Games for functions

Baire classes, Weihrauch degrees, transfinite computations, and ranks

de Holanda Cunha Nobrega, H.

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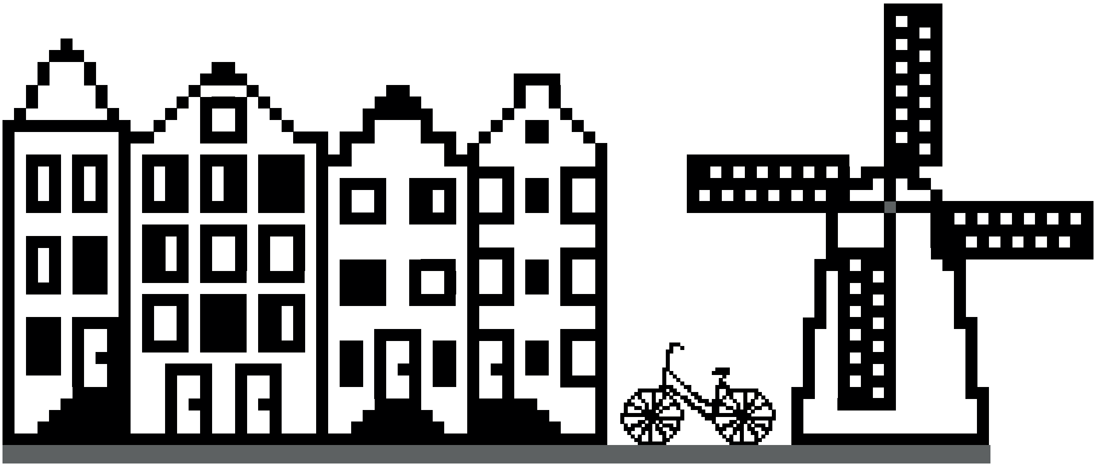

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Games for functions

Baire classes, Weihrauch degrees, transfinite computations, and ranks

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para Diana e Bernardo

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Chapter 1

Introduction

1.1 General introduction

In its early history, the relationship between mathematics and games was limited to mathematical methods being used in the analysis of games. Famous examples are the work on games of chance by Cardano, in the 16^{th} century, and by Fermat and Pascal, in the 17^{th}, originating what is now known as probability theory (cf., e.g., [8]), and Zermelo's analysis of chess and other similar two-player adversarial games which do not depend on chance (cf. [90]).

This relationship only became one of true symbiotic mutualism in the 20^{th} century, when games began being used as tools in the study of mathematics—which is how games feature in this thesis. One illustrious example is the work of Ehrenfeucht and Fraïssé in developing *back and forth* games, which are used to determine the level of similarity between structures and still a fundamental tool in current-day model theory (cf. [48, § 3.2]). However, this is far from the only important example; games have found applications in a wide range of areas of mathematical logic, such as the semantics of logics (e.g., the modal μ-calculus [98, § 2.2]), characterizations of classes of problems in complexity theory (e.g., [75]), and descriptive set theory (e.g., the Choquet game characterizing Baire spaces and the Banach-Mazur game characterizing meager sets; cf. [54, 8.C & 8.H]), among others (cf. [49] for a survey of applications of games to logic).

In many cases, the use of games can help one succinctly express properties of the underlying objects which might otherwise be considerably more cumbersome. In this way, games can also be an important tool in simplifying proofs (e.g., Blackwell's elegant game proof of Kuratowski's coreduction principle for analytic sets [9]), and can therefore also play an important didactic role in the teaching of logic (cf., e.g., [7, 32, 96]).

One case in which games have been truly essential to the development of mathematical logic is the work of Gale and Stewart in the 1950s [39]. The game they introduced started what might be fairly described as a revolution in set theory and still bears fruit as an active topic of research to this day. The Gale-Stewart game concerns subsets of *Baire space*, which is the topological space composed of infinite sequences of natural numbers and the fundamental space of descriptive set theory. Given a subset A of Baire space, the *Gale-Stewart game for A* is played by two players who take turns in ω rounds, each player choosing a natural number at each round. In this way they collectively build an element of Baire space, and the first player wins the game iff the element thus built is in A, with the second player winning otherwise. We say A is *determined* if one of the two players has a strategy for playing which guarantees a win; thus the determinacy of A expresses the infinitary De Morgan law

$$\neg(\exists x_0 \in \omega\, \forall x_1 \in \omega\, \exists x_2 \in \omega\, \forall x_3 \in \omega \cdots (\langle x_0, x_1, x_2, x_3, \ldots \rangle \in A))$$
$$\Leftrightarrow$$
$$\forall x_0 \in \omega\, \exists x_1 \in \omega\, \forall x_2 \in \omega\, \exists x_3 \in \omega \cdots (\langle x_0, x_1, x_2, x_3 \ldots \rangle \notin A)).$$

Postulating the determinacy of Gale-Stewart games for a certain class of sets has far-reaching consequences for that class, in particular in terms of so-called *regularity* properties such as Lebesgue measurability, property of Baire, perfect set property, etc. (cf. [53, Chapter 6] for a thorough account of this topic). It should be noted that, by the time of Gale and Stewart's seminal paper, the Polish mathematicians in Lwów (modern day Lviv, Ukraine) already had a decades-long tradition of applying infinite games to set theory and topology at their famous *Scottish Café* discussions [63]. It is widely accepted that by the 1930s they were aware of what would later become known as the Gale-Stewart game and its basic results, such as the determinacy of all open or closed sets and the existence of non-determined sets as a consequence of the axiom of choice; cf., e.g., [53, p. 371].

A crucial development came with the work of Wadge [99, 100], who introduced a game similar to the one of Gale and Stewart in order to study the relation of continuous reducibility between subsets of Baire space. Given such sets A and B, in the *Wadge game* for A and B two players play as in the Gale-Stewart game, except that the second player is allowed to postpone making a move for finitely many rounds at a time. Thus the two players build elements x and y of Baire space, respectively, and the second player wins the game exactly in case it holds that $x \in A$ iff $y \in B$. Wadge proved that the second player has a winning strategy in the game for A and B iff A is continuously reducible to B, and that if the game for A and B is determined, then either A is reducible to B or B is reducible to the complement of A, a result which came to be known as *Wadge's lemma*. Thus the determinacy of the Wadge game for a class of sets implies that the class is stratified into a highly granular hierarchy in which antichains have size at most 2, which has been described as "the ultimate analysis of [subsets of Baire space] in terms of topological complexity" by Andretta and Louveau [4, p.8].

Viewed from a slightly different perspective, the Wadge game can be seen as a game characterizing the class of continuous functions on Baire space. Given such a function f, we abuse nomenclature slightly and call *Wadge game for f* the game in which two players play exactly as in the Wadge game for sets, thus building elements x and y of Baire space, respectively, with the second player winning iff $f(x) = y$. It follows that the second player has a winning strategy in the game iff f is continuous. The two versions of the Wadge game are of course closely connected: if a function f reduces A to B, then a winning strategy for the second player in the Wadge game for f also wins the Wadge game for A and B for her, and conversely, any winning strategy for the second player in the Wadge game for A and B induces a continuous function f reducing A to B, and that strategy is also winning for the second player in the Wadge game for f.

By giving the second player more freedom in how she builds her sequence y, larger classes of functions can be characterized. In the *backtrack game*, usually attributed to Van Wesep (cf., e.g., [3, p. 4]), the second player can decide to restart her definition of y from scratch an arbitrary finite number of times; the resulting game characterizes the class of $\mathbf{\Delta}^0_2$-functions, i.e., those for which the preimage of any $\mathbf{\Delta}^0_2$ set[†] is also a $\mathbf{\Delta}^0_2$ set [3, Theorem 21]. In the *eraser game*, usually attributed to Duparc [29, 31], the second player can erase each element of y an arbitrary finite number of times; the resulting game characterizes the Baire class 1 functions. In his PhD thesis, Semmes [92] introduced four new games characterizing the Borel measurable functions, the Baire class 2 functions, the $\mathbf{\Delta}^0_3$-functions, and the class of functions for which the preimage of any $\mathbf{\Delta}^0_2$ set is a $\mathbf{\Delta}^0_3$ set.

Because they express the amount of *power* one needs in order to build the output of a function when given access to longer and longer portions of its input, such game characterizations intuitively measure the level of discontinuity of the functions in question. Via the foundational result of effective descriptive set theory that oracle computability of functions on Baire space coincides with the notion of continuity for that space exactly (cf. Theorem 2.8 below), it follows that game characterizations of classes of functions are closely related to questions of computability for that class. Therefore, it should perhaps not come as a surprise that the tools of *computable analysis*, the area of mathematical logic concerned with questions of computability of functions, sets, and other objects from classical analysis, can be fruitfully brought to bear on questions related to such game characterizations.

In computable analysis, the study of computability over a structure X is done via a *representation* of X, which is a surjection from the Baire space onto X. In this way, this analysis is completely analogous to the Ershov-style analysis of computability over countable structures via coding functions from the natural numbers. The fundamental tool of computable analysis which we use in this thesis is that of *Weihrauch reducibility*, which captures the notion that a function is *at most as uncomputable as* another; f is Weihrauch-reducible to g iff the existence of a method for computing g would also imply the computability of f.

[†]These terms are defined precisely in § 1.2.3.

Organization of the text

In this thesis we study several aspects of game characterizations of classes of functions. The text is organized as follows.

We close the present chapter with a review of the background material from descriptive set theory and computable analysis which will be needed for the rest of the thesis.

In Chapter 2 we review games for functions from the literature and define modifications of Semmes's game for Borel functions which characterize the Baire class α functions for each fixed $\alpha < \omega_1$. We also define a construction of games which transforms a game characterizing a class Λ of functions into a game characterizing the class of functions whose domains can be partitioned into countably many relatively $\mathbf{\Pi}^0_\alpha$ sets, in such a way that the restriction of the function to each part is in Λ. Our main results in this chapter are

> **2.31** Theorem (p. 38). *The α-tree game characterizes the class of Baire class α functions.*
>
> **2.61** Theorem (p. 54). *Let $\alpha > 0$. If the Ψ-tree game characterizes a class Λ of functions, then the (α, Ψ)-tree game characterizes the class of functions which are piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ partition.*

In Chapter 3, we use tools from computable analysis to define a general framework of game characterizations of function classes. Concretely, this is done by introducing two parameters into (an appropriate modification of) the Wadge game, and we show how by a particular choice of parameters one can precisely control what class of functions is characterized by the resulting game. As an application, we recast the games characterizing the Baire classes from Chapter 2 into this framework, in the process defining several representations of certain relevant spaces of trees and operations over those represented spaces which might be of independent interest. Our main results in this chapter are

> **3.17** Theorem (p. 65). *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a transparent cylinder and let $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ be a probe. For any multi-valued function f between represented spaces, we have that player $\mathbf{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for f iff $f \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$ ($f \leqslant_{\mathfrak{W}} \Xi$).*
>
> **3.73** Corollary (p. 85). *Let $\alpha < \omega_1$. We have that* Prune^α *is a transparent cylinder which is Weihrauch-complete for the Baire class α functions. Therefore the* $(\mathrm{Prune}^\alpha, \mathrm{Label})$*-Wadge game characterizes the Baire class α functions.*

In Chapter 4, we work in generalized Baire spaces κ^κ for infinite cardinals κ satisfying $\kappa^{<\kappa} = \kappa$, and show that our game characterizations of each fixed Baire class α generalize to that setting. Furthermore, we show how the notion of computability on Baire space, which is the foundation stone of classical computable analysis, can be generalized to the setting of generalized Baire spaces. We show that this is indeed appropriate for generalized computable analysis by defining a representation of Galeotti's generalized real line [42] and analyzing the Weihrauch degree of the intermediate value theorem for that space. Our main results in this chapter are

4.13 Corollary (p. 94). *The (α, κ)-tree game characterizes the κ-Baire class α functions on κ^κ.*

4.34 Theorem (p. 105).

(a) *If there exists an effective enumeration of a dense subset of* $\mathbb{R}_\kappa$, *then* $\mathrm{IVT}_\kappa \leqslant_{\kappa\mathfrak{W}} \mathrm{B}^\kappa_\mathrm{I}$.

(b) *We have* $\mathrm{B}^\kappa_\mathrm{I} \leqslant_{\kappa\mathfrak{W}} \mathrm{IVT}_\kappa$.

(c) *We have* $\mathrm{IVT}_\kappa \leqslant^\mathrm{t}_{\kappa\mathfrak{W}} \mathrm{B}^\kappa_\mathrm{I}$, *and therefore* $\mathrm{IVT}_\kappa \equiv^\mathrm{t}_{\kappa\mathfrak{W}} \mathrm{B}^\kappa_\mathrm{I}$.

Finally, in Chapter 5, we show how the game characterizations of function classes discussed in previous chapters naturally lead to a stratification of each class into a hierarchy, intuitively measuring the complexity of functions in that class. We concretely study some properties of the ranks thus defined for the Wadge, backtrack, and eraser games, showing in each case that they are proper non-collapsing hierarchies of length ω_1, and that the backtrack rank matches a rank on $\mathbf{\Delta}^0_2$-functions introduced by Motto Ros [67, § 5.2], and also matches the well-known Hausdorff-Kuratowski rank in the case of characteristic functions of $\mathbf{\Delta}^0_2$ sets. This idea and the results presented open new paths for further research. Our main contribution in this chapter is

5.1 Definition (p. 110). Let $G_\star$ be a game characterizing a class $\mathscr{C}$ of functions, and let $\mathcal{B}_\star$ be a function associating to each $f \in \mathscr{C}$ and each winning strategy ϑ for $\mathbb{2}$ in $G_\star$ for f a wellfounded structure $\mathcal{B}_\star(\vartheta, f)$. Given such f and ϑ, the $\mathcal{B}_\star$ *rank* of ϑ for f, denoted $\mathrm{rk}_\star(\vartheta, f)$, is the wellfounded rank of $\mathcal{B}_\star(\vartheta, f)$, and the $\mathcal{B}_\star$ *rank* of f is the minimum $\mathcal{B}_\star$ rank of its winning strategies, i.e.,

$$\mathrm{rk}_\star(f) := \min\{\mathrm{rk}_\star(\vartheta, f)\,;\, \vartheta \text{ is a winning strategy for } \mathbb{2} \text{ in } G_\star \text{ for } f\}.$$

Remarks on co-authorship

Many of the results presented in Chapters 3–5 of this thesis were obtained in collaboration with other researchers. Because of this, and in the interest of full transparency, we have strived to meticulously attribute definitions and theorems in the thesis to their original authors, denoting our own contributions by "N." and attributing definitions and theorems to "folklore" in case the original reference is unknown (at least to us). In many cases the attribution to folklore is due to the fact that the notion being defined or the result being proven is simple enough so as not to warrant attribution, but we have chosen to err on the side of caution and attribute them to folklore to avoid giving any impression of claiming ownership of material we did not develop ourselves. In the beginning of each chapter of the thesis we make more detailed remarks on the co-authorship of the material presented in that chapter.

1.2 Preliminaries and notation

1.2.1 Functions

1.1 Definition (Folklore). A *multi-valued function* between given sets X and Y is a binary relation $f \subseteq X \times Y$ whose domain $\mathrm{dom}(f) := \{x \in X\,;\, \exists y \in Y((x,y) \in f)\}$ is X. We write $f : X \rightrightarrows Y$ to denote that f is a multi-valued function between X and Y, and if $f \subseteq X \times Y$ has $\mathrm{dom}(f) \subseteq X$ then we say f is a *multi-valued partial function* between X and Y and write $f : X \rightrightarrows Y$. Given such an $f : X \rightrightarrows Y$ and $x \in X$, we write $f(x) := \{y \in Y\,;\, (x,y) \in f\}$, and if $f(x) = \{y\}$ for some $y \in Y$ then we write $f(x) = y$, as usual.

When we want to stress that a given object f is a function in the usual sense, as opposed to a multi-valued function, then we will say f is a *single-valued* function. We denote the fact that f is a partial single-valued function between X and Y by $f : X \dashrightarrow Y$. In general, it may happen that we omit the word "partial" or that we call a "function" an object which is only a multi-valued function, but we will always be precise with the use of the notation $\to$, $\dashrightarrow$, $\rightrightarrows$, and $\rightrightarrows$.

1.2 Convention. Throughout this thesis, whenever we talk about a multi-valued partial function $f : X \rightrightarrows Y$ without specifying its domain, it is to be considered with its so-called *natural domain* $\mathrm{dom}(f) := \{x \in X\,;\, f(x) \neq \varnothing\}$.

We use the nomenclature *multi-valued (partial) function* instead of *binary relation* to stress that we view these objects from an input-output perspective, considering any $y \in f(x)$ to be an equally valid *output* of the multi-valued function f for the *input* x. This paradigm plays a role in the definitions of many concepts involving multi-valued functions. For example, the composition of multi-valued (partial) functions is not what is usually defined as the composition of binary relations:

1.3 Definition (Folklore). Given multi-valued partial functions $f : X \rightrightarrows Y$ and $g : Y \rightrightarrows Z$, we define their *composition* $g \circ f : X \rightrightarrows Z$ by letting $\mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f)\,;\, f(x) \subseteq \mathrm{dom}(g)\}$ and $g \circ f(x) = \bigcup\{g(y)\,;\, \exists x \in \mathrm{dom}(f)(y \in f(x))\}$.

Thus, the difference with the usual relational composition is that for that notion of composition we would have $\mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f)\,;\, f(x) \cap \mathrm{dom}(g) \neq \varnothing\}$.

According to the same paradigm, the following is the appropriate generalization for multi-valued functions of the notion of (single-valued) function extension.

1.4 Definition (Weihrauch [102, Definition 7]). A multi-valued function f *tightens* g or is a *tightening* of g, denoted by $f \preceq g$, if for every $x \in \mathrm{dom}(g)$ we have that both $x \in \mathrm{dom}(f)$ and $f(x) \subseteq g(x)$ hold.

1.2.2 Trees, wellfounded structures, and their ranks

1.5 Definition (Folklore). A *sequence* is a function whose domain is an ordinal number, and the *length* of a sequence s, denoted by $|s|$, is its domain. Note that this is a slight abuse of notation, since $|X|$ usually denotes the *cardinality* of the set X, and if $\mathrm{dom}(s)$ is not a cardinal then of course it is not the cardinality of the sequence s as a set. However, in this thesis we will not be interested in cardinalities of sequences, so the notation $|s|$ will never be ambiguous. We use angle brackets when extensionally or intensionally denoting the elements in a sequence; thus, e.g., $\langle\rangle$ is the empty sequence and $\langle 2n\,;\, n \in \omega\rangle$ is the sequence of all even natural numbers. However, so as to avoid being cumbersome with notation, when a sequence appears as part of an expression and is itself wrapped in round or square brackets, then we will often omit the angle brackets, e.g., writing $f(x_0, \dots, x_{n-1})$ instead of $f(\langle x_0, \dots, x_{n-1}\rangle)$ to denote the application of a function f to a sequence $\langle x_0, \dots, x_{n-1}\rangle$. Given a sequence s and an ordinal α, we denote by $s\restriction\alpha$ the *restriction* of s to domain $\min\{|s|, \alpha\}$. We say a sequence s is an *initial segment* or *prefix* of a sequence t, denoted $s \subseteq t$, if $|s| \leqslant |t|$ and $s = t\restriction|s|$, and we say s is a *proper* initial segment or prefix of t, denoted $s \subset t$, if $s \subseteq t$ and $s \neq t$. Two sequences are called *compatible* if one is a prefix of the other, and *incompatible* otherwise. If a sequence s is such that $|s|$ is a successor ordinal, then we denote by $\perp(s)$ the last element of s, i.e., the element $s(|s| - 1)$, and by s^- the sequence $s\restriction(|s| - 1)$.

Let κ and μ be cardinal numbers. Following another common abuse of notation, we use μ^κ, $\mu^{<\kappa}$, and $\mu^{\leqslant\kappa}$ to respectively denote not only the cardinalities of the sets of sequences with domain κ and range in μ, with domain some ordinal strictly less than κ and range in μ, and with domain some ordinal at most κ and range in μ, but also the respective sets of sequences themselves. It will always be clear from the context whether the symbol used is meant to denote the set of sequences or the cardinality thereof. We will usually denote elements of $\mu^{<\kappa}$ by σ, τ, ξ, elements of μ^κ by x, y, z, and elements of $\mu^{\leqslant\kappa}$ by s, t, in each case possibly using sub- and/or superscripts. We denote by $\mu^{<\kappa}_{\neq 0}$ and $\mu^{<\kappa}_{\mathrm{succ}}$ the subsets of $\mu^{<\kappa}$ formed of sequences of nonzero and of successor lengths, respectively. Given ordinals $\alpha < \mu$ and $\beta \leqslant \kappa$, by yet another abuse of notation and only when it is clear from the context that we are talking about elements of one of μ^κ, $\mu^{<\kappa}$, or $\mu^{\leqslant\kappa}$, we denote by α^β the sequence with length β and constant value α, i.e., the sequence we would usually denote by $\{\alpha\}^\beta$. Finally, given $s \in \mu^{<\kappa}$ and $t \in \mu^{\leqslant\kappa}$, the *concatenation* of s with

t, denoted by $s^\frown t$, is the element of $\mu^{\leqslant\kappa}$ with length $|s|+|t|$ satisfying

$$(s^\frown t)(\alpha) = \begin{cases} s(\alpha), & \text{if } \alpha < |s| \\ t(\beta), & \text{if } \alpha \geqslant |s| \text{ and } |s|+\beta = \alpha. \end{cases}$$

We will also frequently denote the concatenation of s and t by the juxtaposition st. If $s \in \mu^{\leqslant\kappa}$ has $|s| > 0$, we call the *left shift* of s, denoted by $\mathrm{shift}(s)$, the unique $t \in \mu^{\leqslant\kappa}$ such that $s = \langle s(0)\rangle^\frown t$ holds.

1.6 Definition (Folklore). Given cardinals κ and μ, a *κ-tree on μ* is a subset of $\mu^{<\kappa}$ closed under taking initial segments. Now let T be a κ-tree on μ. If T is not empty, then $\langle\rangle \in T$ is called its *root*. Given an ordinal $\alpha \leqslant \kappa$, the α^{th} *level* of T, denoted by $\mathrm{Level}(T,\alpha)$, is the set of nodes of T of length α, and the *height* of T, denoted by $\mathrm{ht}(T)$, is the least ordinal α for which $\mathrm{Level}(T,\alpha) = \varnothing$ holds. A *κ-branch* of T is an element $x \in \mu^\kappa$ satisfying $x\restriction\alpha \in T$ for every $\alpha < \kappa$, and the *body* of T, denoted by $[T]$, is the set of its κ-branches. Given a cardinal $\alpha \leqslant \kappa$, we say T is *$<\alpha$-branching* if for every $\sigma \in T$ the set $\{\sigma^\frown\langle\beta\rangle \in T\,;\, \beta < \mu\}$ of immediate children of σ in T has cardinality strictly less than α. We say T is *finitely branching* if it is $<\omega$-branching, and *linear* if it is <2-branching, i.e., if every node $\sigma \in T$ has at most one immediate child in T. We will use the same nomenclature to describe the branching at a given node of T; thus, e.g., we say T is *finitely branching at* σ if $\sigma \in T$ has only finitely many immediate children in T. We use the expression *outside of* σ to collectively talk about the branching properties of nodes of T which are not σ. Figure 1.1 shows some examples of branching properties of trees. Given $\sigma \in \mu^{<\kappa}$, we denote

(1) by $\mathrm{Com}(T,\sigma)$ the κ-tree $\{\tau \in T\,;\, \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}$ on μ composed of nodes of T which are compatible with σ;

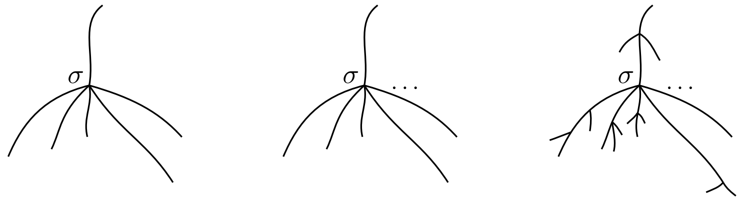
(2) by $\mathrm{Ext}(T,\sigma)$ the set $\{\tau \in T\,;\, \sigma \subseteq \tau\}$ of extensions of σ in T;

(3) by $\mathrm{Conc}(T,\sigma)$ the κ-tree $\{\tau \in \mu^{<\kappa}\,;\, \sigma^\frown\tau \in T\}$ on μ composed of those elements of $\mu^{<\kappa}$ whose pre-concatenations with σ result in nodes of T.

If $\kappa = \mu$ then we omit "on μ" and just call T a *κ-tree*,† and if $\kappa = \omega$ then we omit the prefix "κ-" and just call T a *tree on* μ, or just a *tree* if $\mu = \omega$ as well. An ω-branch of a tree is called an *infinite branch* of T.

1.7 Definition (Folklore). Let X be a set and $\prec$ be an irreflexive binary relation on X. We say $(X,\prec)$ is *wellfounded*, or simply that $\prec$ is wellfounded if there is no risk of ambiguity, if for any nonempty $Y \subseteq X$ there exists $y \in Y$ such that for all $x \in Y$ we have $x \not\prec y$. Such an element y is called a $\prec$-minimal element of Y. We say $(X,\prec)$ is *illfounded* if it is not wellfounded.

†Note that this nomenclature is often used with a different meaning in the literature. For example, for Kanamori [53, p. 75] and Kunen [61, Definition 5.5], a κ-tree is a tree of height κ whose levels have cardinality strictly less than κ.



(a) Finitely branching at σ and linear elsewhere (b) Linear outside of σ (c) Finitely branching outside of σ

Figure 1.1: Examples of branching properties of trees.

The axiom of choice implies that the illfoundedness of $(X, \prec)$ is equivalent to the existence of an infinite descending chain in X, i.e., an infinite sequence $\langle x_n \in X\,;\, n \in \omega\rangle$ such that $x_{n+1} \prec x_n$ holds for all n.

1.8 Convention. Given ordinal numbers $\alpha, \beta < \kappa$, let $(\alpha_0, \beta_0) \prec (\alpha_1, \beta_1)$ iff $(\max(\alpha_0, \beta_0), \alpha_0, \beta_0)$ is lexicographically-less than $(\max(\alpha_1, \beta_1), \alpha_1, \beta_1)$. It is a standard fact of set theory that, for any infinite cardinal κ, the relation $\prec$ restricted to κ^2 is a wellordering of κ^2 of order type κ. The *Gödel pairing function* $\ulcorner \textvisiblespace \urcorner$ is given by $\ulcorner \alpha, \beta \urcorner = \gamma$ iff (α, β) is the γ^{th} element in $\prec$, i.e., if the order type of $\{(\alpha', \beta') \in \kappa^2\,;\, (\alpha', \beta') \prec (\alpha, \beta)\}$ is exactly γ; thus $\ulcorner \textvisiblespace \urcorner \restriction \kappa^2$ is a bijection between κ^2 and κ. Cf. Figure 1.2 for a depiction of some values of $\ulcorner \textvisiblespace \urcorner$. It is easy to see that the tupling function $\ulcorner \textvisiblespace \urcorner$ restricted to ω, and its inverse, are computable; we will prove in Lemma 4.25 that the same holds true for uncountable κ under a suitable notion of transfinite computability. Restricting ourselves to the $\kappa = \omega$ case, we can define tupling functions of all finite arities by iterated applications of pairing, associating, say, to the right. We overload notation and denote all such tupling functions by $\ulcorner \textvisiblespace \urcorner$; thus $\ulcorner k_0, \ldots, k_n \urcorner := \ulcorner k_0, \ulcorner k_1, \ldots, \ulcorner k_{n-1}, k_n \urcorner \ldots \urcorner\urcorner$. For $n > 0$, we denote the inverse of the n-ary tupling function thus defined by bij_n; therefore for any natural $n > 0$, $k \in \omega$, and $\sigma \in \omega^n$, we have

$$\mathsf{bij}_n(k) = \sigma \quad \text{iff} \quad |\sigma| = n \text{ and } k = \ulcorner \sigma(0), \sigma(1), \ldots, \sigma(n-1) \urcorner.$$

From this we can define a bijection $\mathsf{bij} : \omega \to \omega^{<\omega}$ by letting $\mathsf{bij}(n) = \sigma$ iff $n = 0$ and $\sigma = \langle\rangle$, or $n > 0$, $\sigma \neq \langle\rangle$, $n - 1 = \ulcorner \ell, k \urcorner$, and $\mathsf{bij}_{\ell+1}(k) = \sigma$. It follows that bij is computable, has a computable inverse, and satisfies that $\mathsf{bij}(n) \subset \mathsf{bij}(m)$ implies $n < m$. As usual, the tupling functions on ω come associated with natural projections $(\textvisiblespace)^n_k$ for each $k < n \in \omega$, defined by

$$\begin{aligned}(m)^n_k := \text{ the unique } \ell \in \omega \text{ for which } m = \ulcorner m_0, \ldots, m_{k-1}, \ell, m_{k+1}, \ldots, m_{n-1} \urcorner \\ \text{holds for some } m_0, \ldots, m_{k-1}, m_{k+1}, \ldots, m_{n-1} \in \omega.\end{aligned}$$

Tupling functions and their associated projections allow us to make definitions *implicitly*—given $n, m \in \omega$ we can *define* numbers $k_0, \ldots, k_m \in \omega$ by the equation

$$n = \ulcorner k_0, \ldots, k_m \urcorner,$$

which is to be understood as merely shorthand for the definition $k_i := (n)_i^{m+1}$ for each $i \leqslant m$.

	0	1	2	3	$\cdots$	ω	$\cdots$	β	$\cdots$
0	0	1	4	9	$\cdots$	ω			
1	2	3	5	10	$\cdots$	$\omega+1$			
2	6	7	8	11	$\cdots$	$\omega+2$			
3	12	13	14	15	$\cdots$	$\omega+3$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$			
ω	$\omega\cdot 2$	$\omega\cdot 2+1$	$\omega\cdot 2+2$	$\omega\cdot 2+3$	$\cdots$	$\omega\cdot 3$			
$\vdots$									
α								$\ulcorner\alpha,\beta\urcorner$	
$\vdots$									

Figure 1.2: Ordinal pairing.

A fundamental feature of wellfounded structures is that they allow one to define functions with domain X by *wellfounded recursion*, i.e., by first specifying the value of the function on the $\prec$-minimal elements of X, and then recursively defining the value of the function on a given element $x \in X$ assuming that the function is already defined on every element of the set of $\prec$-predecessors of x—i.e., the set $\{y \in X\,;\, y \prec x\}$—and possibly using those values in the definition of $f(x)$. For our purposes the main example of such a function is the following.

1.9 Definition (Folklore). Let $\mathcal{S} = (X, \prec)$ be a wellfounded structure, with X a set. We define a function $\mathrm{rk}_{\mathcal{S}} : X \to \mathrm{On}$, called the *rank function of* $\mathcal{S}$, by the recursion

$$\mathrm{rk}_{\mathcal{S}}(x) = \sup\{\mathrm{rk}_{\mathcal{S}}(y) + 1\,;\, y \prec x\}$$

(note that $\sup\varnothing = 0$). Therefore the rank of x in $\mathcal{S}$ is the least ordinal greater than the rank of every $\prec$-predecessor of x. We define the *rank of* $\mathcal{S}$, denoted $\mathrm{rk}(\mathcal{S})$, as the ordinal number $\sup\{\mathrm{rk}_{\mathcal{S}}(x)\,;\, x \in X\}$.

Note that, depending on the wellfounded structure $\mathcal{S} = (X, \prec)$, it may be the case that $\mathrm{rk}(\mathcal{S}) = \mathrm{rk}_{\mathcal{S}}(x)$ holds for some $x \in X$. We always consider trees as structures with the relation $\supset$, so that, e.g., a tree is wellfounded iff it contains no infinite $\supset$-decreasing (i.e., $\subset$-increasing) chains, and if the tree T is nonempty then its rank is equal to $\mathrm{rk}_T(\langle\rangle)$.

1.10 Theorem (Folklore). *For every wellfounded* $\mathcal{S} = (X, \prec)$ *we have* $\mathrm{rk}(\mathcal{S}) < |X|^+$*, and for every cardinal* κ *and every* $\alpha < \kappa^+$ *there exists a wellfounded tree* T *on* κ *such that* $\mathrm{rk}(T) = \alpha$.

Proof. Recall that successor cardinals are regular, so the supremum of at most $|X|$-many ordinals, each of which is less than $|X|^+$, is also less than $|X|^+$. Therefore the first claim is equivalent to the rank of each $x \in X$ being less than $|X|^+$, and this follows by an easy induction on the ranks of elements of X—the limit case again using the regularity of $|X|^+$.

The second claim can be established with a simple recursive construction. For $\alpha = 0$, just take $T := \{\langle\rangle\}$. If T is a tree on κ of rank $\alpha < \kappa^+$, then $T' := \{\langle\rangle\} \cup \{\langle 0\rangle^\frown t \,;\, t \in T\}$ is easily seen to be a tree on κ of rank $\alpha + 1$. Finally, suppose $\lambda < \kappa^+$ is a limit ordinal and that for each $\alpha < \lambda$ we have a wellfounded tree T_α on κ with rank α. Let $f : \kappa \to \lambda$ be cofinal in λ. Now $T := \{\langle\rangle\} \cup \{\langle\alpha\rangle^\frown t \,;\, \alpha < \lambda \text{ and } t \in T_{f(\alpha)}\}$ is also easily seen to be a tree on κ of rank $\sup\{f(\alpha) + 1 \,;\, \alpha < \kappa\} = \lambda$. $\square$

1.11 Theorem (Folklore). *Let $\mathcal{S} = (X, \prec)$ be wellfounded. Then* $\mathrm{rk}(\mathcal{S})$ *is the least ordinal α for which there exists a $\prec$-increasing function $f : X \to \alpha + 1$.*

Proof. We proceed by induction on $\mathrm{rk}(\mathcal{S})$, with the base case $\mathrm{rk}(\mathcal{S}) = 0$ being immediate. Now suppose $\mathrm{rk}(\mathcal{S}) > 0$, and for each $\gamma < \mathrm{rk}(\mathcal{S})$ let $\mathcal{S}_\gamma = (X_\gamma, \prec_\gamma)$ be the substructure of $\mathcal{S}$ composed of the elements of $\mathcal{S}$ of rank at most γ. In particular $\mathrm{rk}(\mathcal{S}_\gamma) = \gamma$. Since $f{\restriction}X_\gamma : X_\gamma \to \beta + 1$ is $\prec_\gamma$-increasing, it follows that $\beta \geqslant \gamma$. In particular, if $\mathrm{rk}(S)$ is a limit ordinal then we are done. Otherwise, if $\mathrm{rk}(S)$ is a successor, say $\mathrm{rk}(S) = \gamma + 1$, then by what we just saw we have $\beta \geqslant \gamma$. But in this case there exists $x \in X$ with rank $\gamma + 1$, and therefore there also exists $y \prec x$ with rank γ. Since f is $\prec$-increasing, it follows that $\beta \geqslant \gamma + 1 = \mathrm{rk}(S_\gamma)$. $\square$

1.12 Corollary. *If $(X, \prec_X)$ and $(Y, \prec_Y)$ are wellfounded structures and there exists $f : X \to Y$ such that $x \prec_X x'$ implies $f(x) \prec_Y f(x')$, then* $\mathrm{rk}(X, \prec_X) \leqslant \mathrm{rk}(Y, \prec_Y)$.

It will be convenient to also have rank functions for illfounded structures, as follows.

1.13 Definition (Folklore). Let X be a set, $\prec$ be an irreflexive binary relation on X, and $\mathcal{S} = (X, \prec)$. The *wellfounded part* of $\mathcal{S}$, denoted by $\mathrm{WF}(\mathcal{S})$, is the substructure of $\mathcal{S}$ composed of the elements of X which are not part of any infinite descending chains of $\mathcal{S}$. We now define a function $\mathrm{rk}_\mathcal{S} : X \to \mathrm{On} \cup \{\infty\}$, called the *rank function of* $\mathcal{S}$, by letting

$$\mathrm{rk}_\mathcal{S}(x) = \begin{cases} \mathrm{rk}_{\mathrm{WF}(\mathcal{S})}(x), & \text{if } x \in \mathrm{WF}(\mathcal{S}) \\ \infty, & \text{otherwise,} \end{cases}$$

where ∞ is not an ordinal number.

If we stipulate that ∞ is strictly greater than any ordinal number and that $\infty + 1 = \infty$ holds, then the rank function of $\mathcal{S} = (X, \prec)$ also satisfies

$$\mathrm{rk}_\mathcal{S}(x) = \sup\{\mathrm{rk}_\mathcal{S}(y) + 1 \,;\, y \prec x\}.$$

1.14 Definition (Folklore). A *labeled κ-tree* is a pair $\Upsilon = (T, \varphi)$ of a κ-tree $T \subseteq \kappa^{<\kappa}$ and a *labeling function* $\varphi : (T \cap \kappa^{<\kappa}_{\text{succ}}) \to \kappa$. Let $\Upsilon = (T, \varphi)$ be a labeled κ-tree. We define the function $\tilde{\varphi} : T \to \kappa^{<\kappa}$ by $|\tilde{\varphi}(\sigma)| = |\sigma|$ and $\tilde{\varphi}(\sigma)(\alpha) = \varphi(\sigma\restriction(\alpha+1))$ for all $\sigma \in T$ and $\alpha < |\sigma|$. The sequence $\tilde{\varphi}(\sigma)$ is called the *running label* of σ in Υ. Given a second labeled κ-tree $\Upsilon' = (T', \varphi')$, we say Υ is a *subtree* of Υ', denoted $\Upsilon \subseteq \Upsilon'$, if both $T \subseteq T'$ and $\varphi \subseteq \varphi'$ hold. Given a subtree S of T, the labeled κ-tree $(S, \varphi\restriction S)$ is called the *subtree of* Υ *induced by* S.

Again, if $\kappa = \omega$ we will omit the prefix "κ-" and thus talk about labeled trees, subtrees, etc.

1.15 Remark. We will in general overload notation from unlabeled to labeled κ-trees; whenever some such notation is used without prior introduction, the intended meaning will be intuitive. For example, for $\Upsilon = (T, \varphi)$ we will write $\sigma \in \Upsilon$ to mean $\sigma \in T$, or $\mathrm{rk}_\Upsilon(\sigma)$ instead of $\mathrm{rk}_T(\sigma)$, etc. Furthermore, whenever we define an operation F on κ-trees which assigns to a κ-tree T a subtree $F(T) \subseteq T$, we extend F to labeled κ-trees by letting $F(T, \varphi)$ be the subtree of (T, φ) induced by $F(T)$.

Some of our results will involve the following notion, which is well-known in certain areas of logic and computer science, where it models the notion that two systems have equivalent behavior. We only give the definition for labeled trees, i.e., labeled ω-trees.

1.16 Definition (Folklore). Given labeled trees $\Upsilon_0 = (T_0, \varphi_0)$ and $\Upsilon_1 = (T_1, \varphi_1)$, a nonempty relation $B \subseteq T_0 \times T_1$ is called a *bisimulation* between Υ_0 and Υ_1 in case $\sigma\ B\ \tau$ implies

$$\tilde{\varphi}_0(\sigma) = \tilde{\varphi}_1(\tau) \tag{label}$$
$$\forall \sigma' \in T_0(\sigma \subset \sigma' \;\Rightarrow\; \exists \tau' \in T_1(\tau \subset \tau' \wedge \sigma'\ B\ \tau')) \tag{forth}$$
$$\forall \tau' \in T_1(\tau \subset \tau' \;\Rightarrow\; \exists \sigma' \in T_0(\sigma \subset \sigma' \wedge \sigma'\ B\ \tau')) \tag{back}$$
$$|\sigma| > 0 \;\Rightarrow\; \sigma^-\ B\ \tau^- \tag{parent}$$

We say Υ_0 and Υ_1 are *bisimilar*, denoted $\Upsilon_0 \rightleftarrows \Upsilon_1$, if there exists a bisimulation between Υ_0 and Υ_1.

It is straightforward to check that $\rightleftarrows$ is an equivalence relation. It is also easy to see that the union of any family of bisimulations between given labeled trees is also a bisimulation between those trees. Therefore, between any pair of bisimilar labeled trees Υ_0 and Υ_1 there always exists a largest bisimulation, denoted $\rightleftarrows_{\Upsilon_0,\Upsilon_1}$ or simply $\rightleftarrows$ when there is no risk of ambiguity. A particular case of a bisimulation between $\Upsilon_0 = (T_0, \varphi_0)$ and $\Upsilon_1 = (T_1, \varphi_1)$ is an *isomorphism* between those trees, i.e., a bijection $\iota : T_0 \to T_1$ satisfying, for any $\sigma, \tau \in T_0$:

(1) $\sigma \subseteq \tau$ iff $\iota(\sigma) \subseteq \iota(\tau)$,

(2) $|\sigma| = |\iota(\sigma)|$, and

(3) $\varphi_0(\sigma) = \varphi_1(\iota(\sigma))$.

The trees Υ_0 and Υ_1 are *isomorphic*, denoted $\Upsilon_0 \simeq \Upsilon_1$, if there exists an isomorphism between them.

1.17 LEMMA (N.). *If $B \subseteq \Upsilon \times \Upsilon'$ is a bisimulation and $\sigma\ B\ \tau$ holds, then* $\mathrm{rk}_\Upsilon(\sigma) = \mathrm{rk}_{\Upsilon'}(\tau)$.

Proof. If $\mathrm{rk}_\Upsilon(\sigma) = \infty$, i.e., if σ is on an infinite branch of Υ, then it is easy to see that τ is on an infinite branch of Υ' and therefore $\mathrm{rk}_{\Upsilon'}(\tau) = \infty$ as well. By the same argument, we have that if $\mathrm{rk}_{\Upsilon'}(\tau) = \infty$ then $\mathrm{rk}_\Upsilon(\sigma) = \infty$. If $\sigma \in \mathrm{WF}(\Upsilon)$, then we proceed by induction on $\mathrm{rk}_\Upsilon(\sigma)$. For the base case, note that $\mathrm{rk}_\Upsilon(\sigma) = 0$ iff σ is a leaf of Υ, and in this case $\sigma\ B\ \tau$ implies that τ is also a leaf of Υ' and therefore also has rank 0. Now suppose the result holds for every node of rank $< \mathrm{rk}_\Upsilon(\sigma)$. For each $\beta < \mathrm{rk}_\Upsilon(\sigma)$ there exists some descendant σ' of σ in Υ such that $\mathrm{rk}_\Upsilon(\sigma') = \beta$. Since B is a bisimulation, there exists a descendant τ' of τ in Υ' such that $\sigma'\ B\ \tau'$. By induction hypothesis we have $\mathrm{rk}_{\Upsilon'}(\tau') = \beta$, and since $\beta < \mathrm{rk}_\Upsilon(\sigma)$ was arbitrary we have $\mathrm{rk}_{\Upsilon'}(\tau) \geqslant \mathrm{rk}_\Upsilon(\sigma)$. Analogously we can prove $\mathrm{rk}_{\Upsilon'}(\tau) \leqslant \mathrm{rk}_\Upsilon(\sigma)$, so the result follows. □

1.2.3 Baire space

Definitions and notation which we use without prior introduction are standard and can be found, e.g., in [54].

As usual, we endow ω^ω with the product topology, considering ω with the discrete topology, and call the resulting topological space *Baire space*. Its topology is generated by the basis of sets of the form $[\sigma] := \{x \in \omega^\omega\,;\ \sigma \subset x\}$ for $\sigma \in \omega^{<\omega}$.

Given an element $x \in \omega^\omega$ and $n \in \omega$, we define an element $(x)_n \in \omega^\omega$ by letting $(x)_n(k) := x(\ulcorner n, k \urcorner)$ for every $n, k \in \omega$. The map $x \mapsto \langle (x)_n\,;\ n \in \omega \rangle$ is easily seen to be a homeomorphism between ω^ω and the product space $(\omega^\omega)^\omega$.

The collection of Borel subsets of ω^ω is stratified into the *Borel hierarchy*, which is defined as follows.

$$\begin{aligned}
\boldsymbol{\Sigma}^0_0 &:= \{\varnothing, \omega^\omega\} \cup \{\omega^\omega \smallsetminus [\sigma]\,;\ \sigma \in \omega^{<\omega}\} \\
\boldsymbol{\Pi}^0_0 &:= \{\varnothing, \omega^\omega\} \cup \{[\sigma]\,;\ \sigma \in \omega^{<\omega}\} \\
\boldsymbol{\Delta}^0_0 &:= \{\varnothing, \omega^\omega\},
\end{aligned}$$

and recursively for, $\alpha > 0$,

$$\begin{aligned}
\boldsymbol{\Sigma}^0_{<\alpha} &:= \textstyle\bigcup_{\beta<\alpha} \boldsymbol{\Sigma}^0_\beta \\
\boldsymbol{\Pi}^0_{<\alpha} &:= \textstyle\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta \\
\boldsymbol{\Sigma}^0_\alpha &:= \{\textstyle\bigcup_{n\in\omega} A_n\,;\ \forall n \in \omega(A_n \in \boldsymbol{\Pi}^0_{<\alpha})\} \\
\boldsymbol{\Pi}^0_\alpha &:= \{\textstyle\bigcap_{n\in\omega} A_n\,;\ \forall n \in \omega(A_n \in \boldsymbol{\Sigma}^0_{<\alpha})\} \\
\boldsymbol{\Delta}^0_\alpha &:= \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha.
\end{aligned}$$

It is a classical result that for every $\alpha < \omega_1$ we have

(1) $\mathbf{\Delta}^0_\alpha \subset \mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subset \mathbf{\Delta}^0_{\alpha+1}$ and

(2) $\mathbf{\Sigma}^0_{\omega_1} = \mathbf{\Pi}^0_{\omega_1} = \mathbf{\Sigma}^0_{<\omega_1} = \mathbf{\Pi}^0_{<\omega_1}$,

i.e., the Borel hierarchy is a proper hierarchy of length ω_1 (cf., e.g., [65, Exercises 1F.5 & 1F.6]).

Given a subset $X \subseteq \omega^\omega$ we can define the Borel hierarchy *relativized to* X as usual, e.g., setting $\mathbf{\Sigma}^0_\alpha(X) := \{Y \cap X\,;\, Y \in \mathbf{\Sigma}^0_\alpha\}$. Due to the basic laws of how $\bigcup$, $\bigcap$, and $\omega^\omega \smallsetminus \_$ interact with one another, this definition behaves as expected. For example, $Y \in \mathbf{\Sigma}^0_\alpha(X)$ holds iff there exists $\langle Y_n \in \mathbf{\Pi}^0_{<\alpha}(X)\,;\, n < \omega\rangle$ such that $Y = \bigcup_{n<\omega} Y_n$. We will usually omit "$(X)$" from the notation, e.g., writing just $\mathbf{\Sigma}^0_\alpha$ instead of $\mathbf{\Sigma}^0_\alpha(X)$, whenever the context in which the notation appears makes it clear which hierarchy is meant.

Recall that for every ordinal α there exist a unique limit ordinal λ and a unique natural number n such that $\alpha = \lambda + n$. We call α *even* or *odd* according to the parity of n.

1.18 Definition (Folklore). Given an ordinal $\alpha < \omega_1$ and a sequence $\langle A_\beta \subseteq \omega^\omega\,;\, \beta \leqslant \alpha\rangle$ with $A_\alpha = \omega^\omega$, we define $\operatorname{diff}\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ as the set of those $x \in \omega^\omega$ for which the least $\beta \leqslant \alpha$ such that $x \in A_\beta$ holds has parity opposite to that of α.

Given $1 \leqslant \xi < \omega_1$, an increasing sequence $\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ of sets in $\mathbf{\Sigma}^0_\xi$ with $\alpha < \omega_1$ and $A_\alpha = \omega^\omega$ is called a $\mathbf{\Sigma}^0_\xi$*-resolution* for the set $\operatorname{diff}\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$.

Figure 1.3 presents a depiction of $\operatorname{diff}\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ where α is odd and $\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ is an increasing sequence.

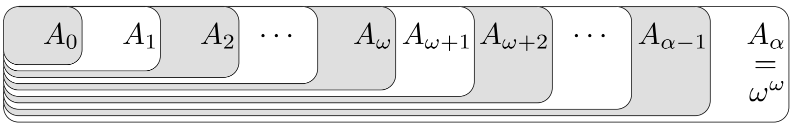


Figure 1.3: The set $\operatorname{diff}\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ (shaded region) for odd α and increasing $\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$.

1.19 Theorem (Hausdorff-Kuratowski, cf. [54, Theorem 22.27]). *For each $1 \leqslant \xi < \omega_1$, we have $A \in \mathbf{\Delta}^0_{\xi+1}$ iff there exists a $\mathbf{\Sigma}^0_\xi$-resolution for A.*

Proof of ($\Leftarrow$). If $\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ is a $\mathbf{\Sigma}^0_\xi$-resolution for A, then

$$A = \bigcup\{A_\beta \smallsetminus \bigcup_{\gamma<\beta} A_\gamma\,;\, \beta \leqslant \alpha \text{ has different parity from that of } \alpha\},$$

and since the difference of two $\mathbf{\Sigma}^0_\xi$ sets is a $\mathbf{\Delta}^0_{\xi+1}$ set, it follows that $A \in \mathbf{\Sigma}^0_{\xi+1}$. Likewise,

$$\omega^\omega \smallsetminus A = \bigcup\{A_\beta \smallsetminus \bigcup_{\gamma<\beta} A_\gamma\,;\, \beta \leqslant \alpha \text{ has the same parity as } \alpha\},$$

so $\omega^\omega \smallsetminus A \in \mathbf{\Sigma}^0_{\xi+1}$ as well. $\square$

We omit the general proof of the converse direction; the case $\xi = 1$ will be obtained as a corollary of Theorem 5.9.

For $A \in \mathbf{\Delta}^0_{\xi+1}$, the least $\alpha < \omega_1$ for which A has a $\mathbf{\Sigma}^0_\xi$-resolution of length $\alpha+1$ is called the *Hausdorff-Kuratowski rank* of A and denoted by $\mathrm{rk}_{\xi\mathrm{HK}}(A)$. It is also a classical result that for every $\alpha < \omega_1$ there exists a $\mathbf{\Delta}^0_{\xi+1}$ set of Hausdorff-Kuratowski rank α.

1.20 Definition (Folklore). Let $\langle s_n\,;\, n \in \omega\rangle$ be a sequence of elements of $\omega^{\leqslant\omega}$, and let $s \in \omega^{\leqslant\omega}$. We say $\langle s_n\,;\, n \in \omega\rangle$

(1) *converges pointwise* to s, denoted $s = \lim_{n\in\omega} s_n$, if for every $n < |s|$ there exists $M \in \omega$ such that for every $m \geqslant M$ we have $n < |s_m|$ and $s_m(n) = s(n)$, and for every $n \in \omega$ for which there exists $M \in \omega$ such that for every $m \geqslant M$ we have $n < |s_m|, |s_M|$ and $s_m(n) = s_M(n)$, we have $n < |s|$ and $s(n) = s_M(n)$.

(2) *converges monotonically* to s if $s_n \in \omega^{<\omega}$ and $s_n \subseteq s_{n+1}$ hold for every $n \in \omega$, and $s = \bigcup_{n\in\omega} s_n$.

We say s is the *pointwise* or *monotone limit* of $\langle s_n\,;\, n \in \omega\rangle$ respectively in each case. We say a function $f : \omega^\omega \dashrightarrow \omega^\omega$ is the *pointwise limit* of a sequence of functions $\langle f_n : \omega^\omega \dashrightarrow \omega^\omega\,;\, n \in \omega\rangle$, denoted $f = \lim_{n\in\omega} f_n$, if $\mathrm{dom}(f) = \mathrm{dom}(f_n)$ holds for every $n \in \omega$ and $f(x) = \lim_{n\in\omega} f_n(x)$ holds for every $x \in \mathrm{dom}(f)$.

1.21 Definition (Folklore). Let $f : \omega^\omega \dashrightarrow \omega^\omega$.

(1) We say f is of *Baire class 0* if it is continuous, and recursively, for $\alpha > 0$ we say f is of *Baire class* α if it is the pointwise limit of a countable sequence of functions, each of which is of Baire class less than α.

(2) Given a class Γ of subsets of ω^ω, we say f is Γ-measurable if the preimage of any open set under f is relatively in Γ, i.e., if for any open set $U \subseteq \omega^\omega$ there exists $V \in \Gamma$ such that $f^{-1}[U] = V \cap \mathrm{dom}(f)$.

(3) We say f is a $\mathbf{\Delta}^0_\alpha$-function if the preimage of any $\mathbf{\Delta}^0_\alpha$ set under f is also a (relative) $\mathbf{\Delta}^0_\alpha$ set.

It is easy to see that a function $f : \omega^\omega \dashrightarrow \omega^\omega$ is a $\mathbf{\Delta}^0_\alpha$-function iff the preimage of any $\mathbf{\Sigma}^0_\alpha$ set under f is also a $\mathbf{\Sigma}^0_\alpha$ set iff the preimage of any $\mathbf{\Pi}^0_\alpha$ set under f is also a $\mathbf{\Pi}^0_\alpha$ set.

1.22 Theorem (Lebesgue, Hausdorff, & Banach; cf. [54, Theorem 24.3]). *A function $f : \omega^\omega \dashrightarrow \omega^\omega$ is Baire class α iff it is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable.*

Proof of ($\Rightarrow$). By induction on α, with the $\alpha = 0$ case following by definition. For $\alpha > 0$, suppose f is the pointwise limit of $\langle f_n\,;\, n \in \omega\rangle$ where each f_n is of Baire class less than α. Since $\mathbf{\Sigma}^0_{\alpha+1}$ is closed under countable unions, it suffices to show that for each $\sigma \in \omega^{<\omega}$ the preimage of $[\sigma]$ under f is in $\mathbf{\Sigma}^0_{\alpha+1}$. The fact that $f = \lim_{n\in\omega} f_n$ implies $f^{-1}[\sigma] = \bigcup_{n\in\omega}\bigcap_{n\leqslant m<\omega} f_m^{-1}[\sigma]$. Since $[\sigma]$ is a closed set and by induction each f_m is $\mathbf{\Sigma}^0_\alpha$-measurable, it follows that $f_m^{-1}[\sigma] \in \mathbf{\Pi}^0_\alpha$. Therefore f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable. □

The proof of the converse direction is more involved, and will be given as a corollary of our game characterizations of the Baire classes, Theorem 4.12.

1.23 Corollary (Folklore). *A function is Borel measurable iff it is of Baire class α for some $\alpha < \omega_1$.*

Proof. The right-to-left direction is immediate, and for the converse, if $f : \omega^\omega \dashrightarrow \omega^\omega$ is Borel measurable then for each $\sigma \in \omega^{<\omega}$ there exists $\alpha_\sigma < \omega_1$ such that $f^{-1}[\sigma] \in \mathbf{\Sigma}^0_{\alpha_\sigma}$. Thus, for $\alpha := \sup_{\sigma\in\omega^{<\omega}} \alpha_\sigma$, we have that f is $\mathbf{\Sigma}^0_\alpha$-measurable. Since ω_1 is a regular cardinal, it follows that $\alpha < \omega_1$. □

1.24 Definition (Folklore). Given a class Λ of partial functions on ω^ω and a class Γ of subsets of ω^ω, we say a function $f : \omega^\omega \dashrightarrow \omega^\omega$ is *piecewise* Λ *on a* Γ *partition* if there exists a sequence $\langle A_n\,;\, n \in \omega\rangle$ of subsets of ω^ω such that

(1) for each $n \in \omega$ we have $A_n \in \Gamma$;

(2) the sequence $\langle A_n \cap \mathrm{dom}(f)\,;\, n \in \omega\rangle$ is a partition of $\mathrm{dom}(f)$;

(3) for each $n \in \omega$ we have $f{\upharpoonright}A_n \in \Lambda$.

The class of functions which are piecewise Λ on a Γ cover is defined in the obvious analogous way, substituting "cover" in the place of "partition" everywhere in Definition 1.24.

1.25 Lemma (Folklore). *Let $\alpha > 0$ and $A \subseteq \omega^\omega$. Any set in $\mathbf{\Sigma}^0_\alpha(A)$ can be partitioned into countably many $\mathbf{\Pi}^0_{<\alpha}(A)$ sets.*

Proof. By induction on α. For the base case $\alpha = 1$, let $X \subseteq \omega^{<\omega}$ be such that $B = \bigcup_{\sigma\in X}([\sigma] \cap A)$, and let Y be the set of $\subseteq$-minimal elements of X. Now we have that the sets $\{[\sigma] \cap A\,;\, \sigma \in Y\}$ form a $\mathbf{\Pi}^0_0(A)$ partition of B.

Suppose the result holds for each $\beta < \alpha$, and let $B \in \mathbf{\Sigma}^0_\alpha(A)$. Then $B = \bigcup_k B_k$ where each B_k is in $\mathbf{\Pi}^0_{<\alpha}(A)$. Let $C_0 := B_0$ and $C_{k+1} = B_{k+1} \cap (\omega^\omega \smallsetminus \bigcup_{m\leqslant k} C_m)$. Note that these sets partition B, and that for each k there exists $\beta < \alpha$ such that $D_k = (\omega^\omega \smallsetminus \bigcup_{m\leqslant k} C_m) \in \mathbf{\Sigma}^0_\beta(A)$. Hence, by the induction hypothesis each D_k can be partitioned into countably many $\mathbf{\Pi}^0_{<\beta}(A)$ sets, and intersecting each part of this with B_{k+1} we get a $\mathbf{\Pi}^0_{<\beta}(A)$ partition of each C_{k+1}. Putting these together we get a $\mathbf{\Pi}^0_{<\alpha}(A)$ partition of B. □

1.26 Corollary (Folklore). *For any $A \subseteq \omega^\omega$ and $\alpha > 0$, any $\mathbf{\Pi}^0_\alpha(A)$ cover of A can be refined to a $\mathbf{\Pi}^0_\alpha$ partition of A. In other words, for any sequence $\langle A_n \,;\, n \in \omega\rangle$ of $\mathbf{\Pi}^0_\alpha$ sets such that $A = \bigcup_{n\in\omega} A_n$, there exists a partition $\langle A^k_n \,;\, n, k \in \omega\rangle$ of A into $\mathbf{\Pi}^0_\alpha(A)$ sets such that $A^k_n \subseteq A_n$ for each $n, k \in \omega$.*

Proof. Let $B_0 := A_0$, $B_{n+1} := A_{n+1} \smallsetminus \bigcup_{m\leqslant n} B_m$, so that $\langle B_n \,;\, n \in \omega\rangle$ is a partition of A into $\mathbf{\Sigma}^0_{\alpha+1}(A)$ sets such that $B_n \subseteq A_n$ for each $n \in \omega$. Now, by the previous result each B_n can be partitioned into countably many $\mathbf{\Pi}^0_\alpha(A)$ sets $\langle A^k_n \,;\, k \in \omega\rangle$, so we are done. □

Figure 1.4 illustrates the proof of Corollary 1.26.

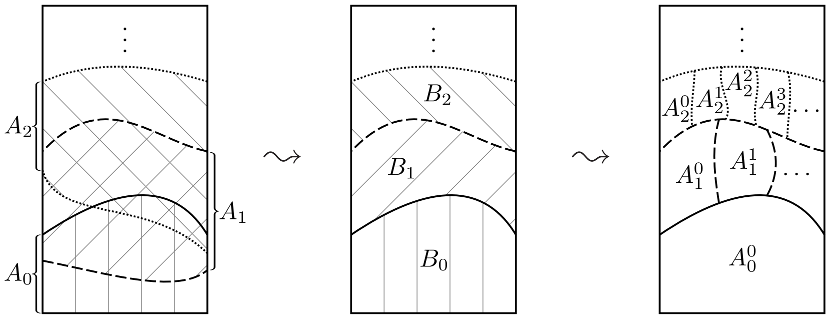


Figure 1.4: Refining a $\mathbf{\Pi}^0_\alpha(A)$ cover of a set A first into a $\mathbf{\Sigma}^0_{\alpha+1}(A)$ partition of A, then refining that into a $\mathbf{\Pi}^0_\alpha(A)$ partition of A, as in the proof of Corollary 1.26.

1.27 Corollary (Folklore). *Let Λ be a class of partial functions on ω^ω closed under taking restrictions, i.e., such that if $f \in \Lambda$ then $f{\restriction}A \in \Lambda$ for any $A \subseteq$ $\mathrm{dom}(f)$. Then, for any $\alpha > 0$ and any $f : \omega^\omega \dashrightarrow \omega^\omega$, the following are equivalent:*

(1) *f is piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ partition;*

(2) *f is piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ cover;*

(3) *f is piecewise Λ on a $\mathbf{\Sigma}^0_{\alpha+1}$ partition.*

Proof. (1) $\Rightarrow$ (2) is immediate, we have (2) $\Rightarrow$ (3) because any $\mathbf{\Pi}^0_\alpha(\mathrm{dom}(f))$ cover of $\mathrm{dom}(f)$ can be refined to a $\mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$ partition of $\mathrm{dom}(f)$, and likewise we have (3) $\Rightarrow$ (1) because any $\mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$ partition of $\mathrm{dom}(f)$ can be refined to a $\mathbf{\Pi}^0_\alpha(\mathrm{dom}(f))$ partition of $\mathrm{dom}(f)$. □

1.28 Theorem (Jayne & Rogers [51]). *A function $f : \omega^\omega \dashrightarrow \omega^\omega$ is piecewise continuous on a $\mathbf{\Pi}^0_1$ partition iff it is a $\mathbf{\Delta}^0_2$-function.*

The Jayne-Rogers theorem as proved in [51] applies to more general topological spaces, but for our interests in this thesis the case of ω^ω suffices. The Jayne-Rogers theorem and its generalizations are an important and active topic of research in descriptive set theory and generalized computability theory [23, 25, 52, 57, 70, 71, 81, 84, 92, 93]. However, since it is not the focus of this thesis, we will not give its (quite involved) proof. We should like to mention that Semmes has given a proof of that theorem based on an in-depth analysis of the eraser and backtrack games [92, § 3.4], and that Kačena, Motto Ros, and Semmes have also given another proof of the Jayne-Rogers theorem which is simpler than the original [52, 71].

Semmes also proved the following analogues of the Jayne-Rogers for functions preserving higher levels of the Borel hierarchy.

1.29 Theorem (Semmes's decomposition theorem [92, Theorems 4.3.7 and 5.2.8]). *A function $f : \omega^\omega \dashrightarrow \omega^\omega$ is*

(1) *piecewise continuous on a $\mathbf{\Pi}^0_2$ partition iff it is a $\mathbf{\Delta}^0_3$-function;*

(2) *piecewise Baire class 1 on a $\mathbf{\Pi}^0_2$ partition iff the preimage of any $\mathbf{\Delta}^0_2$ set under f is a $\mathbf{\Delta}^0_3$ set.*

1.2.4 Represented spaces and Weihrauch reducibility

Represented spaces and continuous or computable maps between them form the setting for computable analysis. The classical reference for computable analysis is the textbook by Weihrauch [101]; for a comprehensive introduction more in line with the style of this thesis we refer to [79].

A *represented* space $\mathbb{X} = (X, \delta_\mathbb{X})$ is given by a set X and a partial surjection $\delta_\mathbb{X} : \omega^\omega \dashrightarrow X$. We will always consider ω^ω as represented by $\mathrm{id}_{\omega^\omega}$, and ω as represented by the function $\delta_\omega(p) = n$ iff $p = 0^n 1^\omega$. Given a represented space $\mathbb{X}$ and $n \in \omega$, $\mathbb{X}^n$ is the represented space of n-tuples represented in the natural way since ω^ω inherits particularly nice tupling functions $\ulcorner \textvisiblespace \urcorner$ of all finite arities from ω. The *coproduct* of a family of represented spaces $\{\mathbb{Y}_x \,;\, x \in \mathbb{X}\}$ indexed by $\mathbb{X}$ is the represented space denoted by $\coprod_{x\in\mathbb{X}} \mathbb{Y}_x$ which is composed of pairs (x, y) such that $y \in \mathbb{Y}_x$, with the representation given in the natural way, letting a name for (x, y) be a ω^ω-pair of a name for x and one for y. We denote by $\mathbb{X}^{<\omega}$ the represented space $\coprod_{n\in\omega} \mathbb{X}^n$; thus $\omega^{<\omega}$ can be seen as the represented space in which σ is named by p iff p encodes the length $|\sigma|$ of σ as well as the $|\sigma|$ elements of σ Finally, $\mathbb{X}^\omega$ is the represented space in which tuples $\langle x_n \rangle_{n\in\omega}$ are named by infinite tuples composed of a name for each x_n—recall that ω^ω has a countable tupling function $\ulcorner \textvisiblespace \urcorner : (\omega^\omega)^\omega \to \omega^\omega$ given by $\ulcorner p_n \urcorner_{n\in\omega} = p$ iff $p(\ulcorner n, k \urcorner) = p_n(k)$ for each $n, k \in \omega$.

A (multi-valued) function between represented spaces is just a (multi-valued) function between the underlying sets. We say that a partial function $F : \omega^\omega \dashrightarrow \omega^\omega$ is a *realizer* for a multi-valued function $f : \mathbb{X} \rightrightarrows \mathbb{Y}$, denoted by $F \vdash f$, if

$\delta_{\mathbb{Y}}(F(p)) \in f(\delta_{\mathbb{X}}(p))$ holds for all $p \in \operatorname{dom}(f\delta_{\mathbb{X}})$; note that realizers are always required to be single-valued. Then, given a class Λ of partial functions in ω^ω, we say $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ is in $(\delta_{\mathbb{X}}, \delta_{\mathbb{Y}})$-$\Lambda$ if it has a realizer in Λ. When $\mathbb{X}$ and $\mathbb{Y}$ are clear from the context, we will just say f is in Λ; thus we have computable, continuous, etc., functions between represented spaces. Note that, for any represented space $\mathbb{X}$, both $\delta_{\mathbb{X}}$ and $\delta_{\mathbb{X}}^{-1}$ are realized by $\mathrm{id}_{\omega^\omega}$. In particular, viewed as functions between represented spaces, $\delta_{\mathbb{X}}$ and $\delta_{\mathbb{X}}^{-1}$ are always computable.

Of course, if $\mathbb{X}$ and $\mathbb{Y}$ are also topological spaces, then it is ambiguous to say a function $f : \mathbb{X} \to \mathbb{Y}$ is, e.g., continuous. However, for the following standard representation of Polish spaces there is no such ambiguity. Let (X, τ) be a Polish space, fix a countable dense sequence $\langle a_i \,;\, i \in \omega\rangle$ in X and a compatible metric d. Now define $\delta_{\mathbb{X}}$ by $\delta_{\mathbb{X}}(p) = x$ iff $d(a_{p(i)}, x) < 2^{-i}$ holds for all $i \in \omega$. In other words, we represent a point by a sequence of basic points converging to it with a prescribed speed. It is a foundational result in computable analysis that the notion of continuity for the represented space $(X, \delta_{\mathbb{X}})$ coincides with that for the Polish space (X, τ).

1.30 Definition (Folklore). Let f and g be multi-valued partial functions between represented spaces. We say that f is *Weihrauch-reducible* to g, in symbols $f \leqslant_{\mathfrak{W}} g$, if there are computable functions $H, K : \omega^\omega \dashrightarrow \omega^\omega$ such that for any $G \vdash g$, the function $p \mapsto H(\ulcorner p, G \circ K(p)\urcorner)$ is a realizer for f. We say that f is *strongly Weihrauch-reducible* to g, in symbols $f \leqslant_{\mathrm{s}\mathfrak{W}} g$, if there are computable functions $H, K : \omega^\omega \dashrightarrow \omega^\omega$ such that for any $G \vdash g$, the function HGK is a realizer for f. We call *t-Weihrauch reducibility* and *strong t-Weihrauch reducibility*, in symbols $\leqslant^{\mathrm{t}}_{\mathfrak{W}}$ and $\leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}}$, the variations where "computable" is replaced with "continuous". We denote by $\equiv_{\mathfrak{W}}$, $\equiv_{\mathrm{s}\mathfrak{W}}$, $\equiv^{\mathrm{t}}_{\mathfrak{W}}$, and $\equiv^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}}$ the corresponding equivalence relations, i.e., defined by letting $f \equiv_{\mathfrak{W}} g$ if both $f \leqslant_{\mathfrak{W}} g$ and $g \leqslant_{\mathfrak{W}} f$, and likewise in the other cases.

Figure 1.5 shows an intuitive graphical depiction of (strong) Weihrauch reducibility.

(a) $f \leqslant_{\mathfrak{W}} g$ (b) $f \leqslant_{\mathrm{s}\mathfrak{W}} g$

Figure 1.5: Weihrauch reducibility of f to g; in each case, the depicted function realizes f for whatever realizer $G \vdash g$ that is plugged in.

We refer the reader to [16] for a recent comprehensive survey on Weihrauch reducibility.

For any represented space $\mathbb{X}$, let $\Delta_{\mathbb{X}} : \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ be the total computable function given by $\Delta_{\mathbb{X}}(x) = (x, x)$. The following result illustrates how the notion of tightening is a good tool for expressing concepts in Weihrauch reducibility theory.

1.31 PROPOSITION (Folklore; cf., e.g., [76, Chapter 4]). *Let* $F : \omega^\omega \dashrightarrow \omega^\omega$, $f : \mathbb{X} \rightrightarrows \mathbb{Y}$, *and* $g : \mathbb{Z} \rightrightarrows \mathbb{W}$.

(1) *The following are equivalent:*

  (a) $F \vdash f$;

  (b) $\delta_\mathbb{Y} F \preceq f \delta_\mathbb{X}$;

  (c) $\delta_\mathbb{Y} F \delta_\mathbb{X}^{-1} \preceq f$.

(2) *The following are equivalent:*

  (a) $f \leqslant_{\mathrm{s}\mathfrak{W}} g$ *(*$f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} g$*)*

  (b) *there exist computable (continuous)* $k : \mathbb{X} \rightrightarrows \mathbb{Z}$ *and* $h : \mathbb{W} \rightrightarrows \mathbb{Y}$ *such that* $hgk \preceq f$.

(3) *The following are equivalent:*

  (a) $f \leqslant_{\mathfrak{W}} g$ *(*$f \leqslant^{\mathrm{t}}_{\mathfrak{W}} g$*)*;

  (b) *there exist computable (continuous)* $k : \mathbb{X} \rightrightarrows \mathbb{Z}$ *and* $h : \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{Y}$ *such that* $f \succeq h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X}$;

  (c) *there exist computable (continuous)* $k : \mathbb{X} \rightrightarrows \mathbb{Z}$ *and* $h : \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{Y}$ *such that* $f = h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X}$.

Proof. We will only prove (3), since (1) is a simple unpacking of definitions and (2) is similar to (3) but simpler.

(3a $\Rightarrow$ 3b) Suppose $f \leqslant_{\mathfrak{W}} g$. Then there exist computable $K : \omega^\omega \dashrightarrow \omega^\omega$ and $H : \omega^\omega \times \omega^\omega \dashrightarrow \omega^\omega$ such that $H \circ (\mathrm{id}_{\omega^\omega} \times GK) \vdash f$ whenever $G \vdash g$. Let $k : \mathbb{X} \rightrightarrows \mathbb{Z}$ and $h : \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{Y}$ be defined by $k = \delta_\mathbb{Z} K \delta_\mathbb{X}^{-1}$ and $h = \delta_\mathbb{Y} H \circ (\delta_\mathbb{X}^{-1} \times \delta_\mathbb{W}^{-1})$. Clearly both k and h are computable. Now suppose $x \in \mathrm{dom}(f)$, and let us show $x \in \mathrm{dom}(h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X})$ and $h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X}(x) \subseteq f(x)$.

Since $H \circ (\mathrm{id}_{\omega^\omega} \times GK) \vdash f$ whenever $G \vdash g$, it follows that $x \in \mathrm{dom}(g\delta_\mathbb{Z} K \delta_\mathbb{X}^{-1})$, i.e., $x \in \mathrm{dom}(k)$ and $k(x) \subseteq \mathrm{dom}(g)$. Note that for any $y \in gk(x)$, any $\delta_\mathbb{W}$-name p for y, and any $\delta_\mathbb{X}$-name q for x, there exists some realizer G of g such that $p = GK(q)$. Therefore $(q,p) \in \mathrm{dom}(\delta_\mathbb{Y} H)$, i.e., $(x, gk(x)) \in \mathrm{dom}(\delta_\mathbb{Y} H \circ (\delta_\mathbb{X}^{-1} \times \delta_\mathbb{W}^{-1})) = \mathrm{dom}(h)$. Hence $x \in \mathrm{dom}(h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X})$. Finally, we have

$$\begin{aligned} h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X}(x) &= h(x, gk(x)) \\ &= \delta_\mathbb{Y} H \circ (\delta_\mathbb{X}^{-1}(x), \delta_\mathbb{W}^{-1} g \delta_\mathbb{Z} K \delta_\mathbb{X}^{-1}(x)). \end{aligned}$$

Let $y \in h \circ (\mathrm{id}_\mathbb{X} \times gk)\Delta_\mathbb{X}(x)$, i.e., let $p_0, p_1 \in \delta_\mathbb{X}^{-1}$, let $w \in g\delta_\mathbb{Z} K(p_1)$, let $p_2 \in \delta_\mathbb{W}^{-1}(w)$, and let $y = \delta_\mathbb{Y} H(p_0, p_2)$. Again, since there exists a realizer $G \vdash g$ such that

$G(K(p_1)) = G(K(p_0)) = p_2$, from $H \circ (\mathrm{id}_{\omega^\omega} \times GK) \vdash f$ it now follows that $y \in f(x)$ as desired.

(3b $\Rightarrow$ 3c) Suppose $h \circ (\mathrm{id}_{\mathbb{X}} \times gk)\Delta_{\mathbb{X}} \preceq f$. Let $k' = k{\restriction}\mathrm{dom}(f)$, so that $\mathrm{dom}(h \circ (\mathrm{id}_{\mathbb{X}} \times gk')\Delta_{\mathbb{X}}) = \mathrm{dom}(f)$. Now let $h' : \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{Y}$ with $\mathrm{dom}(h') = \mathrm{dom}(h)$ be given by $h'(x, y) = f(x) \cup h(x, y)$. Since h is computable, so is h'. Now, for any $x \in \mathrm{dom}(f) = \mathrm{dom}(h' \circ (\mathrm{id}_{\mathbb{X}} \times gk')\Delta_{\mathbb{X}})$ we have $h' \circ (\mathrm{id}_{\mathbb{X}} \times gk')\Delta_{\mathbb{X}}(x) = h'(x, gk(x)) = f(x) \cup h(x, gk(x)) = f(x)$, as desired.

(3c $\Rightarrow$ 3a) Suppose $f = h \circ (\mathrm{id}_{\mathbb{X}} \times gk)\Delta_{\mathbb{X}}$. Let $H \vdash h$ and $K \vdash k$ be computable. Then, also using that $\mathrm{id}_{\omega^\omega} \vdash \mathrm{id}_{\mathbb{X}}$ and $\Delta_{\omega^\omega} \vdash \Delta_{\mathbb{X}}$, we have

$$\begin{array}{rcl} f & = & h \circ (\mathrm{id}_{\mathbb{X}} \times gk)\Delta_{\mathbb{X}} \\ & \succeq & \delta_{\mathbb{Y}} H \circ (\delta_{\mathbb{X}}^{-1} \times \delta_{\mathbb{W}}^{-1}) \circ (\delta_{\mathbb{X}} \mathrm{id}_{\omega^\omega} \delta_{\mathbb{X}}^{-1} \times g\delta_{\mathbb{Z}} K \delta_{\mathbb{X}}^{-1}) \circ (\delta_{\mathbb{X}} \times \delta_{\mathbb{X}})\Delta_{\omega^\omega}\delta_{\mathbb{X}}^{-1} \\ & = & \delta_{\mathbb{Y}} H \circ (\delta_{\mathbb{X}}^{-1}\delta_{\mathbb{X}} \mathrm{id}_{\omega^\omega} \delta_{\mathbb{X}}^{-1}\delta_{\mathbb{X}} \times \delta_{\mathbb{W}}^{-1} g\delta_{\mathbb{Z}} K \delta_{\mathbb{X}}^{-1}\delta_{\mathbb{X}})\Delta_{\omega^\omega}\delta_{\mathbb{X}}^{-1} \\ & \succeq & \delta_{\mathbb{Y}} H \circ (\mathrm{id}_{\omega^\omega} \times \delta_{\mathbb{W}}^{-1} g\delta_{\mathbb{Z}} K)\Delta_{\omega^\omega}\delta_{\mathbb{X}}^{-1}. \end{array}$$

Therefore, for any $G : \omega^\omega \dashrightarrow \omega^\omega$, we have that if $G \vdash g$, i.e., if $G \preceq \delta_{\mathbb{W}}^{-1} g \delta_{\mathbb{Z}}$, then $\delta_{\mathbb{Y}} H \circ (\mathrm{id}_{\omega^\omega} \times GK)\Delta_{\omega^\omega}\delta_{\mathbb{X}}^{-1} \preceq f$, i.e., $H \circ (\mathrm{id}_{\omega^\omega} \times GK)\Delta_{\omega^\omega} \vdash f$, as desired. □

1.32 Corollary. *If f is tightened by g and g is (strongly) Weihrauch-reducible to h, then f is (strongly) Weihrauch-reducible to h.*

Although the Weihrauch degrees form a very rich algebraic structure (cf., e.g., [14, 17, 47] for surveys covering this aspect of the Weihrauch lattice), in this thesis we only need two operations on the Weihrauch degrees, parallelization and sequential composition. Given a map $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ between represented spaces, its *parallelization* is the map $\hat{f} : \mathbb{X}^\omega \rightrightarrows \mathbb{Y}^\omega$ given by $\langle y_n \rangle_{n\in\omega} \in \hat{f}(\langle x_n \rangle_{n\in\omega})$ iff $y_n \in f(x_n)$ for each $n \in \omega$. We say that f is *parallelizable* if $f \equiv_{\mathfrak{W}} \hat{f}$. It is not hard to see that parallelization is a closure operator in the Weihrauch degrees. Rather than defining the sequential composition operator $\star$ explicitly as in [17], we will make use of the following characterization:

1.33 Theorem (Brattka & Pauly [17, Corollary 18]). *For any f and g, we have*

$$f \star g \equiv_{\mathfrak{W}} \max_{\leqslant_{\mathfrak{W}}} \{ f' \circ g' \,;\, f' \leqslant_{\mathfrak{W}} f \wedge g' \leqslant_{\mathfrak{W}} g \wedge f' \circ g' \textit{ is defined}\}.$$

Represented spaces of continuous functions

It is a foundational result in effective descriptive set theory that the continuous single-valued partial functions on ω^ω are exactly those which are computable relative to some oracle (cf. also Theorem 2.8 below), and that therefore the continuous multi-valued partial functions between two represented spaces $\mathbb{X}$ and $\mathbb{Y}$ are exactly those possessing some realizer which is computable relative to some oracle. With this in mind, it is a natural idea to attempt to define a represented space of the continuous multi-valued partial functions between $\mathbb{X}$ and $\mathbb{Y}$ by letting p be a name for f iff $p = 0^n 1 q$ and the n^{th} Turing machine computes some realizer

F_p for f when given oracle q. However, with this naive definition the representation fails to be a single-valued function, and this happens for two reasons. First, since a realizer of a function also realizes any restriction of that function, it follows that p as above would also be a name for any proper restriction of f. And second, it could be that there exists a different multi-valued function $g : \mathbb{X} \rightrightarrows \mathbb{Y}$, say with $\mathrm{dom}(g) = \mathrm{dom}(f)$, such that $f(x) \cap g(x) \neq \varnothing$ holds for all $x \in \mathrm{dom}(f)$. Then if the realizer F_p of f happens to satisfy $\delta_{\mathbb{Y}} \circ F_p(r) \in (f \circ \delta_{\mathbb{X}}(r)) \cap (g \circ \delta_{\mathbb{X}}(r))$, we will have that p is also a name for g.

There are (at least) two ways of solving these problems, and they result in two different represented spaces. First, since the problems pointed out came from the fact that f was partial and multi-valued, the idea works completely if f is total and single-valued. We thus define the represented space $\mathcal{C}(\mathbb{X}, \mathbb{Y})$ of the *total single-valued* continuous functions between $\mathbb{X}$ and $\mathbb{Y}$, as above.

For the second solution, intuitively we want to change the way the function F_p simulates f by making it capture *all* values in the range of f, and *only* computing on valid inputs for f. Since realizers are required to be single-valued, the first of these requirements is met by using a second oracle for the Turing machine, so that varying this oracle will allow us to capture the different outputs of f for a same fixed input. For the second requirement, we stipulate that the Turing machine must only compute names of elements of $\mathbb{Y}$ on inputs which are names of elements of $\mathrm{dom}(f)$. Concretely, the space $\mathcal{M}(\mathbb{X}, \mathbb{Y})$ of the *strongly continuous functions* between $\mathbb{X}$ and $\mathbb{Y}$ is defined by letting p be a name for f iff $p = 0^n 1q$ and, letting M be the n^{th} Turing machine, we have

(1) for every $r \in \mathrm{dom}(f\delta_{\mathbb{X}})$ and every $r' \in \omega^\omega$ we have that M computes a $\delta_{\mathbb{Y}}$-name for an element of $f\delta_{\mathbb{X}}(r)$ when given $\ulcorner r, r' \urcorner$ as input and q as oracle;

(2) for every $x \in \mathrm{dom}(f)$ and every $y \in f(x)$ there exist a $\delta_{\mathbb{X}}$-name r for x and an $r' \in \omega^\omega$ such that M computes a $\delta_{\mathbb{Y}}$-name for y when given $\ulcorner r, r' \urcorner$ as input and q as oracle;

(3) for every $x \in \mathbb{X} \setminus \mathrm{dom}(f)$, every $\delta_{\mathbb{X}}$-name r for x, and every $r' \in \omega^\omega$, we have that M does not compute an element in $\mathrm{dom}(\delta_{\mathbb{Y}})$ when given $\ulcorner r, r' \urcorner$ as input and q as oracle.

In this case we also say that f is *strongly continuous*, and that M *strongly computes* f with oracle p. As expected, if the oracle $q \in \omega^\omega$ in the definition above is computable then f is called *strongly computable*.

Every strongly continuous or strongly computable function is of course continuous or computable, respectively. The converse fails, since the set of names for elements in the domain of a strongly continuous function is a $\mathbf{\Pi}^0_2$ set—an element is in that set iff for every length n there exists some stage of the computation of the corresponding Turing machine with corresponding oracle at which at least n numbers are written on the output tape—, and of course not every partial continuous or computable function has that property. However, for our purposes, these strong notions capture the corresponding weaker notions well enough, due to the following result.

1.34 THEOREM (Brattka & Pauly [17, Lemma 13]). *Every computable or continuous $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ has a strongly computable or strongly continuous, respectively, tightening $g : \mathbb{X} \rightrightarrows \mathbb{Y}$.*

Proof. We can assign to each Turing machine M and oracle q a function $g_{M,q} : \mathbb{X} \rightrightarrows \mathbb{Y}$ given by $\mathrm{dom}(g_{M,q}) = \{\delta_{\mathbb{X}}(r)\,;\, M$ produces an element of $\mathrm{dom}(\delta_{\mathbb{Y}})$ when run on input r with oracle $q\}$ and $g_{M,q}(x) = \{\delta_{\mathbb{Y}}(q')\,;\,$ there exists a $\delta_{\mathbb{X}}$-name r for x such that q' is the output of M when run with input r and oracle $q\}$. Now, if M with oracle q computes a realizer for $f : \mathbb{X} \rightrightarrows \mathbb{Y}$, then it immediately follows that $g_{M,q} \preceq f$. Finally, to see that $g_{M,q}$ is strongly continuous or strongly computable (in case q is computable), let M' be the Turing machine which, on input $\ulcorner r, r' \urcorner$ and with oracle q, simply runs the Turing machine M on input r and oracle q. We now have that M' strongly computes $g_{M,q}$ with oracle q. $\square$

Chapter 2

Games for functions on Baire space

Summary. In this chapter, we provide game characterizations of several classes of functions. In § 2.1, we review games characterizing the class of functions which are piecewise continuous on a countable partition of their domains with closed parts, the Baire class 1 functions, the class of functions which are piecewise continuous on a countable partition of their domains with $\mathbf{\Pi}^0_2$ parts, the class of functions which are piecewise Baire class 1 on a countable partition of their domains with $\mathbf{\Pi}^0_2$ parts, the Baire class 2 functions, and the Borel measurable functions. Then, in § 2.2, we present the main result of this chapter (Theorem 2.31), viz. the definition of a family of games which characterize the Baire class α functions for each $\alpha < \omega_1$. Finally, in § 2.3, we present a construction which transforms a game characterizing a subclass Γ of the Borel measurable functions into a game characterizing the class of functions which are piecewise Γ on a countable partition of their domains with $\mathbf{\Pi}^0_\alpha$ parts (Theorem 2.61). This construction generalizes games due to Andretta and Semmes from the literature (Corollary 2.64).

Remarks on authorship

Unless stated otherwise, definitions and results presented in this chapter are due to the author.

2.1 Definitions and known results

Our general definitions of games and strategies, Definitions 2.1–2.3 below, are slight modifications of the corresponding definitions given by Motto Ros in [69, § 3.1].

2.1 Definition. A *game for functions*, or simply a *game* when there is no risk of ambiguity, is a tuple $G = (\mathrm{M}_{\mathbb{1}}, \mathrm{R}_{\mathbb{1}}, \iota_{\mathbb{1}}, \mathrm{M}_{\mathbb{2}}, \mathrm{R}_{\mathbb{2}}, \iota_{\mathbb{2}})$ where, for $p \in \{\mathbb{1}, \mathbb{2}\}$, the set

$\mathrm{M}_p \neq \varnothing$ is called the set of *moves* for player p, the set $\mathrm{R}_p \subseteq \mathrm{M}_p^\omega$ is called the set of *rules* for p, and the function $\iota_p : \mathrm{R}_p \to \omega^\omega$ is called the *interpretation* function for p.

Informally, we think of G as being played in ω rounds by two players, a male player $\mathbb{1}$ and a female player $\mathbb{2}$, who are both given knowledge of a partial function $f : \omega^\omega \dashrightarrow \omega^\omega$ beforehand. A *run* of G for f is a sequence of ω alternating moves $\langle \mathfrak{m}_{\mathbb{1}}^0, \mathfrak{m}_{\mathbb{2}}^0, \mathfrak{m}_{\mathbb{1}}^1, \mathfrak{m}_{\mathbb{2}}^1, \mathfrak{m}_{\mathbb{1}}^2, \mathfrak{m}_{\mathbb{2}}^2, \ldots, \mathfrak{m}_{\mathbb{1}}^n, \mathfrak{m}_{\mathbb{2}}^n, \ldots \rangle$ by $\mathbb{1}$ and $\mathbb{2}$, starting with $\mathbb{1}$. We then say that $\mathbb{2}$ *wins* the run if one of the following conditions is satisfied:

(1) $\mathfrak{m}_{\mathbb{1}} := \langle \mathfrak{m}_{\mathbb{1}}^n \,;\, n \in \omega \rangle \notin \iota_{\mathbb{1}}^{-1}[\mathrm{dom}(f)]$.

(2) $\mathfrak{m}_{\mathbb{1}} \in \iota_{\mathbb{1}}^{-1}[\mathrm{dom}(f)]$, $\mathfrak{m}_{\mathbb{2}} := \langle \mathfrak{m}_{\mathbb{2}}^n \,;\, n \in \omega \rangle \in \mathrm{R}_{\mathbb{2}}$, and $\iota_{\mathbb{2}}(\mathfrak{m}_{\mathbb{2}}) = f \circ \iota_{\mathbb{1}}(\mathfrak{m}_{\mathbb{1}})$.

We depict a run of G for f in Figure 2.1.

	Round 0		Round 1		Round 2		$\cdots$	After ω rounds
Player $\mathbb{1}$	$\mathfrak{m}_{\mathbb{1}}^0$		$\mathfrak{m}_{\mathbb{1}}^1$		$\mathfrak{m}_{\mathbb{1}}^2$		$\longrightarrow$	$\mathfrak{m}_{\mathbb{1}} \in \iota_{\mathbb{1}}^{-1}[\mathrm{dom}(f)]$
Player $\mathbb{2}$		$\mathfrak{m}_{\mathbb{2}}^0$		$\mathfrak{m}_{\mathbb{2}}^1$		$\mathfrak{m}_{\mathbb{2}}^2$	$\longrightarrow$	$\mathfrak{m}_{\mathbb{2}} \in \mathrm{R}_{\mathbb{2}}$

Figure 2.1: Depiction of a run of G for f which $\mathbb{2}$ wins iff $\iota_{\mathbb{2}}(\mathfrak{m}_{\mathbb{2}}) = f \circ \iota_{\mathbb{1}}(\mathfrak{m}_{\mathbb{1}})$.

Still informally, a *strategy* for player p is a function which takes as input a partial run of the game at which it is now player p's turn to play, and outputs a move for p. A strategy ϑ for p in G for f is a *winning strategy* if p wins any run of the game in which he or she plays according to ϑ. Finally, we say that G *characterizes* a class Λ of partial functions on ω^ω if, for any $f : \omega^\omega \dashrightarrow \omega^\omega$, we have $f \in \Lambda$ iff $\mathbb{2}$ has a winning strategy in G for f.

We make these concepts mathematically precise as follows.

2.2 Definition. Let $G = (\mathrm{M}_{\mathbb{1}}, \mathrm{R}_{\mathbb{1}}, \iota_{\mathbb{1}}, \mathrm{M}_{\mathbb{2}}, \mathrm{R}_{\mathbb{2}}, \iota_{\mathbb{2}})$ be a game. A *run* of G is a sequence $r \in (\mathrm{M}_{\mathbb{1}} \cup \mathrm{M}_{\mathbb{2}})^{\leqslant\omega}$ such that, for any $n < |r|$, we have $r(n) \in \mathrm{M}_{\mathbb{1}}$ iff n is even. Given a run r of G, we denote by $r_{\mathbb{1}}$ the sequence given by $r_{\mathbb{1}}(n) = r(2n)$ for each n such that $2n < |r|$, and by $r_{\mathbb{2}}$ the sequence given by $r_{\mathbb{2}}(n) = r(2n+1)$ for each n such that $2n+1 < |r|$. A *strategy* for player p in G is a function which associates to each finite run of G of even length, if $p = \mathbb{1}$, or of odd length, if $p = \mathbb{2}$, an element of M_p. We usually denote strategies for $\mathbb{1}$ by ζ and for $\mathbb{2}$ by ϑ. Given a strategy ζ for $\mathbb{1}$ and sequence $\mathfrak{m}_{\mathbb{2}} \in \mathrm{M}_{\mathbb{2}}^\omega$, we define a run $\zeta * \mathfrak{m}_{\mathbb{2}}$ of G by recursion, letting $(\zeta * \mathfrak{m}_{\mathbb{2}})(0) = \zeta(\langle\rangle)$, and $(\zeta * \mathfrak{m}_{\mathbb{2}})(2n+1) = \mathfrak{m}_{\mathbb{2}}(n)$ and $(\zeta * \mathfrak{m}_{\mathbb{2}})(2n+2) = \zeta((\zeta * \mathfrak{m}_{\mathbb{2}}) \restriction (2n+2))$ for each $n \in \omega$. Analogously, given a strategy ϑ for $\mathbb{2}$ and a sequence $\mathfrak{m}_{\mathbb{1}} \in \mathrm{M}_{\mathbb{1}}^\omega$, we define a run $\mathfrak{m}_{\mathbb{1}} * \vartheta$ of G by the recursion $(\mathfrak{m}_{\mathbb{1}} * \vartheta)(2n) = \mathfrak{m}_{\mathbb{1}}(n)$ and $(\mathfrak{m}_{\mathbb{1}} * \vartheta)(2n+1) = \vartheta((\mathfrak{m}_{\mathbb{1}} * \vartheta) \restriction (2n+1))$ for each $n \in \omega$.

Note that the definitions of a game, of a run of a game, and of a strategy for a player in a game do not depend on any particular given function in order to make sense. However, the situation is different for the next definition.

2.3 Definition. Let $G = (\mathrm{M}_{\mathbb{1}}, \mathrm{R}_{\mathbb{1}}, \iota_{\mathbb{1}}, \mathrm{M}_{\mathbb{2}}, \mathrm{R}_{\mathbb{2}}, \iota_{\mathbb{2}})$ be a game and $f : \omega^\omega \dashrightarrow \omega^\omega$ be a function. A strategy ζ for $\mathbb{1}$ is *legal in G for f* if $(\zeta * \mathfrak{m}_{\mathbb{2}})_{\mathbb{1}} \in \iota_{\mathbb{1}}^{-1}[\mathrm{dom}(f)]$ holds for any $\mathfrak{m}_{\mathbb{2}} \in \mathrm{M}_{\mathbb{2}}^\omega$, and a legal ζ is a *winning strategy* for $\mathbb{1}$ in G for f if $f \circ \iota_{\mathbb{1}}((\zeta * \mathfrak{m}_{\mathbb{2}})_{\mathbb{1}}) \neq \iota_{\mathbb{2}}(\mathfrak{m}_{\mathbb{2}})$ holds for any $\mathfrak{m}_{\mathbb{2}} \in \mathrm{M}_{\mathbb{2}}^\omega$. A strategy ϑ for $\mathbb{2}$ in G is *legal for* $\mathfrak{m}_{\mathbb{1}} \in \mathrm{M}_{\mathbb{1}}^\omega$ if $(\mathfrak{m}_{\mathbb{1}} * \vartheta)_{\mathbb{2}} \in \mathrm{R}_{\mathbb{2}}$ holds, and ϑ is *legal for f* if it is legal for any $\mathfrak{m}_{\mathbb{1}} \in \iota_{\mathbb{1}}^{-1}[\mathrm{dom}(f)]$. Finally, if ϑ is legal strategy for $\mathbb{2}$ in G for f, then it is a *winning strategy* if $f \circ \iota_{\mathbb{1}}(\mathfrak{m}_{\mathbb{1}}) = \iota_{\mathbb{2}}((\mathfrak{m}_{\mathbb{1}} * \vartheta)_{\mathbb{2}})$ holds for any $\mathfrak{m}_{\mathbb{1}} \in \iota_{\mathbb{1}}^{-1}[\mathrm{dom}(f)]$.

2.4 Convention. In all games we will consider from now on in this chapter, the sets of moves and rules, as well as the interpretation function for $\mathbb{1}$, will be fixed as $\mathrm{M}_{\mathbb{1}} = \omega$, $\mathrm{R}_{\mathbb{1}} = \omega^\omega$, and $\iota_{\mathbb{1}} = \mathrm{id}_{\omega^\omega}$, the identity function on ω^ω. In such a setting, we can simplify matters considerably from the point of view of player $\mathbb{2}$: a strategy ϑ for $\mathbb{2}$ is legal in $G = (\omega, \omega^\omega, \mathrm{id}_{\omega^\omega}, \mathrm{M}_{\mathbb{2}}, \mathrm{R}_{\mathbb{2}}, \iota_{\mathbb{2}})$ for f if $(x * \vartheta)_{\mathbb{2}} \in \mathrm{R}_{\mathbb{2}}$ holds for any $x \in \mathrm{dom}(f)$, and a legal ϑ is a *winning strategy* for $\mathbb{2}$ in G for f if $f(x) = \iota_{\mathbb{2}}((x * \vartheta)_{\mathbb{2}})$ holds for any $x \in \mathrm{dom}(f)$.

The prototypical example of a game for functions is the following.

2.5 Definition (Folklore). The *Wadge game* is the game in which $\mathbb{2}$ plays elements of $\omega^{<\omega}$, with the rule that they must converge monotonically to some element of ω^ω. Formally, the Wadge game is the game $(\omega, \omega^\omega, \mathrm{id}_{\omega^\omega}, \mathrm{M}_{\mathbb{2}}^{\mathcal{W}}, \mathrm{R}_{\mathbb{2}}^{\mathcal{W}}, \iota_{\mathbb{2}}^{\mathcal{W}})$, with $\mathrm{M}_{\mathbb{2}}^{\mathcal{W}} = \omega^{<\omega}$, $\mathrm{R}_{\mathbb{2}}^{\mathcal{W}} = \{\mathfrak{m}_{\mathbb{2}} \in (\mathrm{M}_{\mathbb{2}}^{\mathcal{W}})^\omega \,;\, \forall n < k(\mathfrak{m}_{\mathbb{2}}(n) \subseteq \mathfrak{m}_{\mathbb{2}}(m)) \wedge \forall \ell \exists n(|\mathfrak{m}_{\mathbb{2}}(n)| \geqslant \ell)\}$, and $\iota_{\mathbb{2}}^{\mathcal{W}}(\mathfrak{m}_{\mathbb{2}}) = \bigcup_{n\in\omega} \mathfrak{m}_{\mathbb{2}}(n)$.

2.6 Theorem (Folklore). *The Wadge game characterizes the class of continuous functions.*

Proof. Given a function $f : \omega^\omega \dashrightarrow \omega^\omega$, define a strategy ϑ for $\mathbb{2}$ in the Wadge game for f by letting $\vartheta(\sigma)$ be the longest $\tau \in \omega^{<\omega}$ of length at most $|\tau|$ such that $f[\sigma] \subseteq [\tau]$. If f is continuous, then for each $x \in \mathrm{dom}(f)$ we have that $\langle |\vartheta(x{\restriction}n)| \,;\, n \in \omega \rangle$ is unbounded in ω, implying that ϑ is a winning strategy for $\mathbb{2}$ in the Wadge game for f.

Conversely, given a winning strategy ϑ for $\mathbb{2}$ in the Wadge game for f, for any $\sigma \in \omega^\omega$ we have that $f^{-1}[\sigma] = \{x \in \mathrm{dom}(f) \,;\, \exists n \in \omega(\sigma \subseteq \vartheta(x{\restriction}n))\}$ is a $\mathbf{\Sigma}^0_1(\mathrm{dom}(f))$ set. $\square$

In what follows, games for functions will usually be defined only informally, with the formal definitions being easily deduced in each case. One important motivation for considering games for functions is related to the following notion, which goes back to the work of Turing.†

2.7 Definition. A function $f : \omega^\omega \dashrightarrow \omega^\omega$ is *type-two computable*, or simply *computable*, if there exists a Turing machine M with the property that for any $x \in \mathrm{dom}(f)$, when M is started with x written on its input tape it writes all of $f(x)$ on its output tape without ever halting. In this case we also say that M *computes* f. The function f is *computable with an oracle* if there exist an element $z \in \omega^\omega$ and a Turing machine M which computes f when given access to z as an oracle.

The following result is a straightforward consequence of the definition; note that a strategy for $\mathbb{2}$ in the Wadge game is a function of type $\vartheta : \omega^{<\omega} \dashrightarrow \omega^{<\omega}$.

2.8 Theorem (Folklore). *A function $f : \omega^\omega \dashrightarrow \omega^\omega$ is*

(1) *type-two computable iff $\mathbb{2}$ has a winning strategy in the Wadge game for f which is computable by a Turing machine.*

(2) *continuous iff it is computable with an oracle.*

Therefore, in an intuitive sense, the Wadge game is what results from a Turing machine when one relaxes the requirement that a program—i.e., the analogue of a strategy—be computable. Following this paradigm, a game which characterizes a class of functions makes explicit how much more powerful we have to make Turing machines so that the resulting machines compute the functions in that class, and in this way the complexity of that class is succinctly encapsulated. Furthermore, also in the spirit of Theorem 2.8, one can use a game which characterizes a class of functions to *define* a computable counterpart of that class in a natural way by requiring computable winning strategies in the game.

2.9 Definition (Duparc, implicit in [29–31]). The *eraser game* is the game in which $\mathbb{2}$ plays elements of $\omega^{<\omega}$, with the rule that they must converge pointwise to some element of ω^ω.

2.10 Theorem (Duparc, implicit in [29–31]). *The eraser game characterizes the class of Baire class 1 functions.*

† Although computability theory has classically mostly been associated with the study of the computable functions of natural numbers and other countable structures, it is interesting to note that Turing's seminal papers on the subject are in fact concerned with what we have called type-two computations of real numbers and real-valued functions. In fact, the very opening sentence of [94] reads 'The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means,' i.e., the computable real numbers are those with decimal expressions which are writable by a Turing machine in the type-two sense. (See also the correcting addendum to [94] showing that there are better ways of representing real numbers than by their decimal expansions [95].) We refer the reader to [6, § 2.3] for a thorough historical account of Turing's interest and involvement in computable real numbers and computable real analysis.

Proof. Given a Baire class 1 function $f : \omega^\omega \dashrightarrow \omega^\omega$, there exists a family $\{A_{\sigma,n} \in \mathbf{\Sigma}^0_1(\mathrm{dom}(f))\,;\, \sigma \in \omega^{<\omega}_{\neq 0} \wedge n \in \omega\}$ of sets such that $f^{-1}[\sigma] = \bigcup_{n\in\omega} A_{\sigma,n}$ holds for each $\sigma \in \omega^{<\omega}_{\neq 0}$. For each $\ell > 0$, fix some enumeration b_ℓ of all pairs (σ, n) where $\sigma \in \omega^\ell$ and $n \in \omega$. We define a strategy ϑ for $\mathbb{2}$ in the eraser game for f as follows. Given $\sigma \in \omega^{<\omega}_{\neq 0}$, for each $i \in \{1, \ldots, |\sigma|\}$ let (τ_i, n_i) be the least element according to the enumeration b_i satisfying $[\sigma] \cap A_{\tau_i,n_i} \neq \varnothing$. Then let $N \leqslant |\sigma|$ be greatest such that $\tau_1 \subset \tau_2 \subset \cdots \subset \tau_N$, and define $\vartheta(\sigma) := \tau_N$. Let $x \in \mathrm{dom}(f)$ and $\ell > 0$. There exists some least $i \in \omega$ such that $b^\ell(i) = (\tau, n)$ satisfies $x \in A_{\tau,n}$. In particular $\tau = f(x){\restriction}|\tau|$. It now follows that there exists some m such that $\tau \subseteq \vartheta(x{\restriction}k)$ for all $k \geqslant m$, and we are done.

Conversely, if ϑ is a winning strategy for $\mathbb{2}$ in the eraser game for $f : \omega^\omega \dashrightarrow \omega^\omega$, then for each $\sigma \in \omega^{<\omega}_{\neq 0}$ we have that $f^{-1}[\sigma] = \{x \in \mathrm{dom}(f)\,;\, \exists n \forall m \geqslant n(\sigma \subseteq \vartheta(x{\restriction}m))\}$ is a $\mathbf{\Sigma}^0_2(\mathrm{dom}(f))$ set. □

We can now prove the $\alpha = 1$ case of the missing direction of Theorem 1.22.

2.11 Corollary (Folklore). *If a function $f : \omega^\omega \dashrightarrow \omega^\omega$ is $\mathbf{\Sigma}^0_2$-measurable then it is the pointwise limit of a sequence of continuous functions.*

Proof. If ϑ is a winning strategy for $\mathbb{2}$ in the eraser game for f, then for each $n \in \omega$ let $f_n : \mathrm{dom}(f) \to \omega^\omega$ be given by $f_n(x) = \vartheta(x{\restriction}n)^\frown 0^\omega$. Clearly each f_n is a continuous function, and since $\langle \vartheta(x{\restriction}n)\,;\, n \in \omega\rangle$ converges pointwise to $f(x)$ for each $x \in \mathrm{dom}(f)$, the same holds true for $\langle f_n(x)\,;\, n \in \omega\rangle$. □

The proof of Corollary 2.11 is a simple but illustrative example of how games can be useful in proving results about classes of functions on ω^ω.

2.12 Definition (Van Wesep [97] *apud* Andretta [3]). The *backtrack game* is the game in which $\mathbb{2}$ plays elements of $\omega^{<\omega}$, with the rule that after some finite number of rounds the finite sequences of natural numbers played by $\mathbb{2}$ must converge monotonically to some element of ω^ω.

Equivalently, the backtrack game is the game in which $\mathbb{2}$ builds finitely many sequences of elements of $\omega^{<\omega}$, without any *a priori* bound on how many, each sequence monotonically converging to an element of $\omega^{\leqslant\omega}$. The rule is that exactly one of these sequences must converge to an element of ω^ω.

We refer to the elements of the n^{th} sequence of elements of $\omega^{<\omega}$ built by $\mathbb{2}$ during a run of the backtrack game as the contents of the n^{th} *tape*.

2.13 Theorem (Van Wesep [97] *apud* Andretta [3]). *The backtrack game characterizes the class of functions which are piecewise continuous on a $\mathbf{\Pi}^0_1$ partition.*

Proof. Let f be piecewise continuous on the $\mathbf{\Pi}^0_1$ partition $\langle A_n\,;\, n \in \omega\rangle$. For each n, let ϑ_n be a winning strategy for $\mathbb{2}$ in the Wadge game for $f{\restriction}A_n$. Then it is easy to check that a winning strategy ϑ for $\mathbb{2}$ in the backtrack game for f can be given by $\vartheta(\sigma) = \vartheta_n(\sigma)$ for n least such that $[\sigma] \cap A_n \neq \varnothing$.

Conversely, if ϑ is a winning strategy for $\mathbb{2}$ in the backtrack game for f, then letting $A_n = \{x \in \mathrm{dom}(f)\,;\, \langle \vartheta(x{\restriction}m)\,;\, m \geqslant n\rangle$ is monotone$\}$ it follows that $f{\restriction}A_n$ is continuous. Indeed, a winning strategy for $\mathbb{2}$ in the Wadge game for $f{\restriction}A_n$ is simply to play $\langle\rangle$ for n rounds, then follow ϑ. Since $\langle A_n\,;\, n \in \omega\rangle$ is a $\mathbf{\Pi}^0_1$ cover of $\mathrm{dom}(f)$, by Corollary 1.27 we are done. □

By the Jayne-Rogers theorem, Theorem 1.28, the backtrack game also characterizes the $\mathbf{\Delta}^0_2$ functions.

The study of games for functions on ω^ω was greatly advanced by Semmes's PhD thesis [92] containing (among other things) the multitape game, the multitape eraser game, the tree game, and the game he called $G_{1,3}$.

2.14 Definition (Semmes [91, § 6]). In the *multitape game*, player $\mathbb{2}$ builds countably-many sequences of elements of $\omega^{<\omega}$, each sequence monotonically converging to an element of $\omega^{\leqslant\omega}$. The rule is that exactly one of these sequences must converge to an element of ω^ω.

2.15 Theorem (Andretta & Semmes [91, Theorem 6.1]). *The multitape game characterizes the class of functions which are piecewise continuous on a $\mathbf{\Pi}^0_2$ partition.*

Proof. Let f be piecewise continuous on the $\mathbf{\Pi}^0_2$ partition $\langle A_n\,;\, n \in \omega\rangle$. For each n, let $\langle A_{n,m}\,;\, m \in \omega\rangle$ be $\mathbf{\Sigma}^0_1(\mathrm{dom}(f))$ sets such that $A_n = \bigcap_{m\in\omega} A_{n,m}$. Also, for each $n \in \omega$ let ϑ_n be a winning strategy for $\mathbb{2}$ in the Wadge game for $f{\restriction}A_n$. We define a strategy ϑ for $\mathbb{2}$ in the multitape game for f as follows. We have ω-many tapes; on the n^{th} we play according to ϑ_n. However, if $\vartheta_n(\sigma)$ tells $\mathbb{2}$ to play a different sequence on tape n for the m^{th} time, we will only do this if $[\sigma] \subseteq \bigcap_{k\leqslant m} A_{n,k}$. In this way exactly one tape—the n^{th} tape where n is such that the sequence $x \in \mathrm{dom}(f)$ that player $\mathbb{1}$ is building belongs to A_n—contains a sequence of elements of $\omega^{<\omega}$ which converges to an element of ω^ω, and it is now easy to see that ϑ is a winning strategy.

Conversely, if ϑ is a winning strategy for $\mathbb{2}$ in the multitape game for f, then letting $A_n = \{x \in \mathrm{dom}(f)\,;$ the n^{th} tape is infinite when $\mathbb{2}$ follows ϑ against $x\}$ it follows that $f{\restriction}A_n$ is continuous. Indeed, a winning strategy for $\mathbb{2}$ in the Wadge game for $f{\restriction}A_n$ is simply to copy the moves $\mathbb{2}$ makes on her n^{th} tape when following ϑ and ignore all other tapes. Since $\langle A_n\,;\, n \in \omega\rangle$ is a partition of $\mathrm{dom}(f)$, all that remains is to show that each $A_n \in \mathbf{\Pi}^0_2$. This follows by noting that $A_n = \{x \in \mathrm{dom}(f)\,;$ for each $\ell \in \omega$ there exists some round r of the game when $\mathbb{2}$ follows ϑ against x at which $\mathbb{2}$ adds an element of $\omega^{<\omega}$ of length at least ℓ to her n^{th} tape$\}$. □

By Semmes's decomposition theorem, Theorem 1.29, the multitape game also characterizes the class of $\mathbf{\Delta}^0_3$-functions.

2.16 Definition (Semmes [91, § 7]). In the *multitape eraser game*, player $\mathbb{2}$ builds ω-many sequences of elements of $\omega^{<\omega}$, each sequence converging pointwise to an element of $\omega^{\leqslant\omega}$. The rule is that exactly one of these sequences must converge to an element of ω^ω.

2.17 Theorem (Semmes [91, Theorem 7.1]). *The multitape eraser game characterizes the class of functions which are piecewise Baire class 1 on a $\mathbf{\Pi}^0_2$ partition.*

Proof. Analogous to the proof of Theorem 2.15, using the eraser game instead of the Wadge game. □

Again by Semmes's decomposition theorem, the multitape eraser game also characterizes the class of functions for which the preimage of any $\mathbf{\Delta}^0_2$ set is a $\mathbf{\Delta}^0_3$ set.

2.18 Definition (Semmes [92]). In the *tree game*, player $\mathbb{2}$ plays finite labeled trees, with the rule that the sequence of labeled trees played be a chain with respect to the labeled subtree relation and that the union of this chain, which we call the *final tree*, have exactly one infinite branch. The interpretation function associates to the final tree the sequence of labels along its unique infinite branch.

Thus, in a typical run of the tree game for a function $f : \omega^\omega \dashrightarrow \omega^\omega$, player $\mathbb{1}$ builds some $x \in \mathrm{dom}(f)$ one element at a time, as usual, and $\mathbb{2}$ responds at each round by adding finitely many new nodes and their labels to the tree she is building. Intuitively, with the running labels of the labeled tree she is building, $\mathbb{2}$ is trying to approximate the sequence $f(x)$. It is her task to ensure that these approximations cohere correctly, in the sense that the final tree she is building towards has only one infinite branch, and that this branch has as running label exactly $f(x)$.

Since the tree game and restrictions thereof will be central to the discussion for the remainder of this chapter and most of this thesis, it will be convenient to introduce the following notation. Given $x \in \omega^\omega$ and a strategy ϑ for $\mathbb{2}$ in the tree game which is legal for x, we denote by Υ^ϑ_x the final tree built by $\mathbb{2}$ in a run of the tree game in which player $\mathbb{1}$ plays x and $\mathbb{2}$ follows ϑ. In other words, if $(x * \vartheta)_{\mathbb{2}} = \langle (T_n, \varphi_n)\,;\, n \in \omega\rangle$ then $\Upsilon^\vartheta_x := (\bigcup_{n\in\omega} T_n, \bigcup_{n\in\omega} \varphi_n)$.

2.19 Theorem (Semmes [92, Theorem 2.0.9]). *The tree game characterizes the class of all Borel measurable functions.*

2.20 Definition (Semmes [92]). The game $G_{1,3}$ is the tree game with additional rule that the final tree must be finitely branching outside of its infinite branch.

2.21 Theorem (Semmes [92, Theorem 4.1.1]). *The game $G_{1,3}$ characterizes the Baire class 2 functions.*

We omit the proofs of Theorems 2.19 and 2.21 since they are fairly involved and follow from our main results of this chapter, Theorems 2.31 and 2.50. For now, let us note that these theorems naturally suggest the following avenue for investigation.

2.22 PROBLEM. Given a subclass Λ of the class of all Borel measurable functions, find a rule that when added to the tree game results in a game characterizing Λ.

It is not hard to restate the game characterizations we have seen in this section in the framework of Problem 2.22.

2.23 THEOREM (Folklore). *The tree game with additional rule that the final tree must be*

(1) *linear*

(2) *finitely branching*

(3) *finitely branching at the root and linear elsewhere*

(4) *linear outside the root*

(5) *finitely branching outside the root*

characterizes the class of

(1′) *continuous functions*

(2′) *Baire class 1 functions*

(3′) *functions which are piecewise continuous on a $\mathbf{\Pi}^0_1$ partition*

(4′) *functions which are piecewise continuous on a $\mathbf{\Pi}^0_2$ partition*

(5′) *functions which are piecewise Baire class 1 on a $\mathbf{\Pi}^0_2$ partition,*

respectively.

Proof. In each case, we want to prove that a game G which we know characterizes the class in question, e.g., the Wadge game, the eraser game, etc., is equivalent to the corresponding restricted tree game, i.e., that $\mathbb{2}$ has a winning strategy in one game iff she has a winning strategy in the other.

Starting from a winning strategy ϑ for $\mathbb{2}$ in G, a winning strategy ϑ' for her in the restricted tree game can be obtained as follows. For the Wadge and eraser games, at any round of the game ϑ' tells $\mathbb{2}$ to play the least tree which contains all of the finite sequences that ϑ has told her to play up to and including that round, with $\perp$ as the labeling function (i.e., so that each node of the tree is equal to its own running label). For the backtrack, multitape, and multitape eraser games, whenever ϑ tells $\mathbb{2}$ to add some element $\tau \in \omega^{<\omega}_{\neq 0}$ to her n^{th} tape, ϑ' tells her to add $\langle n\rangle^\frown(\text{shift}(\tau)\restriction m)$ to her tree with label $\tau(m-1)$, for each non-zero $m \leqslant |\tau|$. It is now routine to check that the rules of the Wadge, eraser, backtrack, multitape, and multitape eraser games, respectively, imply that ϑ' builds a final tree as specified in (1)–(5), and that ϑ' is a winning strategy follows directly from the fact that ϑ is.

Conversely, given a winning strategy ϑ for $\mathbf{2}$ in one of the restricted tree games, one can define a winning strategy ϑ' for $\mathbf{2}$ in the corresponding game among the Wadge, eraser, backtrack, multitape, and multitape eraser game, respectively, as follows.

For the Wadge and eraser games, for each σ let σ^* be the longest running label of $\vartheta(\sigma)$ with the property that all running labels of $\vartheta(\sigma)$ which were not present in $\vartheta(\sigma^-)$ are compatible with σ^*, and then define $\vartheta'(\sigma) = \sigma^*$. It is easy to see that, for any $x \in \mathrm{dom}(f)$, the sequence $\langle \vartheta'(x \restriction n)\,;\, n \in \omega\rangle$ converges monotonically or pointwise to the unique infinite running label of Υ_x^ϑ, when ϑ is a legal strategy for $\mathbf{2}$ in the Wadge or eraser game, respectively, for f.

For the backtrack, multitape and multitape eraser games, we just repeat the idea above for each tree $\mathrm{Com}(\vartheta(\sigma), \langle n\rangle)$, i.e., for each σ and $n \in \omega$ let σ_n^* be the longest running label of $\vartheta(\sigma)$ of nodes compatible with $\langle n\rangle$ with the property that all running labels of $\vartheta(\sigma)$ of nodes compatible with $\langle n\rangle$ which were not present in $\vartheta(\sigma^-)$ are compatible with σ_n^*. Then add σ_n^* to the n^{th} tape. It is again routine to check that, for any $x \in \mathrm{dom}(f)$,

(i) when ϑ is a legal strategy for $\mathbf{2}$ in the backtrack, multitape, or multitape eraser game for f, then each sequence $\langle (x \restriction m)_n^*\,;\, m \in \omega\rangle$ converges monotonically, in the backtrack and multitape cases, or pointwise, in the multitape eraser case, to some element of $\omega^{\leqslant\omega}$;

(ii) exactly one such sequence converges in the appropriate way to an element of ω^ω;

(iii) that element of ω^ω is exactly the unique infinite running label of Υ_x^ϑ.

Since ϑ is assumed to be a winning strategy for $\mathbf{2}$ in the corresponding game for f, in any of the five cases the infinite running label of Υ_x^ϑ is exactly $f(x)$, so we are done. $\square$

Labeled *versus* unlabeled trees

It is natural to question[†] whether the use of labeled trees in Theorems 2.19, 2.21, and 2.23 is in some way essential, or if they can be done away with in favor of the conceptually simpler unlabeled trees.

This is indeed the case for the classes of continuous and Baire class 1 functions. For continuous functions this is immediate, since it is intuitive that labels are of no use on a linear tree: instead of adding a node σ with running label τ to her tree, $\mathbf{2}$ can simply add τ to her tree directly. For finitely branching trees, we have that a labeled tree Υ is a finitely branching tree with a unique infinite branch labeled $y \in \omega^\omega$ iff the set of running labels occurring in Υ is a finitely branching tree with y as its unique infinite branch, from which the result follows.

[†]Indeed, it is reported that this question was asked by many readers of Semmes's thesis.

However, labeled trees are necessary even for the tree version of the backtrack game as stated. Consider, for example, the set A of those $x \in \omega^\omega$ for which there exists a unique $n \in \omega$ such that $n \notin \operatorname{ran}(x)$, let A_n be the closed subset of A formed by those x which do not contain n in their range, and finally let $f : A \to \omega^\omega$ be given by $f(x) = 0^{n+1\frown}1^\omega$ iff $x \in A_n$. Then f is constant on the closed partition $\langle A_n\,;\, n \in \omega\rangle$ of its domain, but $\mathbb{1}$ has a winning strategy in the unlabeled tree game with the additional rule that the final tree must be finitely branching at the root and linear elsewhere: for example, $\mathbb{1}$ can start by playing $r+1$ at each round r, so that it looks like 0 will be left out, until a round R at which $\mathbb{2}$ adds the sequence $\langle 0, 1\rangle$ to her tree—note that if no such round exists then $\mathbb{1}$ wins. At the next round, i.e., round $R+1$, player $\mathbb{1}$ plays 0 then goes back to playing $r+1$ at each subsequent round $r > R+1$, so that only $R+2$ is left out of her infinite sequence. But now $\mathbb{2}$ cannot make $0^{R+3\frown}1^\omega$ an infinite branch of her tree without adding a non-root branching node to her tree, so she loses.

One could argue that the requirement in the tree version of the backtrack game, viz. that the final tree must be finitely branching only at the root and linear elsewhere, is too restrictive. For example, the (unlabeled) tree game with additional rule that the final tree must be finitely branching with only finitely many branching nodes *does* characterize the class of functions which are piecewise continuous on a closed partition, as we will see in Proposition 2.24 below.

However, the situation with proper superclasses of the Baire class 1 functions shows that labels are necessary in a deeper sense. Consider, for example, the class of functions characterized by the multitape game. There exist functions g in that class which are not of Baire class 1 but such that $\operatorname{ran}(g) \subseteq 2^\omega$. Thus a winning strategy for $\mathbb{2}$ in the unlabeled tree game for g would always build finitely branching final trees, a contradiction since g is not Baire class 1. To see an example of such a function g, let us say that $x \in \omega^\omega$ *hits* $n \in \omega$ *infinitely often* if $x(m) = n$ for infinitely many m. Now let A be the set of those $x \in \omega^\omega$ for which there exists a unique $n \in \omega$ which x hits infinitely often, let A_n be the $\mathbf{\Pi}^0_2(A)$ subset of A formed by those x which hit n infinitely often, and finally let $g : A \to \omega^\omega$ be given by $g(x) = 0^{n\frown}1^\omega$ iff $x \in A_n$. A winning strategy for $\mathbb{1}$ in the finitely branching unlabeled tree game can be defined by playing 1 repeatedly until $\mathbb{2}$ adds $\langle 0, 1\rangle$ to her tree, at which point $\mathbb{1}$ plays a 0 then plays 2 repeatedly until $\mathbb{2}$ adds $\langle 0, 0, 1\rangle$ to her tree, at which point $\mathbb{1}$ plays a 0 then plays 3 repeatedly until $\mathbb{2}$ adds $\langle 0, 0, 0, 1\rangle$ to her tree, and so on. Then either $\mathbb{2}$ builds a tree with infinitely many branching nodes or she does not build a tree with the correct infinite branch, losing either way.

2.24 Proposition (N.). *The unlabeled tree game with additional rules that the final tree must be finitely branching and have only finitely many branching nodes characterizes the class of functions which are piecewise continuous on a* $\mathbf{\Pi}^0_1$ *partition.*

Proof. Given a function f which is piecewise continuous on a $\mathbf{\Pi}^0_1$ partition $\langle A_n\,;\, n \in$

$\omega\rangle$, let ϑ be a winning strategy for $\mathbb{2}$ in the (labeled) tree game version of the backtrack game. Then the strategy ϑ' defined by letting $\vartheta'(\sigma)$ be the collection of all running labels of $\vartheta(\sigma)$ is easily seen to be winning for $\mathbb{2}$ in the unlabeled tree game described in question.

Conversely, let ϑ be a winning strategy for $\mathbb{2}$ in the unlabeled tree game described in question for f. Given $\sigma \in \omega^{<\omega}_{\neq 0}$, we define σ^* as the longest element of $\vartheta(\sigma)$ such that all elements of $\vartheta(\sigma)$ which are not elements of $\vartheta(\sigma \restriction n)$ for any $n < |\sigma|$ are compatible with σ^*. Let us say that ϑ *changes mind* about σ if $(\sigma^-)^* \nsubseteq \sigma^*$. Since ϑ is a winning strategy for $\mathbb{2}$, it follows that for any $x \in \mathrm{dom}(f)$ we have that ϑ changes mind about x some finite number N of times, i.e., ϑ changes mind about $x \restriction m$ for exactly N natural numbers m. Now let $A_n := \{x \in \mathrm{dom}(f)\,;\, \vartheta$ changes mind about x exactly n times$\}$. Then $f \restriction A_n$ is continuous, since a winning strategy for $\mathbb{2}$ in the Wadge game for $f \restriction A_n$ is to play $\langle\rangle$ until ϑ has changed mind n times, then follow ϑ. Finally, each A_n set is the intersection of an open and a closed set, and therefore a $\mathbf{\Sigma}^0_2(\mathrm{dom}(f))$ set, so by Corollary 1.27 we are done. □

In the remainder of the present chapter we will give a solution to Problem 2.22 for the class of Baire class α functions for each $\alpha < \omega_1$ (Theorem 2.31), and a construction which transforms a solution of Problem 2.22 for a class Λ into solution for the class of functions which are piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ partition, for any $\alpha > 0$ (Theorem 2.61). In particular we get games characterizing the class of functions which are piecewise Baire class β on a $\mathbf{\Pi}^0_\alpha$ partition, for any $\alpha, \beta < \omega_1$ with $\alpha > 0$ (Corollary 2.64).

In [69], Motto Ros also defined games characterizing these classes; however, he describes his game characterizing the Borel measurable functions as being "less informative than [the tree game]", and makes similar comments about his game construction which entails games characterizing the functions of any fixed Baire class, calling them "quite trivial" [69, p. 105]. Since they are direct modifications of Semmes's tree game, the games and constructions we define are certainly more in the spirit of what Motto Ros would call informative.

We should also like to mention that Louveau has announced that, in collaboration with Semmes, he has independently obtained a level-by-level analysis of Semmes's tree game (such an announcement was made, e.g., at a talk given by Louveau in Amsterdam in 2010 [62]); thus it is possible that game characterizations of the Baire classes were independently proved by Louveau and Semmes. Unfortunately, we do not know any more details about the Louveau-Semmes results, since their results are as yet unpublished and no written version has been made publicly available.

2.2 Games for each fixed Baire class

2.2.1 The relaxed tree game

Our first step towards modifying the tree game to characterize any given Baire class is noting that the requirement that the final tree have a *unique* infinite branch can be somewhat loosened, which will make some of the proofs to come easier, in particular when characterizing the class of Baire class α functions for even α. This modification is not essential; in Section 2.2.3 we describe how to modify our proofs so as to obtain corresponding results using restrictions of Semmes's original tree game.

2.25 Definition. The *relaxed tree game* is the modification of the tree game obtained by requiring that the final tree played by $\mathbb{2}$ must have at least one infinite branch but only one infinite running label. The interpretation function associates to the final tree its unique infinite running label.

To avoid confusion, in what follows we will refer to Semmes's original tree game, Definition 2.18, as the *strict* tree game.

2.26 Theorem. *The relaxed tree game characterizes the class of Borel measurable functions.*

Proof. If f is Borel measurable, then by Semmes's theorem (Theorem 2.19) it follows that there exists a winning strategy for $\mathbb{2}$ in the strict tree game for f, and that same strategy is of course also a winning strategy for $\mathbb{2}$ in the relaxed tree game for f.

Conversely, if ϑ is a winning strategy for $\mathbb{2}$ in the relaxed tree game for f, then for each $\sigma \in \omega^{<\omega}$ we have that

$$f^{-1}[\sigma] = \{x \in \mathrm{dom}(f)\,; \text{ there exists an infinite branch of } \Upsilon_x^\vartheta \\ \text{whose running label extends } \sigma\}$$

is an analytic set. Since $[\sigma]$ is clopen, this suffices to show that f is Borel measurable. □

We will give a *standalone* proof of Theorem 2.26, i.e., one that does not appeal to Semmes's theorem, as a consequence of our main theorem of this section, Theorem 2.31.

2.2.2 The pruning derivative

We now define the main operation on trees which will be used through most of this thesis.

2.27 Definition. Given a tree $T \subseteq \omega^{<\omega}$, the *pruning derivative* of T, denoted $\mathrm{PD}(T)$, is the tree

$$\mathrm{PD}(T) = \{\sigma \in T\,;\ \text{the subtree of } T \text{ rooted at } \sigma \text{ has infinite rank}\}.$$

As usual, we can define transfinite iterations of the derivative operation by the following recursion.

$$\begin{aligned} \mathrm{PD}^\star(T, 0) &:= T \\ \mathrm{PD}^\star(T, \alpha+1) &:= \mathrm{PD}(\mathrm{PD}^\star(T, \alpha)) \\ \mathrm{PD}^\star(T, \lambda) &:= \bigcap_{\alpha<\lambda} \mathrm{PD}^\star(T, \alpha) \qquad \text{for limit } \lambda > 0. \end{aligned}$$

Note that $\mathrm{PD}(T) = T$ iff T is a pruned tree and that for any tree $T \subseteq \omega^{<\omega}$ there exists some countable ordinal α such that $\mathrm{PD}^\star(T, \alpha)$ is a pruned tree.

2.28 Lemma (N. & Pauly). *For $\sigma \in T$, we have $\sigma \in \mathrm{PD}^\star(T, \alpha)$ iff $\mathrm{rk}_T(\sigma) \geqslant \omega \cdot \alpha$.*

Proof. By induction on α, the base case $\alpha = 0$ being immediate. For the limit case, we have

$$\begin{aligned} \sigma \in \mathrm{PD}^\star(T, \alpha) \ \text{ iff } \ & \sigma \in \mathrm{PD}^\star(T, \beta) \ \text{ for every } \beta < \alpha \\ \text{iff } \ & \mathrm{rk}_T(\sigma) \geqslant \omega \cdot \beta \ \text{ for every } \beta < \alpha \\ \text{iff } \ & \mathrm{rk}_T(\sigma) \geqslant \omega \cdot \alpha. \end{aligned}$$

Finally, for the successor step, let $\alpha = \beta + 1$. Then

$$\begin{aligned} \sigma \in \mathrm{PD}^\star(T, \alpha) \ \text{ iff } \ & \forall n \in \omega \exists \tau \in \mathrm{PD}^\star(T, \beta)(\sigma \subset \tau \wedge |\tau| \geqslant |\sigma| + n) \\ \text{iff } \ & \forall n \in \omega \exists \tau \in T(\mathrm{rk}_T(\tau) \geqslant \omega \cdot \beta \wedge \sigma \subset \tau \wedge |\tau| \geqslant |\sigma| + n) \\ \text{iff } \ & \mathrm{rk}_T(\sigma) \geqslant \omega \cdot \beta + \omega = \omega \cdot \alpha, \end{aligned}$$

which concludes the proof. □

As indicated in Remark 1.15, we can extend these operations to labeled trees by letting $\mathrm{PD}^\star((T, \varphi), \alpha)$ be the subtree of (T, φ) induced by $\mathrm{PD}^\star(T, \alpha)$, for each ordinal α.

2.29 Corollary. *If $\Upsilon \rightleftarrows \Upsilon'$ then $\mathrm{PD}^\star(\Upsilon, \alpha) \rightleftarrows \mathrm{PD}^\star(\Upsilon', \alpha)$ for any $\alpha < \omega_1$.*

Proof. Follows directly from Lemmas 1.17 and 2.28. □

2.2.3 The α-tree game

Given an ordinal $\alpha = \lambda + n$, we define two ordinals $\alpha_{\downarrow}$ and $\alpha_{\twoheaddownarrow}$ by letting

$$\begin{aligned} \alpha_{\downarrow} &= \lambda + \lceil \tfrac{n}{2} \rceil \\ \alpha_{\twoheaddownarrow} &= \lambda + \lfloor \tfrac{n}{2} \rfloor. \end{aligned}$$

Clearly, $\alpha_{\twoheaddownarrow} \leqslant \alpha_{\downarrow} \leqslant \alpha_{\twoheaddownarrow} + 1$, and α is even iff $\alpha_{\twoheaddownarrow} = \alpha_{\downarrow}$.

We are now ready to define the game which characterizes the class of Baire class α functions.

2.30 DEFINITION. The *α-tree game* is the relaxed tree game with additional rule that the final tree Υ built by player $\mathbf{2}$ must satisfy that

(1) $\mathrm{PD}^\star(\Upsilon, \alpha{\downarrow}))$ is bisimilar to a finitely branching tree; and

(2) $\mathrm{PD}^\star(\Upsilon, \alpha{\downarrow}))$ is bisimilar to a linear tree.

2.31 THEOREM. *The α-tree game characterizes the class of Baire class α functions.*

As usual for game characterizations, one direction of Theorem 2.31 is significantly easier to prove.

2.32 LEMMA. *Given a strategy ϑ for $\mathbf{2}$ in the relaxed tree game, a limit ordinal λ, a natural number n, and $\sigma \in \omega^{<\omega}$, let $X(\vartheta, \sigma, \lambda+n) := \{x \in \omega^\omega\,;\, \sigma \in \mathrm{PD}^\star(\Upsilon^\vartheta_x, \lambda+n)\}$. Then $X(\vartheta,\sigma,0) \in \mathbf{\Sigma}^0_1$ and $X(\vartheta,\sigma,\lambda+n) \in \mathbf{\Pi}^0_{\lambda+2n}$ for $\lambda+n>0$.*

Proof. We have $x \in X(\vartheta,\sigma,0)$ iff there exists a round r at which ϑ tells $\mathbf{2}$ to add σ to her tree when following ϑ against x, which is a $\mathbf{\Sigma}^0_1$ condition.

The other cases are done by induction on $\lambda+n$. For the base case $\lambda+n=1$, note that $\sigma \in \mathrm{PD}^\star(\Upsilon^\vartheta_x, 1)$ iff for all $d \in \omega$ there exists $\sigma' \in \omega^{<\omega}$ such that $\sigma \subseteq \sigma'$, $|\sigma'| \geqslant d$, and $x \in X(\vartheta,\sigma',0)$. By the previous case of the proof, it then follows that $X(\vartheta,\sigma,1) \in \mathbf{\Pi}^0_2$. For the limit case $\lambda>0$ and $n=0$, note that $x \in X(\vartheta,\sigma,\lambda)$ iff $x \in X(\vartheta,\sigma,\gamma)$ for all $\gamma<\lambda$. By induction hypothesis we have $X(\vartheta,\sigma,\gamma) \in \mathbf{\Pi}^0_{<\lambda} = \mathbf{\Sigma}^0_{<\lambda}$ for each $\gamma<\lambda$, so $X(\vartheta,\sigma,\lambda) \in \mathbf{\Pi}^0_\lambda$. The successor step $\lambda+n+1$ is similar to the base case; we have $x \in X(\vartheta,\sigma,\lambda+n+1)$ iff for all d there exists σ' such that $\sigma \subseteq \sigma'$, $|\sigma'| \geqslant d$, and $x \in X(\vartheta,\sigma',\lambda+n)$. By induction hypothesis we have $X(\vartheta,\sigma',\lambda+n) \in \mathbf{\Pi}^0_{\lambda+2n}$, so it now follows that $X(\vartheta,\sigma,\lambda+n+1) \in \mathbf{\Pi}^0_{\lambda+2n+2}$ as desired. $\square$

Partial proof of Theorem 2.31. Let ϑ be a winning strategy for $\mathbf{2}$ in the α-tree game, and let us prove that $f^{-1}[\sigma] \in \mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$ holds for each $\sigma \in \omega^{<\omega}_{\neq 0}$.

Note that Lemma 2.32 immediately gives the desired result in case $\alpha = \lambda+2n$, since then $\alpha{\downarrow} = \lambda+n$ and since by assumption we have that $\mathrm{PD}^\star(\Upsilon^\vartheta_x, \alpha{\downarrow})$ is bisimilar to an infinite linear tree, it follows that for each $\sigma \in \omega^{<\omega}$ we have $\sigma \subset f(x)$ iff there exists $\tau \in \mathrm{PD}^\star(\Upsilon^\vartheta_x, \lambda+n)$ such that $\tilde{\varphi}_x(\tau) = \sigma$. From this we get $f^{-1}[\sigma] \in \mathbf{\Sigma}^0_{\lambda+2n+1}$, as desired.

For the case $\alpha = \lambda+2n+1$, note that since $\overline{\Upsilon^\vartheta_x} := \mathrm{PD}^\star(\Upsilon^\vartheta_x, \alpha{\downarrow})$ is bisimilar to a finitely branching tree for each $x \in \mathrm{dom}(f)$, it follows that for each such x and $m \in \omega$ there exists $H \in \omega$ such that any node of $\overline{\Upsilon^\vartheta_x}$ of length m is either on the infinite branch of $\overline{\Upsilon^\vartheta_x}$ or has no descendants in $\overline{\Upsilon^\vartheta_x}$ of length at least H. Therefore we have $\sigma \subset f(x)$ iff there exists $H \in \omega$ such that for any $\tau \in \mathrm{Level}(\Upsilon^\vartheta_x, H)$, either $\sigma \subset \tilde{\varphi}_x(\tau)$ or $\tau \notin \overline{\Upsilon^\vartheta_x}$. From this we get $f^{-1}[\sigma] \in \mathbf{\Sigma}^0_{\lambda+2n+2}$, as desired. $\square$

The remainder of this section is dedicated to the proof of the converse direction, so let $f : \omega^\omega \dashrightarrow \omega^\omega$ be a given function of Baire class α and let us define a winning strategy for $\mathbf{2}$ in the α-tree game for f.

Idea of the proof. The heart of the idea behind the proof is the following. Given $\sigma \in \omega^{<\omega}$ and $x \in \operatorname{dom}(f)$, deciding whether $\sigma \subset f(x)$ holds is equivalent to deciding the membership of x in a certain $\mathbf{\Sigma}^0_{\alpha+1}$ set which we denote $[\![\sigma]\!]$. Via diligent coding, we make each node τ in $\omega^{<\omega}$ with length $n > 0$ encode a guess for a node $\sigma \in \omega^n$, intentionally claiming that $\sigma \subset f(x)$, i.e., that $x \in [\![\sigma]\!]$ will hold for the x that player $\mathbb{1}$ is building. If $[\![\sigma]\!]$ is an open set, then we can make sure to only add τ to the tree Υ we are building when this claim proves to be true; otherwise, if $[\![\sigma]\!]$ is more complex, then there exist countably many sets from lower levels of the Borel hierarchy, which we denote $[\![\langle\sigma\rangle \oplus n]\!]$ for $n \in \omega$, such that $[\![\sigma]\!] = \bigcup_{n\in\omega}[\![\langle\sigma\rangle \oplus n]\!]$. Then our diligent coding will also ensure that τ encodes a guess for some $n \in \omega$, again intentionally claiming that $x \in [\![\langle\sigma\rangle \oplus n]\!]$ will hold. If $[\![\langle\sigma\rangle \oplus n]\!]$ is a closed set and at some stage of the construction of Υ we find that $x \notin [\![\langle\sigma\rangle \oplus n]\!]$ holds, then we will cease adding extensions of τ to Υ. Otherwise, if $[\![\langle\sigma\rangle \oplus n]\!]$ is more complex, then again there exist countably many sets from lower levels of the Borel hierarchy, which we denote $[\![\langle\sigma\rangle \oplus n \oplus m]\!]$ for $m \in \omega$, such that $[\![\langle\sigma\rangle \oplus n]\!] = \bigcap_{m\in\omega}[\![\langle\sigma\rangle \oplus n \oplus m]\!]$. In this case, the task of verifying or falsifying the claim $x \in [\![\langle\sigma\rangle \oplus n]\!]$ is distributed among the descendants of τ in Υ. For each $m \in \omega$, if $[\![\langle\sigma\rangle \oplus n \oplus m]\!]$ is an open set, then we only add descendants of τ of length $|\tau| + m + 1$ to the tree after $x \in [\![\langle\sigma\rangle \oplus n \oplus m]\!]$ is proven to be true. Otherwise, if $[\![\langle\sigma\rangle \oplus n \oplus m]\!]$ is more complex, then again there exist sets from lower in the Borel hierarchy, which we denote $[\![\langle\sigma\rangle \oplus n \oplus m \oplus k]\!]$, such that $[\![\langle\sigma\rangle \oplus n \oplus m]\!] = \bigcup_{k\in\omega}[\![\langle\sigma\rangle \oplus n \oplus m \oplus k]\!]$. Via coding we make it so that each descendant τ' of τ in $\omega^{<\omega}$ of length $|\tau| + m + 1$ encodes a guess for some k, intentionally claiming $x \in [\![\langle\sigma\rangle \oplus n \oplus m \oplus k]\!]$, and the whole process is then iterated. Then we will have that only nodes of Υ which only make correct guesses survive $\alpha_\downarrow$ pruning derivatives, which will imply that $\mathrm{PD}^\star(\Upsilon, \alpha_\downarrow)$ is bisimilar to a linear tree.

In case α is odd, one additional caveat is that we want to make sure that the first guess made by τ, viz. of some σ and n intentionally claiming $x \in [\![\sigma \oplus n]\!]$, is actually the least witness for $x \in \bigcup_{n'\in\omega}[\![\sigma \oplus n']\!]$. Thus, for each $n' < n$, denoting $[\![\sigma \ominus n']\!] = \omega^\omega \smallsetminus [\![\sigma \oplus n']\!]$, the node τ also makes the claim that $x \in [\![\sigma \ominus n']\!]$. Once more, if $[\![\sigma \ominus n']\!]$ is an open set, then we only add τ to the tree after this claim is proven true; otherwise, there exist sets from lower levels of the Borel hierarchy, which we denote $[\![\sigma \ominus n' \oplus m]\!]$, satisfying $[\![\sigma \ominus n']\!] = \bigcup_{m\in\omega}[\![\sigma \ominus n' \oplus m]\!]$, and τ encodes a guess for some m intentionally claiming $[\![\sigma \ominus n' \oplus m]\!]$. This idea is then iterated, as above. As we will see, if $x \in [\![\sigma \ominus n']\!]$ actually holds for some $n' < n$, then τ will not survive $\alpha_\downarrow$ derivatives of the tree. This will imply that $\mathrm{PD}^\star(\Upsilon, \alpha_\downarrow)$ is bisimilar to a finitely branching tree.

Let us start by defining

$$\mathbb{T} := \{\langle\sigma, n_0, \ldots, n_{k-1}\rangle \,;\, \sigma \in \omega^{<\omega},\ k \in \omega, \text{ and } n_i \in \omega \text{ for each } i < k\}.$$

Elements of $\mathbb{T}$ are called *trails*, and for a nonempty trail $t \in \mathbb{T}$ the sequence $t(0)$

is called the *owner* of t. Given $u \in \mathbb{T} \cup \omega^{<\omega}$ and $n \in \omega$, let

$$\begin{aligned} u \oplus n &= u^\frown\langle 2n\rangle \\ u \ominus n &= u^\frown\langle 2n+1\rangle. \end{aligned}$$

In an intuitive sense, trails provide *paths* through certain families of Borel codes in the following definition. Let $g : \omega^\omega \dashrightarrow \omega^\omega$. A *complete system* for g is a triple $\mathscr{C} = (\mathscr{C}^\Sigma, \mathscr{C}^\Pi, [\![\cdot]\!])$ where $\mathscr{C}^\Sigma$ and $\mathscr{C}^\Pi$ are disjoint subsets of $\mathbb{T}$ and $[\![\cdot]\!] : \mathscr{C}^\Sigma \cup \mathscr{C}^\Pi \to \mathcal{P}(\omega^\omega)$, such that $\mathscr{C}$ satisfies the following properties.

(C1) For each $t \in \mathscr{C}^\Sigma \cup \mathscr{C}^\Pi$ the set $[\![t]\!]$ is a Borel subset of ω^ω; we define the *Borel rank* of t, denoted $\mathrm{rk}(t)$, as the least ordinal β such that $[\![t]\!] \in \mathbf{\Sigma}^0_\beta$, if $t \in \mathscr{C}^\Sigma$, or such that $[\![t]\!] \in \mathbf{\Pi}^0_\beta$, if $t \in \mathscr{C}^\Pi$.

(C2) For each $\sigma \in \omega^{<\omega}_{\neq 0}$ we have $\langle\sigma\rangle \in \mathscr{C}^\Sigma$, with $[\![\sigma]\!]$ of least possible Borel rank such that $g^{-1}[\sigma] = [\![\sigma]\!] \cap \mathrm{dom}(g)$.

(C3) For each $\sigma \in \omega^{<\omega}_{\neq 0}$, if $\mathrm{rk}(\sigma) = 1$ then for each $n \in \omega$ we have $\langle\sigma\rangle \oplus n \in \mathscr{C}^\Sigma$ and $[\![\langle\sigma\rangle \oplus n]\!] = [\![\sigma]\!]$.

(C4) For each $t \in \mathscr{C}^\Sigma$ with $\mathrm{rk}(t) > 1$ and each $n \in \omega$ we have $t \oplus n \in \mathscr{C}^\Pi$ with $\mathrm{rk}(t \oplus n) < \mathrm{rk}(t)$ and $[\![t]\!] = \bigcup_{n\in\omega} [\![t \oplus n]\!]$.

(C5) For each $t \in \mathscr{C}^\Pi$ with $\mathrm{rk}(t) > 1$ and each $n \in \omega$ we have $t \oplus n \in \mathscr{C}^\Sigma$ with $\mathrm{rk}(t \oplus n) < \mathrm{rk}(t)$ and $[\![t]\!] = \bigcap_{n\in\omega} [\![t \oplus n]\!]$.

(C6) If α is odd, then for each σ in $\omega^{<\omega}_{\neq 0}$ and each $n \in \omega$ we have $\langle\sigma\rangle \ominus n \in \mathscr{C}^\Sigma$ with $[\![\langle\sigma\rangle \ominus n]\!] = \omega^\omega \smallsetminus [\![\langle\sigma\rangle \oplus n]\!]$.

(C7) For any $\mathscr{C}^\Sigma_0 \subseteq \mathscr{C}^\Sigma$ and $\mathscr{C}^\Pi_0 \subseteq \mathscr{C}^\Pi$ such that $\mathscr{C}^\Sigma_0 \cup \mathscr{C}^\Pi_0 \subset \mathscr{C}^\Sigma \cup \mathscr{C}^\Pi$, one of the conditions (C1)–(C6) above is not satisfied for $(\mathscr{C}^\Sigma_0, \mathscr{C}^\Pi_0, [\![\cdot]\!]\restriction(\mathscr{C}^\Sigma_0 \cup \mathscr{C}^\Pi_0))$.

Using the axiom of choice, the fact that there exists a complete system for any Borel measurable function is straightforward. However, without assuming (countable) choice, the very definition of Borel sets and the Borel hierarchy is more involved, since, e.g., it is consistent with ZF that every subset of ω^ω is a countable union of countable sets [36]. Since singletons are $\mathbf{\Pi}^0_1$ sets, countable sets are $\mathbf{\Sigma}^0_2$, and countable unions of countable sets are therefore $\mathbf{\Sigma}^0_4$—recall that without the axiom of countable choice we cannot assume that the class $\mathbf{\Sigma}^0_2$ is closed under countable unions. Hence, with the naive definition of the Borel hierarchy, in the Feferman-Lévy model of [36] every Borel measurable function would be of Baire class 3. In this context, the more interesting notion is that of a Borel *code* of a set (cf., e.g., [54, Section 35.B]), and indeed in our case we have complete systems for those functions for which the preimage of any basic open has a $\mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$ Borel code, and our proof goes through in that setting.

We will use a fixed complete system $\mathscr{C} = (\mathscr{C}^\Sigma, \mathscr{C}^\Pi, [\![\cdot]\!])$ for f in the remainder of this section.

The associating, coding, guessing, unraveling, and witnessing functions

In this section we will define the main technical machinery which will be used to define the winning strategy for player $\mathbb{2}$ in the α-tree game for f. Our main objective here is to define a *guessing function* $\mathfrak{g} : \mathbb{I} \dashrightarrow \omega$, where $\mathbb{I} := \{t \in \mathbb{T}\,;\, |t| \geqslant 2 \text{ and } 0 < t(1) \leqslant |t(0)|\}$ is called the set of *inherited trails*, the intuition being that a trail $t \in \mathbb{I}$ is built from other trails which were owned by predecessors of $t(0)$, ultimately having its origin in some trail owned by $t(0)\restriction t(1)$. However, in defining $\mathfrak{g}$ we will define and use several auxiliary functions, viz. an *associating function* $\mathfrak{a} : \mathbb{I} \dashrightarrow \mathscr{C}$, a *coding function* $\mathfrak{c}_n : \{\sigma \in \omega^{<\omega}\,;\, |\sigma| \geqslant n\} \to \omega$ for each $n > 0$, a *witnessing function* $\mathfrak{w} : \omega^{<\omega}_{\neq 0} \to \omega$, and a function $\varphi : \omega^{<\omega}_{\neq 0} \to \omega$, whose restrictions will be used as the labeling functions for the trees player $\mathbb{2}$ plays when following the strategy we define in the next section. If α is odd, then we will also define an *unraveling function* $\mathfrak{u} : \omega^{<\omega}_{\neq 0} \to \omega$.

Recall from Convention 1.8 that we have fixed bijections $\mathsf{bij} : \omega \to \omega^{<\omega}$ and $\mathsf{bij}_n : \omega \to \omega^n$ for each $n > 0$, and that the inverses of the bijections bij_n are collectively denoted $\ulcorner\,\lrcorner$. We denote by $\lhd$ the strict linear order induced on $\omega^{<\omega}$ by bij^{-1}, i.e., the linear order defined by $\sigma \lhd \tau$ iff $\mathsf{bij}^{-1}(\sigma) < \mathsf{bij}^{-1}(\tau)$.

We start by defining the functions $\mathfrak{c}_n$ by $\perp(\sigma) = \ulcorner \mathfrak{c}_1(\sigma), \mathfrak{c}_2(\sigma), \ldots, \mathfrak{c}_{|\sigma|}(\sigma)\urcorner$, and defining φ, $\mathfrak{w}$, and $\mathfrak{u}$ (in case α is odd) by

$$\mathfrak{c}_{|\sigma|}(\sigma) = \begin{cases} \ulcorner \varphi(\sigma), \mathfrak{w}(\sigma)\urcorner, & \text{if } \alpha \text{ is even} \\ \ulcorner\ulcorner \varphi(\sigma), \mathfrak{w}(\sigma)\urcorner, \mathfrak{u}(\sigma)\urcorner, & \text{if } \alpha \text{ is odd.} \end{cases} \tag{2.1}$$

The functions $\mathfrak{a}$ and $\mathfrak{g}$ will be defined by $\subset$-recursion on the first coordinate of their arguments, so let $\sigma \in \omega^{<\omega}_{\neq 0}$ and suppose all values of $\mathfrak{a}$ and $\mathfrak{g}$ for arguments starting with proper nonempty initial segments of σ have already been defined. We start by letting $\langle \sigma, |\sigma|\rangle \in \operatorname{dom}(\mathfrak{a}) \cap \operatorname{dom}(\mathfrak{g})$, setting

$$\begin{aligned} \mathfrak{a}(\sigma, |\sigma|) &:= \langle \tilde{\varphi}(\sigma)\rangle \\ \mathfrak{g}(\sigma, |\sigma|) &:= \ulcorner \varphi(\sigma), \mathfrak{w}(\sigma)\urcorner. \end{aligned}$$

We also let $\langle \sigma, |\sigma|\rangle \oplus \mathfrak{g}(\sigma, |\sigma|) \in \operatorname{dom}(\mathfrak{a})$, setting

$$\mathfrak{a}(\langle \sigma, |\sigma|\rangle \oplus \mathfrak{g}(\sigma, |\sigma|)) := \mathfrak{a}(\sigma, |\sigma|) \oplus \mathfrak{w}(\sigma).$$

If α is odd, then for each $m = \ulcorner p, q\urcorner < \mathfrak{g}(\sigma, |\sigma|)$ let $\langle \sigma, |\sigma|\rangle \ominus m \in \operatorname{dom}(\mathfrak{a})$, setting $\mathfrak{a}(\langle \sigma, |\sigma|\rangle \ominus m) := \langle \mathsf{bij}_{|\sigma|}(p)\rangle \ominus q$. Let

$$\mathcal{U}(\sigma) := \{m < \mathfrak{g}(\sigma, |\sigma|)\,;\, \operatorname{rk}(\mathfrak{a}(\langle \sigma, |\sigma|\rangle \ominus m)) > 1\}, \tag{2.2}$$

and if $\mathcal{U}(\sigma) \neq \varnothing$ then let $e_\sigma : |\mathcal{U}(\sigma)| \to \mathcal{U}(\sigma)$ be an increasing enumeration of $\mathcal{U}(\sigma)$ and define new values of $\mathfrak{a}$ and $\mathfrak{g}$ from $\mathfrak{u}(\sigma)$ as follows. First, for each $i < |\mathcal{U}(\sigma)|$ let $t_i := \langle \sigma, |\sigma|\rangle \ominus e_\sigma(i)$. Then read

$$\mathfrak{u}(\sigma) = \ulcorner \mathfrak{g}(t_0), \mathfrak{g}(t_1), \ldots, \mathfrak{g}(t_{|\mathcal{U}(\sigma)|-1})\urcorner. \tag{2.3}$$

We call the process of starting from the guess $\mathfrak{g}(\sigma, |\sigma|)$ and recursively defining new values of $\mathfrak{g}$ from $\mathfrak{u}$ the *unraveling* of the guess $\mathfrak{g}(\sigma, |\sigma|)$. Now, for each $i < |\mathcal{U}(\sigma)|$ let $t_i \oplus \mathfrak{g}(t_i) \in \mathrm{dom}(\mathfrak{a})$, setting $\mathfrak{a}(t_i \oplus \mathfrak{g}(t_i)) := \mathfrak{a}(t_i) \oplus \mathfrak{g}(t_i)$.

If $|\sigma| > 1$ then let $\tau \subset \sigma$ be nonempty and let $\ell := |\sigma| - |\tau| - 1$. Let

$$\mathrm{inh}(\sigma, \tau) := \{t \in \mathrm{dom}(\mathfrak{a}) \,;\, t(0) = \tau,\, \mathfrak{a}(t) \in \mathscr{C}^{\Pi},\, \mathrm{rk}(\mathfrak{a}(t)) > 1, \\ \text{and } \mathrm{rk}(\mathfrak{a}(t) \oplus \ell) > 1\}. \quad (2.4)$$

This notation is due to the fact that the elements of $\mathrm{inh}(\sigma, \tau)$ are those that σ *inherits* from τ, in the sense that σ must make new guesses about those elements in order to help verify or falsify some guess made by τ. Note that $|\mathrm{inh}(\sigma, \tau)|$ is finite by construction; therefore there exists a bijection $e_{\sigma,\tau} : |\mathrm{inh}(\sigma, \tau)| \to \mathrm{inh}(\sigma, \tau)$ which is increasing with respect to $\lhd$. For each $i < |\mathrm{inh}(\sigma, \tau)|$ let $t_i := \langle \sigma \rangle^\frown \mathrm{shift}(e_{\sigma,\tau}(i)) \oplus \ell \in \mathrm{dom}(\mathfrak{a})$, setting $\mathfrak{a}(t_i) := \mathfrak{a}(e_{\sigma,\tau}(i)) \oplus \ell$, and read new values of $\mathfrak{g}$ from $\mathfrak{c}_{|\tau|}(\sigma)$ by

$$\mathfrak{c}_{|\tau|}(\sigma) = \ulcorner \mathfrak{g}(t_0), \mathfrak{g}(t_1), \ldots, \mathfrak{g}(t_{|\mathrm{inh}(\sigma,\tau)|-1}) \urcorner. \quad (2.5)$$

Now, for each $i < |\mathrm{inh}(\sigma, \tau)|$ let $t_i \oplus \mathfrak{g}(t_i) \in \mathrm{dom}(\mathfrak{a})$, setting $\mathfrak{a}(t_i \oplus \mathfrak{g}(t_i)) := \mathfrak{a}(t_i) \oplus \mathfrak{g}(t_i)$.

This concludes the definition of $\mathfrak{a}$ and $\mathfrak{g}$ for arguments starting with σ; note that there are only finitely many of these.

Definition of the strategy ϑ

Suppose we are at round r of a run of the α-tree game for f, let τ be the sequence of natural numbers played by $\mathbb{1}$ so far, and let $T_{-1} := \{\langle\rangle\}$. Assume we have already defined the notion σ *is active at round* r' for all $\sigma \in \omega^{<\omega}_{\neq 0}$ and $r' < r$, and let us now define what it means for σ to be active at round r. Suppose $\sigma \in \omega^{<\omega}_{\neq 0}$ has been active at exactly N rounds strictly before r. We say that $t \in \mathrm{dom}(\mathfrak{g})$ *demands attention of* σ if

(D1) $t(0) = \sigma$ and $\mathfrak{a}(t \oplus \mathfrak{g}(t)) \in \mathscr{C}^{\Pi}$ has Borel rank 1; in this case, we say t is *satisfied at round* r if $[\tau{\restriction}(N+1)] \cap [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!] \neq \varnothing$;

(D2) $t(0) = \sigma$ and $\mathfrak{a}(t \oplus \mathfrak{g}(t)) \in \mathscr{C}^{\Sigma}$ has Borel rank 1; in this case, we say t is *satisfied at round* r if $[\tau] \subseteq [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!]$;

(D3) α is odd, $t = \langle \sigma, |\sigma| \rangle$, and there exists $n < \mathfrak{g}(t)$ such that $\mathfrak{a}(t \ominus n) \in \mathscr{C}^{\Sigma}$ has Borel rank 1; in this case, we say t is *satisfied at round* r if $[\tau] \subseteq [\![\mathfrak{a}(t \ominus n)]\!]$ holds for each such n; or

(D4) $t(0) \subset \sigma$ and with $\ell = |\sigma| - |t(0)| - 1$ we have $\mathfrak{a}(t \oplus \mathfrak{g}(t) \oplus \ell) \in \mathscr{C}^{\Sigma}$ has Borel rank 1; in this case, we say t is *satisfied at round* r if $[\tau] \subseteq [\![\mathfrak{a}(t \oplus \mathfrak{g}(t) \oplus \ell)]\!]$.

Then we say that σ is *active at round* r if $\sigma = \langle\rangle$, or every $t \in \mathrm{dom}(\mathfrak{g})$ that demands attention of σ is satisfied at round r and every predecessor of σ has been active for at least $N+1$ rounds before (and possibly including) r.

We now define $\vartheta(\tau) := (T_r, \varphi_r)$, where $\varphi_r = \varphi\restriction T_r$ as usual and

$$T_r := T_{r-1} \cup \{\sigma \in \omega^{<\omega} \,;\, \mathsf{bij}^{-1}(\sigma) \leqslant r \text{ and } \sigma \text{ is active at round } r\}. \tag{2.6}$$

Analysis

Consider a run of the game in which $\mathbb{1}$ builds $x \in \mathrm{dom}(f)$ and $\mathbb{2}$ follows the strategy ϑ defined above, at each round r playing the labeled tree (T_r, φ_r), and let $T := \bigcup_r T_r$ and $\varphi := \bigcup_r \varphi_r$. The following is an easy consequence of the definitions.

2.33 Lemma. *The tree T is composed exactly of the elements of $\omega^{<\omega}$ which were active at some round of the game.*

We say that σ *makes a guess about* $t \in \mathrm{dom}(\mathfrak{g})$ if $t(0) = \sigma$, and in this case we say the guess is *right* or *wrong* depending whether x belongs or does not belong, respectively, to $[\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!]$, and the *rank* of the guess is the Borel rank of $\mathfrak{a}(t \oplus \mathfrak{g}(t))$.

2.34 Lemma. *If $\sigma \in T$ makes a wrong guess of rank β about $t \in \mathrm{dom}(\mathfrak{g})$, then $\sigma \notin \mathrm{PD}^\star(T, \beta{\downarrow})$. In particular $\sigma \notin \mathrm{PD}^\star(T, \alpha{\downarrow})$.*

Proof. Let $t' := t \oplus \mathfrak{g}(t)$. Recall that if $\beta = \mathrm{rk}(\mathfrak{a}(t')) > 1$ then $[\![\mathfrak{a}(t')]\!] = \bigcap_{n\in\omega}[\![\mathfrak{a}(t' \oplus n)]\!]$. Hence in case $\beta > 1$ there exists some least $n \in \omega$ such that $x \notin [\![\mathfrak{a}(t' \oplus n)]\!]$, and we let $\gamma = \mathrm{rk}(t' \oplus n) < \beta$. Note that if $\beta > 1$ and $\gamma = 1$, i.e., if $[\![\mathfrak{a}(t' \oplus n)]\!]$ is an open set, then for every $r \in \omega$ we have $[x\restriction r] \not\subseteq [\![\mathfrak{a}(t' \oplus n)]\!]$. Since t' demands attention of all descendants $\sigma' \in \omega^{<\omega}_{\neq 0}$ of σ with $|\sigma'| = |\sigma| + n + 1$ but is never satisfied, it follows that no such descendant is ever active. Therefore the subtree of T rooted at σ has rank at most $|\sigma| + n$, and thus $\sigma \notin \mathrm{PD}^\star(T, 1) \supseteq \mathrm{PD}^\star(T, \beta{\downarrow})$.

From now on we assume $\gamma > 1$ in the cases where $\beta > 1$, and the result will be proved by induction on β. The base case is $\beta = 1$, i.e., when $[\![\mathfrak{a}(t')]\!]$ is a closed set. Since by assumption we have $x \notin [\![\mathfrak{a}(t')]\!]$, there exists $r \in \omega$ such that $[x\restriction r] \cap [\![\mathfrak{a}(t')]\!] = \varnothing$. Thus from round r onwards σ is never active, implying that the subtree of T rooted at σ is finite, and therefore that $\sigma \notin \mathrm{PD}^\star(T, 1)$. For the induction step, let $\beta > 1$ and suppose the result holds for all ordinals $\xi < \beta$. Since $\gamma > 1$ and $x \notin [\![\mathfrak{a}(t' \oplus n)]\!]$, it follows that each descendant τ of σ of length $|\sigma| + n + 1$ makes a wrong guess about $t' \oplus n$. Therefore by the induction hypothesis for each such descendant τ there exists an ordinal $\xi_\tau < \gamma$ such that $\tau \notin \mathrm{PD}^\star(T, \xi_\tau{\downarrow}) \supseteq \mathrm{PD}^\star(T, \gamma{\downarrow})$. If $\gamma{\downarrow} < \beta{\downarrow}$ then we are done, for in this case $\sigma \notin \mathrm{PD}^\star(T, \gamma{\downarrow} + 1) \supseteq \mathrm{PD}^\star(T, \beta{\downarrow})$ as desired. Otherwise, if $\gamma{\downarrow} = \beta{\downarrow}$, then we have $\beta = \lambda + 2m + 2$ and $\gamma = \lambda + 2m + 1$ for some limit λ and natural m. In this case, for every $\xi < \gamma$ we have $\xi_\downarrow < \gamma_\downarrow$, thus no descendant τ of σ of length $|\sigma| + n + 1$ is in $\mathrm{PD}^\star(T, \gamma_\downarrow - 1)$, and we can conclude that $\sigma \notin \mathrm{PD}^\star(T, \gamma_\downarrow) = \mathrm{PD}^\star(T, \beta_\downarrow)$ as desired. □

If α is odd, then we say that σ *overshoots* if there exists $n < \mathfrak{g}(\sigma, |\sigma|)$ such that $x \notin [\![\mathfrak{a}(\langle \sigma, |\sigma| \rangle \ominus n)]\!]$. If α is even, then we stipulate by definition that no σ overshoots.

2.35 LEMMA. *If α is odd and σ overshoots then $\sigma \notin \mathrm{PD}^\star(T, \alpha_\downarrow)$.*

Proof. Let $\beta = \mathrm{rk}(\mathfrak{a}(\langle \sigma, |\sigma| \rangle \ominus n))$. If $\beta > 1$ then σ makes a wrong guess about $t' := \langle \sigma, |\sigma| \rangle \ominus n$, so letting $\gamma := \mathrm{rk}(\mathfrak{a}(t' \oplus \mathfrak{g}(t'))) < \alpha$ by Lemma 2.34 we have $\sigma \notin \mathrm{PD}^\star(T, \gamma_\downarrow) \supseteq \mathrm{PD}^\star(T, \alpha_\downarrow)$. Otherwise, if $\beta = 1$, then $t = \langle \sigma, |\sigma| \rangle$ demands attention of σ but is never satisfied, and therefore $\sigma \notin T \supseteq \mathrm{PD}^\star(T, \alpha_\downarrow)$. □

We say that $\sigma \in \omega^{<\omega}$ is *exact* if it makes no wrong guesses and does not overshoot, and we say that σ is *hereditarily exact* if σ and all its proper predecessors are exact.

2.36 LEMMA. *If σ is hereditarily exact then σ is activated at some round and never deactivated thereafter. In particular $\sigma \in T$.*

Proof. We prove this by induction on $|\sigma|$. Therefore suppose the result holds for all $\sigma \restriction n$ with $0 < n < |\sigma|$, and suppose σ is hereditarily exact. Let r_0 be the least round at which all $\sigma \restriction n$ with $0 < n < |\sigma|$ are already active. Suppose $t \in \mathrm{dom}(\mathfrak{g})$ demands attention of σ.

Case (D1): $t(0) = \sigma$ and $\mathfrak{a}(t \oplus \mathfrak{g}(t)) \in \mathscr{C}^\Pi$ has Borel rank 1. Then $x \in [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!]$ holds since σ is exact, therefore we have that t is satisfied at all rounds. In this case, let $r_t := r_0$.

Case (D2): $t(0) = \sigma$ and $\mathfrak{a}(t \oplus \mathfrak{g}(t)) \in \mathscr{C}^\Sigma$ has Borel rank 1. By construction, this can only happen in case $t = \langle \sigma, |\sigma| \rangle$ and $[\![\mathfrak{a}(t)]\!] = [\![\tilde{\varphi}(\sigma)]\!] = [\![\tilde{\varphi}(f(x) \restriction (n+1))]\!]$ is an open set. But then $[\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!] = [\![\tilde{\varphi}(f(x) \restriction (n+1))]\!]$ as well, so there exists a least $r_t \geqslant r_0$ such that t is satisfied at all rounds $r \geqslant r_t$.

Case (D3): α is odd, $t = \langle \sigma, |\sigma| \rangle$, and there exists $n < \mathfrak{g}(t)$ such that $\mathfrak{a}(t \ominus n) \in \mathscr{C}^\Sigma$ has Borel rank 1. Then since σ does not overshoot we have $x \in [\![\mathfrak{a}(t \ominus n)]\!]$. Hence there exists some least $r_t \geqslant r_0$ such that t is satisfied at all rounds $r \geqslant r_t$.

Case (D4): $t(0) \subset \sigma$ and with $\ell = |\sigma| - |t(0)| - 1$ we have $\mathfrak{a}(t \oplus \mathfrak{g}(t) \oplus \ell) \in \mathscr{C}^\Sigma$ has Borel rank 1. Then let $n < |\sigma|$ be such that $t(0) = \sigma \restriction n$. Since $\sigma \restriction n$ is exact, we have that

$$\begin{aligned} x &\in [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!] \\ &= \bigcap_{p \in \omega} [\![\mathfrak{a}(t \oplus \mathfrak{g}(t) \oplus p)]\!], \end{aligned}$$

so in particular $x \in [\![\mathfrak{a}(t \oplus \mathfrak{g}(t) \oplus \ell)]\!]$. Hence there exists some least $r_t \geqslant r_0$ such that t is satisfied at all rounds $r \geqslant r_t$. It now follows that σ is active at all rounds $r \geqslant \min\{r_t\,;\, t \text{ demands attention of } \sigma\}$. □

2.37 LEMMA. *If σ is hereditarily exact, then σ is part of some infinite branch of T. In particular, T has at least one infinite branch.*

Proof. By induction on $|\sigma|$, we will show that if σ is hereditarily exact then σ has an exact child in T. Since $\langle\rangle \in T$ is hereditarily exact, this suffices.

Thus suppose the result holds for any sequence shorter than σ, and suppose σ is hereditarily exact. Let $d := |\sigma| + 1$. We want to define a child σ' of σ, so our task is to define $\perp(\sigma')$. To this end, we define certain objects $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{c}}_n$ for $n \in \{1, \ldots, d\}$, as follows. The function $\overline{\mathfrak{g}}$ will be defined by a recursion mimicking the definition of $\mathfrak{g}$.

We start by defining $\overline{\mathfrak{c}}_d$. Let $m = f(x)(|\sigma|)$ and let $k \in \omega$ be least such that $x \in [\![\langle f(x)\restriction d\rangle \oplus k]\!]$. Then let $\langle d\rangle \in \mathrm{dom}(\overline{\mathfrak{g}})$, setting $\overline{\mathfrak{g}}(d) := \ulcorner m, k\urcorner$. Now, if α is odd let $\overline{\mathrm{S}} := \{\ulcorner p, q\urcorner < \overline{\mathfrak{g}}(d)\,;\, \mathrm{rk}(\langle \mathsf{bij}_d(p)\rangle \ominus q) > 1\}$. Note that since σ is hereditarily exact, in particular we have for any $\ulcorner p, q\urcorner \in \overline{\mathrm{S}}$

$$\begin{aligned} x &\notin [\![\langle \mathsf{bij}_d(p)\rangle \oplus q]\!] \\ &= \omega^\omega \smallsetminus [\![\langle \mathsf{bij}_d(p)\rangle \ominus q]\!], \end{aligned}$$

i.e.,

$$\begin{aligned} x &\in [\![\langle \mathsf{bij}_d(p)\rangle \ominus q]\!] \\ &= \bigcup_{r\in\omega} [\![\langle \mathsf{bij}_d(p)\rangle \ominus q \oplus r]\!]. \end{aligned}$$

Now, for each $\ulcorner p, q\urcorner \in \overline{\mathrm{S}}$ let $\langle d\rangle \ominus \ulcorner p, q\urcorner \in \mathrm{dom}(\overline{\mathfrak{g}})$, with $\overline{\mathfrak{g}}(\langle d\rangle \ominus \ulcorner p, q\urcorner)$ set to be any $r \in \omega$ such that $x \in [\![\langle \mathsf{bij}_d(p)\rangle \ominus q \oplus r]\!]$. If $\overline{\mathrm{S}} = \varnothing$ then let $\overline{\mathfrak{u}}$ be any natural number. Otherwise, let $e : |\overline{\mathrm{S}}| \to \overline{\mathrm{S}}$ be an increasing enumeration of $\overline{\mathrm{S}}$ and define $\overline{\mathfrak{u}} := \ulcorner \overline{\mathfrak{g}}(\langle d\rangle \ominus e(0)), \overline{\mathfrak{g}}(\langle d\rangle \ominus e(1)), \ldots, \overline{\mathfrak{g}}(\langle d\rangle \ominus e(|\overline{\mathrm{S}}| - 1))\urcorner$. Finally, let $\overline{\mathfrak{c}}_d := \ulcorner \overline{\mathfrak{g}}(d), \overline{\mathfrak{u}}\urcorner$.

Let $n \in \{1, \ldots, |\sigma|\}$ and $\ell := d - n - 1 = |\sigma| - n$. We will now define $\overline{\mathfrak{c}}_n$. Let

$$\begin{aligned} \overline{\mathrm{inh}}(n) := \{t \in \mathrm{dom}(\mathfrak{a})\,;\, t(0) = \sigma\restriction n,\ \mathfrak{a}(t) \in \mathscr{C}^\Pi,\ \mathrm{rk}(\mathfrak{a}(t)) > 1,& \\ \text{and } \mathrm{rk}(\mathfrak{a}(t) \oplus \ell) > 1\}.& \end{aligned}$$

If $\overline{\mathrm{inh}}(n) = \varnothing$ then let $\overline{\mathfrak{c}}_n$ be any natural number. Otherwise let $e_n : |\overline{\mathrm{inh}}(n)| \to \overline{\mathrm{inh}}(n)$ be an increasing bijection with respect to $\lhd$. For each $i < |\overline{\mathrm{inh}}(n)|$ let $\delta_i \in \omega^{<\omega}$ be such that $e_n(i) = \langle \sigma\restriction n\rangle^\frown \delta_i$. Now let $\delta'_i := \delta_i \oplus \ell \in \mathrm{dom}(\overline{\mathfrak{g}})$, letting $\overline{\mathfrak{g}}(\delta'_i)$ be any natural number such that $x \in [\![\mathfrak{a}(e_n(i) \oplus \ell) \oplus \overline{\mathfrak{g}}(\delta'_i)]\!]$. Note that such a natural number must indeed exist, since $\sigma\restriction n$ is exact. Finally, let $\overline{\mathfrak{c}}_n = \ulcorner \overline{\mathfrak{g}}(\delta'_0), \overline{\mathfrak{g}}(\delta'_1), \ldots, \overline{\mathfrak{g}}(\delta'_{|\overline{\mathrm{inh}}(n)|-1})\urcorner$ and $\sigma' := \sigma^\frown\langle \ulcorner \overline{\mathfrak{c}}_1, \overline{\mathfrak{c}}_2, \ldots, \overline{\mathfrak{c}}_d\urcorner\rangle$. Note that this gives

$$\begin{aligned} \mathfrak{c}_n(\sigma') &= \overline{\mathfrak{c}}_n && \text{for all } n \in \{1, \ldots, d\} \\ \mathfrak{g}(\langle\sigma'\rangle^\frown\delta) &= \overline{\mathfrak{g}}(\delta) && \text{for all } \delta \in \mathrm{dom}(\overline{\mathfrak{g}}). \end{aligned}$$

This concludes the definition of σ', which is easily seen to be exact by construction. Finally, by Lemma 2.36, it follows that $\sigma' \in T$. $\square$

2.38 THEOREM. *The tree* $\mathrm{PD}^\star((T,\varphi),\alpha{\downarrow})$ *is bisimilar to an infinite linear tree.*

Proof. By Lemma 2.37 we have that T has infinite branches, and by Lemma 2.34 all of them are labeled $f(x)$, since any element of T whose running label is not a prefix of $f(x)$ will necessarily make a wrong guess, and thus will not be a part of any infinite branches of T. It remains to be shown that $\mathrm{PD}^\star(T,\alpha{\downarrow})$ is pruned, so suppose $\sigma \in \mathrm{PD}^\star(T,\alpha{\downarrow})$ is not part of any infinite branches of $\mathrm{PD}^\star(T,\alpha{\downarrow})$. Therefore, σ is also not a part of any infinite branches of T, and thus by Lemma 2.37 we have that σ is not hereditarily exact. But then either σ or a proper predecessor σ' of σ makes a wrong guess or overshoots, and by Lemmas 2.34 and 2.35 it follows that $\sigma \notin \mathrm{PD}^\star(T,\alpha{\downarrow})$, a contradiction. Thus $\mathrm{PD}^\star((T,\varphi),\alpha{\downarrow})$ is an infinite pruned tree all of whose infinite branches have $f(x)$ as label, and it is easy to see that any such tree is bisimilar to an infinite linear tree. $\square$

In particular, if α is even then $\mathrm{PD}^\star((T,\varphi),\alpha\text{⤈}) = \mathrm{PD}^\star((T,\varphi),\alpha{\downarrow})$ and thus ϑ is a winning strategy for player $\mathbb{2}$ in the α-tree game for f, as desired. Therefore, from now on we assume that α is odd.

Define an equivalence relation $\equiv$ on $\omega^{<\omega}$ by $\sigma \equiv \tau$ iff

(1) $|\sigma| = |\tau|$;

(2) $\mathfrak{g}(\sigma,|\sigma|) = \mathfrak{g}(\tau,|\tau|)$;

(3) for all $\delta \in \omega^{<\omega}$ such that $|\delta| > 1$ and $t_\sigma := \langle\sigma\rangle^\frown\delta \in \mathrm{dom}(\mathfrak{g})$, letting $t_\tau := \langle\tau\rangle^\frown\delta$ one of the following holds:

   (a) $t_\tau \notin \mathrm{dom}(\mathfrak{g})$ and $x \in [\![\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma))]\!]$; or

   (b) $t_\tau \in \mathrm{dom}(\mathfrak{g})$, and $\mathfrak{g}(t_\sigma) = \mathfrak{g}(t_\tau)$ or $x \in [\![\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma))]\!] \cap [\![\mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau))]\!]$;

(4) for all $\delta \in \omega^{<\omega}_{\neq 0}$ such that $|\delta| > 1$ and $t_\tau := \langle\tau\rangle^\frown\delta \in \mathrm{dom}(\mathfrak{g})$, letting $t_\sigma := \langle\sigma\rangle^\frown\delta$ one of the following holds:

   (a) $t_\sigma \notin \mathrm{dom}(\mathfrak{g})$ and $x \in [\![\mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau))]\!]$; or

   (b) $t_\sigma \in \mathrm{dom}(\mathfrak{g})$, and $\mathfrak{g}(t_\sigma) = \mathfrak{g}(t_\tau)$ or $x \in [\![\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma))]\!] \cap [\![\mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau))]\!]$;

   and

(5) for all $n < |\sigma|$, recursively we have $\sigma{\restriction}n \equiv \tau{\restriction}n$.

Note that $\sigma \equiv \tau$ implies $\tilde{\varphi}(\sigma) = \tilde{\varphi}(\tau)$.

2.39 LEMMA. *If* $\sigma \equiv \tau$ *and* $\sigma' \supset \sigma$, *then there exists* $\tau' \supset \tau$ *such that* $\sigma' \equiv \tau'$.

Proof. It is enough to prove the result in case σ' is a direct child of σ, since the general case will follow by induction on $|\sigma'| - |\sigma|$. Therefore our task is to define $\perp(\tau')$. To this end, we proceed similarly to the proof of Lemma 2.37 and define certain objects $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{c}}_n$ for $n \in \{1,\dots,|\sigma'|\}$, as follows. First let $\bar{\mathfrak{c}}_{|\sigma'|} = \mathfrak{c}_{|\sigma'|}(\sigma')$.

The function $\overline{\mathfrak{g}}$ will be defined by a recursion mimicking the definition of $\mathfrak{g}$ and using σ'.

Let $n \in \{1, \ldots, |\tau|\}$ and $\ell := |\sigma'| - n - 1 = |\tau| - n$. Let

$$\begin{aligned}\overline{\mathrm{inh}}(n) := \{t \in \mathrm{dom}(\mathfrak{a})\,;\, t(0) = \tau\restriction n,\, \mathfrak{a}(t) \in \mathscr{C}^{\Pi},\, \mathrm{rk}(\mathfrak{a}(t)) > 1,\\ \text{and } \mathrm{rk}(\mathfrak{a}(t) \oplus \ell) > 1\}.\end{aligned}$$

If $\overline{\mathrm{inh}}(n) = \varnothing$ then let $\overline{\mathfrak{c}}_n$ be any natural number. Otherwise let $e_n : |\overline{\mathrm{inh}}(n)| \to \overline{\mathrm{inh}}(n)$ be an increasing bijection with respect to $\lhd$. For each $i < |\overline{\mathrm{inh}}(n)|$ let $\delta_i \in \omega^{<\omega}$ be such that $e_n(i) = \langle \tau\restriction n\rangle^\frown\delta_i$. Now let $\delta'_i := \delta_i \oplus \ell \in \mathrm{dom}(\overline{\mathfrak{g}})$, letting $\overline{\mathfrak{g}}(\delta'_i) := \mathfrak{g}(\langle\sigma'\rangle^\frown\delta'_i)$, if $\langle\sigma'\rangle^\frown\delta'_i \in \mathrm{dom}(\mathfrak{g})$, or letting $\overline{\mathfrak{g}}(\delta'_i)$ be any natural number such that $x \in [\![\mathfrak{a}(e_n(i) \oplus \ell) \oplus \overline{\mathfrak{g}}(\delta'_i)]\!]$, otherwise. Note that such a natural number must indeed exist, since in this case we have $e_n(i) = \langle\tau\restriction n\rangle^\frown\delta \oplus \mathfrak{g}(\langle\tau\restriction n\rangle^\frown\delta)$ for some $\delta \in \omega^{<\omega}_{\neq 0}$. Thus, since $\langle\sigma'\rangle^\frown\delta'_i \notin \mathrm{dom}(\mathfrak{g})$, we must either have $\langle\sigma\restriction n\rangle^\frown\delta \notin \mathrm{dom}(\mathfrak{g})$ or $\mathfrak{g}(\langle\tau\restriction n\rangle^\frown\delta) \neq \mathfrak{g}(\langle\sigma\restriction n\rangle^\frown\delta)$. Hence, since $\tau\restriction n \equiv \sigma\restriction n$, in either case we have

$$\begin{aligned} x &\in [\![\mathfrak{a}(e_n(i))]\!] \\ &= \bigcap_{m\in\omega} [\![\mathfrak{a}(e_n(i) \oplus m)]\!], \end{aligned}$$

and thus in particular

$$\begin{aligned} x &\in [\![\mathfrak{a}(e_n(i) \oplus \ell)]\!] \\ &= \bigcup_{m\in\omega} [\![\mathfrak{a}(e_n(i) \oplus \ell) \oplus m]\!] \end{aligned}$$

as desired. Finally, let

$$\begin{aligned} \overline{\mathfrak{c}}_n &= \ulcorner \overline{\mathfrak{g}}(\delta'_0), \overline{\mathfrak{g}}(\delta'_1), \ldots, \overline{\mathfrak{g}}(\delta'_{|\overline{\mathrm{inh}}(n)|-1})\urcorner \\ \tau' &= \tau^\frown\langle \ulcorner\overline{\mathfrak{c}}_1, \overline{\mathfrak{c}}_2, \ldots, \overline{\mathfrak{c}}_{|\sigma'|}\urcorner\rangle. \end{aligned}$$

Note that this gives

$$\begin{aligned} \mathfrak{c}_n(\tau') &= \overline{\mathfrak{c}}_n && \text{for all } n \in \{1, \ldots, |\tau'|\} \\ \mathfrak{g}(\langle\tau'\rangle^\frown\delta) &= \overline{\mathfrak{g}}(\delta) && \text{for all } \delta \in \mathrm{dom}(\overline{\mathfrak{g}}). \end{aligned}$$

This concludes the definition of τ'.

2.40 Claim. *$\sigma' \equiv \tau'$.*

Indeed, first note that $|\sigma'| = |\tau'|$, that $\mathfrak{g}(\sigma', |\sigma'|) = \mathfrak{g}(\tau', |\tau'|)$, and that $\sigma'\restriction n \equiv \tau'\restriction n$ for all $n < |\sigma'|$ hold immediately by construction.

Now let $\delta \in \omega^{<\omega}_{\neq 0}$ be such that $t_{\sigma'} := \langle\sigma'\rangle^\frown\delta \in \mathrm{dom}(\mathfrak{g})$ and let $t_{\tau'} := \langle\tau'\rangle^\frown\delta$. If $t_{\tau'} \in \mathrm{dom}(\mathfrak{g})$, then by construction we have $\mathfrak{g}(t_{\sigma'}) = \mathfrak{g}(t_{\tau'})$. Otherwise, there exist $n < |\sigma'|$ and $\delta' \in \omega^{<\omega}$ such that $\delta = \delta' \oplus \mathfrak{g}(\langle\sigma'\restriction n\rangle^\frown\delta') \oplus (|\sigma'| - n - 1)$. Thus, since $t_{\tau'} \notin \mathrm{dom}(\mathfrak{g})$, we must either have $\langle\tau'\restriction n\rangle^\frown\delta' \notin \mathrm{dom}(\mathfrak{g})$ or $\mathfrak{g}(\langle\sigma'\restriction n\rangle^\frown\delta') \neq$

$\mathfrak{g}(\langle\tau'\restriction n\rangle^\frown\delta')$. Hence, since $\tau'\restriction n \equiv \sigma'\restriction n$, in either case we have $x \in [\![\mathfrak{a}(t_{\sigma'} \oplus \mathfrak{g}(t_{\sigma'}))]\!]$, as desired.

Finally, let $\delta \in \omega^{<\omega}_{\neq 0}$ be such that $t_{\tau'} := \langle\tau'\rangle^\frown\delta \in \mathrm{dom}(\mathfrak{g})$ and let $t_{\sigma'} := \langle\sigma'\rangle^\frown\delta$. Now, we either have $t_{\sigma'} \in \mathrm{dom}(\mathfrak{g})$, in which case $\mathfrak{g}(t_{\tau'}) = \mathfrak{g}(t_{\sigma'})$ by construction, or $t_{\sigma'} \notin \mathrm{dom}(\mathfrak{g})$, in which case again by construction we have $x \in [\![\mathfrak{a}(t_{\tau'} \oplus \mathfrak{g}(t_{\tau'}))]\!]$, as desired. □

2.41 Lemma. *Let $\sigma \equiv \tau$ and suppose $\delta \in \omega^{<\omega}_{\neq 0}$ is such that $t_\sigma := \langle\sigma\rangle^\frown\delta \in \mathrm{dom}(\mathfrak{g})$, $t_\tau := \langle\tau\rangle^\frown\delta \in \mathrm{dom}(\mathfrak{g})$, and $\mathfrak{g}(t_\sigma) = \mathfrak{g}(t_\tau)$. Then*

$$\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) = \mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau)).$$

Proof. By induction on $|\sigma|$, the base case being vacuous. For the induction step, if $\delta(0) = |\sigma|$, then either $\delta = \langle|\sigma|\rangle$, in which case

$$\begin{aligned} \mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) &= \langle\tilde{\varphi}(\sigma)\rangle \oplus \mathfrak{w}(\sigma) \\ &= \langle\tilde{\varphi}(\tau)\rangle \oplus \mathfrak{w}(\tau) \\ &= \mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau)), \end{aligned}$$

or $\delta = \langle|\sigma|\rangle \ominus m$ for some $m = \ulcorner p, q\urcorner < \mathfrak{g}(\sigma, |\sigma|) = \mathfrak{g}(\tau, |\tau|)$. Note that $\mathfrak{a}(t_\sigma) = \langle\mathsf{bij}_{|\sigma|}(p)\rangle \ominus q = \mathfrak{a}(t_\tau)$, so

$$\begin{aligned} \mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) &= \mathfrak{a}(t_\sigma) \oplus \mathfrak{g}(t_\sigma) \\ &= \mathfrak{a}(t_\tau) \oplus \mathfrak{g}(t_\tau) \\ &= \mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau)). \end{aligned}$$

Otherwise, if $\delta(0) = n < |\sigma|$, then we have $\delta = \delta' \oplus \mathfrak{g}(\langle\sigma\restriction n\rangle^\frown\delta') \oplus (|\sigma| - n - 1)$, and since $t_\tau \in \mathrm{dom}(\mathfrak{g})$ we must have $\langle\tau\restriction n\rangle^\frown\delta' \in \mathrm{dom}(\mathfrak{g})$ and $\mathfrak{g}(\langle\sigma\restriction n\rangle^\frown\delta') = \mathfrak{g}(\langle\tau\restriction n\rangle^\frown\delta')$. Thus, by induction hypothesis, we get

$$\mathfrak{a}(\langle\sigma\restriction n\rangle^\frown\delta' \oplus \mathfrak{g}(\langle\sigma\restriction n\rangle^\frown\delta')) = \mathfrak{a}(\langle\tau\restriction n\rangle^\frown\delta' \oplus \mathfrak{g}(\langle\tau\restriction n\rangle^\frown\delta')),$$

and therefore

$$\begin{aligned} \mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) &= \mathfrak{a}(t_\sigma) \oplus \mathfrak{g}(t_\sigma) \\ &= \mathfrak{a}(\langle\sigma\restriction n\rangle^\frown\delta' \oplus \mathfrak{g}(\langle\sigma\restriction n\rangle^\frown\delta')) \oplus (|\sigma| - n - 1) \oplus \mathfrak{g}(t_\sigma) \\ &= \mathfrak{a}(\langle\tau\restriction n\rangle^\frown\delta' \oplus \mathfrak{g}(\langle\tau\restriction n\rangle^\frown\delta')) \oplus (|\tau| - n - 1) \oplus \mathfrak{g}(t_\tau) \\ &= \mathfrak{a}(t_\tau \oplus \mathfrak{g}(t_\tau)), \end{aligned}$$

as desired. □

2.42 Lemma. *If $\sigma \equiv \tau$ and σ is active for at least n rounds, then τ is also active for at least n (possibly different) rounds.*

Proof. By induction on $|\sigma|$, the base case being trivial since $\langle\rangle$ is always active. Now suppose the result holds for all $\sigma\restriction m$ with $m < |\sigma|$ and all $n \in \omega$, and let us prove that it holds for σ. This is also done by induction, the base case $n = 0$ being trivially true. Now let $n > 0$ and suppose σ is active for at least n rounds. Using both induction hypotheses there are a least round r_0 at which all predecessors of τ have been active at least n times and a round r_1 at which τ was active for the $(n-1)^{\text{th}}$ time, with $r_1 = -1$ in case $n = 1$. Suppose $t := \langle \tau\restriction m\rangle^\frown\delta \in \operatorname{dom}(\mathfrak{g})$ demands attention of τ, with $m \leqslant |\tau|$. If $t_\sigma := \langle \sigma\restriction m\rangle^\frown\delta \notin \operatorname{dom}(\mathfrak{g})$ or $t_\sigma \in \operatorname{dom}(\mathfrak{g})$ but $\mathfrak{g}(t_\sigma) \neq \mathfrak{g}(t)$, then since $\sigma\restriction m \equiv \tau\restriction m$ we must have $x \in [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!]$, and thus there is some least number r_t such that t is satisfied at all rounds $r \geqslant r_t$. Otherwise, if $\mathfrak{g}(t_\sigma) = \mathfrak{g}(t)$, then by Lemma 2.41 we have $\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) = \mathfrak{a}(t \oplus \mathfrak{g}(t))$. It then easily follows that t_σ demands attention of σ, and we now consider the four cases (D1)–(D4).

Case (D1), i.e., $t_\sigma(0) = \sigma$ and $\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) \in \mathscr{C}^\Pi$ has Borel rank 1. Then since σ is active for at least $n > 0$ rounds it follows that

$$\begin{array}{rcl} \varnothing & \neq & [x\restriction(n+1)] \cap [\![\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma))]\!] \\ & = & [x\restriction(n+1)] \cap [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!], \end{array}$$

so t is satisfied at all rounds up to the round at which τ is active for the n^{th} time, if such a round exists at all, otherwise t is always satisfied. Define $r_t := 0$.

Case (D2), i.e., $t_\sigma(0) = \sigma$ and $\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma)) \in \mathscr{C}^\Sigma$ has Borel rank 1. Then since σ is active for at least $n > 0$ rounds it follows that

$$\begin{array}{rcl} x & \in & [\![\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma))]\!] \\ & = & [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!], \end{array}$$

so again there is some least r_t such that t is satisfied at all rounds $r \geqslant r_t$.

Case (D3), i.e., α is odd, $t_\sigma = \langle \sigma, |\sigma|\rangle$, and there exists $\ulcorner p, q\urcorner < \mathfrak{g}(t_\sigma)$ such that $\mathfrak{a}(t_\sigma \ominus \ulcorner p, q\urcorner) \in \mathscr{C}^\Sigma$ has Borel rank 1. Note that $\mathfrak{a}(t_\sigma \ominus \ulcorner p, q\urcorner) = \langle \mathsf{bij}_{|\sigma|}(p)\rangle \ominus q = \mathfrak{a}(t \ominus \ulcorner p, q\urcorner)$, so since σ is active for at least $n > 0$ rounds it follows that there is some least r_t such that t_σ and t are satisfied at all rounds $r \geqslant r_t$.

Case (D4), i.e., $t_\sigma(0) = \sigma\restriction m$ with $m < |\tau|$ and setting $\ell := |\sigma| - m - 1$ we have $\mathfrak{a}(t_\sigma \oplus \mathfrak{g}(t_\sigma) \oplus \ell) \in \mathscr{C}^\Sigma$ with Borel rank 1. Then since σ is active for at least $n > 0$ rounds it follows that there is some least r_t such that t_σ and t are satisfied at all rounds $r \geqslant r_t$.

Now let $R := \max(\{r_0, r_1 + 1\} \cup \{r_t\,;\, t \text{ demands attention of } \tau\})$. Then τ is active for the n^{th} time at round R. □

2.43 Lemma. *For every ordinal β, the tree $\mathrm{PD}^\star(T, \beta)$ is closed under $\equiv$.*

Proof. For the base case $\beta = 0$, note that by Lemmas 2.33 and 2.42 we have that $\sigma \equiv \tau$ and $\sigma \in T$ imply $\tau \in T$. The limit step is immediate, so let us prove the successor step. Suppose $\sigma \equiv \tau$ and $\sigma \in \mathrm{PD}^\star(T, \beta + 1)$. Thus $\sigma \in \mathrm{PD}^\star(T, \beta)$, and therefore by induction hypothesis we have $\tau \in \mathrm{PD}^\star(T, \beta)$. Now let $n \in \omega$. Since $\sigma \in \mathrm{PD}^\star(T, \beta + 1)$, there exists $\sigma' \in \mathrm{PD}^\star(T, \beta)$ such that $\sigma \subset \sigma'$ and $|\sigma| + n + 1$. By Lemma 2.39, there exists $\tau' \supset \tau$ such that $\sigma' \equiv \tau'$, and thus by the induction hypothesis we have $\tau' \in \mathrm{PD}^\star(T, \beta)$, which concludes the proof. □

Now let $\Upsilon' := \mathrm{PD}^\star((T,\varphi),\alpha_\downarrow)$.

2.44 Lemma. *The restriction of $\equiv$ to $(\Upsilon' \times \Upsilon')$ is a bisimulation of Υ' with itself.*

Proof. We have already seen (label), condition (parent) is immediate, and by symmetry we have that (back) and (forth) are equivalent. So suppose $\sigma, \tau \in \Upsilon'$ are such that $\sigma \equiv \tau$ and let $\sigma' \in \Upsilon'$ be a child of σ. By Lemmas 2.39 and 2.43, it follows that there is a child τ' of τ in Υ' such that $\sigma' \equiv \tau'$, as desired. $\square$

2.45 Lemma. *For every $\sigma \in \Upsilon'$ and every child σ' of σ in Υ', there are only finitely many children σ'' of σ in Υ' such that $\sigma' \not\equiv \sigma''$.*

Proof. Let $n := |\sigma| + 1$, let m be such that $\mathsf{bij}_n(m) = f(x)\restriction n$, and let k be least such that $x \in [\![\langle f(x)\restriction n\rangle \oplus k]\!]$. Since by Lemma 2.35 no element of Υ' overshoots, we must have $\mathfrak{g}(\tau, n) \leqslant \ulcorner m, k \urcorner$ for any child τ of σ in Υ'. Now, by Lemma 2.34, for any $t \in \mathrm{dom}(\mathfrak{g})$ such that $t(0) \in \Upsilon'$ and $\mathrm{rk}(\mathfrak{a}(t \oplus \mathfrak{g}(t))) < \alpha$ we have $x \in [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!]$. Thus, all but possibly one clause in the definition of $\equiv$ are satisfied for any pair of children of σ in Υ'. Hence, if $\sigma' \not\equiv \sigma''$ are children of σ in Υ' then we must have $\mathfrak{g}(\sigma', n) \neq \mathfrak{g}(\sigma'', n)$, and since $\mathfrak{g}(\sigma', n), \mathfrak{g}(\sigma'', n) \leqslant \ulcorner m, k \urcorner$ this concludes the proof. $\square$

2.46 Lemma. *Let $\Upsilon = (T, \varphi)$ be a labeled tree and for simplicity denote $\rightleftarrows_{\Upsilon,\Upsilon}$ by $\rightleftarrows$. Define $\Upsilon_{\rightleftarrows} := (T_{\rightleftarrows}, \varphi_{\rightleftarrows})$ as the subtree of Υ induced by $T_{\rightleftarrows} = \{\sigma \in T\,;\, \nexists \tau \in T(\tau \rightleftarrows \sigma$ and $\tau <_{\mathrm{lex}} \sigma)\}$. Then $\Upsilon_{\rightleftarrows}$ is a labeled tree and $\Upsilon \rightleftarrows \Upsilon_{\rightleftarrows}$.*

Proof. We first prove that all initial segments of any $\sigma \in T_{\rightleftarrows}$ are in $T_{\rightleftarrows}$ by induction on $|\sigma|$. The cases $|\sigma| = 0$ and $|\sigma| = 1$ are immediate, so let $|\sigma| = n > 1$. By the induction hypothesis, it is enough to prove $\sigma^- \in T_{\rightleftarrows}$. So suppose for a contradiction that $\sigma^- \notin T_{\rightleftarrows}$, and let $\tau \rightleftarrows \sigma^-$ be such that $\tau <_{\mathrm{lex}} \sigma^-$. Then by (forth) for $\rightleftarrows$ there exists some child τ' of τ in T such that $\tau' \rightleftarrows \sigma$, a contradiction since $\tau' <_{\mathrm{lex}} \sigma$.

Finally, let B be the restriction of $\rightleftarrows$ to $T \times T_{\rightleftarrows}$ and suppose $\sigma \; B \; \tau$. Conditions (label), (parent), and (back) are easily seen to hold. To see that (forth) holds, let $\sigma' \in T$ be such that $\sigma \subset \sigma'$. By the (forth) condition for $\rightleftarrows$, the set $X = \{\tau' \in T\,;\, \tau \subset \tau'$ and $\sigma' \rightleftarrows \tau'\}$ is not empty, so let τ' be the $<_{\mathrm{lex}}$-least element in X.

2.47 Claim. $\tau' \in T_{\rightleftarrows}$.

Indeed, suppose not, and let $\tau'' \in T$ be such that $\tau'' \rightleftarrows \tau'$ and $\tau'' <_{\mathrm{lex}} \tau'$. Now (parent) for $\rightleftarrows$ implies that $\tau' \rightleftarrows \tau''\restriction(|\tau|)$, but $\tau \not\subset \tau''$ since $\tau'' \notin X$. Therefore we have $\tau''\restriction(|\sigma|) <_{\mathrm{lex}} \tau$ which contradicts the fact that $\tau \in T_{\rightleftarrows}$.

Hence $\sigma' \; B \; \tau'$ and that concludes the proof. $\square$

2.48 Corollary. *If α is odd then $\mathrm{PD}^\star((T,\varphi),\alpha_\downarrow)$ is bisimilar to a finitely branching tree.*

This concludes the proof of Theorem 2.31.

We can now give a new proof of Theorem 2.26 without appealing to Semmes's theorem, Theorem 2.19.

2.49 Corollary (Theorem 2.26). *The relaxed tree game characterizes the class of Borel measurable functions.*

Proof. By Corollary 1.23, any Borel measurable function is of Baire class α for some $\alpha < \omega_1$, so $\mathbb{2}$ has a winning strategy in the α-tree game for f, which of course also wins the relaxed tree game for f. The converse direction of the original proof of Theorem 2.26 did not use Semmes's theorem, so we are done. $\square$

Requiring unique infinite branches

The more forgiving condition for the infinite branches of the final tree in the relaxed tree game is merely a convenience; the proof of Theorem 2.31 can be modified in a straightforward way to show the following.

2.50 Theorem. *The class of Baire class α functions is characterized by the strict tree game with the additional rule that the final tree Υ built by $\mathbb{2}$ must have* $\mathrm{PD}^\star(\Upsilon, \alpha_\downarrow)$ *finitely branching and* $\mathrm{PD}^\star(\Upsilon, \alpha_\downarrow)$ *linear.*

Sketch of proof. The modification is a natural, if slightly cumbersome, extension of an idea already present in the odd case of the proof of Theorem 2.31, viz. that of unraveling guesses. Recall that, in that proof, we get $\mathrm{PD}^\star(\Upsilon, \alpha_\downarrow)$ to be bisimilar to a finitely branching tree by controlling guesses of rank α, using condition (C6) and the unraveling function $\mathfrak{u}$ as defined in (2.1). A node σ of Υ encodes a guess $\mathfrak{g}(\sigma, |\sigma|) = \ulcorner \phi(\sigma), \mathfrak{w}(\sigma) \urcorner$ for its own label $\phi(\sigma)$ and corresponding witness, intentionally claiming that

(1) we have $x \in [\![\langle \tilde{\varphi}(\sigma) \rangle \oplus \mathfrak{w}(\sigma)]\!]$, and

(2) for each $p, q \in \omega$ such that $\ulcorner p, q \urcorner < \mathfrak{g}(\sigma, |\sigma|)$, we have $x \notin [\![\mathsf{bij}_{|\sigma|}(p) \oplus q]\!]$, i.e., $x \in [\![\mathsf{bij}_{|\sigma|}(p) \ominus q]\!]$

Thus, for each such $\ulcorner p, q \urcorner < \ulcorner \varphi(\sigma), \mathfrak{w}(\sigma) \urcorner$ for which $[\![\mathsf{bij}_{|\sigma|}(p) \ominus q]\!]$ has Borel rank > 1, the node σ also encodes a guess of a natural number n intentionally claiming that $x \in [\![\mathsf{bij}_{|\sigma|}(p) \ominus q \oplus n]\!]$. The collection of all these guesses is coded into $\mathfrak{u}(\sigma)$ by (2.3). This ensures that, if $x \in [\![\mathsf{bij}_{|\sigma|}(p) \oplus q]\!]$ actually holds for some $\ulcorner p, q \urcorner < \ulcorner \varphi(\sigma), \mathfrak{w}(\sigma) \urcorner$, then σ makes a wrong guess of rank $< \alpha$, and therefore $\sigma \notin \mathrm{PD}^\star(\Upsilon, \alpha_\downarrow)$.

To prove Theorem 2.50 we unravel *all* guesses made by nodes of Υ. There will no longer be a case distinction between even and odd α. We start by tweaking the definition of a complete system: (C6) becomes

(C6′) For each $t \in \mathscr{C}^\Sigma$ with $\mathrm{rk}(t) > 1$ and each $n \in \omega$ we have $t \ominus n \in \mathscr{C}^\Sigma$ with $[\![t \ominus n]\!] = \omega^\omega \smallsetminus [\![t \oplus n]\!]$.

We will change $\mathfrak{u}$ so as to have $\mathrm{dom}(\mathfrak{u}) = \mathrm{dom}(\mathfrak{g})$; the definition is by recursion, with the basic cases coming from changing (2.1) and (2.5) to

$$\begin{aligned} \mathfrak{c}_{|\sigma|}(\sigma) &= \ulcorner\ulcorner\phi(\sigma), \mathfrak{w}(\sigma)\urcorner, \mathfrak{u}(\sigma, |\sigma|)\urcorner \\ &= \ulcorner\mathfrak{g}(\sigma, |\sigma|), \mathfrak{u}(\sigma, |\sigma|)\urcorner, \end{aligned} \tag{2.1$'$}$$

$$\mathfrak{c}_{|\tau|}(\sigma) = \ulcorner\mathfrak{g}(t_0), \mathfrak{u}(t_0), \mathfrak{g}(t_1), \mathfrak{u}(t_1), \dots, \mathfrak{g}(t_{|\mathrm{inh}(\sigma,\tau)|-1}), \mathfrak{u}(t_{|\mathrm{inh}(\sigma,\tau)|-1})\urcorner. \tag{2.5$'$}$$

Since before we were only using $\mathfrak{u}$ to unravel one guess for each σ, in (2.2) we only needed to define a single set $\mathcal{U}(\sigma)$; now, for each $t \in \mathrm{dom}(g)$ we define

$$\mathcal{U}(t) = \{m < \mathfrak{g}(t)\,;\, \mathrm{rk}(\mathfrak{a}(t) \ominus m) > 1\}. \tag{2.2$'$}$$

If $\mathcal{U}(t) = \varnothing$, then the value of $\mathfrak{u}(t)$ has no bearing on the definition of the winning strategy for $\mathbb{2}$. However, in order to have a tree with a unique infinite branch, we stipulate that if $\mathcal{U}(t) = \varnothing$ but $\mathfrak{u}(t) \neq 0$, then the owner σ of t is never active at any round of the game (and is therefore never added to the tree). If $\mathcal{U}(t) \neq \varnothing$, then we define $t_0, \dots, t_{|\mathcal{U}(t)|-1}$ as before, and (2.3) becomes

$$\mathfrak{u}(t) = \ulcorner\mathfrak{g}(t_0), \mathfrak{u}(t_0), \mathfrak{g}(t_1), \mathfrak{u}(t_1), \dots, \mathfrak{g}(t_{|\mathcal{U}(t)|-1}), \mathfrak{u}(t_{|\mathcal{U}(t)|-1})\urcorner. \tag{2.3$'$}$$

Of course, these newly defined guesses also get unraveled, but since $\mathrm{rk}(\mathfrak{a}(t')) < \mathrm{rk}(\mathfrak{a}(t))$ holds for any $t' \in \mathcal{U}(t)$, the whole recursion is well defined. Finally, we also need to adjust the definition of when a trail $t \in \mathrm{dom}(g)$ demands attention of a node σ: (D3) becomes

(D3$'$) $t(0) = \sigma$ and there exists $n < \mathfrak{g}(t)$ such that $\mathfrak{a}(t \ominus n) \in \mathscr{C}^{\Sigma}$ has Borel rank 1; in this case, we say t is *satisfied at round* r if $[\tau] \subseteq [\![\mathfrak{a}(t \ominus n)]\!]$ holds for each such n.

Now consider a fixed run of the game in which $\mathbb{1}$ plays $x \in \mathrm{dom}(f)$ and $\mathbb{2}$ follows her winning strategy ϑ defined using these modified definitions, therefore playing the labeled tree $\Upsilon := \Upsilon^{\vartheta}_{x}$. We say that a node σ *overshoots* if there exists a trail $t \in \mathrm{dom}(\mathfrak{g})$ owned by σ and some $n < \mathfrak{g}(t)$ such that $x \notin [\![t \ominus n]\!]$.

It is straightforward to check that only the hereditarily exact nodes survive $\alpha{\downarrow}$ pruning derivatives of Υ, and in particular $\mathrm{PD}^{\star}(\Upsilon, \alpha{\downarrow})$ is an infinite linear tree. To see that $\mathrm{PD}^{\star}(\Upsilon, \alpha_{\downarrow})$ is finitely branching we now no longer use Lemmas 2.39–2.45, but prove it directly.

2.51 Claim. *The tree* $\mathrm{PD}^{\star}(\Upsilon, \alpha_{\downarrow})$ *is finitely branching.*

For each $n > 0$ let $\ell_n = \ulcorner p, q\urcorner$ be least such that $\mathsf{bij}_n(p) = f(x){\restriction}n$ and $x \in [\![(f(x){\restriction}n) \oplus q]\!]$. For any node $\sigma \in \mathrm{PD}^{\star}(\Upsilon, \alpha_{\downarrow})$, we have that $\sigma{\restriction}n$ makes no wrong guesses of rank $< \alpha$ nor overshoots on guesses of any rank, for each $n \in \{1, \dots, |\sigma|\}$. In other words, for each $n \in \{1, \dots, |\sigma|\}$ we must have $\mathfrak{g}(\sigma{\restriction}n, n) \leqslant \ell_n$, with $\mathfrak{g}(t)$ the least number such that $x \in [\![\mathfrak{a}(t \oplus \mathfrak{g}(t))]\!]$ for any other trail $t \in \mathrm{dom}(\mathfrak{g})$ owned by $\sigma{\restriction}n$. It now follows that any node $\sigma \in \mathrm{PD}^{\star}(\Upsilon, \alpha_{\downarrow})$ can have at most $\ell_{|\sigma|+1}$ children in $\mathrm{PD}^{\star}(\Upsilon, \alpha_{\downarrow})$. □

2.3 Games for partition classes

2.3.1 The (α, Ψ)-tree game

Given $\sigma, \tau \in \omega^{<\omega}$ with $|\sigma| = |\tau|$, we denote by $\ulcorner\sigma, \tau\urcorner$ the element of $\omega^{<\omega}$ with $|(\ulcorner\sigma, \tau\urcorner)| = |\sigma|$ and $\ulcorner\sigma, \tau\urcorner(n) = \ulcorner\sigma(n), \tau(n)\urcorner$ for every $n < |\sigma|$.

2.52 Lemma. (1) *For any $\xi \in \omega^{<\omega}$ there exist unique $\sigma, \tau \in \omega^{<\omega}$ such that $\xi = \ulcorner\sigma, \tau\urcorner$.*

(2) *If $|\sigma| = |\tau|$ and $|\sigma'| = |\tau'|$, then $\ulcorner\sigma, \tau\urcorner \subseteq \ulcorner\sigma', \tau'\urcorner$ iff $\sigma \subseteq \sigma'$ and $\tau \subseteq \tau'$.*

Given a tree T and $n \in \omega$, we define the *subtree of T restricted to n*, denoted $T{\restriction}\langle n\rangle$, as the least subtree of T containing $\{\ulcorner\sigma, \tau\urcorner \in T\,;\, \sigma(0) = n\}$.

2.53 Lemma. *For every α we have $(\mathrm{PD}^\star(T, \alpha)){\restriction}\langle n\rangle = \mathrm{PD}^\star(T{\restriction}\langle n\rangle, \alpha)$.*

Proof. If $\sigma \in S = T{\restriction}\langle n\rangle$ then $\mathrm{rk}_S(\sigma) = \mathrm{rk}_T(\sigma)$, so the result follows from Lemma 2.28. □

2.54 Definition. Given trees S and T, we define their *product* as the tree $S \otimes T := \{\ulcorner\sigma, \tau\urcorner;\, \sigma \in S \text{ and } \tau \in T\}$, and given a tree S and a labeled tree $\Upsilon = (T, \varphi)$, we define their *product* as the labeled tree $S \otimes \Upsilon := (S \otimes T, \varphi')$, where $\varphi'(\ulcorner\sigma, \tau\urcorner) = \varphi(\tau)$.

2.55 Lemma. $\mathrm{rk}_{S\otimes T}(\ulcorner\sigma, \tau\urcorner) = \min\{\mathrm{rk}_S(\sigma), \mathrm{rk}_T(\tau)\}$.

Proof. By induction, we have $\mathrm{rk}_{S\otimes T}(\ulcorner\sigma, \tau\urcorner) \geqslant \alpha$ iff for every $\beta < \alpha$ there exists $\ulcorner\sigma', \tau'\urcorner \in S \otimes T$ such that $\ulcorner\sigma, \tau\urcorner \subseteq \ulcorner\sigma', \tau'\urcorner$ and $\mathrm{rk}_{S\otimes T}(\ulcorner\sigma', \tau'\urcorner) = \beta$ iff for every $\beta < \alpha$ there exist $\sigma' \in S$ and $\tau' \in T$ such that $\sigma \subseteq \sigma'$, $\tau \subseteq \tau'$, and $\min\{\mathrm{rk}_S(\sigma'), \mathrm{rk}_T(\tau')\} = \beta$ iff $\min\{\mathrm{rk}_S(\sigma), \mathrm{rk}_T(\tau)\} \geqslant \alpha$. □

2.56 Corollary. *For any α we have $\mathrm{PD}^\star(S \otimes T, \alpha) = \mathrm{PD}^\star(S, \alpha) \otimes \mathrm{PD}^\star(T, \alpha)$.*

Proof. This follows directly from Lemmas 2.28 and 2.55. □

2.57 Definition. Let S be a tree with a unique infinite branch and let Ψ be a property of labeled trees. We say a labeled tree Υ is Ψ *over S* if

(1) for each $n \in \omega$ there exists a labeled tree Υ_n with $\Upsilon{\restriction}\langle n\rangle = \mathrm{Com}(S, \langle n\rangle) \otimes \Upsilon_n$;

(2) there exists a unique $n \in \omega$ such that $\Upsilon{\restriction}\langle n\rangle$ is illfounded, and for this n we have that Υ_n satisfies Ψ.

2.58 Definition. A tree T is a $\mathbf{\Pi}^0_\alpha$ *partition tree* if

(PT1) $\mathrm{PD}^\star(T, \alpha{\downarrow})$ is infinite and linear;

(PT2) if α is a successor ordinal, then $\mathrm{PD}^\star(T, \alpha{\downarrow} - 1)$ is linear outside the root;

(PT3) $\mathrm{PD}^\star(T, \alpha\!\downarrow)$ is finitely branching.

In other words, given a limit ordinal λ and $n \in \omega$, a tree T which has a unique infinite branch is

(1) a $\mathbf{\Pi}^0_\lambda$ partition tree iff $\mathrm{PD}^\star(T, \lambda)$ is linear;

(2) a $\mathbf{\Pi}^0_{\lambda+2n+1}$ partition tree iff $\mathrm{PD}^\star(T, \lambda + n)$ is finitely branching at the root and linear elsewhere;

(3) a $\mathbf{\Pi}^0_{\lambda+2n+2}$ partition tree iff $\mathrm{PD}^\star(T, \lambda + n)$ is linear outside the root.

2.59 Definition. Let Ψ be a property of labeled trees.

(1) The Ψ*-tree game* is the relaxed tree game with addional rule that the final tree must satisfy Ψ.

(2) The (α, Ψ)-tree game is the relaxed tree game with additional rule that the final tree must be Ψ over some $\mathbf{\Pi}^0_\alpha$ partition tree.

2.60 Lemma. *Given a strategy ϑ for $\mathbb{2}$ in the relaxed tree game, a limit ordinal λ, and natural numbers n and h, we have $\{x \in \omega^\omega\,;\, \mathrm{PD}^\star(\Upsilon^\vartheta_x, \lambda + n)$ has height $\geqslant h\} \in \mathbf{\Sigma}^0_{\lambda+2n+1}$.*

Proof. Just note that, with the sets $X(\sigma, \lambda + n)$ as defined in Lemma 2.32, we have $\{x \in \omega^\omega\,;\, \mathrm{PD}^\star(\Upsilon^\vartheta_x, \lambda + n)$ has height $\geqslant h\} = \{x \in \omega^\omega\,;$ there exists $\sigma \in \omega^{<\omega}$ such that $|\sigma| = h$ and $x \in X(\sigma, \lambda + n)\} \in \mathbf{\Sigma}^0_{\lambda+2n+1}$. □

2.61 Theorem. *Let $\alpha > 0$. If the Ψ-tree game characterizes a class Λ of functions, then the (α, Ψ)-tree game characterizes the class of functions which are piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ partition.*

Proof sketch. Let ϑ be a winning strategy for $\mathbb{2}$ in the (α, Ψ)-tree game for f, and let us prove that f is piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ partition. For each $n \in \omega$, let $A_n := \{x \in \mathrm{dom}(f)\,;\, \Upsilon^\vartheta_x{\restriction}\langle n\rangle$ is illfounded$\}$. Note that $\langle A_n\,;\, n \in \omega\rangle$ is a partition of $\mathrm{dom}(f)$ on which f is Λ. Indeed, we can define a strategy ϑ' for $\mathbb{2}$ in the Ψ-tree game for $f{\restriction}A_n$ by letting $\vartheta'(\xi)$ be the least labeled tree containing the nodes $\tau \in \omega^{<\omega}$ for which there exists some $\sigma \in \omega^{<\omega}$ with $\sigma(0) = n$ and $\ulcorner\sigma, \tau\urcorner \in \vartheta(\xi)$, letting the label of τ in $\vartheta'(\xi)$ be equal to the label of any such node $\ulcorner\sigma, \tau\urcorner$ in $\vartheta(\xi)$. Then, given $x \in A_n$, since $\Upsilon^\vartheta_x{\restriction}\langle n\rangle$ is illfounded it follows that $\Upsilon^{\vartheta'}_x$ is an illfounded labeled tree satisfying Ψ whose infinite branches have the same running labels as those of Υ^ϑ_x. Therefore ϑ' is a winning strategy for $\mathbb{2}$ in the Ψ-tree game for f. By Corollary 1.27, all that remains to be proved is that each A_n is a $\mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$ set, for which we use Lemma 2.60 repeatedly. If α is a limit ordinal, then $A_n = \{x \in \mathrm{dom}(f)\,;\, (\mathrm{PD}^\star(\Upsilon^\vartheta_x, \alpha)){\restriction}\langle n\rangle$ has height $\geqslant 1\} \in \mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$. If $\alpha = \lambda + 2k + 1$ for some limit λ and $k \in \omega$, then $\alpha\!\downarrow = \lambda + k$ and $A_n = \{x \in \mathrm{dom}(f)\,;$ there exists h such that $(\mathrm{PD}^\star(\Upsilon^\vartheta_x, \lambda + k)){\restriction}\langle n\rangle$

has height at least h and for every $n' \neq n$ the height of $(\mathrm{PD}^\star(\Upsilon_x^\vartheta, \lambda + k))\lceil\langle n\rangle$ is less than $h)\} \in \mathbf{\Sigma}^0_{\alpha+1}(\mathrm{dom}(f))$ Finally, if $\alpha = \lambda + 2k + 2$ for some limit λ and $k \in \omega$, then $\alpha{\downarrow} - 1 = \lambda + k$ and $A_n = \{x \in \mathrm{dom}(f)\,;$ for every h the height of $(\mathrm{PD}^\star(\Upsilon_x^\vartheta, \lambda + k))\lceil\langle n\rangle$ is at least $h\} \in \mathbf{\Pi}^0_\alpha(\mathrm{dom}(f))$.

Conversely, let $\langle A_n\rangle_{n\in\omega}$ be a $\mathbf{\Pi}^0_\alpha(\mathrm{dom}(f))$ partition of $\mathrm{dom}(f)$ on which f is Λ, and let us define a winning strategy for $\mathbb{2}$ in the (α, Ψ)-tree game for f. We can use the idea of guessing as in the proof of Theorem 2.50 to define a strategy ϑ' that builds a $\mathbf{\Pi}^0_\alpha$ partition tree $\Upsilon_x^{\vartheta'}$, as follows†. Nodes of length 1 in $\Upsilon_x^{\vartheta'}$ make guesses for the n such that $x \in A_n$ will hold, and as in the proof of Theorem 2.50, these guesses get recursively unraveled. The task of verifying guesses is left for the elements of $\omega^{<\omega}$ of length more than 1, and in each case we also recursively unravel all guesses. The specifics of how nodes are added to $\Upsilon_x^{\vartheta'}$ are the same as those in the proof of Theorem 2.50, i.e., at round r of the game we add to $\Upsilon_x^{\vartheta'}$ the nodes σ which are active at that round and satisfy $\mathsf{bij}^{-1}(\sigma) \leqslant r$.

2.62 Claim. *The tree $\Upsilon_x^{\vartheta'}$ is a $\mathbf{\Pi}^0_\alpha$ partition tree.*

Let us call a node $\sigma \in \Upsilon_x^{\vartheta'}$ *exact* if σ makes no wrong guesses, and *hereditarily* exact if $\sigma\lceil n$ is exact for every $n \in \{1, \ldots, |\sigma|\}$. Thus, as in the proof of Theorem 2.50, it follows that the hereditarily exact nodes form the unique infinite branch of $\Upsilon_x^{\vartheta'}$.

To see that (PT1) holds, since all guesses by nodes of $\Upsilon_x^{\vartheta'}$ have rank $\leqslant \alpha$, the nodes which make some wrong guess are not in $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\downarrow})$. It follows that $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\downarrow})$ is composed of the hereditarily exact nodes of $\Upsilon_x^{\vartheta'}$, i.e., $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\downarrow})$ is infinite and linear.

For (PT2), suppose $\alpha = \beta + 1$. Note that nodes of $\Upsilon_x^{\vartheta'}$ of length more than 1 only make guesses of rank $< \beta$, so any such node which makes a wrong guesses is not in $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \gamma{\downarrow}) \supseteq \mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\downarrow} - 1)$. Hence $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\downarrow} - 1)$ is linear outside the root.

For (PT3), if α is even, i.e., if $\alpha{\twoheaddownarrow} = \alpha{\downarrow}$, then the result follows from (PT1). If α is odd, i.e., if $\alpha{\twoheaddownarrow} = \alpha{\downarrow} - 1$, then by (PT2) it is enough to show that the root has only finitely many children in $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\twoheaddownarrow})$. Let n be such that $x \in A_n$. If a node σ of length 1 in $\Upsilon_x^{\vartheta'}$ makes a guess that $x \in A_{n'}$ for some $n' > n$, via unraveling σ also makes a guess for a witness to the formula $x \notin A_n$, if A_n has Borel rank > 1, or σ is only added to $\Upsilon_x^{\vartheta'}$ when it becomes clear that $x \notin A_n$, in case A_n is a closed set. In either case, we have $\sigma \notin \mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\twoheaddownarrow})$. Thus the nodes of length 1 in $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\twoheaddownarrow})$ only make guesses for n' such that $x \in A_{n'}$ for $n \leqslant n'$, and furthermore the guesses resulting from unraveling this guess are all exact. Together, these facts imply that $\mathrm{PD}^\star(\Upsilon_x^{\vartheta'}, \alpha{\twoheaddownarrow})$ has only finitely many nodes of length 1, as desired.

†Since we went into detail in the proofs of Theorems 2.31 and 2.50, we will only give a sketch here. Working out the details would involve defining trails, complete systems, active nodes, etc., in a manner analogous to that of the proofs of Theorems 2.31 and 2.50.

We can finally define a strategy ϑ for $\mathbb{2}$ in the (α, Ψ)-tree game for f. For each n, let ϑ_n be a winning strategy for $\mathbb{2}$ in the Ψ-tree game for $f \restriction A_n$. Now let $\vartheta(\xi)$ be the least tree containing the nodes $\{\ulcorner \sigma, \tau \urcorner ; \sigma \in \vartheta'(\xi) \wedge \sigma \restriction 1$ makes the guess $x \in A_n \wedge \tau \in \vartheta_n(\xi)\}$, with the label for $\ulcorner \sigma, \tau \urcorner$ coming from $\vartheta_n(\xi)$, where n is such that $\sigma \restriction 1$ guesses $x \in A_n$.

Let $x \in \mathrm{dom}(f)$. The tree $\Upsilon_x^{\vartheta'}$ is a $\mathbf{\Pi}^0_\alpha$ partition tree and therefore has a unique infinite branch, say, one whose initial segment of length 1 makes the correct guess $x \in A_n$. In this case we have that $\Upsilon_x^{\vartheta_n}$ satisfies Ψ and is an illfounded labeled tree whose only infinite running label is $f(x)$. Therefore ϑ is a winning strategy for $\mathbb{2}$ in the (α, Ψ)-tree game for f. □

2.3.2 A sharper (α, Ψ)-tree game

For a large class of properties Ψ, we can get a sharper version of Theorem 2.61 with the same proof, as follows.

If a labeled tree Υ is Ψ over a tree S, then we say Υ is *strongly* Ψ over S if for each $n \in \omega$ there exists a labeled tree Υ_n satisfying Ψ such that $\Upsilon \restriction \langle n \rangle = \mathrm{Com}(S, \langle n \rangle) \otimes \Upsilon_n$. We say that a strategy ϑ for $\mathbb{2}$ in the Ψ-tree game for f is *strong* if Υ_x^ϑ satisfies Ψ for every $x \in \omega^\omega$. Finally, we say that the Ψ-tree game *strongly characterizes* a class Λ of partial functions on ω^ω if, for any $f : \omega^\omega \dashrightarrow \omega^\omega$, we have $f \in \Lambda$ iff $\mathbb{2}$ has a strong winning strategy in the Ψ-tree game for f.

2.63 Theorem. *If the Ψ-tree game strongly characterizes a class Λ of functions, then the class of functions which are piecewise Λ on a $\mathbf{\Pi}^0_\alpha$ partition is characterized by the strict tree game with additional rule that the final tree must be strongly Ψ over a $\mathbf{\Pi}^0_\alpha$ partition tree.*

As mentioned above, the proof of Theorem 2.63 is the same as that of Theorem 2.61.

It is straightforward to check that the winning strategy for $\mathbb{2}$ in the strict α-tree game given in the proof of Theorem 2.50 is strong. In particular, we get the following generalizations of Theorems 2.13, 2.15, and 2.17.

2.64 Corollary. *Let λ be a limit ordinal and $n \in \omega$.*

(1) *The class of functions which are piecewise Baire class $\lambda + 2n$ on a $\mathbf{\Pi}^0_{\lambda+2n+1}$ partition is characterized by the strict tree game with additional rule that the final tree Υ must satisfy that $\mathrm{PD}^\star(\Upsilon, \lambda + n)$ is finitely branching at the root and linear elsewhere.*

(2) *The class of functions which are piecewise Baire class $\lambda + 2n$ on a $\mathbf{\Pi}^0_{\lambda+2n+2}$ partition is characterized by the strict tree game with additional rule that the final tree Υ must satisfy that $\mathrm{PD}^\star(\Upsilon, \lambda + n)$ is linear outside the root.*

(3) *The class of functions which are piecewise Baire class* $\lambda + 2n + 1$ *on a* $\mathbf{\Pi}^0_{\lambda+2n+2}$ *partition is characterized by the strict tree game with additional rule that the final tree* Υ *must satisfy that* $\mathrm{PD}^\star(\Upsilon, \lambda + n)$ *is finitely branching outside the root.*

Chapter 3

Parametrized Wadge games

Summary. In this chapter, we show that game characterizations and Weihrauch degrees correspond closely to each other, so that we can employ the results and techniques developed for Weihrauch reducibility to study function classes in descriptive set theory and vice versa. In § 3.1, we develop some of the theory of transparent cylinders, one of the main tools from Weihrauch reducibility theory which we use in this chapter. In § 3.2, we define our generalization of the Wadge game: a general definition of a Wadge-style game taking two parameters (Definition 3.12), and resulting in a game which characterizes a class of functions related to the Weihrauch degrees of the parameters (Theorem 3.17). These games are equipped to deal with multi-valued functions between arbitrary represented spaces, and we show that they provide characterizations for many classes of functions of interest, such as continuous, Baire class 1, and $\mathbf{\Delta}^0_2$-multi-valued functions between represented spaces (i.e., the classes of the multi-valued functions which have a realizer in the corresponding class). Then, in § 3.3 we briefly discuss issues regarding the determinacy of parametrized Wadge games and relations with a generalization of Wadge reducibility. Finally, in § 3.4, as a further *proof of concept* of our definition we show how the games for functions of a fixed Baire class, from Chapter 2, can also be obtained as parametrized Wadge games (Corollary 3.73).

An early version of the material presented in §§ 3.1–3.3 has appeared in the proceedings volume of *Computability in Europe 2017*, held 12–16 June 2017 in Turku, Finland [73].

Remarks on co-authorship

The material presented in this chapter is the result of a collaboration with the author's copromotor Arno Pauly, building on the work of the author presented

in Chapter 2. This collaboration was funded by the Royal Society International Exchange Grant *Infinite games in logic and Weihrauch degrees*, and developed in part while both collaborators were Visiting Fellows at the Isaac Newton Institute for Mathematical Sciences in the program *Mathematical, Foundational and Computational Aspects of the Higher Infinite*.

Unless stated otherwise, definitions and results in this chapter are jointly due to the author and Pauly.

3.1 Transparent cylinders

In this section we study properties of *transparent cylinders*, which were introduced by Brattka and Gherardi [13], and which will play a central role in our parametrized version of the Wadge game. While the traditional scope of descriptive set theory is that of Polish spaces, their subsets, and functions between them, these restrictions are immaterial for the approach presented here; our results naturally hold for multi-valued functions between represented spaces. Therefore this work is part of a larger trend of extending descriptive set theory to more general settings, cf., e.g., [18, 58, 77, 80, 82, 85].

3.1 Definition (Brattka & Gherardi [13]). Let $f : \mathbb{X} \rightrightarrows \mathbb{Y}$. We call f

(1) a *cylinder* if $\mathrm{id}_{\omega^\omega} \times f \leqslant_{\mathrm{s}\mathfrak{W}} f$;

(2) *transparent* iff for any computable or continuous $g : \mathbb{Y} \rightrightarrows \mathbb{Y}$ there is a computable or continuous, respectively, $f_g : \mathbb{X} \rightrightarrows \mathbb{X}$ such that $f \circ f_g \preceq g \circ f$.

Note that f is a cylinder iff $g \leqslant_{\mathfrak{W}} f$ and $g \leqslant_{\mathrm{s}\mathfrak{W}} f$ are equivalent for all g. The transparent (single-valued) functions on Baire space were studied by de Brecht under the name *jump operator* in [19]. One of the reasons for their relevance is that they induce endofunctors on the category of represented spaces, which in turn can characterize function classes in descriptive set theory [80]. The term *transparent* was coined by Brattka, Gherardi, and Marcone in [15]. Our extension of the concept to multi-valued functions between represented spaces is rather straightforward, but requires the use of the notion of tightening. Note that the definition of computability and continuity for functions between represented spaces implies that $\delta_{\mathbb{X}}$ is transparent for any $\mathbb{X}$.

Two examples of transparent cylinders which will be relevant in what follows are the functions lim and $\lim_\Delta : \omega^\omega \dashrightarrow \omega^\omega$ defined by letting $\lim(p) = \lim_{n\in\omega}(p)_n$ and letting $\lim_\Delta(p)$ be the restriction of lim to the domain $\{p \in \omega^\omega\,;\, \exists n \in \omega \forall k \geqslant n((p)_k = (p)_n)\}$. To see a further example, related to Semmes's tree game characterizing the Borel functions, one first needs to define the appropriate represented space of labeled trees. For this, it is best to work in a quotient space of labeled trees under bisimilarity. The resulting quotient space can be thought of as the space of labeled trees in which the order of the subtrees rooted at the children of a node, and possible repetitions among these subtrees, are abstracted away. Then the map Prune, which removes from (any representative of the equivalence class of) a labeled tree which has one infinite branch all of the nodes which are not part of that infinite branch, is a transparent cylinder. This idea will be developed in full in § 3.4 below.

3.2 Proposition. *Let $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $g : \mathbb{Y} \rightrightarrows \mathbb{Z}$ be cylinders. If f is transparent then $g \circ f$ is a cylinder and $g \circ f \equiv_{\mathfrak{W}} g \star f$. Furthermore, if g is also transparent then so is $g \circ f$.*

Proof. ($g \circ f$ is a cylinder) As g is a cylinder, there are computable $h : \mathbb{Z} \rightrightarrows \omega^\omega \times \mathbb{Z}$ and $k : \omega^\omega \times \mathbb{Y} \rightrightarrows \mathbb{Y}$ such that $\mathrm{id}_{\omega^\omega} \times g \succeq h \circ g \circ k$. Likewise, there are computable $h' : \mathbb{Y} \rightrightarrows \omega^\omega \times \mathbb{Y}$ and $k' : \omega^\omega \times \mathbb{X} \rightrightarrows \mathbb{X}$ such that $\mathrm{id}_{\omega^\omega} \times f \succeq h' \circ f \circ k'$. As composition respects tightening [83, Lemma 2.4(1b)], we conclude that $(\mathrm{id}_{\omega^\omega} \times g) \circ (\mathrm{id}_{\omega^\omega} \times f) = \mathrm{id}_{\omega^\omega} \times (g \circ f) \succeq h \circ g \circ k \circ h' \circ f \circ k'$. Note that $(k \circ h') : \mathbb{Y} \rightrightarrows \mathbb{Y}$ is computable, and as f is transparent, there is some computable $f_{k \circ h'} : \mathbb{X} \rightrightarrows \mathbb{X}$ with $(k \circ h') \circ f \succeq f \circ f_{k \circ h'}$. But then $\mathrm{id}_{\omega^\omega} \times (g \circ f) \succeq h \circ g \circ k \circ h' \circ f \circ k' \succeq h \circ g \circ f \circ f_{k \circ h'} \circ k'$, thus h and $f_{k \circ h'} \circ k'$ witness that $\mathrm{id}_{\omega^\omega} \times (g \circ f) \leqslant_{\mathrm{s}\mathfrak{W}} (g \circ f)$, i.e., $g \circ f$ is a cylinder.

($g \circ f \equiv_{\mathfrak{W}} g \star f$) The direction $g \circ f \leqslant_{\mathfrak{W}} g \star f$ is immediate. Let f' and g' be such that $f' \leqslant_{\mathfrak{W}} f$, $g' \leqslant_{\mathfrak{W}} g$, and $g' \circ f'$ is defined. We need to show that $g' \circ f' \leqslant_{\mathfrak{W}} g \circ f$. As g and f are cylinders, we find that already $g' \leqslant_{\mathrm{s}\mathfrak{W}} g$ and $f' \leqslant_{\mathrm{s}\mathfrak{W}} f$. Let h, k witness the former and h', k' the latter. We conclude $h \circ g \circ k \circ h' \circ f \circ k' \preceq g' \circ f'$. As above, there then is some computable $f_{k \circ h'}$ with $k \circ h' \circ f \succeq f \circ f_{k \circ h'}$. Then h and $f_{k \circ h'} \circ k'$ witness that $g' \circ f' \leqslant_{\mathrm{s}\mathfrak{W}} g \circ f$.

Now suppose that g is also transparent.

($g \circ f$ is transparent) Let $h : \mathbb{Z} \rightrightarrows \mathbb{Z}$ be computable. By assumption that g is transparent, there is some computable $g_h : \mathbb{Y} \rightrightarrows \mathbb{Y}$ such that $g \circ g_h \preceq h \circ g$. Then there is some computable $f_h : \mathbb{X} \rightrightarrows \mathbb{X}$ with $f \circ f_h \preceq g_h \circ f$. As composition respects tightening [83, Lemma 2.4.1.b], we find that $h \circ g \circ f \succeq g \circ g_h \circ f \succeq g \circ f \circ f_h$, which is what we need. □

3.3 Definition. Given a function $f : A \rightrightarrows B$ and $C \subseteq B$, the *corestriction* of f to C, denoted $f \lfloor C$, is the restriction of f to domain $\{x \in \mathrm{dom}(f)\,;\, f(x) \subseteq C\}$. This notion extends to functions between represented spaces in a natural way. A represented space $(X, \delta_{\mathbb{X}})$ is a *subspace* of (Y, δ_Y), denoted $(X, \delta_{\mathbb{X}}) \subseteq (Y, \delta_Y)$, if $X \subseteq Y$ and $\delta_{\mathbb{X}} = \delta_Y \lfloor X$.

3.4 Proposition. *If $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\mathbb{Z} \subseteq \mathbb{W} \subseteq \mathbb{Y}$, then $f \lfloor \mathbb{Z} \leqslant_{\mathrm{s}\mathfrak{W}} f \lfloor \mathbb{W}$.*

3.5 Proposition. *Any corestriction of a transparent map is transparent.*

Proof. Let $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ be transparent, and let $\mathbb{Z}$ be a subspace of $\mathbb{Y}$. Let $g : \mathbb{Z} \rightrightarrows \mathbb{Z}$ be computable. Then $g : \mathbb{Y} \rightrightarrows \mathbb{Y}$, and therefore there exists a computable $f_g : \mathbb{X} \rightrightarrows \mathbb{X}$ such that $f \circ f_g \preceq g \circ f$. Note that $\mathrm{dom}(g \circ f) \subseteq \mathrm{dom}((f \lfloor \mathbb{Z}) \circ f_g)$. Indeed, if $x \in \mathrm{dom}(g \circ f)$, then $f \circ f_g(x) \subseteq g \circ f(x) \subseteq \mathbb{Z}$, so

$f_g(x) \subseteq \mathrm{dom}(f\lfloor\mathbb{Z})$ and therefore $x \in \mathrm{dom}((f\lfloor\mathbb{Z}) \circ f_g)$ as desired. From this it immediately follows that $((f\lfloor\mathbb{Z}) \circ f_g)\restriction\mathrm{dom}(g \circ f) = (f \circ f_g)\restriction\mathrm{dom}(g \circ f)$, from which we conclude $(f\lfloor\mathbb{Z}) \circ f_g \preceq g \circ (f\lfloor\mathbb{Z})$. $\square$

3.6 Theorem. *Any multi-valued function between represented spaces is strongly-Weihrauch-equivalent to a multi-valued function on ω^ω.*

Proof. Let $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ be given. Define $f' : \omega^\omega \rightrightarrows \omega^\omega$ by $\mathrm{dom}(f') = \mathrm{dom}(f \circ \delta_\mathbb{X})$ and $q \in f'(p)$ iff $\delta_\mathbb{Y}(q) \in f \circ \delta_\mathbb{X}(p)$. To see that $f \equiv_{\mathrm{s}\mathfrak{W}} f'$, first suppose $F \vdash f$, i.e., for any $p \in \mathrm{dom}(f \circ \delta_\mathbb{X})$ we have $\delta_\mathbb{Y} \circ F(p) \in f \circ \delta_\mathbb{X}(p)$. Then $F(p) \in f'(p)$, so $F \vdash f'$. Conversely, suppose for any $p \in \mathrm{dom}(f') = \mathrm{dom}(f \circ \delta_\mathbb{X})$ we have $F(p) \in f'(p)$. But this happens iff $\delta_\mathbb{Y} \circ F(p) \in f \circ \delta_\mathbb{X}(p)$, i.e., $F \vdash f$. $\square$

3.7 Theorem (Brattka & Pauly, implicit in [17, § 3.2]). *Every multi-valued function f is strongly Weihrauch-equivalent to some transparent cylinder f^{tc}, which can furthermore be taken to have codomain ω^ω.*

Proof. By Theorem 3.6, it is enough to prove the result for $f : \omega^\omega \rightrightarrows \omega^\omega$. Let $f^{\mathrm{tc}} : \mathcal{M}(\omega^\omega, \omega^\omega) \times \omega^\omega \rightrightarrows \omega^\omega$ be given by $f^{\mathrm{tc}}(h, x) = h \circ f(x)$.

That $f^{\mathrm{tc}} \leqslant_{\mathrm{s}\mathfrak{W}} f$ holds is of course immediate, and conversely we have $f \leqslant_{\mathrm{s}\mathfrak{W}} f^{\mathrm{tc}}$ since $\mathrm{id}_{\omega^\omega}$ has a computable name in $\mathcal{M}(\omega^\omega, \omega^\omega)$, so the function $K(x) = (\mathrm{id}_{\omega^\omega}, x)$ is computable and $f = f^{\mathrm{tc}} \circ K$.

To see that f^{tc} is a cylinder, define computable $K : \omega^\omega \times (\mathcal{M}(\omega^\omega, \omega^\omega) \times \omega^\omega) \dashrightarrow \omega^\omega$ and $H : \omega^\omega \times \omega^\omega \dashrightarrow \omega^\omega \times \omega^\omega$ by $K(p, (h, x)) = (h_p, x)$ where $h_p(y) = \ulcorner p, h(y)\urcorner$ and $H(\ulcorner p, y\urcorner) = (p, y)$. Then $H \circ f^{\mathrm{tc}} \circ K(p, (h, x)) = H(h_p f(x)) = H(\ulcorner p, f(x)\urcorner) = (p, f(x))$, so $\mathrm{id}_{\omega^\omega} \times f \leqslant_{\mathrm{s}\mathfrak{W}} f^{\mathrm{tc}}$. Since $f \equiv_{\mathrm{s}\mathfrak{W}} f^{\mathrm{tc}}$, this suffices.

Finally, to see that f^{tc} is transparent, let $g : \omega^\omega \rightrightarrows \omega^\omega$ be continuous or computable. Define $g' : \mathcal{M}(\omega^\omega, \omega^\omega) \times \omega^\omega \rightrightarrows \mathcal{M}(\omega^\omega, \omega^\omega) \times \omega^\omega$ by $g'(h, x) = (g \circ h, x)$. Note that g' is continuous or computable, respectively, since g is. Furthermore, we have $f^{\mathrm{tc}} \circ g'(h, x) = f^{\mathrm{tc}}(g \circ h, x) = g \circ h \circ f(x) = g \circ f^{\mathrm{tc}}(h, x)$, i.e., $f^{\mathrm{tc}} \circ g' = g \circ f^{\mathrm{tc}}$ as desired. $\square$

3.8 Definition. We say that a represented space $\mathbb{X}$ *(strongly) encodes ω^ω* if any $f : \omega^\omega \rightrightarrows \omega^\omega$ is (strongly) Weihrauch-equivalent to some $f' : \omega^\omega \rightrightarrows \mathbb{X}$.

Note that if $\mathbb{X}$ has a subspace which is computably isomorphic to ω^ω, then $\mathbb{X}$ strongly encodes ω^ω.

3.9 Theorem. *Let $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a transparent cylinder. If $\mathbb{Z} \subseteq \mathbb{Y}$ (strongly) encodes ω^ω, then $f\lfloor\mathbb{Z}$ is transparent and (strongly) Weihrauch-equivalent to f. In the strong case, $f\lfloor\mathbb{Z}$ is also a cylinder.*

Proof. Note that $f\lfloor\mathbb{Z} \leqslant_{\mathrm{s}\mathfrak{W}} f$ holds for any f and $\mathbb{Z}$, and if f is transparent then so is $f\lfloor\mathbb{Z}$. Now, by Theorem 3.6, there is some $g : \omega^\omega \rightrightarrows \omega^\omega$ which is strongly Weihrauch-equivalent to f. Therefore, by assumption, there exists $g' : \omega^\omega \rightrightarrows \mathbb{Z}$ such that g' is (strongly) Weihrauch-equivalent to f. Since f is a transparent cylinder, there exists a computable $h : \omega^\omega \dashrightarrow \omega^\omega$ such that $g' \succeq f \circ h$. Hence, since the codomain of g' is $\mathbb{Z}$, it follows that $g' \succeq f\lfloor\mathbb{Z} \circ h$, i.e., $g' \leqslant_{\mathrm{s}\mathfrak{W}} f\lfloor\mathbb{Z}$ and therefore f is (strongly) Weihrauch-reducible to $f\lfloor\mathbb{Z}$. Finally, if $\mathbb{Z}$ strongly encodes ω^ω, then we have $\mathrm{id}_{\omega^\omega} \times f\lfloor\mathbb{Z} \leqslant_{\mathrm{s}\mathfrak{W}} \mathrm{id}_{\omega^\omega} \times f \leqslant_{\mathrm{s}\mathfrak{W}} f \equiv_{\mathrm{s}\mathfrak{W}} f\lfloor\mathbb{Z}$, so $f\lfloor\mathbb{Z}$ is a cylinder. $\square$

3.2 Parametrized Wadge games

In order to deal with general multi-valued functions between represented spaces, we need to adapt the definition of a game, as follows.

3.10 Definition. A *game for multi-valued functions*, or simply a *game* when there is no risk of ambiguity, is a tuple $G = (\mathrm{M}_{\mathbb{1}}, \mathrm{R}_{\mathbb{1}}, \iota_{\mathbb{1}}, \mathrm{M}_{\mathbb{2}}, \mathrm{R}_{\mathbb{2}}, \iota_{\mathbb{2}})$ where, for $p \in \{\mathbb{1}, \mathbb{2}\}$, the set $\mathrm{M}_p \neq \varnothing$ is called the set of *moves* for player p, the set $\mathrm{R}_p \subseteq \mathrm{M}_p^\omega$ is called the set of *rules* for p, and the function $\iota_p : \mathrm{R}_p \rightrightarrows \omega^\omega$ is called the *interpretation* function for p.

Runs and *strategies* in games for multi-valued functions are defined in exactly the same way as in games for functions.

3.11 Definition. Let $G = (\mathrm{M}_{\mathbb{1}}, \mathrm{R}_{\mathbb{1}}, \iota_{\mathbb{1}}, \mathrm{M}_{\mathbb{2}}, \mathrm{R}_{\mathbb{2}}, \iota_{\mathbb{2}})$ be a game for multi-valued functions and let $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ be given. A strategy ζ for $\mathbb{1}$ is *legal in G for f* if $(\zeta * \mathfrak{m}_{\mathbb{2}})_{\mathbb{1}} \in (\delta_{\mathbb{X}} \circ \iota_{\mathbb{1}})^{-1}[\mathrm{dom}(f)]$ holds for all $\mathfrak{m}_{\mathbb{2}} \in \mathrm{M}_{\mathbb{2}}^\omega$. A legal ζ is a *winning strategy* for $\mathbb{1}$ in G for f if $\delta_{\mathbb{Y}} \circ \iota_{\mathbb{2}}(\mathfrak{m}_{\mathbb{2}}) \not\subseteq f \circ \delta_{\mathbb{X}}\iota_{\mathbb{1}}((\zeta * \mathfrak{m}_{\mathbb{2}})_{\mathbb{1}})$ holds for any $\mathfrak{m}_{\mathbb{2}} \in \mathrm{dom}(\delta_{\mathbb{Y}} \circ \iota_{\mathbb{2}})$. A strategy ϑ for $\mathbb{2}$ is *legal in G for f* if $(\mathfrak{m}_{\mathbb{1}} * \vartheta)_{\mathbb{2}} \in \iota_{\mathbb{2}}^{-1}[\mathrm{dom}(\delta_{\mathbb{Y}})]$ holds for any $\mathfrak{m}_{\mathbb{1}} \in (\delta_{\mathbb{X}} \circ \iota_{\mathbb{1}})^{-1}[\mathrm{dom}(f)]$. Finally, a legal ϑ is a *winning strategy* for $\mathbb{2}$ in G for f if $\delta_{\mathbb{Y}} \circ \iota_{\mathbb{2}}((\mathfrak{m}_{\mathbb{1}} * \vartheta)_{\mathbb{2}}) \subseteq f \circ \delta_{\mathbb{X}} \circ \iota_{\mathbb{1}}(\mathfrak{m}_{\mathbb{1}})$ holds for any $\mathfrak{m}_{\mathbb{1}} \in (\delta_{\mathbb{X}} \circ \iota_{\mathbb{1}})^{-1}[\mathrm{dom}(f)]$.

We are ready to define our parametrization of the Wadge game.

3.12 Definition. Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$. The *Wadge game parametrized by Ξ and π*, in short the (Ξ, π)-*Wadge game*, is the game for multi-valued functions $(\omega, \omega^\omega, \mathrm{id}_{\omega^\omega}, \mathrm{M}_{\mathbb{2}}^{\mathcal{W}}, \mathrm{R}_{\mathbb{2}}^{\mathcal{W}}, \iota_{\mathbb{2}})$, where $\iota_{\mathbb{2}} = \pi \circ \Xi \circ \delta_{\mathbb{X}} \circ \iota_{\mathbb{2}}^{\mathcal{W}}$ and $\mathrm{M}_{\mathbb{2}}^{\mathcal{W}}$, $\mathrm{R}_{\mathbb{2}}^{\mathcal{W}}$ are the moves and rules from the original Wadge game, Definition 2.5.

Thus, the (Ξ, π)-Wadge game is like the Wadge game but, instead of $\mathbb{1}$ building an element $x \in \mathrm{dom}(f)$ and $\mathbb{2}$ trying to build $f(x)$, now $\mathbb{1}$ builds a name for some element $x \in \mathrm{dom}(f)$ and $\mathbb{2}$ tries to build a name for some element $y \in \mathbb{Y}$ which is transformed by $\pi\Xi$ into a name for an element in $f(x)$. Intuitively, the idea is that the main transformation is done by Ξ, but because fixing a parametrized game entails fixing Ξ, in order for a fixed game to be able to deal with functions between different represented spaces there needs to be some map which will work as an intermediary between the target space of Ξ and the source space, say $\mathbb{Z}$, of the function in question. This role will be played by $\delta_{\mathbb{Z}}\pi$, which in the cases we are interested in will be a computable map.

It is easy to see that, restricted to single-valued functions on ω^ω, the original Wadge game is the $(\mathrm{id}_{\omega^\omega}, \mathrm{id}_{\omega^\omega})$-Wadge game, the eraser game is the $(\lim, \mathrm{id}_{\omega^\omega})$-Wadge game, and the backtrack game is the $(\lim_\Delta, \mathrm{id}_{\omega^\omega})$-Wadge game. Semmes's tree game for the Borel functions is the $(\mathrm{Prune}, \mathrm{Label})$-Wadge game, where Label is the function extracting the infinite running label from (any representative of the equivalence class of) a pruned labeled tree consisting of exactly one infinite branch. The details of this last example, including the definitions of the represented spaces involved, will be given in § 3.4 below.

3.13 Lemma. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$. If $\mathbf{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for $f : \mathbb{Z} \rightrightarrows \mathbb{W}$, then there exists a continuous (computable) $k : \omega^\omega \dashrightarrow \omega^\omega$ such that $\delta_\mathbb{W} \pi \Xi \delta_\mathbb{X} k \delta_\mathbb{Z}^{-1} \preceq f$. In particular $f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$ ($f \leqslant_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$).*

Proof. Any (computable) legal strategy ϑ for player $\mathbf{2}$ in the game for f gives rise to a continuous (computable) function $k : \omega^\omega \dashrightarrow \omega^\omega$ with domain $\mathrm{dom}(k) = \mathrm{dom}(f\delta_\mathbb{Z})$, defined by $k(x) = \iota_{\mathbf{2}}^{\mathcal{W}}(x * \vartheta)$. Now suppose ϑ is winning, i.e., suppose $\delta_\mathbb{W} \pi \Xi \delta_\mathbb{X} \iota_{\mathbf{2}}^{\mathcal{W}}(x * \vartheta) \subseteq f\delta_\mathbb{Z}(x)$ holds for any $x \in \mathrm{dom}(f\delta_\mathbb{Z}) = \mathrm{dom}(k)$. This means $\delta_\mathbb{W} \pi \Xi \delta_\mathbb{X} k \preceq f\delta_\mathbb{Z}$, which implies $\delta_\mathbb{W} \pi \Xi \delta_\mathbb{X} k \delta_\mathbb{Z}^{-1} \preceq f\delta_\mathbb{Z}\delta_\mathbb{Z}^{-1} = f$. Thus the continuous (computable) maps $\delta_\mathbb{W}$ and $\delta_\mathbb{X} k \delta_\mathbb{Z}^{-1}$ witness that $f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$ ($f \leqslant_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$). □

In order to see under which conditions we have the converse of Lemma 3.13, first we need the following notion, which is just the dual notion to being an admissible representation as in [87].

3.14 Definition. A computable partial function $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ is a *probe* for $\mathbb{Y}$ if for every computable or continuous $f : \mathbb{Y} \rightrightarrows \omega^\omega$ there is a computable or continuous, respectively, $g : \mathbb{Y} \rightrightarrows \mathbb{Y}$ such that $\pi g \preceq f$.

Note that a probe is always surjective, since constant functions $\mathbb{Y} \to \omega^\omega$ are continuous, and that a probe is also always transparent.

3.15 Theorem. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ be transparent and let $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ be a probe. For any multi-valued function f between represented spaces, we have that player $\mathbf{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for f iff $f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \Xi$ ($f \leqslant_{\mathrm{s}\mathfrak{W}} \Xi$).*

Proof. If $\mathbf{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for f, then by Lemma 3.13 we have $f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$ ($f \leqslant_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$), and we have $\pi \circ \Xi \leqslant_{\mathrm{s}\mathfrak{W}} \Xi$ since π is computable.

Conversely, suppose $f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \Xi$ ($f \leqslant_{\mathrm{s}\mathfrak{W}} \Xi$). Thus, there are continuous (computable) h, k with $h \circ \Xi \circ k \preceq f$. As $\delta_\mathbb{W} \circ \delta_\mathbb{W}^{-1} = \mathrm{id}_\mathbb{W}$, we have $\delta_\mathbb{W} \circ \delta_\mathbb{W}^{-1} \circ h \circ \Xi \circ k \preceq f$. Now $\delta_\mathbb{W}^{-1} \circ h : \mathbb{Y} \rightrightarrows \omega^\omega$ is continuous (computable), so by definition of a probe, there is some continuous (computable) $g : \mathbb{Y} \rightrightarrows \mathbb{Y}$ with $\delta_\mathbb{W} \circ \pi \circ g \circ \Xi \circ k \preceq f$. As Ξ is transparent, there is some continuous (computable) Ξ_g with $\Xi \circ \Xi_g \preceq g \circ \Xi$, thus $\delta_\mathbb{W} \circ \pi \circ \Xi \circ \Xi_g \circ k \preceq f$. As $\Xi_g \circ k : \mathbb{Z} \rightrightarrows \mathbb{X}$ is continuous (computable), it has some (continuous) computable realizer $K : \omega^\omega \dashrightarrow \omega^\omega$. By Theorem 2.6, player $\mathbb{2}$ has a (computable) winning strategy in the Wadge game for K, and it is easy to see that this strategy also wins the (Ξ, π)-Wadge game for f for her. □

We also have the following partial converse of Theorem 3.15, where for convenience we call the set of multi-valued functions which is (strongly) Weihrauch-reducible to a given multi-valued function f the *lower cone of f in the (strong) Weihrauch degrees*.

3.16 Proposition. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$. If the (Ξ, π)-Wadge game characterizes a lower cone in the strong Weihrauch degrees, then it is the lower cone of $\pi \circ \Xi$, and $\pi \circ \Xi$ is transparent.*

Proof. Whenever player $\mathbb{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for f, by Lemma 3.13 we have $f \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$ ($f \leqslant_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$). Furthermore, player $\mathbb{2}$ obviously has a winning strategy in the (Ξ, π)-Wadge game for $\pi \circ \Xi$: just copy all the moves made by $\mathbb{1}$. Since by assumption the game characterizes a lower cone in the strong Weihrauch degrees, this establishes the first part of the result.

To see that $\pi \circ \Xi : \mathbb{X} \rightrightarrows \omega^\omega$ is transparent, let $g : \omega^\omega \rightrightarrows \omega^\omega$ be continuous (computable). Then $g \circ \pi \circ \Xi \leqslant^{\mathrm{t}}_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$ ($g \circ \pi \circ \Xi \leqslant_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$), hence player $\mathbb{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for $g \circ \pi \circ \Xi$. By Lemma 3.13, this strategy induces some continuous (computable) $k : \omega^\omega \dashrightarrow \omega^\omega$ satisfying $g \circ \pi \circ \Xi \succeq \pi \circ \Xi \circ \delta_{\mathbb{X}} \circ k \circ \delta_{\mathbb{X}}^{-1}$. Therefore $\delta_{\mathbb{X}} \circ k \circ \delta_{\mathbb{X}}^{-1}$ is the desired witness. □

3.17 Theorem. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a transparent cylinder and let $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ be a probe. For any multi-valued function f between represented spaces, we have that player $\mathbb{2}$ has a (computable) winning strategy in the (Ξ, π)-Wadge game for f iff $f \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$ ($f \leqslant_{\mathfrak{W}} \Xi$).*

Proof. Follows immediately from Theorem 3.15 since the fact that Ξ is a cylinder means that $f \leqslant_{\mathrm{s}\mathfrak{W}} \Xi$ and $f \leqslant_{\mathfrak{W}} \Xi$ are equivalent. □

3.18 Corollary. *Suppose $\Lambda \subseteq \Lambda'$ are classes of multi-valued functions between represented spaces for which there exists some multi-valued function f between represented spaces satisfying $\Lambda = \{g \in \Lambda' \,;\, g \leqslant_{\mathfrak{W}} f\}$. Then there exists a parametrized Wadge game characterizing Λ as a subclass of Λ', i.e., there exist $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ such that, for any $g \in \Lambda'$, we have $g \in \Lambda$ iff $\mathbb{2}$ has a winning strategy in the (Ξ, π)-Wadge game for g.*

Proof. By Theorem 3.7, there exists a transparent cylinder $f^{\mathrm{tc}} : \mathbb{Z} \rightrightarrows \omega^\omega$ such that $f \equiv_{\mathrm{s}\mathfrak{W}} f^{\mathrm{tc}}$. Therefore for any g we have $g \leqslant_{\mathfrak{W}} f$ iff $g \leqslant_{\mathfrak{W}} f^{\mathrm{tc}}$. Now, by Theorem 3.17, for any $g \in \Lambda'$ we have $g \in \Lambda$ iff $\mathbb{2}$ has a winning strategy in the $(f^{\mathrm{tc}}, \mathrm{id}_{\omega^\omega})$-Wadge game for g. □

3.19 Corollary. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\Xi' : \mathbb{Y} \rightrightarrows \mathbb{Z}$ be transparent cylinders, and let $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ and $\pi' : \mathbb{Z} \rightrightarrows \omega^\omega$ be probes. If the (Ξ, π)-Wadge game characterizes the class Λ and the (Ξ', π')-Wadge game characterizes the class Λ', then the $(\Xi' \circ \Xi, \pi')$-Wadge game characterizes the class $\Lambda' \circ \Lambda := \{f' \circ f \,;\, f' \in \Lambda' \wedge f \in \Lambda\}$.*

Proof. If $\mathbf{2}$ has a (computable) winning strategy in the $(\Xi' \circ \Xi, \pi')$-Wadge game for $f : \mathbb{A} \rightrightarrows \mathbb{B}$, then by Theorem 3.17 we have $f \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi'\Xi$ ($f \leqslant_{\mathfrak{W}} \Xi'\Xi$). Thus, by Proposition 1.31, there exist continuous (computable) $k : \mathbb{A} \rightrightarrows \mathbb{X}$ and $h : \mathbb{A} \times \mathbb{Z} \rightrightarrows \mathbb{B}$ such that $h \circ (\mathrm{id}_{\mathbb{A}} \times \Xi'\Xi k)\Delta_{\mathbb{A}} = f$. Now let $g' = h \circ (\mathrm{id}_{\mathbb{A}} \times \Xi')$ and $g = (\mathrm{id}_{\mathbb{A}} \times \Xi k)\Delta_{\mathbb{A}}$. Then $f = g'g$, and since $g' \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi'$ ($g' \leqslant_{\mathfrak{W}} \Xi'$) and $g \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$ ($g \leqslant_{\mathfrak{W}} \Xi$), we have $g' \in \Lambda'$ and $g \in \Lambda$, as desired.

Conversely, if $g = f' \circ f$ with $f' \in \Lambda'$ and $f \in \Lambda$, then by Theorem 3.17 we have $f' \leqslant_{\mathfrak{W}} \Xi'$ and $f \leqslant_{\mathfrak{W}} \Xi$. Now, by Proposition 3.2, it follows that $f' \circ f \leqslant_{\mathfrak{W}} \Xi' \circ \Xi$. Finally, since by Proposition 3.2 we have that $\Xi' \circ \Xi$ is a transparent cylinder, again by Theorem 3.17 it follows that $\mathbf{2}$ has a winning strategy in the $(\Xi' \circ \Xi, \pi')$-Wadge game for g. □

We thus get game characterizations of many classes of functions, including, e.g., ones not covered by Motto Ros's constructions in [69]. For example, consider the function $\mathrm{Sort} : 2^\omega \to 2^\omega$ given by $\mathrm{Sort}(p) = 0^n 1^\omega$ if p contains exactly n occurrences of 0 and $\mathrm{Sort}(p) = 0^\omega$ otherwise. This map was introduced and studied by Neumann and Pauly in [72]. From the results in [72] it follows that the class Λ of total functions on ω^ω which are Weihrauch-reducible to Sort is neither the class of pointwise limits of functions in some other class, nor the class of $\mathbf{\Gamma}$-measurable functions for any boldface pointclass $\mathbf{\Gamma}$ of subsets of ω^ω closed under countable unions and finite intersections. By Theorem 3.7, Sort is Weihrauch-equivalent to some transparent cylinder $\mathrm{Sort}^{\mathrm{tc}}$ with codomain ω^ω. Thus, by Theorem 3.17, Λ is characterized by the $(\mathrm{Sort}^{\mathrm{tc}}, \mathrm{id}_{\omega^\omega})$-Wadge game.

As before, we also have the following partial converse of Theorem 3.17.

3.20 Proposition. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$. If the (Ξ, π)-Wadge game characterizes a lower cone in the Weihrauch degrees, then it is the lower cone of $\pi \circ \Xi$, and $\pi \circ \Xi$ is a transparent cylinder.*

Proof. By Proposition 3.16, we only have to prove that $\pi \circ \Xi$ is a cylinder. As $\mathrm{id}_{\omega^\omega} \times (\pi \circ \Xi) \leqslant_{\mathfrak{W}} \pi \circ \Xi$, the assumption that the (Ξ, π)-Wadge game characterizes a lower cone in the Weihrauch degrees implies that player $\mathbf{2}$ wins the (Ξ, π)-Wadge game for $\mathrm{id}_{\omega^\omega} \times (\pi \circ \Xi)$. Thus, again by Lemma 3.13 we have $\mathrm{id}_{\omega^\omega} \times (\pi \circ \Xi) \leqslant_{\mathrm{s}\mathfrak{W}} \pi \circ \Xi$, and we find $\pi \circ \Xi$ to be a cylinder. □

3.3 Using game characterizations

3.3.1 Determinacy

One application of having game characterizations of a property is realized together with determinacy: either by choosing the set-theoretic axioms accordingly, or by restricting to simple cases and invoking, e.g., Martin's Borel determinacy theorem (cf., e.g., [53, Theorem 31.3]), we can conclude that if the property is false, i.e., player $\mathbb{2}$ has no winning strategy, then player $\mathbb{1}$ has a winning strategy. In this way, player $\mathbb{1}$'s winning strategies serve as explicit, positive witnesses of the failure of the property. Applying this line of reasoning to our parametrized Wadge games, we obtain the following corollaries of Theorem 3.17:

3.21 Corollary (ZFC). *Let Ξ be a transparent cylinder and π be a probe such that $\pi \circ \Xi$ is single-valued and $\operatorname{dom}(\pi \circ \Xi)$ is Borel. Then for any $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ such that $\operatorname{dom}(\delta_{\mathbb{X}})$ and $f(x)$ are Borel for any $x \in \mathbb{X}$, we find that $f \not\leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$ iff player $\mathbb{1}$ has a winning strategy in the (Ξ, π)-Wadge game for f.*

3.22 Corollary (ZF + DC + AD). *Let Ξ be a transparent cylinder and π be a probe. Then $f \not\leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$ iff player $\mathbb{1}$ has a winning strategy in the (Ξ, π)-Wadge game for f.*

Unfortunately, as determinacy fails in a computable setting (e.g., Blass [10, Theorem 2] has shown there exist recursive sets A—therefore, necessarily determined—such that neither player has a computable winning strategy in the Gale-Stewart game for A), we do not retain the computable counterparts of these results. More generally, we lack a clear grasp on the connections between winning strategies of player $\mathbb{1}$ in the (Ξ, π)-Wadge game for a function f and *positive* witnesses of the fact that f is not in the class characterized by the game. As pointed out by Carroy and Louveau in private communication, this is true even for the original Wadge game for functions, i.e., the $(\mathrm{id}_{\omega^\omega}, \mathrm{id}_{\omega^\omega})$-Wadge game. Here we already have a notion of positive witnesses for discontinuity, viz. *points of discontinuity*, and can therefore make this discussion mathematically precise:

3.23 Question. Let a point of discontinuity of a function $f : \omega^\omega \to \omega^\omega$ be given as a sequence $(x_n)_{n \in \omega}$, a point $x \in \omega^\omega$, and $\sigma \in \omega^{<\omega}$ with $\sigma \subseteq f(x)$ such that $x_n \restriction (n+1) = x \restriction (n+1)$ holds for every $n \in \omega$ but $\sigma \not\subseteq f(x_n)$. Let DiscPoint be the multi-valued map that takes as input a winning strategy for player $\mathbb{1}$ in the $(\mathrm{id}_{\omega^\omega}, \mathrm{id}_{\omega^\omega})$-Wadge game for some function $f : \omega^\omega \to \omega^\omega$, and outputs a point of discontinuity for that function. Is DiscPoint computable? More generally, what is the Weihrauch degree of DiscPoint?

3.3.2 Generalized Wadge reducibility

Recall that given $A, B \subseteq \omega^\omega$, we say that A is *Wadge-reducible* to B, in symbols $A \leqslant_{\mathcal{W}} B$, if there exists a continuous $f : \omega^\omega \to \omega^\omega$ such that $f^{-1}[B] = A$ (we use the notation $\leqslant_{\mathcal{W}}$ instead of the more established $\leqslant_W$ in order to help avoid confusion with Weihrauch reducibility, $\leqslant_{\mathfrak{W}}$). Equivalently, we could consider the multi-valued total function $\frac{B}{A} : \omega^\omega \rightrightarrows \omega^\omega$ defined by $\frac{B}{A}(x) = B$ if $x \in A$ and $\frac{B}{A}(x) = (\omega^\omega \setminus B)$ if $x \notin A$. It is easy to see that we have $A \leqslant_{\mathcal{W}} B$ iff $\frac{B}{A}$ is continuous. By Wadge's famous lemma (cf. § 1.1) and Borel determinacy, it follows that for any Borel $A, B \subseteq \omega^\omega$, either $A \leqslant_{\mathcal{W}} B$ or $\omega^\omega \setminus B \leqslant_{\mathcal{W}} A$. In particular, the Wadge hierarchy on the Borel sets is a strict weak order of width 2.[†]

Both definitions generalize in a natural way to the case where $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$ for represented spaces $\mathbb{X}$, $\mathbb{Y}$: $A \leqslant'_{\mathcal{W}} B$ iff there exists a continuous $f : \mathbb{X} \to \mathbb{Y}$ such that $A = f^{-1}[B]$, and $\frac{B}{A} : \mathbb{X} \rightrightarrows \mathbb{Y}$ is defined by letting $\frac{B}{A}(x) = B$, if $x \in A$, and $\frac{B}{A}(x) = \mathbb{Y} \setminus B$, otherwise. It is easy to see that if $A \leqslant'_{\mathcal{W}}$ then $\frac{B}{A}$ is continuous, since if $f : \mathbb{X} \to \mathbb{Y}$ is such that $A = f^{-1}[B]$, then any realizer of f also realizes $\frac{B}{A}$. However, since not every continuous multi-valued function has a continuous uniformization, the converse does not hold in general. As noted, e.g., by Hertling [46], the relation $\leqslant'_{\mathcal{W}}$ restricted to $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ already introduces infinite antichains in the resulting degree structure, and Ikegami showed that in fact the partial order $(\wp(\omega), \subseteq_{\mathrm{fin}})$ can be embedded into that degree structure [50, Theorem 5.1.2]. The generalization of $\frac{B}{A}$ was proposed by Pequignot [85] as an alternative,[‡] and we adopt it for the remainder of this section. Thus, we define $A \leqslant_{\mathcal{W}} B$ iff $\frac{B}{A}$ is continuous.

3.24 Definition. Given a multi-valued function Ξ and $A \subseteq \mathbb{X}$, $B \subseteq \mathbb{Y}$ for represented spaces $\mathbb{X}$ and $\mathbb{Y}$, let $A \leqslant_\Xi B$ iff $\frac{B}{A} \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$.

3.25 Proposition. *If $\Xi \star \Xi \equiv_{\mathfrak{W}} \Xi$, then $\leqslant_\Xi$ is a quasiorder.*

3.26 Theorem. *Let $\Xi : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a transparent cylinder and $\pi : \mathbb{Y} \rightrightarrows \omega^\omega$ be a probe, and let $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{W}$ be such that the (Ξ, π)-Wadge game for $\frac{B}{A}$ is determined. Then either $A \leqslant_\Xi B$ or $B \leqslant_{\mathcal{W}} \omega^\omega \setminus A$.*

Proof. If player $\mathbb{2}$ has a winning strategy in the (Ξ, π)-Wadge game for $\frac{B}{A}$, then by Theorem 3.17, we find that $\frac{B}{A} \leqslant^{\mathrm{t}}_{\mathfrak{W}} \Xi$, hence $A \leqslant_\Xi B$. Otherwise, player $\mathbb{1}$ has a winning strategy in that game. This winning strategy induces a continuous function $s : \omega^\omega \to \omega^\omega$, such that if player $\mathbb{2}$ plays $y \in \omega^\omega$, then player $\mathbb{1}$ plays $s(y) \in \omega^\omega$. As Ξ is a transparent cylinder and π a probe, since $\mathrm{id}_{\omega^\omega} \leqslant_{\mathfrak{W}} \Xi$, by Theorem 3.17 player $\mathbb{2}$ has a winning strategy in the (Ξ, π)-Wadge game for $\mathrm{id}_{\omega^\omega}$. Thus, by Lemma 3.13, there is a continuous function $t : \omega^\omega \dashrightarrow \omega^\omega$ such that $(\pi \circ \Xi \circ \delta_{\mathbb{X}} \circ t) \preceq \mathrm{id}_{\omega^\omega}$, and since $\mathrm{id}_{\omega^\omega}$ is total and single-valued, we have

[†] A relation R is a *strict weak order* on a set X if there exists some ordinal number α and a partition $\langle X_\beta \,;\, \beta < \alpha \rangle$ of X such that $x \mathrel{R} y$ holds iff $x \in X_\beta$, $y \in X_\gamma$, and $\beta < \gamma$. The *width* of R is the supremum of the cardinalities of the parts in the partition.

[‡] While Pequignot only introduces the notion for second countable T_0 spaces, the extension to all represented spaces is immediate. Note that one needs to take into account that for general represented spaces, the Borel sets can show unfamiliar properties, e.g., even singletons can fail to be Borel (cf. also [88, 89]).

that t is total and $(\pi \circ \Xi \circ \delta_{\mathbb{X}} \circ t) = \mathrm{id}_{\omega^\omega}$ Now we consider $s \circ t : \omega^\omega \to \omega^\omega$. If $\delta_{\mathbb{Z}}(x) \in A$, then if player $\mathbb{2}$ plays $t(x)$, player $\mathbb{1}$ needs to play some $s(t(x))$ such that $\delta_{\mathbb{W}}(s(t(x))) \notin B$. Likewise, if $\delta_{\mathbb{Z}}(x) \notin A$, then for player $\mathbb{1}$ to win, it needs to be the case that $\delta_{\mathbb{W}}(s(t(x))) \in B$. Thus, $s \circ t$ is a continuous realizer of $\frac{B}{\omega^\omega \setminus A}$, and $B \leqslant_{\mathcal{W}} \omega^\omega \setminus A$ follows. $\square$

3.27 Corollary (ZF + DC + AD). *Suppose $\Xi \star \Xi \equiv_{\mathfrak{W}} \Xi$. Then $<_\Xi$ is a strict weak order of width at most 2.*

In [68], Motto Ros has identified sufficient conditions on a generalized reducibility (although in a different formalism) to ensure that its degree structure is equivalent to the Wadge degrees. We leave for future work the task of determining precisely for which Ξ the degree structure of $\leqslant_\Xi$ (restricted to subsets of ω^ω) is equivalent to the Wadge degrees, and which other structure types are realizable.

3.4 Games for functions of a fixed Baire class revisited

3.4.1 Spaces of trees

Given $p \in \omega^\omega$ and $\sigma \in \omega^{<\omega}$, we say that σ is a *path through* p if $p(0) \neq 0$ and recursively $\mathrm{shift}(\sigma)$ is a path through $(\mathrm{shift}(p))_{\sigma(0)}$ in case $\sigma \neq \langle\rangle$. We define $\mathbb{UT}$ as the space of (unlabeled) trees represented by the total function $\delta_{\mathbb{UT}}$ given by

$$\delta_{\mathbb{UT}}(p) := \{\sigma \in \omega^{<\omega}\,;\, \sigma \text{ is a path through } p\}.$$

If $\sigma \neq \langle\rangle$ is a path through p, then its *label according to* p is $p(\sigma(0)) - 1$, if $|\sigma| = 1$, or the label of $\mathrm{shift}(\sigma)$ according to $(\mathrm{shift}(p))_{\sigma(0)}$, otherwise. We now define $\mathbb{LT}$ as the space of labeled trees represented by the total function $\delta_{\mathbb{LT}}$ given by $\delta_{\mathbb{LT}}(p) = (\delta_{\mathbb{UT}}(p), \varphi)$, where $\varphi(\sigma)$ is the label of σ according to p. It is not hard to see that there exist a computable enumeration $\mathrm{e}_{\mathbb{LT}} : \omega \to \mathbb{LT}$ of all finite labeled trees and a computable function $\mathrm{size} : \omega \to \omega$ such that $\mathrm{e}_{\mathbb{LT}}(m) \subset \mathrm{e}_{\mathbb{LT}}(n)$ implies $m < n$, and such that $\mathrm{e}_{\mathbb{LT}}(n)$ has exactly $\mathrm{size}(n)$ nodes.

An *abstract labeled tree*, or simply *abstract tree* is an equivalence class of labeled trees under the relation of bisimilarity. We define $\mathbb{AT}$ as the space of abstract trees represented by the total function $\delta_{\mathbb{AT}}$ given by $\delta_{\mathbb{AT}}(p) = \delta_{\mathbb{LT}}(p)/\rightleftarrows$, i.e., the equivalence class of $\delta_{\mathbb{LT}}(p)$ under bisimilarity. We typically denote abstract trees by $\mathcal{A}$, with or without sub- or superscripts.

As usual with quotient constructions, any property of labeled trees can be extended to abstract trees by stipulating that an abstract tree has the property in question if one of its representatives does. Note that for some properties this extension behaves better than for some others. For example, the property of *having rank* α behaves well, since by Lemma 1.17 any two bisimilar labeled trees have the same rank. On the other hand, the property of *being finitely branching* does not behave as well, since every finitely branching labeled tree is bisimilar to an infinitely branching one.

Informal trees†

Note that, according to our definition, formally speaking an abstract tree is not itself a tree but only a certain type of set of labeled trees. However, for the sake of intuition it can be helpful to think of an abstract tree as an unordered tree without any concrete underlying set of vertices, as follows. We call an *informal tree* a (possibly empty) countable set I of objects of the form (n, J), where n is a natural number and J is again an informal tree. The intuition is that such a tree I is the tree for which each such object (n, J) represents a child of the root of I with label n and whose subtree is exactly J. See Figure 3.1 for the depiction of a simple informal tree.

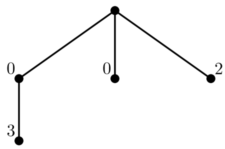


Figure 3.1: Depiction of the informal tree $\{(0, \{(3, \varnothing)\}), (0, \varnothing), (2, \varnothing)\}$.

To see how informal trees correspond to abstract trees, let $\delta_{\mathbb{IT}}$ be the partial function defined by corecursion with

$$\delta_{\mathbb{IT}}(p) = \{(n, \delta_{\mathbb{IT}}(q))\,;\, \exists k((p)_k = \langle n+1\rangle^\frown q)\}.$$

Then we say an informal tree I *corresponds* to an abstract tree $\mathcal{A}$ if there exists $p \in \mathrm{dom}(\delta_{\mathbb{IT}})$ with $\delta_{\mathbb{IT}}(p) = I$ and $\delta_{\mathbb{AT}}(p) = \mathcal{A}$.

3.28 Proposition. *In ZFC, the domain of $\delta_{\mathbb{IT}}$ is the set of $p \in \omega^\omega$ for which $\delta_{\mathbb{AT}}(p)$ is wellfounded. Therefore, in ZFC no informal tree corresponds to an illfounded abstract tree.*

This is, of course, because if p is such that $\delta_{\mathbb{AT}}(p)$ is illfounded, then in order for $p \in \mathrm{dom}(\delta_{\mathbb{IT}})$ to hold there would have to exist an infinite $\in$-descending chain of sets starting at $\delta_{\mathbb{IT}}(p)$, contradicting the axiom of foundation.

However—as is often the case with definitions by corecursion [66]—, this definition and the correspondence would also work for illfounded trees if one were to work in a system of non-wellfounded set theory such as ZFC$^-$ + AFA, where AFA is the axiom of anti-foundation first formulated by Forti and Honsell [37]

†This section concerns a connection of abstract trees with alternative set theories and has no mathematical relevance for the rest of this chapter.

and later popularized by Aczel [1]—in the style of Aczel [1, Chapter 6], in ZFC$^-$ + AFA the set of informal trees can be defined as the greatest fixed point of the class operator Φ defined by letting $\Phi(X)$ be the class of all countable sets of elements of the form (n, T), with $n \in \omega$ and T a countable subset of X. Thus in ZFC$^-$ + AFA the set of informal trees is exactly

$$\bigcup\{x \in V\,;\, x \subseteq \Phi(x)\}.$$

3.29 Proposition (ZFC$^-$ + AFA). *The correspondence between abstract and informal trees is a bijection.*

We will not pursue this line of investigation any further; we thus now move back to our setting of ZFC for the remainder of the thesis.

Computable functions between abstract trees

We denote by $\mathcal{O}(\omega)$ the represented space of subsets of ω given by *enumeration*, i.e., so that p is a name for $X \subseteq \omega$ iff $X = \{n \in \omega\,;\, \exists k \in \omega(p(k) = n+1)\}$. Note that any computable function of type $\mathcal{O}(\omega) \rightrightarrows \mathcal{O}(\omega)$ has a computable realizer which uses only *positive information*, by which we mean that it works by only following rules of the form "enumerate a certain natural number into the output set only after having seen some finite set of natural numbers enumerated into the input set", i.e., via the so-called *enumeration operators* (cf., e.g., [74, Chapter XIV]). Let $\mathrm{SubTrees} : \mathbb{LT} \to \mathcal{O}(\omega)$ be defined by letting $\mathrm{SubTrees}(\Upsilon) = \{n \in \omega\,;\, \mathrm{e}_{\mathbb{LT}}(n)$ is a subtree of $\Upsilon\}$. It is easy to see that SubTrees is computable.

3.30 Lemma. *There exists a computable map* $\mathrm{ConsTree} : \mathcal{O}(\omega) \rightrightarrows \mathbb{LT}$ *such that* $\mathrm{ConsTree} \circ \mathrm{SubTrees}(\Upsilon)$ *is total and* $\Upsilon' \in \mathrm{ConsTree} \circ \mathrm{SubTrees}(\Upsilon)$ *implies* $\Upsilon' \simeq \Upsilon$.

Proof. The function ConsTree can be defined as follows. Suppose we are at stage k of the construction, when some $n \in \omega$ is enumerated into the input. If some m has been enumerated at some earlier stage such that $\mathrm{e}_{\mathbb{LT}}(m) \supset \mathrm{e}_{\mathbb{LT}}(n)$, then we proceed to the next stage. Otherwise let X be the set of $m \in \omega$ such that $\mathrm{e}_{\mathbb{LT}}(m)$ is a maximal subtree of $\mathrm{e}_{\mathbb{LT}}(n)$ among those m which have been enumerated at earlier stages. By construction, for each $m \in X$ we have defined an associated $a(m) \in \omega$ and an isomorphism $\iota_m : \mathrm{e}_{\mathbb{LT}}(m) \to \mathrm{e}_{\mathbb{LT}}(a(m))$, and we have guaranteed that $\mathrm{e}_{\mathbb{LT}}(a(m))$ will be a subtree of the output tree we are constructing. Let $N \geqslant n$ be least such that there exists an isomorphism $\iota_n : \mathrm{e}_{\mathbb{LT}}(n) \to \mathrm{e}_{\mathbb{LT}}(N)$ extending ι_m for each $m \in X$ (in particular $\mathrm{e}_{\mathbb{LT}}(a(m)) \subset \mathrm{e}_{\mathbb{LT}}(N)$ for every $m \in X$) and such that no node of $\mathrm{e}_{\mathbb{LT}}(N)$ which is not in $\bigcup_{m \in X} \mathrm{e}_{\mathbb{LT}}(a(m))$ has been promised to be part of our current partial output. Then let $a(n) := N$ and guarantee that $\mathrm{e}_{\mathbb{LT}}(N)$ will be a subtree of our output tree.

It is now straightforward to check that running the algorithm above on a name for $\mathrm{SubTrees}(\Upsilon)$, we have $\Upsilon = \bigcup_{n \in \mathrm{dom}(a)} \mathrm{e}_{\mathbb{LT}}(n)$ and that $\iota := \bigcup_{n \in \mathrm{dom}(a)} \iota_n$ is an isomorphism between Υ and $\Upsilon' := \bigcup_{n \in \mathrm{dom}(a)} \mathrm{e}_{\mathbb{LT}}(a(n))$. $\square$

3.31 Lemma. *Let $G \vdash g : \mathbb{AT} \rightrightarrows \mathbb{AT}$. Suppose $F : \omega^\omega \dashrightarrow \omega^\omega$ and $H : \omega^\omega \dashrightarrow \omega^\omega$ are such that $\delta_{\mathbb{LT}} F(p) \rightleftarrows \delta_{\mathbb{LT}}(p)$ and $\delta_{\mathbb{LT}} H(q) \rightleftarrows \delta_{\mathbb{LT}}(q)$ for any $p \in \mathrm{dom}(F)$ and $q \in \mathrm{dom}(H)$, and $\mathrm{dom}(G) \subseteq \mathrm{dom}(HGF)$. Then $HGF \vdash g$.*

Proof. We have $\delta_{\mathbb{AT}} HGF(p) = \delta_{\mathbb{AT}} GF(p)$ and $\delta_{\mathbb{AT}} F(p) = \delta_{\mathbb{AT}}(p)$, from which it follows that $\delta_{\mathbb{AT}} HGF(p) = \delta_{\mathbb{AT}} G(p)$. □

3.32 Corollary. *Let F, H be computable realizers of* ConsTree *and* SubTrees*, respectively. If G is a computable realizer of some $g : \mathbb{AT} \rightrightarrows \mathbb{AT}$, then so is $FHGFH$.*

Proof. Indeed, we have $\delta_{\mathbb{LT}} FH(p) \in \mathrm{ConsTree} \circ \mathrm{SubTrees} \circ \delta_{\mathbb{LT}}(p)$, and therefore $\delta_{\mathbb{LT}} FH(p) \rightleftarrows \delta_{\mathbb{LT}}(p)$. Lemma 3.31 now applies. □

Note that HGF is a computable realizer of some function $g' : \mathcal{O}(\omega) \rightrightarrows \mathcal{O}(\omega)$; thus we can assume that $FHGFH$ works by only following rules of the form "make a certain finite labeled tree a subtree of the output only after having seen some finite set of finite labeled trees as subtrees of the input", which is to say, "make a certain finite labeled tree a subtree of the output only after having seen a certain finite labeled tree as a subtree of the input".

3.4.2 The Weihrauch degree of the pruning derivative

Having defined the represented spaces of trees, labeled trees, and abstract trees, we immediately have the corresponding operations of pruning derivative $\mathrm{PD} : \mathbb{UT} \to \mathbb{UT}$, $\mathrm{PD} : \mathbb{LT} \to \mathbb{LT}$, and $\mathrm{PD} : \mathbb{AT} \to \mathbb{AT}$. Note that Corollary 2.29 guarantees that PD as defined on abstract trees is a well-defined operation.

In order to see $\mathrm{PD}^\star$ as an operation on represented spaces, let us first recall the definition from [78] of the space $\mathbb{CO}$ of countable ordinals, represented by the function δ_{nK} defined recursively by

(1) $\delta_{\mathrm{nK}}(0p) = 0$

(2) $\delta_{\mathrm{nK}}(1p) = \delta_{\mathrm{nK}}(p) + 1$

(3) $\delta_{\mathrm{nK}}(2\ulcorner p_n \urcorner_{n\in\omega}) = \sup_{n\in\omega} \delta_{\mathrm{nK}}(p_n)$.

We now have the operations $\mathrm{PD}^\star : \mathbb{UT} \times \mathbb{CO} \to \mathbb{UT}$, $\mathrm{PD}^\star : \mathbb{LT} \times \mathbb{CO} \to \mathbb{LT}$, and $\mathrm{PD}^\star : \mathbb{AT} \times \mathbb{CO} \to \mathbb{AT}$. The fact that the notation PD and $\mathrm{PD}^\star$ is so overloaded will not be a problem, since which version is meant will usually be clear from the context. Whenever this is not the case, PD and $\mathrm{PD}^\star$ will refer to the operations on abstract trees.

In order to analyze the Weihrauch degree of $\mathrm{PD}^\star$, we will first introduce and analyze several operations on trees. We introduce and analyze them as modularly as possible, in the hope that this will increase the clarity of the presentation and the potential for applicability of the operations in other situations.

3.33 Definition. Given $\sigma_0, \dots, \sigma_{n-1} \in \omega^{<\omega}$ such that $|\sigma_i| = |\sigma_j| = \ell$ for each $i, j < n$, let $\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner \in \omega^\ell$ be defined by

$$\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner(m) = \ulcorner \sigma_0(m), \dots, \sigma_{n-1}(m) \urcorner$$

for each $m < \ell$. Note that $\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner \subseteq \ulcorner \tau_0, \dots, \tau_{n-1} \urcorner$ iff $\sigma_i \subseteq \tau_i$ for every $i < n$. Now, given trees $T_0, \dots, T_{n-1}$, let their *product* be the tree $\bigotimes_{i<n} T_i := \{\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner ; \forall i < n(\sigma_i \in T_i)\}$. If $n = 2$ then we use the smaller infix notation $T_0 \otimes T_1$ to denote the product, thus matching Definition 2.54 exactly.

3.34 Proposition. *The operation* $\bigotimes : \mathbb{UT}^{<\omega} \to \mathbb{UT}$ *is computable and*

(1) $\bigotimes_{i<n} T_i = \varnothing$ *iff* $T_i = \varnothing$ *for some* $i < n$.

(2) $\mathrm{PD}(\bigotimes_{i<n} T_i) = \bigotimes_{i<n} \mathrm{PD}(T_i)$.

(3) $\bigcap_{\beta<\alpha} \bigotimes_{i<n} T_i^\beta = \bigotimes_{i<n} \bigcap_{\beta<\alpha} T_i^\beta$ *for any ordinal* α.

In particular, $\mathrm{PD}^\star(\bigotimes_{i<n} T_i, \alpha) = \bigotimes_{i<n} \mathrm{PD}^\star(T_i, \alpha)$ for any ordinal α.

Recall that we extend the binary product $\otimes$ to type $\mathbb{LT} \times \mathbb{UT} \to \mathbb{LT}$ by letting $(T, \varphi) \otimes S = (T \otimes S, \varphi')$, where $\varphi'(\ulcorner \sigma, \tau \urcorner) = \varphi(\sigma)$.

3.35 Proposition. *If* S *is pruned and nonempty then* $(T, \varphi) \rightleftarrows (T, \varphi) \otimes S$.

Proof. Let $B \subseteq T \times (T \otimes S)$ be given by $\sigma \; B \; \tau$ iff $\tau = \ulcorner \sigma, \xi \urcorner$ for some $\xi \in S$. It is easy to see that B satisfies conditions (label) and (parent). Suppose $\sigma \; B \; \tau$, and let $\xi \in S$ be such that $\tau = \ulcorner \sigma, \xi \urcorner$. For (forth), let σ' be a child of σ in (T, φ). Since S is pruned, ξ has a child ξ' in S, and therefore $\tau' := \ulcorner \sigma', \xi' \urcorner$ is a child of τ in $(T, \phi) \otimes S$. Now $\sigma' \; B \; \tau'$ follows. For (back), let τ' be a child of τ in $(T, \phi) \otimes S$. Thus σ' is a child of σ in (T, φ), from which $\sigma' \; B \; \tau'$ follows. □

For our next operation on trees, let us first define some auxiliary notation $[\sigma_0, \dots, \sigma_{n-1}]$, for $\sigma_0, \dots, \sigma_{n-1} \in \omega^{<\omega}$.

3.36 Definition. We define $[\,] := \langle\rangle$. Then, given $\sigma_0, \dots, \sigma_{n-1} \in \omega^{<\omega}$ such that $n = |\sigma_0| > 0$ and $|\sigma_i| = n - i$ for each $i < n$, let $[\sigma_0, \dots, \sigma_{n-1}]$ be defined by

$$[\sigma_0, \dots, \sigma_{n-1}](m) = \ulcorner \sigma_0(m), \sigma_1(m), \dots, \sigma_m(m) \urcorner$$

for each $m < n$. Note that $[\sigma_0, \dots, \sigma_{n-1}] \subseteq [\tau_0, \dots, \tau_{m-1}]$ iff $n \leqslant m$ and $\sigma_i \subseteq \tau_i$ for each $i < n$. Now, given trees $\langle T_n \rangle_{n\in\omega}$, let their *countable product* be the tree

$$\bigboxtimes_{n\in\omega} T_n := \{[\sigma_0, \dots, \sigma_{n-1}] ; \forall n \in \omega(\sigma_n \in T_n)\}.$$

Note that $\langle\rangle \in \boxtimes_{n\in\omega} T_n$ always holds. In particular, we will *not* always have $\mathrm{PD}^\star(\boxtimes_{n\in\omega} T_n, \alpha) = \boxtimes_{n\in\omega} \mathrm{PD}^\star(T_n, \alpha)$ for all α, as we had for finite products of trees.

3.37 PROPOSITION. *The operation* $\boxtimes : \mathbb{UT}^\omega \to \mathbb{UT}$ *is computable and*

(1) *For* $m > 0$ *we have* $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \leqslant \min_{i<m} \mathrm{rk}_{T_i}(\sigma_i)$, *with equality in case* $\mathrm{rk}(T_j) \geqslant \min_{i<m} \mathrm{rk}_{T_i}(\sigma_i)$ *holds for each* $j \geqslant m$. *As a consequence, we have* $\mathrm{rk}(S) \leqslant \min_{n\in\omega}(\mathrm{rk}(T_n) + n)$.

(2) *For every* α *we have* $\mathrm{PD}^\star(\boxtimes_{n\in\omega} T_n, \alpha) \subseteq \boxtimes_{n\in\omega} \mathrm{PD}^\star(T_n, \alpha)$, *with equality in case* $\alpha = 0$ *or* $\mathrm{PD}^\star(T_n, \alpha) \neq \varnothing$ *for all* $n \in \omega$.

(3) *If all* T_n *are pruned and nonempty then so is* $\boxtimes_{n\in\omega} T_n$.

Proof. The computability of $\boxtimes$ is straightforward.

(1) By induction on $\mathrm{rk}_{T_i}(\sigma_i)$, we show that $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \leqslant \mathrm{rk}_{T_i}(\sigma_i)$. If $\mathrm{rk}_{T_i}(\sigma_i) = 0$, this is easy to see. For $\mathrm{rk}_{T_i}(\sigma_i) > 0$ we have that every descendant of σ_i in T_i has rank less than $\mathrm{rk}_{T_i}(\sigma_i)$, so by inductive hypothesis every descendant of $[\sigma_0, \dots, \sigma_{m-1}]$ in S has rank less than $\mathrm{rk}_{T_i}(\sigma_i)$, and therefore $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \leqslant \mathrm{rk}_{T_i}(\sigma_i)$. Conversely, by induction on α we show that if $\mathrm{rk}_{T_i}(\sigma_i), \mathrm{rk}(T_j) \geqslant \alpha$ holds for all $i < m$ and $j \geqslant m$, then $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \geqslant \alpha$ as well. The case $\alpha = 0$ is clear. Now suppose $\alpha > 0$. Given $\beta < \alpha$, for each $i < m$ let σ'_i be an immediate child of σ_i in T_i of rank at least β, and let $\sigma'_m \in T_m$ have length 1 and rank at least β. Then $[\sigma'_0, \dots, \sigma'_m]$ is an immediate child of $[\sigma_0, \dots, \sigma_{m-1}]$ in S. Since $\mathrm{rk}(T_j) \geqslant \beta$ for each $j \geqslant m + 1$, by induction hypothesis we get $\mathrm{rk}_S([\sigma'_0, \dots, \sigma'_m]) \geqslant \beta$. Therefore $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) > \beta$, and since $\beta < \alpha$ was arbitrary we get $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \geqslant \alpha$, as desired. Finally $\tau \in S$ has length $n + 1$, then $\tau = [\sigma_0, \dots, \sigma_n]$ where $\sigma_i \in T_i$ for every $i \leqslant n$. In particular, $\mathrm{rk}_S(\tau) \leqslant \mathrm{rk}_{T_n}(\sigma_n) < \mathrm{rk}(T_n)$, so $\mathrm{rk}(S) = \mathrm{rk}_S(\langle\rangle) \leqslant \mathrm{rk}(T_n) + n$.

(2) Follows by combining (1) with Lemma 2.28.

(3) Follows from (1) since a tree is pruned and nonempty iff all its nodes have rank ∞. □

3.38 DEFINITION. Given trees $T_0, \dots, T_{n-1}$, let their *mix* be the tree $\bigoplus_{i<n} T_i$ such that $\bigoplus_{i<n} T_i = \varnothing$ iff $T_i = \varnothing$ for some $i < n$, and otherwise $\langle\rangle \in \bigoplus_{i<n} T_i$ and $\mathrm{Conc}(\bigoplus_{i<n} T_i, \langle\ulcorner m, k\urcorner\rangle) = \mathrm{Conc}(T_m, \langle k\rangle)$ for each $m < n$ and $k \in \omega$. Intuitively, the mix of $T_0, \dots, T_{n-1}$ is the tree obtained by merging the roots of those trees into a single root. If $n = 2$ then we use the smaller infix notation $T_0 \oplus T_1$ to denote the mix.

3.39 PROPOSITION. *The operation* $\bigoplus : \mathbb{UT}^{<\omega} \to \mathbb{UT}$ *is computable and*

(1) $\mathrm{PD}(\bigoplus_{i<n} T_i) = \bigoplus_{i<n} \mathrm{PD}(T_i)$.

(2) $\bigcap_{\beta<\alpha} \bigoplus_{i<n} T_i^\beta = \bigoplus_{i<n} \bigcap_{\beta<\alpha} T_i^\beta$ *for any ordinal* α.

In particular, $\mathrm{PD}^\star(\bigoplus_{i<n} T_i, \alpha) = \bigoplus_{i<n} \mathrm{PD}^\star(T_i, \alpha)$ *for any ordinal* α

3.40 DEFINITION. Given trees $\langle T_n\rangle_{n\in\omega}$, let their *countable mix* be the tree $\boxplus_{n\in\omega} T_n$ such that $\langle\rangle \in \boxplus_{n\in\omega} T_n$ and $\mathrm{Conc}(\boxplus_{n\in\omega} T_n, \langle\ulcorner m,k\urcorner\rangle) = \mathrm{Conc}(T_m,\langle k\rangle)$ for each $m,k \in \omega$. As before with countable products, $\boxplus$ will not commute with $\mathrm{PD}^\star(\,\_\,,\alpha)$ for all α in general.

3.41 PROPOSITION. *The operation* $\boxplus : \mathbb{UT}^\omega \to \mathbb{UT}$ *is computable and satisfies* $\mathrm{PD}^\star(\boxplus_{n\in\omega} T_n, \alpha) \subseteq \boxplus_{n\in\omega} \mathrm{PD}^\star(T_n,\alpha)$, *with equality in case* $\alpha = 0$ *or* $\mathrm{PD}^\star(T_n,\alpha) \neq \varnothing$ *for some* $n \in \omega$.

To proceed, we need the notion of a Borel truth value. This represented space was introduced in [44, Definition 5.8] (built on ideas from [65]), and further investigated in [78]. Our definition differs slightly from the one given in the literature, but is easily seen to be equivalent.

3.42 DEFINITION. A *Borel truth value* is a pair $b = (T,\mu)$ such that T is a wellfounded tree and μ is a function, called a *tagging* function, assigning to each node of T one of the *tags* $\bot,\top,\forall,\exists$, in such a way that each leaf is tagged $\top$ or $\bot$, and each non-leaf node is tagged $\forall$ or $\exists$ (in alternating fashion, i.e., so that if a node tagged $\forall$ has a parent, then the parent is tagged $\exists$ and vice versa). A name for a Borel truth value (T,μ) is an element $p \in 5^\omega$ which is a $\delta_{\mathbb{UT}}$-code for T and such that if $\sigma \in T$, i.e., if σ is a path through p, then

$$\mu(\sigma) = \begin{cases} \bot, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 1, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 1 \\ \top, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 2, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 2 \\ \forall, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 3, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 3 \\ \exists, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 4, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 4 \\ \mu'(\mathrm{shift}(\sigma)), & \text{if } |\sigma| > 1, \end{cases}$$

where (T',μ') is the Borel truth value named by $(\mathrm{shift}(p))_{\sigma(0)}$. In other words, intuitively in a name for a Borel truth value, zeroes indicate absence of the corresponding node, and nonzero values indicate both presence of the corresponding node and its tag. The *value* $\mathrm{Val}(b) \in \{\top,\bot\}$ of a Borel truth value b is defined by recursion on the rank of b in a straightforward way. The space of Borel truth values is denoted by $\mathbb{S}(\mathcal{B})$. The Σ^0_α-*truth values* (denoted $\mathbb{S}(\Sigma^0_\alpha)$) are those with rank $\leqslant \alpha$ and root tagged $\exists$, and the Π^0_α-*truth values* (denoted $\mathbb{S}(\Pi^0_\alpha)$) are those with rank $\leqslant \alpha$ and root tagged $\forall$.

Given an ordinal α, with $\alpha = \lambda + n$ for some limit ordinal λ and $n \in \omega$, let $\alpha{\uparrow} = \lambda + 2n$ and recall that $\alpha{\downarrow} = \lambda + \lceil \frac{n}{2} \rceil$.

3.43 PROPOSITION. *The map* $\mathrm{isPresent} : \mathbb{LT} \times \mathbb{CO} \times \omega \rightrightarrows \coprod_{\alpha\in\mathbb{CO}} \mathbb{S}(\Pi^0_\alpha)$, *mapping* (Υ,α,ℓ) *such that* $\mathrm{e}_{\mathbb{LT}}(\ell)$ *is linear to* $(\max\{1,\alpha{\uparrow}\}, b)$ *where* $\mathrm{Val}(b) = \top$ *iff* $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$, *is computable.*

Proof. It is straightforward to see that $\alpha \mapsto \max\{1, \alpha{\uparrow}\} : \mathbb{CO} \to \mathbb{CO}$ is computable. Computability of the second component is shown by induction over the δ_{nK}-name q of α provided.

If $q = 0q'$, then we check whether $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \Upsilon$ and return either the tree of rank 1 with root tagged $\forall$ and children tagged $\top$ (if yes), or with root tagged $\forall$ and children tagged $\bot$ (if no).

If $q = 1q'$, then $\alpha = \beta + 1$ and q' is a name for β. Let h be the height of $\mathrm{e}_{\mathbb{LT}}(\ell)$. We start searching for confirmation that $\beta > 0$. Until we find it, we add children with tag $\exists$ to the root tagged $\forall$, and then for each $\ell' \in \omega$ such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $h' > h$ extending $\mathrm{e}_{\mathbb{LT}}(\ell)$, we add a grandchild tagged $\top$ or $\bot$ to the $(h' - h)^{\text{th}}$ child, depending on whether or not $\mathrm{e}_{\mathbb{LT}}(\ell') \subseteq \Upsilon$. If we do receive confirmation that $\beta > 0$, we add a grandchild tagged $\top$ to each $\exists$-child produced so far, and then ignore these children. Then, for each $\ell' \in \omega$ such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $h' > h$ extending $\mathrm{e}_{\mathbb{LT}}(\ell)$, we compute $b_{\ell'} \in \mathbb{S}(\Pi^0_{\max\{1,\beta\uparrow\}})$ denoting whether or not $\mathrm{e}_{\mathbb{LT}}(\ell') \subseteq \mathrm{PD}^\star(\Upsilon, \beta)$. Then we add each $b_{\ell'}$ as a grandchild of the root of b via the $(h' - h)^{\text{th}}$ new child tagged $\exists$.

If $q = 2^{\ulcorner} q_i {}^{\urcorner}{}_{i\in\omega}$, then $\alpha = \sup_{i\in\omega} \alpha_i$ and each q_i is a name for α_i. For each $i \in \omega$, we compute whether $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha_i)$ as $b_i \in \mathbb{S}(\Pi^0_{\max\{1,\alpha_i\uparrow\}})$. By induction, each b_i has root tagged $\forall$, and we now obtain the answer b as the mix of the b_i.

3.44 Claim. $\mathrm{Val}(b) = \top$ *iff* $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$.

By induction on the name q of α. If $q = 0q'$ then it is immediate to see that the claim holds. Suppose the claim holds for q' and let $q = 1q'$. Let $\beta = \delta_{\mathrm{nK}}(q')$ and suppose $\mathrm{e}_{\mathbb{LT}}(\ell)$ has height h. Then $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ iff for every $n \in \omega$ there exists some $\ell' \in \omega$ such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $h+n$ extending $\mathrm{e}_{\mathbb{LT}}(\ell)$, with $\mathrm{e}_{\mathbb{LT}}(\ell') \subseteq \mathrm{PD}^\star(\Upsilon, \beta)$. By induction, the result of iterating the algorithm for (Υ, β, ℓ') gives the correct output. Therefore $\mathrm{Val}(b) = \top$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. Finally, suppose the claim holds for each q_n and let $q = 2^{\ulcorner} q_n {}^{\urcorner}{}_{n\in\omega}$. Let $\alpha_n = \delta_{\mathrm{nK}}(q_n)$. Again, by induction the result of iterating the algorithm for $(\Upsilon, \alpha_n, \ell)$ gives the correct output. Therefore $\mathrm{Val}(b) = \top$ iff all children of the roots of all b_n have value $\top$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha_n)$ for all $n \in \omega$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$, as desired.

3.45 Claim. *The Borel truth value b has rank $\leqslant \max\{1, \alpha{\uparrow}\}$.*

We again proceed by induction on the name q of α. If $q = 0q'$ then by construction b has rank 1. If $q = 1q'$ and $\beta := \delta_{\mathrm{nK}}(q') = 0$, then again by construction b has rank $\leqslant 2 = 1{\uparrow}$. If $q = 1q'$ and $\beta := \delta_{\mathrm{nK}}(q') > 0$, then by induction each b_τ as defined in the algorithm has rank $\leqslant \beta{\uparrow}$, and therefore b has rank $\leqslant \beta{\uparrow} + 2 = \alpha{\uparrow}$. Finally, if $q = 2^{\ulcorner} q_i {}^{\urcorner}{}_{i\in\omega}$, for each i let $\alpha_i = \delta_{\mathrm{nK}}(q_i)$. By induction, each b_i as defined in the algorithm has rank $\leqslant \max\{1, \alpha_i{\uparrow}\}$, and by construction b has rank $\leqslant \max\{1, \sup_{i\in\omega} \alpha_i{\uparrow}\} = \max\{1, \alpha{\uparrow}\}$. ☐

3.46 Proposition. *The map* Witness : $\mathbb{S}(\mathcal{B}) \rightrightarrows \mathbb{UT}$*, mapping b of rank $\alpha > 0$ to some T such that if* $\mathrm{Val}(b) = \top$ *then* $\mathrm{PD}^\star(T, \alpha{\downarrow})$ *is a nonempty pruned tree, and if* $\mathrm{Val}(b) = \bot$ *then* $\mathrm{PD}^\star(T, \alpha{\downarrow}) = \varnothing$*, is computable.*

Proof. If b is composed of a single node, then we output $\omega^{<\omega}$ or $\varnothing$ according to whether $\mathrm{Val}(b) = \top$ or $\mathrm{Val}(b) = \bot$. Otherwise, we iteratively compute trees $(T_n)_{n\in\omega}$ for all the subtrees rooted at the children of the root of b, and output $\boxtimes_{n\in\omega} T_n$ if the root of b is tagged $\forall$, or output $\boxplus_{n\in\omega} T_n$ if the root of b is tagged $\exists$.

3.47 Claim. *Suppose $\beta > 0$ is such that $\mathrm{PD}^\star(T_n, \beta)$ is pruned for each $n \in \omega$. Then $\mathrm{PD}^\star(T, \beta)$ is pruned, if the root of b is tagged $\forall$, and $\mathrm{PD}^\star(T, \beta + 1)$ is pruned, if the root of b is tagged $\exists$. Furthermore, if $\mathrm{PD}^\star(T, \delta)$ is pruned, then it is nonempty in case $\mathrm{Val}(b) = \top$ and empty in case $\mathrm{Val}(b) = \bot$.*

If b is composed of a single node then the claim follows easily. Otherwise, suppose the root of b is tagged $\forall$, so that $T = \boxtimes_{n\in\omega} T_n$. If $\mathrm{Val}(b) = \top$ then each $\mathrm{PD}^\star(T_n, \beta)$ is pruned and nonempty, and therefore the same holds for $\mathrm{PD}^\star(T, \beta)$. Conversely, if $\mathrm{Val}(b) = \bot$ then $\mathrm{PD}^\star(T_{n_0}, \beta) = \varnothing$ for some $n_0 \in \omega$. Thus there is some $\gamma < \beta$ and $H \in \omega$ such that $\mathrm{PD}^\star(T_{n_0}, \gamma)$ has height $\leqslant H < \omega$. Then $\mathrm{PD}^\star(T, \gamma)$ has height $\leqslant H' < \omega$ for some H' depending on H and n_0, and therefore $\mathrm{PD}^\star(T, \beta) = \varnothing$. Now suppose the root of b is tagged $\exists$, so that $T = \boxplus_{n\in\omega} T_n$. If $\mathrm{Val}(b) = \top$ then some $\mathrm{PD}^\star(T_n, \beta)$ is pruned and nonempty. Therefore the same holds for $\mathrm{PD}^\star(T, \beta)$. Otherwise, if $\mathrm{Val}(b) = \bot$ then each $\mathrm{PD}^\star(T_n, \beta)$ is empty. Therefore $\mathrm{PD}^\star(T, \beta) \subseteq \{\langle\rangle\}$, and thus $\mathrm{PD}^\star(T, \beta + 1)$ is empty.

3.48 Claim. *Let b' be a Borel truth value and let T' be the result of applying the algorithm above to b'. For $\beta = \mathrm{rk}(b')$, we have that if the root of b' has tag $\forall$ then $\mathrm{PD}^\star(T', \beta{\downarrow})$ is pruned, and if the root of b' has tag $\exists$ then $\mathrm{PD}^\star(T', \beta{\downarrow} + 1)$ is pruned.*

By induction on β. If $\beta = 0$, i.e., if b' is a single node, then by construction T' is pruned. Now suppose $\beta > 0$, and let the n^{th} child σ_n of the root of b' have rank $\beta_n < \beta$. Suppose the root of b' has tag $\exists$, so that each σ_n is either a leaf or has tag $\forall$. By induction, the result T_n of applying the algorithm to the subtree of b' rooted at σ_n is such that $\mathrm{PD}^\star(T_n, \beta{\downarrow}_n)$ is pruned. Since $\sup_{n\in\omega} \beta{\downarrow}_n \leqslant \beta{\downarrow}$, by the preceding claim it follows that $\mathrm{PD}^\star(T, \beta{\downarrow} + 1)$ is pruned. Finally, suppose the root of b' has tag $\forall$, so that each σ_n is either a leaf or has tag $\exists$. By induction, the result T_n of applying the algorithm to the subtree of b' rooted at σ_n is such that $\mathrm{PD}^\star(T_n, \beta{\downarrow}_n + 1)$ is pruned. If $\sup_{n\in\omega}(\beta{\downarrow}_n + 1) \leqslant \beta{\downarrow}$ for each n, then by the preceding claim we are done. Otherwise, say $\beta{\downarrow}_n = \beta{\downarrow}$ for some n. Then β_n is odd and $\beta = \beta_n + 1$. In particular $\beta_n = \gamma_n + 1$ for some γ_n, and therefore $\delta{\downarrow} \leqslant \gamma{\downarrow}_n < \beta{\downarrow}_n$ for each $\delta < \beta_n$. Hence, since by induction the result S of applying the algorithm to any subtree of β' rooted at some child of σ_n is such that $\mathrm{PD}^\star(S, \delta{\downarrow})$ is pruned for some $\delta < \beta_n$, by the preceding claim it follows that $\mathrm{PD}^\star(T_n, \gamma{\downarrow}_n + 1) = \mathrm{PD}^\star(T_n, \beta{\downarrow}_n)$ is pruned. Thus, again by the preceding claim, $\mathrm{PD}^\star(T, \beta{\downarrow})$ is pruned. □

3.49 Proposition. *For each $\alpha \in \mathbb{CO}$ we have that $\mathrm{PD}^\star(\,\_\,, \alpha)$ is parallelizable.*

Proof. Given abstract trees $\langle \mathcal{A}_n \rangle_{n\in\omega}$ with respective representatives $\langle \Upsilon_n \rangle_{n\in\omega}$, let $\mathcal{A}$ be the abstract tree represented by the tree Υ in which the root has a child σ_n labeled n for each $n \in \omega$ such that $\Upsilon_n \neq \varnothing$, and such that $\mathrm{Conc}(\Upsilon, \sigma_n) = \Upsilon_n$ in the positive case. It is now straightforward to see that each $\mathrm{PD}^\star(\mathcal{A}_n, \alpha)$ can be reconstructed from $\mathrm{PD}^\star(\mathcal{A}, \alpha)$. $\square$

Let $\underline{2}$ be the represented space composed of two elements, $\top$ and $\bot$, the first represented by 1^ω and the latter by 0^ω.

3.50 Corollary. *For each $\alpha > 0$ we have that $\mathrm{PD}^\star(\,\_\,, \alpha)$ is Weihrauch-equivalent to the parallelization of $\mathrm{id}_\alpha : \mathbb{S}(\Pi^0_{\alpha\uparrow}) \to \underline{2}$. Furthermore, the reductions in both directions can be taken to be uniform in α.*

Proof. To reduce $\mathrm{PD}^\star(\,\_\,, \alpha)$ to the parallelization of id_α, note that we can use isPresent from Proposition 3.43 to compute for each finite linear labeled tree whether or not it is present in $\mathrm{PD}^\star(\Upsilon, \alpha)$ as a $\mathbb{S}(\Pi^0_{\alpha\uparrow})$-truth value. We then use the parallelization of $\mathrm{id}_\alpha : \mathbb{S}(\Pi^0_{\alpha\uparrow}) \to \underline{2}$ to convert all of these into booleans, and can thus construct $\mathrm{PD}^\star(\Upsilon, \alpha)$.

For the converse, we use $\mathrm{Witness}(\alpha_\uparrow, b)$ from Proposition 3.46 to obtain some Υ such that $\mathrm{PD}^\star(\Upsilon, \alpha) = \varnothing$ if $\mathrm{Val}(b) = \bot$ and $\mathrm{PD}^\star(\Upsilon, \alpha) \neq \varnothing$ if $\mathrm{Val}(b) = \top$. As $\{\varnothing\}$ is a decidable subset of $\mathbb{LT}$, we can therefore write a $\underline{2}$-name for b if given an $\mathbb{LT}$-name for $\mathrm{PD}^\star(\Upsilon, \alpha)$. $\square$

3.51 Theorem (Folklore). *If $A \subseteq \omega^\omega$ is Wadge-complete for $\mathbf{\Pi}^0_\alpha$, then $\hat{\chi_A}$ is t-Weihrauch-complete for Baire class α, where $\chi_A : \omega^\omega \to \underline{2}$ is given by $\chi_A(x) = \top$ iff $x \in A$.*

Proof. That $\hat{\chi_A}$ is Baire class α follows from noticing that

$$\hat{\chi_A}^{-1}[\sigma] = \bigcap_{\substack{n<|\sigma| \\ \sigma(n)=0}} \{x \in \omega^\omega \,;\, (x)_n \in A\} \cap \bigcap_{\substack{n<|\sigma| \\ \sigma(n)=1}} \{x \in \omega^\omega \,;\, (x)_n \notin A\},$$

which is the intersection of a $\mathbf{\Pi}^0_\alpha$ set with a $\mathbf{\Sigma}^0_\alpha$ set.

Now let $F : \omega^\omega \dashrightarrow \omega^\omega$ be a Baire class α realizer of $f : \mathbb{X} \rightrightarrows \mathbb{Y}$. Let $\langle \sigma_n \,;\, n \in \omega \rangle$ be some enumeration of $\omega^{<\omega}$. Since F is Baire class α, there exists some countable collection $\langle X_{n,m} \,;\, n, m \in \omega \rangle$ of $\mathbf{\Sigma}^0_\alpha$ sets such that $F^{-1}[\sigma_n] = \bigcup_{m\in\omega} X_{n,m}$. Since A is Wadge-complete for $\mathbf{\Pi}^0_\alpha$, for each $n, m \in \omega$ there exists a continuous $f_{n,m} : \omega^\omega \to \omega^\omega$ such that $X_{n,m} = f^{-1}_{n,m}[\omega^\omega \smallsetminus A]$. Now, defining a continuous $K : \omega^\omega \to \omega^\omega$ by $(K(x))_{\ulcorner n,m \urcorner} = f_{n,m}(x)$, we have $\sigma_n \subseteq F(x)$ iff $x \in X_{n,m}$ for some m iff $\hat{\chi_A}(K(x))(\ulcorner n, m \urcorner) = 1$ for some m. Finally, defining a continuous $H : \omega^\omega \dashrightarrow \omega^\omega$ by $H(x) = \bigcup\{\sigma_n \,;\, \exists m(x(\ulcorner n, m \urcorner) = 1\}$ with its natural domain, we have $H\hat{\chi_A}K \preceq F$. Therefore $F \leqslant_{\mathrm{s}\mathfrak{W}} \hat{\chi_A}$, and $f \leqslant_{\mathrm{s}\mathfrak{W}} \hat{\chi_A}$ as well. $\square$

3.52 Corollary. *For each $\alpha > 0$ the parallelization of the map $\mathrm{id}_\alpha : \mathbb{S}(\Pi^0_{\alpha\uparrow}) \to \underline{2}$ is t-Weihrauch-complete for Baire class α.*

Proof. It is enough to show that for each $\alpha > 0$ the characteristic function of any $\mathbf{\Pi}^0_\alpha$ set is Weihrauch-reducible to id_α, and that id_α is Weihrauch-reducible to the characteristic function of some $\mathbf{\Pi}^0_\alpha$ set. Both of these claims can be easily proved by induction. $\square$

3.53 Corollary. *The operation* $\mathrm{PD}^\star(\,\lrcorner\,,\alpha)$ *is Weihrauch-complete for Baire class* $\alpha{\uparrow}$.

3.4.3 The transparency of the pruning derivative

3.54 Proposition. *The map* $\mathrm{isAbsent} : \mathbb{LT}\times\mathbb{CO}\times\omega \rightrightarrows \coprod_{\alpha\in\mathbb{CO}} \mathbb{S}(\Pi^0_\alpha)$*, mapping* (Υ,α,ℓ) *such that* $\mathrm{e}_{\mathbb{LT}}(\ell)$ *is linear to* $(\max\{1,\alpha{\uparrow}\},b)$ *where* $\mathrm{Val}(b)=\top$ *iff* $\mathrm{e}_{\mathbb{LT}}(\ell)\not\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$*, is computable.*

Proof. We just run isPresent on (Υ,α,ℓ) then dualize the output by exchanging tags $\forall$ with $\exists$ and $\top$ with $\bot$. $\square$

3.55 Corollary. *The operation* $\mathrm{Neg} : \mathbb{LT}\times\mathbb{CO}\times\omega \rightrightarrows \mathbb{UT}$*, given by* $S\in \mathrm{Neg}(\Upsilon,\alpha,m)$ *iff* $\mathrm{PD}^\star(S,\alpha)$ *is a pruned tree and* $\mathrm{PD}^\star(S,\alpha)\neq\varnothing$ *iff* $\mathrm{e}_{\mathbb{LT}}(m)\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$*, is computable.*

Proof. Let $\Upsilon_0,\dots,\Upsilon_n$ be the linear subtrees of $\mathrm{e}_{\mathbb{LT}}(m)$, and for each $i\leqslant n$ let $\ell_i\in\omega$ be such that $\mathrm{e}_{\mathbb{LT}}(\ell_i)=\Upsilon_i$. By Propositions 3.43 and 3.46, letting $S_i\in \mathrm{Witness}\circ\mathrm{isPresent}(\Upsilon,\alpha,\ell_i)$, we have that $\mathrm{PD}^\star(S_i,\alpha)$ is pruned and $\mathrm{PD}^\star(S_i,\alpha)\neq\varnothing$ iff $\Upsilon_i\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$. Now letting $S=\bigotimes_{i\leqslant n} S_i$ we have that $\mathrm{PD}^\star(S,\alpha)$ is pruned and that $\mathrm{PD}^\star(S,\alpha)\neq\varnothing$ iff $\mathrm{e}_{\mathbb{LT}}(m)\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$, as desired. $\square$

3.56 Corollary. *The operation* $\mathrm{WitnessAbsence} : \mathbb{LT}\times\mathbb{CO}\times\omega^\omega \rightrightarrows \mathbb{UT}$*, given by* $S\in \mathrm{WitnessAbsence}(\Upsilon,\alpha,x)$ *iff in case* $\alpha>0$ *then* $\mathrm{PD}^\star(S,\alpha)$ *is a pruned tree and* $\mathrm{PD}^\star(S,\alpha)=\varnothing$ *iff* $\mathrm{e}_{\mathbb{LT}}(x(m))\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$ *for some* $m\in\omega$*, is computable.*

Proof. For each m and each linear subtree $\Upsilon^m_0,\dots,\Upsilon^m_{n_m}$ of $\mathrm{e}_{\mathbb{LT}}(x(m))$, let $\ell^m_i\in\omega$ be such that $\mathrm{e}_{\mathbb{LT}}(\ell^m_i)=\Upsilon^m_i$. By Propositions 3.54 and 3.46, letting $S^m_i\in \mathrm{Witness}\circ\mathrm{isAbsent}(\Upsilon,\alpha,\ell^m_i)$, we have that $\mathrm{PD}^\star(S^m_i,\alpha)$ is a pruned tree and $\mathrm{PD}^\star(S^m_i,\alpha)=\varnothing$ iff $\Upsilon^m_i\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$. Now letting $S^m=\bigotimes_{i\leqslant n_m} S^m_i$ we have that $\mathrm{PD}^\star(S^m,\alpha)$ is pruned and that $\mathrm{PD}^\star(S^m,\alpha)=\varnothing$ iff $\mathrm{e}_{\mathbb{LT}}(x(m))\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$. Now let $S=\boxtimes_{m\in\omega} S^m$.

Suppose $\alpha>0$, and first suppose that $\mathrm{e}_{\mathbb{LT}}(x(m))\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$ for some $m\in\omega$. Then $\mathrm{PD}^\star(S^m,\alpha)=\varnothing$, so for some $\beta<\alpha$ we have that $\mathrm{PD}^\star(S^m,\beta)$ has some finite height H. Hence $\mathrm{PD}^\star(S,\beta)$ also has some finite height H' (which depends on H and m), and therefore $\mathrm{PD}^\star(S,\alpha)=\varnothing$, as desired. Now suppose $\mathrm{e}_{\mathbb{LT}}(x(m))\not\subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$ for all $m\in\omega$. Then each $\mathrm{PD}^\star(S^m,\alpha)$ is pruned and nonempty, and therefore the same holds for $\mathrm{PD}^\star(S,\alpha)$. $\square$

3.57 Proposition (Pauly [78, Theorem 31]). *The function* $\min : \mathbb{CO}\times\mathbb{CO}\to\mathbb{CO}$ *is computable.*

3.58 Proposition. *The function* TreeWithRank : $\mathbb{CO} \rightrightarrows \mathbb{UT}$*, given by*

$$T \in \mathrm{TreeWithRank}(\alpha) \quad \textit{iff} \quad T \textit{ is a wellfounded tree and } \mathrm{rk}(T) = \alpha,$$

is computable.

Proof. We will define a computable $F \vdash \mathrm{TreeWithRank}$. Given $p \in \mathrm{dom}(\delta_{\mathrm{nK}})$, if $p(0) = 0$ we let $F(p) = 10^\omega$, i.e., a code for the tree $\{\langle\rangle\}$. If $p = 1q$, we let $F(p)$ be a code for the tree $T := \{\langle\rangle\} \cup \{\langle 0\rangle^\frown\sigma\,;\, \sigma \in \delta_{\mathbb{UT}} \circ F(p)\}$. Finally, if $p = 2q_0q_1\ldots$, we let $F(p)$ be a code for the mix of the trees coded by the $F(q_n)$. It is now routine to check that $F \vdash \mathrm{TreeWithRank}$. □

3.59 Corollary. *The operation* Pos : $\mathbb{LT} \times \mathbb{CO} \times \omega \rightrightarrows \mathbb{UT}$*, given by* $S \in \mathrm{Pos}(\Upsilon, \alpha, n)$ *iff*

$$\mathrm{PD}^\star(S, \alpha) = \begin{cases} \{\langle\rangle\}, & \textit{if } \mathrm{e}_{\mathbb{LT}}(n) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha) \\ \varnothing, & \textit{otherwise,} \end{cases}$$

is computable.

Proof. Given a labeled tree $\Upsilon = (T, \varphi)$, a countable ordinal α, and a natural number n, we output a tree S of rank $\beta := \min(\{\omega \cdot \alpha\} \cup \{\mathrm{rk}_T(\sigma)\,;\, \sigma \in \mathrm{e}_{\mathbb{LT}}(n)\})$.

We have $\mathrm{PD}^\star(S, \alpha) = \{\langle\rangle\}$ iff $\beta = \omega \cdot \alpha$ iff $\mathrm{rk}_T(\sigma) \geqslant \omega \cdot \alpha$ for each $\sigma \in \mathrm{e}_{\mathbb{LT}}(n)$ iff $\sigma \in \mathrm{PD}^\star(\Upsilon, \alpha)$ for each $\sigma \in \mathrm{e}_{\mathbb{LT}}(n)$ iff $\mathrm{e}_{\mathbb{LT}}(n) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$, and $\mathrm{PD}^\star(S, \alpha) = \varnothing$ otherwise. □

3.60 Definition. We define Graft : $\mathbb{LT} \times \mathbb{UT} \times \mathbb{UT} \times \omega^{<\omega} \to \mathbb{LT}$ by

(1) $\mathrm{Graft}(\Upsilon, S, U, \sigma) \smallsetminus \mathrm{Ext}(\omega^{<\omega}, \sigma) = \Upsilon \smallsetminus \mathrm{Ext}(\omega^{<\omega}, \sigma)$

(2) $\mathrm{Conc}(\mathrm{Graft}(\Upsilon, S, U, \sigma), \sigma) = (\mathrm{Conc}(\Upsilon, \sigma) \otimes S) \oplus U$

3.61 Definition. We define Aux : $\mathbb{LT} \times \mathbb{LT} \times \mathbb{UT} \times \mathbb{CO} \times \omega^{<\omega} \times \omega^{<\omega} \rightrightarrows \mathbb{LT}$ as follows. Given $\Upsilon, \Upsilon_{\mathrm{aux}} \in \mathbb{LT}$, $\alpha \in \mathbb{CO}$, and $\sigma, \tau \in \omega^{<\omega}$ such that $|\sigma| = |\tau| > 0$, let $\Upsilon' \in \mathrm{Aux}(\Upsilon, \Upsilon_{\mathrm{aux}}, U, \alpha, \sigma, \tau)$ iff $\Upsilon' = \mathrm{Graft}(\Upsilon, S_{\mathrm{N}}, S_{\mathrm{P}} \otimes U, \sigma)$ for some

$$\begin{aligned} S_{\mathrm{N}} &\in \mathrm{Neg}(\Upsilon_{\mathrm{aux}}, \alpha, \bot(\tau)) \\ S_{\mathrm{P}} &\in [\textstyle\bigotimes_{n<|\tau|-1}(\mathrm{Neg}(\Upsilon_{\mathrm{aux}}, \alpha, \tau(n)))] \otimes \mathrm{Pos}(\Upsilon_{\mathrm{aux}}, \alpha, \bot(\tau)). \end{aligned}$$

Recall from Theorem 1.34 that every computable or continuous multi-valued function between represented spaces is tightened by a strongly computable or strongly continuous, respectively, multi-valued function between the same spaces. Therefore, in order to conclude that $\mathrm{PD}^\star(\,\_\,, \alpha)$ is transparent for each α, it is enough to prove the following stronger result.

3.62 Theorem. *There is a computable operation* Trans : $\mathcal{M}(\mathbb{AT}, \mathbb{AT}) \times \mathbb{CO} \rightrightarrows \mathcal{M}(\mathbb{AT}, \mathbb{AT})$ *such that* $g \in \mathrm{Trans}(f, \alpha)$ *iff* $\mathrm{dom}(f \circ \mathrm{PD}^\star(\,\_\,, \alpha)) \subseteq \mathrm{dom}(g)$ *and* $\mathrm{PD}^\star(\mathcal{A}_g, \alpha) \in f(\mathrm{PD}^\star(\mathcal{A}, \alpha))$ *for any* $\mathcal{A} \in \mathrm{dom}(f \circ \mathrm{PD}^\star(\,\_\,, \alpha))$ *and* $\mathcal{A}_g \in g(\mathcal{A})$*.*

Proof. Let $f \in \mathcal{M}(\mathbb{AT}, \mathbb{AT})$ be given in the form of a Turing machine M which strongly computes f with some given oracle q. Let $F : \omega^\omega \dashrightarrow \omega^\omega$ be defined with $\mathrm{dom}(F) = \mathrm{dom}(f\delta_{\mathbb{AT}})$ by letting $F(p)$ be the output of M on input $\ulcorner p, 0^\omega \urcorner$ and oracle q. Thus F is a computable realizer of f, so by Corollary 3.32 we can assume that for each m there exists a computable subset $X_m \subseteq \omega$ such that $\mathrm{e}_{\mathbb{LT}}(m) \subseteq \delta_{\mathbb{LT}} F(p)$ iff $\mathrm{e}_{\mathbb{LT}}(n) \subseteq \delta_{\mathbb{LT}}(p)$ for some $n \in X_m$. Thus we can construct a computable labeled tree Υ_F which *represents* F, as follows. The nodes of length 1 of Υ_F are bijectively associated to the pairs (n, ℓ) such that $\mathrm{e}_{\mathbb{LT}}(\ell)$ is a linear tree of height 1 and $n \in X_\ell$. If $\sigma \in \Upsilon_F$ is associated to (n, ℓ), then

(1) the label of σ in Υ_F is the label of the node of $\mathrm{e}_{\mathbb{LT}}(\ell)$ at height $|\sigma|$, and

(2) the children of σ in Υ_F are bijectively associated to the pairs (n', ℓ') such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $|\sigma| + 1$, $n' \in X_{\ell'}$, and $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{e}_{\mathbb{LT}}(\ell')$.

It is now straightforward to check that if $\delta_{\mathbb{LT}} \circ F(p)$ is not empty then it is bisimilar to the subtree Υ_p of Υ_F composed of the root plus those σ which are associated to (n, ℓ) with $\mathrm{e}_{\mathbb{LT}}(n) \subseteq \delta_{\mathbb{LT}}(p)$.

Formally, our goal now is to computably define a Turing machine M' from M, q, and α, such that the function $g := g_{M',q}$ from the proof of Theorem 1.34 has the desired properties. To simplify the presentation, we will define g directly and leave the definition of M' implicit. Thus, we want to define a computable $g : \mathbb{LT} \rightrightarrows \mathbb{LT}$ such that for any $p \in \mathrm{dom}(F)$ and any $\Upsilon' \in g(\delta_{\mathbb{LT}}(p))$, letting $\delta_{\mathbb{LT}}(p') = \mathrm{PD}^\star(\delta_{\mathbb{LT}}(p), \alpha)$, we have:

(1) if $\delta_{\mathbb{LT}} \circ F(p) \neq \varnothing$ then $\mathrm{PD}^\star(\Upsilon', \alpha) \rightleftarrows \Upsilon_{p'}$;

(2) if $\delta_{\mathbb{LT}} \circ F(p) = \varnothing$ then $\mathrm{PD}^\star(\Upsilon', \alpha) = \varnothing$.

Again, since F is computable, there exists a computable $z \in \omega^\omega$ such that $\delta_{\mathbb{LT}} \circ F(p) = \varnothing$ iff $\mathrm{e}_{\mathbb{LT}}(z(n)) \subseteq \delta_{\mathbb{LT}}(p)$ holds for some $n \in \omega$. Given $p \in \mathrm{dom}(f \circ \mathrm{PD}^\star(\,\_\,, \alpha) \circ \delta_{\mathbb{AT}})$, let $\Upsilon := \delta_{\mathbb{LT}}(p)$ and $U \in \mathrm{WitnessAbsence}(\Upsilon, \alpha, z)$. Therefore, if $\delta_{\mathbb{LT}} \circ F(p) \neq \varnothing$ then $\mathrm{e}_{\mathbb{LT}}(z(m)) \not\subseteq \delta_{\mathbb{LT}}(p)$ for all $m \in \omega$ and thus $\mathrm{PD}^\star(U, \alpha)$ is pruned and nonempty, and if $\delta_{\mathbb{LT}} \circ F(p) = \varnothing$ then $\mathrm{e}_{\mathbb{LT}}(z(m)) \subseteq \delta_{\mathbb{LT}}(p)$ for some $m \in \omega$ and thus $\mathrm{PD}^\star(U, \alpha) = \varnothing$. Let $V \in \mathrm{TreeWithRank}(\omega \cdot \alpha)$, so that $\mathrm{PD}^\star(V, \alpha) = \{\langle\rangle\}$. Let $\Upsilon_0 = (\Upsilon_F \oplus V) \otimes U$, so that if $\delta_{\mathbb{LT}} \circ F(p) \neq \varnothing$ then $\mathrm{PD}^\star(\Upsilon_0, \alpha) \rightleftarrows \mathrm{PD}^\star(\Upsilon_F, \alpha)$, and if $\delta_{\mathbb{LT}} \circ F(p) = \varnothing$ then $\mathrm{PD}^\star(\Upsilon_0, \alpha) = \varnothing$. We let any node in Υ_0 coming from Υ_F be associated to the same pair (n, ℓ) as the corresponding node in Υ_F.

Now suppose we are at stage $s > 0$ of the construction, so that we have already built a tree Υ_{s-1}. Let $\sigma := \mathsf{bij}(s)$, with $\mathsf{bij} : \omega \to \omega^{<\omega}$ the bijection fixed in Convention 1.8. If $\sigma \notin \Upsilon_{s-1}$ or $\sigma \in \Upsilon_{s-1}$ but is not associated to any (n, ℓ), then let $\Upsilon_s = \Upsilon_{s-1}$. Otherwise suppose $\sigma \restriction (m+1)$ is associated to some (n_m, ℓ_m) for each $m < |\sigma|$. Let $*\sigma := \langle n_0, \ldots, n_{|\sigma|-1} \rangle$ and define $\Upsilon_s := \mathrm{Aux}(\Upsilon_{s-1}, \Upsilon, U, \alpha, \sigma, *\sigma)$. Recall that in this case we have

$$\mathrm{Conc}(\Upsilon_s, \sigma) = (\mathrm{Conc}(\Upsilon_{s-1}, \sigma) \otimes S_{\mathrm{N}}) \oplus (S_{\mathrm{P}} \otimes U)$$

for some $S_{\mathrm{N}}, S_{\mathrm{P}}$ as in the definition of Aux. Hence we let each descendant $\sigma^\frown\tau$ of σ in Υ_s in which τ comes from the $\mathrm{Conc}(\Upsilon_{s-1},\sigma)\otimes S_{\mathrm{N}}$ component of $\oplus$ above be associated to the same (n,ℓ) as the corresponding node in Υ_{s-1}.

We then define Υ' by letting $\sigma\in\Upsilon'$ iff $\sigma\in\Upsilon_s$ for $s=\mathsf{bij}^{-1}(\sigma)$, with the label for σ being its label in Υ_s in the positive case.

3.63 Claim. *Every node of* $\mathrm{PD}^\star(\Upsilon',\alpha)$ *other than the root is associated to some pair* (n,ℓ).

Indeed, it is easy to see that this is true of Υ_0. Thus if a node $\xi\in\mathrm{PD}^\star(\Upsilon',\alpha)$ is not associated to some such pair, this means that ξ was added to Υ' at some stage $s>0$ of the construction. Let $\sigma=\mathsf{bij}(s)$. In this case we have $\Upsilon_s:=\mathrm{Aux}(\Upsilon_{s-1},\Upsilon,U,\alpha,\sigma,*\sigma)$, i.e., $\Upsilon_s=\mathrm{Graft}(\Upsilon_{s-1},S_{\mathrm{N}},S_{\mathrm{P}}\otimes U,\sigma)$ for some

$$\begin{array}{rcl} S_{\mathrm{N}} & \in & \mathrm{Neg}(\Upsilon,\alpha,\bot(*\sigma)) \text{ and} \\ S_{\mathrm{P}} & \in & [\bigotimes_{m<|\sigma|-1}(\mathrm{Neg}(\Upsilon,\alpha,*\sigma(m)))]\otimes\mathrm{Pos}(\Upsilon,\alpha,\bot(*\sigma)). \end{array}$$

The fact that ξ is not associated to any pair (n,ℓ) implies that $\xi=\sigma^\frown\eta$ for some $\eta\neq\langle\rangle$ coming from $S_{\mathrm{P}}\otimes U$. By construction the subtree of ξ in Υ' is the same as in Υ_s, since for any $s'>s$ such that $\mathsf{bij}(s')\supseteq\xi$ we have $\Upsilon_{s'}=\Upsilon_{s'-1}$, and for any $s'>s$ such that $\sigma'=\mathsf{bij}(s')\not\supseteq\xi$ we have $\Upsilon_{s'}\setminus\mathrm{Ext}(\omega^{<\omega},\sigma')=\Upsilon_{s'-1}\setminus\mathrm{Ext}(\omega^{<\omega},\sigma')$. Hence $\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon',\alpha),\xi)\subseteq\mathrm{Conc}(\mathrm{PD}^\star(S_{\mathrm{P}}\otimes U,\alpha),\eta)=\varnothing$, i.e., $\xi\notin\mathrm{PD}^\star(\Upsilon',\alpha)$.

3.64 Claim. *If* $\delta_{\mathbb{LT}}\circ F(p)=\varnothing$ *then* $\mathrm{PD}^\star(\Upsilon',\alpha)=\varnothing$.

Indeed, if $\delta_{\mathbb{LT}}\circ F(p)=\varnothing$ then $\mathrm{PD}^\star(U,\alpha)=\varnothing$. Hence $\mathrm{PD}^\star(\Upsilon_0,\alpha)=\varnothing$, and at each stage $s>0$ we either keep $\Upsilon_s=\Upsilon_{s-1}$, or else Υ_s differs from Υ_{s-1} only in that

$$\mathrm{Conc}(\Upsilon_s,\sigma)=(\mathrm{Conc}(\Upsilon_{s-1},\sigma)\otimes S_{\mathrm{N}})\oplus(S_{\mathrm{P}}\otimes U)$$

for some $S_{\mathrm{N}}, S_{\mathrm{P}}$ as in the definition of $\mathrm{Aux}(\Upsilon_{s-1},\Upsilon,U,\alpha,\sigma,*\sigma)$, where $\sigma=\mathsf{bij}(s)$. But then we have that

$$\begin{aligned} &\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_s,\alpha),\sigma)\\ &\qquad=(\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_{s-1},\alpha),\sigma)\otimes\mathrm{PD}^\star(S_{\mathrm{N}},\alpha))\oplus\mathrm{PD}^\star(S_{\mathrm{P}}\otimes U,\alpha)\\ &\qquad=\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_{s-1},\alpha),\sigma)\otimes\mathrm{PD}^\star(S_{\mathrm{N}},\alpha), \end{aligned}$$

so assuming by induction that $\mathrm{PD}^\star(\Upsilon_{s-1},\alpha)=\varnothing$ holds, it follows that we have $\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_s,\alpha),\sigma)=\varnothing$ as well. But then $\mathrm{PD}^\star(\Upsilon_s,\alpha)=\varnothing$, as desired. Therefore we have $\mathrm{PD}^\star(\Upsilon',\alpha)=\varnothing$.

For the rest of the proof we assume that $\delta_{\mathbb{LT}}\circ F(p)\neq\varnothing$, which implies that $\mathrm{PD}^\star(U,\alpha)$ is a pruned and nonempty tree. Furthermore, since $\mathrm{PD}^\star(V,\alpha)=\{\langle\rangle\}$, we have $\langle\rangle\in\mathrm{PD}^\star(\Upsilon',\alpha)$.

3.65 Claim. *Suppose* $\sigma\in\Upsilon'\setminus\{\langle\rangle\}$. *Then* $\sigma\in\mathrm{PD}^\star(\Upsilon',\alpha)$ *iff* $\mathrm{e}_{\mathbb{LT}}(*\sigma(m))\subseteq\mathrm{PD}^\star(\Upsilon,\alpha)$ *for each* $m<|\sigma|$.

Let $s = \mathsf{bij}^{-1}(\sigma)$. Suppose $\mathrm{e}_{\mathbb{LT}}(*\sigma(m)) \not\subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for some $m < |\sigma|$. Let $s' = \mathsf{bij}^{-1}(\sigma\restriction(m+1))$. Note that $\mathrm{PD}^\star(S_\mathrm{N}, \alpha) = \mathrm{PD}^\star(S_\mathrm{P} \otimes U, \alpha) = \varnothing$ for any $S_\mathrm{N} \in \mathrm{Neg}(\Upsilon, \alpha, *\sigma(m))$ and $S_\mathrm{P} \in \mathrm{Pos}(\Upsilon, \alpha, *\sigma(m))$. Thus we also have $\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_s, \alpha), \sigma\restriction(m+1)) = \varnothing$. Put together, and also considering the preceding claim, the two last statements imply $\tau \notin \mathrm{PD}^\star(\Upsilon_{s''}, \alpha)$ for any $\tau \supseteq \sigma\restriction(m+1)$. Thus $\sigma \notin \mathrm{PD}^\star(\Upsilon', \alpha)$. Conversely, suppose $\mathrm{e}_{\mathbb{LT}}(*\sigma(m)) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for every $m < |\sigma|$. Then for any

$$S_\mathrm{P} \in [\bigotimes_{m<|\sigma|-1} (\mathrm{Neg}(\Upsilon, \alpha, *\sigma(m)))] \otimes \mathrm{Pos}(\Upsilon, \alpha, \perp(*\sigma))$$

we have $\mathrm{PD}^\star(S_\mathrm{P} \otimes U, \alpha) = \mathrm{PD}^\star(S_\mathrm{P}, \alpha) = \{\langle\rangle\}$. In particular, since $\Upsilon_s = \mathrm{Aux}(\Upsilon_{s-1}, \Upsilon, U, \alpha, \sigma, *\sigma)$, it follows that the descendants of σ in Υ_s which are not associated to any (n, ℓ) already guarantee that $\sigma \in \mathrm{PD}^\star(\Upsilon', \alpha)$, as desired.

Let p' be such that $\delta_{\mathbb{LT}}(p') = \mathrm{PD}^\star(\Upsilon, \alpha)$.

3.66 Claim. *The trees* $\mathrm{PD}^\star(\Upsilon', \alpha)$ *and* $\Upsilon_{p'}$ *are bisimilar.*

Define $B \subseteq \mathrm{PD}^\star(\Upsilon', \alpha) \times \Upsilon_{p'}$ by letting $\sigma\ B\ \tau$ iff $\sigma = \tau = \langle\rangle$ or $|\sigma| = |\tau|$, $\sigma\restriction n\ B\ \tau\restriction n$ for each $n < |\sigma|$, and σ and τ are associated to the same pair (n, ℓ). In order to verify that B is a bisimulation, the only nontrivial properties to check are (back) and (forth). So suppose $\sigma\ B\ \tau$ and for (back) let τ' be a child of τ in $\Upsilon_{p'}$. Then τ' is associated to some (n', ℓ') where $\mathrm{e}_{\mathbb{LT}}(n') \subseteq \delta_{\mathbb{LT}}(p') = \mathrm{PD}^\star(\Upsilon, \alpha)$. But then by construction σ has some child σ' in Υ' associated to (n', ℓ'). By Claim 3.65 we have $\sigma' \in \mathrm{PD}^\star(\Upsilon', \alpha)$, and $\sigma'\ B\ \tau'$ follows. Finally, for (forth) let σ' be a child of σ in $\mathrm{PD}^\star(\Upsilon', \alpha)$. Again by Claim 3.65 we get that σ' is associated to some (n', ℓ') such that $\mathrm{e}_{\mathbb{LT}}(n') \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. But then τ must have a child τ' in $\Upsilon_{p'}$ which is also associated to (n', ℓ'), and therefore $\sigma'\ B\ \tau'$.

Our assumption that $\delta_{\mathbb{LT}} \circ F(p) \neq \varnothing$ implies that both $\mathrm{PD}^\star(\Upsilon', \alpha)$ and $\Upsilon_{p'}$ are nonempty trees, and $B \neq \varnothing$. Hence we have $\mathrm{PD}^\star(\Upsilon', \alpha) \rightleftarrows \Upsilon_{p'}$ as desired. □

3.67 Theorem. *The operation* $\mathrm{PD}^\star(\,\_\,, \alpha)$ *is a transparent cylinder.*

Proof. Transparency follows directly from Theorem 3.62. To see that $\mathrm{PD}^\star(\,\_\,, \alpha)$ is a cylinder, given a code p of an abstract tree $\mathcal{A}$, let $\mathcal{A}_p$ be the abstract tree obtained from $\mathcal{A}$ by changing each of its labels ℓ to $\ulcorner 1, \ell \urcorner$ plus adding an infinite path with induced label $\langle \ulcorner 0, p(n) \urcorner \rangle_{n\in\omega}$. Then $\mathrm{PD}^\star(\mathcal{A}_p, \alpha)$ is obtained from $\mathrm{PD}^\star(\mathcal{A}, \alpha)$ by the same change of labels as above plus the addition of the same infinite path. Now both p and $\mathrm{PD}^\star(\mathcal{A}, \alpha)$ can easily be reconstructed from $\mathrm{PD}^\star(\mathcal{A}_p, \alpha)$ without needing direct access to p; in other words, $\mathrm{id}_{\omega^\omega} \times \mathrm{PD}^\star(\,\_\,, \alpha) \leqslant_{\mathrm{s}\mathfrak{W}} \mathrm{PD}^\star(\,\_\,, \alpha)$. □

Let $\mathbb{AT}_{\mathrm{lin}}$ be the subspace of $\mathbb{AT}$ composed of the linear abstract trees, and let $\mathbb{AT}^*_{\mathrm{lin}}$ be the subspace of $\mathbb{AT}_{\mathrm{lin}}$ composed of the linear abstract trees which have a unique infinite induced label. The spaces $\mathbb{AT}_{\mathrm{fb}}$ and $\mathbb{AT}^*_{\mathrm{fb}}$ are defined analogously for finitely branching trees. Note that $\mathbb{AT}^*$lin is composed exactly of the nonempty pruned linear trees. Let $\mathrm{Prune}_{\mathrm{fb}}$ be the restriction of PD to $\mathbb{AT}^*_{\mathrm{fb}}$, and note that $\mathrm{Prune}_{\mathrm{fb}} : \mathbb{AT}^*_{\mathrm{fb}} \to \mathbb{AT}^*_{\mathrm{lin}}$.

3.68 LEMMA. *The operation* $\mathrm{Prune}_{\mathrm{fb}}$ *is Weihrauch-equivalent to* $\lim$.

Proof. ($\lim \leqslant_{\mathfrak{W}} \mathrm{Prune}_{\mathrm{fb}}$) Given $p \in \mathrm{dom}(\lim)$, we can build an abstract finitely branching tree whose induced labels are exactly the sequences of the form $(p)_n \restriction n$. Since $\lim(p)$ is well defined, this tree is in the domain of $\mathrm{Prune}_{\mathrm{fb}}$; applying this map to this tree results in a linear tree with an infinite branch labeled $\lim(p)$.

($\mathrm{Prune}_{\mathrm{fb}} \leqslant_{\mathfrak{W}} \lim$) Given a name p of an abstract tree $\mathcal{A}$ in the domain of $\mathrm{Prune}_{\mathrm{fb}}$, let $\Upsilon = \delta_{\mathbb{LT}}(p)$ be one of its representatives. Since Υ is bisimilar to a finitely branching tree, for each $\sigma \in \Upsilon$ by Kőnig's lemma we have that $\sigma \in \mathrm{PD}(\Upsilon)$ iff $\mathrm{Conc}(\Upsilon, \sigma)$ has infinite height. Therefore deciding whether $\sigma \in \mathrm{PD}(\Upsilon)$ holds can be done with a single use of lim, and since lim is parallelizable, one application of lim suffices to decide this for all $\sigma \in \Upsilon$ at once. With this information we can construct $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$. □

3.69 THEOREM. *The operation* $\mathrm{Prune}_{\mathrm{fb}}$ *is transparent.*

Proof. The proof is a simplified version of the proof of Theorem 3.62.

Let $f : \mathbb{AT}^*_{\mathrm{lin}} \rightrightarrows \mathbb{AT}^*_{\mathrm{lin}}$ be computable. Then f has a realizer F such that for each $\tau \in \omega^{<\omega}$ there exists a computable $X_\tau \subseteq \omega^{<\omega}$ such that τ is an induced label of the tree $\delta_{\mathbb{AT}} \circ F(p)$ iff ξ is an induced label of $\delta_{\mathbb{AT}}(p)$ for some $\xi \in X_\tau$.

Let p be given and $\Upsilon := \delta_{\mathbb{LT}}(p)$. We can computably define a labeled tree Υ_G with the following properties. The nodes at level 1 of Υ_G are bijectively associated to the pairs (ξ, τ) such that $|\tau| = 1$ and $\xi \in X_\tau$ is an induced label of Υ. Recursively, if $\sigma \neq \langle\rangle$ is in Υ_G and is associated to a pair (ξ, τ), then we have:

(1) The induced label of σ in Υ_G is τ.

(2) If some node of Υ with induced label ξ has rank at least $|\tau| + 1$, then the children of σ in Υ_G are bijectively associated to the pairs (ξ', τ') such that $|\tau'| = |\tau| + 1$, $\tau' \supset \tau$, and $\xi' \in X_{\tau'}$ is an induced label of Υ; otherwise σ is a leaf of Υ_G.

3.70 CLAIM. *The trees* $\mathrm{PD}(\Upsilon_G)$ *and* $\Upsilon_{FH} := \delta_{\mathbb{LT}} \circ F \circ H(p)$ *are bisimilar.*

Let $H \vdash \mathrm{PD} : \mathbb{LT} \to \mathbb{LT}$. To see that $\mathrm{PD}(\Upsilon_G) \rightleftarrows \Upsilon_{FH}$, let $\sigma \; B \; \tau$ iff $\sigma = \tau = \langle\rangle$ or σ and τ have the same induced labels in $\mathrm{PD}(\Upsilon_G)$ and Υ_{FH}, respectively. Now suppose $\sigma \; B \; \tau$, and let σ be associated to (ξ_0, τ_1). Let σ' be a child of σ in $\mathrm{PD}(\Upsilon_G)$. It follows that σ' is associated to some pair (ξ_1, τ_1) such that $\tau_0 \subseteq \tau_1$. Since σ' is in the pruning derivative of $\delta_{\mathbb{LT}} \circ G(p)$, by condition 2 of the construction it follows that there are nodes of Υ of arbitrary length whose labels extend ξ_1. Since Υ is bisimilar to a finitely branching tree, this implies that some node ν of Υ with induced label ξ_1 is the root of a subtree of Υ of infinite height. Thus ν is in $\delta_{\mathbb{LT}} \circ H(p)$, and since $\xi_1 \in X_{\tau_1}$ it follows that some node with induced label τ_1 is in Υ_{FH}. Finally, since Υ_{FH} is linear, it follows that τ has a child τ' with induced label τ_1, and thus $\sigma' \; B \; \tau'$. Conversely, let τ' be a child of τ in Υ_{FH}, and let τ_1

be its induced label. Therefore, some $\xi_1 \in X_{\tau_1}$ is an induced label in $\mathrm{PD}(\Upsilon)$, and thus some node ν of Υ has τ_1 as its induced label and is the root of a subtree of Υ of infinite height. This implies that some child σ' of σ in Υ_G is associated to (ξ_1, τ_1), and that such σ' is also in $\mathrm{PD}(\Upsilon_G)$. Therefore $\sigma' \mathrel{B} \tau'$. Finally, note that Υ_{FH} contains an infinite path since $f : \mathbb{AT}^*_{\mathrm{lin}} \rightrightarrows \mathbb{AT}^*_{\mathrm{lin}}$, which implies that Υ_G and $\mathrm{PD}(\Upsilon_G)$ also contain an infinite path. Therefore $B \neq \varnothing$ and $\mathrm{PD}(\Upsilon_G)$ is bisimilar to Υ_{FH}.

3.71 Claim. *The tree Υ_G is bisimilar to a finitely branching tree.*

By construction, nodes of Υ_G which have the same induced label have bisimilar (indeed, isomorphic) subtrees. Thus if some node σ of Υ_G has infinitely many children σ_n which are roots of non-bisimilar subtrees, then the labels of the σ_n are pairwise distinct. Therefore the σ_n must be associated to elements (ξ_n, τ_n) such that the τ_n are pairwise $\subseteq$-incomparable. But Υ is bisimilar to a finitely branching tree; thus in particular only finitely many different labels occur on each of its levels. This implies that $\lim_{n\in\omega} |\tau_n| = \infty$, and therefore arbitrarily long prefixes of the infinite induced label of Υ occur among the prefixes of the τ_n. But then we cannot have that all ξ_n have the same length $|\sigma| + 1$, a contradiction. □

3.72 Lemma. *The operation* $\mathrm{Prune}_{\mathrm{fb}}$ *is a cylinder.*

Proof. Given a name p of an abstract tree $\mathcal{A} \in \mathrm{dom}(\mathrm{Prune}_{\mathrm{fb}})$, let $\mathcal{A}_p$ be the tree obtained from $\mathcal{A}$ by changing the label ℓ of any node $\sigma \neq \langle\rangle$ to $\ulcorner \ell, p(|\sigma|)\urcorner$. Then $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A}_p)$ is obtained from $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$ via the same transformation, and since $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$ has an infinite branch, it is easy to reconstruct both p and $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$ from $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A}_p)$. In other words, $\mathrm{id}_{\omega^\omega} \times \mathrm{Prune}_{\mathrm{fb}} \leqslant_{\mathrm{s}\mathfrak{W}} \mathrm{Prune}_{\mathrm{fb}}$. □

3.4.4 Games for functions of a fixed Baire class

For an even ordinal α , let Prune^α be the corestriction of $\mathrm{PD}^\star(\,\_\,, \alpha{\downarrow})$ to $\mathbb{AT}^*_{\mathrm{lin}}$, let $\mathrm{Prune}^\alpha_{\mathrm{fb}}$ be the corestriction of $\mathrm{PD}^\star(\,\_\,, \alpha{\downarrow})$ to $\mathbb{AT}^*_{\mathrm{fb}}$, and finally let $\mathrm{Prune}^{\alpha+1} = \mathrm{Prune}_{\mathrm{fb}} \circ \mathrm{Prune}^\alpha_{\mathrm{fb}}$.

3.73 Corollary. *Let $\alpha < \omega_1$. We have that* Prune^α *is a transparent cylinder which is Weihrauch-complete for the Baire class α functions. Therefore the* $(\mathrm{Prune}^\alpha, \mathrm{Label})$*-Wadge game characterizes the Baire class α functions.*

Proof. Suppose $\alpha = \lambda + 2n$. We have that $\mathrm{PD}^\star(\,\_\,, \lambda + n)$ is a transparent cylinder which is Weihrauch-complete for the Baire class $\lambda + 2n$ functions, so to see that the same holds for $\mathrm{Prune}^{\lambda+2n}$, by Theorem 3.9 it is enough to show that $\mathbb{AT}^*_{\mathrm{lin}}$ strongly encodes ω^ω. But any $F : \omega^\omega \rightrightarrows \omega^\omega$ is easily seen to be strongly Weihrauch-equivalent to the map $F' : \omega^\omega \rightrightarrows \mathbb{AT}^*_{\mathrm{lin}}$ which assigns $x \in \mathrm{dom}(F)$ to any linear abstract tree whose unique infinite label is in $F(x)$.

Now suppose $\alpha = \lambda + 2n + 1$. Since $\mathbb{AT}^*_{\mathrm{lin}} \subseteq \mathbb{AT}^*_{\mathrm{fb}} \subseteq \mathbb{AT}$, by Proposition 3.4 and the fact that $\mathrm{Prune}^{\lambda+2n}$ is Weihrauch-complete for Baire class $\lambda + 2n$ it follows that $\mathrm{Prune}_{\mathrm{fb}}^{\lambda+2n}$ also has this property. Now, since $\mathrm{Prune}_{\mathrm{fb}}$ is a transparent cylinder which is Weihrauch-complete for the Baire class 1 functions, the result follows. $\square$

In other words, for even α, the restriction of the tree game in which the final tree built by player $\mathbb{2}$ must have $\alpha{\downarrow}^{\text{th}}$ pruning derivative bisimilar to a linear tree characterizes the Baire class α functions, and for odd α, the restriction of the tree game in which the final tree built by player $\mathbb{2}$ must have $\alpha{\downarrow}^{\text{th}}$ pruning derivative bisimilar to a finitely branching tree characterizes the Baire class α functions.

Chapter 4

Games and computable analysis on generalized Baire spaces

Summary. In this chapter we generalize some of the results from the previous chapters from the setting of Baire space ω^ω to that of *generalized Baire spaces* κ^κ with $\kappa > \omega$. In § 4.1 we briefly review definitions and results concerning generalized Baire spaces from the literature, as well as introducing a couple of our own. In § 4.2, we generalize the α-tree game, from Chapter 2, to κ^κ (Definition 4.9) and show that it characterizes the generalized version of the Baire class α functions (Corollary 4.13). Finally, in § 4.3, we introduce a notion of type-two computability for κ^κ and 2^κ, and generalize computable analysis to the generalization of the real line of cardinality 2^κ, defined by Galeotti using Conway's surreal numbers. As a *proof of concept* of our definitions, we analyze the Weihrauch degree of the intermediate value theorem for the generalized real line (Theorem 4.34).

An early version of the material presented in § 4.3 has appeared in the proceedings volume of *Computability in Europe 2017*, held 12–16 June 2017 in Turku, Finland [42], where it was awarded Best Student Paper.

Remarks on co-authorship

The material presented in § 4.3 is the result of a collaboration with Lorenzo Galeotti, building on Galeotti's Master's thesis written under the supervision of the author and Benedikt Löwe at the University of Amsterdam. This collaboration was further developed while both collaborators were Visiting Fellows at the Isaac Newton Institute for Mathematical Sciences in the program *Mathematical, Foundational and Computational Aspects of the Higher Infinite.*

Unless stated otherwise, definitions and results in § 4.2 are due to the author, and those in § 4.3 are jointly due to Galeotti and the author.

4.1 Introduction and definitions

The interest in generalized Baire spaces has been growing considerably among set theorists in recent years. Although much of the earlier interest in the topic stemmed from applications to other areas, such as classification problems in model theory (cf., e.g., [38] for a detailed account of this application), the area has evolved to a point where it is considered interesting in its own right, with a large part of the research internally motivated. The recent paper [56] contains a community-produced list of open questions in several directions of research involving generalized Baire spaces.

Given an infinite cardinal κ, we consider the space κ^κ as endowed with the so-called *bounded topology*, viz. the topology generated by the basis $\{[\sigma]\,;\,\sigma \in \kappa^{<\kappa}\}$, where as usual $[\sigma] := \{x \in \kappa^\kappa\,;\,\sigma \subset x\}$. Note that for $\kappa > \omega$ this does not coincide with the product topology of κ copies of the discrete space κ. As is customary for generalized Baire spaces, we assume $\kappa^{<\kappa} = \kappa$. This assumption guarantees the space κ^κ is reasonably *well-behaved*. For example, under this assumption it follows that each level κ-$\mathbf{\Sigma}^0_\alpha$ of the appropriate generalized version of the Borel hierarchy is closed under unions of length κ, cf. Theorem 4.1 below. Cf. [38, § 2.1] for more detailed motivations for this assumption. Since for any infinite cardinal κ we have $\kappa^{\mathrm{cf}(\kappa)} > \kappa$ (cf., e.g., [61, Lemma I.10.40]), the assumption $\kappa^{<\kappa} = \kappa$ implies that κ is regular.

We now proceed to define generalizations of some classical notions from Baire space, as laid out in § 1.2.3, to generalized Baire space κ^κ.

Given a sequence $\langle w_\alpha \rangle_{\alpha<\kappa}$ of elements in $\kappa^{<\kappa}$, we define an element $[w_\alpha]_{\alpha<\kappa} \in \kappa^\kappa$ as the concatenation of the w_α. Furthermore, as in the $\kappa = \omega$ case, given $p \in \kappa^\kappa$ and $\alpha < \kappa$, we define $(p)_\alpha \in \kappa^\kappa$ by letting $(p)_\alpha(\beta) = p(\ulcorner \alpha, \beta \urcorner)$ for all $\beta < \kappa$, where $\ulcorner \_ \urcorner$ is the Gödel pairing function from Convention 1.8.

The collection of *κ-Borel sets* is the least collection of subsets of κ^κ which contains all open sets and is closed under complementation (relative to κ^κ) and unions of length κ. The collection of κ-Borel sets is stratified into the *κ-Borel hierarchy*, in a manner entirely analogous to the $\kappa = \omega$ case (cf. § 1.2.3), including the relativized hierarchy for any $X \subseteq \kappa^\kappa$. Thus, concretely we have

$$
\begin{aligned}
\kappa\text{-}\mathbf{\Sigma}^0_0(X) &= \{\varnothing, X\} \cup \{X \smallsetminus [\sigma]\,;\,\sigma \in \kappa^{<\kappa}\}\\
\kappa\text{-}\mathbf{\Pi}^0_\alpha(X) &= \{\kappa^\kappa \smallsetminus Y\,;\,Y \in \kappa\text{-}\mathbf{\Sigma}^0_\alpha(X)\}\\
\kappa\text{-}\mathbf{\Sigma}^0_\alpha(X) &= \{\textstyle\bigcup_{\beta<\kappa} X_\beta\,;\,\forall\beta < \kappa(X_\beta \in \kappa\text{-}\mathbf{\Pi}^0_{<\alpha})\} \quad \text{for } \alpha > 0,
\end{aligned}
$$

where κ-$\mathbf{\Pi}^0_{<\alpha} = \bigcup_{\gamma<\alpha} \kappa$-$\mathbf{\Pi}^0_\gamma$.

The regularity of κ^+ implies that the κ-Borel hierarchy has length at most κ^+, and in fact with some more work than in the $\kappa = \omega$ case one can show that its length is exactly κ^+ (cf., e.g., [5, Proposition 6.2(g)]).

4.1 Theorem (Folklore). *Let $\alpha < \kappa^+$. If α is a successor ordinal, then κ-$\mathbf{\Sigma}^0_\alpha$ is closed under intersections of length less than κ. If α is a limit ordinal, then κ-$\mathbf{\Sigma}^0_\alpha$ is closed under intersections of length less than the cofinality of α.*

Proof. First note that if $\beta < \kappa$ then the intersection of β-many basic open sets is either $\varnothing$ or a basic open set. Indeed, let $\langle \sigma_\gamma \,;\, \gamma < \beta \rangle$ be a sequence of elements of $\kappa^{<\kappa}$. If there exist $\delta, \delta' < \beta$ such that σ_δ and $\sigma_{\delta'}$ are incompatible, then $\bigcap_{\delta<\beta}[\sigma_\delta] = \varnothing$. Otherwise, the regularity of κ implies that $\bigcup_{\delta<\beta} \sigma_\delta$ is an element of $\kappa^{<\kappa}$. Since $\bigcap_{\delta<\beta}[\sigma_\delta] = [\bigcup_{\delta<\beta} \sigma_\delta]$, this establishes the claim.

The general result now follows by using Skolemization as follows. Let $A_\gamma \in \kappa$-$\mathbf{\Sigma}^0_\alpha$ for each $\gamma < \beta < \kappa$, where if α is a limit ordinal we have that β is less than the cofinality of α. For each $\gamma < \beta$ let $\langle A^\gamma_\delta \in \kappa$-$\mathbf{\Pi}^0_{<\alpha} \,;\, \delta < \kappa \rangle$ be such that $A_\gamma = \bigcup_{\delta<\kappa} A^\gamma_\delta$. Then, by Skolemization, we have

$$\bigcap_{\gamma<\beta} A_\gamma = \bigcup_{f\in\kappa^\beta} \bigcap_{\gamma<\beta} A^\gamma_{f(\gamma)}.$$

If α is a successor ordinal, then either using the claim above and the closure of κ-$\mathbf{\Sigma}^0_1$ under arbitrary unions, in case $\alpha = 1$, or the closure of κ-$\mathbf{\Pi}^0_{\alpha-1}$ under intersections of length at most κ, the fact that $\kappa^{<\kappa} = \kappa$ implies $|\kappa^\beta| = \kappa$, and the closure of κ-$\mathbf{\Sigma}^0_\alpha$ under unions of length κ, in case $\alpha > 1$, it follows that $\bigcap_{\gamma<\beta} A_\gamma \in \kappa$-$\mathbf{\Sigma}^0_\alpha$ as well. Finally, if α is a limit ordinal, then since β is less than the cofinality of α it follows that for each $f : \beta \to \kappa$ we have $\bigcap_{\gamma<\beta} A^\gamma_{f(\gamma)} \in \kappa$-$\mathbf{\Pi}^0_{<\alpha}$. Therefore $\bigcap_{\gamma<\beta} A_\gamma \in \kappa$-$\mathbf{\Sigma}^0_\alpha$. $\square$

4.2 Definition. Given a class Γ of subsets of κ^κ, we say a function $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ is (relatively) Γ-*measurable* if for any open set $X \subseteq \kappa^\kappa$ there exists $Y \in \Gamma$ such that $f^{-1}[X] = Y \cap \operatorname{dom}(f)$.

We will usually omit the word "relatively", considering it to be implied by the context. Clearly, if Γ is closed under unions of length κ, then we can equivalently substitute "open set" for "basic open set" in this definition. Recall that we denote by $\kappa^{<\kappa}_{\text{succ}}$ the subset of $\kappa^{<\kappa}$ composed of the elements whose lengths are successor ordinals. It now follows that if Γ is closed under intersections of length less than κ, then $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ is Γ-measurable iff $f^{-1}[\sigma] \in \Gamma$ holds for each $\sigma \in \kappa^{<\kappa}_{\text{succ}}$. Indeed, if $\sigma \in \kappa^{<\kappa}$ has limit length, then $f^{-1}[\sigma] = \bigcap\{f^{-1}[\sigma{\restriction}\alpha] \,;\, \alpha < |\sigma| \text{ is a successor ordinal}\}$. Therefore if each such $f^{-1}[\sigma{\restriction}\alpha]$ is in Γ then since $|\sigma| < \kappa$ it follows that $f^{-1}[\sigma] \in \Gamma$ as well.

4.3 Definition. Let $\lambda \leqslant \kappa$ be a limit ordinal. We say a sequence $s = \langle x_\alpha \in \kappa^\kappa \,;\, \alpha < \lambda \rangle$ *converges pointwise* to $x \in \kappa^\kappa$, or that x is the *pointwise limit* of s, denoted $x = \lim_{\alpha<\lambda} x_\alpha$, if for every $\beta < \kappa$ there exists $\alpha_\beta < \lambda$ such that for all α with $\alpha_\beta \leqslant \alpha < \lambda$ we have $x_\alpha{\restriction}\beta = x{\restriction}\beta$. We say a sequence $s = \langle f_\alpha : \kappa^\kappa \dashrightarrow \kappa^\kappa \,;\, \alpha < \lambda \rangle$ of functions *converges pointwise* to $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$, or that f is the *pointwise limit* of s, denoted $f = \lim_{\alpha<\lambda} f_\alpha$, if $\operatorname{dom}(f) = \operatorname{dom}(f_\alpha)$ holds for every $\alpha < \lambda$ and $f(x) = \lim_{\alpha<\lambda} f_\alpha(x)$ holds for every $x \in \operatorname{dom}(f)$.

If $\lambda < \kappa$, the regularity of κ implies that $x = \lim_{\alpha<\lambda} x_\alpha$ iff there exists $\beta < \lambda$ such that $x = x_\alpha$ holds whenever $\beta \leqslant \alpha < \lambda$, so that the convergence is trivial in this case. Similarly, we have $f = \lim_{\alpha<\lambda} f_\alpha$ iff for every $x \in \mathrm{dom}(f)$ there exists $\beta_x < \lambda$ such that $f(x) = f_\alpha(x)$ holds whenever $\beta_x \leqslant \alpha < \lambda$, but since β_x depends on x it can now be the case that $f \neq f_\alpha$ for every $\alpha < \lambda$.

4.4 Definition. Let $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ be given. We say f is of *κ-Baire class 0* if it is continuous, and recursively for $\alpha > 0$ we say f is of *κ-Baire class α* if f is the pointwise limit of a sequence $\langle f_\alpha\,;\, \alpha \leqslant \lambda\rangle$ of functions, each of which is of κ-Baire class less than α, with $\lambda \leqslant \kappa$.

The following is a completely straightforward generalization of the corresponding direction of Theorem 1.22.

4.5 Proposition (Folklore). *If $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ is κ-Baire class α, then it is κ-$\mathbf{\Sigma}^0_{\alpha+1}$-measurable.*

For the converse direction, one could generalize the classical work of Lebesgue, Hausdorff, and Banach (Theorem 1.22), but we will instead obtain that result as a corollary of our game characterization of that class of functions in the next section.

4.2 Games for functions of a fixed κ-Baire class

4.6 Definition. Let $T \subseteq \kappa^{<\kappa}$ be a κ-tree. We call T a *κ-simple* tree if for any $\alpha < \kappa$ there exists $\beta < \kappa$ such that, for any $\sigma \in \mathrm{Level}(T, \alpha)$, we have that σ has descendants at level β of T iff σ is on a κ-branch of T.

The κ-simple trees will be used as the analogues of the finitely branching trees from the $\kappa = \omega$ case. Of course, a more immediate candidate for such an analogue would be the trees in which each node has fewer than κ immediate successors, or the trees whose levels have cardinality strictly less than κ. These last two conditions are of course equivalent for $\kappa = \omega$, but this is not the case in general. For example, if CH holds and $\kappa = \omega_1$ (in particular $\kappa^{<\kappa} = \kappa$), then in the tree $T = 2^{<\kappa}$ each node has two immediate successors, but there are $2^\omega = \kappa$ nodes of length ω in T. Now, by Kőnig's lemma, any finitely branching ω-tree is ω-simple. However, for $\kappa > \omega$ the situation is more complex, and depending on the value of κ and the ambient set theory, the relationship between a tree being κ-simple and having levels with cardinality strictly less than κ can vary. A κ-tree which has fewer than κ-many nodes at each level, height κ, and no κ-branches is an instance of what is called a *κ-Aronszajn* tree in the literature. Thus a κ-Aronszajn tree is not κ-simple, even though it has levels with cardinality strictly less than κ. The nonexistence of κ-Aronszajn trees for a given cardinal κ is called the *tree property* for κ, and an inaccessible cardinal for which the tree property holds is called *weakly compact*, a well-studied and important large cardinal property. Now, Kőnig's lemma is precisely the statement of the tree property for ω, it is provable in ZFC that the tree property for ω_1 fails (cf., e.g., [61, Theorem II.5.9]), and if CH holds then the tree property for ω_2 fails as well (cf., e.g., [61, Exercise II.37]). On the other hand, Mitchell has shown that the tree property for ω_2 is independent of $\mathsf{ZFC} + 2^\omega = \omega_2$ [64]. Since it is actually ω-simplicity that plays a key role in the proof of Theorem 2.31, for the purposes of this section it is the κ-simple trees which are important.

4.7 DEFINITION. Given a κ-tree T, we define its *κ-pruning derivative* by

$$\mathrm{PD}_\kappa(T) := \{\sigma \in T\,;\, \forall\alpha < \kappa\, \exists\tau \in \mathrm{Conc}(T,\sigma)(|\tau| \geqslant \alpha)\}.$$

As before, we define the iterated variant $\mathrm{PD}^\star_\kappa(\smile,\smile)$ by the recursion

(1) $\mathrm{PD}^\star_\kappa(T,0) = T$;

(2) $\mathrm{PD}^\star_\kappa(T,\alpha+1) = \mathrm{PD}_\kappa(\mathrm{PD}^\star_\kappa(T,\alpha))$;

(3) $\mathrm{PD}^\star_\kappa(T,\lambda) = \bigcap_{\alpha<\lambda} \mathrm{PD}^\star_\kappa(T,\alpha)$, for λ a limit ordinal.

Again, it is easy to see that for any κ-tree T we have $T = \mathrm{PD}_\kappa(T)$ iff T is pruned, i.e., if every node of T lies on some κ-branch of T. Furthermore, for any κ-tree T there exists an ordinal $\alpha < \kappa^+$ such that $\mathrm{PD}^\star_\kappa(T,\alpha) = \mathrm{PD}^\star_\kappa(T,\alpha+1)$.

4.8 DEFINITION. A *game for functions on κ^κ*, or simply a *game* in the remainder of this section, is the natural generalization of a game for function on ω^ω, Definition 2.1, to the setting of κ^κ. The crucial difference is that sets of rules for a player are sets of κ-sequences of moves for that player. The notions of *runs*, *strategies*, etc., are entirely analogous to the ω case.

4.9 DEFINITION. Given $\alpha < \kappa^+$, the (α,κ)-tree game is the game where player $\mathbb{1}$ plays elements of κ, with rule set κ^κ and interpretation function $\mathrm{id}_{\kappa^\kappa}$, and $\mathbb{2}$ plays labeled κ-trees of cardinality strictly less than κ, with the rules that the sequence of labeled κ-trees played must be a chain with respect to $\subseteq$, that the limit Υ of this chain, which we call the *final tree*, must have exactly one κ-branch, and that

(1) the tree $\mathrm{PD}^\star_\kappa(\Upsilon,\alpha{\downarrow})$ must be linear;

(2) the tree $\mathrm{PD}^\star_\kappa(\Upsilon,\alpha{\downdownarrows})$ must be κ-simple.

The interpretation function for $\mathbb{2}$ associates to the final tree the sequence of labels along its unique κ-branch.

4.10 THEOREM. *The (α,κ)-tree game characterizes the class of κ-$\mathbf{\Sigma}^0_{\alpha+1}$-measurable functions on κ^κ.*

The proof of Theorem 4.10 is a generalization of the proof of Theorem 2.50, which is itself a modified version of the proof of Theorem 2.31. In general, the generalization goes through in a straightforward way, with, e.g., the role of ω being played by κ and that of $\omega^{<\omega}_{\neq 0}$ by $\kappa^{<\kappa}_{\mathrm{succ}}$. In the interest of avoiding tedious repetitions, we will provide a sketch of the proof of Theorem 4.10 by only going through the parts of the proofs of Theorems 2.50 and 2.31 where the generalization is not completely straightforward.

The proof that $\mathbb{2}$ can only have a winning strategy in the (α, κ)-tree game for $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ in case f is κ-$\mathbf{\Sigma}^0_{\alpha+1}$-measurable is entirely analogous to the proof of the corresponding direction of Theorem 2.31. For odd α, the requirement that the final tree Υ must satisfy that $\mathrm{PD}^\star_\kappa(\Upsilon, \alpha_\downarrow)$ is κ-simple is precisely what allows the proof of the corresponding case of Theorem 2.31 to generalize directly.

Moving on to the proof of the converse direction, let $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ be a fixed function of κ-$\mathbf{\Sigma}^0_{\alpha+1}$-measurable.

We begin by fixing bijections $\mathsf{bij}_\kappa : \kappa \to \kappa^{<\kappa}$ and $\mathsf{bij}_{\kappa,\alpha} : \kappa \to \kappa^\alpha$ for each $0 < \alpha < \kappa$; such bijections exist because of the assumption $\kappa^{<\kappa} = \kappa$. Following the same overload of notation from the $\kappa = \omega$ case, we denote all of the inverses of the bijections $\mathsf{bij}_{\kappa,\alpha}$ by $\ulcorner \_ \urcorner$, calling these inverses *tupling functions*; thus

$$\mathsf{bij}_{\kappa,\alpha}(\beta) = \sigma \quad \text{iff} \quad |\sigma| = \alpha \text{ and } \beta = \ulcorner \sigma(\gamma)\, ;\, \gamma < \alpha \urcorner.$$

The definitions of the set of *trails* $\mathbb{T}$ and of the operations $\oplus$ and $\ominus$ now become

$$\begin{aligned} \mathbb{T} &:= \{\langle \sigma, \beta_0, \ldots, \beta_{k-1}\rangle\, ;\, \sigma \in \kappa^{<\kappa}_{\mathrm{succ}},\ k \in \omega, \text{ and } \beta_i \in \kappa \text{ for each } i < k\} \\ u \oplus (\lambda + n) &:= u^\frown\langle \lambda + 2n\rangle \\ u \ominus (\lambda + n) &:= u^\frown\langle \lambda + 2n + 1\rangle, \end{aligned}$$

for any $u \in \mathbb{T} \cup \kappa^{<\kappa}$, limit $\lambda < \kappa$, and $n \in \omega$ (in particular note that trails are still finite sequences). We use the tupling function $\ulcorner \_ \urcorner$ to define the associating, coding, guessing, unraveling, and witnessing functions as before. However, there are now κ-many coding functions: for each successor ordinal $\alpha < \kappa$ we define $\mathfrak{c}_\alpha : \{\sigma \in \kappa^{<\kappa}_{\mathrm{succ}}\, ;\, |\sigma| \geqslant \alpha\} \to \kappa$ by letting

$$\perp(\sigma) = \ulcorner \mathfrak{c}_\alpha(\sigma)\, ;\, \alpha \leqslant |\sigma| \text{ is a successor ordinal} \urcorner.$$

The remaining definitions and lemmas from the proofs of Theorems 2.31 and 2.50 are generalized in a straightforward way; one small caveat is that, because the bijection bij_κ is arbitrary, we of course do not know if $\mathsf{bij}_\kappa(\alpha) \subset \mathsf{bij}_\kappa(\beta)$ implies $\alpha < \beta$. Therefore, in defining the strategy, equation (2.6) becomes

$$T_\alpha := \bigcup_{\beta<\alpha} T_\beta \cup \{\sigma \in \kappa^{<\kappa}\, ;\, \text{every } \tau \subseteq \sigma \text{ satisfies that } \mathsf{bij}_\kappa^{-1}(\tau) \leqslant \alpha \text{ and that } \tau \text{ is active at round } \alpha\}. \quad (2.6')$$

Thus we get a strategy ϑ for $\mathbb{2}$ such that if $\mathbb{1}$ plays $x \in \mathrm{dom}(f)$ and $\mathbb{2}$ follows ϑ, then $\mathbb{2}$ builds a tree Υ with the property that, e.g., if $\sigma \in \Upsilon$ makes a wrong guess about any trail then $\sigma \notin \mathrm{PD}^\star_\kappa(\Upsilon, \alpha\downarrow)$. Furthermore, if $\sigma \in \Upsilon$ makes a guess about a trail $t \in \mathrm{dom}(\mathfrak{g})$ and there exists $\beta < \mathfrak{g}(t)$ such that $x \notin [\![\mathfrak{a}(t \ominus \beta)]\!]$, then also as before we get $\sigma \notin \mathrm{PD}^\star_\kappa(\Upsilon, \alpha\downarrow)$. Any node which is hereditarily exact lies on a κ-branch of Υ. It follows that Υ has a unique κ-branch and that $\mathrm{PD}^\star_\kappa(\Upsilon, \alpha\downarrow)$ is a linear tree. The last lemma which is significantly different from the corresponding result in the proofs of Theorems 2.50 and 2.31, and which also wraps up the proof of Theorem 4.10, is the following.

4.11 LEMMA. *The tree* $\mathrm{PD}^\star_\kappa(\Upsilon, \alpha\downarrow)$ *is* κ*-simple.*

Proof. Let $\Upsilon' := \mathrm{PD}^\star_\kappa(\Upsilon, \alpha\downarrow)$. By induction on $\beta < \kappa$ we define $\gamma_\beta < \kappa$ such that, for any $\sigma \in \mathrm{Level}(\Upsilon', \beta)$, we have that σ has descendants at level γ_β of Υ' iff σ is on a κ-branch of Υ'.

If β is a limit ordinal then setting $\gamma_\beta = \sup_{\delta<\beta} \gamma_\delta$ works, since $\gamma_\beta < \kappa$ because κ is regular and furthermore if $\sigma \in \mathrm{Level}(\Upsilon', \beta)$ then σ is on a κ-branch of Υ' iff $\sigma\restriction\delta$ is on a κ-branch of Υ' for every $\delta < \beta$.

Now suppose β is a successor ordinal. Let $\mu < \kappa$ be least such that $x \in (f(x)\restriction\beta) \oplus \mu$, let $\nu < \kappa$ be such that $\mathsf{bij}_{\kappa,\beta}(\nu) = f(x)\restriction\beta$, and let $\delta = \ulcorner \nu, \mu \urcorner$. Thus for any $\delta' = \ulcorner \nu', \mu' \urcorner < \delta$ we have $x \notin [\![\mathsf{bij}_{\kappa,\beta}(\nu') \oplus \mu']\!]$. If $\mathsf{bij}_{\kappa,\beta}(\nu') \oplus \mu'$ has Borel rank 1, let $\varepsilon_{\delta'} = 0$; otherwise, let $\varepsilon_{\delta'}$ be least such that $x \notin [\![\mathsf{bij}_{\kappa,\beta}(\nu') \oplus \mu' \oplus \varepsilon_{\delta'}]\!]$. Finally, define $\gamma_\beta := \sup(\{\gamma_{\beta'}\,;\, \beta' < \beta\} \cup \{\beta + \varepsilon_{\delta'} + 1\,;\, \delta' < \delta\})$.

Suppose $\sigma \in \mathrm{Level}(\Upsilon', \beta)$ has a descendant at level γ_β of Υ'. Then, since $\gamma_\beta \geqslant \gamma_{\beta'}$ for each $\beta' < \beta$, by induction each proper initial segment of σ of successor length lies on the κ-branch of Υ', and is in particular exact. Now let ν' be such that $\mathsf{bij}_{\kappa,\beta}(\nu') = \tilde{\varphi}(\sigma)$ and let $\delta' = \ulcorner \nu', \mathfrak{w}(\sigma) \urcorner$. We have $\delta' \leqslant \delta$, since otherwise σ would overshoot and we would not have $\sigma \in \Upsilon'$. But by what we saw above, if $\delta' < \delta$ then σ has no descendant at level $\beta + \varepsilon_{\delta'} + 1 \leqslant \gamma_\beta$ of Υ', since every such descendant would have to make a wrong guess of rank less than α. Therefore $\delta' = \delta$, i.e., σ is exact as well and therefore lies on a κ-branch of Υ'. □

We can now prove the converse of Proposition 4.5.

4.12 THEOREM. *If* $f : \kappa^\kappa \dashrightarrow \kappa^\kappa$ *is* κ-$\mathbf{\Sigma}^0_{\alpha+1}$*-measurable then it is* κ*-Baire class* α*.*

Proof. Suppose f is κ-$\mathbf{\Sigma}^0_{\alpha+1}$-measurable and let ϑ be a winning strategy for $\mathbb{2}$ in the (α, κ)-tree game for f. For each $x \in \mathrm{dom}(f)$ let $(T_x, \varphi_x) = \Upsilon^\vartheta_x$.

If α is a limit ordinal, then for each $\beta < \alpha$ define $f_\beta : \mathrm{dom}(f) \to \kappa^\kappa$ by letting $\sigma \subset f_\beta(x)$ iff σ is the running label of the lexicographically-least node of length $|\sigma|$ in $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \beta)$, i.e., $\sigma \subset f(x)$ iff there exists $\tau \in \mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \beta)$ with $\tilde{\varphi}_x(\tau) = \sigma$ such that for every $\eta \in \kappa^{|\tau|}$, either $\eta \geqslant_{\mathrm{lex}} \tau$ or $\eta \notin \mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \beta)$. By the generalized version of Lemma 2.32, it follows that $f_\beta^{-1}[\sigma] \in \kappa$-$\mathbf{\Sigma}^0_{\beta'}$ for some $\beta' < \alpha$ which depends on β but not on σ. Hence f_β is κ-$\mathbf{\Sigma}^0_{<\alpha}$-measurable, and by induction f_β

is of κ-Baire class less than α. Since $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \alpha) = \bigcap_{\beta<\alpha} \mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \beta)$ is a linear tree, it follows that $f(x) = \lim_{\beta<\alpha} f_\beta(x)$. Therefore f is κ-Baire class α.

Otherwise, if $\alpha = \lambda + 2n + 1$ for some limit λ and natural n, then given $\beta < \kappa$ define $f_\beta : \mathrm{dom}(f) \to \kappa^\kappa$ as follows. For each $x \in \mathrm{dom}(f)$, by the κ-simplicity of $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda+n)$ there exists $\delta^x_\beta < \kappa$ such that any node of length β with descendants of length δ^x_β in $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$ is on the κ-branch of $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$. Now let $f_\beta(x) =$ the running label of the node of length β which has a descendant at level δ^x_β of $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$, concatenated with κ-many 0s. Thus for any $\sigma \in \kappa^{<\kappa}$ and $x \in \mathrm{dom}(f)$ we have $\sigma \subseteq f_\beta(x)$ iff $\sigma(\xi) = 0$ for any ξ such that $\beta \leqslant \xi < |\sigma|$, and furthermore there exists $\tau \in \Upsilon^\vartheta_x$ whose running label is $\sigma\restriction\beta$ and which has some descendant of length δ^x_β in $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$. Again by the generalized version of Lemma 2.32, it follows that $f^{-1}_\beta[\sigma] \in \kappa\text{-}\mathbf{\Sigma}^0_{\lambda+2n+1}$, so by induction f_β is κ-Baire class $< \alpha$. Since $f = \lim_{\beta<\kappa} f_\beta$, it follows that f is κ-Baire class α, as desired.

Finally, if $\alpha = \lambda + 2n + 2$ for some limit λ and natural n, then given $\beta < \kappa$ define $f_\beta : \mathrm{dom}(f) \to \kappa^\kappa$ by letting $\sigma \subset f_\beta(x)$ iff σ is the running label of the lexicographically-least node of length $|\sigma|$ in $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$ which has a descendant of length $|\sigma| + \beta + 1$ in $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$, i.e., $\sigma \subset f(x)$ iff there exists $\tau \in \mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$ with $\tilde\varphi_x(\tau) = \sigma$ such that

(1) the node τ has a descendant of length $|\tau| + \beta + 1$ in $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$;

(2) for every $\eta \in \kappa^{|\tau|}$, either $\eta \geqslant_{\mathrm{lex}} \tau$ or for every descendant ξ of η with length $|\eta| + \beta + 1$ we have $\xi \notin \mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n)$.

By the generalized version of Lemma 2.32, it follows that $f^{-1}_\beta[\sigma] \in \kappa\text{-}\mathbf{\Sigma}^0_{\lambda+2n+2}$. Therefore f_β is $\kappa\text{-}\mathbf{\Sigma}^0_\alpha$-measurable, and by induction has κ-Baire class less than α. Since $\mathrm{PD}^\star_\kappa(\Upsilon^\vartheta_x, \lambda + n + 1)$ is linear, it follows that $f(x) = \lim_{\beta<\kappa} f_\beta(x)$. Therefore f is κ-Baire class α. □

4.13 Corollary. *The (α, κ)-tree game characterizes the κ-Baire class α functions on κ^κ.*

4.3 Towards computable analysis on the generalized real line

4.3.1 Introduction

As mentioned in the introduction, computable analysis is largely about the study of the computational content of theorems in classical real analysis. This is done by inducing a computability notion on spaces of cardinality at most $2^{\aleph_0}$, such as $\mathbb{R}$, through coding such spaces with ω^ω or 2^ω as the space of codes, an approach similar to the theory of *numberings* in classical computability theory (cf., e.g., [35]). In [41], Galeotti provided the foundational basis for the study of *generalized* *computable analysis*, the generalization of computable analysis where the spaces of codes are generalized Baire or Cantor spaces. In particular, using Conway's surreal numbers, Galeotti introduced $\mathbb{R}_\kappa$, a generalized version of the real line suitable for doing analysis, and proved a version of the intermediate value theorem for that space.

The work presented in this section is a continuation of [40, 41], strengthening their results and answering in the positive the open question from [41] of whether a natural notion of computability exists for 2^κ. This is done by generalizing the framework of type two computability to uncountable cardinals κ such that $\kappa^{<\kappa} = \kappa$. Then we use this framework to induce a well-behaved notion of computability over the generalized real line $\mathbb{R}_\kappa$, for example showing that, as in the classical case, the field operations on $\mathbb{R}_\kappa$ are computable. Finally we generalize Weihrauch reducibility to spaces of cardinality 2^κ and give an example as a proof of concept, showing that the generalized version of the intermediate value theorem introduced in [41] is Weihrauch equivalent to a generalized version of the boundedness principle $\mathrm{B_I}$.

4.3.2 The surreal numbers

4.14 Definition (Conway [27]). A *surreal number* is a function from an ordinal α to $\{+,-\}$, i.e., a sequence of pluses and minuses of ordinal length. We denote the class of surreal numbers by No, and the set of surreal numbers of length strictly less than α by $\mathrm{No}_{<\alpha}$. The *length* of a surreal number x, denoted $|x|$, is its domain. For surreal numbers x and y, we say x is *less than* y, denoted $x < y$, if there exists $\alpha \leqslant \min\{|x|, |y|\}$ such that $x(\beta) = y(\beta)$ for all $\beta < \alpha$, and (i) $x(\alpha) = -$ and either $\alpha = |y|$ or $y(\alpha) = +$, or (ii) $\alpha = |x|$ and $y(\alpha) = +$.

In Conway's original idea, every surreal number is generated by filling some *gap* between shorter numbers. The following theorem connects this intuition to the surreal numbers as we have defined them. First, given sets of surreal numbers X and Y, we write $X < Y$ if for all $x \in X$ and $y \in Y$ we have $x < y$. In particular, if $X = \varnothing$ or $Y = \varnothing$ then $X < Y$ holds vacuously. Furthermore, given a surreal number x we write $x < X$ instead of $\{x\} < X$.

4.15 Theorem (Simplicity theorem; Conway [27, Theorem 11]). *If L and R are two sets of surreal numbers such that $L < R$, then there exists a unique surreal x of minimal length such that $L < x < R$, denoted by $[\,L \mid R\,]$. Furthermore, for every $x \in \mathrm{No}$ we have $x = [\,L \mid R\,]$ for $L = \{y \in \mathrm{No}\,;\, x > y \wedge y \subset x\}$ and $R = \{y \in \mathrm{No}\,;\, x < y \wedge y \subset x\}$. The pair L, R is called the* canonical cut *of x.*

Using the simplicity theorem Conway defined the field operations $\stackrel{s}{+}$, $\stackrel{s}{\cdot}$, $\stackrel{s}{-}$, and $\frac{1}{\cdot}{}_{s}$ (the multiplicative inverse over No), and proved that these operations satisfy the axioms of real closed fields. These operations satisfy the following, where for any (unary or binary) operation $*$, surreal number z, and sets X, Y of surreal numbers we use the notations $*X := \{*x\,;\, x \in X\}$, $z*X := \{z*x\,;\, x \in X\}$, and $X*Y := \{x*y\,;\, x \in X \text{ and } y \in Y\}$.

4.16 Theorem (Conway [27, p. 5], Gonshor [43, § 3C]). *Let $x = [\,L_x \mid R_x\,]$, $y = [\,L_y \mid R_y\,]$ be surreal numbers. We have*

$$\begin{aligned} x \overset{s}{+} y &= [\,(L_x \overset{s}{+} y) \cup (x \overset{s}{+} L_y) \mid (R_x \overset{s}{+} y) \cup (x \overset{s}{+} R_y)\,] \\ \overset{s}{-}x &= [\,\overset{s}{-}R_x \mid \overset{s}{-}L_x\,] \\ x \overset{s}{\cdot} y &= [((L_x \overset{s}{\cdot} y) \overset{s}{+} (x \overset{s}{\cdot} L_y) \overset{s}{-} (L_x \overset{s}{\cdot} L_y)) \cup ((R_x \overset{s}{\cdot} y) \overset{s}{+} (x \overset{s}{\cdot} R_y) \overset{s}{-} (R_x \overset{s}{\cdot} R_y)) \\ &\qquad \mid ((L_x \overset{s}{\cdot} y) \overset{s}{+} (x \overset{s}{\cdot} R_y) \overset{s}{-} (L_x \overset{s}{\cdot} R_y)) \cup ((R_x \overset{s}{\cdot} y) \overset{s}{+} (x \overset{s}{\cdot} L_y) \overset{s}{-} (R_x \overset{s}{\cdot} L_y))] \end{aligned}$$

Now let $z = [\,L \mid R\,]$ be a positive surreal number and assume $L > 0$. Let $r_{\langle\rangle} := 0$ and recursively for every $z_0, \dots, z_n \in (L \cup R) \smallsetminus \{0\}$ let

$$r_{\langle z_0,\dots,z_n\rangle} := [1 \overset{s}{+} ((z_n \overset{s}{-} z) \overset{s}{\cdot} r_{\langle z_0,\dots,z_{n-1}\rangle})] \overset{s}{\cdot} \tfrac{1}{z_n}{}^{s}.$$

Then we have $\frac{1}{z}{}^{s} = [\,L' \mid R'\,]$, where $L' = \{r_{\langle z_0,\dots,z_n\rangle}\,;\, n \in \omega$ and $z_i \in L$ for even-many $i \leqslant n\}$ and $R' = \{r_{\langle z_0,\dots,z_n\rangle}\,;\, n \in \omega$ and $z_i \in L$ for odd-many $i \leqslant n\}$.

By considering the sequences with constant value $+$, it is easy to see that the class of ordinal numbers can be embedded into No. Restricted to ordinals, the operations $\overset{s}{+}$ and $\overset{s}{\cdot}$ are not the usual ordinal operations, but rather the so-called *natural* or *Hessenberg* operations. It is a straightforward consequence of the fact that every ordinal numbers has a unique Cantor normal form that for each pair α, β of ordinals there exist a unique decreasing sequence of ordinals $\gamma_0 > \gamma_1 > \cdots > \gamma_n$ and unique sequences of natural numbers $m_0, \dots, m_n$ and $k_0, \dots, k_n$ such that $m_i + k_i > 0$ for every $i < n$ and

$$\begin{aligned} \alpha &= \omega^{\gamma_0} m_0 + \omega^{\gamma_1} m_1 + \cdots + \omega^{\gamma_n} m_n \\ \beta &= \omega^{\gamma_0} k_0 + \omega^{\gamma_1} k_1 + \cdots + \omega^{\gamma_n} k_n. \end{aligned}$$

Then we have

$$\begin{aligned} \alpha \overset{s}{+} \beta &= \omega^{\gamma_0}(m_0 + k_0) + \omega^{\gamma_1}(m_1 + k_1) + \cdots + \omega^{\gamma_n}(m_n + k_n) \\ \alpha \overset{s}{\cdot} \beta &= \textstyle\sum_{i,j<n} \omega^{(\gamma_i \overset{s}{+} \gamma_j)} m_i k_j, \end{aligned}$$

where the summation is to be taken in non-increasing order w.r.t. the parameter $\gamma_i \overset{s}{+} \gamma_j$. Note that, in particular, we have $\alpha \overset{s}{+} n = \alpha + n$ for every ordinal α and natural number n. The Hessenberg operations can also be characterized order-theoretically as follows.

4.17 Fact (Carruth [24, Theorems 1 and 2]). Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be disjoint wellordered sets with respective order types α and β. Then $\alpha \overset{s}{+} \beta$ is the maximum of the order types of wellorders on $X \cup Y$ which extend $\leqslant_X \cup \leqslant_Y$, and $\alpha \overset{s}{\cdot} \beta$ is the maximum of the order types of wellorders on $X \times Y$ which extend the product partial order $(x, y) \leqslant (x', y')$ iff $x \leqslant_X x'$ and $y \leqslant_Y y'$.

In the remainder of this chapter, all operations on ordinals will be the surreal operations. For this reason, and to improve readability, we will denote surreal addition by $+$, subtraction by $-$, multiplication by $\cdot$, and the multiplicative inverse by $\frac{1}{\smile}$.

4.3.3 The generalized real line

A crucial property of the real line is its Dedekind completeness, forming a cornerstone of many theorems in real analysis. However, it is a classical theorem that there exist no real closed fields which properly extend $\mathbb{R}$ and which are Dedekind complete (cf., e.g., [26, Theorem 8.7.3]). We therefore need to replace Dedekind completeness with a weaker property. This was done in [40, 41], and we repeat the central definitions here.

Let K be an ordered field. We call $\langle L, R\rangle$ a *cut* over K if $L, R \subseteq K$ and $L < R$. Moreover we say that $\langle L, R\rangle$ is a *Cauchy cut* if it is a cut and L has no maximum, R has no minimum and for each positive $\varepsilon \in K$ there exist $\ell \in L$ and $r \in R$ such that $r < \ell + \varepsilon$. We say that K is *Cauchy complete* if for each Cauchy cut $\langle L, R\rangle$ there exists $x \in K$ such that $L < x < R$. Note that Cauchy completeness as just defined is a reformulation of standard, sequential Cauchy completeness in terms of cuts (cf., e.g., [33], where Cauchy cuts are called *Veronese cuts*), so we can define the Cauchy completion of $\mathrm{No}_{<\kappa}$ as follows.

4.18 DEFINITION (Galeotti [41]). $\mathbb{R}_\kappa := \mathrm{No}_{<\kappa} \cup \{[\,L \mid R\,]\,;\, \langle L, R\rangle$ is a Cauchy cut over $\mathrm{No}_{<\kappa}\}$.

Let $(X, <)$ be an ordered set and κ be a cardinal. We say that X is an *η_κ-set* if whenever $L, R \subseteq X$ are such that $L < R$ and $|L|, |R| < \kappa$, there exists $x \in X$ such that $L < x < R$.

4.19 THEOREM (Galeotti [41]). *The field $\mathbb{R}_\kappa$ is the unique Cauchy-complete real closed field extension of $\mathbb{R}$ which is an η_κ-set of cardinality 2^κ, whose smallest dense subset has cardinality κ, and in which $\mathrm{No}_{<\kappa}$ can be densely embedded.*

In view of the previous theorem, from now on we will call $\mathrm{No}_{<\kappa}$ the *κ-rational numbers* and use the symbol $\mathbb{Q}_\kappa$ instead of $\mathrm{No}_{<\kappa}$. Note that $\mathbb{Q}_\omega$ is *not* the set of rational numbers but rather the set of dyadic rational numbers, i.e., those of the form $\frac{n}{2^m}$ where n, m are integers and $m \neq 0$.

Note that, since $\mathbb{R}_\kappa$ is not Dedekind-complete, the order topology on $\mathbb{R}_\kappa$ is not a good tool for doing analysis on that space. This is because many basic theorems of analysis for the order topology of an ordered field, such as the intermediate value theorem, actually imply the Dedekind completeness of the ordered field in question. Indeed, let K be an ordered field which is not Dedekind-complete, which is equivalent to the existence of a bounded nonempty subset $X \subset K$ without a least upper bound. Let $f : K \to K$ be defined by $f(x) = 1$ if x is an upper bound of X, with $f(x) = -1$ otherwise. Now, f is continuous, for if x is an upper bound of X then there exists some $\delta > 0$ such that every element of $(x - \delta, x + \delta)$ is also an upper bound of X, and likewise if x is not an upper bound then for some $\delta > 0$ no element of $(x - \delta, x + \delta)$ is an upper bound of X either. But obviously f has no zeroes, so the intermediate value theorem does not hold for K. Thus what one needs is a suitably *coarser* notion than that of a topology.

4.20 Definition (Alling [2, § 0.02]). A *κ-topology* over a set X is a collection of subsets τ of X satisfying:

(1) both $\varnothing$ and X are in τ;

(2) for any $\alpha < \kappa$, if $\langle A_i ; i \in \alpha \rangle$ is a sequence of sets in τ then $\bigcup_{i<\alpha} A_i \in \tau$; and

(3) for all $A, B \in \tau$, we have $A \cap B \in \tau$.

With κ-topologies one can define direct analogues of many topological notions. We refer to these with the prefix "κ-"; thus we have κ-open sets, κ-continuous functions, κ-topologies generated by families of subsets of a set, etc. Note that, unlike the classical case of the interval topology over $\mathbb{R}$, the interval κ-topologies over $\mathbb{R}_\kappa$ in which the intervals have endpoints in $\mathbb{R}_\kappa \cup \{-\infty, +\infty\}$ or in $\mathbb{Q}_\kappa \cup \{-\infty, +\infty\}$ are different in general. In what follows we will only consider the generalized real line $\mathbb{R}_\kappa$ equipped with the former.

4.21 Theorem (IVT$_\kappa$; Galeotti [41]). *Let $a, b \in \mathbb{R}_\kappa$ and $f : [0,1] \to \mathbb{R}_\kappa$ be a κ-continuous function. Then for every $r \in [f(0), f(1)]$ there exists $c \in [0,1]$ such that $f(c) = r$.*

Additionally, a generalized version of the extreme value theorem [40] and a generalized version of the Bolzano-Weierstraß theorem (for κ weakly compact) [20] have been proved to hold for $\mathbb{R}_\kappa$. Thus $\mathbb{R}_\kappa$ is a suitable setting for generalizing results from classical analysis.

4.3.4 Computability on generalized Cantor spaces

The idea of generalizing Turing machines to the transfinite is due to Hamkins and Kidder in the late 1980s, whose results were only published years later after further developments by Hamkins and Lewis [45]. Of course, the idea of extending the classical notion of computability to the transfinite is much older, but an underlying notion of *computation* was missing until the Hamkins-Kidder-Lewis machine ("no machine model for admissible recursion theory was elaborated. Levy announced a generalization of Turing machines working on regular cardinals ..., but no further details were published. Other approaches to ordinal recursion theory were based on recursion schemata" [60, p. 311]). In the Hamkins-Kidder-Lewis machines, called *infinite time Turing machines*, the "machinery" (tapes, heads, states, etc.) is the same as for the classical Turing machines, but running times of machines are allowed to be of any ordinal type. Building on this idea, Koepke [59] introduced a further generalization into the model by stipulating that the tapes of the machine also have unbounded ordinal length, with the resulting machines called *ordinal Turing machines*. Infinite time Turing machines and ordinal Turing machines have become the standard basic models of transfinite computations, from which new models have arisen by stipulating different tape lengths and/or running times (e.g., [28, 60, 86]).

In this section we define a generalized version of type two computability for 2^κ by building on the ordinal Turing machines whose tapes have length κ and whose successful computations halt in time less than κ; for concreteness, we are going to follow the definition of Koepke and Seyfferth [60, § 2]. We restrict ourselves to 2^κ, instead of κ^κ, in the interest of avoiding cumbersome details of coding. Our presentation will be fairly high level; in other words, we will not go into full detail in defining the machines, which would involve talking about states, transition functions, etc., in particular because these are the same as for the classical Turing machine model.

A *κ-Turing machine* has the following tapes of length κ: finitely many read-only tapes for the input, finitely many read and write scratch tapes and one write-only tape for the output. Each cell of each tape has either 0 or 1 written in it at any given time, with the default value being 0. These machines can run for infinite time of ordinal type κ; at successor stages of a computation a κ-Turing machine behaves exactly like a classical Turing machine, while at limit stages the contents of each cell of each tape and the positions of the heads are computed using inferior limits†.

A partial function $f : 2^{<\kappa} \to 2^{<\kappa}$ is *computed* by a κ-Turing machine M if whenever M is given $x \in \mathrm{dom}(f)$ as input, its computation halts after fewer than κ steps with $f(x)$ written on the output tape. As in the classical case $\kappa = \omega$, the notion of type two computability for functions on 2^κ also uses the machinery of κ-Turing machines.

4.22 Definition. A partial function $f : 2^\kappa \to 2^\kappa$ is *type two-computed* by a κ-Turing machine M, or simply *computed* by M, if whenever M is given $x \in \mathrm{dom}(f)$ as input, for every $\alpha < \kappa$ there exists a stage $\beta < \kappa$ of the computation at which $f(x) \upharpoonright \alpha$ is written on the output tape. An *oracle κ-Turing machine* is a κ-Turing machine with an additional read-only input tape of length κ, called its *oracle tape*. A partial function $f : 2^\kappa \to 2^\kappa$ is *computable with an oracle* if there exists an oracle κ-Turing machine M and $x \in 2^\kappa$ such that M computes f when x is written on the oracle tape at the start of the computation.

By minor modifications of classical proofs one can prove that κ-Turing machines are closed under recursion and composition, and that there exists a universal κ-Turing machine.

The *κ-Wadge game* is the direct generalization of the Wadge game from Definition 2.5 to the setting of κ^κ. Thus, it is the game for functions on κ^κ in which player $\mathbb{1}$ plays elements of κ, with rule set κ^κ and interpretation function $\mathrm{id}_{\kappa^\kappa}$, and $\mathbb{2}$ plays elements of $\kappa^{<\kappa}$, with the rule that they must converge monotonically

†Given a limit ordinal λ and a non-decreasing sequence $s = \langle x_\alpha\,;\, \alpha < \lambda\rangle$ of ordinals, the *limit* of s is the unique ordinal γ such that for every $\gamma' < \gamma$ there exists $\lambda' < \lambda$ such that $\gamma' < x_\alpha \leqslant \gamma$ holds whenever $\lambda' \leqslant \alpha < \lambda$. Now, given an arbitrary sequence $s' = \langle y_\alpha\,;\, \alpha < \lambda\rangle$ of ordinals, the *inferior limit* of s', denoted $\liminf s'$, is the limit of the non-decreasing sequence $\langle \inf\{y_\beta\,;\, \alpha \leqslant \beta < \lambda\}\,;\, \alpha < \lambda\rangle$.

to some element of κ^κ. The following is a consequence of the case $\alpha = 0$ of Theorem 4.10, since that proof also works for functions on 2^κ, or from 2^κ to κ^κ, or vice versa. Of course, a direct proof can also easily be obtained by adapting the proof of Theorem 2.6.

4.23 Lemma (Folklore). *Let $\lambda, \gamma \in \{2, \kappa\}$. The class of continuous partial functions between λ^κ and γ^κ is characterized by the κ-Wadge game.*

The following is a fundamental result about the notion of computability for functions $f : 2^\kappa \dashrightarrow 2^\kappa$; in the $\kappa = \omega$ case, it is precisely this result that underlies the rich interplay between effective and classical descriptive set theory.

4.24 Theorem. *A partial function $f : 2^\kappa \to 2^\kappa$ is continuous iff it is computable with some oracle.*

Proof. Suppose that M is a κ-Turing machine that computes f when given oracle $x \in 2^\kappa$. Define a strategy ϑ for $\mathbb{2}$ in the κ-Wadge game for f by letting $\vartheta(\sigma) = \tau$ iff when M is given x as an oracle, after reading σ on the input tape, the output tape contains exactly τ. It is easy to see that ϑ is a winning strategy.

For the converse direction, for each $\sigma \in 2^{<\kappa}$, let us denote

$$\nu(\sigma) := 10^{\sigma(0)+1}10^{\sigma(1)+1}10^{\sigma(2)+1} \cdots 1.$$

Let M be an oracle κ-Turing machine which writes τ on the output tape after having read σ on the input tape iff there exists some even $\alpha < \kappa$ such that $w_\alpha = \sigma$ and $w_{\alpha+1} = \tau$, where $[\nu(w_\alpha)]_{\alpha<\kappa} \in 2^\kappa$ is the sequence on the oracle tape. Let ϑ be a winning strategy for $\mathbb{2}$ in the κ-Wadge game for f. Since $2^{<\kappa} = \kappa$, let $\mathsf{bij}'_\kappa : \kappa \to 2^{<\kappa}$ be a bijection. Then M computes f when given as oracle $[(\nu \circ \mathsf{bij}'_\kappa(\alpha))(\nu \circ \vartheta \circ \mathsf{bij}'_\kappa(\alpha))]_{\alpha<\kappa}$ ☐

Theorem 4.24 is therefore a strong justification to the claim that the notion of type two computability by κ-Turing machines we introduced is the *correct* one for computability of functions on 2^κ.

4.3.5 Represented spaces

The basic notions of represented space theory generalize in an entirely straightforward way from ω^ω or 2^ω to κ^κ or 2^κ. Since we have defined a notion of computability on 2^κ, it will be convenient to use that as our space of codes as well. Thus, e.g., a *κ-represented space* is a pair $\mathbb{X} = (X, \delta)$ of a nonempty set X and a partial surjection $\delta : 2^\kappa \dashrightarrow X$, for functions $f : \mathbb{X} \dashrightarrow \mathbb{Y}$ and $g : \mathbb{Z} \dashrightarrow \mathbb{W}$ between κ-represented spaces, we say that f is *strongly κ-Weihrauch reducible* to

g, in symbols $f \leqslant_{\kappa s\mathfrak{W}} g$, if there exist computable functions $H, K : 2^\kappa \dashrightarrow 2^\kappa$ such that $HGK \vdash f$ whenever $G \vdash g$, and so on†.

We represent κ and κ^κ by the functions δ_κ and δ_{κ^κ}, respectively, given by $\delta_\kappa(p) = \alpha$ iff $p = 0^\alpha 1 0^\kappa$ and $\delta_{\kappa^\kappa}(p) = x$ iff $p = [0^{\alpha_\beta+1}1]_{\beta<\kappa}$ and $x = \langle \alpha_\beta \rangle_{\beta<\kappa}$. It is straightforward to see that a function $f : \kappa \to \kappa$ is δ_κ-computable iff it is computable by a κ-machine as in [60, Definition 2].

4.25 Lemma. *The mutually inverse bijections $\ulcorner \_ \urcorner \restriction \kappa^2 : \kappa^2 \to \kappa$ and $\mathsf{bij}_\kappa : \kappa \to \kappa^2$ are δ_κ-computable.*

Proof. Note that it is enough to prove that bij_κ is δ_κ-computable, since then $\ulcorner \_ \urcorner \restriction \kappa^2$ can be δ_κ-computed by simulating the program for bij_κ with each ordinal in increasing order as input until the correct output is found (note that whether an output is correct or not can be recognized in time less than κ, and thus this whole process also takes time less than κ). To compute bij_κ, given γ the idea is to enumerate the first γ pairs of ordinals less than κ in the order $\prec$. This can be done computably as follows. At successor stages, having listed $\langle \alpha, \beta \rangle$ in the previous stage, the next pair to be listed is $\langle \alpha+1, \beta \rangle$, if $\alpha + 1 < \beta$; $\langle \beta, 0 \rangle$, if $\alpha + 1 = \beta$; and $\langle \alpha, \beta+1 \rangle$, if $\beta + 1 \leqslant \alpha$. At limit stages, the counters keeping track of the values of α and β along the computation get set to lim inf of those values. This information allows us to decide the next pair to be listed by a straightforward case distinction. For example, suppose the lim inf of the values of α is α' and the lim inf of the values of β is also α'. If it is the first time that we have reached the pair $\langle \alpha', \alpha' \rangle$ in this way, then the next pair to be listed is $\langle \alpha', 0 \rangle$; otherwise the next pair is indeed $\langle \alpha', \alpha' \rangle$. The complete program is described in Algorithm 1 on p. 107. □

Let $\delta : 2^\kappa \to X$ and $\delta' : 2^\kappa \to X$ be two representations of a space X. Then we say that δ *continuously reduces* to δ', in symbols $\delta \leqslant_{\mathrm{t}} \delta'$, if there exists a continuous function $h : 2^\kappa \to 2^\kappa$ such that for every $p \in \mathrm{dom}(\delta)$ we have $\delta(p) = \delta'(h(p))$. Similarly we say that δ *computably reduces* to δ', in symbols $\delta \leqslant \delta'$, if h above can be taken computable. If $\delta \leqslant_{\mathrm{t}} \delta'$ and $\delta' \leqslant_{\mathrm{t}} \delta$ we say that δ and δ' are *continuously equivalent*, and similarly for the computable case. Note that as in classical computable analysis if $\delta \leqslant \delta'$ and f is δ-computable then f is also δ'-computable.

4.26 Proposition. *The representation δ_{κ^κ} is $\leqslant$-maximal ($\leqslant_{\mathrm{t}}$-maximal) among the computable (continuous) representations of κ^κ.*

†Carl has also introduced a notion of generalized (strong) Weihrauch reducibility in [21]. Because his goal is to investigate multi-valued (class) functions on V, the space of codes he uses is the class of ordinal numbers, considered with the ordinal Turing machines of Koepke [59]. Therefore his approach is significantly different from ours, and we do not know of any connections between the two.

Proof. We will prove the computable case, since the continuous case can easily be obtained with a similar proof. Let δ be a computable representation of κ^κ. We want to show that there exists a computable function $f : 2^\kappa \to 2^\kappa$ such that $\delta_{\kappa^\kappa}(f(p)) = \delta(p)$ for every $p \in \mathrm{dom}(\delta)$. Since δ is computable, there exists a computable winning strategy ϑ for $\mathbf{2}$ in the κ-Wadge game for δ. Given $\sigma \in 2^{<\kappa}$ let $\vartheta'(\sigma)$ be the concatenation of the sequences $0^{\vartheta(\sigma)(\alpha)+1}1$ for $\alpha < |\vartheta(\sigma)|$. Thus ϑ' is computable, and trivially it is a winning strategy for $\mathbf{2}$ in the κ-Wadge game for the function $f : \mathrm{dom}(\delta) \to 2^{<\kappa}$ given by $f(p) = \bigcup_{\alpha<\kappa} \vartheta'(p\restriction\alpha)$. It is now easy to see that $\delta_{\kappa^\kappa}(f(p)) = \delta(p)$ holds for every $p \in \mathrm{dom}(\delta)$, so we are done. $\square$

Representing $\mathbb{Q}_\kappa$ and $\mathbb{R}_\kappa$

In classical computable analysis one can show that many of the natural representations of $\mathbb{R}$ are well behaved with respect to type two computability, although, e.g., the naive decimal expansion representation is not (for example, multiplication by 3 fails to be computable under that representation, cf., e.g., [101, Example 2.1.4.7]). In this section we show that some of these results extend to the $\kappa > \omega$ case. First we introduce representations for generalized rational numbers, which will serve as a starting point to representing $\mathbb{R}_\kappa$. As we have seen in the introduction, surreal numbers can be expressed as binary sequences and, because of the simplicity theorem, as cuts. It is then natural to introduce two representations which reflect this fact. Let $p \in 2^\kappa$ and $q \in \mathbb{Q}_\kappa$. We define $\delta_{\mathbb{Q}_\kappa}(p) = q$ iff $p = [w_\alpha]_{\alpha<\kappa}$ where $w_\alpha := 00$ if $\alpha \in \mathrm{dom}(q)$ and $q(\alpha) = -$, $w_\alpha := 01$ if $\alpha \notin \mathrm{dom}(q)$, and finally $w_\alpha := 11$ if $\alpha \in \mathrm{dom}(q)$ and $q(\alpha) = +$. In order to define a representation of $\mathbb{Q}_\kappa$ in terms of cuts, we start by defining a sequence $\langle X_\alpha \rangle_{\alpha<\kappa}$ of subsets of 2^κ, letting $X_0 := \{10^\kappa\}$, $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for limit $\lambda > 0$, and $X_{\alpha+1} = X_\alpha \cup \{1p \in 2^\kappa\,;$ for every $\beta < \kappa$, if $(p)_\beta(0) = 1$ then $(p)_\beta \in X_\alpha$, and if $(p)_\beta(0) = 0$ then $(p)_\gamma(0) = 0$ for all $\gamma \geqslant \beta$ of the same parity as $\beta\}$. We define $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$ with domain $\bigcup_{\alpha<\kappa} X_\alpha$ by recursion, letting $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}(1p) = [\,L \mid R\,]$, where

$$L = \{\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}((p)_\alpha)\,;\, \alpha < \kappa \text{ is even and } (p)_\alpha(0) = 1\}$$
$$R = \{\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}((p)_\alpha)\,;\, \alpha < \kappa \text{ is odd and } (p)_\alpha(0) = 1\}.$$

The structure of the sets X_α ensures that this is a well-defined recursion. Thus, e.g., we have that 10^κ is a $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$-name for $[\,\varnothing \mid \varnothing\,] = 0$ and $1p$ with $(p)_0 = 10^\kappa$, $(p)_\alpha = 0^\kappa$ for all $\alpha > 0$ is a $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$-name for $[\,\{0\} \mid \varnothing\,] = 1$, etc.

4.27 Lemma. *The representations $\delta_{\mathbb{Q}_\kappa}$ and $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$ are computably equivalent.*

Proof. ($\delta_{\mathbb{Q}_\kappa} \leqslant \delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$) Let $p \in \mathrm{dom}(\delta_{\mathbb{Q}_\kappa})$. If p is a $\delta_{\mathbb{Q}_\kappa}$-name for 0 (note that this is decidable) we just return 10^κ, which is a $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$-name for $[\,\varnothing \mid \varnothing\,] = 0$. Otherwise we compute two subsets $L := \{p'01\,;\, p'11 \subset p\}$ and $R := \{p'01\,;\, p'00 \subset p\}$. Note that there exists some even $\alpha < \kappa$ such that $p(\alpha')p(\alpha'+1) = 01$ holds for all even $\alpha' \geqslant \alpha$, so the entire computation of L and R can be done in time less than κ.

Then we recursively compute the cuts for the elements of L and R and return them respectively as the left and right sets of the cut representation of p. It easy to see that the algorithm computes a code for the canonical cut of $\delta_{\mathbb{Q}_\kappa}(p)$ as in Theorem 4.15.

($\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa} \leqslant \delta_{\mathbb{Q}_\kappa}$) Let $p \in \mathrm{dom}(\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa})$. If p is a code for the cut $[\,\varnothing \mid \varnothing\,]$ we return a representation of the empty sequence. Now suppose p is the code for the cut $[\,L \mid R\,]$, with at least one of L, R not equal to $\varnothing$. We first recursively compute the sequences for the elements of L and R, call the sets of these sequences L_s and R_s respectively. Now suppose $\alpha < \kappa$ is even and we want to compute the value at α and $\alpha+1$ of the output sequence q. If for some even $\alpha' < \alpha$ we had $q(\alpha') = 0$ and $q(\alpha'+1) = 1$, then likewise we set $q(\alpha) = 0$ and $q(\alpha+1) = 1$. Therefore now suppose otherwise. We first compute M_L and m_R respectively the minimal and maximal elements in $\{00, 01, 11\}$ such that for every $p' \in L_s$ and $p'' \in R_s$ we have $p'(\alpha)p'(\alpha+1) <_{\mathrm{lex}} M_L$ and $m_R <_{\mathrm{lex}} p''(\alpha)p''(\alpha+1)$. Then by a case distinction on M_L and m_R we can decide the values of $q(\alpha)$ and $q(\alpha+1)$. For example, if the output is already smaller than R_s, $M_L = 00$ (i.e., $-$) and $m_R = 11$ (i.e., $+$) then we set $q(\alpha)q(\alpha+1) = 01$ (i.e., undefined). All of the other combinations can be treated similarly. □

4.28 Lemma. *The operations $+$, $-$, $\cdot$, $\frac{1}{\smile}$ and the order $<$ are $\delta^{\mathrm{c}}_{\mathbb{Q}_\kappa}$-computable.*

Proof. For the operations, this is an easy consequence of their inductive definitions in terms of cuts, Theorem 4.16. For the order, just note that for $\delta_{\mathbb{Q}_\kappa}$-names p_x and p_y of x and y, respectively, we have $x < y$ iff p_x is lexicographically-less than p_y. □

Given that $\mathbb{R}_\kappa$ is the Cauchy completion of $\mathbb{Q}_\kappa$, the following is a natural representation of $\mathbb{R}_\kappa$. We let $\delta_{\mathbb{R}_\kappa}(p) = x$ iff for each $\alpha < \kappa$ we have $(p)_\alpha \in \mathrm{dom}(\delta_{\mathbb{Q}_\kappa})$, $\delta_{\mathbb{Q}_\kappa}((p)_\alpha) < x + \frac{1}{\alpha+1}$, and $x < \delta_{\mathbb{Q}_\kappa}((p)_\alpha) + \frac{1}{\alpha+1}$.

4.29 Theorem. *The field operations $+$, $-$, $\cdot$, and $\frac{1}{\smile}$ are $\delta_{\mathbb{R}_\kappa}$-computable.*

Proof. Let us do the proof for $\cdot$, the others being similar. Given codes p and q for $x, y \in \mathbb{R}_\kappa$ respectively, let $x_\alpha = \delta_{\mathbb{Q}_\kappa}((p)_\alpha)$ and $y_\alpha = \delta_{\mathbb{Q}_\kappa}((q)_\alpha)$. Note that for each ordinal $\alpha < \kappa$ we can compute some ordinal $\alpha' < \kappa$ such that $\frac{1}{\alpha'+1}(x_0+y_0+3) \leqslant \frac{1}{\alpha+1}$. We then output $r \in 2^\kappa$, where $(r)_\alpha$ is a $\delta_{\mathbb{Q}_\kappa}$-name for $x_{\alpha'}y_{\alpha'}$.

We have $xy - x_{\alpha'}y_{\alpha'} = x(y - y_{\alpha'}) + y_{\alpha'}(x - x_{\alpha'}) < (x_0+1)\frac{1}{\alpha'+1} + (y_0+2)\frac{1}{\alpha'+1} \leqslant \frac{1}{\alpha+1}$, as desired, and likewise we can prove $x_{\alpha'}y_{\alpha'} - xy < \frac{1}{\alpha+1}$. □

On the other hand, the following is suggested by the definition of $\mathbb{R}_\kappa$ as the collection of Cauchy cuts over $\mathbb{Q}_\kappa$. We let $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}(p) = x$ iff for each $\alpha < \kappa$ we have $(p)_\alpha \in \mathrm{dom}(\delta_{\mathbb{Q}_\kappa})$ and $x = [\,L \mid R\,]$, with $L = \{\delta_{\mathbb{Q}_\kappa}((p)_\alpha)\ ;\ \alpha < \kappa \text{ is even}\}$; $R = \{\delta_{\mathbb{Q}_\kappa}((p)_\alpha)\ ;\ \alpha < \kappa \text{ is odd}\}$; and for each even $\alpha < \kappa$ we have $\delta_{\mathbb{Q}_\kappa}((p)_{\alpha+1}) < \delta_{\mathbb{Q}_\kappa}((p)_\alpha) + \frac{1}{\alpha+1}$.

4.30 Theorem. *The representations $\delta_{\mathbb{R}_\kappa}$ and $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}$ are computably equivalent.*

Proof. To reduce $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}$ to $\delta_{\mathbb{R}_\kappa}$, given p, we output $q =$ by making $(q)_\alpha$ equal to $(p)_\beta$, where β is the α^{th} even ordinal. It is now easy to see that q is a $\delta_{\mathbb{R}_\kappa}$-name for $\delta^{\mathrm{V}}_{\mathbb{R}_\kappa}(p)$.

For the converse reduction, given p, we output $q =$ where for each even α we let $(q)_\alpha$ be a $\delta_{\mathbb{Q}_\kappa}$-name for $\delta_{\mathbb{Q}_\kappa}((p)_{2\alpha+2}) - \frac{1}{2\alpha+3}$ and $(q)_{\alpha+1}$ be a $\delta_{\mathbb{Q}_\kappa}$-name for $\delta_{\mathbb{Q}_\kappa}((p)_{2\alpha+2}) + \frac{1}{2\alpha+3}$. Then letting $L := \{\delta_{\mathbb{Q}_\kappa}((p)_\alpha)\ ;\ \alpha < \kappa \text{ is even}\}$ and $R := \{\delta_{\mathbb{Q}_\kappa}((p)_\alpha)\ ;\ \alpha < \kappa \text{ is odd}\}$ we have $L < x < R$ and for each even $\alpha < \kappa$ we have $\delta_{\mathbb{Q}_\kappa}((q)_{\alpha+1}) = \delta_{\mathbb{Q}_\kappa}((p)_{2\alpha+2}) + \frac{1}{2\alpha+3} = \delta_{\mathbb{Q}_\kappa}((q)_\alpha) + \frac{2}{2\alpha+3} < \delta_{\mathbb{Q}_\kappa}((q)_\alpha) + \frac{1}{\alpha+1}$, as desired. $\square$

4.3.6 Generalized boundedness principles and the intermediate value theorem

As shown in, e.g., [11,12], the so-called *boundedness principles* and *choice principles* are important building blocks in characterizing the Weihrauch degrees of interest in computable analysis. In this section, as a proof of concept for the notions introduced in the previous sections, we focus on the study of the intermediate value theorem and its relationship with the boundedness principle $\mathrm{B_I}$. Concretely, we generalize a classical result from Brattka and Gherardi [12], proving that IVT_κ is Weihrauch equivalent to a generalized version of $\mathrm{B_I}$. This strengthens a result from [41], viz. that $\mathrm{B_I}$ is continuously reducible to IVT_κ.

The theorem IVT_κ as stated in Theorem 4.21 can be considered as the multi-valued partial function $\mathrm{IVT}_\kappa : \mathcal{C}([0,1], \mathbb{R}_\kappa) \rightrightarrows [0,1]$ defined by $\mathrm{IVT}_\kappa(f) = \{c \in [0,1]\,;\, f(c) = 0\}$, with $f \in \mathrm{dom}(\mathrm{IVT}_\kappa)$ iff f is κ-continuous and $f(0) \cdot f(1) < 0$ (note that any κ-continuous function on $\mathbb{R}_\kappa$ is also continuous).

To introduce the boundedness principle $\mathrm{B}^\kappa_{\mathrm{I}}$, we will need the following represented spaces. Let $\mathbb{S}^{\uparrow}_{\mathrm{b}}$ be the space of bounded increasing sequences of κ-rationals, represented by letting p be a name for $\langle x_\alpha \rangle_{\alpha<\kappa}$ iff $(p)_\alpha \in \mathrm{dom}(\mathbb{Q}_\kappa)$ and $\delta_{\mathbb{Q}_\kappa}((p)_\alpha) = x_\alpha$ for each $\alpha < \kappa$. The represented space $\mathbb{S}^{\downarrow}_{\mathrm{b}}$ is defined analogously, with bounded decreasing sequences of κ-rationals. Note that, unlike the classical case of the real line, not all limits of bounded monotone sequences of length κ exist in $\mathbb{R}_\kappa$. Therefore, although for the real line the spaces $\mathbb{S}^{\uparrow}_{\mathrm{b}}$ and $\mathbb{S}^{\downarrow}_{\mathrm{b}}$ naturally correspond to the spaces of *lower reals* $\mathbb{R}_<$ and *upper reals* $\mathbb{R}_>$, respectively, in our generalized setting the correspondence fails. We define $\mathrm{B}^\kappa_{\mathrm{I}}$ as the principle which, given an increasing sequence $\langle q_\alpha \rangle_{\alpha<\kappa}$ and decreasing sequence $\langle q'_\alpha \rangle_{\alpha<\kappa}$ in $\mathbb{Q}_\kappa$ for which there exists $x \in \mathbb{R}_\kappa$ such that $\{q_\alpha\,;\,\alpha < \kappa\} \leqslant x \leqslant \{q'_\alpha\,;\,\alpha < \kappa\}$, picks one such x. Formally we have the multi-valued partial function $\mathrm{B}^\kappa_{\mathrm{I}} : \mathbb{S}^{\uparrow}_{\mathrm{b}} \times \mathbb{S}^{\downarrow}_{\mathrm{b}} \rightrightarrows \mathbb{R}_\kappa$ with $x \in \mathrm{B}^\kappa_{\mathrm{I}}(s, s')$ iff $\{s(\alpha)\,;\,\alpha < \kappa\} \leqslant x \leqslant \{s'(\alpha)\,;\,\alpha < \kappa\}$.

4.31 Lemma. *Let $f : [0,1] \to \mathbb{R}_\kappa$ and $x \in \mathbb{R}_\kappa$. Suppose there exists a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ of pairwise distinct elements of $[0,1]$ such that $f(x_\alpha) = x$ if $\alpha < \kappa$ is even and $f(x_\alpha) \neq x$ otherwise, and such that for any odd $\alpha, \beta < \kappa$ there exists an even $\gamma < \kappa$ such that x_γ is between x_α and x_β. Then f is not κ-continuous.*

Proof. If such a sequence exists, then either the preimage of the κ-open set $(x,+\infty)$ or of the κ-open set $(-\infty,x)$ under f must contain x_α for κ-many of the odd $\alpha<\kappa$, but not contain x_β for any even $\beta<\kappa$. Such a set cannot be κ-open. $\square$

4.32 Lemma. *Let $f:[0,1]\to\mathbb{R}_\kappa$ be κ-continuous, $\beta,\beta'<\kappa$, $y\in\mathbb{R}_\kappa$, and let $\langle r_\alpha\rangle_{\alpha<\beta}$ and $\langle r'_\alpha\rangle_{\alpha<\beta'}$ be two sequences in $[0,1]$ such that $\{r_\alpha\,;\,\alpha<\beta\}<\{r'_\alpha\,;\,\alpha<\beta'\}$ and $\{f(r_\alpha)\,;\,\alpha<\beta\}<y<\{f(r'_\alpha)\,;\,\alpha<\beta'\}$. Then there exists $x\in[0,1]$ such that $\{r_\alpha\,;\,\alpha<\beta\}<x<\{r'_\alpha\,;\,\alpha<\beta'\}$ and $f(x)=y$.*

Proof. Assume not. Note that, by the IVT_κ, if there exist $x,x'\in[0,1]$ such that $\{r_\alpha\,;\,\alpha<\beta\}<\{x,x'\}<\{r'_\alpha\,;\,\alpha<\beta'\}$ and $f(x)<y<f(x')$, then there exists x'' between x and x' such that $f(x')=y$, and we are done. Therefore assume otherwise. Without loss of generality, we can assume that for every x such that $\{r_\alpha\,;\,\alpha<\beta\}<x<\{r'_\alpha\,;\,\alpha<\beta'\}$ we have $f(x)>y$ (a similar proof works for $f(x)<y$). Note that the set $\{r_\alpha\,;\,\alpha<\beta\}$ has cofinality at most $\beta<\kappa$ and, since $\mathbb{R}_\kappa$ is an η_κ-set, it follows that $X=\{x\in[0,1]\,;\,\forall\alpha<\beta(r_\alpha<x)\}$ has coinitiality κ. Therefore X is not κ-open. Now since f is κ-continuous we have that $f^{-1}[(y,+\infty)]$ is κ-open, say $f^{-1}[(y,+\infty)]=\bigcup_{\alpha\in\gamma}(y_\alpha,z_\alpha)$ with $\gamma<\kappa$ and $y_\alpha,z_\alpha\in[0,1]$ for every $\alpha<\gamma$. Now consider the set $I:=\{\alpha\in\gamma\,;\,(y_\alpha,z_\alpha)\cap X\neq\varnothing\}$. We have that $X\subset\bigcup_{\alpha\in I}(y_\alpha,z_\alpha)$. Note that since X is not κ-open we have $X\neq\bigcup_{\alpha\in I}(y_\alpha,z_\alpha)$. Now let $x\in\bigcup_{\alpha\in I}(y_\alpha,z_\alpha)\smallsetminus X$, so that there exists $\alpha\in I$ such that $x\in(y_\alpha,z_\alpha)$. Take $x'\in(y_\alpha,z_\alpha)\cap X$. By the fact that $x\notin X$, there exists $\alpha'<\beta$ such that $x<r_{\alpha'}$ and by IVT_κ there exists a root of f between $r_{\alpha'}$ and x', but this is a contradiction because $(y_\alpha,z_\alpha)\subset f^{-1}[(y,+\infty)]$. $\square$

4.33 Corollary. *Let $f:[0,1]\to\mathbb{R}_\kappa$ be κ-continuous, and let $x\in[0,1]$, $\langle r_\alpha\rangle_{\alpha<\kappa}$ and $\langle r'_\alpha\rangle_{\alpha<\kappa}$ be respectively increasing and decreasing sequences in $[0,1]$ such that for all $\alpha<\kappa$ we have $f(r_\alpha)<x$ and $f(r'_\alpha)>x$. Then there exists $y\in[0,1]$ such that $f(y)=x$ and $\{r_\alpha\,;\,\alpha<\kappa\}<y<\{r'_\alpha\,;\,\alpha<\kappa\}$.*

Proof. We construct a sequence $\langle x_\alpha\rangle_{\alpha<\gamma}$ for some $\gamma<\kappa$ as follows. First let $\delta_0=1$. Having constructed $\langle x_\beta\rangle_{\beta<\alpha}$ for some even $\alpha<\kappa$, by Lemma 4.32 there exists $x_\alpha\in[0,1]$ such that $f(x_\alpha)=x$ and $\{r_\beta\,;\,\beta<\sup_{\nu<\alpha}\delta_\nu\}<x_\alpha<\{r'_\beta\,;\,\beta<\sup_{\nu<\alpha}\delta_\nu\}$. If $\{r_\beta\,;\,\beta<\kappa\}<x_\alpha<\{r'_\beta\,;\,\beta<\kappa\}$, then we are done and $\gamma=\alpha$. Otherwise there exists $\beta<\kappa$ such that $r_\beta>x$ or $r'_\beta<x$, so we let $x_{\alpha+1}=r_\beta$ or $x_{\alpha+1}=r'_\beta$ accordingly, and let $\delta_\alpha=\beta+1$. If the construction goes on for κ steps, then $\langle x_\alpha\rangle_{\alpha<\kappa}$ is as in Lemma 4.31, a contradiction. Hence the construction ends at some stage $\gamma<\kappa$, and therefore $\{r_\beta\,;\,\beta<\kappa\}<x_\gamma<\{r'_\beta\,;\,\beta<\kappa\}$. $\square$

4.34 Theorem. (1) *If there exists an effective enumeration of a dense subset of $\mathbb{R}_\kappa$, then $\mathrm{IVT}_\kappa\leqslant_{\kappa\mathfrak{W}}\mathrm{B}^\kappa_\mathrm{I}$.*

(2) *We have $\mathrm{B}^\kappa_\mathrm{I}\leqslant_{\kappa\mathfrak{W}}\mathrm{IVT}_\kappa$.*

(3) *We have $\mathrm{IVT}_\kappa\leqslant^{\mathrm{t}}_{\kappa\mathfrak{W}}\mathrm{B}^\kappa_\mathrm{I}$, and therefore $\mathrm{IVT}_\kappa\equiv^{\mathrm{t}}_{\kappa\mathfrak{W}}\mathrm{B}^\kappa_\mathrm{I}$.*

Proof. For item 1, let the κ-continuous function $f : [0,1] \to \mathbb{R}_\kappa$ be given, $\mathbb{D}$ be a dense subset of $\mathbb{R}_\kappa$ and $\langle d_\gamma \rangle_{\gamma<\kappa}$ be an effective enumeration of $[0,1] \cap \mathbb{D}$. Without loss of generality we can assume $f(0) < 0$ and $f(1) > 0$, and start setting $r_0 = 0$ and $r'_0 = 1$. Now assume that for $0 < \alpha < \kappa$ we have already defined an increasing sequence $\langle r_\beta \rangle_{\beta<\alpha}$ and a decreasing sequence $\langle r'_\beta \rangle_{\beta<\alpha}$ of elements of $[0,1] \cap \mathbb{D}$ with $\{r_\beta\,;\, \beta < \alpha\} < \{r'_\beta\,;\, \beta < \alpha\}$ and $\{f(r_\beta)\,;\, \beta < \alpha\} < 0 < \{f(r'_\beta)\,;\, \beta < \alpha\}$. By Lemma 4.32 there still exists a root of f between the two sequences. Note that, since $\mathbb{R}_\kappa$ is an η_κ-set and again by applying Lemma 4.32, there exist $r_L, r_R \in \mathbb{D}$ such that $\{r_\beta\,;\, \beta < \alpha\} < r_L < r_R < \{r'_\beta\,;\, \beta < \alpha\}$ and $f(r_L) < 0$, $f(r_R) > 0$. Therefore, by searching in the sequence $\langle d_\gamma \rangle_{\gamma<\kappa}$ and running the corresponding algorithms in parallel, we can find such a pair r_L, r_R in fewer than κ computation steps. Let β, γ, δ be such that $\ulcorner \beta, \ulcorner \gamma, \delta \urcorner \urcorner = \alpha$, where $\ulcorner \_ \urcorner$ is the Gödel pairing function, which has a computable inverse by Lemma 4.25. If $r_L < d_\gamma < d_\delta < r_R$, $f(d_\gamma) < 0$, and $f(d_\delta) > 0$, where the last two comparisons are decided in fewer than β steps of computation, then let $r_\alpha = d_\gamma$ and $r'_\alpha = d_\delta$; otherwise let $r_\alpha = r_L$ and $r'_\alpha = r_R$.

By Corollary 4.33 we have that there exists $x \in [0,1]$ such that $\{r_\alpha\,;\, \alpha < \kappa\} < x < \{r'_\alpha\,;\, \alpha < \kappa\}$. It remains to be proved that $f(x) = 0$ for any such x. Suppose not, say $f(x) > 0$ for some such x. Then also $f(y) > 0$ for some $y \in \mathbb{D}$ such that $\{r_\alpha\,;\, \alpha < \kappa\} < y < \{r'_\alpha\,;\, \alpha < \kappa\}$. Now let $\beta, \gamma, \delta < \kappa$ be such that $d_\gamma = y$, $d_\delta = r_\nu$ for some ν such that $y - r_\nu < \{r'_\alpha - r_\beta\,;\, \alpha, \beta < \kappa\}$ and $f(y) < 0$, $f(r_\nu) > 0$ are decided in fewer than β computation steps. Then at stage $\alpha = \ulcorner \beta, \ulcorner \gamma, \delta \urcorner \urcorner$ of the computation we define a pair r_α, r'_α such that $r'_\alpha - r_\alpha \leqslant y - r_\nu$, a contradiction. This ends the proof of 1.

Item 2 can be proved by a straightforward generalization of the proof of [12, Theorem 6.2], and the proof of item 3 is the same as that of item 1 without the requirement that the enumeration $\langle d_\gamma \rangle_{\gamma<\kappa}$ of the dense subset of $[0,1] \cap \mathbb{D}$ be effective. $\square$

Note that the antecedent of item 1 of Theorem 4.34 is satisfied, e.g., in the constructible universe **L**. We leave for future work the task of investigating the set-theoretic properties of that condition more deeply.

Algorithm 1: Pairing for κ.

```
Input: (A δ_κ-name for) γ < κ.
Output: (δ_κ-names for) the pair ⟨α, β⟩ at position γ in the order ≺.
Initialize α := 0, β := 0, η := 0, ξ := 0, ζ := 1, μ := 1, ν := 0
/* ⟨α, β⟩ iterates over all pairs of elements of κ in the order ≺,
   η keeps track of what to do in case α = β at some stage,
   ξ keeps track of the position of the current pair ⟨α, β⟩,
   ζ is the least ordinal that has not appeared as a component in a pair
   listed before,
   μ and ν are counters for detecting limit stages.                    */
while true do
    if μ > ν then                              // successor stage or beginning
        if α = β = η = 0 then
            η := 1
        else if α = β and η = 1 then               // ⟨α, α⟩ was already listed
            α := 0,   η := 0,   β := β + 1
            if β = ζ then
                ζ := ζ + 1
        else if α < β then
            α := α + 1
            if α = β then
                β := 0
        else if α > β then
            β := β + 1
            if α = β then
                η := 1
    else if α = β then                            // limit stage and α = β
        if α = 0 then
            β := ζ,   ζ := ζ + 1
        else if η = 0 then                        // not time to list ⟨α, α⟩ yet
            η := 1,   β := 0
    if ξ = γ then
        return α, β
    else                                          // proceed to next stage
        ξ := ξ + 1,   μ := μ + 2,   ν := ν + 1
```

Chapter 5

Ranks defined from games

Summary. In this chapter, we introduce a general framework for defining a *rank* for a class of functions from a game characterizing that class (Definition 5.1), apply it to the Wadge, backtrack, eraser, multitape, 2-tree and 3-tree games, and study some basic properties of the ranks thus obtained (§§ 5.2–5.4).

Remarks on co-authorship

The material presented in this chapter constitutes the starting point of a collaboration of the author with Márton Elekes, Viktor Kiss, and Zoltán Vidnyánszky. This collaboration began while the author and Elekes were Visiting Fellows at the Isaac Newton Institute for Mathematical Sciences in the program *Mathematical, Foundational and Computational Aspects of the Higher Infinite*, and was further developed during two visits of the author to Budapest in 2017. However, the results of this collaboration are at an early stage, and are not included in this thesis.

Therefore, unless stated otherwise, definitions and results in this chapter are due solely to the author.

5.1 Introduction

Ranks for classes of functions have been widely studied in the literature of descriptive set theory and functional analysis. As a non-exhaustive list, we mention the works of Motto Ros, for $\mathbf{\Delta}^0_2$-functions on Baire space [67, § 5]; Carroy, building on the work of Motto Ros and also mainly discussing $\mathbf{\Delta}^0_2$-functions and other subclasses of the Baire class 1 functions on Baire space [22], Kechris and Louveau, for Baire class 1 functions on compact metrizable spaces [55], and Elekes, Kiss, and Vidnyánszky, for Baire class α functions on Polish spaces, for any $\alpha < \omega_1$ [34].

In each game for functions on ω^ω we have considered in this thesis, the structure of the game is such that the legal strategies for $\mathbf{2}$ in the game for some function f are those that only tell $\mathbf{2}$ to make a certain type of *bad* move (from the point of view of player $\mathbf{2}$) *finitely often* when following the strategy against any $x \in \mathrm{dom}(f)$. For example, in the Wadge game the bad moves are the consecutive passes, and in the backtrack game the bad moves are the backtracks, i.e., those moves which do not extend the move made immediately before in a run. This requirement that a certain type of move only be made finitely often corresponds to the wellfoundedness of a certain related countable structure, which therefore has countable rank, from which we can define a rank for f in a natural way.

Concretely, we have the following.

5.1 Definition. Let $G_\star$ be a game characterizing a class $\mathscr{C}$ of functions, and let $\mathcal{B}_\star$ be a function associating to each $f \in \mathscr{C}$ and each winning strategy ϑ for $\mathbf{2}$ in $G_\star$ for f a wellfounded structure $\mathcal{B}_\star(\vartheta, f)$. Given such f and ϑ, the $\mathcal{B}_\star$ *rank* of ϑ for f, denoted $\mathrm{rk}_\star(\vartheta, f)$, is the wellfounded rank of $\mathcal{B}_\star(\vartheta, f)$, and the $\mathcal{B}_\star$ *rank* of f is the minimum $\mathcal{B}_\star$ rank of its winning strategies, i.e.,

$$\mathrm{rk}_\star(f) := \min\{\mathrm{rk}_\star(\vartheta, f)\,;\, \vartheta \text{ is a winning strategy for } \mathbf{2} \text{ in } G_\star \text{ for } f\}.$$

Winning strategies for $\mathbf{2}$ of $\mathcal{B}_\star$-rank exactly $\mathrm{rk}_\star(f)$ are called *optimal* strategies.

Since it will be used repeatedly in the remainder of this chapter, for convenience let us denote the set $\mathrm{dom}(\vartheta) \cap \{(f(x))\restriction n\,;\, x \in \mathrm{dom}(f) \wedge n \in \omega\}$ by $\mathrm{dom}(\vartheta, f)$. In the cases we will consider in this chapter, to each node $s \in \mathcal{B}_\star(\vartheta, f)$ there will correspond an element $\sigma_s \in \mathrm{dom}(\vartheta, f)$, thus in particular satisfying $[\sigma_s] \cap \mathrm{dom}(f) \neq \varnothing$, in such a way that if s is less than s' in the order of $\mathcal{B}_\star(\vartheta, f)$ then $\sigma_s \supset \sigma_{s'}$ holds. Therefore, if $\mathrm{dom}(f)$ is a closed subset of ω^ω and there exists an infinite descending chain $\langle s_n\,;\, n \in \omega\rangle$ in $\mathcal{B}_\star(\vartheta, f)$, then $x = \bigcup_{n\in\omega} \sigma_{s_n}$ is in $\mathrm{dom}(f)$ and will cause $\mathbf{2}$ to make *bad moves* infinitely often when following ϑ against x. From this it will follow that ϑ is not a winning strategy for $\mathbf{2}$ in $G_\star$ for f, so by contraposition, if ϑ *is* a winning strategy for $\mathbf{2}$ in $G_\star$ for f then $\mathcal{B}_\star(\vartheta, f)$ is wellfounded. For this reason, in this chapter we only consider classes of functions whose members have closed domains.

5.2 The Wadge game rank

As indicated above, in the Wadge game the *bad moves* are the consecutive passes, i.e., the moves which do not properly extend the previous move in a run. Concretely, for any continuous function $f : \omega^\omega \dashrightarrow \omega^\omega$ and winning strategy ϑ for $\mathbf{2}$ in the Wadge game for f we define

$$\begin{aligned} \mathcal{B}_{\mathcal{W}}(\vartheta, f) &= (\mathrm{dom}(\vartheta, f), \prec^\vartheta_{\mathcal{W}}) \\ \sigma \prec^\vartheta_{\mathcal{W}} \sigma' &\text{ iff } \sigma \supset \sigma' \text{ and } \vartheta(\sigma) = \vartheta(\sigma'). \end{aligned}$$

The rank function $\mathrm{rk}_{\mathcal{W}}$ obtained from $\mathcal{B}_{\mathcal{W}}$ as in Definition 5.1 is called *Wadge game rank*.

It is easy to see that a function f has Wadge game rank 0 iff f is a Lipschitz function with constant 1, i.e., iff for any $x, y \in \mathrm{dom}(f)$ and $n \in \omega$, if $x \restriction n = y \restriction n$ then $f(x) \restriction n = f(y) \restriction n$, and more generally, f has Wadge game rank $k \in \omega$ iff for any $x, y \in \mathrm{dom}(f)$ and $n \in \omega$, if $x \restriction (n(k+1)) = y \restriction (n(k+1))$ then $f(x) \restriction n = f(y) \restriction n$.

5.2 Theorem. *For each $\alpha < \omega_1$ there exists a continuous function with rank greater than α.*

Proof. Given a wellfounded tree $T \subseteq \omega^{<\omega}$ we define a continuous function $f : \omega^\omega \to \omega^\omega$ as follows. For each $s \in \omega^{\leqslant\omega}$ such that $s \notin T \cup [T]$ there exists $n_s \in \omega$ such that $\mathsf{bij}(n_s)$ is the longest node of T which is an initial segment of s, with $\mathsf{bij} : \omega \to \omega^{<\omega}$ the bijection fixed in Convention 1.8. We also define $\ell_s := |\mathsf{bij}(n_s)|$. Now let $f(x) = \langle \ulcorner n_x, x(\ell_x) \urcorner \rangle ^\frown 0^\omega$.

5.3 Claim. *The game rank of f is at least* $\mathrm{rk}(T \setminus \{\langle\rangle\}, \supset)$.

Indeed, suppose ϑ is a winning strategy for $\mathbb{2}$ in the Wadge game for f. Since for any $\xi \in T \setminus \{\langle\rangle\}$ there exist $x, y \in [\xi]$ with $f(x)(0) \neq f(y)(0)$, the fact that ϑ is a winning strategy for $\mathbb{2}$ implies that $\vartheta(\xi) = \langle\rangle$. Thus if $\xi \supset \xi'$ are in $T \setminus \{\langle\rangle\}$ then $\xi \prec^{\vartheta}_{\mathcal{W}} \xi'$ in $\mathcal{B}_{\mathcal{W}}(\vartheta, f)$, implying the claim.

The definition of f immediately suggests a winning strategy ϑ for $\mathbb{2}$ in the Wadge game for f: pass (i.e., play $\langle\rangle$) up to (but not including) the round at which the sequence $\mathbb{1}$ is building goes out of T; at that point the value of the function at the input player $\mathbb{1}$ it is building is completely determined, and $\mathbb{2}$ will not have to pass again.

5.4 Claim. *The strategy ϑ has Wadge game rank* $\mathrm{rk}(T \setminus \{\langle\rangle\}, \supset)$*, and therefore the same holds for f.*

If $\sigma \notin T \setminus \{\langle\rangle\}$ is not the empty sequence, then f has constant value $\langle \ulcorner n_\sigma, \sigma(\ell_\sigma \urcorner \rangle ^\frown 0^\omega$ on $[\sigma]$, so in particular $\mathbb{2}$ does not pass again when facing σ following ϑ. Therefore the nodes of rank greater than 0 in $\mathcal{B}_{\mathcal{W}}(\vartheta, f)$ are also in $T \setminus \{\langle\rangle\}$, so $\mathrm{rk}_{\mathcal{W}}(\vartheta, f) \leqslant \mathrm{rk}(T \setminus \{\langle\rangle\}, \supset)$ as desired.

Since the ranks of countable trees are cofinal in ω_1 (cf. Theorem 1.10), the same holds for the Wadge game ranks of continuous functions. □

5.3 The backtrack game rank

In the backtrack game, a bad move is—of course—backtracking, i.e., a move which does not extend the previous move in a run. Concretely, we define

$$\begin{aligned} \mathcal{B}_{\mathrm{bt}}(\vartheta, f) &= (\mathrm{dom}(\vartheta, f), \prec^{\vartheta}_{\mathrm{bt}}) \\ \sigma \prec^{\vartheta}_{\mathrm{bt}} \sigma' &\text{ iff } \sigma \supset \sigma' \text{ and } \vartheta(\sigma) \not\supseteq \vartheta(\sigma'). \end{aligned}$$

The rank function $\mathrm{rk}_{\mathrm{bt}}$ obtained from $\mathcal{B}_{\mathrm{bt}}$ is called *backtrack game rank*.

It is easy to see that a function has backtrack game rank 0 iff it is continuous. The backtrack game rank is also closely related to the following family of modified backtrack games introduced by Motto Ros.

5.5 DEFINITION (Motto Ros [67, § 5.2]). Let $\alpha < \omega_1$. The *α-backtrack game* is the game in which player $\mathbb{1}$ plays as in the Wadge game, and player $\mathbb{2}$ plays pairs $(\sigma, \beta) \in \omega^{<\omega} \times (\alpha + 1)$, with rule set composed of the infinite sequences $\langle(\sigma_n, \beta_n)\,;\, n \in \omega\rangle$ of moves such that

(1) if $n < m$ then $\beta_m \leqslant \beta_n$,

(2) if $\sigma_n \not\subseteq \sigma_m$ then $\beta_m < \beta_n$, and

(3) if $N \in \omega$ is such that $\beta_n = \beta_N$ holds for all $n \geqslant N$ then $\bigcup_{n \geqslant N} \sigma_n \in \omega^\omega$.

The interpretation associates to a rule-respecting sequence of moves $\langle(\sigma_n, \beta_n)\,;\, n \in \omega\rangle$ the element $\bigcup_{n \geqslant N} \sigma_n \in \omega^\omega$, where N is least such that $\beta_n = \beta_N$ holds for all $n \geqslant N$.

Our definition differs slightly from the one given by Motto Ros, but the differences are not essential.

5.6 THEOREM. *Suppose $f : \omega^\omega \dashrightarrow \omega^\omega$ is a $\mathbf{\Delta}^0_2$-function. Then $\mathrm{rk}_{\mathrm{bt}}(f)$ is the least ordinal α such that $\mathbb{2}$ has a winning strategy in the α-backtrack game for f.*

Proof. To see that $\mathbb{2}$ has a winning strategy in the $\mathrm{rk}_{\mathrm{bt}}(f)$-backtrack game for f, let ϑ be an optimal strategy for her in the backtrack game for f. Let ϑ' be the strategy for $\mathbb{2}$ in the $\mathrm{rk}_{\mathrm{bt}}(f)$-backtrack game for f given by $\vartheta'(\sigma) = (\vartheta(\sigma), \beta)$, where β is the rank of σ in $\mathcal{B}_{\mathrm{bt}}(\vartheta, f)$. It is easy to see that ϑ' is a winning strategy.

Conversely, if ϑ is a winning strategy in the α-backtrack game for f, then let ϑ' be the strategy for her in the backtrack game for f given by $\vartheta'(\sigma) = \tau$ iff $\vartheta(\sigma) = (\tau, \beta)$ for some $\beta \leqslant \alpha$. Since the function $g : \mathcal{B}_{\mathrm{bt}}(\vartheta', f) \to \alpha + 1$ given by $g(\sigma) = \beta$ iff $\vartheta(\sigma) = (\tau, \beta)$ for some $\tau \in \omega^{<\omega}$ clearly satisfies $g(\sigma) < g(\sigma')$ whenever $\sigma \prec^{\vartheta'}_{\mathrm{bt}} \sigma'$, by Theorem 1.11 it follows that the rank of $\mathcal{B}_{\mathrm{bt}}(\vartheta', f)$, and thus also $\mathrm{rk}_{\mathrm{bt}}(f)$, is at most α, as desired. $\square$

Given $A \subseteq \omega^\omega$, we define its *characteristic function* $\chi_A : \omega^\omega \to \omega^\omega$ by

$$\chi_A(x) = \begin{cases} 1^\omega & \text{if } x \in A \\ 0^\omega & \text{if } x \notin A. \end{cases}$$

5.7 THEOREM (Folklore). *A set $A \subseteq \omega^\omega$ is in $\mathbf{\Delta}^0_2$ iff χ_A is a $\mathbf{\Delta}^0_2$-function.*

Proof. ($\Leftarrow$) Immediate, since $[\langle 1\rangle] \in \mathbf{\Delta}^0_2$ and $A = \chi_A^{-1}[\langle 1\rangle]$.
($\Rightarrow$) Let $\langle F_n \in \mathbf{\Pi}^0_1\,;\, n \in \omega\rangle$ and $\langle G_n \in \mathbf{\Sigma}^0_1\,;\, n \in \omega\rangle$ be such that $A = \bigcup_{n\in\omega} F_n = \bigcap_{n\in\omega} G_n$. We define a winning strategy ϑ for $\mathbb{2}$ in the backtrack game for χ_A as follows. Given $\sigma \in \omega^{<\omega}_{\neq 0}$, let $\vartheta(\sigma) = 1^{|\sigma|}$ if the least n such that $[\sigma] \cap F_n \neq \varnothing$ holds is less than or equal to the least m such that $[\sigma] \not\subseteq G_m$ holds, with $\vartheta(\sigma) = 0^{|\sigma|}$ otherwise.

5.8 Claim. *The strategy ϑ is winning for $\mathbb{2}$.*

Indeed, if $x \in A$ then $x \in F_n$ holds for some least n and $x \in G_m$ holds for all m. In this case, the strategy ϑ cannot tell $\mathbb{2}$ to play incompatible sequences more than $2n$ times, since at some round m we will have that n is least such that $[x\restriction m] \cap F_n$ holds and for every $n' < n$ we have $[x\restriction m] \not\subseteq G_{n'}$. The case $x \notin A$ is similar, since then $x \notin F_n$ holds for all n and $x \notin G_m$ holds for some least m. $\square$

5.9 Theorem. *For every $A \in \mathbf{\Delta}^0_2$ there exists $n \in \omega$ such that* $\mathrm{rk}_{\mathrm{bt}}(\chi_A) \leqslant \mathrm{rk}_{1\mathrm{HK}}(A) \leqslant \mathrm{rk}_{\mathrm{bt}}(\chi_A) + n$. *In particular the backtrack ranks of $\mathbf{\Delta}^0_2$-functions are cofinal in ω_1.*

Proof. Let $A \in \mathbf{\Delta}^0_2$ have Hausdorff-Kuratowski rank α.

Let $\langle A_\beta\,;\, \beta \leqslant \alpha\rangle$ be a $\mathbf{\Sigma}^0_1$-resolution for A. Define a strategy ϑ for $\mathbb{2}$ in the backtrack game for χ_A by letting $\vartheta(\sigma) = 0^{|\sigma|}$ or $\vartheta(\sigma) = 1^{|\sigma|}$ according to whether the parity of the least $\beta \leqslant \alpha$ such that $[\sigma] \subseteq A_\beta$ holds is equal or different, respectively, to that of α. Clearly ϑ is a winning strategy, since by associating each node $\sigma \in \mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A)$ to the least $\beta_\sigma \leqslant \alpha$ such that $[\sigma] \subseteq A_{\sigma_\beta}$, it follows that if $\sigma \prec^\vartheta_{\mathrm{bt}} \sigma'$ in $\mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A)$ then $\beta_\sigma < \beta_{\sigma'}$. Therefore $\prec^\vartheta_{\mathrm{bt}}$-chains in $\mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A)$ are finite, and by Theorem 1.11 the association $\sigma \in \mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A) \mapsto \beta_\sigma$ also implies that the backtrack game rank of ϑ, and thus also of χ_A, is at most α. In particular the backtrack game rank of χ_A is at most the Hausdorff-Kuratowski rank of A.

Conversely, let ϑ be a winning strategy for $\mathbb{2}$ in the backtrack game for χ_A. For each limit λ and natural n such that $\lambda + n \leqslant \mathrm{rk}_{\mathrm{bt}}(\vartheta, \chi_A)$, let $A_{\lambda+2n} = \bigcup_{\gamma<\lambda+2n} A_\gamma \cup \bigcup\{[\sigma]\,;\, \sigma \text{ has rank at most } \lambda+n \text{ in } \mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A) \text{ and } \vartheta(\sigma)(0) = 0\}$ and $A_{\lambda+2n+1} = A_{\lambda+2n} \cup \bigcup\{[\sigma]\,;\, \sigma \text{ has rank at most } \lambda+n \text{ in } \mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A) \text{ and } \vartheta(\sigma)(0) = 1\}$. Finally, for the limit γ and $n \in \omega$ such that $\mathrm{rk}_{\mathrm{bt}}(\vartheta, \chi_A) = \gamma + n$, let $A_{\gamma+2n+2} = \omega^\omega$.

5.10 Claim. *The sequence $\langle A_\beta\,;\, \beta \leqslant \gamma + 2n + 2\rangle$ is a $\mathbf{\Sigma}^0_1$-resolution for A.*

By construction $\langle A_\beta\,;\, \beta \leqslant \gamma + 2n + 2\rangle$ is an increasing sequence of open sets, so all that remains to be shown is $A = \mathrm{diff}\langle A_\beta\,;\, \beta \leqslant \gamma + 2n + 2\rangle$, i.e., that $x \in A$ iff the least $\beta \leqslant \gamma + 2n + 2$ for which $x \in A_\beta$ holds is odd. Suppose $x \in A$. If for some $m \in \omega$ we have $\vartheta(x\restriction m) = 0$, then since $\chi_A(x) = 1^\omega$ there exists some $k > m$ such that $x\restriction k \prec^\vartheta_{\mathrm{bt}} x\restriction m$. In particular the initial segment σ of x of lowest rank in $\mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A)$, say rank $\lambda + m$ for some limit λ and $m \in \omega$, satisfies $\vartheta(\sigma)(0) = 1$. Therefore $\lambda + 2m + 1$ is the least ordinal β such that $x \in A_\beta$ holds. Conversely, if the least $\beta \leqslant \gamma + 2n + 2$ for which $x \in A_\beta$ holds is odd, then for the initial segment σ of x of lowest rank in $\mathcal{B}_{\mathrm{bt}}(\vartheta, \chi_A)$ we have $\vartheta(\sigma)(0) = 1$. Since ϑ is a winning strategy, this implies $\chi_A(x) = 1^\omega$ as desired.

In particular the Hausdorff-Kuratowski rank α of A satisfies $\alpha \leqslant \mathrm{rk}_{\mathrm{bt}}(\chi_A) + k$ for some $k \in \omega$. □

Note that the proof of Theorem 5.9 provides a proof of Theorem 1.19 for the case $\xi = 1$ (also using Theorem 5.7).

5.4 The eraser game rank and beyond

In the eraser game, there are ω many types of bad moves: for each $n \in \omega$, changing the n^{th} position of the output. Concretely, we define

$$\begin{aligned} \mathcal{B}_{\mathrm{e}}(\vartheta, f) &= (\{(\sigma, n) \in \mathrm{dom}(\vartheta, f) \times \omega\,;\, |\vartheta(\sigma)| > n\}, \prec_{\mathrm{e}}^{\vartheta}) \\ (\sigma, n) \prec_{\mathrm{e}}^{\vartheta} (\sigma', n') &\text{ iff } \sigma \supset \sigma',\ n = n', \text{ and } \vartheta(\sigma)(n) \neq \vartheta(\sigma')(n). \end{aligned}$$

The rank function rk_{e} obtained from $\mathcal{B}_{\mathrm{e}}$ is called *eraser game rank*.

It is easy to see that the continuous functions are those with eraser game rank 0, but for $\mathbf{\Delta}^0_2$-functions the ranks are already cofinal in ω_1, as the next result shows.

5.11 Proposition. *For any $A \in \mathbf{\Delta}^0_2$ we have $\mathrm{rk}_{\mathrm{e}}(\chi_A) = \mathrm{rk}_{\mathrm{bt}}(\chi_A)$. In particular the eraser game ranks of $\mathbf{\Delta}^0_2$ functions are cofinal in ω_1.*

Proof. First note that if ϑ is a winning strategy for $\mathbb{2}$ in the backtrack game for χ_A, then for $\sigma, \sigma' \in \mathrm{dom}(\vartheta, \chi_A)$ and $n \in \omega$ such that $|\vartheta(\sigma)|, |\vartheta(\sigma')| > n$, we have $(\sigma, n) \prec_{\mathrm{e}}^{\vartheta} (\sigma', n)$ iff $\sigma \prec_{\mathrm{bt}}^{\vartheta} \sigma'$, so Corollary 1.12 implies $\mathrm{rk}_{\mathrm{bt}}(\vartheta, \chi_A) \leqslant \mathrm{rk}_{\mathrm{e}}(\vartheta, \chi_A)$.

Conversely, since the range of χ_A is $\{0^\omega, 1^\omega\}$, any winning strategy ϑ for $\mathbb{2}$ in the eraser game for χ_A can be transformed into a winning strategy ϑ' for $\mathbb{2}$ in the backtrack game for χ_A by letting

$$\vartheta'(\sigma) = \begin{cases} \langle\rangle, & \text{if } \vartheta(\sigma) = \langle\rangle \\ 1^{|\vartheta(\sigma)|}, & \text{if } \vartheta(\sigma)(0) = 1 \\ 0^{|\vartheta(\sigma)|}, & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathrm{rk}_{\mathrm{bt}}(\vartheta', \chi_A) \leqslant \mathrm{rk}_{\mathrm{e}}(\vartheta, \chi_A)$. □

We say a sequence $\langle f_n\,;\, n \in \omega\rangle$ of functions with the same domain X *converges uniformly* to a function $f : X \to \omega^\omega$ if

$$\forall n \exists n^* \forall m \geqslant n^* \forall x \in X (f(x)(n) = f_m(x)(n)).$$

5.12 Theorem. *If $\langle f_n\,;\, n \in \omega\rangle$ is a sequence of Baire class 1 functions converging uniformly to a function f, then $\mathrm{rk}_{\mathrm{e}}(f) \leqslant \sup\{\mathrm{rk}_{\mathrm{e}}(f_n)\,;\, n \in \omega\}$.*

Proof. For each $n \in \omega$ let n^* be such that $f(x)(n) = f_m(x)(n)$ holds for all $m \geqslant n^*$ and all $x \in \mathrm{dom}(f)$, and let ϑ_n be an optimal strategy for $\mathbb{2}$ in the eraser game for f_n. Define a strategy ϑ for $\mathbb{2}$ in the eraser game for f by letting $\vartheta(\sigma)(n) = \vartheta_{n^*}(\sigma)(n)$ with $|\vartheta(\sigma)| \leqslant |\sigma|$ maximum such that for all $n < |\vartheta(\sigma)|$ we have $n < |\vartheta_{n^*}(\sigma)|$.

5.13 Claim. *The strategy ϑ is winning for $\mathbb{2}$ in the eraser game for f.*

Indeed, for every $n \in \omega$ and every $x \in \mathrm{dom}(f)$ there exists $m \in \omega$ such that $\vartheta_{n^*}(x{\restriction}k)(n) = f(x)(n)$ holds for all $k \geqslant m$. Now let $m' \geqslant m$ be least such that $n < |\vartheta_{\ell^*}(x{\restriction}k)|$ holds for all $\ell \leqslant n$ and $k \geqslant m'$. It now follows that $\vartheta(x{\restriction}k)(n) = f(x)(n)$ holds for all $k \geqslant m'$.

5.14 Claim. $\mathrm{rk}_{\mathrm{e}}(\vartheta, f) \leqslant \sup\{\mathrm{rk}_{\mathrm{e}}(f_n)\,;\, n \in \omega\}$

Indeed, $(\sigma', n) \prec_{\mathrm{e}}^{\vartheta} (\sigma, n)$ implies $(\sigma', n) \prec_{\mathrm{e}}^{\vartheta_{n^*}} (\sigma, n)$, so the rank of (σ, n) in $\mathcal{B}_{\mathrm{e}}(\vartheta, f)$ is bounded by the rank of (σ, n) in $\mathcal{B}_{\mathrm{e}}(\vartheta_{n^*}, f_{n^*})$. Therefore $\mathrm{rk}_{\mathrm{e}}(\vartheta, f)$, which is the supremum of the ranks of the elements (σ, n) in $\mathcal{B}_{\mathrm{e}}(\vartheta, f)$, is bounded by $\sup\{\mathrm{rk}_{\mathrm{e}}(\vartheta_{n^*})\,;\, n \in \omega\}$, which is itself bounded by $\sup\{\mathrm{rk}_{\mathrm{e}}(f_n)\,;\, n \in \omega\}$ because each strategy ϑ_n is assumed to be optimal for f_n. □

In [34, Theorem 6.1], Elekes, Kiss, and Vidnyánszky isolate five conditions whose conjunction guarantees that an arbitrary rank function on the bounded Baire class 1 functions on the real numbers is *essentially* the same as several other rank functions known from the literature (corresponding to well-known characterizations of Baire class 1 functions and also discussed in depth by Kechris and Louveau in [55]). Theorem 5.9, Proposition 5.11, and Theorem 5.12 imply that two of these conditions are satisfied by the eraser game rank. However, because in [34] the authors are interested in real-valued functions, some of the other conditions presented there use the field structure of the real numbers in a seemingly fundamental way. Therefore, a natural line of investigation—which is being pursued by the author, in collaboration with Elekes, Kiss, and Vidnyánszky—is the following.

5.15 Question. (1) What is the analogue of [34, Theorem 6.1] for ranks of Baire class 1 functions on Baire space?

(2) We can define a rank function for the real-valued Baire class 1 functions by letting the rank of f be the least eraser rank of a realizer of f. Does this rank function satisfy the conditions of [34, Theorem 6.1]?

We can define a rank for $\mathbf{\Delta}^0_3$-functions based on the multitape game as follows.

$$\mathcal{B}_{\mathrm{mt}}(\vartheta, f) = (\{(\sigma, n, m) \in \mathrm{dom}(\vartheta, f) \times \omega \times \omega\,;\, n \neq m \text{ and } \langle n\rangle, \langle m\rangle \in \vartheta(\sigma)\}, \prec_{\mathrm{mt}}^{\vartheta})$$

$$(\sigma, n, m) \prec_{\mathrm{mt}}^{\vartheta} (\sigma', n', m') \quad \text{iff} \quad \sigma \supset \sigma',\ n = n',\ m = m', \text{ and } \min\{\mathrm{rk}_{\vartheta(\sigma')}(\langle n\rangle), \mathrm{rk}_{\vartheta(\sigma')}(\langle m\rangle)\} < \min\{\mathrm{rk}_{\vartheta(\sigma)}(\langle n\rangle), \mathrm{rk}_{\vartheta(\sigma)}(\langle m\rangle)\}$$

It is not hard to see that a set is $\mathbf{\Delta}^0_3$ iff its characteristic function is a $\mathbf{\Delta}^0_3$-function.

5.16 THEOREM (Folklore). *A set $A \subseteq \omega^\omega$ is in $\mathbf{\Delta}^0_3$ iff χ_A is a $\mathbf{\Delta}^0_3$-function.*

Sketch of proof. ($\Leftarrow$) Immediate, since $[\langle 1\rangle] \in \mathbf{\Delta}^0_3$ and $A = \chi_A^{-1}[\langle 1\rangle]$.
($\Rightarrow$) Let $\langle B_{n,m} \in \mathbf{\Sigma}^0_1\,;\, n,m \in \omega\rangle$ and $\langle C_{n,m} \in \mathbf{\Pi}^0_1\,;\, n,m \in \omega\rangle$ be such that $A = \bigcup_{n\in\omega}\bigcap_{m\in\omega} B_{n,m} = \bigcap_{n\in\omega}\bigcup_{m\in\omega} C_{n,m}$. We can now define a winning strategy for $\mathbb{2}$ in the tree version of the multitape game (cf. Theorem 2.23(4)) for χ_A by employing the notion of *guessing* as in the proof of Theorem 2.31. Each node of length 1 with label 1 that $\mathbb{2}$ adds to her tree corresponds to a guess for the least n such that $x \in \bigcap_m B_{n,m}$, where x is the sequence $\mathbb{1}$ is building in a given run of the game, and each node of length 1 with label 0 that $\mathbb{2}$ adds to her tree corresponds to a guess for the least n such that $x \in \bigcap_m(\omega^\omega \smallsetminus C_{n,m})$. In order to satisfy the linearity condition of Theorem 2.23(4), we can stipulate that the nodes of length greater than 1 be of the form $\langle k\rangle^\frown 0^{m+1}$, with the same label as $\langle k\rangle$. (The guessing structure is so that such a node is only added to the tree if $x \in B_{n,m}$ or $x \in \omega^\omega \smallsetminus C_{n,m}$ holds, depending on the case.)

If the guess n encoded by k is either wrong or not the least one that is correct, then there will be a finite bound on the lengths of the descendants of $\langle k\rangle$ in the tree, and if n is exact, i.e., the least one that is correct, then $\langle k\rangle^\frown 0^\omega$ is an infinite branch of Υ^ϑ_x with running label χ_A. $\square$

5.17 QUESTION. What is the relationship between the Hausdorff-Kuratowski rank of a $\mathbf{\Delta}^0_3$ set and the multitape game rank of its characteristic function?

We can of course also define ranks for the Baire class α functions for any given $\alpha < \omega_1$ by using the α-tree game we defined in Chapter 2. For $\alpha = 0$ and $\alpha = 1$ these coincide with the Wadge game rank and eraser game rank, respectively. For, e.g., $\alpha \in \{2,3\}$, concretely we have

$$\begin{aligned}
\mathcal{B}_2(\vartheta, f) &= (\{(\sigma,\tau,\xi) \in \mathrm{dom}(\vartheta,f) \times \omega^{<\omega} \times \omega^{<\omega}\,;\, \tau \text{ and } \xi \text{ are} \\
&\qquad\qquad \text{incompatible nodes of } \vartheta(\sigma)\}, \prec^\vartheta_2) \\
(\sigma,\tau,\xi) \prec^\vartheta_2 (\sigma',\tau',\xi') &\text{ iff } \sigma \supset \sigma',\ \tau = \tau',\ \xi = \xi', \text{ and} \\
&\quad \min\{\mathrm{rk}_{\vartheta(\sigma')}(\tau), \mathrm{rk}_{\vartheta(\sigma')}(\xi)\} < \min\{\mathrm{rk}_{\vartheta(\sigma)}(\tau), \mathrm{rk}_{\vartheta(\sigma)}(\xi)\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_3(\vartheta, f) &= (\{(\sigma,s) \in \mathrm{dom}(\vartheta,f) \times (\omega^{<\omega})^{<\omega}\,;\, \text{elements of } s \text{ are} \\
&\qquad\qquad \text{pairwise incompatible nodes of } \vartheta(\sigma)\}, \prec^\vartheta_3) \\
(\sigma,s) \prec^\vartheta_3 (\sigma',s') &\text{ iff } \sigma \supset \sigma',\ s \supset s', \text{ and } \min\{\mathrm{rk}_{\vartheta(\sigma')} s'(n)\,;\, n < |s'|\} \\
&\qquad\qquad < \min\{\mathrm{rk}_{\vartheta(\sigma)} s(n)\,;\, n < |s'|\}
\end{aligned}$$

We leave for future work the task of analyzing the properties of these ranks.

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Samenvatting

Spellen voor functies:
Baire klasses, Weihrauchgraden, transfinite berekeningen, en rang

Spelkarakteriseringen van functieklasses in de descriptieve verzamelingenleer vinden hun oorsprong in het werk van Wadge en werden verder ontwikkeld door onder andere Van Wesep, Andretta, Duparc, Motto Ros en Semmes. In dit proefschrift worden deze karakteriseringen vanuit verschillende perspectieven belicht.

We definieren aanpassingen van Semmes's spelkarakterisering van de Borel functies, om zo spelkarakteriseringen van de Baire klasse α functies voor elke $\alpha < \omega_1$ te verkrijgen. Sommige van deze resultaten zijn gelijktijdig bewezen door Louveau en Semmes in nog niet gepubliceerd werk. Ook definiëren we een constructie die een spelkarakterisering van een klasse Λ omvormt tot een karakterisering van de klasse van functies die stuksgewijs Λ zijn op een aftelbare deelpartitie bestaande uit $\mathbf{\Pi}^0_\alpha$ verzamelingen, voor iedere $0 < \alpha < \omega_1$.

Vervolgens definiëren we met behulp van technieken uit de berekenbare analyse een geparametriseerde versie van het Wadge spel, en laten we zien hoe de parameterkeuze gebruikt kan worden om af te stellen welke functieklasse wordt gekarakteriseerde door het resulterende spel. Het proefschrift beschrijft een toepassing die de spelkarakterisering van de Baire klasses omvormt naar dit framework.

Verder wordt een generalisatie van de spelkarakterisering van functieklasses naar gegeneraliseerde Baire-ruimtes beschreven. We laten ook zien hoe de notie van berekenbaarheid kan worden uitgebreid naar gegeneraliseerde Baire-ruimtes, en tonen aan dat dit geschikt is voor een algemenere vorm van berekenbare analyse door een representatie van Galeotti's gegeneraliseerde reële lijn te definiëren en de Weihrauchgraad van de tussenwaardestelling in die ruimte te analyseren.

In het laatste gedeelte van dit proefschrift demonstreren we hoe uit de besproken spelkarakteriseringen van functieklasses op een natuurlijke wijze tot een hiërarchie leidt, die op een intuïtieve manier de complexiteit van de functies in de bijbehorende klasses aangeeft. Dit idee en de genoemde resultaten openen nieuwe wegen voor vervolgonderzoek.

Abstract

Games for functions:
Baire classes, Weihrauch degrees, transfinite computations, and ranks

Game characterizations of classes of functions in descriptive set theory have their origins in the seminal work of Wadge, with further developments by Van Wesep, Andretta, Duparc, Motto Ros, and Semmes, among others. In this thesis we study such characterizations from several perspectives.

We define modifications of Semmes's game characterization of the Borel functions, obtaining game characterizations of the Baire class α functions for each fixed $\alpha < \omega_1$. Some of our results were independently proved by Louveau and Semmes in unpublished work. We also define a construction of games which transforms a game characterizing a class Λ of functions into a game characterizing the class of functions which are piecewise Λ on a countable partition of their domains by $\mathbf{\Pi}^0_\alpha$ sets, for each $0 < \alpha < \omega_1$.

We then define a framework of parametrized Wadge games by using tools from computable analysis, and show how the choice of parameters can be used to fine-tune what class of functions is characterized by the resulting game. As an application, we recast our games characterizing the Baire classes into this framework.

Furthermore, we generalize our game characterizations of function classes to generalized Baire spaces, i.e., the spaces of functions from an uncountable cardinal to itself. We also show how the notion of computability on Baire space can be generalized to the setting of generalized Baire spaces, and show that this is indeed appropriate for developing a generalized version of computable analysis by defining a representation of Galeotti's generalized real line and analyzing the Weihrauch degree of the intermediate value theorem for that space.

In the final part of the thesis, we show how the game characterizations of function classes discussed lead in a natural way to a stratification of each class into a hierarchy, intuitively measuring the complexity of functions in that class. This idea and the results presented open new paths for further research.

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