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Search for long-living topological solutions of the nonlinear ϕ^4 field theory

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We look for long-living topological solutions of classical nonlinear (1 + 1)-dimensional ϕ^4 field theory. To that effect we use the well-known cut-and-match method. In this framework, new long-living states are obtained in both topological sectors. In particular, in one case a highly excited state of a kink is found. We discover several ways of energy reset. In addition to the expected emission of wave packets (with small amplitude), for some selected initial conditions the production of kink-antikink pairs results in a large energy reset. Also, the topological number of a kink in the central region changes in the contrast of conserving full topological number. At lower excitation energies there is a long-living excited vibrational state of the kink; this phenomenon is the final stage of all considered initial states. Over time this excited state of the kink changes to a well-known linearized solution—a discrete kink excitation mode. This method yields a qualitatively new way to describe the large-amplitude bion, which was detected earlier in the kink-scattering processes in the nontopological sector.

DOI: [10.1103/PhysRevD.95.056009](https://doi.org/10.1103/PhysRevD.95.056009)**I. INTRODUCTION**

We consider the $\lambda\phi^4$ theory with a real scalar field $\phi(t, x)$ in (1 + 1) dimensions [1–3]. Its dynamics is determined by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2} \left(\frac{\partial\phi}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial\phi}{\partial x} \right)^2 - U(\phi), \quad (1)$$

where $U(\phi)$ is a potential, defining the self-interaction of the field in the considered model [1],

$$U(\phi) = \frac{\lambda}{4} \left(\frac{m^2}{\lambda} - \phi^2 \right)^2. \quad (2)$$

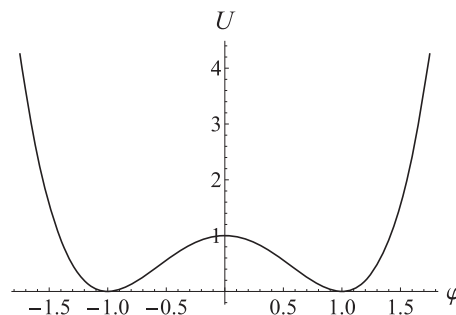
The plot of Eq. (2) is shown in Fig. 1. We analyze a model with a non-negative potential with two minima, so all static solutions with finite energy split into disjoint classes, so-called topological sectors, according to their asymptotic behavior at very large x . Solutions with $\phi(-\infty) \neq \phi(+\infty)$ are called topological, while those with $\phi(-\infty) = \phi(+\infty)$ are nontopological. Both types of solutions are of growing interest in physics. In particular, they arise in the questions of three- or two-dimensional domain walls. However, the one-dimensional case is also curious and was considered in different works for diverse models [4–6]. In the $\lambda\phi^4$ model there is a soliton solution called a kink; the phenomenon of

a “wobbling kink” was studied in [7,8]. Moreover, a three- or two-dimensional domain wall presents a one-dimensional kink interpolating two different vacua of the model. In some cases these can be solved approximately [9]. The domain walls in the $\lambda\phi^4$ model can be applied to some cosmological models, for example, during discussions of dark matter and dark energy [10]. The results of numerical simulations in other models [9] can be applied to solid-state physics [11].

The Lagrangian (1) with (2) yields the equation of motion for $\phi(t, x)$. After transition to dimensionless variables it reads

$$\phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0. \quad (3)$$

As a next step, we find and study the analytical solutions of Eq. (3). Note that the vacua of this model $\phi_{\text{vac}}^{(1)} = -1$

FIG. 1. The dimensionless potential (2) of the $\lambda\phi^4$ model.

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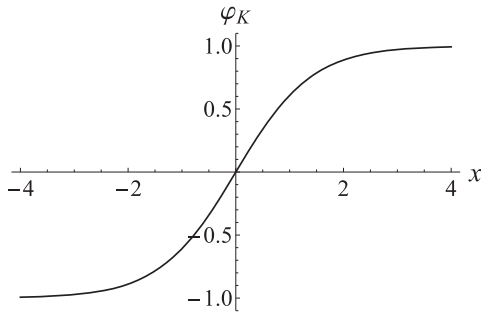


FIG. 2. The kink solution [Eq. (4)].

and $\varphi_{\text{vac}}^{(2)} = +1$ are stable solutions of (3). Moreover, there is the unstable permanent solution $\varphi = 0$ with infinite energy.

In addition to the previous solutions, there is also a static, nontrivial, topological, solitary wave-like solution. It can be easily found by solving the static limit of Eq. (3),

$$K \equiv \varphi_K(x - x_0) = \tanh \frac{(x - x_0)}{\sqrt{2}}. \quad (4)$$

The antikink \bar{K} is given by minus K . The energy functional for the Lagrangian (1), in the static case (4), is called the mass of the kink $M_K = 2\sqrt{2}/3$. The plot of Eq. (4) is presented by Fig. 2.

A. Excitation spectrum of the kink

In order to analyze the excitation spectrum of the static kink, we add a small perturbation $\delta\varphi$ to it. In other words, we make the ansatz

$$\varphi(t, x) = \varphi_K(x) + \delta\varphi(t, x) = \varphi_K(x) + e^{i\omega t}\psi(x).$$

By taking the terms in Eq. (3) linear in $\delta\varphi$, we obtain the following equation:

$$\hat{H}\psi = E\psi, \quad \hat{H} = -\frac{d^2}{dx^2} - 3\cosh^{-2}\frac{x}{\sqrt{2}},$$

$$E = \omega^2 - 2. \quad (5)$$

The eigenvalue $\omega_0 = 0$ belongs to the discrete part of the excitation spectrum (5) [1], but also there is one vibrational excitation given by

$$\delta\varphi_1 = \psi_1(x)e^{i\omega_1 t}, \quad \omega_1 = \sqrt{3/2},$$

$$\psi_1(x) = \left(\frac{3}{2\sqrt{2}}\right)^{1/2} \tanh \frac{x}{\sqrt{2}} \cosh^{-1} \frac{x}{\sqrt{2}}. \quad (6)$$

B. Analytical solution, depending on x

The above solutions are not a full set of solutions to the φ^4 model. Let us consider a static wave solution with infinite energy. We consider the static limit of Eq. (3),

$$\varphi_{xx} = -\varphi + \varphi^3. \quad (7)$$

This equation is analogous to Newton's equation

$$\ddot{x} = F(x) = -\nabla V(x), \quad \text{where}$$

$$V(x \rightarrow \varphi) = \frac{\varphi^2}{2} - \frac{\varphi^4}{4} + \text{const}. \quad (8)$$

Figure 3 shows a plot of a $V(x \rightarrow \varphi)$. In this case $\varphi(x)$ describes a trajectory of an oscillatory movement between the points $-\varphi_0$ and φ_0 . In the range $0 < \varphi_0 < 1$ oscillations are periodic. In the limiting case $\varphi_0 = 1$ the fluctuations disappear, because the time necessary to return to the starting point $\varphi_0 = 1$ reaches infinity.

Defining the dimensionless variable $\varphi(x) = \varphi_0\chi(x)$ and the constants

$$k^2 = \frac{\varphi_0^2/2}{1 - \varphi_0^2/2}, \quad b^2 = 1 - \frac{\varphi_0^2}{2},$$

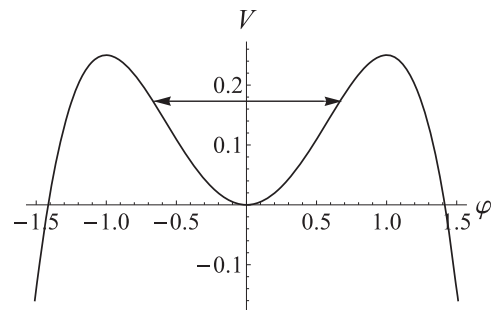
where $0 \leq k^2 \leq 1$ and $1/2 \leq b^2 \leq 1$, leads us to

$$\int_0^{\chi(x)} \frac{d\chi}{\sqrt{(1 - \chi^2)(1 - k^2\chi^2)}} = \langle \chi = \sin \psi \rangle = \int_0^{\arcsin \chi} \frac{d\psi}{\sqrt{(1 - k^2 \cos^2 \psi)}}.$$

The last integral is nothing but the elliptic integral of the first kind [$F(\arcsin \chi, k) = bx$] [12]. Then, the static periodic solution of Eq. (7) can be written as

$$\varphi_{\text{el}}(x) = \varphi_0 \text{sn}(bx, k), \quad (9)$$

where $\text{sn}(bx, k)$ is the elliptic sine [12]. At small k (corresponding to $\varphi_0 \ll 1$) there is a concordance $\text{sn}(z) \approx \sin(z)$. At $\varphi_0 \rightarrow 0$ the solution (9) becomes a permanent unstable solution $\varphi \sim 0$, as previously noted. Plots of Eq. (9) are shown in Figs. 4 and 5 for different values of the parameter φ_0 . The elliptic sine period is calculated using the following formula [12]:


 FIG. 3. The potential in Eq. (8). The arrow shows the domain of the solution $\varphi(x)$ for an arbitrarily chosen value $0 < \varphi_0 < 1$.

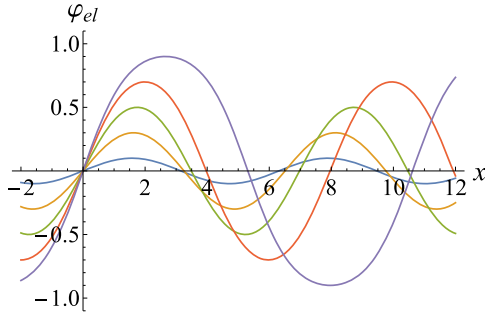


FIG. 4. The dependence of the solution $\varphi_{el}(x)$ on the parameter φ_0 . The plot is for different φ_0 from 0.1 to 0.9 with step 0.2. The curves on the plot are ordered by the degree to which the parameter φ_0 grows.

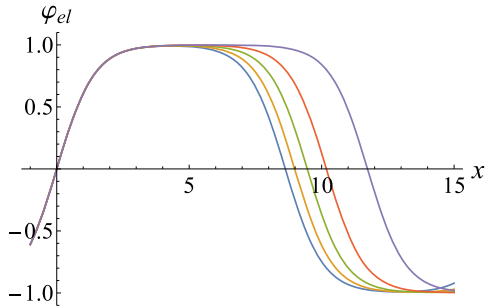


FIG. 5. The dependence of the solution $\varphi_{el}(x)$ on the parameter φ_0 . The plot is for different φ_0 from 0.991 to 0.999 with step 0.002. The curves on the plot are ordered by the degree to which the parameter φ_0 grows.

$$T = \frac{4F(\pi/2, k)}{\sqrt{1 - 0.5\varphi_0^2}}. \quad (10)$$

II. FORMULATION OF THE PROBLEM

The long-living solutions with high amplitude are of increasing interest in classical field theory. This type of solution, called the bion or breather, was found early in the kink-antikink collisions in the φ^4 model both in one- and three- dimensional cases [13–18].

Here, we propose to use the popular cut-and-match method to find a long-living field configuration, using previously found solutions $\varphi = 0$, Eq. (4), and Eq. (9). In this case, a part of the initial state is composed by the kink (4), which is divided in two equal pieces at $x = 0$. These halves of the kink are fixed at $\pm x_0$. Then, one of the solutions ($\varphi = 0$ or φ_{el} on the finite interval) of (3) is placed in the space between these two halves. An initial state constructed in the manner described is shown in Fig. 6.

Note that if we take $\varphi = 0$, the initial state will become unstable. Its energy linearly increases with growing distance $2x_0$.

In another case we make a solution in terms of the elliptic solution (9) for a fixed value $0 < \varphi_0 < 1$. For a smooth gluing of selected solutions one defines the value of x_0 as a

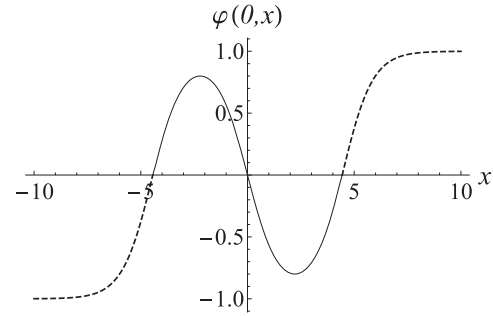


FIG. 6. The plot of the initial state, which is constructed with the cut-and-match method. The dashed line shows the half-kinks function (4), while the solid line shows a solution in terms of elliptic function φ_{el} for $\varphi_0 = 0.8$.

half of the period T of the elliptic function φ_{el} . Thus we obtain an initial configuration

$$(-1, \varphi_0, 0, -\varphi_0, 1)$$

which we define to mean the following: in the area $-\infty < x < -T/2$ the initial state consists of the left half of (4), in $-T/2 < x < +T/2$ it is given by (9) [such that $\varphi(x = -T/4) = \varphi_0$, $\varphi(x = 0) = 0$, and $\varphi(x = T/4) = -\varphi_0$], and in the area $T/2 < x < \infty$ the solution consists of the right part of (4). There T is the period of the elliptic function (10). The profile of this type of initial state is shown in Fig. 6 (for $\varphi_0 = 0.8$). First, we consider a “static initial state” ($\partial_t \varphi_0 = 0$), but later we take into account some configurations with dynamics defined by

$$\frac{\partial \varphi_0(0, x)}{\partial t} = \frac{\varphi_0(\tau, x) - \varphi_0(0, x)}{\tau} = \delta \varphi_0 \neq 0.$$

A. Numerical solution of the equation of motion

We solve the partial differential equation (3) using a convergent difference scheme and with nonfixed boundary conditions, while derivatives are approximated by finite differences. The steps are taken as $h = 0.04$ (space step) and $\tau = 0.02$ (time step), while the equation is solved from $t = 0$ to $t = 100$. This choice of steps helps to optimize a ratio accuracy of the obtained results and the duration of computing. During the evolution a check of the conservation of energy is performed by taking into account a flow of energy from fixed boundaries. The initial states are compiled with the use of the computer algebra system *Mathematica* 8.

B. Result for unstable vacuum $\varphi = 0$

The initial condition consists of two halves of the kink, placed in $\pm x_0$, and the unstable zero solution $\varphi = 0$ between them. The energy increases linearly with growing value of x_0 . Two parts in the evolution are observed. First, there is a convergence of both halves of kink with velocity equal to the speed of light. When the halves finally meet each other, two processes alternate: a formation of loops and an emission of

waves from the kink (the so-called “wobbling kink”). The obtained solution φ_{sol} is close to the linearized solution of Eq. (3), where $\varphi_{\text{sol}} \approx \varphi_K + \delta\varphi$, which is very long lived and is characterized by a small emission of waves. These waves carry off some energy from the area of localization. Let us explain this phenomenon. At small values of $k(\varphi_0)$ the solution changes from $\varphi_{\text{el}} \approx \text{sn}x$ to $\varphi \approx \sin x$. As the sin is a periodic function, when $2x_0 \leq 2\pi$ the initial condition (with loops) does not cause an excitation like an excited mode of an elliptic function, but instead an excitation like a high-amplitude vibration of a kink. So, the evolution of the initial state can be described qualitatively by

$$\varphi_{\text{sol}} \approx \tanh\left(\frac{x}{\sqrt{2}}\right) \left(1 + \frac{A(t)}{\cosh(x/\sqrt{2})}\right),$$

$$A(t) = A_0 \cos \omega t. \tag{11}$$

The evolution can be described by this equation as there are two modes in the kink spectrum. One of them, which correlates with Eq. (6), is responsible for small vibrations across the solution. We have an idea that even if the observed vibrations stop being small, they still can be described with a periodic function like the $\cos \omega t$ [we take $\omega = \sqrt{3/2}$ like in Eq. (6)]. In Eq. (11) the parameter A_0 is taken constant, but it is not a constant in the numerical simulations because there is a small emission from the area of localization of the solution.

Moreover, there is one precondition to describe qualitatively an obtained solution accurately by Eq. (11). The function (11) equals zero in $x = 0$ one time if $A_0 \geq -1$, and three times if $A_0 < -1$. Note that a quasiperiodic formation of the loops with period equals $\approx 2\pi$; it is also one reason for using the proposed phenomenological description. This period correlates with $\cos \omega t$,

$$T = \frac{2\pi}{\omega} \approx 2\pi, \quad \text{as } \omega = \sqrt{\frac{3}{2}} \approx 1.$$

In [7], it is shown that considering the substitution of $\varphi_K + \delta\varphi$ in Eq. (3) in the quadratic approximation by $\delta\varphi$ gives us an asymptotically stable solution. Its large amplitude vibrations are characterized by strong suppression. In Figs. 7 and 8 two parts of the evolution and a comparison with the analytical solution (11) are shown for two chosen moments of time.

For high values of x_0 the observed loops in the evolution are characterized by not-small amplitudes. In this case the

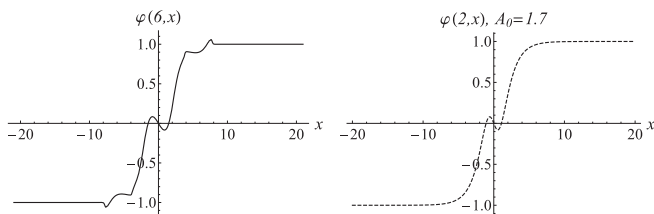


FIG. 7. The profile of $\varphi(t, x)$ at $t = 6$ (left) and mapped plot of the solution (11) for $A_0 = 1.7$ at $t = 2$ (right), $x_0 = 2$.

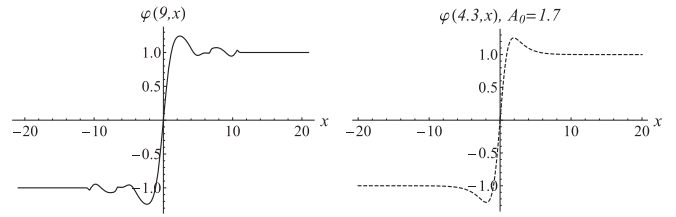


FIG. 8. The profile of $\varphi(t, x)$ at $t = 9$ (left) and mapped plot of the solution (11) for $A_0 = 1.7$ at $t = 4.3$ (right), $x_0 = 2$.

final states of evolution can be identify with the elliptic solution φ_{el} between two halves of the kink.

C. Result for an elliptic function with $0 < \varphi_0 < 1$

1. Dynamical initial state ($\delta\varphi_0 < 0$), configuration $(-1, \varphi_0, -1)$

In previous works a long-living configuration has been found, the so-called bion [1]. However, an analytical description of the observed process has not been given. In our work we take an initial condition $(-1, \varphi_0, -1)$, composed of one half-kink K and one half-antikink \bar{K} as well as a half of period of φ_{el} with fixed φ_0 . This initial state is dynamical ($\varphi_0 + \delta\varphi_0, \delta\varphi_0 < 0$) and is shown in Fig. 9. We obtain a long-living state with oscillation of the amplitude of φ at $x = 0$ (see Fig. 10). The observing oscillations in terms of an amplitude $\varphi_0(t)$ are called a

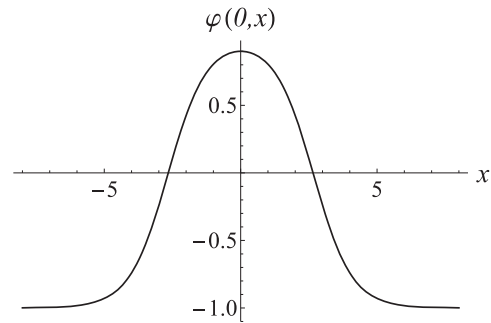


FIG. 9. The profile of $\varphi(t, x)$ at $t = 0$, with parameters $\varphi_0 = 0.8$ and $\delta\varphi_0 = -0.001$.

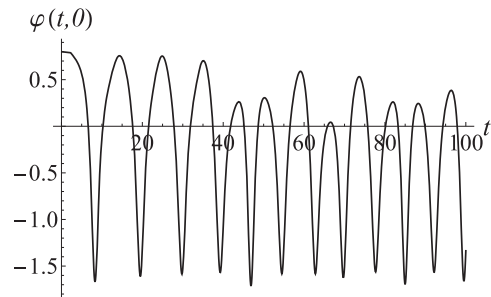


FIG. 10. The profile of $\varphi(t, 0)$, with parameters $\varphi_0 = 0.8$ and $\delta\varphi_0 = -0.001$.

regular bion. They also can be a new description of early found bion [1].

2. Statistic initial state ($\delta\varphi_0=0$), configuration

$$(-1, \varphi_0, 0, -\varphi_0, 1)$$

We take an initial condition $(-1, \varphi_0, 0, -\varphi_0, 1)$ for $\varphi_0 > 0.7$ (for smoother stitching), while the observed evolution does not qualitatively depend on φ_0 . We also show the results for the case $\varphi_0 = 0.8$.

We find two phases in the evolution: the external phase (a loop of high-amplitude formation) and the internal phase (a highly deformed kink). After some time the loops continue forming, but with smaller amplitude. After 4–5 cycles the external phase ends and the solution starts to resemble a long-living excited kink with the wave packet emission from the area of localization. This phenomenon is called a wobbling kink. This state is a final step of the evolution, which is observed for other variants of initial states. The profiles of $\varphi(t, x)$ for $\varphi_0 = 0.8$ at some particular time are shown in Figs. 11 and 12.

3. Dynamical initial state ($\delta\varphi_0 < 0$), configuration

$$(-1, \varphi_0, 0, -\varphi_0, 1)$$

An addition of $\delta\varphi_0 < 0$ to the initial state of the configuration $(-1, \varphi_0, 0, -\varphi_0, 1)$ leads to a faster reduction of the amplitude. At low values of $|\delta\varphi_0|$, where $\delta\varphi_0 < 0$, the loops arise. For the first time during the evolution, the kink-antikink pairs $K\bar{K}$ turn up. This phenomenon has a threshold. The increase of $|\delta\varphi_0|$ gives us a qualitatively new type of the evolution ($-0.0013 < \delta\varphi_0 < -0.0044$ for $\varphi_0 = 0.9$). In the system, we achieve

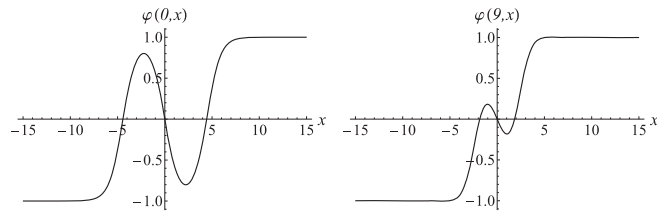


FIG. 11. The profiles of $\varphi(t, x)$ at $t = 0$ (left) and at $t = 9$ (right), with parameters $\varphi_0 = 0.8$ and $\delta\varphi_0 = 0$.

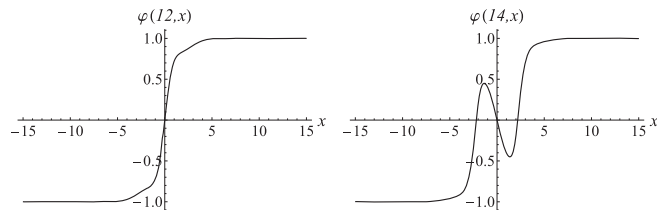


FIG. 12. The profiles of $\varphi(t, x)$ at $t = 12$ (left) and at $t = 14$ (right), with parameters $\varphi_0 = 0.8$ and $\delta\varphi_0 = 0$.

$$K \rightarrow K\bar{K}K, \tag{12}$$

where in the center of the configuration a topological number is changing. The initial state for $\varphi_0 = 0.9$ is shown in Fig. 13. The transition (12) is shown in Fig. 14 for $\delta\varphi_0 = -0.0040$. The next increasing of $|\delta\varphi_0|$ ($-0.0045 \leq \delta\varphi_0 < \dots$ for $\varphi_0 = 0.9$) gives us the next transition,

$$K \rightarrow K\bar{K}K\bar{K}K. \tag{13}$$

In this case we observe a conservation of topological number in the center. These transitions are observed for different φ_0 .

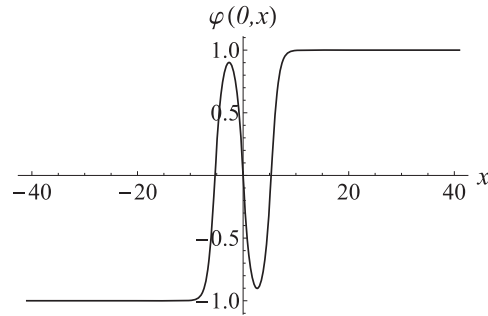


FIG. 13. The profile of $\varphi(t, x)$ at $t = 0$, with parameter $\varphi_0 = 0.9$.

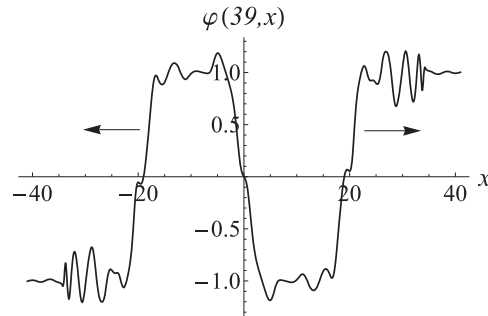


FIG. 14. The profile of $\varphi(t, x)$ at $t = 39$ and the formation of (12), with parameters $\varphi_0 = 0.9$ and $\delta\varphi_0 = -0.0040$. The arrows indicate the direction of movement of the formed kinks.

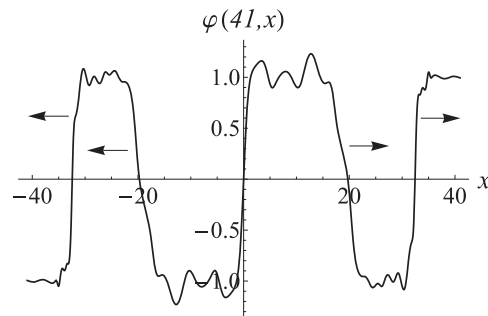


FIG. 15. The profile of $\varphi(t, x)$ at $t = 41$ and the formation of (13), with parameters $\varphi_0 = 0.9$ and $\delta\varphi_0 = -0.0045$. The arrows indicate the direction of movement of the formed (anti)kinks.

The transition (13) is shown in Fig. 15 for $\delta\varphi_0 = -0.0045$. We expect that with increasing value of $|\delta\varphi_0|$, similar qualitative changes will be observed in the evolution of the initial state.

III. CONCLUSIONS

In this work we study new long-living solutions in the classical $\lambda\varphi^4$ field theory model in $(1+1)$ dimensions.

We use the cut-and-match method for forming initial states for numerical simulations. Using this method gives us new long-living solutions both for vacuum solutions and solutions with nontrivial topological number.

In previous work [13], a long-living configuration was observed in the kink-antikink scattering and was called a bion. In current work the cut-and-match method gives us an opportunity to describe a bion formation in a new qualitative way.

Furthermore, the highly excited states of the kink are observed in a sector with nontrivial topological number. We find a number of ways to reset this energy from this state. Except for emission of wave packet with small amplitude, firstly, an arising of kink-antikink pairs has been observed. This phenomenon can be perceived as a way to reset the energy. At the same time there is a change of the topological number of the kink located in the central zone in the area. At lower excitation energies there is a long-living excited vibrational state of the kink.

The phenomenon called the wobbling kink is a final state for all considered initial conditions. After some time the excited state of a kink turns to a linearized one, which was formerly known as a discrete mode of exciting kink.

Despite the large number of new results, the cut-and-match method has a number of remaining issues in its application to the $\lambda\varphi^4$ model. In particular, a more detailed study of the dynamic of the initial conditions for the case of $\delta\varphi_0 < 0$ will be interesting, because in the last case there is the phenomenon of the birth of new kink-antikink pairs.

In conclusion, we note that this research could be useful in different area of physics and, in particular, could be implemented in the description of the early stages of the evolution of the Universe.

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- [1] T. I. Belova and A. E. Kudryavtsev, *Usp. Fiz. Nauk* **167**, 377 (1997) [*Sov. Phys. Usp.* **40**, 359 (1997)].
- [2] D. Bazeia, [arXiv:hep-th/0507188](https://arxiv.org/abs/hep-th/0507188).
- [3] R. Rajaraman, *Solitons and Instantons* (North Holland, Amsterdam, 1982).
- [4] V. A. Gani and A. E. Kudryavtsev, *Phys. Rev. E* **60**, 3305 (1999).
- [5] V. A. Gani, V. Lensky, and M. A. Lizunova, *J. High Energy Phys.* **08** (2015) 147.
- [6] V. A. Gani, M. A. Lizunova, and R. V. Radomskiy, *J. High Energy Phys.* **04** (2016) 043.
- [7] O. F. Oxtoby and I. V. Barashenkov, *Teor. Mat. Fiz.* **159**, 527 (2009) [*Theor. Math. Phys.* **159**, 863 (2009)]; *Phys. Rev. E* **80**, 026608 (2009); **80**, 026609 (2009).
- [8] M. J. Rice and E. J. Mele, *Solid State Commun.* **35**, 487 (1980); M. J. Rice, *Phys. Rev. B* **28**, 3587 (1983); H. Segur, *J. Math. Phys. (N.Y.)* **24**, 1439 (1983); O. M. Kiselev, *Russian Journal of Mathematical Physics* **5**, 29 (1997); *Sibirskii Matematicheskii Zhurnal* **41**, 345 (2000) [*Siberian Mathematical Journal* **41**, 281 (2000)].
- [9] V. A. Gani, A. E. Kudryavtsev, and M. A. Lizunova, *Phys. Rev. D* **89**, 125009 (2014).
- [10] A. Friedland, H. Murayama, and M. Perelstein, *Phys. Rev. D* **67**, 043519 (2003).
- [11] Yu. N. Gornostyrev, M. I. Katsnelson, A. V. Kravtsov, and A. V. Trefilov, *Phys. Rev. B* **60**, 1013 (1999); *Phys. Rev. E* **66**, 027201 (2002).
- [12] E. Yanke, F. Emde, and F. Lesh, *Special Functions* (Nauka, Moscow, 1977) (in Russian).
- [13] A. E. Kudryavtsev, *Pis'ma Zh. Eksp. Teor. Fiz.* **22**, 178 (1975) [*JETP Lett.* **22**, 82 (1975)].
- [14] T. I. Belova, *Yad. Fiz.* **58**, 130 (1995) [*Sov. J. Nucl. Phys.* **58**, 124 (1995)].
- [15] B. S. Getmanov, *Pis'ma Zh. Eksp. Teor. Fiz.* **24**, 323 (1976) [*JETP Lett.* **24**, 291 (1976)].
- [16] N. S. Manton and H. Merabet, Report No. DAMTP 96-45, 1966.
- [17] T. I. Belova, N. A. Voronov, I. U. Kobzarev, and N. B. Konyukhova, Report No. ITEP-170, 1976 (in Russian).
- [18] I. L. Bogolyubsky and V. G. Makhankov, *Pis'ma Zh. Eksp. Teor. Fiz.* **24**, 15 (1976) [*JETP Lett.* **24**, 12 (1976)].