, concisely expressed in the terms of two generalised functions defined by (3.2.41). ■

3.5 Portfolios of ZCIIS's with Netting

In analogy to the previous result for the YYIIS CVA we can derive a similar formula for a netted portfolio of ZCIIS instruments. However, because of the moments matching technique, we must impose a restriction: we can consider either only payer instruments or only receiver instruments.

A modified technique was employed in the existing literature to overcome this limitation for the case of interest rate swaps with netting, for example by Brigo and Masetti (2005). An extra parameter was introduced to add a shift to the lognormal distributions to allow for the possibility of negative values. However, in our case it is not possible to proceed in a similar manner since our inflation CVA pricing formula is of the Margrabe type (i.e. we exchange one asset with a lognormal price distribution for another) whereas the interest rate swap CVA pricing formula is of the classical Black type. In our case, including such a shift to allow for negative netting coefficients would lead to a pricing formula where the Margrabe formula can no longer be applied. Instead, it would lead to a pricing formula of spread option type (see e.g. Brigo and Mercurio (2006), Appendix B, page 921).

Starting from the valuation equation of a single ZCIIS instrument (3.3.3), we now derive the valuation equation for a portfolio of ZCIIS instruments, if they are all payer or receiver swaps.

A portfolio of m payer instruments (which we will call PZCIIS) would lead to the following present value function

$$\begin{aligned}\text{PZCIIS}\left(t, I(0), \{T_j, N_j, K_j\}_{j=1,\dots,m}\right) &= \sum_{j=1}^{m} N_j \left(\frac{I(t)}{I(0)} P_r(t, T_j) - (1+K_j)^{T_j} \cdot P_n(t, T_j)\right) \\ &= \frac{I(t)}{I(0)} \sum_{j=1}^{m} N_j P_r(t, T_j) - \sum_{j=1}^{m} N_j \widehat{K}_j P_n(t, T_j), \quad (3.5.1)\end{aligned}$$

where $\widehat{K}_j \equiv (1+K_j)^{T_j}$, $\forall j = 1, ..., m$. The instruments are ordered by increasing maturity, so T_m represents the longest maturity. We choose buckets for the default time using a partition up to the longest instrument maturity in the portfolio:

$$(0 = t_0, t_1], (t_1, t_2], ..., (t_{n-2}, t_{n-1}], (t_{n-1}, t_n = T_m],$$

to write down the approximation as introduced earlier

$$\begin{aligned}CVA &\approx (1-R) \sum_{i=1}^{n} \mathbb{Q}\{\tau \in (t_i, t_{i-1}]\} EE(t_i), \\ EE(\tau) &= \mathbb{E}_0^{\mathbb{Q}_n}\left[D(0,\tau)\left(\omega\left(\frac{I(\tau)}{I(0)}\sum_{j=1}^{m} N_j P_r(\tau, T_j) - \sum_{j=1}^{m} N_j \hat{K}_j P_n(\tau, T_j)\right)\right)^+\right]. \quad (3.5.2)\end{aligned}$$

We can then use moment matching techniques to calculate approximations for the Expected Exposures, since the N_j and $\hat{K}_j$ are all of the same sign. We take them positive, and use $\omega = 1$ for payer, and $\omega = -1$ for receiver swaps.

Proposition 3.3. *Under the Jarrow Yildirim model* (3.2.11) *the moment matching approximation for the Expected Exposure at default time τ of a portfolio of netted Zero-Coupon Inflation Indexed Swaps* (3.5.1) *that are all in the same direction (either all payer when $\omega = 1$, or all receiver when $\omega = -1$) is*

$$EE(\tau) = \omega \left(a\Phi \left(\omega \frac{\ln \frac{a}{b} + \frac{\varsigma(\tau)^2}{2}}{\varsigma(\tau)} \right) - b\Phi \left(\omega \frac{\ln \frac{a}{b} - \frac{\varsigma(\tau)^2}{2}}{\varsigma(\tau)} \right) \right),$$

with coefficients

$$a = \sum_{j=1}^{m} N_j P_r(0, T_j), \qquad b = \sum_{j=1}^{m} N_j \widehat{K}_j P_n(0, T_j),$$

and variance

$$\varsigma(\tau)^2 = \ln \frac{\mathbb{E}\left[\bar{G}^2\right] \mathbb{E}\left[\bar{H}^2\right]}{\mathbb{E}\left[\bar{G}\bar{H}\right]^2}, \tag{3.5.3}$$

where the second moments are

$$\begin{aligned}
\mathbb{E}\left[\bar{G}^2\right] &= \sum_{p=1}^{m} \sum_{q=1}^{m} N_p N_q P_r(0, T_p) P_r(0, T_q) e^{\sigma_r^2 V_{r,r}(t,T_p,T_q) - \rho_{rI}\sigma_r\sigma_I(U_r(0,\tau,T_p)+U_r(0,\tau,T_q)) + \sigma_I^2\tau},\\
\mathbb{E}\left[\bar{H}^2\right] &= \sum_{p=1}^{m} \sum_{q=1}^{m} N_p N_q \widehat{K}_p \widehat{K}_q P_n(0, T_p) P_n(0, T_q) e^{\sigma_n^2 V_{n,n}(t,T_p,T_q)},\\
\mathbb{E}\left[\bar{G}\bar{H}\right] &= \sum_{p=1}^{m} \sum_{q=1}^{m} N_p \widehat{K}_p N_q P_n(0, T_p) P_r(0, T_q) e^{\rho_{nr}\sigma_n\sigma_r V_{n,r}(\tau,T_p,T_q) - \rho_{nI}\sigma_n\sigma_I U_n(0,\tau,T_p) - \rho_{rI}\sigma_r\sigma_I U_r(0,\tau,T_q)},
\end{aligned}$$

with functions $V_{k,l}(t, T_1, T_2)$, and $U_k(s, t, T)$ defined in (3.2.41).

Proof. As before, by looking at the expression for $EE(\tau)$ in equation (3.5.2), we can define the following quantities:

$$\bar{G} \equiv e^{-\int_0^\tau n(s)ds} \frac{I(\tau)}{I(0)} \sum_{j=1}^{m} N_j P_r(\tau, T_j), \tag{3.5.4}$$

$$\bar{H} \equiv e^{-\int_0^\tau n(s)ds} \sum_{j=1}^{m} N_j \widehat{K}_j P_n(\tau, T_j),$$

which leads to an expectation $EE(\tau)$ in the form $\mathbb{E}_0^{\mathbb{Q}_n}\left[\left(\omega\left(\bar{G} - \bar{H}\right)\right)^+\right]$ with, as before,

$$\bar{G} \approx ae^X, \quad \bar{H} \approx be^Y,$$

for bivariate Gaussian (X, Y).

The *netting coefficients* N_j and $N_j\widehat{K}_j$ are all of the same sign, giving us

$$\mathbb{E}_0^{\mathbb{Q}_n}\left[\left(\omega\left(\bar{G} - \bar{H}\right)\right)^+\right] \approx \mathbb{E}\left[\left(\omega\left(ae^X - be^Y\right)\right)^+\right].$$

As before, we only need to calculate

$$\mathrm{Var}(X-Y) = \ln \frac{\mathbb{E}\left[\bar{G}^2\right]\mathbb{E}\left[\bar{H}^2\right]}{\mathbb{E}\left[\bar{G}\bar{H}\right]^2},$$

and a and b to apply the Margrabe formula. Both $\bar{G}$ and $\bar{H}$ are similar to the quantity H in the previous YYIIS case, and we are going to rely heavily on our earlier derivations. The form of the quantities $\bar{G}$ and $\bar{H}$ allows us to apply the same method of separating deterministic and stochastic parts as was done previously for G and H, so similar expressions are found here:

$$\bar{G} = \sum_{j=1}^{m} \bar{c}_j\left(\tau\right) e^{\bar{X}_j(\tau)}, \tag{3.5.5}$$

$$\bar{H} = \sum_{j=1}^{m} \bar{d}_j\left(\tau\right) e^{\bar{Y}_j(\tau)},$$

with deterministic parts

$$\bar{c}_j\left(\tau\right) = N_j P_r(0,T_j) e^{\mathbb{E}[\bar{X}_j]}, \tag{3.5.6}$$

$$\bar{d}_j\left(\tau\right) = N_j \hat{K}_j P_n(0,T_j) e^{\mathbb{E}[\bar{Y}_j]},$$

and the zero-mean stochastic parts

$$\begin{aligned} \bar{X}_j - \mathbb{E}[\bar{X}_j] &= -\sigma_r \int_0^\tau \int_0^s e^{-a_r(s-u)} dW_r\left(u\right) ds - B_r(\tau,T_j)\sigma_r \int_0^\tau e^{-a_r(\tau-u)} dW_r\left(u\right) + \sigma_I W_I\left(\tau\right), \\ \bar{Y}_j - \mathbb{E}[\bar{Y}_j] &= -\sigma_n \int_0^\tau \int_0^s e^{-a_n(s-u)} dW_n\left(u\right) ds - B_n(\tau,T_j)\sigma_r \int_0^\tau e^{-a_n(\tau-u)} dW_n\left(u\right). \end{aligned} \tag{3.5.7}$$

After using identity (3.3.11) those stochastic processes become

$$\begin{aligned} \bar{X}_j - \mathbb{E}[\bar{X}_j] &= -\sigma_r \int_0^\tau B_r\left(u,T_j\right) dW_r\left(u\right) + \sigma_I W_I\left(\tau\right), \\ \bar{Y}_j - \mathbb{E}[\bar{Y}_j] &= -\sigma_n \int_0^\tau B_n\left(u,T_j\right) dW_n\left(u\right). \end{aligned} \tag{3.5.8}$$

Once again, we proceed using the moment matching technique. The moments are

$$
\begin{aligned}
\mathbb{E}[\bar{G}] &= \sum_{j=1}^{m} \bar{c}_j(\tau)\, e^{\frac{1}{2}\mathrm{Var}\left(\bar{X}_j(\tau)\right)} = \sum_{j=1}^{m} N_j P_r(0,T_j) e^{\mathbb{E}[\bar{X}_j]} e^{\frac{1}{2}\mathrm{Var}\left(\bar{X}_j\right)}, \\
\mathbb{E}\left[\bar{G}^2\right] &= \sum_{p=1}^{m}\sum_{q=1}^{m} \bar{c}_p(\tau)\,\bar{c}_q(\tau)\, e^{\frac{1}{2}\mathrm{Var}\left(\bar{X}_p(\tau)+\bar{X}_q(\tau)\right)} \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} N_p P_r(0,T_p) N_q P_r(0,T_q) e^{\mathbb{E}[\bar{X}_p]+\mathbb{E}[\bar{X}_q]} e^{\frac{1}{2}\mathrm{Var}\left(\bar{X}_p+\bar{X}_q\right)}, \\
\mathbb{E}[\bar{H}] &= \sum_{j=1}^{m} \bar{d}_j(\tau)\, e^{\frac{1}{2}\mathrm{Var}\left(\bar{Y}_j(\tau)\right)} = \sum_{j=1}^{m} N_j \widehat{K}_j P_n(0,T_j) e^{\mathbb{E}[\bar{Y}_j]} e^{\frac{1}{2}\mathrm{Var}\left(\bar{Y}_j\right)}, \\
\mathbb{E}\left[\bar{H}^2\right] &= \sum_{p=1}^{m}\sum_{q=1}^{m} \bar{d}_p(\tau)\,\bar{d}_q(\tau)\, e^{\frac{1}{2}\mathrm{Var}\left(\bar{Y}_p(\tau)+\bar{Y}_q(\tau)\right)} \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} N_p \widehat{K}_p P_n(0,T_p) N_q \widehat{K}_q P_n(0,T_q) e^{\mathbb{E}[\bar{Y}_p]+\mathbb{E}[\bar{Y}_q]} e^{\frac{1}{2}\mathrm{Var}\left(\bar{Y}_p+\bar{Y}_q\right)}, \\
\mathbb{E}\left[\bar{G}\bar{H}\right] &= \sum_{p=1}^{m}\sum_{q=1}^{m} \bar{d}_p(\tau)\,\bar{c}_q(\tau)\, e^{\frac{1}{2}\mathrm{Var}\left(\bar{Y}_p(\tau)+\bar{X}_q(\tau)\right)} \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} N_p \widehat{K}_p P_n(0,T_p) N_q P_r(0,T_q) e^{\mathbb{E}[\bar{Y}_p]+\mathbb{E}[\bar{X}_q]} e^{\frac{1}{2}\mathrm{Var}\left(\bar{Y}_p(\tau)+\bar{X}_q(\tau)\right)}.
\end{aligned}
\tag{3.5.9}
$$

We again have that lognormals $e^{\bar{X}_j(\tau)}$, and $e^{\bar{Y}_j(\tau)}$ are discounted tradable asset price processes and thus, the fact that $\mathbb{E}\left[e^{\bar{X}_j(\tau)}\right] = \mathbb{E}\left[e^{\bar{Y}_j(\tau)}\right] = 1$ gives us $\mathbb{E}\left[\bar{X}_j\right] = -\frac{1}{2}\mathrm{Var}(\bar{X}_j)$ and $\mathbb{E}\left[\bar{Y}_j\right] = -\frac{1}{2}\mathrm{Var}(\bar{Y}_j)$, for every j, as was the case for X and Y in (3.3.10).[12] After applying this fact to the moment expressions (3.5.9), they reduce to

$$
\begin{aligned}
\mathbb{E}[\bar{G}] &= \sum_{j=1}^{m} N_j P_r(0,T_j), \\
\mathbb{E}\left[\bar{G}^2\right] &= \sum_{p=1}^{m}\sum_{q=1}^{m} N_p N_q P_r(0,T_p) P_r(0,T_q) e^{\mathrm{Cov}\left(\bar{X}_p,\bar{X}_q\right)}, \\
\mathbb{E}[\bar{H}] &= \sum_{j=1}^{m} N_j \widehat{K}_j P_n(0,T_j), \\
\mathbb{E}\left[\bar{H}^2\right] &= \sum_{p=1}^{m}\sum_{q=1}^{m} N_p N_q \widehat{K}_p \widehat{K}_q P_n(0,T_p) P_n(0,T_q) e^{\mathrm{Cov}\left(\bar{Y}_p,\bar{Y}_q\right)}, \\
\mathbb{E}\left[\bar{G}\bar{H}\right] &= \sum_{p=1}^{m}\sum_{q=1}^{m} N_p \widehat{K}_p N_q P_n(0,T_p) P_r(0,T_q) e^{\mathrm{Cov}\left(\bar{Y}_p,\bar{X}_q\right)}.
\end{aligned}
\tag{3.5.10}
$$

[12]Please note that here X and Y denote different quantities from $\bar{X}_j(\tau)$ and $\bar{Y}_j(\tau)$. Here X and Y are auxiliary variables to approximate processes $\frac{1}{a}\ln\bar{G}$ and $\frac{1}{b}\ln\bar{H}$, and not the original processes themselves as was the case in Section 3.3.

We observe that as before, we are going to need expressions for the covariances of mixed terms:

$$\mathrm{Cov}\left(\bar{X}_p, \bar{X}_q\right) = \mathrm{Cov}\left\{-\sigma_r \int_0^\tau B_r\left(u, T_p\right) dW_r\left(u\right) + \sigma_I W_I\left(\tau\right),\ -\sigma_r \int_0^\tau B_r\left(u, T_q\right) dW_r\left(u\right) + \sigma_I W_I\left(\tau\right)\right\}$$

$$= \sigma_r^2 \int_0^\tau B_r\left(u, T_p\right) B_r\left(u, T_q\right) du - \rho_{rI}\sigma_r\sigma_I \int_0^\tau B_r(u, T_p) du - \rho_{rI}\sigma_r\sigma_I \int_0^\tau B_r(u, T_q) du + \sigma_I^2\tau$$

$$= \sigma_r^2 V_{r,r}(t, T_p, T_q) - \rho_{rI}\sigma_r\sigma_I U_r(0, \tau, T_p) - \rho_{rI}\sigma_r\sigma_I U_r(0, \tau, T_q) + \sigma_I^2\tau,$$

$$\mathrm{Cov}\left(\bar{Y}_p, \bar{Y}_q\right) = \mathrm{Cov}\left\{-\sigma_n \int_0^\tau B_n\left(u, T_p\right) dW_n\left(u\right),\ -\sigma_n \int_0^\tau B_n\left(u, T_q\right) dW_n\left(u\right)\right\}$$

$$= \sigma_n^2 \int_0^\tau B_n\left(u, T_p\right) B_n\left(u, T_q\right) du = \sigma_n^2 V_{n,n}(t, T_p, T_q), \tag{3.5.11}$$

$$\mathrm{Cov}\left(\bar{Y}_p, \bar{X}_q\right) = \mathrm{Cov}\left\{-\sigma_n \int_0^\tau B_n\left(u, T_p\right) dW_n\left(u\right),\ -\sigma_r \int_0^\tau B_r\left(u, T_q\right) dW_r\left(u\right) + \sigma_I W_I\left(\tau\right)\right\}$$

$$= \rho_{nr}\sigma_n\sigma_r \int_0^\tau B_n(u, T_p) B_r(u, T_q) du - \rho_{nI}\sigma_n\sigma_I \int_0^\tau B_n(u, T_p) du - \rho_{rI}\sigma_r\sigma_I \int_0^\tau B_r(u, T_q) du$$

$$= \rho_{nr}\sigma_n\sigma_r V_{n,r}(\tau, T_p, T_q) - \rho_{nI}\sigma_n\sigma_I U_n(0, \tau, T_p) - \rho_{rI}\sigma_r\sigma_I U_r(0, \tau, T_q),$$

which are expressed in terms of the functions $V_{k,l}(t, T_1, T_2)$, and $U_k(s, t, T)$ defined in (3.2.41).

Now, when looking at the right hand sides of the equations in (3.5.10), from the form of the first moments we can conclude that in this case for both auxiliary variables X and Y the convenient property that $\mathbb{E}\left[X\right] = -\frac{1}{2}\mathrm{Var}(X)$ and $\mathbb{E}\left[Y\right] = -\frac{1}{2}\mathrm{Var}(Y)$ holds. This was to be expected since the sum of discounted tradables is still a discounted tradable so $\mathbb{E}\left[e^X\right] = \mathbb{E}\left[e^Y\right] = 1$ holds, i.e. $\mathbb{E}\left[\bar{G}/a\right] = \mathbb{E}\left[\bar{H}/b\right] = 1$. This gives us the two unknowns a and b:

$$\sum_{j=1}^{m} N_j P_r(0, T_j) = a, \tag{3.5.12}$$

$$\sum_{j=1}^{m} N_j \widehat{K}_j P_n(0, T_j) = b, \tag{3.5.13}$$

which together with the Margrabe type formula (3.2.43) gives the final closed form approximation for the expected exposure at default of the portfolio of ZCIIS instruments. ■

3.6 Numerical Studies

3.6.1 ZCIIS CVA Results

In this subsection we will compare profiles of the Expected Exposure of the ZCIIS instruments given by the exact formula (3.3.5) to numerically obtained approximations based on Monte Carlo simulations. As an example, we take a ZCIIS instrument of payer type, with a notional of 2.79 million euros, and a fixed rate of 2.49%, which was priced at its inception on 10-Dec-2010 and expires on 03-Dec-2040. We will look at other strikes and maturities afterwards.

The maturity of the contract is therefore $T = 30$ years. The parameters used were $a_n = 0.0498$, $a_r = 0.2273$, $\sigma_n = 0.0108$, $\sigma_r = 0.0078$, $\sigma_I = 0.0339$ and $\rho_{nr} = -0.8755$, $\rho_{nI} = -0.3861$ and $\rho_{rI} = 0.7799$. The nominal rate term structure was taken to be flat with $n_0 = 1\%$, which together with the term structure of the ZCIIS strikes gives an initial flat real rate term structure with $r_0 = -1.46\%$. The initial value of the inflation index was taken to be $I_0 = 117$ and the default recovery rate $REC = 0.4$.

Figure 3.1 shows the results. We can clearly see that the Monte Carlo result is close to our closed form solution when enough paths have been simulated. To calculate the CVA, the profiles for the Expected Exposure were combined with counterparty default probabilities, which for simplicity have been based on a flat hazard rate curve which gives a flat curve with a daily probability of default of 10^{-5}. This produces a CVA of 226.68 basis points (63,244 euros) for our analytic solution versus 226.54 basis points (63,204 euros) for the Monte Carlo simulation with 10.000 paths. The relative error is thus of the order of magnitude of 10^{-4}.

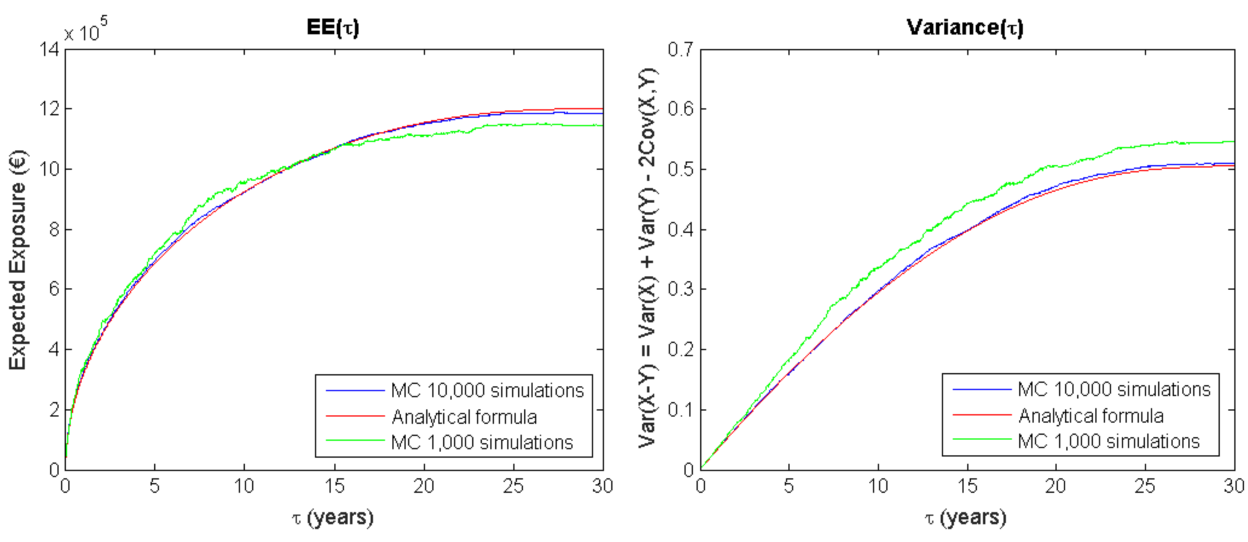

Figure 3.1: The expected exposure profile $EE(\tau)$ (left) and squared volatility $\varsigma(\tau)^2$ (right) for the ZCIIS, obtained by Monte Carlo simulations with 1,000 paths (green), 10.000 paths (blue) and our formula (red).

In the same figure, the expected exposure profile obtained by the Monte Carlo method with 1,000 simulations was added to show that results are still not very reliable for that number of simulations. On the right hand side of the figure we show the time evolution of the relevant variance. We notice that for a small number of Monte Carlo simulations the Expected Exposure may be underestimated at large maturities while the variance is systematically overestimated in this case. Table 3.1 shows CPU times and relative errors for our method and the Monte Carlo simulations for another strike and another maturity as well. In both cases daily time steps were used for the calculations. We see that our formula outperforms simulations in terms of computational time needed. For errors of a basis point or less, Monte Carlo simulation CPU time is 15 times higher than for our method. Figure 3.2 shows the relationship between speed and accuracy for Monte Carlo methods on a logarithmic scale.

		Number of Monte Carlo simulations						
	Analytical	250	500	1,000	2,000	4,000	8,000	16,000
T=30, K=2.49%								
CVA (bp)	226.6807	209.6532	214.1824	229.1274	225.0112	226.5375	226.5790	226.7129
relative error (10^{-3})	0	-75.1	-55.4	10.8	-7.4	-0.63	-0.45	0.14
CPU time (s)	2.62	5.12	8.09	12.75	22.72	46.65	128.32	205.10
T=10, K=2.49%								
CVA (bp)	49.7244	55.0273	45.5484	47.5670	48.2522	50.8175	49.3963	49.6539
relative error (10^{-3})	0	106.6	-84.0	-43.4	-29.6	22.0	-6.6	-1.4
CPU time (s)	2.52	4.87	6.34	11.60	20.15	39.97	130.10	210.53
T=10, K=1.49%								
CVA (bp)	91.9073	95.7819	88.6794	89.8195	89.6964	90.7518	92.5880	91.4571
relative error (10^{-3})	0	42.2	-35.1	-22.7	-24.1	-12.6	-7.4	-4.9
CPU time (s)	2.94	5.17	9.39	15.25	32.15	46.65	128.32	205.10

Table 3.1: The ZCIIS CVA example: values for an increasing number of simulations.

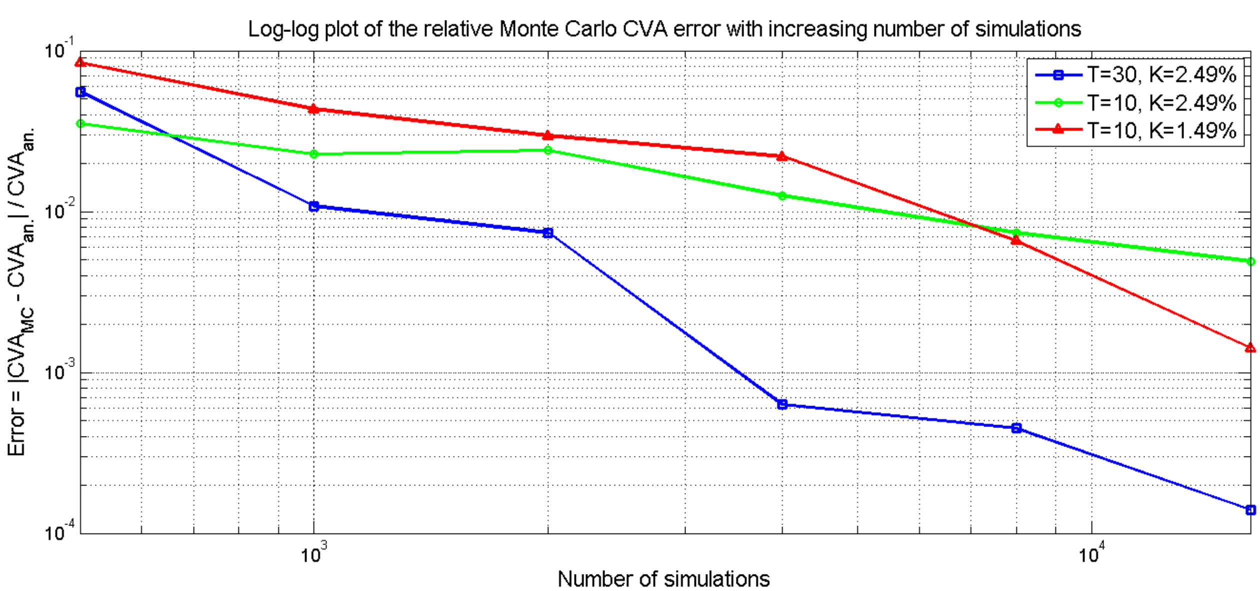

Figure 3.2: The log-log graph of the Monte Carlo error for the ZCIIS CVA value for an increasing number of simulations and different maturity/strike combinations.

3.6.2 YYIIS CVA Results

We will now compare the analytically obtained approximation for the Expected Exposure profile of the YYIIS instruments given by (3.4.25) to the numerically obtained values using Monte Carlo simulations.

The YYIIS instrument was chosen to be similar to the ZCIIS contract from the previous section, so it is of payer type, with a notional of 2.79 million euros, with year-on-year fixed rate at 2.44%. This rate was determined such that the contract has zero value at its inception on 10-Dec-2010 on which nominal and real rate term structures were set flat at $n_0 = 1\%$ and $r_0 = -1.46\%$, respectively. The fixing dates are $T_i = i$ for $i = 1 \ldots 30$ for 30 years until its maturity date on 03-Dec-2040. The other model parameters are the same as before.

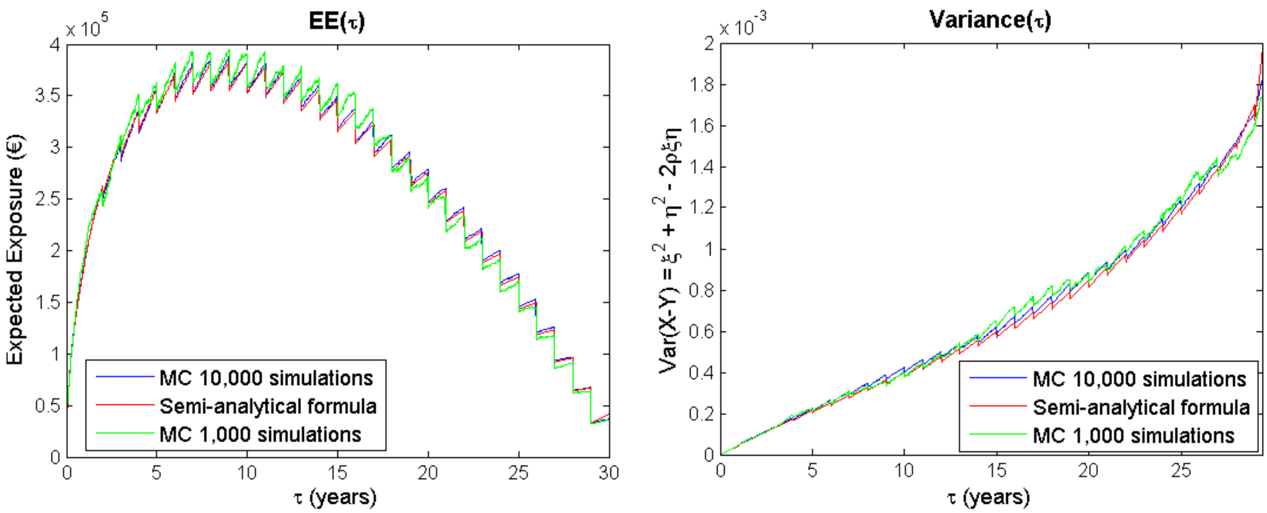

Figure 3.3: The expected exposure profile $EE(\tau)$ (left) and total variance $\varsigma(\tau)^2$ (right) for the YYIIS product, obtained by Monte Carlo simulations with 1,000 paths (green), 10.000 paths (blue) and our formula (red).

The time profile of the expected exposure is given in Figure 3.3 (left) and the time evolution of the variance is shown in Figure 3.3 (right). An almost perfect match is observed between the Monte Carlo result with 10.000 scenarios and the closed form solution. The two expected profiles combined with the same daily counterparty default as in the previous section lead to CVA values of 61.90 basis points (17,271 euros) for our formula and 61.76 basis points (17,232 euros) for the Monte Carlo simulation with 10.000 paths.[13] Table 3.2 summarises another set of CVA calculations together with the respective CPU times, for a varying number of Monte Carlo scenarios and different strikes and fixing dates.

[13]Please note that in this case we do not refer to the difference between Monte Carlo results and our formula as "error" but as a "difference". This is due to the fact that our closed form solution in this case was derived by using a moment matching approximation and as such cannot be regarded as the "true" value. However in order to compare results they are presented (analogous to the case of ZCIIS CVA in previous subsection) by showing the relative difference of Monte Carlo results with respect to the value obtained by our closed form approximate solution. The same is the case in the following subsection which presents results of the Portfolio of ZCIIS CVA.

		Number of Monte Carlo simulations						
	Analytical	250	500	1,000	2,000	4,000	8,000	12,000
T={$T_1, ..., T_{30}$}, K=2.44%								
CVA (bp)	61.9030	58.0250	63.3920	62.5657	62.6567	62.1411	62.1615	61.8255
relative difference (10^{-3})	0	-62.6	24.1	10.7	12.2	3.8	4.2	-1.3
CPU time (s)	85.36	28.39	41.77	79.77	130.2	154.75	589.45	1841.7
T={$T_1, ..., T_{10}$}, K=2.48%								
CVA (bp)	8.5462	8.0912	8.2534	8.2413	8.3257	8.4136	8.4817	8.5279
relative difference (10^{-3})	0	-53.2	-34.3	-35.7	-25.8	-15.5	-7.5	-2.1
CPU time (s)	5.8	8.78	13.94	16.43	20.48	26.91	50.11	173.2
T={$T_1, ..., T_{10}$}, K=1.48%								
CVA (bp)	15.4998	14.7891	16.0172	15.3403	15.4338	15.6062	15.5302	15.5039
relative difference (10^{-3})	0	-45.9	33.4	-10.3	-4.3	6.9	2.0	0.3
CPU time (s)	5.26	7.02	8.32	18.45	29.47	41.83	43.81	181.77

Table 3.2: The YYIIS CVA example: values for an increasing number of simulations.

The graph showing speed versus accuracy on a logarithmic scale is presented in Figure 3.4 and it clearly demonstrates the usefulness of our method when accurate CVA calculations are required. Another advantage of the analytically obtained result can be seen in the situation when we have YYIIS contracts with the same fixing time structure $T_1, T_2, ..., T_M$ that differ only in strikes K. Then we can actually retain most of the earlier calculations, and in particular the expressions for variance-covariance terms given by (3.4.8), when we consider products with the same structure but different strikes. In this case the reduction in CPU times when compared to Monte Carlo methods is even larger. We tested our method for many different fixing dates and strikes and it performed well in all cases.

We have also tested the method using actual term structure of nominal and real interest rates as of 10-Dec-2010, given in Figure 3.5, as opposed to flat term structure which was assumed in all numerical studies above. Figure 3.6 demonstrates that the method works equally well also in this more realistic case. Exactly the same YYIIS instrument from the beginning of this subsection was used and the CVA in this case was 37.81 basis points (10,549 euros) for our formula and 38.06 basis points (10,618 euros) for the Monte Carlo simulation with 10.000 paths, which amounts to a relative difference of $6.5 \cdot 10^{-3}$. For different parameters of the YYIIS instrument both relative differences in the obtained CVA and the CPU times are in line with the previous (flat term) structure results presented in Table 3.2, which shows that our closed form solution performs equally well for this more realistic term structure and as such it can be used in practice.

There are cases where our approximation works less well. An example is a long maturity, already in life YYIIS contract initiated at times of low (high) inflation. Thus having a low (high) year-on-year fixed rate K, combined with a present term structure which implies extreme inflation (deflation). This market expectation of periods of extreme inflation (deflation) would mean increasing (decreasing) ZCIIS rates $K(T)$, from formula (3.3.4), which would lead to out-of-the-money Margrabe options in Proposition 3.2, for which the two-moment matching approximation to a lognormal distribution overprices the option. This is consistent with findings of Milevsky and Posner (1998b) that moment matching may overprice out-of-the-money basket call options (for moneyness around 0.8 or less). However in order for this effect to be significant in our case we must have strong divergence of nominal and real rates (think of them approximately as the quantities on the second panel of Figure 3.5) to be of the order of magnitude $0.2K$ per year, and for our example case of $K = 2.44\%$, this would amount to around 0.5% difference between every two year points. We see that in the realistic case presented here the difference between those two curves was almost constant.

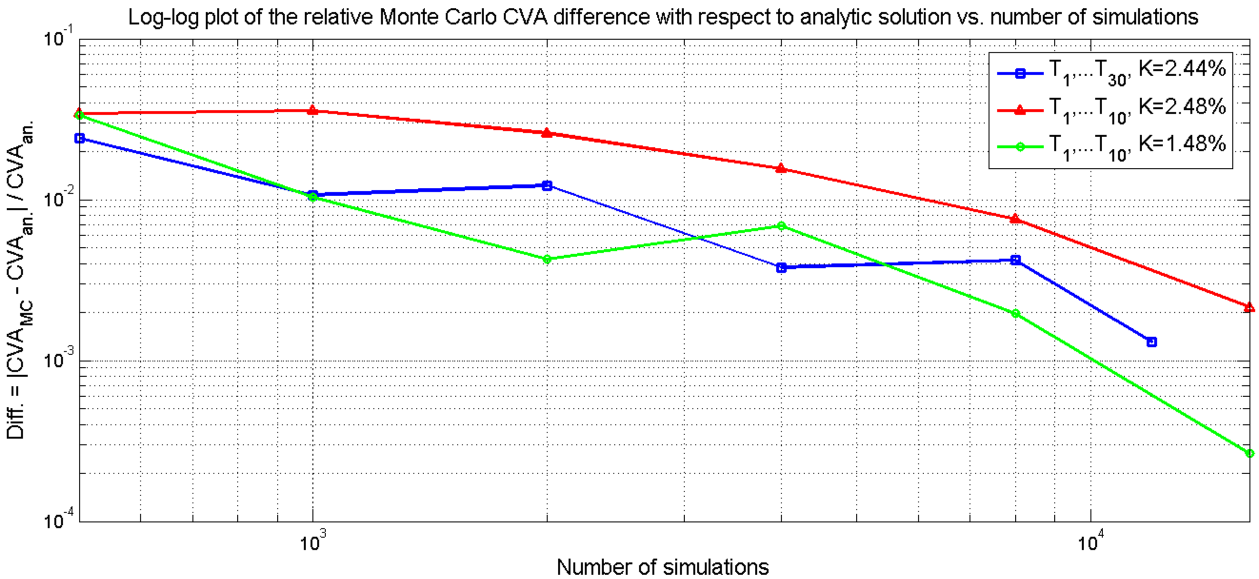

Figure 3.4: The log-log graph of the difference between Monte Carlo and our formula for the YYIIS CVA value for an increasing number of simulations and different strike/fixing-time combinations.

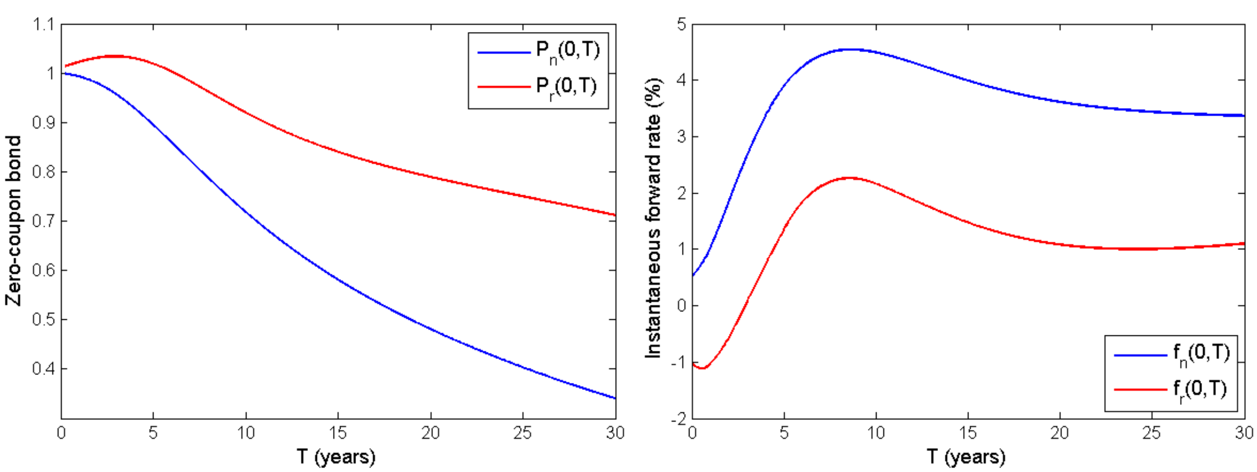

Figure 3.5: The actual term structure of nominal (blue) and real (red) interest rate prevailing at the inception of the YYIIS contract.

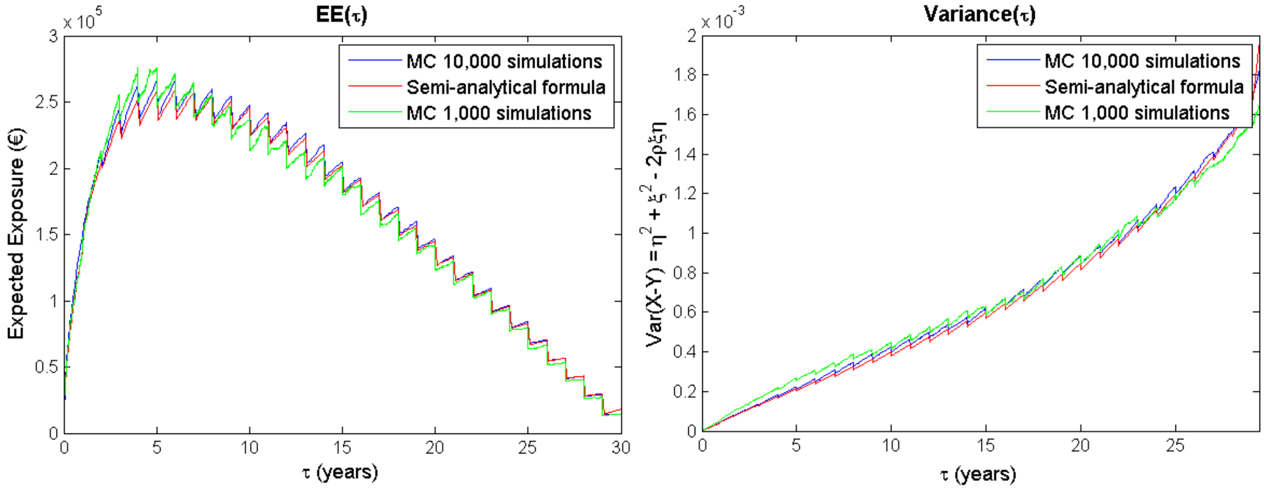

Figure 3.6: The expected exposure profile $EE(\tau)$ (left) and total variance $\varsigma(\tau)^2$ (right) for the YYIIS product, obtained by Monte Carlo simulations with 10.000 paths (blue) and our formula (red), using actual (non-flat) initial term structure of nominal and real rates.

3.6.3 Portfolios of ZCIIS's with Netting CVA Results

We now consider a portfolio of thirty ZCIIS swaps which are all of the payer type, with equal notionals of 2.79 million euros, maturities ranging from 1 to 30 years and fixed rates (strikes) at [1.04%, 1.09%, ..., 2.44%, 2.49%]. These have been priced from their inception on 10-Dec-2010 until the expiration of the last contract on 03-Dec-2040. The Expected Exposure and variance results are depicted in Figure 3.7.

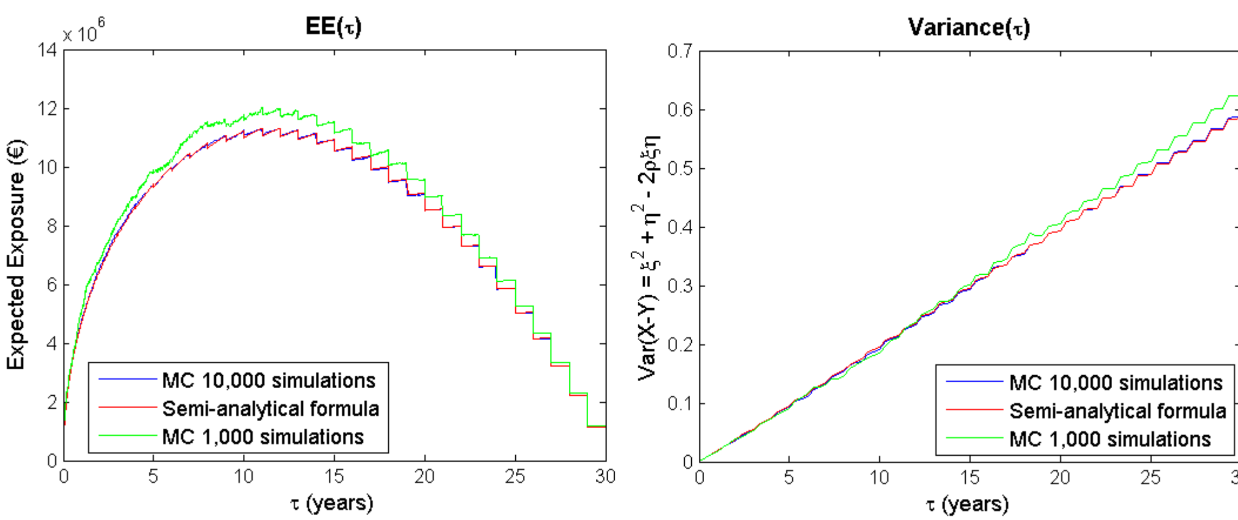

Figure 3.7: The expected exposure profile $EE(\tau)$ (left) and total variance $\varsigma(\tau)^2$ (right) for the portfolio of ZCIIS's, obtained by Monte Carlo simulations with 1,000 paths (green), 10.000 paths (blue) and our formula (red).

		Number of Monte Carlo simulations						
	Analytical	250	500	1,000	2,000	4,000	8,000	12,000
T_m=**30, K**=[1.04%,...,2.49%]								
CVA (bp)	63.3918	68.9087	66.3526	66.7889	62.9987	64.0150	63.2245	63.4176
relative difference (10^{-3})	0	87.0	46.7	53.6	-6.2	9.8	-2.6	0.4
CPU time (s)	10.14	15.40	17.60	23.92	54.83	127.65	259.25	405.28
T_m=**10, K**=[1.04%,...,1.49%]								
CVA (bp)	6.6903	6.1501	6.2623	6.8864	6.5409	6.5663	6.7037	6.6871
relative difference (10^{-3})	0	-80.7	-64.0	29.3	-22.3	-18.5	2.0	0.5
CPU time (s)	1.28	7.16	9.96	11.86	24.87	20.36	46.24	116.91
T_m=**10, K**=[0.04%,...,0.49%]								
CVA (bp)	2.0615	2.2326	1.6711	2.187	2.0493	2.0567	2.0630	2.0621
relative difference (10^{-3})	0	83.0	-189.4	60.9	-5.9	-2.3	0.7	0.3
CPU time (s)	0.95	5.32	6.88	7.54	19.66	26.78	36.42	108.12

Table 3.3: The portfolio of ZCIIS's CVA example: values for an increasing number of simulations.

The CVA obtained from our approximating profile is 63.39 basis points (530,589 euros) versus 63.42 basis points (530,826 euros) from the simulation with 10.000 scenarios. The Monte Carlo result with only 1,000 simulations shows significant overestimation in the profiles, and thus it leads to a higher CVA value of 66.79 bp (559,022 euros). Table 3.3 gives further examples for different maturities and strikes of the ZCIIS portfolio and the performance graph is shown in Figure 3.8.

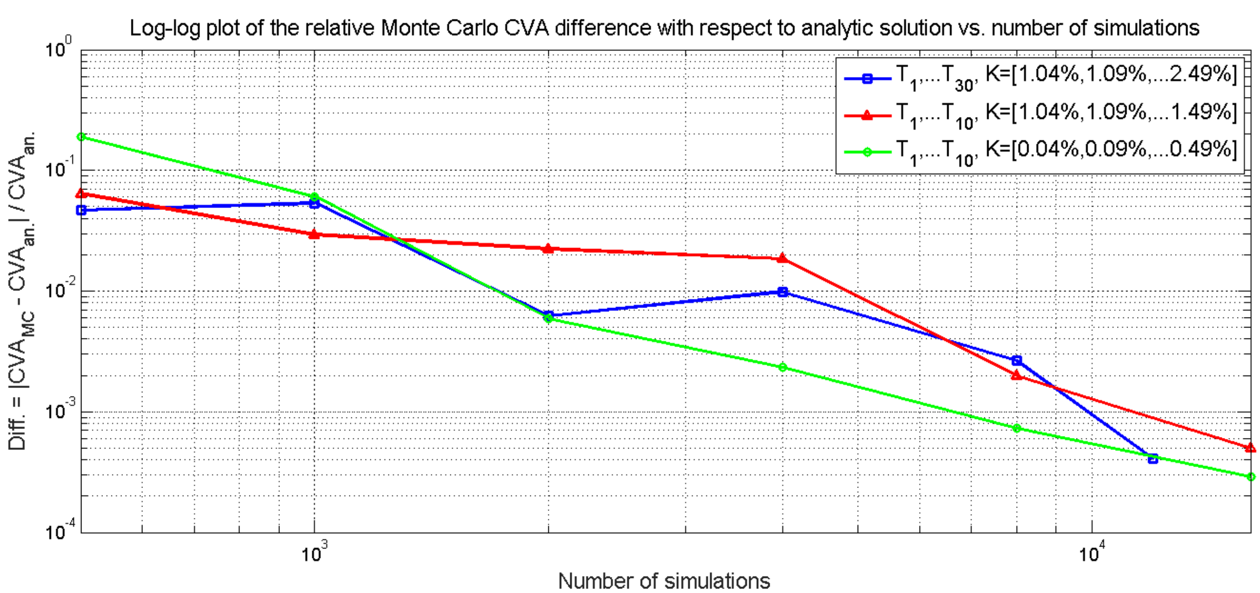

Figure 3.8: The log-log graph of the difference between Monte Carlo and our formula for the portfolio of ZCIIS's CVA value for an increasing number of simulation and different combinations of strikes and maturities.

3.7 Conclusion

We have given a derivation of an exact analytical solution for the ZCIIS CVA under the Jarrow-Yildirim model. We have compared it with a standard Monte Carlo CVA calculation approach and have shown in a simulation study that Monte Carlo estimates indeed converge to values that are close to our analytic solution, but at less efficient computation times.

Furthermore, closed form approximations for the Expected Exposure of the Year-on-Year Inflation Indexed Swap and for a portfolio of Zero-Coupon Inflation Indexed Swap instruments were derived using moment matching techniques.

The substantial CPU time reduction (see the Tables 1-3 that report example CPU times for different cases) show that our methods are much faster than Monte Carlo simulations. If we aim to obtain an accuracy of say 0.1%, this can mean a considerable saving of time for institutions that have many inflation linked derivative instruments in their portfolios.

An obvious limitation of our approach is due to the fact that financial institutions will have many derivative positions towards the same counterparty and it is thus crucial to consider netting agreements. The CVA is a portfolio-based value and it is less meaningful as a stand alone quantity for a singe derivative position. We derived a formula for netted portfolios of ZCIIS instruments but we had to impose that notionals are all of the same sign, i.e. they are all either only payer or only receiver instruments.[14] This is necessary because of the moments matching technique used here, in order to stay within the domain of application of the Margrabe formula. As mentioned before, it is possible to employ a modified moments matching technique to overcome this limitation as in Brigo and Masetti (2005) for the case of interest rate swaps with netting. An extra parameter is then introduced to add a shift to the lognormal distributions to allow for the possibility of negative values. The quantities $\bar{G}$ and $\bar{H}$ in Proposition 3.3 would remain the sums of lognormal random variables as defined in (3.5.4), but they would now be approximated with

$$\bar{G} \approx ae^{X} - c, \quad \bar{H} \approx be^{Y} - d,$$

to ensure that each *netting coefficient* N_j in the sums can be of arbitrary sign. However, the CVA pricing formula (3.5.2) will no longer be of the Margrabe type but will instead contain the term $\mathbb{E}_0^{\mathbb{Q}_n}\left[\left(\omega\left(ae^X - be^Y - c + d\right)\right)^+\right]$, with X and Y correlated normal random variables, which is the pricing formula for a spread option. The approximate solutions for this problem have been developed starting with Kirk (1995) and Pearson (1995) and later followed by Li, Deng and Zhou (2006). Brigo and Mercurio (2006) give a pseudo-analytical formula for this expectation in terms of improper integrals. Therefore, by building on that work it should be possible to further extend our approach and arrive at a general CVA formula for a portfolio of different inflation derivatives altogether under netting agreements.

A second limitation of our CVA formulae is that they do not account for the possibility of *wrong way risk*, which acknowledges that the likelihood of a counterparty's default event may depend on the market value of the expected exposure towards that counterparty. Whether this is a realistic assumption can vary from case to case, mostly depending on the

[14] A naive adjustment would be to simply add those two CVA numbers at the end, which is indeed an acceptable estimate in the absence of a more exact method, but it would always over-estimate the real CVA value since $(X+Y)^+ \leq X^+ + Y^+$, for any two real numbers X and Y.

asset class.[15] Properly addressing this issue would require an underlying model for credit (which is not needed in the usual approach under which the default probabilities are given exogenously) which could then be coupled to the interest rate/inflation model through appropriately correlated Brownian motions. The book of Brigo, Morini, Pallavicini (2013) gives a detailed theoretical treatment and an overview of current industry practice to deal with this issue. Pioneered by Brigo and Pallavicini (2007) work on wrong way risk developed further and besides Brigo and Vrins (2017), who price the wrong way risk CVA via a change of measure and drift adjustment, notable more recent contributions include Glasserman and Yang (2016), who present an adjustment to bound the wrong way risk and Feng and Oosterlee (2016), who study wrong way risk CVAs for European and Bermudan options based on an intensity model and present very efficient numerical algorithms.

Our approach can, however, be directly applied to another important counterparty risk adjustment - Debt Value Adjustment (DVA), which is an adjustment to the price of the derivative to account for the fact that our institution is also not default-free, i.e. this is the discount we are expected to give to our counterparty since we may default as well. The final value of the derivative actually becomes $Price_{Default-free} - CVA + DVA$. This is easily obtained from (3.2.5) by "flipping" sides (recovery rate and default probabilities in that equation represent those of the counterparty, and the expected exposure is ours, while for DVA it is the opposite), i.e.

$$\text{DVA} \approx (1 - R_{our}) \sum_{i=1}^{n} \mathbb{Q}_{our}\left(\tau \in (t_i, t_{i-1}]\right) EE_{cpt.}(t_i), \tag{3.7.1}$$

$$EE_{cpt.}(t_i) = NEE_{.}(t_i) = \mathbb{E}_0^{\mathbb{Q}}\left[D(0,t_i)(V(t_i))^{-}\right] = -\mathbb{E}_0^{\mathbb{Q}}\left[D(0,t_i)(-V(t_i))^{+}\right].$$

The problem boils down to determining the counterparty's (positive) Expected Exposure towards us. That is actually our Negative Expected Exposure[16] $NEE_{.}(t_i)$, which can still be obtained by formulae presented in this thesis using the sign flipping above, which would only affect the corresponding ω.

A final limitation worth mentioning is in the Jarrow-Yildirim model itself. It is analytically tractable and very intuitive due to its similarity to the classical short rate model of Hull and White, but it is difficult to calibrate and it may not be sufficiently sophisticated for the more advanced inflation derivatives. However, our methods cannot be directly applied to more complex inflation models such as the ones in Mercurio (2005), Kazziha (1999) and Belgrade, Benhamou and Koehler (2004). Nonetheless, for those more complicated cases it is reasonable to resort back to Monte Carlo based methods[17] and employ our method to generate a convenient control variate in order to improve the accuracy of the calculations.

[15] Consider for example an oil producing company as a counterparty to which we have significant exposure in the form of commodity derivatives. It would be very erroneous to assume that the probability of default of that company is totally independent of our exposure to the commodities market (i.e. oil prices).

[16] $V(t_i)$ is still value of the derivative position for us, as given by the second equation in (3.2.4).

[17] The LIBOR market model is a multifactor model often used by practitioners under which derivatives are almost exclusively priced by Monte Carlo.

Chapter 4

Equilibrium Bid-Ask Spread of European Derivatives in Dry Markets*

4.1 Introduction

Financial markets present equilibrium bid-ask spreads that can only be explained by market imperfections. The related literature has focused on equity markets and on market imperfections such as information asymmetries and transaction costs. In this paper we are interested in the equilibrium bid-ask spread of derivatives. There is a microstructure literature that tries to explain the spread of derivatives. It includes some papers,[1] all based on either the inventory approach[2] or the information based approach.[3] This paper shows that equilibrium bid-ask spreads for derivatives may be generated by an alternative imperfection, namely by a dry (illiquid) market for the underlying asset, whereby transactions may not be permitted at all times. Such illiquid situations are of practical importance. As an example, exchanges are closed many hours per day and stocks cannot be traded during that period. Also, there is strong empirical evidence of large trading volume in the beginning and end of trading days, and relatively light trading in the middle of that period, see Jain and Joh (1986), among others. Such pattern reflects that markets are essentially dry away from opening and closing hours. These are typical situations where adding assets will not help completing the market. Admati and Pfleiderer (1988, 1989) seminal papers started modelling such intraday dryness for stock markets. Such literature was expanded to markets such as the FX, see Bollerslev

*This chapter extends earlier unpublished work by Amaro de Matos and Lacerda (2006). We will cite, with permission from the authors, from that report throughout this chapter, without continuously repeating this reference.

[1]Biais, Foucault and Salanié (1998) analyse three different market structures and the ways the associated restrictions lead to differences in prices, bid-ask spreads, trades and risk-sharing. There are also a few empirical studies that examine bid-ask spreads in the derivatives markets, such as George and Longstaff (1993), Chan, Chung and Johnson (1995) and Etling and Miller (2000).

[2]Among others, Stoll (1978) and Amihud and Mendelson (1980) study bid-ask spreads and stock inventory. Lee, Mucklow and Ready (1993), Hasbrouck and Sofianos (1993), Madhavan and Smidt (1993) and Manaster and Mann (1996) also find evidence on the relationship of bid-ask spreads to market maker inventory costs.

[3]Some authors discussing the topic are Copeland and Galai (1983), Glosten and Milgrom (1985), Admati and Pfleiderer (1992) and Foster and Viswanathan (1994). More recently, Morrison (2004), Bagnoli *et al.* (2001) and Vayanos (2001).

and Domowitz (1993). Other common examples where trading may not be possible include refracting periods for swing options and vesting periods for employee stock options.

An illiquid underlying asset implies that markets become incomplete, in the sense that perfect hedging of the derivative in all states of nature is no longer possible. Under such incompleteness a unique non-arbitrage price for the derivative does not exist. Rather, there is a range of values within which all possible equilibrium prices must lie. Thus, the equilibrium bid-ask spread must fall within such range.

A particular characterisation of no-arbitrage bounds on prices for European derivatives is based on the construction of a super-replicating portfolio as in El Karoui and Quenez (1991, 1995), Edirisinghe, Naik and Uppal (1993) and Karatzas and Kou (1996), and Föllmer and Leukert (1999), among others. Amaro de Matos and Antão (2001) characterise super-replicating bounds for European option pricing under dry markets. In most cases the super-replication bounds produce too broad intervals, and certainly not equilibrium values. Other works using super-replication strategies under different no-arbitrage criteria obtain narrower super-replication bounds.[4] Different bounds can be obtained through utility indifference pricing, as introduced by Hodges and Neuberger (1989). Despite being utility dependent, this method has a meaningful economic interpretation. However, as Davis, Panas and Zariphopoulou (1993) point out, utility indifference pricing does not determine equilibrium prices either, but rather define an interval within which the trading values must lie. Alternatively, we may introduce the marginal price as the utility indifference price for an infinitesimal quantity as in Davis (1997), Karatzas and Kou (1996), and Kallsen (2002). Although the marginal price can be shown to be unique as in Karatzas and Kou (1996), and Hugonnier, Kramkov and Schachermayer (2005), it is not yet an equilibrium price, reflecting simply the willingness to pay for a marginal (infinitesimal) amount.

The value of a derivative can also be seen as its expected discounted payoff under the risk-neutral probability measure. As in incomplete markets there are many such admissible measures, authors have proposed different criteria to select a number of them, *e.g.* the minimal martingale measure by Follmer and Schweizer (1990), the variance optimal measure by Schweizer (1996) and the minimal entropy measure by Rouge and El Karoui (2000) and Frittelli (2000a, 2000b). An alternative approach was taken by Jarrow, Protter and Shimbo (2010) who explain the existence of bubbles assuming that incomplete markets may exhibit different local martingale measures across time, a non-trivial extension of the classical martingale pricing framework. The utility-maximisation price dependence on the choice of the distance metric can be found in Henderson (2005) and Henderson *et al.* (2003). Coherent risk-neutral probability measures were studied by Artzner *et al.* (1999), and convex risk-neutral measures by Follmer and Schied (2002). Such measures were introduced to axiomatise measures of risk and to generalise the properties of utility indifference prices.

With the exception of the marginal price, all the methodologies proposed above either establish a range of variation for the value of the derivative or use an *ad hoc* criterion in order to get the price without any economic insight. One exception related to our approach is the work of Barrieu and El Karoui (2002), where derivative pricing rules are characterised in incomplete markets taking into account the demand and supply sides of the markets.

Another notable exception is Duffie, Garleanu and Pedersen (2005), where the bid-ask

[4] See the works of Bernardo and Ledoit (2000), Cochrane and Saá-Requejo (2000) and Bondarenko (2003).

spread is also explained by illiquidity, basing their result in the theory of intermediation. They obtain a bid-ask equilibrium modeling a market where investors meet randomly according to their search efforts in the presence of market makers. In their model illiquidity - and thus, market incompleteness - is associated with counterparty's search effort and bargaining power. In other words, the bid-ask spread depends on how difficult it is for an investor to find other investors to transact, or have easier (or not) access to multiple market makers. Results for a monopolistic market maker are then characterised as a special case. Our model differs substantially from this approach as we do not delve into any modelling of the mechanics of searching for a trading counterparty. In our work market incompleteness is generated by dry markets and the equilibrium bid-ask spread is derived from utility (or profit) maximisation first principles, providing a different perspective than the results in Duffie *et al.* (2005).

In this work we characterise equilibrium bid and ask prices for our model of dry markets. Equilibrium must verify that all agents maximise their utilities and markets clear. Hence, the determination of equilibrium aggregates all agents' decisions. Using the utility maximisation approach we construct the market demand and supply for the derivative from the point of view of a representative trader. We then introduce the market makers, who are intermediaries responsible for setting the bid and ask prices. We first analyse equilibrium in a context where there is one monopolistic market maker, followed by a similar analysis of the case where many market makers compete in prices. Our results show that under risk-neutral market makers, an equilibrium bid-ask spread is obtained for the monopolistic case, whereas no equilibrium exists under price competition. This leads us to conclude that for bid-ask spreads to exist under competition, we should either have risk-averse market makers or consider a mixed strategy Nash equilibrium in a game between risk-neutral market makers. Our results assume neither asymmetric information nor optimal inventory strategies and still explains the existence of an equilibrium bid-ask spread for derivatives. It is related to the failure of the continuous hedging portfolio rebalancing hypothesis since we assume, as in Longstaff (2001), that the underlying asset can not be transacted at all points in time.

Our work is organised as follows. Section 4.2 characterises the demand and supply for derivatives, and simulations are performed for different types of utilities. Section 4.3 states the problem of the market maker(s), presenting first the monopoly case and then the oligopoly case. Section 4.4 concludes. Our main technical proofs are presented in the Appendix B.

4.2 The Model

A two-period recombining binomial model not allowing for transactions at the intermediate point in time is formally equivalent to a one period trinomial model. In the remaining of this work we thus consider a one-period economy, with dates 0 and t, where transactions are only possible at those two discrete points in time. Due to liquidity, constraints transactions are only possible at those two discrete times. At time t there are three possible states of nature, labelled by $i = 1, 2, 3$, with respective probabilities p_i. In this economy there are three different assets being transacted: a risk free asset with unitary initial value, providing a certain total return R per period; a risky asset (the underlying asset) with initial value S_0 and uncertain final values S_i, for $i = 1, 2, 3$; and a European derivative, written on the underlying asset, with expiration date t. In particular, we number the states in an order such that $S_1 > S_2 > S_3$. Notice also that, in order to avoid arbitrage opportunities, we must have $S_1 > RS_0 > S_3$. The possible payoffs of the derivative at time t are denoted G_i, for $i = 1, 2, 3$, and depend only on the final state of nature. We also assume that the payoffs of each considered European derivative are ordered according to the states' labels, in a monotonic way (i.e. decreasing for the call-like, increasing for the put-like derivative).

Every agent builds a portfolio composed of shares, risk free asset and derivatives. Each agent can influence neither the market price of the underlying asset, nor the market price of the derivative. Each representative agent maximises a von Neumann-Morgenstern utility $\mathbb{E}U$, of the wealth at time t, where the utility function U is increasing and concave in wealth.

In the following Section 4.2.1 we derive the individual and the market aggregate demand and supply functions for the derivative. Section 4.2.2 characterises its no-arbitrage price bounds and reservation price. In Section 4.2.3 the individual demand and supply for two different types of utility functions are presented in order to illustrate the method.

4.2.1 Demand/Supply

Consider a representative agent that maximises the expected value of wealth at the terminal date t. The problem that the agent faces is to choose at time 0 the number of shares of the underlying asset Δ_0, the amount B_0 invested in the risk-free asset and how many units of the derivative he is going to *buy* or *sell* (q_d and q_s respectively) for a given price of the derivative. This portfolio will be held until time t. The problem that the representative agent solves is essentially the same for demand or supply (apart from the change of sign). We thus use the notation[5] $q_{d/s}$ for the demanded or supplied amount, and formalise the problem as follows:

$$\max_{\Delta_0, B_0, q_{d/s}} \mathbb{E}U\left[\Delta_0 S + RB_0 \pm q_{d/s} G\right] = \max_{\Delta_0, B_0, q_{d/s}} \sum_{i=1}^{3} p_i U\left[\Delta_0 S_i + RB_0 \pm q_{d/s} G_i\right],$$

subject to

$$\Delta_0 S_0 + B_0 \pm q_{d/s} P_{d/s} \leq y, \tag{4.2.1}$$

$$\Delta_0 S_i + RB_0 \pm q_{d/s} G_i \geq 0, \qquad i = 1, 2, 3 \tag{4.2.2}$$

and $q_{d/s} \geq 0$.

The set of solutions for the problem is implicitly characterised by the following result.

Proposition 4.1. *The maximisation problem above is solved by every Δ_0^* and $q_{d/s}^*$ satisfying*

$$\begin{aligned} S_0 &= \frac{1}{R} \sum_{i=1}^{3} \alpha\left(w_{i,d/s}\right) S_i, \\ P_{d/s} &\geq (\leq) \frac{1}{R} \sum_{i=1}^{3} \alpha\left(w_{i,d/s}\right) G_i, \quad \text{if } q_{d/s}^* > 0 \end{aligned} \tag{4.2.3}$$

[5] Together with " $\pm$ ", " $\geq (\leq)$" and " $\mp$ " symbols further below, meaning " $+$ ", " $\geq$ " and " $-$ " signs for the demand problem, i.e. with corresponding q_d, P_d and $w_{i,d}$ quantities, and " $-$ ", " $\leq$ " and " $+$ " signs for the supply problem, i.e. with corresponding q_d, P_d and $w_{i,s}$.

with

$$\alpha\left(w_{i,d/s}\right)=\frac{p_iU'\left(w_{i,d/s}\right)}{\sum_{j=1}^{3}p_jU'\left(w_{j,d/s}\right)},\qquad i=1,2,3$$

and

$$w_{i,d/s}=\Delta_0^*\left(S_i-RS_0\right)+Ry\pm q_{d/s}^*\left(G_i-RP_{d/s}\right). \tag{4.2.4}$$

Proof. As utility is increasing in wealth, constraint (4.2.1) is satisfied as an equality, so

$$B_0=y-\Delta_0S_0\mp q_{d/s}P_{d/s} \tag{4.2.5}$$

and the problem above can be rewritten as

$$\max_{\Delta_0,q_{d/s}}\ \sum_{i=1}^{3}p_iU\left[\Delta_0\left(S_i-RS_0\right)+Ry\pm q_{d/s}\left(G_i-RP_{d/s}\right)\right],$$

subject to

$$w_{i,d/s}\equiv\Delta_0\left(S_i-RS_0\right)+Ry\pm q_{d/s}\left(G_i-RP_{d/s}\right)\geq 0,\qquad i=1,2,3$$

and $q_{d/s}\geq 0$.

Ignoring the positivity constraints, the first order conditions are:

$$\left\{\begin{array}{l}\frac{\partial\mathbb{E}\left[U\left(w_{i,d/s}\right)\right]}{\partial\Delta_0}=0,\\ \\ \frac{\partial\mathbb{E}\left[U\left(w_{i,d/s}\right)\right]}{\partial q_{d/s}}\leq 0;\end{array}\right.\Rightarrow\left\{\begin{array}{l}\displaystyle\sum_{i=1}^{3}p_i(S_i-RS_0)U'(w_{i,d/s})=0,\\ \\ \displaystyle\pm\sum_{i=1}^{3}p_i(G_i-RP_{d/s})U'(w_{i,d/s})\leq 0;\end{array}\right. \tag{4.2.6}$$

leading to equation (4.2.3). The maximum is guaranteed since the second order conditions are satisfied. See Appendix B.1.1 for details. ■

In the next proposition, a necessary and sufficient condition for the individual demand $\phi_d\left(P_d\right)$ to be decreasing in prices, and for the individual supply $\phi_s\left(P_s\right)$ to be increasing in prices is established.

Theorem 4.1. *A necessary and sufficient condition to obtain both $\phi_d'\left(P_d\right)<0$ and $\phi_s'\left(P_s\right)>0$, is that*

$$\begin{aligned}&\sum_{i=1}^{3}p_i\left(S_i-RS_0\right)^2U''\left(w_{i,d/s}\right)\sum_{i=1}^{3}p_iU'\left(w_{i,d/s}\right)\\&+q\left\{\sum_{i=1}^{3}p_i\left(S_i-RS_0\right)\left(G_i-RP_{d/s}\right)U''\left(w_{i,d/s}\right)\sum_{i=1}^{3}p_i\left(S_i-RS_0\right)U''\left(w_{i,d/s}\right)\right.\\&\left.-\sum_{i=1}^{3}p_i\left(S_i-RS_0\right)^2U''\left(w_{i,d/s}\right)\sum_{i=1}^{3}p_i\left(G_i-RP_{d/s}\right)U''\left(w_{i,d/s}\right)\right\}\leq 0.\end{aligned}$$

Proof. See Appendix B.1.2 for details. ■

From the maximisation problem faced by a representative buyer/seller of the derivative, we obtain the optimal amount $q_{d/s} = \phi_{d/s}\left(P_{d/s}\right)$. If this function is monotonic, i.e., if the condition of the above proposition is satisfied, $\phi_{d/s}\left(P_{d/s}\right)$ may be inverted in order to obtain an individual market demand/supply

$$P_{d/s} = \phi_{d/s}^{-1}\left(q_{d/s}\right).$$

Assuming that there are n equal agents buying/selling the derivative in this economy, the market demand/supply ($Q_{d/s}$) and the inverse market demand/supply can be written as

$$Q_{d/s} = nq_{d/s} = n\phi_{d/s}\left(P_{d/s}\right) \Rightarrow P_{d/s} = \phi_{d/s}^{-1}\left(\frac{Q_{d/s}}{n}\right). \tag{4.2.7}$$

4.2.2 Arbitrage Bounds, Reservation Price and Fair Price

4.2.2.1 Arbitrage Bounds and Finite Utility

In order to guarantee that there are no arbitrage opportunities in this market, we must ensure that the price of the derivative is within the super-replication (no-arbitrage) bounds.

The upper bound of the arbitrage-free range of variation is given by

$$P^u = \min_{\Delta_0, B_0} \Delta_0 S_0 + B_0,$$

subject to

$$\Delta_0 S_i + RB_0 \geq G_i, \quad i = 1, 2, 3.$$

The lower bound of the arbitrage-free range of variation is given by

$$P^l = \max_{\Delta_0, B_0} \Delta_0 S_0 + B_0,$$

subject to

$$\Delta_0 S_i + RB_0 \leq G_i, \quad i = 1, 2, 3.$$

The upper and lower bounds can be written in a shorter way, if we introduce some simplifying notation. Let the vector of parameters of our model be $\pi \equiv \left(S_0, S_1, S_2, S_3, G_1, G_2, G_3, R\right)$. We further define

$$\begin{aligned}
\mathcal{G}^+ &= \left\{\pi : G_1\left(S_2 - S_3\right) + G_2\left(S_3 - S_1\right) + G_3\left(S_1 - S_2\right) \geq 0\right\}, \\
\mathcal{G}^- &= \left\{\pi : G_1\left(S_2 - S_3\right) + G_2\left(S_3 - S_1\right) + G_3\left(S_1 - S_2\right) \leq 0\right\}, \\
\mathcal{H}^+ &= \left\{\pi : S_2 - RS_0 \geq 0\right\}, \\
\mathcal{H}^- &= \left\{\pi : S_2 - RS_0 \leq 0\right\},
\end{aligned}$$

and

$$P_{i,j} = \frac{G_j\left(S_i - RS_0\right) - G_i\left(S_j - RS_0\right)}{R\left(S_i - S_j\right)}. \tag{4.2.8}$$

We then have

$$P^u = P_{1,3} \text{ and } P^l = P_{2,3}$$

if $\pi \in \mathcal{G}^+ \cap \mathcal{H}^+$,

$$P^u = P_{1,3} \text{ and } P^l = P_{1,2}$$

if $\pi \in \mathcal{G}^+ \cap \mathcal{H}^-$,

$$P^u = P_{2,3} \text{ and } P^l = P_{1,3}$$

if $\pi \in \mathcal{G}^- \cap \mathcal{H}^+$, and

$$P^u = P_{1,2} \text{ and } P^l = P_{1,3}$$

if $\pi \in \mathcal{G}^- \cap \mathcal{H}^-$.[6]

We claim that prices above P^u or below P^l will generate arbitrage opportunities and therefore, infinite utility. The reason is as follows. First consider demand. If the derivative's price is below the lower bound, it would then be possible to buy the derivative with the proceedings obtained by selling a super-replicating portfolio with higher value, assuring a positive wealth at time t, in all possible states of nature. Hence, the utility and the optimal solution would not be finite. In what concerns supply, if the derivative's price is higher than the upper bound, it would then be possible to sell the derivative and buy a cheaper super-replicating portfolio with the proceedings, assuring once again, an unbounded positive wealth as well.

This issue can be analysed going back to the maximisation problem of both buyers and sellers. In fact, there is no restriction to infinite solutions to these problems. If we would have added the restriction $0 < q < \infty$, the optimal solution would immediately imply that $\phi(P^u) > q > \phi(P^l)$. Therefore, the imposition of finite solutions provides an alternative way to characterise the bounds of the no-arbitrage region. Finite q implies that the inequality in the first order condition (4.2.6) can be replaced by a strict equality, reducing the set of solutions to a single solution. This is true both for the demand and supply problem and thus, ignoring whether the investor is buying or selling the derivative i.e. $P_{d/s} \equiv P$, the first order conditions (4.2.6) can be written as:

$$\begin{cases} \sum_{i=1}^{3} \frac{p_i S_i}{R S_0} U'(w_i) = \sum_{i=1}^{3} p_i U'(w_i), \\ \sum_{i=1}^{3} \frac{p_i G_i}{R P} U'(w_i) = \sum_{i=1}^{3} p_i U'(w_i). \end{cases}$$

If there is a finite solution for the maximisation problem then

$$\sum_{i=1}^{3} p_i U'(w_i) \equiv A > 0,$$

where w_i is evaluated at the optimum values of Δ, $q_{d/s}$ and B. Hence, the first order conditions solve the following system

$$\begin{cases} \sum_{i=1}^{3} \frac{p_i S_i}{R S_0} U'(w_i) = A, \\ \sum_{i=1}^{3} p_i U'(w_i) = A, \\ \sum_{i=1}^{3} \frac{p_i G_i}{R P} U'(w_i) = A. \end{cases}$$

[6] See Appendix B.2 for the full derivation of the bounds.

This is a linear system in $U'(w_1)$, $U'(w_2)$ and $U'(w_3)$ with the solution

$$\begin{cases} U'(w_1) = A\frac{RP(S_2-S_3)+G_3(RS_0-S_2)-G_2(RS_0-S_3)}{G_1(S_2-S_3)+G_2(S_3-S_1)+G_3(S_1-S_2)}\frac{1}{p_1} \equiv A\mu_1, \\ U'(w_2) = A\frac{RP(S_3-S_1)+G_1(RS_0-S_3)-G_3(RS_0-S_1)}{G_1(S_2-S_3)+G_2(S_3-S_1)+G_3(S_1-S_2)}\frac{1}{p_2} \equiv A\mu_2, \\ U'(w_3) = A\frac{RP(S_1-S_2)+G_2(RS_0-S_1)-G_1(RS_0-S_2)}{G_1(S_2-S_3)+G_2(S_3-S_1)+G_3(S_1-S_2)}\frac{1}{p_3} \equiv A\mu_3. \end{cases} \tag{4.2.9}$$

As $U'(w_1)$, $U'(w_2)$ and $U'(w_3)$ are strictly positive, we must impose some constraints on the parameters, implying that $P^u > P > P^l$ for P^u and P^l defined as above for the different regions of π.

4.2.2.2 Reservation Price and Fair Price

In this subsection we characterise the behaviour of investors when the price of the derivative is *actuarially fair*.[7] In particular, we establish under which conditions investors would prefer to buy or would prefer to sell the derivative. We also provide conditions under which the investors prefer to buy or prefer to sell the underlying asset. In both cases the conditions do not depend on the preferences. Additionally, we establish the relation between the actuarially fair price $\bar{P}$ and the reservation price $\hat{P}$ of the derivative, where the reservation price is defined as the price at which the optimal transacted quantity is zero. The results are as follows.

If both the derivative and underlying asset have actuarially fair values, investors prefer to assure a risk-free wealth at maturity and do not transact the derivative or the risky asset. Alternatively, if the price of the underlying asset is not actuarially fair, there are two possibilities. First, if the asset is undervalued, the agent will buy it, i.e. $S_0 < \frac{1}{R}\sum_{i=1}^{3} p_i S_i \Rightarrow \Delta^* > 0$. Second, if the asset is overvalued, the agent will sell it, i.e. $S_0 > \frac{1}{R}\sum_{i=1}^{3} p_i S_i \Rightarrow \Delta^* < 0$. Furthermore, investors will buy $(q_d = q^* > 0)$ or sell $(q_s = q^* < 0)$ the derivative, depending on the payoff structure, as characterised in the following theorem.

Theorem 4.2. *Let the price of the derivative P be actuarially fair, $P = \bar{P} \equiv \frac{1}{R}\sum_{i=1}^{3} p_i G_i$. Also, let $\bar{P}$ belong to the arbitrage-free range of variation. Then the sign of Δ^* and q^* are characterised in the table below, as well as the fair price relationship with the reservation price.*

	$G_1 \geq G_2 \geq G_3$	$G_1 \leq G_2 \leq G_3$
$\sum_{i=1}^{3} p_i (S_i - RS_0) > 0$	$\Delta^* > 0; q^* < 0 \Rightarrow \hat{P} < \bar{P}$	$\Delta^* > 0; q^* > 0 \Rightarrow \hat{P} > \bar{P}$
$\sum_{i=1}^{3} p_i (S_i - RS_0) = 0$	$\Delta^* = 0; q^* = 0 \Rightarrow \hat{P} = \bar{P}$	$\Delta^* = 0; q^* = 0 \Rightarrow \hat{P} = \bar{P}$
$\sum_{i=1}^{3} p_i (S_i - RS_0) < 0$	$\Delta^* < 0; q^* > 0 \Rightarrow \hat{P} > \bar{P}$	$\Delta^* < 0; q^* < 0 \Rightarrow \hat{P} < \bar{P}$

The proof of this result is presented in detail in the Appendix B.3 giving the sign of Δ^* and q^*. As for the reservation price $\hat{P}$, we know that it must depend on the investor's utility by definition, as opposed to the exogenous $\bar{P}$.

When $P = \bar{P}$, the sign of the optimal transacted quantity of the derivative is well defined and does not depend on the risk aversion of the investors. Hence, $\hat{P}$ will be larger than $\bar{P}$ if

[7] We say that the price of the derivative is actuarially fair if it equals the expected payoff at maturity discounted at the risk-free rate.

the demanded quantity is positive for $P = \bar{P}$, and $\hat{P}$ will be smaller than $\bar{P}$ if the demanded quantity is negative for $P = \bar{P}$.

Remark 1. Although $\hat{P}$ depends on the utility of the investors, the fact that $\hat{P} \leq \bar{P}$ or $\hat{P} \geq \bar{P}$ depends only on the parameters of the economy, not on the investors' preferences.

Remark 2. If $\hat{P}$ belongs to the arbitrage-free range, $\bar{P}$ must be outside the range defined by $\left(\inf \hat{P}, \sup \hat{P}\right)$, where the infimum and the supremum above are taken over the class of all admissible utility functions.

4.2.3 Illustrations

We now consider the cases of Constant Absolute Risk Aversion (CARA) utility function $U(w) = -e^{-\delta w}$, with absolute risk aversion coefficient $\delta > 0$, and Constant Relative Risk Aversion (CRRA) utility function $U(w) = \frac{w^{1-\gamma}-1}{1-\gamma}$, with relative risk aversion coefficient $\gamma > 0$. Notice that, when $\gamma = 0$, the latter utility becomes $U(w) = w - 1$, characterising a risk neutral agent. Moreover, when $\gamma \to 1$, the utility function converges to $U(w) = \ln(w)$.

4.2.3.1 Explicit Solution for Demand and Supply

For these utility functions it is possible to have an explicit solution for the demand and supply of the derivative. The derivation is presented in Appendix B.4.1. From (4.2.9) we have:

$$\begin{aligned}
\mu_1 &= \tfrac{RP(S_2-S_3)+G_3(RS_0-S_2)-G_2(RS_0-S_3)}{G_1(S_2-S_3)+G_2(S_3-S_1)+G_3(S_1-S_2)} \tfrac{1}{p_1}, \\
\mu_2 &= \tfrac{RP(S_3-S_1)+G_1(RS_0-S_3)-G_3(RS_0-S_1)}{G_1(S_2-S_3)+G_2(S_3-S_1)+G_3(S_1-S_2)} \tfrac{1}{p_2}, \\
\mu_3 &= \tfrac{RP(S_1-S_2)+G_2(RS_0-S_1)-G_1(RS_0-S_2)}{G_1(S_2-S_3)+G_2(S_3-S_1)+G_3(S_1-S_2)} \tfrac{1}{p_3}.
\end{aligned}$$

If a CARA utility function is considered then

$$q = -\frac{1}{\delta} \frac{(S_2 - S_3)\ln \mu_1 + (S_3 - S_1)\ln \mu_2 + (S_1 - S_2)\ln \mu_3}{G_1(S_2 - S_3) + G_2(S_3 - S_1) + G_3(S_1 - S_2)}. \tag{4.2.10}$$

If a CRRA utility function is considered then

$$q = Ry \frac{\mu_1^{-\frac{1}{\gamma}}(S_2 - S_3) + \mu_2^{-\frac{1}{\gamma}}(S_3 - S_1) + \mu_3^{-\frac{1}{\gamma}}(S_1 - S_2)}{\begin{array}{c} \mu_1^{-\frac{1}{\gamma}}\left[(S_3 - RS_0)(G_2 - RP) - (S_2 - RS_0)(G_3 - RP)\right] \\ +\mu_2^{-\frac{1}{\gamma}}\left[(S_1 - RS_0)(G_3 - RP) - (S_3 - RS_0)(G_1 - RP)\right] \\ +\mu_3^{-\frac{1}{\gamma}}\left[(S_2 - RS_0)(G_1 - RP) - (S_1 - RS_0)(G_2 - RP)\right] \end{array}}. \tag{4.2.11}$$

In both cases supply and demand are simply given by

$$\begin{cases} q_d(P_d) = q, & q > 0, \ (\Leftrightarrow P_d \equiv P < \hat{P}), \\ q_s(P_s) = -q, & q < 0, \ (\Leftrightarrow P_s \equiv P > \hat{P}). \end{cases} \tag{4.2.12}$$

4.2.3.2 Properties of Individual Demand and Supply

In what follows we present some properties of the individual demand and supply. If $u(w)$ is a CARA utility function then:[8]

1. The individual demand for the derivative is a decreasing function of the price, i.e.

$$\frac{\partial q_d}{\partial P_d} \leq 0.$$

2. The individual supply for the derivative is an increasing function of the price, i.e.

$$\frac{\partial q_s}{\partial P_s} \geq 0.$$

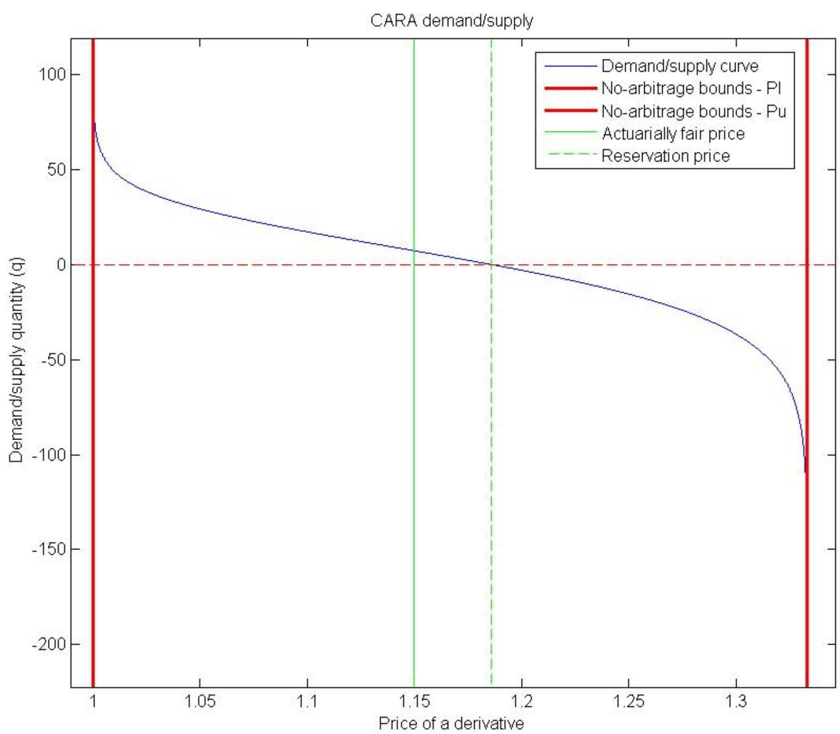

Figure 4.1: Transacted quantity $q(P)$ of the derivative for the CARA utility.

3. The optimal number of options multiplied by δ is constant. Hence, the demand and/or supply will shift downwards when the absolute risk aversion coefficient δ increases, i.e.

$$\frac{\partial q_d}{\partial \delta} \leq 0 \text{ and } \frac{\partial q_s}{\partial \delta} \leq 0.$$

4. The price such that the optimal number of derivatives is equal to zero (i.e. reservation price) is independent of δ.

5. The optimal number of shares is independent of the initial wealth.

[8]See Appendix B.4.2.1 for details on CARA and Appendix B.4.2.2 for details on CRRA properties.

The graph (Figure 4.1) displays the transacted quantities of the derivative as a function of its price $q(P)$. These values are obtained from equation (4.2.12) through MATLAB simulation assuming that the derivative was a European put option with $K = 11$, and that the parameters are: price of the risky asset $S_0 = 10$, and $S = [S_1, S_2, S_3] = [12, 10.5, 9]$; derivative's payoff $G = [G_1, G_2, G_3] = [0, 0.5, 2]$; states' probabilities $p = [0.2, 0.3, 0.5]$; the risk free rate assumed to be zero, and the initial wealth normalised to $y = 1$. The parameter of CARA utility function is $\delta = 0.1$.

The arbitrage-free range for the price is $\left(P^l, P^u\right) = \left(1, \frac{4}{3}\right)$ (thick red lines on the graph), reservation price is $\hat{P} = 1.1859$ (dotted green line) and fair price is $\bar{P} = 1.15$ (solid green line).

If $u\left(w\right)$ is a CRRA utility function then:

1. Numerically, the individual demand for the derivative can be shown to be a decreasing function of the price, i.e.

$$\frac{\partial q_d}{\partial P_d} \leq 0.$$

2. Numerically, the individual supply for the derivative can be shown to be an increasing function of the price, i.e.

$$\frac{\partial q_s}{\partial P_s} \geq 0.$$

3. The demand and supply are increasing functions of the exogenous wealth, i.e.

$$\frac{\partial q_d}{\partial y} > 0 \text{ and } \frac{\partial q_s}{\partial y} > 0.$$

4. The reservation price is an increasing function of γ if $G_1\left(S_2 - S_3\right) + G_2\left(S_3 - S_1\right) + G_3\left(S_1 - S_2\right) < 0$. Otherwise, the reservation price is a decreasing function of γ.

Note that under the CRRA utility function the quantities depend on the level of initial wealth y, whereas under the CARA utility they do not, which is expected and a well known property of their respective Arrow-Prat (1964, 1965) absolute risk aversion coefficients $A(w) = -\frac{U''(w)}{U'(w)}$.

The graph presented in Figure 4.2, is obtained for this case of utility function using the same parameters as above with the addition of CRRA utility function parameter $\gamma = 0.9$.

The arbitrage-free range of variation for the price is the same $\left(P^l, P^u\right) = \left(1, \frac{4}{3}\right)$ (thick red lines on the graph), reservation price is now $\hat{P} = 1.1861$ (dotted green line), whereas fair price is unchanged at $\bar{P} = 1.15$ (solid green line).

4.2.3.3 Other Variables - Explicit Solution and Simulations

We can also get explicit expressions for the transacted amount (Δ_0) of the underlying asset and for the transacted amount (B) of the risk-free asset.

For a CARA utility function

$$\Delta_0 = -\frac{1}{\delta}\frac{\left(G_2 - G_3\right)\ln\mu_1 + \left(G_3 - G_1\right)\ln\mu_2 + \left(G_1 - G_2\right)\ln\mu_3}{G_1\left(S_2 - S_3\right) + G_2\left(S_3 - S_1\right) + G_3\left(S_1 - S_2\right)}.$$

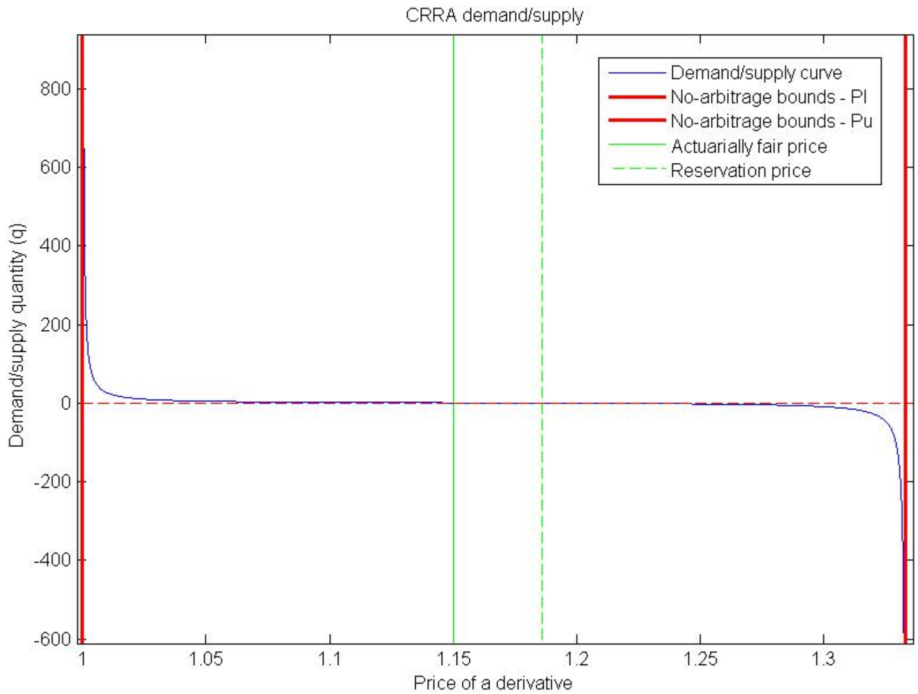

Figure 4.2: Transacted quantity $q(P)$ of the derivative for the CRRA utility.

For a CRRA utility function

$$\Delta_0 = Ry \frac{\mu_1^{-\frac{1}{\gamma}} (G_3 - G_2) + \mu_2^{-\frac{1}{\gamma}} (G_1 - G_3) + \mu_3^{-\frac{1}{\gamma}} (G_2 - G_1)}{\begin{array}{l} \mu_1^{-\frac{1}{\gamma}} [(S_3 - RS_0)(G_2 - RP) - (S_2 - RS_0)(G_3 - RP)] \\ +\mu_2^{-\frac{1}{\gamma}} [(S_1 - RS_0)(G_3 - RP) - (S_3 - RS_0)(G_1 - RP)] \\ +\mu_3^{-\frac{1}{\gamma}} [(S_2 - RS_0)(G_1 - RP) - (S_1 - RS_0)(G_2 - RP)] \end{array}}. \tag{4.2.13}$$

The optimal demand and supply quantity for the underlying asset are simply given by

$$\begin{cases} \Delta_{0,d}(P_d) = \Delta_0, & P_d \equiv P < \hat{P}, \\ \Delta_{0,s}(P_s) = \Delta_0, & P_s \equiv P > \hat{P}. \end{cases} \tag{4.2.14}$$

Using (4.2.12) and (4.2.14) in (4.2.5) we get for both cases (P_d and P_s):

$$B_0 = y - \Delta_0 S_0 - qP, \; \forall P. \tag{4.2.15}$$

From equation (4.2.4) we have the solution for wealth

$$w_i = \Delta_0 S_i + RB + qG_i, \; \forall P, \tag{4.2.16}$$

the argument of any utility function $U(w_i)$. For any given price, the expected utility function, $\mathbb{E}U(.)$ is obtained by taking expectations over the different states of nature.

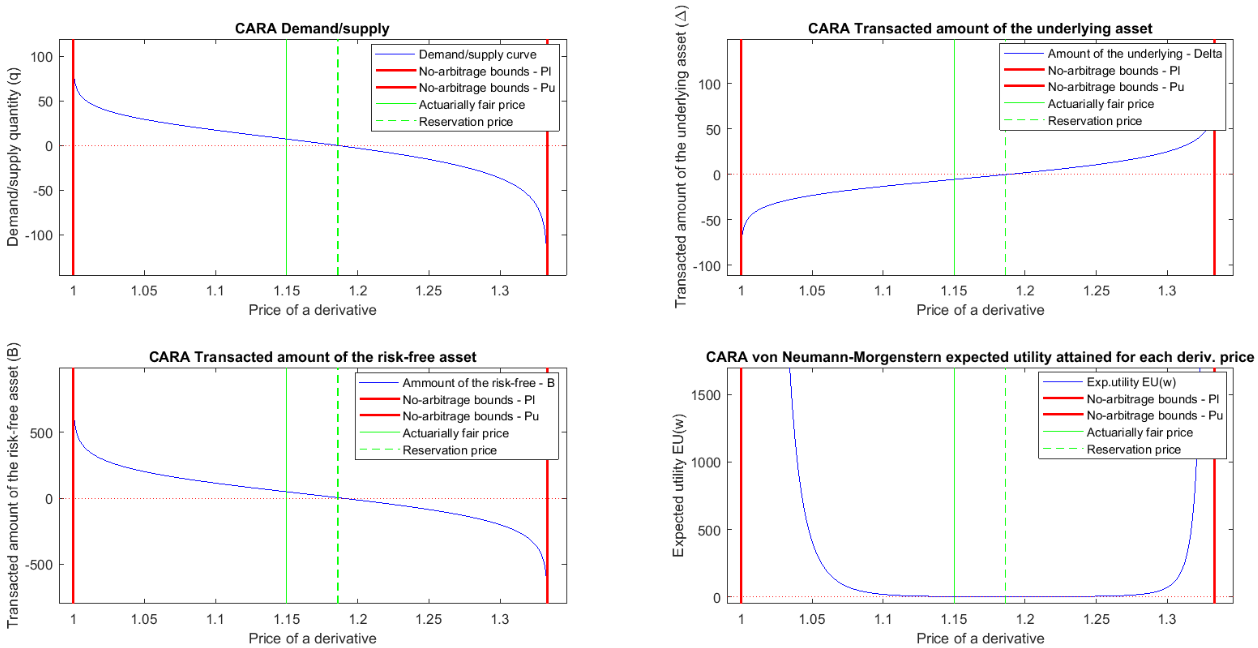

Figure 4.3: For the CARA utility: a) derivative demand and supply curves; b) transacted amount of underlying asset; c) transacted amount of risk-free asset; d) expected utility level attained for different prices.

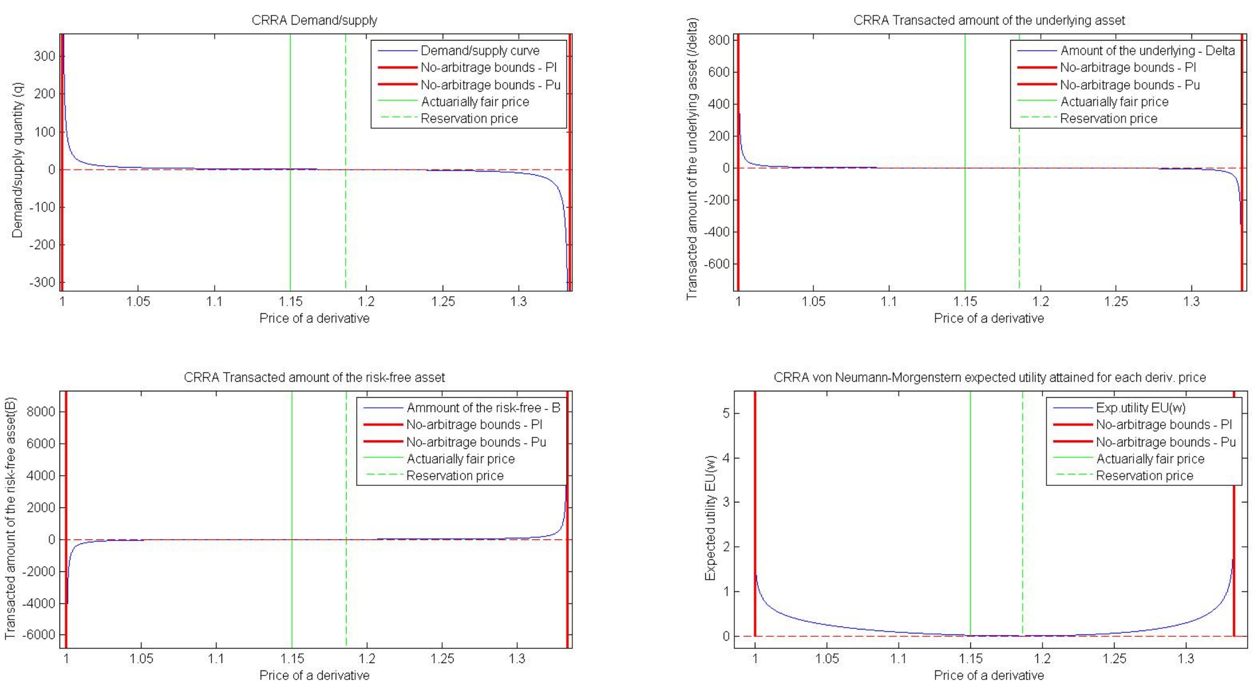

Figure 4.4: For the CRRA utility: a) derivative demand and supply curves; b) transacted amount of underlying asset; c) transacted amount of risk-free asset; d) expected utility level attained for different prices.

For a CARA utility function, the Figure 4.3 shows for the same parameters as before a) the derivative demand and supply curves; b) the transacted amount of underlying asset; c) the transacted amount of risk-free asset; d) the expected utility level attained for different prices. The expected utility level reaches zero exactly at the reservation price. This is in accordance with the utility indifference pricing, e.g. Henderson and Hobson (2004). At the reservation price any investor is exactly indifferent between (a) paying nothing and not having the derivative and (b) paying the reservation price to receive the payoff of the derivative at maturity. If derivative price moves left or right away from the reservation price, the expected utility gradually increases until it reaches a super-replication bound where it is infinite, allowing for an arbitrage opportunity with unbounded wealth. Graphs of intermediate results - quantities w_i and $U(w_i)$, $i = 1, 2, 3$, for the same example are given in Appendix B.4.2.3.

A similar figure for the CRRA utility function is shown in the Figure 4.4.

4.3 Market Makers

In the previous section we presented the optimisation problem of illiquid traders and derived the demand and supply of derivatives as a function of exogenous endowments. In this section we present the problem faced by market makers, given the demand and supply functions for derivatives. The optimal strategy of these intermediaries depends on the level of competition. We shall first consider the case of a monopolistic market maker and then consider the case when they compete.[9] Given optimal prices and quantities, the market maker(s) must also define an optimal hedging strategy in stocks and bonds.

4.3.1 Monopolistic Market Maker

The monopolist market maker's problem consists of choosing the bid and ask prices, together with a hedging strategy, so as to maximise his expected utility. Equivalently, the problem can be solved by choosing the optimal amount of transacted derivatives (sell and buy) and the optimal hedging strategies. This equivalence follows assuming that the market maker must satisfy market demand and supply at the set ask and bid prices. Let Q_A/Q_B be the number of European derivatives that the market maker is selling/buying, Δ be the number of shares in the hedging portfolio and B be the amount invested in the risk-free asset. In what follows we allow the optimal quantity sold Q_A to be different from the optimal quantity bought Q_B.

If the market maker is risk neutral he faces the following problem:

$$\max_{Q_B,Q_A,\Delta,B} \ \mathbb{E}\left[\pi\right] = \sum_{i=1}^{3} p_i \left[Q_B G_i - Q_A G_i + \Delta S_i + RB\right],$$

[9] Important feature of the option trading on the American Stock Exchange is the use of specialists. The option specialist has access to more information than other traders and, therefore, can mantain a monopolistic position. For instance, on many exchanges, only the specialist has information about the orders at the opening of the market. The access to this information allows him to extract some monopolistic profits. In contrast, at the Chicago Board Options Exchange each market maker is required to compete with others. CBOE requires that each transaction be executed at the highest bid and lowest ask prices emerging from the group of market makers participating in the process. These rules induce a strong competition between market makers.

subject to

$$\Delta S_0 + B - Q_A P_d(Q_A) + Q_B P_s(Q_B) \leq y, \tag{4.3.1}$$
$$Q_B G_i - Q_A G_i + \Delta S_i + RB \geq 0, \qquad i = 1,2,3 \text{ and} \tag{4.3.2}$$
$$Q_B \geq 0, \quad Q_A \geq 0. \tag{4.3.3}$$

Several assumptions concerning the market demand and supply are made.

Assumption 1: The supply and the demand functions are, respectively, increasing and decreasing in the transacted quantities,

$$\begin{aligned} \frac{dP_s(Q_B)}{dQ_B} &> 0, \\ \frac{dP_d(Q_A)}{dQ_A} &< 0. \end{aligned}$$

Assumption 2: The function[10]

$$\sum_{i=1}^{3} p_i \left[Q_B \left(G_i - RP_s(Q_B)\right) - Q_A \left(G_i - RP_d(Q_A)\right) + \Delta \left(S_i - RS_0\right) + Ry\right]$$

is concave in Q_A, Q_B and Δ.

In order to simplify the notation let $\bar{X}_k \equiv X_k/R$. Taking into account that, at the optimum, $\Delta S_0 + B - Q_A P_d + Q_B P_s = y$, the problem of the monopolistic risk-neutral market maker can be rewritten as

$$\max_{Q_B, Q_A, \Delta} \sum_{i=1}^{3} p_i \left\{Q_B \left[\bar{G}_i - P_s(Q_B)\right] - Q_A \left[\bar{G}_i - P_d(Q_A)\right] + \Delta \left(\bar{S}_i - S_0\right) + y\right\},$$

subject to

$$-Q_B \left[\bar{G}_i - P_s(Q_B)\right] + Q_A \left[\bar{G}_i - P_d(Q_A)\right] - \Delta \left(\bar{S}_i - S_0\right) - y \leq 0, \qquad i = 1,2,3$$

and

$$Q_B \geq 0, \quad Q_A \geq 0.$$

Since the objective function is quasiconcave and the constraint set is convex,[11] there exists a solution for the problem.[12] Moreover, and provided that all the constraints that hold in equality are independent, the solution of the problem is given by the Kuhn-Tucker solutions.[13]

[10] See Appendix B.5.1 for details on this assumption. This function is simply the objective function of the monopolist, incorporating the first restriction, shown later to be always binding.

[11] See Appendix B.5.2.1.

[12] See theorem MK4, in Mas-Colell *et al.* (1995), page 962.

[13] Once again using a theorem of Mas-Colell *et al.* (1995), theorem MK2, page 959.

The Lagrangean of the problem is given by

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{3} p_i \left\{ Q_B \left[\bar{G}_i - P_s(Q_B)\right] - Q_A \left[\bar{G}_i - P_d(Q_A)\right] + \Delta\left(\bar{S}_i - S_0\right) + y \right\} \\ &\quad - \sum_{i=1}^{3} \lambda_i \left\{ Q_B \left[\bar{G}_i - P_s(Q_B)\right] - Q_A \left[\bar{G}_i - P_d(Q_A)\right] + \Delta\left(\bar{S}_i - S_0\right) + y \right\}.\end{aligned}$$

The first order Kuhn-Tucker conditions are

$$\begin{cases} \frac{d\mathcal{L}}{dQ_B} \leq 0, & Q_B \geq 0, \quad \frac{d\mathcal{L}}{dQ_B} Q_B = 0; \\ \frac{d\mathcal{L}}{dQ_A} \leq 0, & Q_A \geq 0, \quad \frac{d\mathcal{L}}{dQ_A} Q_A = 0; \\ \frac{d\mathcal{L}}{d\Delta} = 0; & \\ \frac{d\mathcal{L}}{d\lambda_i} \leq 0, & \lambda_i \leq 0, \quad \frac{d\mathcal{L}}{d\lambda_i} \lambda_i = 0; \end{cases}$$

for $i = 1$, 2 and 3. If the solution is characterised by $Q_A^* > 0$ and $Q_B^* > 0$, the first order conditions at the optimum are

$$\begin{cases} \frac{d\mathcal{L}}{dQ_B} = 0, \\ \frac{d\mathcal{L}}{dQ_A} = 0, \\ \frac{d\mathcal{L}}{d\Delta} = 0; \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^{3} p_i \left[\bar{G}_i - \Psi(Q_B^*)\right] - \sum_{i=1}^{3} \lambda_i \left[\bar{G}_i - \Psi(Q_B^*)\right] = 0, \\ \sum_{i=1}^{3} p_i \left[\bar{G}_i - \Psi(Q_A^*)\right] - \sum_{i=1}^{3} \lambda_i \left[\bar{G}_i - \Psi(Q_A^*)\right] = 0, \\ \sum_{i=1}^{3} p_i \left[\bar{S}_i - S_0\right] - \sum_{i=1}^{3} \lambda_i \left[\bar{S}_i - S_0\right] = 0; \end{cases}$$

where

$$\Psi(Q^*) = \left. \frac{d\left[QP(Q)\right]}{dQ} \right|_{Q^*}. \tag{4.3.4}$$

If $\lambda_i < 0$, $i = 1, 2, 3$, all constraints would be binding at the optimal point characterizing the solution, and the value of the objective function would be zero. Moreover, it is not possible to have all $\lambda_i = 0$, $i = 1, 2, 3$, unless $\sum_{i=1}^{3} p_i \left[\bar{S}_i - S_0\right] = 0$. Hence, either there is only one value of i such that $\lambda_i = 0$, or there is only one value of i such that $\lambda_i < 0$ (i.e. two values of i such that $\lambda_i = 0$). In the latter case, the first order conditions presented above results in equation (4.3.5) with $\lambda_k = \sigma_k < 0 \Rightarrow k \in \mathcal{K}$. In the former case, the first order conditions lead to equation (4.3.6). Here,

$$\sigma_k \equiv \frac{\sum_{i=1}^{3} p_i \bar{S}_i - S_0}{\bar{S}_k - S_0},$$

and the function of Q[14]

$$\Phi_k(Q) = \frac{\sum_{i=1}^{3} p_i \bar{G}_i - \frac{d[QP(Q)]}{dQ}}{\bar{G}_k - \frac{d[QP(Q)]}{dQ}} \frac{\sum_{i=1}^{3} p_i \bar{G}_i - \Psi(Q)}{\bar{G}_k - \Psi(Q)}.$$

[14] Note that, if we consider the market supply, Φ_k is evaluated at $Q = Q_B$. Alternatively, if we consider the market demand, Φ_k is evaluated at $Q = Q_A$.

The sets $\mathcal{K}$ and $\mathcal{B}_k(Q_A, Q_B, \Delta, B)$ are defined as follows:

$$\mathcal{K} \equiv \left\{ m : sign\left(\sum_{i=1}^{3} p_i \bar{S}_i - S_0\right) = -sign\left(\bar{S}_m - S_0\right)\right\},$$

and

$$\mathcal{B}_k(Q_A, Q_B, \Delta, B) \equiv \left\{ \begin{array}{ll} (Q_A, Q_B, \Delta, B): & \Delta S_0 + B + Q_A P_d(Q_A) - Q_B P_s(Q_B) = y, \\ & Q_B G_i - Q_A G_i + \Delta S_i + RB \geq 0, i \neq k \text{ and} \\ & Q_B G_k - Q_A G_k + \Delta S_k + RB = 0 \end{array} \right\}.$$

The previous derivation and the existence of the bid-ask spread are summarised as follows.

Proposition 4.2. *Under Assumptions 1 and 2, and in the presence of a risk-neutral monopolist market maker, a sufficient condition for the existence of an equilibrium with strictly positive quantities* $\{Q_B^*, Q_A^*\}$, *is characterised by* Q_A^* *and* Q_B^* *satisfying* ***either***

$$\Phi_k(Q_B^*) = \Phi_k(Q_A^*) = \sigma_k; \tag{4.3.5}$$

with $(Q_A^*, Q_B^*, \Delta^*, B^*) \in \mathcal{B}_k(Q_A, Q_B, \Delta, B)$ *and* $k \in \mathcal{K}$, ***or***

$$\left\{ \begin{array}{l} \sum_{i=1}^{3} p_i \bar{G}_i - \Psi(Q_B^*) - \lambda_k \left[\bar{G}_k - \Psi(Q_B^*)\right] - \lambda_j \left[\bar{G}_j - \Psi(Q_B^*)\right] = 0, \\ \sum_{i=1}^{3} p_i \bar{G}_i - \Psi(Q_A^*) - \lambda_k \left[\bar{G}_k - \Psi(Q_A^*)\right] - \lambda_j \left[\bar{G}_j - \Psi(Q_A^*)\right] = 0, \\ \sum_{i=1}^{3} p_i \bar{S}_i - S_0 - \lambda_k \left[\bar{S}_k - S_0\right] - \lambda_j \left[\bar{S}_j - S_0\right] = 0; \end{array} \right. \tag{4.3.6}$$

with $\lambda_k < 0$, $\lambda_j < 0$, $k \neq j$, *with at least one of* k *and* j *in* $\mathcal{K}$, *and* $(Q_A^*, Q_B^*, \Delta^*, B^*) \in \mathcal{B}_k(Q_A, Q_B, \Delta, B) \cap \mathcal{B}_j(Q_A, Q_B, \Delta, B)$. *Moreover, the above conditions are also sufficient to generate a bid-ask spread, i.e.* $P_d(Q_A^*) > P_s(Q_B^*)$.

Proof. The result about the existence of a bid-ask spread under the above conditions is discussed in Appendix B.5.2.2. ■

Now, we present a necessary and sufficient condition to have (4.3.5) fulfilled. Notice that

$$\Phi_k(Q_B = 0) = \frac{\sum_{i=1}^{3} p_i \bar{G}_i - \hat{P}_s}{\bar{G}_k - \hat{P}_s} \text{ and } \Phi_k(Q_A = 0) = \frac{\sum_{i=1}^{3} p_i \bar{G}_i - \hat{P}_d}{\bar{G}_k - \hat{P}_d},$$

where $\hat{P}_s$ and $\hat{P}_d$ are the supply and demand reservation prices, respectively.

Corollary 4.1. *A necessary and sufficient condition to have equation* (4.3.5) *satisfied is that there are reservation prices* $\hat{P}_d$ *and* $\hat{P}_s$ *such that*

$$\Phi_k(Q_A = 0) \leq \sigma_k \leq \Phi_k(Q_B = 0) < 0 \tag{4.3.7}$$

if $\sum_{i=1}^{3} p_i \bar{G}_i - \bar{G}_k \leq 0$, *and*

$$0 > \Phi_k(Q_A = 0) \geq \sigma_k \geq \Phi_k(Q_B = 0) \tag{4.3.8}$$

if $\sum_{i=1}^{3} p_i \bar{G}_i - \bar{G}_k \geq 0$.

Moreover, necessary conditions for (4.3.7) *and* (4.3.8) *to hold are*

$$\sum_{i=1}^{3} p_i \bar{G}_i < \hat{P}_d \leq \hat{P}_s < \bar{G}_k, \tag{4.3.9}$$

$$\sum_{i=1}^{3} p_i \bar{G}_i > \hat{P}_d \geq \hat{P}_s > \bar{G}_k. \tag{4.3.10}$$

Proof. The proof is given in Appendix B.5.3. ■

In equilibrium the market maker gains on one side of the market but loses on the other side.

We illustrate the point here. Consider the second case presented above. In equilibrium, the market maker expects to gain in the long position on the derivative (because $P_s < \sum_{i=1}^{3} p_i \bar{G}_i$), but expects to lose in the short position (because $P_d < \sum_{i=1}^{3} p_i \bar{G}_i$). This results from the fact that (i) in this equilibrium, at least one of the wealth constraints is binding and (ii) the interval defined by the demand and supply reservation prices does not contain the expected value of the derivative's payoff, $\sum_{i=1}^{3} p_i \bar{G}_i$. This latter fact implies that a market maker selling and buying simultaneously the derivative has necessarily a positive expected utility on one side, and a negative expected utility on the other side. The market maker may thus choose to be only on the side of the market that provides a positive expected utility. However, in order to *maximise* the expected utility, the market maker may find an incentive to enter the other side of the market, relaxing the binding restrictions. This only happens provided the improvement of the positive expected utility more than compensates the negative expected utility on the other side of the market. Our result reflects the fact that a bid-ask spread exists only when the market maker faces one such incentive.

Validity of our results can be justified by empirical patterns of intraday bid-ask spreads of CBOE option prices depending on the level of the competition in market-making, observed by Chan *et al.* (1995).

Comparative Statics. Having characterised the equilibrium bid-ask spread in the monopolistic market maker case, let us investigate how it changes with different parameters of our agents' utility functions.

Corollary 4.2. *If the agents' utility is of the CARA type as illustrated before in Section 4.2.3, i.e. individual supply and demand are given by* (4.2.10) *with* (4.2.12)*, and aggregate supply/demand by* (4.2.7)*, the equilibrium ask price is a decreasing function of CARA utility parameter* δ*, whereas the equilibrium bid price is an increasing function of the same parameter i.e.*

$$\begin{aligned} \frac{dP_d\left(Q_A^*\right)}{d\delta} &\leq 0, \\ \frac{dP_s\left(Q_B^*\right)}{d\delta} &\geq 0. \end{aligned}$$

Thus, the bid-ask spread narrows as the coefficient of absolute risk aversion increases.

Proof. The bid-ask spread dependence on the agents' utility function parameters is discussed in Appendix B.5.4. ■

For the other illustrative case, using a CRRA utility function, due to the individual supply and demand (4.2.11), with (4.2.12), and aggregate supply/demand in (4.2.7), we can numerically reach a similar conclusion that the equilibrium ask price decreases with the coefficient of relative risk aversion γ, whereas the equilibrium bid price increases, i.e.

$$\begin{aligned}\frac{dP_d\left(Q_A^*\right)}{d\gamma} &\leq 0,\\ \frac{dP_s\left(Q_B^*\right)}{d\gamma} &\geq 0.\end{aligned}$$

Again, this implies that the bid-ask spread narrows as γ increases.

4.3.2 Competition Between Market Makers

In this section, our model is extended to consider the presence of several market makers. In an oligopoly, the payoffs for one market maker depend on its own actions, as well as on the actions of the other market makers. The strategic interactions between the market makers will determine the equilibrium.

Here, individual market makers simultaneously determine their bid and ask prices, the number of shares and the amount invested in the risk-free asset, behaving in their own interest in a non-cooperative game. The objective is to compute the Nash equilibrium of this game.

Let M be the number of market makers in this market. The i-th market maker, with $i = 1, ..., M$, has expected utility $\mathcal{U}_i\left(P_{i,d}, P_{i,s}, P_{-i,d}, P_{-i,s}, \Delta_i, \Delta_{-i}\right)$, where $P_{i,d}$, $P_{i,s}$ and Δ_i are, respectively, the ask price, the bid price and the number of units of the underlying asset held by market maker i. The values $P_{-i,d}, P_{-i,s}$ and Δ_{-i} correspond to the components of the vector of the analogous variables relative to the remaining $M-1$ market makers, i.e. $\left(P_{-i,d}, P_{-i,s}, \Delta_{-i}\right) = \left(P_{j,d}, P_{j,s}, \Delta_j\right)_{j\in\{1,\ldots,M\},j\neq i}$.

Formally, the expected utility of market maker j corresponds to its expected profit:

$$\begin{aligned}\Pi_j\left(P_{j,d}, P_{j,s}, P_{-j,d}, P_{-j,s}, \Delta_j, \Delta_{-j}\right) = \sum_{i=1}^{3} p_i &\left[Q_{j,B}\left(P_{j,s}, P_{-j,s}\right)\left(G_i - RP_{j,s}\right)\right.\\ &\left.-Q_{j,A}\left(P_{j,d}, P_{-j,d}\right)\left(G_i - RP_{j,d}\right) + \Delta_j\left(S_i - RS_0\right) + Ry\right].\end{aligned}$$

For given prices, the optimal amount transacted Δ_j^*, solves the problem

$$\mathcal{U}_i\left(P_{j,d}, P_{j,s}, P_{-j,d}, P_{-j,s}\right) = \max_{\Delta_j} \Pi_j\left(P_{j,d}, P_{j,s}, P_{-j,d}, P_{-j,s}, \Delta_j, \Delta_{-j}\right),$$

leading to $\Delta_j^* = \Delta_j\left(P_{j,d}, P_{j,s}, P_{-j,d}, P_{-j,s}\right)$ and characterising the expected utility function to be maximised on prices. For that given amount Δ_j^*, prices are set optimally as the solution of

$$\max_{P_{j,s}, P_{j,d}} \mathcal{U}_i\left(P_{j,d}, P_{j,s}, P_{-j,d}, P_{-j,s}\right)$$

subject to

$$\begin{aligned} 0 \;\leq\; & Q_{j,B}\left(P_{j,s},P_{-j,s}\right)\left(G_i - RP_{j,s}\right) - \\ & Q_{j,A}\left(P_{j,d},P_{-j,d}\right)\left(G_i - RP_{j,d}\right) + \Delta_j^*\left(S_i - RS_0\right) + Ry, \end{aligned}$$

for $i = 1,2,3$, where $Q_{j,B}\left(P_{j,s},P_{-j,s}\right)$ and $Q_{j,A}\left(P_{j,d},P_{-j,d}\right)$ are, respectively, the demand and supply functions faced by firm j.

Financial products are generally considered homogeneous goods. Options traded by different intermediaries are taken as perfect substitutes by the investors, who choose to transact with the intermediary setting the best price. In financial markets the best quotes can be easily found. In particular, automated trading systems facilitate the disclosure of the best price. Hence, the homogeneous good assumption gives rise to discontinuity of the demand and supply curves. Market maker j is viewed as facing the demand curve $Q_{j,A}\left(P_{j,d},P_{-j,d}\right)$, a function of the ask price that all market makers quote. Supposing that $P_{i,d} = P_{k,d}$, for all $i, k \neq j$,

$$Q_{j,A}\left(P_{j,d},P_{-j,d}\right) = \begin{cases} \chi_A\left(P_{j,d},P_{i,d}\right), & \text{if } P_{j,d} > P_{i,d}, \text{ for all } i, \\ \frac{1}{M}Q_A\left(P_{j,d}\right), & \text{if } P_{j,d} = P_{i,d}, \text{ for all } i, \\ Q_A\left(P_{j,d}\right), & \text{if } P_{j,d} < P_{i,d}, \text{ for all } i; \end{cases} \tag{4.3.11}$$

where $\chi_A\left(P_{j,d},P_{i,d}\right) \in \left[0, Q_A\left(P_{j,d}\right) - \frac{M-1}{M}Q_A\left(P_{i,d}\right)\right]$.

The last line reflects the fact that, if market maker j quotes the lowest price, he or she will face all the market demand. The second line corresponds to a fair ratio among the market makers. The first line, however, is more subtle. If market maker j quotes a higher price than his competitors, several situations are possible. As competitors have wealth constraints, they may not be able to sell additional units at the lower price to former customers of market maker j. In that case $Q_{j,A}\left(P_{j,d},P_{-j,d}\right) = Q_A\left(P_{j,d}\right) - \frac{M-1}{M}Q_A\left(P_{i,d}\right)$. However, it may also be possible that, even having binding constraints,[15] competitors are able to sell additional units to former customers of market maker j, reducing his or her demand to zero, i.e. $Q_{j,A}\left(P_{j,d},P_{-j,d}\right) = 0$. An intermediate solution is also possible, where the competitors are not able to sell to *all* the former customers of market maker j. Hence, we can conclude that when market maker j sets a price above his competitors, his or her demand will belong to the range $\left[0, Q_A\left(P_{j,d}\right) - \frac{M-1}{M}Q_A\left(P_{i,d}\right)\right]$. This discontinuous demand faced by a competitive market maker is illustrated in Figure 4.5.

On the other hand, market maker j is viewed as facing the following supply curve, also shown in Figure 4.5,

$$Q_{j,B}\left(P_{j,s},P_{-j,s}\right) = \begin{cases} Q_B\left(P_{j,s}\right), & \text{if } P_{j,s} > P_{i,s}, \text{ for all } i, \\ \frac{1}{M}Q_B\left(P_{j,s}\right), & \text{if } P_{j,s} = P_{i,s}, \text{ for all } i, \\ \chi_B\left(P_{j,s},P_{i,s}\right), & \text{if } P_{j,s} < P_{i,s}, \text{ for all } i; \end{cases} \tag{4.3.12}$$

where $\chi_B\left(P_{j,s},P_{i,s}\right) \in \left[0, Q_B\left(P_{j,s}\right) - \frac{M-1}{M}Q_B\left(P_{-j,s}\right)\right]$.

[15]This is the case if $\sum_{i=1}^{3} p_i G_i - RP_{i,d} < 0$ and $G_k - RP_{i,d} < 0$, where k identifies the binding restriction.

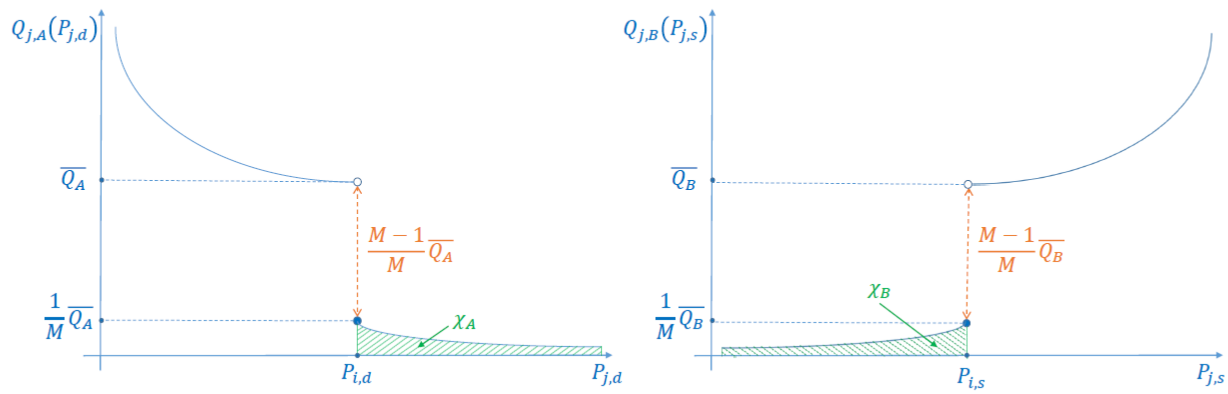

Figure 4.5: Competitive market maker's demand and supply curves.

Definition 4.1. A vector of ask and bid prices $(P_d^*, P_s^*, \Delta^*) = \left(P_{i,d}^*, P_{i,s}^*, \Delta_i^*\right)_{i=1,\dots,M}$ is an equilibrium iff, for all i and all possible prices $(P_d, P_s) = (P_{i,d}, P_{i,s})_{i=1,\dots,M}$,

$$\mathcal{U}_i\left(P_{i,d}^*, P_{i,s}^*, P_{-i,d}^*, P_{-i,s}^*\right) \geq \mathcal{U}_i\left(P_{i,d}, P_{i,s}, P_{-i,d}, P_{-i,s}\right).$$

In other words, a set of prices is a Nash equilibrium if market makers have no incentive to set different prices in order to obtain higher utility.

This result is known as Bertrand paradox, Bertrand (1883), establishing that when there is price competition between identical firms with no constraints, price equals marginal cost, and firms make no profit. As market makers are perfect competitors in prices, it is usually accepted that market makers earn zero profits. However, that is not the case here, since there are positive wealth constraints. Hence, we must investigate further the existence of a pure Nash equilibrium of this game.

For each firm j we define two reaction functions as the optimal demand and supply prices which are functions of the prices quoted by other firms:

$$\begin{aligned} P_{j,s}^* &= \Upsilon_{j,s}\left(P_{-j,s}, P_{j,d}, P_{-j,d}\right), \\ P_{j,d}^* &= \Upsilon_{j,d}\left(P_{-j,d}, P_{j,s}, P_{-j,s}\right). \end{aligned}$$

The symmetric Nash equilibrium of the game is the set of prices that solve the following system

$$\begin{cases} P_{j,s}^* = \Upsilon_{j,s}\left(P_{-j,s}^*, P_{j,d}^*, P_{-j,d}^*\right), & \forall j \in \{1,\dots,M\}, \\ P_{j,d}^* = \Upsilon_{j,d}\left(P_{-j,d}^*, P_{j,s}^*, P_{-j,s}^*\right), & \forall j \in \{1,\dots,M\}. \end{cases}$$

Theorem 4.3. *Under the assumptions of the model, if all market makers are risk-neutral, there is no pure symmetric Nash equilibrium of the game.*

Proof. Let (P_d^*, P_s^*) be an equilibrium candidate. We prove that there is always a profitable deviation and therefore (P_d^*, P_s^*) cannot be an equilibrium. The proof examines the case of zero, one, two or all binding constraints.

If no wealth constraint is binding, a profitable deviation is easily identified. In the case $\sum_{i=1}^3 p_i\left(\bar{S}_i - S_0\right) > 0\,(< 0)$ each market maker could increase the expected profit by increasing (decreasing) Δ.

A solution with three binding constraints would not be an equilibrium of the game. In that case, the expected profit would be zero and a profitable deviation would be, for instance, $P_d = \hat{P}_d$, $P_s = \hat{P}_s$ and $\Delta = 0$, with an expected profit of Ry.

We now consider the more complex case of one binding constraint (say, constraint k). On one hand, if $\sum_{i=1}^{3} p_i \left(\bar{S}_i - S_0\right) > 0$, the constraint that is binding is the one such that $\bar{S}_k - S_0 < 0$, because if that is not the case it would be possible to increase utility by increasing Δ. On the other hand, for the same reasons, if $\sum_{i=1}^{3} p_i \left(\bar{S}_i - S_0\right) < 0$, the constraint that is binding is the one such that $\bar{S}_k - S_0 > 0$.

In order to check for all the profitable deviations for market maker j, assume that all other market makers are playing the hypothetical equilibrium. If market maker j decides to slightly increase the price that he is charging to the demand ($P_{j,d}$), and if the wealth constraints of his competitors do not allow them to sell more units of the derivative, then the impact in the expected profit is

$$-\frac{\partial Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\sum\nolimits_{i=1}^{3} p_i \bar{G}_i - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right). \tag{4.3.13}$$

However, if the market maker j decides to decrease $P_{j,d}$, he will face all the market demand. Hence, the impact in the expected profit is

$$-d_{Q_{j,A}} \left[\sum\nolimits_{i=1}^{3} p_i \bar{G}_i - \left(P^*_{j,d} - \varepsilon\right)\right], \tag{4.3.14}$$

where $d_{Q_{j,A}}$ denotes the variation in the quantity sold by firm j. Ignoring the positivity constraints of wealth at time t, note that, in order to be profitable to slightly increase the price, we must assure that equation (4.3.13) is positive. Moreover, it would be profitable to slightly decrease the price, increasing the quantity sold, if equation (4.3.14) is positive.

Additionally, we consider the wealth constraints. Since the expected profit and the wealth constraints are linear in the amount of bought/sold shares of stock (Δ), at least one of the wealth constraints is binding. Let this constraint be denoted by k. In an analogous way to the case just described, if the market maker decides to slightly increase the price that he is charging to the demand ($P_{j,d}$) then the impact in the wealth constraint that is binding is

$$-\frac{\partial Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\bar{G}_k - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right). \tag{4.3.15}$$

However, if the market maker j decides to slightly decrease $P_{j,d}$, he will face all the market demand. Hence, the impact in the wealth constraint that is binding is

$$-d_{Q_{j,A}} \left[\bar{G}_k - \left(P^*_{j,d} - \varepsilon\right)\right], \tag{4.3.16}$$

where $d_{Q_{j,A}}$ denotes the variation in the quantity sold by firm j, as before.

All the possibilities concerning the sign of equations (4.3.13), (4.3.14), (4.3.15) and (4.3.16) are presented in the next table.

	Eq. (4.3.13)	Eq. (4.3.14)	Eq. (4.3.15)	Eq. (4.3.16)
Case I	≤ 0	> 0	≤ 0	> 0
Case II	≤ 0	> 0	> 0	> 0
Case III	≤ 0	> 0	> 0	< 0
Case IV	≤ 0	> 0	> 0	$= 0$
Case V	> 0	≥ 0	> 0	≥ 0
Case VI	> 0	≥ 0	> 0	≤ 0
Case VII	> 0	> 0	≤ 0	> 0
Case VIII	> 0	$= 0$	≤ 0	> 0
Case IX	> 0	≤ 0	> 0	≥ 0
Case X	> 0	≤ 0	> 0	≤ 0
Case XI	> 0	< 0	< 0	> 0
Case XII	> 0	< 0	$= 0$	> 0

Table 4.1: All possible cases when one constraint is binding.

If cases I and II were considered the market maker j could slightly decrease the price that he is charging, increasing the quantity that he is selling, which would result in an increase of the expected value of the wealth. In case III the market maker can find a profitable deviation by changing $P_{j,d}$ and Δ_j. See Appendix B.6.1. In case IV, if market maker j decides to increase the price the quantity demand will be zero. The reason is that as equation (4.3.14) and equation (4.3.16) are non-negative the other market makers can increase their expected wealth by selling to the investors that used to buy from market maker j. Hence, we can not find a profitable deviation changing $P_{j,d}$. In what follows we will find a profitable equilibrium changing $P_{j,s}$. We begin by noticing that, as $\sum_{i=1}^{3} p_i\bar{G}_i$ does not belong to the interval defined by the reservation price, then $\sum_{i=1}^{3} p_i\bar{G}_i$ does not belong to the interval defined $P_{-j,s}$ and $P_{-j,d}$, with $P_{-j,d} > P_{-j,d}$. Hence, the situation described in this case, $\sum_{i=1}^{3} p_i\bar{G}_i < P_{-j,d} = \bar{G}_k$, implies $\sum_{i=1}^{3} p_i\bar{G}_i < P_{-j,s} \leq \bar{G}_k$. Notice that the impact of decreasing the price $P_{j,s}$ in the expected wealth is

$$-\left\{\frac{\partial Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)}{\partial P_{j,s}}\left[\sum\nolimits_{i=1}^{3} p_i\bar{G}_i - P_{j,s}\right] - Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)\right\} > 0.$$

Moreover, the impact on the constraint is

$$-\left\{\frac{\partial Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)}{\partial P_{j,s}}\left[\bar{G}_k - P_{j,s}\right] - Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)\right\} \gtrless 0.$$

If the impact on the constraint is positive a profitable deviation for market maker j is slightly decrease the price $P_{j,s}$. However, if that is not the case it is possible to find a profitable deviation changing $P_{j,s}$ and Δ_{j}. See Appendix B.6.1.

Case V is analogous to case IV. In what concerns cases VI, IX and X, the market maker j must increase the price that he is charging, decreasing the quantity that he is selling and increase the expected value of the wealth. Case VII is equal to cases I and II. Case VIII is not an admissible possibility because, by equation (4.3.13) and (4.3.15), we conclude that

$\sum_{i=1}^{3} p_i \bar{G}_i > \bar{G}_k$, whereas, by equations (4.3.14) and (4.3.16), we conclude $\sum_{i=1}^{3} p_i \bar{G}_i < \bar{G}_k$, which is a contradiction. The remaining cases XI and XII are presented in the Appendix B.6.1.

Another possibility is that there are two constraints binding. Let them be denoted by m and n. In Appendix B.6.1, all the possibilities concerning the relation between P_d, P_s, $\bar{G}_n$ and $\bar{G}_m$, are presented and a profitable deviation for each case is identified. ■

Hence, if price competition between market makers is introduced in our model, a pure symmetric Nash equilibrium of the game does not exist. This result is similar to the well known Edgeworth paradox, Edgeworth (1897), since in any possible situation, given equal equilibrium prices, each player has an incentive to undercut the others.

As mentioned before this result stems from the fact that market makers have positive wealth constraints (otherwise with no constraints at all we would fall back to the Bertrand paradox). However, those constraints are too rigid and thus we have effectively replicated the Edgeworth result, by failing to find the pure symmetric equilibrium. If we instead assumed more flexible capacity constraints by introducing market makers' convex cost functions, this could potentially help finding equilibria even under the pure strategies.

However, a mixed strategy Nash equilibrium must exist, according to the classical existence theorem result by Dasgupta and Maskin (1986a).[16] The theorem states[17] that a mixed strategy equilibrium exists when the payoff function is discontinuous only at a few points characterised as follows: let a' be an action combination at which the payoff function is discontinuous; then if a'' is an action combination exactly similar to a' except for one player's action, then there must be no discontinuity at a''.

This is exactly what we have in our case, from the discontinuous demand and supply curves (4.3.11) and (4.3.12), depicted in Figure 4.5. The action combinations correspond to the choices of prices of the market makers whereas the payoff is represented by the output demand/supply quantities.[18] We can see that the discontinuity in our equations arises from one player's choices of prices while others are kept fixed. Thus, our action combinations for both demand and supply quantities are valid candidates to apply Dasgupta-Maskin theorem and a mixed strategy equilibrium must exist. Another way to see this is to realise that when proving the Theorem 4.3, i.e. proving that there is no pure strategy Nash equilibrium, we were always able to identify one and only one player that would have incentive to deviate and thus violate the equilibrium. In those cases Dasgupta-Maskin theorem implies that there must be a mixed strategy equilibrium.

In general, a mixed strategy equilibrium is obtained by considering solutions where players are assigned a probability θ, $0 \leq \theta \leq 1$, to choose an action, i.e. the fraction of market makers that choose that action is θ. In our case we can interpret the mixed equilibrium existence theorem in the following way.

[16] See Theorem 5 in Dasgupta and Maskin (1986a).

[17] For completeness, Appendix B.7 presents a version of the theorem taken from Rasmusen and Blackwell (1994) book.

[18] As mentioned before this is a Bertrand game where the actions of the players are to choose prices as opposed to a Cournot game, where the players choose quantities they sell/buy, Cournot (1838).

Conjecture 4.1. *Under the assumptions of the model, if all market makers are risk-neutral, there is a mixed strategy Nash equilibrium of the game. A fraction θ of the total number of market makers will choose to set different prices (either higher or lower) than their competitors, and the bid-ask equilibrium prices will still exist as in the case of monopolistic market maker.*

Comparative Statics. Even without fully characterising the mixed strategy equilibrium bid-ask spread in the competing market maker case, we can say how it will change with different parameters of our agents' utility functions.

Corollary 4.3. *In the case of CARA utility function of the agents, illustrated in Section 4.2.3, i.e. if individual supply and demand are given by* (4.2.10) *with* (4.2.12)*, and aggregate supply/demand by* (4.2.7)*, the mixed strategy Nash equilibrium is as follows: the ask price is a decreasing function of coefficient of absolute risk aversion δ, whereas the bid price is an increasing function of the same parameter i.e.*

$$\begin{aligned} \frac{dP_d\left(Q_A^*\right)}{d\delta} &\leq 0, \\ \frac{dP_s\left(Q_B^*\right)}{d\delta} &\geq 0. \end{aligned}$$

Thus, also in the mixed strategy equilibrium under competing market makers, the bid-ask spread narrows as the coefficient of absolute risk aversion increases.

Proof. Since in the mixed strategy equilibrium, players (market makers) will choose prices and thus face demand and supply given by (4.3.11) and (4.3.12), both the fraction θ of the players who choose to set prices differently and the remaining fraction $1-\theta$ of them who choose not to, will have the similar functional form of demand and supply, i.e. in both cases it will be directly proportional to the market aggregate demand and supply, Q_A and Q_B, given by (4.2.7), with (4.2.10) and (4.2.12). The bid-ask spread dependence on the agents' utility function parameter δ will be exactly the same as in the monopolistic market maker equilibrium case, i.e. Corollary 4.2, whose proof is discussed in Appendix B.5.4. ■

The case of CRRA utility function of the agents can be discussed following the same line of reasoning. In this case we must consider the individual supply and demand (4.2.11) with (4.2.12), and aggregate supply/demand in (4.2.7), which are to be inserted in (4.3.11) and (4.3.12). Once again, there is no reason for the dependence on the coefficient of relative risk aversion γ (parameter of agents' utility function) to be any different from the previous case of a monopolistic market maker equilibrium. Thus, as before, numerically, we can reach a similar conclusion that the mixed strategy Nash equilibrium ask price decreases with γ, whereas the equilibrium bid price increases with the same parameter, i.e.

$$\begin{aligned} \frac{dP_d\left(Q_A^*\right)}{d\gamma} &\leq 0, \\ \frac{dP_s\left(Q_B^*\right)}{d\gamma} &\geq 0. \end{aligned}$$

Again, this implies that also the mixed strategy bid-ask spread narrows as γ increases.

4.4 Conclusion

In this paper we have considered a simple economy where markets are incomplete due to the inexistence of transactions of the underlying asset at some points in time.

Although our two-period economy may be seen as simplified setting, our main contributions are robust with respect to different types of utility functions considered. They may be summarised as follows.

First, we characterise the investment decisions in the risky assets, when the derivative is fairly priced.

Second, we find that if the fair price is in the no-arbitrage region, then it is either above the reservation ask price or below the reservation bid price. The implication is that, for a risk-neutral, monopolistic market maker to transact in both sides of the market, a loss in one side is necessary to justify the gain in the other side.

Third, sufficient conditions for an equilibrium to exist under a risk-neutral, monopolistic market maker are presented.

Finally and interestingly, the imperfection considered here (dry markets) suffices to provide conditions assuring the existence of a bid-ask spread under a monopolistic market maker, although one such equilibrium can be shown not to exist when competition in prices is introduced and only pure strategies are considered.

However, in the mixed strategy game an equilibrium bid-ask spread must exist according to the Dasgupta-Maskin (1986a) theorem. We illustrate how this theorem is applied to our setting of dry markets, and point out its practical implications. The challenge of exactly determining the fraction of market makers θ who will chose to act differently then the others, and predicting how many of them will chose to increase/decrease their prices and what is the most probable equilibrium outcome is a topic of further research.

Furthermore, for some specific and standard utility functions (i.e. CARA and CRRA) we show several additional results.

First, demand and supply curves are derived and they present the desired behaviour.

Second, the reservation prices do not depend on the agents' initial wealth level.

Third, the reservation prices may depend on the agents' risk aversion. In the case of a CARA utility function the reservation price does not depend on the absolute risk aversion coefficient. However, in the case of a CRRA utility function, the reservation price does depend on the relative risk aversion coefficient.

Additionally, the bid-ask spread narrows as the risk aversion coefficient increases, for both monopolistic market maker and when there is a competition. This result holds both for CARA and CRRA utility functions. This is an intuitive result that tells us that risk-neutral market makers use their market power (a monopolistic market maker even more so) to shrink the bid-ask range of prices as the agents in the market become more risk averse.

Appendix A

Auxiliary Derivations for Chapter 2

A.1 Functions to Derive Upper and Lower Bounds

A.1.1 A Lognormal Expectation

Lemma A.1. *If (X, Y) are two bivariate normal random variables with means (μ_x, μ_y), standard deviations (σ_x, σ_y) and correlation ρ then*

$$\mathbb{E}\left[e^X \cdot 1_{\{X<a,\,Y<b\}}\right] = e^{\mu_x+\frac{\sigma_x^2}{2}}\, N_2\left(\frac{a-\mu_x}{\sigma_x}-\sigma_x,\, \frac{b-\mu_y}{\sigma_y}-\rho\sigma_x\right), \tag{A.1.1}$$

where $N_2(\cdot,\cdot)$ is the bivariate standard normal cumulative distribution function given by (2.3.2) with correlation ρ.

Proof. We need to calculate $\mathbb{E}\left[e^X \cdot 1_{\{X<a,\,Y<b\}}\right] = \int_{-\infty}^{a}\int_{-\infty}^{b} e^x p(x,y)\,dy\,dx$, where

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho^2)}}$$

is a bivariate normal probability density function. We complete the square in the following way:

$$\mathbb{E}\left[e^X \cdot 1_{\{X<a,\,Y<b\}}\right] = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{a}\int_{-\infty}^{b} e^{-\frac{F}{2(1-\rho^2)}}\,dy\,dx, \tag{A.1.2}$$

where the expression F in the exponent is

$$F \equiv -2x\left(1-\rho^2\right) + \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho\,(x-\mu_x)\,(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2.$$

We expand the first two terms as follows:

$$\begin{aligned} F &= -2x + 2x\rho^2 + \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho\,(x-\mu_x)\,(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \\ &= \frac{(x-\mu_x)^2 - 2x\sigma_x^2}{\sigma_x^2} - \frac{2\rho\,(x-\mu_x)\,(y-\mu_y) - 2x\rho^2\sigma_x\sigma_y}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}. \end{aligned}$$

We proceed with completing the square in the numerator of the first fraction by adding and subtracting $\sigma_x^4 + 2\mu_x\sigma_x^2$ which gives us:

$$F = \frac{\left(x - (\mu_x + \sigma_x^2)\right)^2}{\sigma_x^2} - \sigma_x^2 - 2\mu_x - \frac{2\rho\,(x-\mu_x)\,(y-\mu_y) - 2x\rho^2\sigma_x\sigma_y}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}.$$

We want to have the same $x - (\mu_x + \sigma_x^2)$ term in the first term of the second fraction so we write

$$\begin{aligned} F &= -2\mu_x - \sigma_x^2 + \frac{\left(x - (\mu_x + \sigma_x^2)\right)^2}{\sigma_x^2} - \frac{2\rho\left(x - (\mu_x + \sigma_x^2)\right)(y-\mu_y) + 2\rho\sigma_x^2\,(y-\mu_y) - 2x\rho^2\sigma_x\sigma_y}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \\ &= -2\mu_x - \sigma_x^2 + \frac{\left(x - (\mu_x + \sigma_x^2)\right)^2}{\sigma_x^2} - \frac{2\rho\left(x - (\mu_x + \sigma_x^2)\right)(y-\mu_y) - 2\rho^2\sigma_x\sigma_y \cdot x}{\sigma_x\sigma_y} - \frac{2\rho\sigma_x\,(y-\mu_y)}{\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}. \end{aligned}$$

We add and subtract $2\rho^2\sigma_x\sigma_y(\mu_x + \sigma_x^2)$ and obtain:

$$\begin{aligned} F = {} & -2\mu_x - \sigma_x^2 + \frac{\left(x - (\mu_x + \sigma_x^2)\right)^2}{\sigma_x^2} \\ & - \frac{2\rho\left(x - (\mu_x + \sigma_x^2)\right)(y - \mu_y - \rho\sigma_x\sigma_y)}{\sigma_x\sigma_y} + 2\rho^2(\mu_x + \sigma_x^2) + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho\sigma_x\,(y-\mu_y)}{\sigma_y}. \end{aligned}$$

Finally, the only thing left to do is to complete the square in the last two terms by adding and subtracting $\rho^2\sigma_x^2$:

$$F = -2\mu_x - \sigma_x^2 + \rho^2(2\mu_x + 2\sigma_x^2) + \frac{\left(x - \mu_x - \sigma_x^2\right)^2}{\sigma_x^2} - \frac{2\rho\left(x - \mu_x - \sigma_x^2\right)(y - \mu_y - \rho\sigma_x\sigma_y)}{\sigma_x\sigma_y} + \frac{(y - \mu_y - \rho\sigma_x\sigma_y)^2}{\sigma_y^2} - \rho^2\sigma_x^2.$$

By grouping other terms in front of the three fractions we finally have

$$\begin{aligned} F &= -(1-\rho^2)\left(2\mu_x + \sigma_x^2\right) + \left(\frac{x-\mu_x}{\sigma_x} - \sigma_x\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x} - \sigma_x\right)\left(\frac{y-\mu_y}{\sigma_y} - \rho\sigma_x\right) + \left(\frac{y-\mu_y}{\sigma_y} - \rho\sigma_x\right)^2 \\ &= -(1-\rho^2)\left(2\mu_x + \sigma_x^2\right) + \overline{x}^2 - 2\rho\overline{x}\overline{y} + \overline{y}^2, \end{aligned}$$

where new variables are introduced:

$$\overline{x} = \frac{x-\mu_x}{\sigma_x} - \sigma_x, \quad \overline{y} = \frac{y-\mu_y}{\sigma_y} - \rho\sigma_x.$$

By substituting the expression for F, back into the integral (A.1.2), and applying this change of integration variables, it reduces to

$$\mathbb{E}\left[e^X \cdot 1_{\{X<a,\,Y<b\}}\right] = e^{\mu_x + \frac{\sigma_x^2}{2}}\,\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\overline{a}}\int_{-\infty}^{\overline{b}} e^{-\frac{\overline{x}^2 - 2\rho\overline{x}\overline{y} + \overline{y}^2}{2(1-\rho^2)}}\,d\overline{y}\,d\overline{x} = e^{\mu_x + \frac{\sigma_x^2}{2}}\,N_2\left(\overline{a}, \overline{b}\right),$$

where we have used the bivariate standard normal cumulative distribution function N_2 given in (2.3.2), and the new limits of integration are

$$\overline{a} = \frac{a-\mu_x}{\sigma_x} - \sigma_x, \quad \overline{b} = \frac{b-\mu_y}{\sigma_y} - \rho\sigma_x.$$

This gives (A.1.1) and completes the proof. ■

A.1.2 Some Conditional Expectations

Let us now derive the functions $f_A^{i,j}$, $f_B^{i,j}$ and $g^{i,j}$ given by (2.3.27), (2.3.28) and (2.3.29). The derivation of $f_B^{i,j}$ is completely analogous to $f_A^{i,j}$, and therefore only one is presented in details. The other is easily obtained by interchanging the quantities d_A^i and $\widetilde{d_B^j}$ with $\widetilde{d_A^i}$ and d_B^j, respectively, (and A with B). In all three cases we can split the function in four parts according to four different regions identified in Figure A.1 below. So

$$\begin{aligned} f_A^{i,j}(t,A_t,B_t) &= \mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}\in[A^{i-1},A^i)\}}1_{\{B_{\tau-}\in[B^{j-1},B^j)\}}\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}<A^i,\,B_{\tau-}<B^j\}}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}<A^{i-1},\,B_{\tau-}<B^j\}}\right] \\ &\quad - \mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}<A^i,\,B_{\tau-}<B^{j-1}\}}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}<A^{i-1},\,B_{\tau-}<B^{j-1}\}}\right], \end{aligned} \tag{A.1.3}$$

$$\begin{aligned} g^{i,j}(t,A_t,B_t) &= \mathbb{E}_t^{\mathbb{Q}}\left[1_{\{A_{\tau-}\in[A^{i-1},A^i]\}}1_{\{B_{\tau-}\in[B^{j-1},B^j]\}}\right] = \mathbb{Q}\left\{A_{\tau-}\in[A^{i-1},A^i),\, B_{\tau-}\in[B^{j-1},B^j)\right\} \\ &= \mathbb{Q}\left\{A_{\tau-}<A^i,\, B_{\tau-}<B^j\right\} - \mathbb{Q}\left\{A_{\tau-}<A^{i-1},\, B_{\tau-}<B^j\right\} \\ &\quad - \mathbb{Q}\left\{A_{\tau-}<A^i,\, B_{\tau-}<B^{j-1}\right\} + \mathbb{Q}\left\{A_{\tau-}<A^{i-1},\, B_{\tau-}<B^{j-1}\right\}. \end{aligned} \tag{A.1.4}$$

From the SDE in (2.3.1), the solution for the geometric Brownian motion process at time τ^-, under the risk neutral measure $\mathbb{Q}$, is

$$A_{\tau-} = A_t e^{\left(r-\frac{1}{2}\sigma_A^2\right)(\tau-t)+\sigma_A\left(W_\tau^{\mathbb{Q},A}-W_t^{\mathbb{Q},A}\right)},$$

where the Brownian motion increment $W_\tau^{\mathbb{Q},A}-W_t^{\mathbb{Q},A}$ is normally distributed with zero mean and variance $\tau-t$, which means that the random variable $X=\ln A_{\tau-}$ is normally distributed with mean $\mu_x=\ln A_t+\left(r-\frac{1}{2}\sigma_A^2\right)(\tau-t)$ and standard deviation $\sigma_x=\sigma_A\sqrt{\tau-t}$. We have an analogous relationship for $Y=\ln B_{\tau-}$.

Now, consider the first term from (A.1.3) and use the previous result (A.1.1) in Lemma A.1:

$$\mathbb{E}_t^{\mathbb{Q}}\left[A_{\tau-}1_{\{A_{\tau-}<A^i,\,B_{\tau-}<B^j\}}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\ln A_{\tau-}}1_{\{\ln A_{\tau-}<\ln A^i,\,\ln B_{\tau-}<\ln B^j\}}\right] \tag{A.1.5}$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^X\cdot 1_{\{X<\ln A^i,\,Y<\ln B^j\}}\right] = e^{\mu_x+\frac{\sigma_x^2}{2}}N_2\left(\frac{\ln A^i-\mu_x}{\sigma_x}-\sigma_x,\ \frac{\ln B^j-\mu_y}{\sigma_y}-\rho\sigma_x\right)$$

$$= A_t e^{r(\tau-t)}N_2\left(\frac{\ln\frac{A^i}{A_t}-\left(r+\frac{1}{2}\sigma_A^2\right)(\tau-t)}{\sigma_A\sqrt{\tau-t}},\ \frac{\ln\frac{B^j}{B_t}-\left(r-\frac{1}{2}\sigma_B^2+\rho\sigma_A\sigma_B\right)(\tau-t)}{\sigma_B\sqrt{\tau-t}}\right).$$

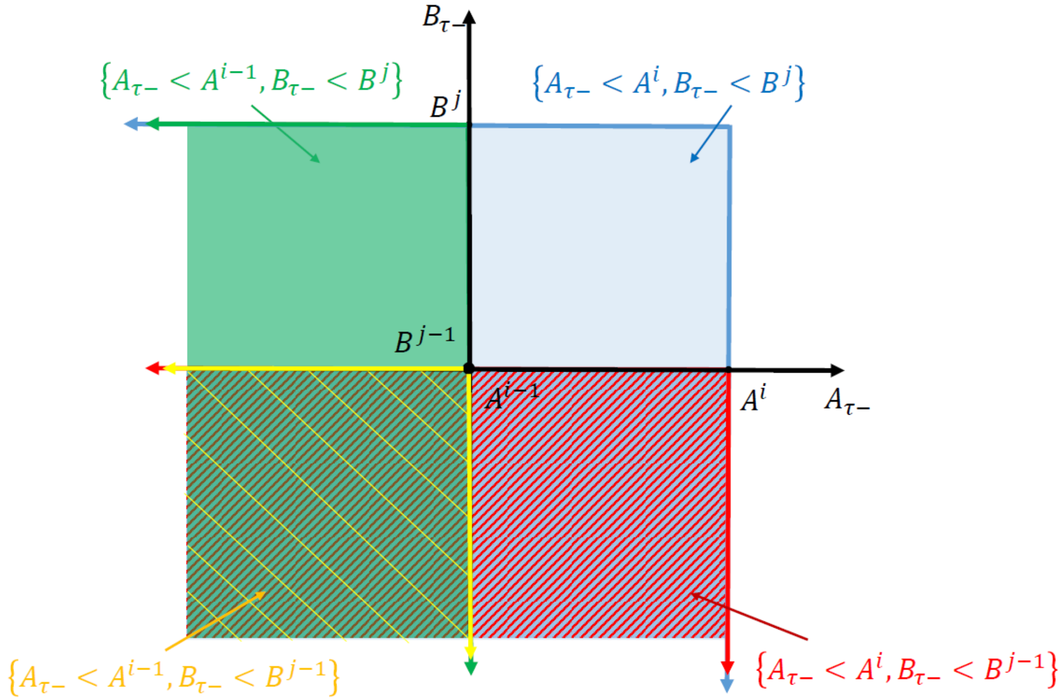

Figure A.1: Regions in (A, B) space required for derivations of functions $f_A^{i,j}$, $f_B^{i,j}$ and $g^{i,j}$.

Looking back to expression (A.1.3) and combining four such terms we finally obtain equation (2.3.27) with the definitions of the quantities d_A^i and $\widetilde{d_B^j}$. Equation (2.3.28) is obtained completely analogously by symmetry.

For expression (A.1.4) the situation is even simpler. By looking at the first term we conclude that

$$\begin{aligned}
&\mathbb{Q}\left\{A_{\tau-} < A^i,\, B_{\tau-} < B^j\right\} = \mathbb{Q}\left\{\ln A_{\tau-} < \ln A^i,\, \ln B_{\tau-} < \ln B^j\right\} \\
&= \mathbb{Q}\left\{X < \ln A^i,\, Y < \ln B^j\right\} = N_2\left(\frac{\ln A^i - \mu_x}{\sigma_x},\, \frac{\ln B^j - \mu_y}{\sigma_y}\right) \qquad \text{(A.1.6)} \\
&= N_2\left(\frac{\ln\frac{A^i}{A_t} - \left(r - \frac{1}{2}\sigma_A^2\right)(\tau - t)}{\sigma_A\sqrt{\tau - t}},\, \frac{\ln\frac{B^j}{B_t} - \left(r - \frac{1}{2}\sigma_B^2\right)(\tau - t)}{\sigma_B\sqrt{\tau - t}}\right).
\end{aligned}$$

Again, by combining four such terms we obtain the final expression (2.3.29). As already pointed out, we see that analogous quantities d_B^j and $\widetilde{d_B^j}$ are obtained from d_A^i and $\widetilde{d_A^i}$ by replacing A with B (and i with j). As a matter of fact, the function $f_B^{i,j}$ given in (2.3.28) follows directly from $f_A^{i,j}$ in (2.3.27), by interchanging the quantities d_A^i and $\widetilde{d_B^j}$ with $\widetilde{d_A^i}$ and d_B^j, respectively, which is obtained by interchanging A and B, interchanging i and j, and using the symmetry in N_2.

A.2 Partial Derivatives of Call-on-max Option Prices

A.2.1 First Order Derivatives

We will present here the derivation of the first order sensitivities of the call-on-max option price of Lemma 2.1, with respect to the underlying assets. From (2.3.3), by applying the chain rule, we have the partial derivative[1]

$$\begin{aligned}\frac{\partial \mathrm{OW}\left(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{\partial A} &= N_2\left(\widetilde{c_A}, c_A;\, \rho_A\right) + A\frac{\partial N_2\left(\widetilde{c_A}, c_A;\, \rho_A\right)}{\partial A} + B\frac{\partial N_2\left(\widetilde{c_B}, c_B;\, \rho_B\right)}{\partial A} \\ &\quad + Ke^{-r(T-t)}\frac{\partial N_2\left(-c_A+\sigma_A\sqrt{T-t}\,, -c_B+\sigma_B\sqrt{T-t}\,;\, \rho\right)}{\partial A}.\end{aligned} \tag{A.2.1}$$

Further, the derivative becomes

$$\begin{aligned}\frac{\partial \mathrm{OW}\left(A, B, K, r, \sigma_A, \sigma_B, \rho, T-t\right)}{\partial A} &= N_2\left(\widetilde{c_A}, c_A; \rho_A\right) + A\left[\frac{\partial N_2}{\partial \widetilde{c_A}}\left(\widetilde{c_A}, c_A; \rho_A\right)\frac{\partial \widetilde{c_A}}{\partial A} + \frac{\partial N_2}{\partial c_A}\left(\widetilde{c_A}, c_A; \rho_A\right)\frac{\partial c_A}{\partial A}\right] \\ &\quad + B\frac{\partial N_2}{\partial \widetilde{c_B}}\left(\widetilde{c_B}, c_B; \rho_B\right)\frac{\partial \widetilde{c_B}}{\partial A} \\ &\quad + Ke^{-r(T-t)}\frac{\partial N_2}{\partial(-c_A)}\left(-c_A+\sigma_A\sqrt{T-t}, -c_B+\sigma_B\sqrt{T-t}; \rho\right)\left(-\frac{\partial c_A}{\partial A}\right),\end{aligned} \tag{A.2.2}$$

with

$$\frac{\partial N_2\left(a, b;\, \rho\right)}{\partial a} = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{b} e^{-\frac{a^2-2\rho a y+y^2}{2\left(1-\rho^2\right)}}\,dy = \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\,N\left(\frac{b-\rho a}{\sqrt{1-\rho^2}}\right), \tag{A.2.3}$$

and

$$\begin{aligned}\frac{\partial \widetilde{c_A}}{\partial A} &= \frac{1}{A\sigma\sqrt{T-t}}, \\ \frac{\partial c_A}{\partial A} &= \frac{1}{A\sigma_A\sqrt{T-t}}, \\ \frac{\partial \widetilde{c_B}}{\partial A} &= -\frac{\partial \widetilde{c_A}}{\partial A}.\end{aligned}$$

This gives us the following result:

$$\begin{aligned}\frac{\partial \mathrm{OW}\left(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{\partial A} &= N_2\left(\widetilde{c_A}, c_A;\, \rho_A\right) + \frac{1}{\sqrt{2\pi\left(T-t\right)}}\left[\frac{e^{-\frac{\widetilde{c_A}^2}{2}}}{\sigma}\,N\left(\frac{c_A-\rho_A\widetilde{c_A}}{\sqrt{1-\rho_A^2}}\right)\right. \\ &\quad + \frac{e^{-\frac{c_A^2}{2}}}{\sigma_A}\,N\left(\frac{\widetilde{c_A}-\rho_A c_A}{\sqrt{1-\rho_A^2}}\right) - \frac{Be^{-\frac{\widetilde{c_B}^2}{2}}}{A\sigma}\,N\left(\frac{c_B-\rho_B\widetilde{c_B}}{\sqrt{1-\rho_B^2}}\right) \\ &\quad \left. - \frac{Ke^{-r(T-t)}e^{-\frac{\left(-c_A+\sigma_A\sqrt{T-t}\right)^2}{2}}}{A\sigma_A}\,N\left(\frac{\rho c_A-c_B+\left(\sigma_B-\rho\sigma_A\right)\sqrt{T-t}}{\sqrt{1-\rho^2}}\right)\right].\end{aligned} \tag{A.2.4}$$

[1] We will again present only the partial derivative with respect to the underlying A. The derivation for B is completely analogous and it is easily obtained by interchanging A and B in the end result.

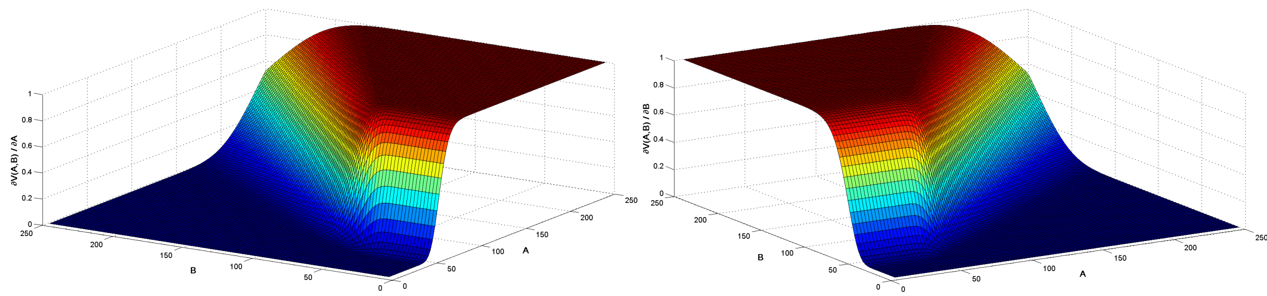

Figure A.2: Deltas of the European call on maximum option.

A.2.1.1 The Limit when $A \to \infty$

For a fixed B, from (2.3.3) we have that c_A, $\widetilde{c_A} \to \infty$ when $A \to \infty$, and thus

$$\lim_{A\to\infty} N_2\left(\widetilde{c_A}, c_A;\, \rho_A\right) = 1, \tag{A.2.5}$$

as well as

$$\lim_{A\to\infty} N\left(\frac{c_A - \rho_A \widetilde{c_A}}{\sqrt{1-\rho_A^2}}\right) = 1, \tag{A.2.6}$$

and

$$\lim_{A\to\infty} N\left(\frac{\widetilde{c_A} - \rho_A c_A}{\sqrt{1-\rho_A^2}}\right) = 1, \tag{A.2.7}$$

since $\rho_A < 1$. Further, terms $\lim\limits_{A\to\infty} e^{-\frac{c_A^2}{2}} = 0$ and $\lim\limits_{A\to\infty} e^{-\frac{\widetilde{c_A}^2}{2}} = 0$. Hence, in the above expression (A.2.4), the first two terms in the brackets go to zero when when $A \to \infty$.

Third term goes to zero both due to A in the denominator, and due to $\lim\limits_{A\to\infty} e^{-\frac{\widetilde{c_B}^2}{2}} = 0$, while

$$\lim_{A\to\infty} N\left(\frac{c_B - \rho_B \widetilde{c_B}}{\sqrt{1-\rho_B^2}}\right) = 1, \tag{A.2.8}$$

because of $\lim\limits_{A\to\infty} \widetilde{c_B} = -\infty$. Similar conclusion holds for the last term since

$$\lim_{A\to\infty} e^{-\frac{\left(-c_A+\sigma_A\sqrt{T-t}\right)^2}{2}} = 0, \tag{A.2.9}$$

and

$$\lim_{A\to\infty} N\left(\frac{\rho c_A - c_B + \left(\sigma_B - \rho\sigma_A\right)\sqrt{T-t}}{\sqrt{1-\rho^2}}\right) = 1. \tag{A.2.10}$$

This leaves us with only the first term in front of the brackets, i.e.

$$\lim_{A\to\infty} \frac{\partial \mathrm{OW}\left(A,\, B,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t\right)}{\partial A} = \lim_{A\to\infty} N_2\left(\widetilde{c_A}, c_A;\, \rho_A\right) = 1.$$

A.2.1.2 The Stronger Limit Result when $A \to \infty$

We now also show that

$$\lim_{A\to\infty}\left(1-\frac{\partial \mathrm{OW}}{\partial A}(A,\,.\,)\right)A=0,$$

i.e. the first derivative of the function converges to 1 relatively fast. From (A.2.4) we have that

$$\begin{aligned}\lim_{A\to\infty}\left(A-A\frac{\partial \mathrm{OW}}{\partial A}(A,\,.)\right) &= \lim_{A\to\infty}A\left(1-N_2\left(\widetilde{c_A},c_A;\,\rho_A\right)\right)+\frac{1}{\sqrt{2\pi\left(T-t\right)}}\left[\lim_{A\to\infty}\frac{Ae^{-\frac{\widetilde{c_A}^2}{2}}}{\sigma}N\left(\frac{c_A-\rho_A\widetilde{c_A}}{\sqrt{1-\rho_A^2}}\right)\right.\\ &+\lim_{A\to\infty}\frac{Ae^{-\frac{c_A^2}{2}}}{\sigma_A}N\left(\frac{\widetilde{c_A}-\rho_A c_A}{\sqrt{1-\rho_A^2}}\right)-\lim_{A\to\infty}\frac{Be^{-\frac{\widetilde{c_B}^2}{2}}}{\sigma}N\left(\frac{c_B-\rho_B\widetilde{c_B}}{\sqrt{1-\rho_B^2}}\right) \qquad \text{(A.2.11)}\\ &\left.-\lim_{A\to\infty}\frac{Ke^{-r(T-t)}e^{-\frac{\left(-c_A+\sigma_A\sqrt{T-t}\right)^2}{2}}}{\sigma_A}N\left(\frac{\rho c_A-c_B+\left(\sigma_B-\rho\sigma_A\right)\sqrt{T-t}}{\sqrt{1-\rho^2}}\right)\right].\end{aligned}$$

Since $0\le N(x)\le 1$ for all x, while for $A\to\infty$ we have $c_A,\,\widetilde{c_A}\to\infty;\ \widetilde{c_B}\to-\infty$, all terms in the bracket converge to zero. This is because

$$Ae^{-\frac{\widetilde{c_A}^2}{2}}=e^{-\frac{1}{2}\frac{(\ln A+\delta_1)^2}{\delta_2}+\ln A}, \qquad \text{(A.2.12)}$$

for constants $\delta_1\in\mathbb{R}$, $\delta_2>0$ which do not depend on A, and similar expressions can be written down for the other three cases. We are thus left only with

$$\begin{aligned}\lim_{A\to\infty}A\left(1-N_2\left(\widetilde{c_A},c_A;\,\rho_A\right)\right) &= \lim_{A\to\infty}\frac{1-N_2\left(\widetilde{c_A},c_A;\,\rho_A\right)}{\frac{1}{A}}\\ &= \lim_{A\to\infty}\frac{-\frac{e^{-\frac{\widetilde{c_A}^2}{2}}}{A\sigma\sqrt{2\pi(T-t)}}N\left(\frac{c_A-\rho_A\widetilde{c_A}}{\sqrt{1-\rho_A^2}}\right)-\frac{e^{-\frac{c_A^2}{2}}}{A\sigma_A\sqrt{2\pi(T-t)}}N\left(\frac{\widetilde{c_A}-\rho_A c_A}{\sqrt{1-\rho_A^2}}\right)}{-\frac{1}{A^2}}\\ &= \lim_{A\to\infty}\left[A\frac{e^{-\frac{\widetilde{c_A}^2}{2}}}{\sigma\sqrt{2\pi\left(T-t\right)}}N\left(\frac{c_A-\rho_A\widetilde{c_A}}{\sqrt{1-\rho_A^2}}\right)\right.\\ &\qquad\left.+A\frac{e^{-\frac{c_A^2}{2}}}{\sigma_A\sqrt{2\pi\left(T-t\right)}}N\left(\frac{\widetilde{c_A}-\rho_A c_A}{\sqrt{1-\rho_A^2}}\right)\right]=0,\end{aligned}$$

where in the first step the limit is transformed so that in the second step L'Hôpital's rule can be applied, while we once again used (A.2.3). But $\lim_{A\to\infty}Ae^{-\frac{\widetilde{c_A}^2}{2}}=0$ and $\lim_{A\to\infty}Ae^{-\frac{c_A^2}{2}}=0$, because of (A.2.12), and this completes the proof.

A.3 PDE Approach for Pricing a Call-on-max Option

Let r, σ_A, σ_B, ρ be given and fix the strike K and time of maturity T. Given the dynamics of the underlying assets (2.3.1), according to standard no arbitrage pricing theory, the price of a European call on maximum bivariate option OW $(A_t,\, B_t,\, K,\, r,\, \sigma_A,\, \sigma_B,\, \rho,\, T-t) \equiv V(t, A, B)$ can also be found as a solution of the following partial differential equation:

$$\frac{\partial V}{\partial t} = rV - rA\frac{\partial V}{\partial A} - rB\frac{\partial V}{\partial B} - \frac{1}{2}\sigma_A^2 A^2 \frac{\partial^2 V}{\partial A^2} - \rho\sigma_A\sigma_B AB\frac{\partial^2 V}{\partial A \partial B} - \frac{1}{2}\sigma_B^2 B^2 \frac{\partial^2 V}{\partial B^2},$$

with the terminal condition $V(T, A, B) = (\max(A, B) - K)^+$. By a suitable change of variables:

$$\theta = T - t,$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$
$$x_1 = \frac{2\rho\sigma_A \ln(B) - 2\sigma_B \ln(A) + (\sigma_A^2\sigma_B - \rho\sigma_A\sigma_B^2 + 2r\rho\sigma_A - 2r\sigma_B)\,\theta}{\sigma_A\sigma_B\sqrt{2\,(1-\rho^2)}},$$
$$x_2 = \frac{2\ln(B) + (2r - \sigma_B^2)\,\theta}{\sigma_B\sqrt{2}},$$

and

$$\varphi(\theta, x) = e^{r\theta}V(t, A, B),$$

this PDE can be transformed into the 2D heat equation[2]

$$\frac{\partial \varphi}{\partial \theta} = \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2}, \tag{A.3.1}$$

with initial condition

$$f(x) \equiv \varphi(0, x) = V(T, A, B).$$

The solution of the heat equation in $\mathbb{R}^2$ is, see for example Haberman (1983):

$$\varphi(\theta, x) = \frac{1}{4\pi\theta}\int\limits_{\mathbb{R}^2} f(x)\exp\left(-\frac{|x-y|^2}{4\theta}\right)dy, \tag{A.3.2}$$

which then yields a solution for our OW call-on-max option price $V(t, A, B) = e^{-r\theta}\varphi(\theta, x)$.

[2]The general form of the heat equation is $\frac{\partial \varphi}{\partial t} = k\nabla^2\varphi$, where k is a constant (i.e. thermal diffusivity) and ∇^2 denotes the Laplace operator, which in the Cartesian coordinate system is $\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$.

Appendix B

Auxiliary Derivations for Chapter 4

B.1 Some Proofs on Demand/Supply

B.1.1 Individual Demand/Supply: Second Order Conditions

In this appendix we present the second order conditions of the problem presented in Section 4.2.1 and show that, using the concavity of the utility function, they are always respected. The second order conditions are:

$$\begin{cases} \frac{\partial^2 \mathbb{E}\left[U\left(w_{i,d/s}\right)\right]}{\partial \Delta_0^2} \leq 0, \\ \\ \frac{\partial^2 \mathbb{E}\left[U\left(w_{i,d/s}.\right)\right]}{\partial q^2} \leq 0, \\ \\ \frac{\partial^2 \mathbb{E}\left[U\left(.w_{i,d/s}\right)\right]}{\partial \Delta_0^2} \frac{\partial^2 \mathbb{E}\left[U\left(.w_{i,d/s}\right)\right]}{\partial q^2} - \left[\frac{\partial^2 \mathbb{E}\left[U\left(w_{i,d/s}.\right)\right]}{\partial q \partial \Delta_0}\right]^2 > 0; \end{cases} \Leftrightarrow$$

$$\begin{cases} \sum_{i=1}^{3} p_i \left(S_i - RS_0\right)^2 U''\left(w_{i,d/s}\right) \leq 0, \\ \sum_{i=1}^{3} p_i \left(G_i - RP_{d/s}\right)^2 U''\left(w_{i,d/s}\right) \leq 0, \\ \left\{\sum_{i=1}^{3} p_i \left(S_i - RS_0\right)^2 U''\left(w_{i,d/s}\right)\right\} \left\{\sum_{i=1}^{3} p_i \left(G_i - RP_{d/s}\right)^2 U''\left(w_{i,d/s}\right)\right\} \\ \quad - \left[\pm \sum_{i=1}^{3} p_i \left(S_i - RS_0\right) \left(G_i - RP_{d/s}\right)^2 U''\left(w_{i,d/s}\right)\right]^2 > 0. \end{cases}$$

The third condition reduces to:

$$\sum \sum_{i \neq j} p_i p_j U''\left(w_{i,d/s}\right) U''\left(w_{j,d/s}\right) \left[\left(S_i - RS_0\right)\left(G_j - RP_{d/s}\right) + \left(G_i - RP_{d/s}\right)\left(S_j - RS_0\right)\right]^2 \geq 0.$$

As, $U''\left(w_{i,d/s}\right) \leq 0$ (utility function concave in wealth), all three second order conditions are always satisfied.

B.1.2 Proof of Theorem 4.1

Proof. Let $F\left(q_{d/s},\Delta,P_{d/s}\right)$ and $G\left(q_{d/s},\Delta,P_{d/s}\right)$ denote the first order conditions, for a positive q_d, of the problem that must be solved to find the market demand, i.e.

$$\begin{aligned} F\left(q_{d/s},\Delta,P_{d/s}\right) &= \sum_{i=1}^{3} p_i\left(S_i - RS_0\right)U'\left(w_{i,d/s}\right), \\ G\left(q_{d/s},\Delta,P_{d/s}\right) &= \sum_{i=1}^{3} p_i\left(G_i - RP_{d/s}\right)U'\left(w_{i,d/s}\right). \end{aligned}$$

Using the implicit function theorem we know

$$\begin{aligned} \frac{dq_{d/s}}{dP_{d/s}} &= -\frac{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial P_{d/s}} - \frac{\partial G}{\partial \Delta}\frac{\partial F}{\partial P_{d/s}}}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_{d/s}} - \frac{\partial F}{\partial q_{d/s}}\frac{\partial G}{\partial \Delta}} \\ &= -\frac{\begin{pmatrix} \pm\sum_{i=1}^{3} p_i\left(S_i-RS_0\right)^2 U''\left(w_{i,d/s}\right) \\ \cdot\left[-R\sum_{i=1}^{3} p_i U'\left(w_{i,d/s}\right) - Rq_{d/s}\sum_{i=1}^{3} p_i\left(G_i-RP_{d/s}\right)U''\left(w_{i,d/s}\right)\right] \\ \mp\sum_{i=1}^{3} p_i\left(G_i-RP_{d/s}\right)\left(S_i-RS_0\right)U''\left(w_{i,d/s}\right) \\ \cdot Rq_{d/s}\sum_{i=1}^{3} p_i\left(S_i-RS_0\right)U''\left(w_{i,d/s}\right) \end{pmatrix}}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_{d/s}} - \frac{\partial F}{\partial q_{d/s}}\frac{\partial G}{\partial \Delta}} \\ &= \pm R\frac{\begin{pmatrix} \sum_{i=1}^{3} p^i\left(S_i-RS_0\right)^2 U''\left(w_{i,d/s}\right)\sum_{i=1}^{3} p_i U'\left(w_{i,d/s}\right) \\ +q\left\{\sum_{i=1}^{3} p_i\left(S_i-RS_0\right)\left(G_i-RP_{d/s}\right)U''\left(w_{i,d/s}\right)\sum_{i=1}^{3} p_i\left(S_i-RS_0\right)U''\left(w_{i,d/s}\right)\right. \\ \left.-\sum_{i=1}^{3} p_i\left(S_i-RS_0\right)^2 U''\left(w_{i,d/s}\right)\sum_{i=1}^{3} p_i\left(G_i-RP_{d/s}\right)U''\left(w_{i,d/s}\right)\right\} \end{pmatrix}}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_{d/s}} - \frac{\partial F}{\partial q_{d/s}}\frac{\partial G}{\partial \Delta}}, \end{aligned}$$

where

$$\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_{d/s}} - \frac{\partial F}{\partial q_{d/s}}\frac{\partial G}{\partial \Delta} = \frac{\partial^2 \mathbb{E}\left[U\left(w_{i,d/s}\right)\right]}{\partial \Delta_0^2}\frac{\partial^2 \mathbb{E}\left[U\left(w_{i,d/s}\right)\right]}{\partial q^2} - \left[\frac{\partial^2 \mathbb{E}\left[U\left(w_{i,d/s}\right)\right]}{\partial q \partial \Delta_0}\right]^2 \geq 0,$$

by using the last second order conditions from previous appendix. Thus, we have that sign of $\frac{dq_{d/s}}{dP_{d/s}}$ depends on the sign of the numerator which proves the inequality of the theorem for both demand (decreasing function of demand price) and supply (increasing function of supply price) cases. ■

B.2 Arbitrage Bounds

The upper bound is given by

$$P^u = \min_{\Delta_0,B_0} \Delta_0 S_0 + B_0,$$

subject to

$$\Delta_0 S_i + RB_0 \geq G_i, \quad i = 1,2,3.$$

In the optimum two of the wealth constraints will be binding. Remember that, by assumption, $S_1 > S_2 > S_3$. Three possibilities must be considered.

1. The constraints binding are the first and the second. In that case the solution would be given by

$$\begin{aligned}\Delta_0 &= \frac{-G_1 + G_2}{S_2 - S_1},\\ B_0 &= -\frac{S_1 G_2 - G_1 S_2}{(S_2 - S_1) R}.\end{aligned}$$

The third constraint will be respected if and only if

$$G_1 (S_2 - S_3) + G_2 (S_3 - S_1) - G_3 (S_2 - S_1) \leq 0.$$

If that is the case the upper bound will be given by

$$P^u = \frac{G_2 (S_1 - RS_0) - G_1 (S_2 - RS_0)}{R (S_1 - S_2)}.$$

2. The constraints binding are the first and the third. In that case the solution would be given by

$$\begin{aligned}\Delta_0 &= \frac{-G_3 + G_1}{S_1 - S_3},\\ B_0 &= \frac{-S_3 G_1 + G_3 S_1}{(S_1 - S_3) R}.\end{aligned}$$

The second constraint will be respected if and only if

$$G_1 (S_2 - S_3) + G_2 (S_3 - S_1) - G_3 (S_2 - S_1) \geq 0.$$

If that is the case the upper bound will be given by

$$P^u = \frac{G_3 (S_1 - RS_0) - G_1 (S_3 - RS_0)}{R (S_1 - S_3)}.$$

3. The constraints binding are the second and the third. In that case the solution would be given by

$$\begin{aligned}\Delta_0 &= \frac{-G_3 + G_2}{S_2 - S_3},\\ B_0 &= -\frac{S_3 G_2 - G_3 S_2}{(S_2 - S_3) R}.\end{aligned}$$

The second constraint will be respected if and only if

$$G_1 (S_2 - S_3) + G_2 (S_3 - S_1) - G_3 (S_2 - S_1) \leq 0.$$

If that is the case the upper bound will be given by

$$P^u = \frac{G_3 (S_2 - RS_0) - G_2 (S_3 - RS_0)}{R (S_2 - S_3)}.$$

Note that if

$$G_1\left(S_2-S_3\right)+G_2\left(S_3-S_1\right)-G_3\left(S_2-S_1\right)\geq 0,$$

the upper bound will be the one described in situation 2. However, if

$$G_1\left(S_2-S_3\right)+G_2\left(S_3-S_1\right)-G_3\left(S_2-S_1\right)\leq 0,$$

there are two possible solutions. The solution described in situation 1 has a higher value than the one described in situation 3 if

$$\frac{G_2\left(S_1-RS_0\right)-G_1\left(S_2-RS_0\right)}{R\left(S_1-S_2\right)}\geq\frac{G_3\left(S_2-RS_0\right)-G_2\left(S_3-RS_0\right)}{R\left(S_2-S_3\right)}$$

$$\Leftrightarrow$$

$$G_1\left(S_2-S_3\right)\left(S_2-RS_0\right)+G_2\left(S_3-S_1\right)\left(S_2-RS_0\right)-G_3\left(S_2-S_1\right)\left(S_2-RS_0\right)\geq 0.$$

Hence, for the vector of parameters $\pi\equiv\left(S_0,S_1,S_2,S_3,G_1,G_2,G_3,R\right)$ and using the definitions

$$\begin{aligned}\mathcal{G}^+ &= \left\{\pi : G_1\left(S_2-S_3\right)+G_2\left(S_3-S_1\right)+G_3\left(S_1-S_2\right)\geq 0\right\},\\ \mathcal{G}^- &= \left\{\pi : G_1\left(S_2-S_3\right)+G_2\left(S_3-S_1\right)+G_3\left(S_1-S_2\right)\leq 0\right\};\end{aligned}$$

and

$$\mathcal{H}^+=\left\{\pi : S_2-RS_0\geq 0\right\},$$

$$\mathcal{H}^-=\left\{\pi : S_2-RS_0\leq 0\right\};$$

we can write the upper bound

$$P^u=\frac{G_3\left(S_1-RS_0\right)-G_1\left(S_3-RS_0\right)}{R\left(S_1-S_3\right)}\equiv P_{1,3}$$

if $\pi\in\mathcal{G}^+$,

$$P^u=\frac{G_3\left(S_2-RS_0\right)-G_2\left(S_3-RS_0\right)}{R\left(S_2-S_3\right)}\equiv P_{2,3}$$

if $\pi\in\mathcal{G}^-\cap\mathcal{H}^+$, and

$$P^u=\frac{G_2\left(S_1-RS_0\right)-G_1\left(S_2-RS_0\right)}{R\left(S_1-S_2\right)}\equiv P_{1,2}$$

if $\pi\in\mathcal{G}^-\cap\mathcal{H}^-$.

The lower bound is given by

$$P^l=\max_{\Delta_0,B_0}\Delta_0 S_0+B_0,$$

subject to

$$\Delta_0 S_i+RB_0\leq G_i,\quad i=1,2,3.$$

Proceeding in the same way we find out that

$$P^l = \frac{G_3\left(S_1 - RS_0\right) - G_1\left(S_3 - RS_0\right)}{R\left(S_1 - S_3\right)} \equiv P_{1,3}$$

if $\pi \in \mathcal{G}^-$,

$$P^l = \frac{G_3\left(S_2 - RS_0\right) - G_2\left(S_3 - RS_0\right)}{R\left(S_2 - S_3\right)} \equiv P_{2,3}$$

if $\pi \in \mathcal{G}^+ \cap \mathcal{H}^+$, and

$$P^l = \frac{G_2\left(S_1 - RS_0\right) - G_1\left(S_2 - RS_0\right)}{R\left(S_1 - S_2\right)} \equiv P_{1,2}$$

if $\pi \in \mathcal{G}^+ \cap \mathcal{H}^-$.

All this gives us in total 4 combinations of lower and upper bounds expressions as presented in Section 4.2.2.1.

B.3 Reservation and Fair Prices - Proof of Theorem 4.2

Proof. As the utility function is strictly concave in Δ and q then

$$\begin{aligned} & \sum_{i=1}^{3} p_i U\left[\Delta_0\left(S_i - RS_0\right) + q\left(G_i - RP\right) + Ry\right] \\ \leq \;& U\left[\Delta_0 \sum_{i=1}^{3} p_i\left(S_i - RS_0\right) + q \sum_{i=1}^{3} p_i\left(G_i - RP\right) + Ry\right]. \end{aligned}$$

If $P = \frac{1}{R}\sum_{i=1}^{3} p_i G_i$ and $S_0 = \frac{1}{R}\sum_{i=1}^{3} p_i S_i$ we have

$$\sum_{i=1}^{3} U\left[\Delta_0\left(S_i - RS_0\right) + q\left(G_i - RP\right) + Ry\right] \leq U\left[Ry\right].$$

Hence, in order to maximise the expected utility, the best strategy is $\Delta^* = q^* = 0$ leading to the maximal expected utility $U\left[Ry\right]$.

In the remainder of this proof we shall concentrate on the case where $P = \frac{1}{R}\sum_{i=1}^{3} p_i G_i$ and $S_0 \neq \frac{1}{R}\sum_{i=1}^{3} p_i S_i$. Regarding the sign of the optimal Δ_0, notice that

$$\begin{aligned} & \sum_{i=1}^{3} p_i U\left[\Delta_0\left(S_i - RS_0\right) + q\left(G_i - RP\right) + Ry\right] \\ \leq \;& U\left[\Delta_0 \sum_{i=1}^{3} p_i\left(S_i - RS_0\right) + Ry\right]. \end{aligned}$$

In order to guarantee a level of expected utility above $U\left[Ry\right]$, which is the utility with $\Delta = q = 0$, we must assure that $\Delta_0 > 0$ if $\sum_{i=1}^{3}\left(S_i - RS_0\right) > 0$ and $\Delta_0 < 0$ if $\sum_{i=1}^{3}\left(S_i - RS_0\right) < 0$.

Regarding the sign of the optimal q, notice that

$$P = \bar{P} \Leftrightarrow p_2 (G_2 - RP) = -p_1 (G_1 - RP) - p_3 (G_3 - RP).$$

implying that the first order condition (second condition from Proposition 4.1)

$$\sum_{i=1}^{3} p_i (G_i - RP) U' (w_i) = 0,$$

may be rewritten as

$$p_1 (G_1 - RP) [U' (w_1) - U' (w_2)] = -p_3 (G_3 - RP) [U' (w_3) - U' (w_2)], \tag{B.3.1}$$

with $w_i = \Delta_0 (S_i - RS_0) + Ry + q (G_i - RP), i = 1, 2$ and 3. Using the assumption about the order of the payoffs and the fairness of P,

$$\begin{aligned} RP &\in \{\min (G_1, G_3), \max (G_1, G_3)\} \\ &\Leftrightarrow \frac{(G_1 - RP)}{(G_3 - RP)} < 0 \\ &\Rightarrow \frac{[U' (w_1) - U' (w_2)]}{[U' (w_3) - U' (w_2)]} < 0, \end{aligned}$$

which implies that

$$w_1 > w_2 \text{ and } w_3 > w_2, \tag{Case 1}$$

or

$$w_1 < w_2 \text{ and } w_3 < w_2. \tag{Case 2}$$

Note that $w_i = \Delta_0 (S_i - RS_0) + Ry + q (G_i - RP)$, implying that

$$w_i > w_j \Leftrightarrow \Delta_0 (S_j - S_i) < q (G_i - G_j). \tag{B.3.2}$$

We now consider the four possible situations, analysing in each one the two cases mentioned above.

1. We first consider the case $G_1 \geq G_2 \geq G_3$ and $\sum_{i=1}^{3} p_i (S_i - RS_0) > 0$.

   (a) If $G_1 > G_2 > G_3$, in Case 1 we must have

   $$w_3 > w_2 \Leftrightarrow \Delta_0 (S_2 - S_3) < q (G_3 - G_2).$$

   As $\Delta > 0$, q must be strictly negative. The same procedure applies if Case 2 is considered, leading to

   $$w_1 < w_2 \Leftrightarrow \Delta_0 (S_2 - S_1) > q (G_1 - G_2), \tag{B.3.3}$$

   and a strictly negative value for q.

   (b) $G_1 > G_2 = G_3$ is incompatible with Case 1. However, Case 2 applies.

   (c) $G_1 = G_2 > G_3$ is incompatible with Case 2, but Case 1 applies.

2. We next consider the case $G_1 \leq G_2 \leq G_3$ and $\sum_{i=1}^{3} p_i (S_i - RS_0) > 0$.

   (a) If $G_1 < G_2 < G_3$, in Case 1 we must assure $\Delta_0 (S_2 - S_1) > q (G_1 - G_2)$. As $\Delta > 0$, q must be strictly positive. A similar reasoning applies in Case 2. Using the relation in (B.3.3), $\Delta > 0 \Rightarrow q > 0$.

   (b) $G_1 < G_2 = G_3$ is incompatible with Case 1. However, Case 2 applies.

   (c) $G_1 = G_2 < G_3$ is incompatible with Case 2, but Case 1 applies.

3. We now consider the case $G_1 \geq G_2 \geq G_3$ and $\sum_{i=1}^{3} p_i (S_i - RS_0) < 0$.

   (a) If $G_1 > G_2 > G_3$ in Case 1 we must have

   $$w_1 > w_2 \Leftrightarrow \Delta_0 (S_2 - S_1) < q (G_1 - G_2),$$

   and $\Delta < 0 \Rightarrow q > 0$. A similar procedure applies in Case 2, where we take

   $$w_3 < w_2 \Leftrightarrow \Delta_0 (S_2 - S_3) > q (G_3 - G_2),$$

   and $\Delta < 0 \Rightarrow q > 0$.

   (b) $G_1 > G_2 = G_3$ is incompatible with Case 1. However, Case 2 applies.

   (c) $G_1 = G_2 > G_3$ is incompatible with Case 2, but Case 1 applies.

4. We finally consider the case $G_1 \leq G_2 \leq G_3$ and $\sum_{i=1}^{3} p_i (S_i - RS_0) < 0$.

   (a) If $G_1 > G_2 > G_3$ in Case 1 we must have $\Delta_0 (S_2 - S_1) < q (G_1 - G_2)$. As $\Delta < 0$, q must be strictly negative. A similar reasoning applies in Case 2. We must assure that $\Delta_0 (S_2 - S_3) > q (G_3 - G_2)$. As $\Delta < 0$, q must be strictly negative.

   (b) $G_1 < G_2 = G_3$ is incompatible with Case 1. However, Case 2 applies.

   (c) $G_1 = G_2 < G_3$ is incompatible with Case 2, but Case 1 applies. ■

B.4 Illustrations

B.4.1 Explicit Solution for the Individual Demand and Supply Functions

A solution for the demand and supply of the derivative for a CARA and a CRRA utility functions can be explicitly obtained. The procedure is the following.

We start from the equation (4.2.9) derived in Section 4.2.2.1. Thus:

$$\begin{aligned}
\mu_1 &= \frac{RP(S_2 - S_3) + G_3 (RS_0 - S_2) - G_2 (RS_0 - S_3)}{G_1 (S_2 - S_3) + G_2 (S_3 - S_1) + G_3 (S_1 - S_2)} \frac{1}{p_1}, \\
\mu_2 &= \frac{RP(S_3 - S_1) + G_1 (RS_0 - S_3) - G_3 (RS_0 - S_1)}{G_1 (S_2 - S_3) + G_2 (S_3 - S_1) + G_3 (S_1 - S_2)} \frac{1}{p_2}, \\
\mu_3 &= \frac{RP(S_1 - S_2) + G_2 (RS_0 - S_1) - G_1 (RS_0 - S_2)}{G_1 (S_2 - S_3) + G_2 (S_3 - S_1) + G_3 (S_1 - S_2)} \frac{1}{p_3};
\end{aligned}$$

and denote by $[U'(.)]^{-1}$ the inverse function of the marginal utility. Hence the system (4.2.9) becomes:

$$U'(w_i) = A\mu_i \Rightarrow w_i = [U']^{-1}(A\mu_i).$$

Moreover, as $w_i = \Delta_0(S_i - RS_0) + Ry + q(G_i - RP)$ the following system is obtained

$$\begin{array}{c}\Delta_0(S_1 - RS_0) + q(G_1 - RP) - [U']^{-1}(A\mu_i) = -Ry,\\ \Delta_0(S_2 - RS_0) + q(G_2 - RP) - [U']^{-1}(A\mu_i) = -Ry,\\ \Delta_0(S_3 - RS_0) + q(G_3 - RP) - [U']^{-1}(A\mu_i) = -Ry.\end{array} \tag{B.4.1}$$

The system presented above is a system of three equations and three variables (q, Δ_0, A), with a unique solution for the CARA and CRRA utility functions. First, we are going to consider the CARA utility functions and then the CRRA utility functions.

If a CARA utility is considered then $U'(w_i) = \delta e^{-\delta w_i}$. Therefore,

$$U'(w_i) = A\mu_i \Rightarrow w_i = [U']^{-1}(A\mu_i) = -\frac{1}{\delta}\ln\left(\frac{A\mu_i}{\delta}\right).$$

Hence, system (B.4.1) can be written as

$$\begin{array}{c}\Delta_0(S_1 - RS_0) + q(G_1 - RP) + \frac{1}{\delta}\ln\left(\frac{A\mu_1}{\delta}\right) = -Ry,\\ \Delta_0(S_2 - RS_0) + q(G_2 - RP) + \frac{1}{\delta}\ln\left(\frac{A\mu_2}{\delta}\right) = -Ry,\\ \Delta_0(S_3 - RS_0) + q(G_3 - RP) + \frac{1}{\delta}\ln\left(\frac{A\mu_3}{\delta}\right) = -Ry.\end{array}$$

Solving for q we obtain the individual demand/supply for the derivative, i.e.,

$$q = -\frac{1}{\delta}\frac{(S_2 - S_3)\ln\mu_1 + (S_3 - S_1)\ln\mu_2 + (S_1 - S_2)\ln\mu_3}{G_1(S_2 - S_3) + G_2(S_3 - S_1) + G_3(S_1 - S_2)}.$$

If the CRRA utility is considered then $U'(w_i) = (w_i)^{-\gamma}$. Therefore,

$$U'(w_i) = A\mu_i \Rightarrow w_i = [U']^{-1}(A\mu_i)^{-1} = (A\mu_i)^{-\frac{1}{\gamma}}.$$

Hence, system (B.4.1) becomes

$$\begin{array}{c}\Delta_0(S_1 - RS_0) + q(G_1 - RP) - (A\mu_1)^{-\frac{1}{\gamma}} = -Ry,\\ \Delta_0(S_2 - RS_0) + q(G_2 - RP) - (A\mu_2)^{-\frac{1}{\gamma}} = -Ry,\\ \Delta_0(S_3 - RS_0) + q(G_3 - RP) - (A\mu_3)^{-\frac{1}{\gamma}} = -Ry.\end{array}$$

Solving for q gives

$$q = Ry\frac{\mu_1^{-\frac{1}{\gamma}}(S_2 - S_3) + \mu_2^{-\frac{1}{\gamma}}(S_3 - S_1) + \mu_3^{-\frac{1}{\gamma}}(S_1 - S_2)}{\left(\begin{array}{c}\mu_1^{-\frac{1}{\gamma}}[(S_3 - RS_0)(G_2 - RP) - (S_2 - RS_0)(G_3 - RP)]\\ +\mu_2^{-\frac{1}{\gamma}}[(S_1 - RS_0)(G_3 - RP) - (S_3 - RS_0)(G_1 - RP)]\\ +\mu_3^{-\frac{1}{\gamma}}[(S_2 - RS_0)(G_1 - RP) - (S_1 - RS_0)(G_2 - RP)]\end{array}\right)}.$$

B.4.2 Properties of the Individual Demand and Supply

B.4.2.1 CARA Utility Function

Properties 1 and 2

For a CARA utility function the first order conditions (4.2.3) are given by:

$$\begin{cases} \sum_{i=1}^{3} p_i \left(S_i - RS_0\right) e^{-\delta\left[\Delta_{0,d/s} S_i + q_{d/s} G_i\right]} = 0, \\ \sum_{i=1}^{3} p_i \left(G_i - RP_{d/s}\right) e^{-\delta\left[\Delta_{0,d/s} S_i + q_{d/s} G_i\right]} \leq 0. \end{cases}$$

In order to prove that $\frac{\partial q_d}{\partial P_d} < 0$ notice that, as $U''\left(w_{i,d/s}\right) = -\delta U'\left(w_{i,d/s}\right)$, we have

$$\sum_{i=1}^{3} p_i \left(S_i - RS_0\right) U''\left(w_{i,d/s}\right) = \sum_{i=1}^{3} p_i \left(G_i - RP_d\right) U''\left(w_{i,d/s}\right) = 0.$$

Using Proposition 4.1 whose proof is given in B.1.2 we have the following for the demand (Property 1) and the supply (Property 2) of the derivative:

$$\frac{\partial q_d}{\partial P_d} = \frac{R\sum_{i=1}^{3} p_i \left(S_i - RS_0\right)^2 U''\left(w_{i,d}\right) \cdot \sum_{i=1}^{3} p_i U'\left(w_{i,d}\right)}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \Delta}} < 0,$$

$$\frac{\partial q_s}{\partial P_s} = -\frac{R\sum_{i=1}^{3} p_i \left(S_i - RS_0\right)^2 U''\left(w_{i,s}\right) \cdot \sum_{i=1}^{3} p_i U'\left(w_{i,s}\right)}{\frac{\partial F}{\partial \Delta}\frac{\partial G}{\partial q_d} - \frac{\partial F}{\partial q_d}\frac{\partial G}{\partial \Delta}} > 0.$$

Property 3

In order to prove that $\frac{d\phi_d(P_d)}{d\delta} \leq 0$ notice that the first and second conditions of the optimisation problem can be written in terms of $\Lambda = \delta\Delta$ and $\Gamma = \delta q$, eliminating δ and q from the first and second order conditions. Hence, we can only find the optimal values of Λ and Γ (Λ^* and Γ^*). The optimal values of the number of shares bought/sold and the number of options bought/sold is given by

$$\Delta = \frac{\Lambda^*}{\delta},$$

$$q = \frac{\Gamma^*}{\delta}.$$

Property 4

This property follows from the fact that δq is constant.

Property 5

Moreover, note that the first and second conditions are independent of y. Hence, the optimal values will also be independent of y.

B.4.2.2 CRRA Utility Function

For a CRRA utility function the first order conditions presented in equations (4.2.3) can be written as

$$\begin{cases} \sum_{i=1}^{3} p_i \left(S_i - RS_0\right) \left[\Delta_{0,d/s}\left(S_i - RS_0\right) + Ry + q_{d/s}\left(G_i - RP_d\right)\right]^{-\gamma} = 0, \\ \sum_{i=1}^{3} p_i \left(G_i - RP_d\right) \left[\Delta_{0,d/s}\left(S_i - RS_0\right) + Ry + q_{d/s}\left(G_i - RP_d\right)\right]^{-\gamma} = 0. \end{cases}$$

Property 1 and 2

See Section 4.2.3. Numerically shown, graph in the Figure 4.2.

Property 3

The property follows straightforwardly using the optimal quantity defined in equation (4.2.11).

Property 4

Let the function $\upsilon\left(P,\gamma\right)$ be defined as[1]

$$\upsilon\left(P\right) = \mu_1^{-\frac{1}{\gamma}}\left(S_3 - S_2\right) + \mu_2^{-\frac{1}{\gamma}}\left(S_1 - S_3\right) + \mu_3^{-\frac{1}{\gamma}}\left(S_2 - S_1\right).$$

Using the optimal quantity, defined in equation (4.2.11), the reservation price is the price $\hat{P}$ such that

$$\upsilon\left(\hat{P},\gamma\right) = 0. \tag{B.4.2}$$

Using the implicit function theorem we have

$$\begin{aligned} \frac{d\hat{P}}{d\gamma} &= -\frac{\frac{\partial\upsilon\left(\hat{P},\gamma\right)}{\partial\gamma}}{\frac{\upsilon\left(\hat{P},\gamma\right)}{\partial\hat{P}}} \\ &= -\frac{G_1\left(S_2 - S_3\right) + G_2\left(S_3 - S_1\right) - G_3\left(S_2 - S_1\right)}{\gamma} \\ &= \frac{\ln\left(\mu_1\right)\mu_1^{-\frac{1}{\gamma}}\left(S_3 - S_2\right) + \ln\left(\mu_2\right)\mu_2^{-\frac{1}{\gamma}}\left(S_1 - S_3\right) + \ln\left(\mu_3\right)\mu_3^{-\frac{1}{\gamma}}\left(S_2 - S_1\right)}{\mu_1^{-\frac{1}{\gamma}-1}\left(S_3 - S_2\right)^2 + \mu_2^{-\frac{1}{\gamma}-1}\left(S_1 - S_3\right)^2 + \mu_3^{-\frac{1}{\gamma}-1}\left(S_2 - S_1\right)^2}. \end{aligned}$$

As the denominator of the second fraction is always positive we have to check the sign of the numerator in order to define the sign of $\frac{d\hat{P}}{d\gamma}$.

Using equation (B.4.2) we can write $\mu_2^{-\frac{1}{\gamma}}$ as an weighted average of $\mu_1^{-\frac{1}{\gamma}}$ and $\mu_3^{-\frac{1}{\gamma}}$, i.e.

$$\mu_2^{-\frac{1}{\gamma}} = \frac{S_2 - S_3}{S_1 - S_3}\mu_1^{-\frac{1}{\gamma}} + \frac{S_3 - S_1}{S_1 - S_3}\mu_3^{-\frac{1}{\gamma}}.$$

[1] Note that each μ_i is a function of P.

Moreover, as $p_1\mu_1 + p_2\mu_2 + p_3\mu_3 = 1$, we must have one of the following situations:

$$\mu_1^{-\frac{1}{\gamma}} > \mu_2^{-\frac{1}{\gamma}} > \mu_3^{-\frac{1}{\gamma}} \Leftrightarrow \mu_1 < \mu_2 < \mu_3 \Leftrightarrow \mu_1 < 1,\ \mu_3 > 1, \tag{B.4.3}$$
$$\mu_1^{-\frac{1}{\gamma}} < \mu_2^{-\frac{1}{\gamma}} < \mu_3^{-\frac{1}{\gamma}} \Leftrightarrow \mu_1 > \mu_2 > \mu_3 \Leftrightarrow \mu_1 > 1,\ \mu_3 < 1. \tag{B.4.4}$$

Additionally, as $\ln(.)$ is a concave function we have

$$\ln(\mu_2) \geq \ln(\mu_1)\frac{S_2 - S_3}{S_1 - S_3} + \ln(\mu_3)\frac{S_2 - S_1}{S_1 - S_3}.$$

Hence,

$$\begin{aligned}
& \ln(\mu_1)\mu_1^{-\frac{1}{\gamma}}(S_3 - S_2) + \ln(\mu_2)\mu_2^{-\frac{1}{\gamma}}(S_1 - S_3) + \ln(\mu_3)\mu_3^{-\frac{1}{\gamma}}(S_2 - S_1) \\
\geq\ & \ln(\mu_1)\mu_1^{-\frac{1}{\gamma}}(S_3 - S_2) + \left[\ln(\mu_1)\frac{S_2 - S_3}{S_1 - S_3} + \ln(\mu_3)\frac{S_1 - S_2}{S_1 - S_3}\right]\mu_2^{-\frac{1}{\gamma}} \\
& + \ln(\mu_3)\mu_3^{-\frac{1}{\gamma}}(S_2 - S_1) \\
=\ & \ln(\mu_1)(S_2 - S_3)\left[\mu_2^{-\frac{1}{\gamma}} - \mu_1^{-\frac{1}{\gamma}}\right] + \ln(\mu_3)(S_1 - S_2)\left[\mu_2^{-\frac{1}{\gamma}} - \mu_3^{-\frac{1}{\gamma}}\right].
\end{aligned}$$

If case given by (B.4.3) is considered

$$\underbrace{\ln(\mu_1)}_{<0}\ \underbrace{(S_2 - S_3)}_{>0}\ \underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_1^{-\frac{1}{\gamma}}\right]}_{<0} + \underbrace{\ln(\mu_3)}_{>0}\ \underbrace{(S_1 - S_2)}_{>0}\ \underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_3^{-\frac{1}{\gamma}}\right]}_{>0} > 0.$$

If case given by (B.4.4) is considered

$$\underbrace{\ln(\mu_1)}_{>0}\ \underbrace{(S_2 - S_3)}_{>0}\ \underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_1^{-\frac{1}{\gamma}}\right]}_{>0} + \underbrace{\ln(\mu_3)}_{<0}\ \underbrace{(S_1 - S_2)}_{>0}\ \underbrace{\left[\mu_2^{-\frac{1}{\gamma}} - \mu_3^{-\frac{1}{\gamma}}\right]}_{<0} > 0,$$

resulting in

$$\ln(\mu_1)\mu_1^{-\frac{1}{\gamma}}(S_3 - S_2) + \ln(\mu_2)\mu_2^{-\frac{1}{\gamma}}(S_1 - S_3) + \ln(\mu_3)\mu_3^{-\frac{1}{\gamma}}(S_2 - S_1) \geq 0.$$

It follows that

$$sign\left(\frac{d\hat{P}}{d\gamma}\right) = -sign\left[G_1(S_2 - S_3) + G_2(S_3 - S_1) - G_3(S_2 - S_1)\right].$$

B.4.2.3 CARA Utility Function Intermediate Results

For the same CARA example presented in Subsection 4.2.3.3 we hereby show two more quantities $w_i,\ i = 1, 2, 3$:

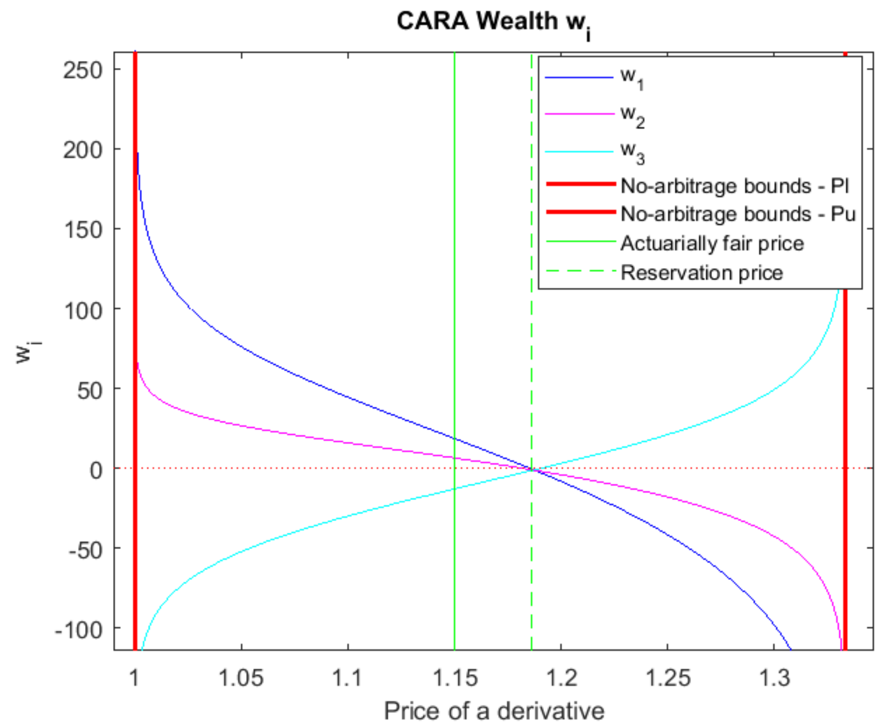

Figure B.1: The wealth attained under different states of nature for different derivative prices

and $U(w_i)$, $i = 1, 2, 3$:

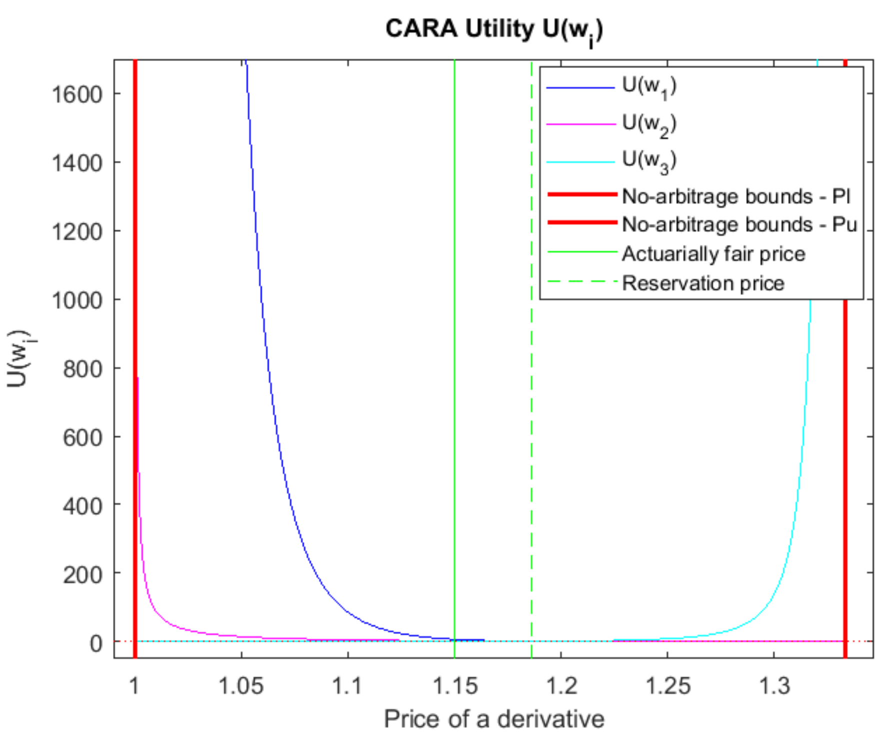

Figure B.2: The utility attained under different states of nature for different derivative prices under different states of nature (with probabilities of occuring $p_i,\ i = 1, 2, 3$), which lead to the final expected utility (von Neumann-Morgenstern) presented in Figure 4.3, bottom right panel.

B.5 Monopolistic Market Maker

B.5.1 Conditions on Assumption 2

A function $f : A \to R$ is concave if and only if for every $x \in A$, the Hessian matrix $D^2 f(x)$ is negative semidefinite. For the function considered, the Hessian matrix is

$$\begin{bmatrix} -\frac{d^2[Q_B P_s(Q_B)]}{dQ_B^2} & 0 & 0 \\ 0 & \frac{d^2[Q_A P_d(Q_A)]}{dQ_A^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, this matrix negative semidefinite if and only if $Q_B P_s(Q_B)$ is a convex function in Q_B and $Q_A P_d(Q_A)$ is a concave function in Q_A, i.e.

$$\begin{aligned} \frac{d^2[Q_B P_s(Q_B)]}{dQ_B^2} &= 2\frac{dP_s(Q_B)}{dQ_B} + \frac{\partial^2 P_s(Q_B)}{\partial Q_B^2} \geq 0, \\ \frac{d^2[Q_A P_d(Q_A)]}{dQ_A^2} &= 2\frac{dP_d(Q_A)}{dQ_A} + \frac{\partial^2 P_d(Q_A)}{\partial Q_A^2} \leq 0. \end{aligned}$$

B.5.2 Proof of Proposition 4.2

B.5.2.1 Convexity of the Constraint Set

In order to check for the convexity of the constraint set consider two possible elements of the constraint set. Let them be (Q_B^1, Q_A^1, Δ^1) and (Q_B^2, Q_A^2, Δ^2). Then, for each constraint $i = 1, 2$ and 3, we have

$$-Q_B^1\left(G_i - RP_s\left(Q_B^1\right)\right) + Q_A^1\left(G_i - RP_d\left(Q_A^1\right)\right) - \Delta^1(S_i - RS_0) - Ry \leq 0 \tag{B.5.1}$$

and

$$-Q_B^2\left(G_i - RP_s\left(Q_B^2\right)\right) + Q_A^2\left(G_i - RP_d\left(Q_A^2\right)\right) - \Delta^2(S_i - RS_0) - Ry \leq 0. \tag{B.5.2}$$

If the element

$$\left(Q_B^3, Q_A^3, \Delta^3\right) = \left(\lambda Q_B^1 + (1-\lambda) Q_B^2, \lambda Q_A^1 + (1-\lambda) Q_A^2, \lambda\Delta^1 + (1-\lambda)\Delta^2\right)$$

respects the three constraints then the constraint set is convex. In order to check for that characteristic of the constraint set, multiply equation (B.5.1) by λ and equation (B.5.2) by $1 - \lambda$. The following equation is obtained

$$\begin{gathered} -Q_B^3 G_i + Q_A^3 G_i - \Delta^3(S_i - RS_0) - Ry + \\ -R\left(\lambda Q_A^1 P_d\left(Q_A^1\right) + (1-\lambda) Q_A^2 P_d\left(Q_A^2\right)\right) + \\ +R\left(\lambda Q_B^1 P_s\left(Q_B^1\right) + (1-\lambda) Q_B^2 P_s\left(Q_B^2\right)\right) \leq 0. \end{gathered}$$

As $Q_A P_d(Q_A)$ is a concave function

$$\begin{aligned} R\left(\lambda Q_A^1 P_d\left(Q_A^1\right) + (1-\lambda) Q_A^1 P_d\left(Q_A^1\right)\right) &\leq RQ_A^3 P_d\left(Q_A^3\right), \\ -R\left(\lambda Q_A^1 P_d\left(Q_A^1\right) + (1-\lambda) Q_A^1 P_d\left(Q_A^1\right)\right) &\geq -RQ_A^3 P_d\left(Q_A^3\right). \end{aligned}$$

Moreover, as $Q_B P_s(Q_B)$ is a convex function

$$R\left[\lambda Q_B^1 P_s\left(Q_B^1\right) + (1-\lambda) Q_B^2 P_s\left(Q_B^2\right)\right] \geq RQ_B^3 P_s\left(Q_B^3\right),$$

we have

$$-Q_B^3\left[G_i - RP_s\left(Q_B^3\right)\right] + Q_A^3\left[G_i - RP_d\left(Q_A^2\right)\right] - \Delta^3\left(S_i - RS_0\right) - Ry \leq 0.$$

B.5.2.2 Existence of the Bid-Ask Spread

In what follows the proof of the second part of Proposition 4.2, that concerns the existence of a bid-ask spread if there is an equilibrium with strictly positive quantities $\{Q_B^*, Q_A^*\}$, is presented.

Proof. First, the optimal condition expressed in equality (4.3.5) is considered. Then, the optimal condition expressed in equality (4.3.6) is also considered.

In the case presented in equations (4.3.5) the proof is done by contradiction. Suppose that $P_d(Q_A^*) \leq P_s(Q_B^*)$. Three cases must be considered concerning the relation between $P_d(Q_A^*)$, $Q_A^* \frac{dP_d\left(Q_A^*\right)}{dQ_A}$, $P_s(Q_B^*)$, $Q_B^* \frac{dP_s\left(Q_B^*\right)}{dQ_B}$ and $\bar{G}_k$. First, consider that $\bar{G}_k > P_s(Q_B^*) + Q_B^* \frac{dP_s\left(Q_B^*\right)}{dQ_B}$. Using equation (4.3.5), we obtain

$$\begin{aligned} \bar{G}_k - P_d(Q_B^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} &\geq \bar{G}_k - P_s(Q_B^*) - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} > 0 \Leftrightarrow \\ \sum_{i=1}^{3} p_i \bar{G}_i - P_d(Q_A^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} &\leq \sum_{i=1}^{3} p_i \bar{G}_i - P_s(Q_B^*) - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} \Leftrightarrow \\ P_s(Q_B^*) - P_d(Q_A^*) &\leq Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}. \end{aligned}$$

As, by Assumption 1,

$$Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} < 0$$

then

$$P_s(Q_B^*) - P_d(Q_A^*) < 0,$$

contradicting $P_d(Q_A^*) \leq P_s(Q_B^*)$.

In the second case, consider that $P_d(Q_A^*) < P_s(Q_B^*)$ and $P_d(Q_A^*) + Q_A^* \frac{dP_d\left(Q_A^*\right)}{dQ_A} < \bar{G}_k <$

$P_s(Q_B^*) + Q_B^* \frac{dP_s(Q_B^*)}{dQ_B}$. From equation (4.3.5) we obtain

$$\begin{aligned}
\bar{G}_k - P_s(Q_B^*) Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} &\leq \bar{G}_k - P_d(Q_A^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} \Leftrightarrow \qquad \text{(B.5.3)} \\
\sum_{i=1}^{3} p_i \bar{G}_i - P_s(Q_B^*) - Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} &\geq \sum_{i=1}^{3} p_i \bar{G}_i - P_d(Q_A^*) - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} \Leftrightarrow \\
P_d(Q_A^*) - P_s(Q_B^*) &\geq Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A}.
\end{aligned}$$

As, by Assumption 1,

$$Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} - Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} > 0$$

then

$$P_d(Q_A^*) - P_s(Q_B^*) > 0,$$

contradicting $P_d(Q_A^*) \leq P_s(Q_B^*)$.

At last, consider that $\bar{G}_k < P_d(Q_A^*) + Q_A^* \frac{dP_d(Q_A^*)}{dQ_A}$. In this case, from equation (4.3.5) we obtain the same relation that is displayed in equation (B.5.3), for the case $P_d(Q_A^*) < \bar{G}_k < P_s(Q_B^*)$. Therefore, by contraction we prove that for the tangency solution $P_d(Q_A^*) > P_s(Q_B^*)$.

Now, consider the optimal conditions expressed in equations (4.3.6). Subtracting the second equation from the first one we have

$$(1 - \lambda_k - \lambda_j) [\Psi(Q_A^*) - \Psi(Q_B^*)] = 0.$$

As $1 - \lambda_k - \lambda_j > 0$, we must assure $\Psi(Q_A^*) - \Psi(Q_B^*) = 0$. Then, using Assumption 1, and noting that $\Psi(Q_A^*) - \Psi(Q_B^*) = 0$ is equivalent to

$$P_d(Q_A^*) - P_s(Q_B^*) = -Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} + Q_B^* \frac{dP_s(Q_B^*)}{dQ_B},$$

we have $P_d(Q_A^*) - P_s(Q_B^*) > 0$. ■

B.5.3 Proof of Corollary 4.1

Proof. In order to study the behaviour of the function $\Phi_k(Q)$ we take the derivative

$$\frac{d\Phi_k(Q)}{dQ} = \frac{\frac{d^2[QP(Q)]}{dQ^2} \left[\sum_{i=1}^{3} p_i \bar{G}_i - \bar{G}_k\right]}{\left[\bar{G}_k - \frac{d[QP(Q)]}{dQ}\right]^2}.$$

The sign of $\frac{d^2[Q_B P_s(Q_B)]}{dQ_B^2}$ and $\frac{d^2[Q_A P_d(Q_A)]}{dQ_A^2}$ is well defined by Assumption 2. Therefore, we identify the regions where $\Phi_k(Q_B)$ is decreasing in Q_B, and $\Phi_k(Q_A)$ is increasing in Q_A,

	$\sum_{i=1}^{3} p_i \bar{G}_i - \bar{G}_k \geq 0$	$\sum_{i=1}^{3} p_i \bar{G}_i - \bar{G}_k \leq 0$
$\frac{d\Phi_k(Q_B)}{dQ_B}$	≥ 0	≤ 0
$\frac{d\Phi_k(Q_A)}{dQ_A}$	≤ 0	≥ 0

Equation (4.3.5) reads $\Phi_k(Q_A^*) = \Phi_k(Q_B^*) = \sigma_k$. Now consider the case $\sum_{i=1}^{3} p_i\bar{G}_i - \bar{G}_k \geq 0$. In that region $\Phi_k(Q_A^*) = \Phi_k(Q_B^*) \Rightarrow \Phi_k(Q_A = 0) \geq \Phi_k(Q_B = 0)$, and the full equation (4.3.5) is satisfied iff

$$\Phi_k(Q_A = 0) \geq \sigma_k \geq \Phi_k(Q_B = 0).$$

As $\Phi_k(Q = 0)$ is an increasing function on P, then $\hat{P}_d \geq \hat{P}_s$.

In what follows we present the relation between $\hat{P}_d$, $\hat{P}_s$, $\sum_{i=1}^{3} p_i\bar{G}_i$ and $\bar{G}_k$ at the optimum of the monopolistic market maker.

Suppose that $\hat{P}_d$ and $\hat{P}_s$ belong to the arbitrage-free range of variation for the value of a European derivative. It is not possible to have both $\hat{P}_d$ and $\hat{P}_s$ above $\sum_{i=1}^{3} p_i\bar{G}_i$ or below $\bar{G}_k$, otherwise $\Phi_k(Q_A = 0) > 0$ and $\Phi_k(Q_B = 0) > 0$, which is incompatible with $\sigma_k < 0$. Moreover, it is not possible to have $\hat{P}_d > \sum_{i=1}^{3} p_i\bar{G}_i$ and $\bar{G}_k < \hat{P}_s < \sum_{i=1}^{3} p_i\bar{G}_i$, because from Theorem 4.2, Remark 1, we know that $\sum_{i=1}^{3} p_i\bar{G}_i \notin \left(\hat{P}_s, \hat{P}_d\right)$. Finally, the situation where $\hat{P}_s < \bar{G}_k$ is also not possible because in that case $\Phi_k(Q_B = 0) > 0$, contradicting the fact that $\sigma_k \geq \Phi_k(Q_B = 0)$ and $\sigma_k \leq 0$. Hence, the case that remains is exactly what is given by equation (4.3.10). Proceeding in the same way for the case $\sum_{i=1}^{3} p_i\bar{G}_i - \bar{G}_k \leq 0$ we find out that (4.3.9) holds. ■

B.5.4 Proof of Corollary 4.2

Proof. From both equilibrium characterisations within Proposition 4.2, i.e. (4.3.5) and (4.3.6) we can conclude that $\Psi(Q_A^*) = c_A$, $\Psi(Q_B^*) = c_B$ where $c_A > 0$ and $c_B > 0$ are positive constants (quantities that depend only on the parameters of the model and are not functions of demand and supply, nor of the prices) and Ψ is a function defined as before in (4.3.4). From this definition and after applying the chain rule we have

$$\begin{cases} P_d(Q_A^*) + Q_A^* \frac{dP_d(Q_A^*)}{dQ_A} = c_A > 0, \\ P_s(Q_B^*) + Q_B^* \frac{dP_s(Q_B^*)}{dQ_B} = c_B > 0; \end{cases} \implies \begin{cases} \frac{dP_d(Q_A^*)}{dQ_A} > 0, \\ \frac{dP_s(Q_B^*)}{dQ_B} > 0. \end{cases} \tag{B.5.4}$$

By using aggregate demand/supply formula (4.2.7) and its differentiation given in (4.2.12), the required quantities become:

$$\begin{aligned} \frac{dP_d(Q_A^*)}{d\delta} &= \frac{dP_d(Q_A^*)}{dQ_A}\frac{dQ_A}{d\delta} = \frac{dP_d(Q_A^*)}{dQ_A} n\left(\frac{dq_d}{d\delta}\right) \leq \quad 0, \\ \frac{dP_d(Q_B^*)}{d\delta} &= \frac{dP_s(Q_B^*)}{dQ_B}\frac{dQ_B}{d\delta} = \frac{dP_s(Q_B^*)}{dQ_B} n\left(-\frac{dq_s}{d\delta}\right) \geq \quad 0. \end{aligned}$$

The final inequalities in the last step above follow from the signs of the quantities just given in (B.5.4) and the CARA utility Property 3, derived in the Subsection 4.2.3.2, which completes the proof. ■

For the case of CRRA utility similar property with respect to parameter γ can only be shown numerically.

B.6 Competition Between Market Makers

B.6.1 Proof of Theorem 4.3

Proof.

Case III
In this case we must consider two possible situations:
Situation 1:

$$\left|\frac{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\sum_{i=1}^{3}p_i\bar{G}_i-P_{j,d}\right]+Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\bar{G}_k-P_{j,d}\right]+Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}\right|<\left|\frac{\sum_{i=1}^{3}p_i\bar{S}_i-S_0}{\bar{S}_k-S_0}\right|.$$

Deviation: Increase $P_{j,d}$ and decrease Δ_0 if $\bar{S}_k-S_0>0$, or increase Δ_0 if $\bar{S}_k-S_0<0$.

Situation 2:

$$\left|\frac{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\sum_{i=1}^{3}p_i\bar{G}_i-P_{j,d}\right]+Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\bar{G}_k-P_{j,d}\right]+Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}\right|\geq\left|\frac{\sum_{i=1}^{3}p_i\bar{S}_i-S_0}{\bar{S}_k-S_0}\right|.$$

As

$$\left|\frac{\sum_{i=1}^{3}p_i\bar{G}_i-P_{j,d}}{\bar{G}_k-(P_{j,d}-\varepsilon)}\right|>\left|\frac{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\sum_{i=1}^{3}p_i\bar{G}_i-P_{j,d}\right]+Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}}\left[\bar{G}_k-P_{j,d}\right]+Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}\right|,$$

we have

$$\left|\frac{\sum_{i=1}^{3}p_i\bar{G}_i-P_{j,d}}{\bar{G}_k-P_{j,d}}\right|>\left|\frac{\sum_{i=1}^{3}p_i\bar{S}_i-S_0}{\bar{S}_k-S_0}\right|.$$

Deviation: Decrease $P_{j,d}\rightarrow$ increase $Q^*_{j,A}$ and increase Δ_0 if $\bar{S}_k-S_0>0$, or decrease Δ_0 if $\bar{S}_k-S_0<0$.

Case IV
Notice that the impact of slightly decrease the price $P_{j,s}$ in the expected wealth is

$$-\left\{\frac{\partial Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)}{\partial P_{j,s}}\left[\sum_{i=1}^{3}p_i\bar{G}_i-P_{j,s}\right]-Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)\right\}>0.$$

Moreover, the impact on the constraint is

$$-\left\{\frac{\partial Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)}{\partial P_{j,s}}\left[\bar{G}_k-P_{j,s}\right]-Q_{j,B}\left(P_{j,s},P^*_{-j,s}\right)\right\}<0.$$

Moreover,

$$\sum_{i=1}^{3} p_i \bar{G}_i - P_{j,s} \leq 0 \text{ and } \bar{G}_k - P_{j,s} \geq 0.$$

Proceeding in an analogous way as in case III we can find a profitable deviation changing $P_{j,s}$ and Δ_j.

Case XI

In this case two possibilities must be considered.

Situation 1:

$$\left| \frac{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\sum_{i=1}^{3} p_i \bar{G}_i - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\bar{G}_k - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)} \right| > \left| \frac{\sum_{i=1}^{3} p_i \bar{S}_i - S_0}{\bar{S}_k - S_0} \right|.$$

Deviation: Increase $P^*_{j,d}$ and increase Δ_0 if $\bar{S}_k - S_0 > 0$, or decrease Δ_0 if $\bar{S}_k - S_0 < 0$.

Situation 2:

$$\left| \frac{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\sum_{i=1}^{3} p_i \bar{G}_i - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\bar{G}_k - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)} \right| \leq \left| \frac{\sum_{i=1}^{3} p_i \bar{S}_i - S_0}{\bar{S}_k - S_0} \right|.$$

Note that

$$-\frac{\partial Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\bar{G}_k - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right) < 0 \Rightarrow \bar{G}_k - P_{j,d} < 0,$$

as

$$\sum_{i=1}^{3} p_i \bar{G}_i - P^*_{j,d} > 0.$$

In this case we can check that

$$\left| \frac{-dQ_{j,A}\left[\sum_{i=1}^{3} p_i \bar{G}_i - P_{j,d}\right]}{-dQ_{j,A}\left[\bar{G}_k - P_{j,d}\right]} \right| < \left| \frac{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\sum_{i=1}^{3} p_i \bar{G}_i - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{-\frac{\partial Q_{j,A}\left(P_{j,d},P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\bar{G}_k - P_{j,d}\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)} \right|,$$

hence, as

$$\left| \frac{-dQ_{j,A}\left[\sum_{i=1}^{3} p_i \bar{G}_i - P_{j,d}\right]}{-dQ_{j,A}\left[\bar{G}_k - P_{j,d}\right]} \right| < \left| \frac{\sum_{i=1}^{3} p_i \bar{S}_i - S_0}{\bar{S}_k - S_0} \right|,$$

we have the following deviation.

Deviation: Decrease $P_{j,d} \rightarrow$ increase $Q^*_{j,A}$ and decrease Δ_0 if $\bar{S}_k - S_0 > 0$, or increase Δ_0 if $\bar{S}_k - S_0 < 0$.

Case XII

In this case note that as

$$-\frac{\partial Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right)}{\partial P_{j,d}} \left[\bar{G}_k - (P_{j,d} - \varepsilon)\right] + Q_{j,A}\left(P_{j,d}, P^*_{-j,d}\right) < 0,$$

case XI applies.

Now, consider the case when two constraints are binding. Let them be constraint m and n. If an agent decides to increase P_s or decrease P_d, the positive alteration in quantities must be such that

$$dQ_{j,B}\left(\bar{G}_m - (P_{j,s}+\varepsilon)\right) - dQ_{j,A}\left(\bar{G}_m - (P_{j,d}-\varepsilon)\right) \geq 0,$$
$$dQ_{j,B}\left(\bar{G}_n - (P_{j,s}+\varepsilon)\right) - dQ_{j,A}\left(\bar{G}_n - (P_{j,d}-\varepsilon)\right) \geq 0.$$

The alteration in the utility is

$$dQ_{j,B}\left(\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s}+\varepsilon)\right) - dQ_{j,A}\left(\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,d}-\varepsilon)\right) > 0,$$

and it is positive or equal to zero if

$$\begin{cases} \frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\sum_{i=1}^{3} p_i\bar{G}_i - \left(P_{j,d}-\varepsilon\right)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s}+\varepsilon)}, & P_{j,d} \leq \sum_{i=1}^{3} p_i\bar{G}_i; \\ \forall Q_{j,B},\ \forall Q_{j,A}, & P_{j,s} < \sum_{i=1}^{3} p_i\bar{G}_i < P_{j,d}; \\ \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\sum_{i=1}^{3} p_i\bar{G}_i - \left(P_{j,d}-\varepsilon\right)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s}+\varepsilon)}, & \sum_{i=1}^{3} p_i\bar{G}_i < P_{j,s}. \end{cases}$$

The constraints will be respected if

$$dQ_{j,B}\left(\bar{G}_k - (P_{j,s}+\varepsilon)\right) - dQ_{j,A}\left(\bar{G}_k - (P_{j,d}-\varepsilon)\right) > 0;$$

$$\begin{cases} \frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\bar{G}_k - \left(P_{j,d}-\varepsilon\right)}{\bar{G}_k - (P_{j,s}+\varepsilon)}, & P_{j,d} \leq \bar{G}_k; \\ \forall Q_{j,B},\ \forall Q_{j,A}, & P_{j,s} < \bar{G}_k < P_{j,d}; \\ \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_k - \left(P_{j,d}-\varepsilon\right)}{\bar{G}_k - (P_{j,s}+\varepsilon)}, & \bar{G}_k < P_{j,s}. \end{cases}$$

Now, for each case of the two constraints binding. Let them be denoted by m and n. All the possibilities concerning the relation between P_d, P_s, $\bar{G}_n$ and $\bar{G}_m$, are presented (Cases I - VI) in Figure B.1, as well as profitable deviation for each case.

	$. < P_s$	$P_s < . < P_d$	$P_d < .$
Case I			$\bar{G}_n$, $\bar{G}_m$
Case II		$\bar{G}_n$	$\bar{G}_m$
Case III	$\bar{G}_n$		$\bar{G}_m$
Case IV		$\bar{G}_n$, $\bar{G}_m$	
Case V	$\bar{G}_n$	$\bar{G}_m$	
Case VI	$\bar{G}_n$, $\bar{G}_m$		

Table B.1: All possible cases when two constraints are binding.

Case I

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \max\left\{\frac{\bar{G}_n-(P_{j,d}-\varepsilon)}{\bar{G}_n-(P_{j,s}+\varepsilon)}, \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}\right\}.$$

Situation A:

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \max\left\{\frac{\bar{G}_n-(P_{j,d}-\varepsilon)}{\bar{G}_n-(P_{j,s}+\varepsilon)}, \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}, \frac{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,d}-\varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,s}+\varepsilon)}\right\}.$$

Situation B:

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \max\left\{\frac{\bar{G}_n-(P_{j,d}-\varepsilon)}{\bar{G}_n-(P_{j,s}+\varepsilon)}, \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}\right\}.$$

Situation C:

$$\max\left\{\frac{\bar{G}_n-(P_{j,d}-\varepsilon)}{\bar{G}_n-(P_{j,s}+\varepsilon)}, \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}\right\} < \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,d}-\varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,s}+\varepsilon)}$$

which is verified.

Case II

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}.$$

Situation A:

In order to have an increase in utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,d}-\varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,s}+\varepsilon)}.$$

Hence, any deviation:

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \max\left\{\frac{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,d}-\varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,s}+\varepsilon)}, \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}\right\},$$

will increase utility.

Situation B:

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\bar{G}_m-(P_{j,d}-\varepsilon)}{\bar{G}_m-(P_{j,s}+\varepsilon)}.$$

Situation C:

In order to have an increase in utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,d}-\varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i-(P_{j,s}+\varepsilon)}.$$

Hence, as

$$\frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)} > \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)},$$

any $\frac{dQ_{j,B}}{dQ_{j,A}}$ such that

$$\frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} < \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)},$$

will increase utility.

Case III

In order to have the wealth constraints respected we must have

$$\frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} < \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}.$$

It is easy to check that there is a non-empty set for $\frac{dQ_{j,B}}{dQ_{j,A}}$.

Situation A:

In order to have an increase in utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)}.$$

Hence, as

$$\frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)},$$

any $\frac{dQ_{j,B}}{dQ_{j,A}}$ such that

$$\max\left\{\frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)}, \frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)}\right\} < \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)},$$

will increase utility.

Situation B:

$$\frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} < \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}.$$

Situation C:

In order to have an increase in utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)}.$$

Hence, as

$$\frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)} > \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)},$$

any $\frac{dQ_{j,B}}{dQ_{j,A}}$ such that

$$\frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} < \frac{dQ_{j,B}}{dQ_{j,A}} < \min\left\{\frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}, \frac{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s} + \varepsilon)}\right\},$$

will increase utility.

Case IV

Any possible $\frac{dQ_{j,B}}{dQ_{j,A}}$ will respect the wealth constraints. Hence, it is possible to find a deviation that increases utility.

Case V

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}.$$

Situation A:

In order to have an increase in utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s} + \varepsilon)}.$$

Hence, as

$$\frac{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s} + \varepsilon)} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)},$$

any $\frac{dQ_{j,B}}{dQ_{j,A}}$ such that

$$\frac{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s} + \varepsilon)} < \frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)},$$

will increase utility.

Situation B:

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}.$$

Situation C:

In order to also have an increase in utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \min\left\{\frac{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i\bar{G}_i - (P_{j,s} + \varepsilon)}, \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}\right\}.$$

Case VI

In order to have the wealth constraints respected we must have

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \min\left\{\frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}, \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)}\right\}.$$

Situation A:
In order to increase utility

$$\frac{dQ_{j,B}}{dQ_{j,A}} > \frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)}.$$

Hence, as

$$\frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)} < \min\left\{ \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}, \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} \right\},$$

a deviation is

$$\frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)} < \frac{dQ_{j,B}}{dQ_{j,A}} < \min\left\{ \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}, \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} \right\}.$$

Situation B:

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \min\left\{ \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}, \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)} \right\}.$$

Situation C:

$$\frac{dQ_{j,B}}{dQ_{j,A}} < \min\left\{ \frac{\bar{G}_n - (P_{j,d} - \varepsilon)}{\bar{G}_n - (P_{j,s} + \varepsilon)}, \frac{\bar{G}_m - (P_{j,d} - \varepsilon)}{\bar{G}_m - (P_{j,s} + \varepsilon)}, \frac{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,d} - \varepsilon)}{\sum_{i=1}^{3} p_i \bar{G}_i - (P_{j,s} + \varepsilon)} \right\},$$

which is verified. ■

B.7 Dasgupta-Maskin (1986a) Discontinuity Equilibrium Existence Theorem

For each player i $(i = 1, ..., n)$ let the action set $A_i \in \mathbb{R}^m$ be convex and compact. Let the payoff function $\pi_i(a_1, ...a_n)$ be continuous except on a subset A_i^* consisting of action combinations such that for some $j \neq i$, some action k, and some integer d, there is some continuous one-to-one function $f_{i,j}^d : \mathbb{R}^1 \to \mathbb{R}^1$ such that

$$f_{i,j}^d = \left(f_{i,j}^d\right)^{-1}, \quad and \quad a_{,jk} = f_{i,j}^d\left(a_{ik}\right).$$

Suppose that $\sum \pi_i(a)$ is upper semicontinuous, and for all i the payoff is bounded and is weakly lower semicontinuous in a_i. Then the game has a mixed strategy Nash equilibrium.

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