Monitoring and control of stochastic systems

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This thesis was prepared within the partnership between the University of Amsterdam and The University of Queensland with the purpose of obtaining a joint doctorate degree. The thesis was prepared in the Faculty of Science at the University of Amsterdam and in the School of Mathematics and Physics at The University of Queensland.

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Monitoring and Control of Stochastic Systems

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Monitoring and Control of Stochastic Systems

Julia Kuhn
Master of Science, Mathematics

A thesis submitted for the degree of Doctor of Philosophy at
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Contents

Notation and Abbreviations

This list is not exhaustive as it only includes basic notation that is valid throughout the entire thesis.

Notation

$\mathbb{1}\{\cdot\}$: Indicator function (thus, $\mathbb{1}\{A\} = 1$ if event A occurs, $\mathbb{1}\{A\} = 0$ otherwise)

a^*: Optimising value of a

$a_i(t)$: Action taken at time t for arm i ($a_i(t) \in \{0, 1\}$)

$\mathcal{B}_n$: $\mathcal{B}_n = \{0/n, 1/n, \ldots, (n-1)/n\}$

ε_t: Innovation at time t

$\eta_i(t)$: Number of time steps ago that arm i was last observed before t

$\iota(\cdot)$: Index function

$I_X(\cdot)$: LD rate function of X (Legendre transform of $\Lambda_X(\cdot)$)

$I(\cdot \,|\, x)$: LD rate function of Pois(x)

L: Likelihood ratio

$\mathscr{L}$: Log-likelihood ratio

$\Lambda_X(\cdot)$: Cumulant-generating function of a random variable X

$M_X(\cdot)$: Moment-generating function of a random variable X

π_ι: Index policy with index function $\iota(\cdot)$

$r_i(\cdot)$: Expected immediate reward: $r_i(s) := \mathbb{E}_s[R_i^1(X_i)]$

$R_i^a(X_i)$: Reward of arm i when it is in state X_i under action a

τ: Stopping time

$\theta_X(a)$: Optimising value of θ such that $I_X(a) = \theta_X(a)\, a - \Lambda_X\big(\theta_X(a)\big)$

ϑ: Belief state

$X^{(r)}$: r-th Sample of the random variable X

Abbreviations

AR:	Gaussian autoregression of order 1
ARMA:	autoregressive moving average
CLT:	Central limit theorem
EV:	Extreme value
GE:	Gilbert-Elliott
i.i.d.:	Independent and identically distributed
LD:	Large deviations
LLR:	Log-likelihood ratio
MA:	moving average
MDP:	Markov decision process
MGF:	Moment-generating function
MMP:	Markov-modulated Poisson
Ph:	Phase-type
POMDP:	Partially observable Markov decision process
RMAB:	Restless multiarmed bandit problem
RORMAB:	Reward-observing restless multiarmed bandit problem
VAR:	Vector autoregressive
VARMA:	Vector autoregressive moving average

Kendall's notation

M/G/m:	m-Server queueing system with stationary Poisson arrivals and stationary random service times
M/M/m:	m-Server queueing system with stationary Poisson arrivals and stationary exponentially distributed service times
M/M/∞:	Infinite-server queueing system with stationary Poisson arrivals and stationary exponentially distributed service times
MMP/Ph/m:	m-Server queueing system with Markov modulated Poisson arrivals and Phase-type distributed service times

Preface

There is nothing certain but the uncertain. This proverb could well be the slogan of the discipline of stochastic modeling, whose main objective it is to incorporate the certainty about the uncertainty into any mathematical model of the real world. For example, in telecommunication systems uncertainty may result from random fluctuations in the number and behaviour of users, the duration of service, and environmental factors. Other common application areas for stochastic modeling and analysis include industrial manufacturing, biology, medicine and traffic.

"Uncertainty is an uncomfortable position", said Voltaire, which may explain why such a large community of researchers all around the globe struggles to make the uncertain tractable despite the seeming absurdity of such an endeavour. To this end, uncertain events are included in the model, weighed by the likelihood with which they occur. In this way it may be hoped that any conclusions drawn from the model are sufficiently robust – at least with respect to this 'predictable randomness' in the dynamic behaviour of the system.

While as mathematicians we strive to answer all our questions fully analytically, delving into this 'philosophy' of stochastic modelling reminded me that there is no guaranteed accuracy to even a fully exact analysis of the model with respect to the real-world system it is describing. From this point of view, it seems to me that any decision guidelines derived from a stochastic model should first of all aim to be practical.

Looking back on all the problems I tackled and could only solve by means of heuristics and approximations, I find this thought a comforting one. In fact, I came to learn that heuristics and approximations often turn out to be more useful than the exact solution. For example, in Part II of this thesis we investigate heuristic policies for a certain class of Markovian decision problems referred to as bandit problems. In Part III we consider procedures for the detection of changes in the probabilistic behaviour of data streams that are based on approximations to the type I error probability. For both classes of problems heuristic solutions have been of interest for decades even though an exact analysis of the problems is actually possible, simply because such an analysis is too cumbersome and computationally expensive to be practical. Asymptotically accurate solutions are in comparison much more tractable, and they often yield good results, particularly in the face of nowadays' data abundance and large-scale networks.

Monitoring and control. The systems to be modelled are increasingly complex. For example, in modern telecommunication networks the presence of many users and large amounts of data requires a top speed processing capability of the system. This is enabled through powerful hardware and software, supplemented by smart architecture, effective control policies, and careful use of the available resources. Questions related to the design

and control of such stochastic systems are therefore of great practical importance.

Static control problems are those that can or must be solved to a large extent a-priori, such as questions related to system architecture, the number and type of servers required, and more generally all issues related to the *design* of the system. Design choices affecting the architecture of the system are often not easily reversible and in this case have to be made off-line. This can apply, for example, in the context of manufacturing networks, where one has to make decisions about the number or the type of machines to be acquired, or the layout of a factory. But even in cases in which a certain choice of design can be adjusted easily, it may be more cost-effective or otherwise desirable to decide such questions a-priori: For example, in the context of staffing it can be impractical to update the staffing rule for a particular day in the course of that day as it incurs additional labour costs for on-call service.

On the other hand, there are circumstances in which it can be worth these additional costs to update the staffing rule throughout the day; for example in hospitals, where it is very crucial that a certain performance standard is being kept. In other cases, it may be comparatively easy to adapt one's initial choices at a later point in time, as is true for example for certain financial portfolios of stocks that can be bought or sold at any point time; or in communication networks, where calls can be rerouted to different servers. Generally, one hopes that a greater flexibility in the control policy yields a better performance of the system.

Dynamic control policies allow for adaptation based on up-to-date information concerning the current state of the system. The latter is identified by performance characteristics which affect the control or decision that is to be taken. Examples include the workload of each node in a computer network, or the signal-to-noise-ratio of a communication channel; for the aforementioned financial portfolio stock prices are an important performance measure.

A good control policy, whether static or dynamic, makes use of the information available concerning the typical dynamic behaviour of the system. This means that the performance of the decision policy crucially depends on the validity of the stochastic model. Consequently, procedures for testing and monitoring the relevant data streams are needed. Since environmental factors may change, for example during times of peak load, one requires procedures that are capable of detecting such changes in an automated fashion.

Outline of the thesis. Paralleling the described trinity of control problems given by static control, dynamic control and monitoring of stochastic systems, this thesis is divided into three parts. Each part begins with an introductory chapter, which explains the motivation and summarises some of the necessary mathematical background. The other chapters are based on the research I conducted together with my supervisors and other co-authors. This research led to a number of articles which I am referencing at the end of the thesis, together with other related literature. To acknowledge the contribution of my co-authors, I provided a clarification of my own contributions in the preamble of this thesis. I briefly summarize the contents of Parts I–III below; more detailed outlines are provided in the introductory chapters preceding each part.

Part I is mainly focussed on performance analysis as an important first step in designing static control policies for stochastic systems. In particular, we derive asymptotic expressions and efficient simulation algorithms for certain performance characteristics that are related to rare events such as network congestion of failure. The study of the asymptotic behaviour of rare event probabilities is commonly known as *large deviations*

theory [38]. Often large deviations asymptotics are derived for the scaled logarithm of the probability of interest. Many of our results in Part I, however, are in terms of *exact* or *sharp* asymptotics, that is, we provide approximations for the rare event probability itself instead of for its logarithm. Such exact approximations can be considerably more accurate. In order to be able to check the quality of the obtained approximations, as well as to enable performance evaluation in cases where the asymptotic regime does not yield a realistic description of the system, we consider efficient methods for the simulation of such performance measures.

In Chapters 2 and 3 of Part I we are concerned with performance metrics relevant for the analysis of particular *queueing models* [131, Chapter 8]. The focus is on rare-event simulation in Chapter 2, and on both exact asymptotics and simulation in Chapter 3. In particular, in these chapters we are interested in the probability that backlog, that is, the number of customers or jobs waiting in the queue, exceeds a given (large) level – as we point out in Chapter 3 knowledge of this probability can aid the system designer for example in devising static rules for staffing. In Chapter 4 we focus on rare-event probabilities related to the comparison between order statistics of two populations of sample means. The derived expressions may be useful for devising static control policies for certain queueing or packing problems. They also have implications regarding the design of testing procedures such as those we consider in Part III (see below).

In Part II we turn to dynamic control problems. We concentrate on a particular class of problems related to the control of stochastic systems known as *restless multiarmed bandits* [48] (RMABs). Bandit problems are a type of *Markov decision problem* [126], in which at every decision time a number of competing arms of a slot machine (also known as one-armed bandit) have to be selected to play on. RMABs are used, for example, to model problems related to the allocation of jobs or customers to different servers, of resources to competing projects, or of messages to transmission channels.

Our objective in this part is to investigate computationally feasible decision making strategies for certain *partially observable* RMABs where the available information concerning the state of each arm is assumed to be incomplete. For example, in the context of channel selection, it is reasonable to assume that due to physical, cost or other constraints the state of a channel is only sensed when that channel is selected and a feedback is received upon transmission. Given this type of incomplete state information, we are interested in structural properties and performance characteristics of so called *index policies*, which form a computationally tractable class of control heuristics [48].

In Chapter 6 we review two types of models that have been proposed for the purpose of modelling the channel selection problem. Traditionally, the focus has been on the Gilbert-Elliott channel model, in which channels are assumed to behave like a two-state Markov chain. A second model which has only recently come to attention in this context is the RMAB with Gaussian autoregressive arms. We concentrate on the latter model in Chapter 7, where we investigate structural properties and propose a heuristic index policy.

Part III of the thesis is about statistical procedures for the testing of data sequences. We distinguish between off-line testing and on-line (sequential) monitoring. In both cases we focus on procedures based on a *log-likelihood ratio* (LLR) test statistic, motivated by the good optimality properties of such tests. We show how one can use logarithmic LD and other asymptotics to control the performance of the testing procedures of interest.

In Chapter 9 we consider an off-line testing problem, where the objective is to identify

the data stream that follows a given target distribution. Since in practice data is often limited and the collection of new data points can be expensive, we assume that the target process is to be identified based on a limited number of samples. The question is then how to allocate the samples in order to obtain an accurate identification procedure. We use LD asymptotics to approximate the probability of a false identification, which allows us to solve the problem by convex optimisation.

In Chapters 10 and 11, we focus on methods for *change point detection* – the detection of changes in the model parameters. A popular method for change point detection is the method of cumulative sums (CUSUM) [118], which proceeds by sequentially evaluating a LLR test statistic and comparing it to a predefined threshold; a change point is detected as soon as the threshold is exceeded. It is desirable to choose the threshold such that the number of false alarms is kept to a specified level. Traditionally, the number of false alarms is measured by the average run length – the expected time until the first false alarm. However, this does not in general allow one to control the number of false alarms at every particular time instance. Therefore, in Chapter 10 we consider two stronger false alarm criteria, for which approximation methods are investigated that facilitate the selection of a threshold. We provide examples featuring change point detection for dependent Gaussian sequences in Chapter 11.

Part I

Static Control Problems

CHAPTER 1

Introduction

Choices regarding the design and static control of a stochastic network are made with the objective to optimise certain relevant performance criteria. This is easily achieved if the stochastic model is simple enough to allow for exact closed-form evaluation of the performance criteria of interest. In many applications, however, adequate models are more complex so that one may need to resort to approximate solutions.

In this part of the thesis, we derive approximations for a number of different performance criteria that are relevant, for example, in the context of communication systems. Specifically, we are interested in probabilities related to certain undesirable events that one may wish to control, such as the probability of a blockage due to overload, or any other type of failure probability that is affected by the network design. One then tries to optimise the design and control of the network in a way that ensures that such events are *rare* in that the probabilities of their occurrence are kept at a desired (small) level.

The study of rare event probabilities is referred to as *large deviations* (LD) theory [38]; we review some of the relevant literature and mathematical background in Section 1.1.1. LD approximations are accurate with respect to some *asymptotic regime*, that is, they converge to the exact value of the quantity to be estimated (often the decay rate of the probability rather than the probability itself) as some scaling parameter grows large. This parameter can, for example, scale the number of customers arriving to the network.

As we shall see, these LD asymptotics often yield relatively simple expressions that can be useful to understand the behaviour of the probability of interest in a certain regime. On the other hand, it is often difficult to find such approximations for the probability itself so that one often has to be content with describing its decay rate. Besides, the assumption that the system is large in some sense is not always realistic in practice, and in such cases asymptotic expressions can yield poor approximations. Estimates obtained from computer-simulated data are therefore also of interest. The simulation-based approach is broadly applicable but can be computationally demanding, especially in the context of rare event probabilities. A common technique to simulate such probabilities relatively efficiently is known as *importance sampling*. We provide some background on rare-event simulation with importance sampling in Section 1.1.2.

An important class of models commonly considered in the study of stochastic systems are referred to as *queueing models*. In Chapters 2 and 3 we study two particular queueing

models in further detail, and we therefore provide some background on queueing theory in Section 1.1.3.

The further outline of this part of the thesis is presented in Section 1.2.

1.1 Background

1.1.1 Large deviations

In this section we state some key results of LD theory; parts of this exposition are taken from [80] and [88]. The theory of LD studies probabilities of rare events. A gentle introduction to LD theory is given by Bucklew [24]. For a mathematically more rigorous treatment we recommend the book by Dembo and Zeitouni [38].

Let us summarise some key ideas in the theory of LD that are needed in the course of this thesis. We start by defining the large deviations principle, which will ease the presentation of some important limit results that are presented later in this section.

Large deviations principle. Let E be a complete separable metric space and for each $n \in \mathbb{N}$ let $(\Omega_n, \mathcal{F}_n, P_n)$ be a probability space and $\boldsymbol{X}_n : \Omega_n \to E$ a random vector. The *LD principle* (LDP) characterises the limiting behaviour of certain sequences of rare event probability measures. Roughly speaking, it states that rare event probabilities decay exponentially, where the exponential rate can be quantified in terms of some *rate function* $\mathscr{I}$. The following definitions are taken from [38] and [44].

Definition 1.1. A rate function $\mathscr{I}$ is a lower semicontinuous mapping $\mathscr{I} : E \to [0, \infty]$, that is, for all finite $a \in \mathscr{I}(E)$, the level set $\Psi(a) := \{x \in E : \mathscr{I}(x) \leq a\}$ is a closed subset of E.

Note that, because E is a metric space, the lower semicontinuity property is satisfied if and only if $x_n \to x$ implies $\liminf_{x_n \to x} \mathscr{I}(x_n) \geq \mathscr{I}(x)$ because $\inf \mathscr{I}(B_\epsilon(x)) = \inf\{\mathscr{I}(y) : y \in B_\epsilon(x)\}$, where $B_\epsilon(x)$ denotes the open ball around x with radius ϵ. This property implies that the rate function is unique (for a proof see [76, Theorem 23.8]).

In this thesis, we are concerned with cases in which $\mathscr{I} : \mathbb{R}^d \to [0, \infty]$ is a *good rate function*, meaning that the level sets defined above are compact subsets of $E = \mathbb{R}^d$. This is convenient because of the following well-known property of the rate function (a proof can, for instance, be found in [76, p. 494]).

Lemma 1.2. *Let $\mathscr{I}$ be a rate function on E. Then, for every non-empty compact subset $K \subset E$, there exists at least one $x^* \in K$ such that $\mathscr{I}(x^*) = \inf_{x \in K} \mathscr{I}(x)$.*

Thus, when $\mathscr{I}$ is a good rate function, for all $a \in \mathscr{I}(E)$, the infimum over the level set $\Psi(a)$ is attained within that set.

Definition 1.3. We say that the sequence $(\boldsymbol{X}_n)_{n \in \mathbb{N}}$ satisfies the LDP with rate function $\mathscr{I}(\cdot)$ if,

(a) For any closed set $F \subseteq E$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_n \left(\boldsymbol{X}_n \in F\right) \leq - \inf_{x \in F} \mathscr{I}(x);$$

(b) For any open set $G \subseteq E$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n \left(\boldsymbol{X}_n \in G \right) \geq - \inf_{x \in G} \mathscr{I}(x).$$

Colloquially speaking, when a sequence satisfies the LDP, the probability of a rare event decreases to zero exponentially fast as n increases. Furthermore, given that the rare event has occurred, it has almost surely occurred in the most likely of the unlikely ways (corresponding to the optimizing x on the right-hand side).

Let B be a Borel subset of E, and denote its closure by B^c and its interior by B°. In this thesis we only consider $E = \mathbb{R}^d$, and B and $\mathscr{I}(\cdot)$ such that $\mathscr{I}(B^c) = \mathscr{I}(B^\circ)$, so that the LDP reduces to [44, Proposition 3.3]

$$\lim_{n\to\infty} \log \mathbb{P}_n \left(\boldsymbol{X}_n \in B \right) = - \inf_{x \in B} \mathscr{I}(x).$$

In the following we state conditions under which a sequence of random vectors satisfies an LDP.

Logarithmic asymptotics. Let $\overline{\boldsymbol{X}}_n := n^{-1} \sum_{t=1}^n \boldsymbol{X}_t$, and assume that $\boldsymbol{X}_t$ are independent and identically distributed (i.i.d.) as a generic random variable $\boldsymbol{X}$. Then provided that the mean is finite, the law of large numbers states that the law of $\overline{\boldsymbol{X}}_n$ tends to $\mathbb{E}\boldsymbol{X}$ as $n \to \infty$. Cramér's theorem [33] asserts that the law of certain sets that do not contain this point of concentration decays exponentially in n. His theorem characterises the logarithmic rate of convergence by a rate function that is given by the *Legendre transform* (also referred to as Fenchel-Legendre transform, or convex conjugate) of the cumulant-generating function (see Definition 1.5).

Definition 1.4. The Legendre transform of a convex function $\psi : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$I(x) := \sup_{\theta \in \mathbb{R}^d} \left(\theta' x - \psi(\theta) \right).$$

Definition 1.5. The moment-generating function (MGF) of $\boldsymbol{X} \in \mathbb{R}^d$ is defined as

$$M_{\boldsymbol{X}}(\boldsymbol{\theta}) := \mathbb{E}\left(e^{\boldsymbol{\theta}' \boldsymbol{X}} \right)$$

wherever this expectation exists; for $\boldsymbol{\theta} \in \mathbb{R}^d$. The cumulant-generating function $\Lambda_{\boldsymbol{X}}(\cdot)$ is defined as

$$\Lambda_{\boldsymbol{X}}(\boldsymbol{\theta}) := \log M_{\boldsymbol{X}}(\boldsymbol{\theta}).$$

We denote the Legendre transform of $\Lambda_{\boldsymbol{X}}(\cdot)$ by $I_{\boldsymbol{X}}(\cdot)$. As an example we compute the Legendre transform $I_X(a)$, $a \in \mathbb{R}$, corresponding to $\Lambda_X(\cdot)$ where X is a univariate Poisson random variable with rate λ (we write $X \sim \text{Pois}(\lambda)$). In this case, the MGF is

$$M_X(\theta) = \exp\left(\lambda(e^\theta - 1) \right).$$

The optimizing θ in the definition of the Legendre transform exists, and is given by $\log(a/\lambda)$. Inserting this, we obtain

$$I_X(a) = I(a \,|\, \lambda) := a \log(a/\lambda) - a + \lambda. \tag{1.1}$$

Cramér's result corresponds to the light-tailed regime, as formalised by the following assumption.

Assumption I.1. The MGF $M_{\boldsymbol{X}}(\boldsymbol{\theta})$ is finite; that is, $D_{M_{\boldsymbol{X}}} := \{\boldsymbol{\theta} : M_{\boldsymbol{X}}(\boldsymbol{\theta}) < \infty\} = \mathbb{R}^d$.

This implies that all moments of $\boldsymbol{X}$ are finite. Under this assumption, it turns out that $I_{\boldsymbol{X}}(\cdot)$ is a good convex rate function [38, Lemma 2.2.31(a)]. We can now state Cramér's famous theorem; the following version is Theorem 4.1 in [44].

Theorem 1.6 (Cramér)**.** *Let $(\boldsymbol{X}_t)_{t\in\mathbb{N}}$ be a sequence of independent random vectors taking values in $\mathbb{R}^d$, distributed as a random vector $\boldsymbol{X}$, and satisfying Assumption I.1. Then the sequence of sample means $\overline{\boldsymbol{X}}_n := \frac{1}{n}\sum_{t=1}^n \boldsymbol{X}_t$ satisfies the LDP with convex rate function $I_{\boldsymbol{X}}(\cdot)$ defined as the Legendre transform of $\Lambda_{\boldsymbol{X}}(\cdot)$.*

For semi-infinite intervals and i.i.d. sequences (X_t) in $\mathbb{R}$, this result can be strengthened to the following corollary, [38, Corollary 2.2.19].

Corollary 1.7. *For any $a \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\overline{X}_n \geq a) = -\inf_{x\geq a} I_X(x).$$

Let us consider some properties of the rate function for the special case where $d = 1$. Obviously, $I_X(x) \geq 0$ since $0 \cdot x - \Lambda(0) = 0$. By Jensen's inequality, $M_X(\theta) \geq \exp(\theta\,\mathbb{E}X)$ so that $\theta\,\mathbb{E}X - \Lambda_X(\theta) \leq 0$. Hence, it holds that $I(\mathbb{E}X) = 0$. Consequently, in accordance with our intuition, $\mathbb{E}X$ is a minimum of the convex non-negative function $I(\cdot)$, which is thus non-decreasing for $x > \mathbb{E}X$ (and non-increasing for $x < \mathbb{E}X$). Therefore, for $a > \mathbb{E}X$, Cramér's theorem actually asserts that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\overline{X}_n \geq a\right) = -I_X(a). \tag{1.2}$$

For example, for i.i.d. Poisson random variables X_i this limit result holds with rate function $I_X(a)$ as given in (1.1).

Formal proofs of Cramér's theorem and its corollary can be found in [38, Chapter 2]. Here we only sketch the main ideas as they will be useful later-on in this thesis.

Sketch of proof. We first note that the upper bound of the LDP is essentially a Chernoff bound: Assuming the setting of the corollary, for $\theta > 0$, we have

$$\mathbb{P}\left(\overline{X}_n \geq a\right) = \mathbb{P}\left(e^{\theta\sum_{i=1}^n X_i} \geq \mathrm{e}^{\theta n a}\right) \leq e^{-\theta n a} M_X(\theta)^n.$$

This holds in particular with θ^* that minimises the upper bound on the right-hand side (which we assume to exist for the moment). It can further be checked that for $a > \mathbb{E}X$ the Legendre transform is optimised at $\theta^* > 0$, hence, taking the logarithm and dividing by n we obtain the upper bound $-I_X(a)$ in this case. (Similarly for the case $a < \mathbb{E}X$, where instead we have $\theta^* < 0$).

It remains to be shown that under the assumptions of Corollary 1.7 this upper bound is attained asymptotically as $n \to \infty$. To this end, a change of measure is applied such that under the alternative measure the event of interest is not rare; then the lower bound can be derived from central limit arguments. For simplicity assume that X has a density function $p(\cdot)$ under the original measure. We then wish to determine an alternative density

$q(\cdot)$ such that under $q(\cdot)$ the sample mean $\overline{X}_n$ has mean value a. This is found by noting that the optimising θ^* satisfies $a - M'_X(\theta)/M_X(\theta) = 0$, and thus, $q(\cdot)$ needs to satisfy

$$\mathbb{E}_q\left[\overline{X}_n\right] = \int_{-\infty}^{\infty} xq(x)\mathrm{d}x = a = \frac{M'_X(\theta^*)}{M_X(\theta^*)} = \frac{\mathbb{E}\left[Xe^{\theta^* X}\right]}{\mathbb{E}\left[e^{\theta^* X}\right]}.$$

It is then easy to see that the *exponentially tilted* (also referred to as *exponentially twisted*) measure

$$q(x) = \frac{e^{\theta^* x}}{M_X(\theta^*)}p(x) \tag{1.3}$$

solves the equation. It turns out that with this change of measure, to obtain the statement of Cramér, it suffices that the law of large numbers applies. Using central limit arguments (Berry-Esseen) a stronger statement can be proven, see Theorem 1.9 below. □

Cramér's theorem is limited to the independent case. However, an extension to dependent sequences is possible, and an important result in this direction is referred to in the literature as Gärtner-Ellis theorem. We will need it in Part III of this thesis.

To state the theorem, we define the limiting cumulant-generating function as

$$\Lambda_{\boldsymbol{X}}(\boldsymbol{\theta}) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(e^{n\,\boldsymbol{\theta}'\boldsymbol{X}_n}\right), \tag{1.4}$$

wherever the limit exists. Note that this reduces to the cumulant-generating function of $\boldsymbol{X}_1$ if $\boldsymbol{X}_i$ are i.i.d.; therefore (with a slight abuse of notation) we denote both functions by $\Lambda_{\boldsymbol{X}}(\cdot)$.

Theorem 1.8 (Gärtner-Ellis)**.** *Assume that for every $\boldsymbol{\theta} \in \mathbb{R}^d$ the limit $\Lambda_{\boldsymbol{X}}(\boldsymbol{\theta})$ as defined in (1.4) exists as a finite number; and that $\Lambda_{\boldsymbol{X}}(\cdot)$ is differentiable. Then $\boldsymbol{X}_n$ satisfies the LDP with rate function $I_{\boldsymbol{X}}(\cdot)$ given by the Legendre transform of $\Lambda_{\boldsymbol{X}}(\cdot)$.*

By the same reasoning as before we obtain that for the specific case $d = 1$, the Legendre transform is non-decreasing for $x \geq \mathbb{E}X$. Theorem 1.8 establishes that the asymptotic relationship (1.2) holds for more general sequences of random variables; for example, $\overline{X}_n$ can be sample means of correlated random variables, provided that the limiting cumulant-generating function exists and is finite.

Exact asymptotics. Theorems 1.6 and 1.8 provide *logarithmic* asymptotics for certain sequences of probability measures. For sequences of sample means $\overline{X}_n$, they suggest approximations of the form

$$\mathbb{P}\left(\overline{X}_n \geq a\right) \approx \exp\left(-nI_X(a)\right). \tag{1.5}$$

Note, however, that, while such approximations are often useful (and indeed we shall use them in Part III of this thesis), they can turn out to be inaccurate for moderate values of n. To illustrate, note that Eq. (1.2) is valid if $\mathbb{P}\left(\overline{X}_n \geq a\right)$ behaves as (i) $10^9 \cdot e^{-nI_X(a)}$, (ii) $n^{-100}e^{-nI_X(a)}$, or (iii) $e^{\sqrt{n}}e^{-nI_X(a)}$, but obviously in none of these cases the 'naïve' approximation (1.5) is accurate; see e.g. [104, p. 40] for a brief exposition on this. Approximations to the probability itself rather than its logarithm are more scarce in the literature, and usually referred to as *strong*, *sharp*, or *exact* LD results. An important result on exact LD asymptotics is due to Bahadur and Rao [13], which we state next. It requires the following assumption.

Assumption I.2. The optimising θ in the definition of $I_X(a)$ exists (and is denoted by $\theta_X(a)$ or sometimes θ^* for short).

In the setting of Theorem 1.8 this assumption holds in fact for any a in the interior of the essential domain $D_{I_X} := \{x \in \mathbb{R} : I_X(x) < \infty\}$. It is well-known [38, Lemma 2.2.5] that if $a > \mathbb{E}\,X$, then $\theta_X(a) > 0$; likewise, if $a < \mathbb{E}\,X$, then $\theta_X(a) < 0$. Furthermore, the optimizing $\theta_X(a)$ is easily seen to satisfy $I'_X(a) = \theta_X(a)$ as well as $\Lambda'(\theta) = a$. These facts we use repeatedly in Section 4.

Theorem 1.9 (Bahadur-Rao)**.** *Fix an $a > \mathbb{E}\,X$, and assume that Assumptions I.1 and I.2 apply. Then for some positive and finite constant $C_X(a)$, as $n \to \infty$,*

$$\mathbb{P}(\overline{X}_n \geq a)\,\sqrt{n}\,e^{nI_X(a)} \sim C_X(a); \tag{1.6}$$

where $f(n) \sim g(n)$ denotes that $f(n)/g(n) \to 1$ as $n \to \infty$.

This is Theorem 3.7.4 in [38]. The idea of the proof is similar as that of Cramér's theorem: We apply a change of measure such that the event of interest is not rare. Then instead of the law of large numbers, we apply a stronger limit result due to Berry and Esseen, which provides bounds on the rate of convergence of the sample mean to a normal distribution. For details see [19].

The precise form of $C_X(a)$ depends on whether X corresponds to a non-lattice or a lattice random variable. A lattice distribution is a discrete distribution concentrated on a set of points of the form $a + bn$ for $n \in \mathbb{Z}$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$. If X is non-lattice (for example, it has a normal or a gamma distribution), then

$$C_X(a) = \frac{1}{\theta_X(a)\sqrt{2\pi\Lambda''_X(\theta_X(a))}}, \tag{1.7}$$

where $\Lambda''_X(\theta_X(a))$ denotes the second derivative of $\Lambda_X(\theta)$ evaluated at $\theta_X(a)$. If X is a lattice random variable (for example, it is Poisson), we instead have

$$C_X(a) = \frac{1}{1 - \exp\big(-\theta_X(a)\big)}\,\frac{1}{\sqrt{2\pi\Lambda''\big(\theta_X(a)\big)}}. \tag{1.8}$$

In Section 3.1 we shall also need a local limit version of (1.6): with $\xi_n(\cdot)$ denoting the density of $\sum_{i=1}^n X_i$, from [123] we have

$$\lim_{n\to\infty} \xi_n(na)\,\mathrm{e}^{nI_X(a)}\sqrt{n} = C_X(a)I'_X(a). \tag{1.9}$$

Further extensions of the Bahadur-Rao result include results on the uniformity of the convergence by Höglund [65]. A version not necessarily requiring the i.i.d. assumption has been proven by Chaganty and Sethuraman in [28]. This result was further extended into a multi-dimensional context by the same authors [29]: there, exact asymptotics are established of the probability that a vector of sample means is in a given rectangular set.

For later reference in Section 4, we informally state Theorem 3.4 by Chaganty and Sethuraman [29] (the precise statement is lengthy, and we will therefore only give relevant details of it when needed): Under a number of technical conditions, it is proved that for positive constants C and I, a sequence of random vectors $\boldsymbol{X}_n \in \mathbb{R}^d$ and a matrix A,

$$\mathbb{P}(A\overline{\boldsymbol{X}}_n \geq \mathbf{0}) \sim \frac{C}{n^{d/2}}e^{-nI}, \tag{1.10}$$

where for two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d$ we define $\boldsymbol{v} \geq \boldsymbol{w}$ to hold if and only if $v_i \geq w_i$ for all $i \in \{1, \ldots, d\}$.

Later in this chapter we make use of the presented LD results to obtain asymptotic expressions for certain rare-event probabilities. In the next section we present a technique for estimating such probabilities using simulated data.

1.1.2 Rare-event simulation

In this section we give a brief introduction on a well-established rare-event simulation technique known as importance sampling. For further details see, for example, [10, 130]. Consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let (A_n) denote an decreasing sequence of subsets of $\mathcal{F}$, such that $\varrho_n := \mathbb{P}(A_n) \to 0$ exponentially fast; that is, ϱ_n satisfies a large deviations principle of the form

$$\limsup_{n\to\infty} \frac{1}{n} \log \varrho_n = -I, \tag{1.11}$$

for some constant $I > 0$. Suppose we want to estimate ϱ_n for n large. The crude (naïve) Monte Carlo estimator is given by

$$\widehat{\varrho}_n^{\mathrm{MC}} = \frac{1}{r} \sum_{i=1}^{r} \mathbb{1}^{(i)}(A_n),$$

where $\mathbb{1}^{(i)}(A_n)$ denotes the i-th sample of $\mathbb{1}(A_n)$. While this produces an unbiased estimator of ϱ_n, it is intuitive that this is not an efficient estimation procedure when ϱ_n is small: most of the samples are zero, but a single observation of a sample equal to one can skew the estimator by some orders of magnitude. Thus, r has to be chosen very large for the obtained estimate to be sufficiently accurate. Mathematically, the *relative error* — the ratio of the estimator's standard deviation to its mean — is given by

$$\frac{\sqrt{r^{-1}\varrho_n(1-\varrho_n)}}{\varrho_n} \frac{r^{-1}}{\sqrt{\varrho_n}},$$

which increases to infinity exponentially fast, as $n \to \infty$ while r remains fixed.

A popular approach that can achieve considerable efficiency improvements upon crude Monte Carlo estimation is a method known as *importance sampling*. The main idea we have already encountered in the sketch of the proof of Cramér's theorem: sample the random variable of interest under an alternative measure $\mathbb{Q}$ such that the variance of the estimator is minimised — typically, this new measure is such that the event A_n is not rare any more. For the obtained estimator to be unbiased, the samples from $\mathbb{Q}$ needs to be re-weighed by the *likelihood ratio* $\mathcal{L}$, which is a version of the Radon-Nikodym derivative $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$.

More formally, consider a new probability measure $\mathbb{Q}$ with $\mathbb{Q}(A_n \cap B) > 0$ whenever $\mathbb{P}(A_n \cap B) > 0$ for any $B \in \mathcal{F}$. Then we have

$$\varrho_n = \int_\Omega \mathbb{1}(A_n) \mathrm{d}\mathbb{P} = \int_\Omega \mathbb{1}(\mathrm{A_n})\, \mathcal{L}\, \mathrm{d}\mathbb{Q} = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}(\mathrm{A_n})\, \mathcal{L}\big],$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation taken with respect to $\mathbb{Q}$. An unbiased estimator for

this quantity is:

$$\widehat{\varrho}_n^{\mathrm{IS}} = \frac{1}{r}\sum_{i=1}^{r} \mathbb{1}^{(i)}(A_n)\,\mathcal{L}^{(i)}.$$

We desire a new measure $\mathbb{Q}$ that minimises the variance of this estimator, which is given by

$$\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}(A_n)\mathcal{L}^2\right] - \varrho_n^2.$$

Thus, in principle a zero-variance estimator is achieved by choosing $\mathbb{Q}$ such that we have

$$\mathrm{d}\mathbb{Q} = \frac{\mathbb{1}(A_n)\,\mathrm{d}\mathbb{P}}{\varrho_n}.$$

This choice, however, is infeasible since $\mathbb{Q}$ would depend on ϱ_n, which is the unknown quantity to be estimated. What can often be done is to find an alternative measure that achieves zero variance asymptotically, on a logarithmic scale. Since the variance is always non-negative, we have

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}(A_n)\,\mathcal{L}^2\right] \geq -2I,$$

by (1.11). This motivates the following notion of *asymptotic efficiency* (or logarithmic efficiency) (see, e.g., [134]):

Definition 1.10. A distribution $\mathbb{Q}$ is asymptotically efficient for estimating ϱ_n if

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}(A_n)\,\mathcal{L}^2\right] \leq -2I.$$

If the estimator in fact has *bounded relative error* (at least for large n), then it is called *strongly efficient*; see Definition 1.11. As the name suggests, strong efficiency implies asymptotic efficiency; see e.g. [135, Lemma 1]).

Definition 1.11. A distribution $\mathbb{Q}$ is strongly efficient for estimating ϱ_n if the relative error is bounded by a constant $c < \infty$ in that

$$\limsup_{n\to\infty} \frac{\sqrt{\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}(A_n)\,\mathcal{L}^2\right] - \varrho_n^2}}{\sqrt{r}\,\varrho_n} \leq c.$$

As an example consider $\varrho_n = \mathbb{P}(\overline{X}_n \geq a)$ for $a > \mathbb{E}X$, where we assume that (X_i) is an i.i.d. sequence satisfying Assumptions I.1 and I.2. Recall from Section 1.1.1 that $\overline{X}_n$ satisfies a LDP with rate function $I_X(a) = \theta^* a - \Lambda_X(\theta^*)$. This suggests that the exponentially twisted measure we used in (1.3) yields an asymptotically efficient importance-sampling distribution. For a twisting parameter $\vartheta \in \mathbb{R}$, we define the exponentially twisted distribution as

$$\mathbb{Q}_\vartheta\left(\overline{X}_n \in \mathrm{d}x\right) := \mathrm{e}^{n\vartheta x - n\Lambda_X(\vartheta)}\,\mathbb{P}\left(\overline{X}_n \in \mathrm{d}x\right). \tag{1.12}$$

It is then easy to check that the resulting importance sampling estimator is asymptotically

efficient when $\vartheta = \theta^*$:

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_{\mathbb{Q}_{\theta^*}} \left[\mathbb{1}\left\{ \overline{X}_n \geq a \right\} \mathcal{L}^2 \right] \\
&\quad = \limsup_{n\to\infty} \frac{1}{n} \log \int_a^\infty \mathrm{e}^{2n\Lambda_X(\theta^*) - 2n\theta^* x} \, \mathbb{Q}_{\theta^*} \left(\overline{X}_n \in \mathrm{d}x \right) \\
&\quad \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{e}^{n\Lambda_X(\theta^*) - n\theta^* a} \, \mathbb{P}\left(\overline{X}_n \geq a \right) \\
&\quad = -2I_X(a).
\end{aligned}$$

To see that the inequality holds, recall that for $a > \mathbb{E}X$ we have that $\theta^* > 0$ by [38, Lemma 2.2.5], and therefore the exponential term is larger if we replace x by a.

In Chapters 2 and 3 we propose efficient importance sampling procedures for estimating probabilities of certain rare events that are of interest in queueing systems, which are introduced in the next section.

1.1.3 Simple Markovian queues

The approximation and estimation techniques described in Sections 1.1.1 and 1.1.2 are often applied in the area of queueing theory. Consider a stochastic system to which customers or jobs arrive according to a particular stochastic process (usually in continuous time), which then have to be served before leaving the system again. This kind of problem is referred to as *queueing problem* because, under the assumption that the number of servers is finite, one or several queues may form whenever customers cannot begin their service immediately because all servers are occupied. Questions of interest to the system designer are for example the average length of the queue, or the probability that more than a certain number of customers are in the system at a particular time instance (i.e., the probability that the *size* of the system exceeds a given level). These questions can be difficult to answer, depending on the assumptions one is willing to impose on the arrival processes and the service time distribution. Traditionally, it is assumed that the arrival process is Poisson with constant arrival rate, service times are i.i.d. exponentially distributed [90, 131]. Using Kendall's notation, we denote this type of queueing system as $M/M/m$, where M stands for 'Markovian' and m is the number of servers.

The long-run steady-state behaviour of such Markovian queueing systems is generally well-understood, and even a transient analysis can be done quite explicitly [96]. However, the assumption that interarrival and service times are exponentially distributed is rather restrictive, and it turns out that it is often not very realistic. For example, it has been found that call arrivals in call centres are typically not stochastically independent, and that they are often *overdispersed*, meaning that the variance of the number of arrivals in an interval of given length is significantly larger than the corresponding mean value [46, 71, 75, 162]. This has led to the idea to use Cox processes [32] instead to model arrivals, i.e., Poisson processes in which the arrival rate follows some (non-negative) stochastic process. The perhaps most well-known process of this type is the Markov-modulated Poisson process that we consider in Chapter 2. A less standard type of Cox process is studied in Chapter 3. For service times it has been established that they do not necessarily follow an exponential distribution but may more closely resemble lognormally distributed data; see [46]. We therefore allow for more general service time distributions in the subsequent chapters.

A cornerstone model in the theory of queues is the infinite-server queue, which can

yield reasonable approximations for communication networks, where typically the number of servers is large. For the infinite-server queueing model the analysis is comparatively tractable because customers do not interact in that they do not vie for access to the available servers, and no queue is formed [90]. Nevertheless, exact analysis quickly becomes difficult when the assumptions of exponential interarrival and service time distributions are relaxed, and much of the available literature is hence concerned with approximations; see e.g. [21, 36, 53] and our paper [63], which is presented in Chapter 3. The analysis complicates even more if the number of servers is assumed to be finite. Approximations for particular generalisations of the $M/M/m$ model have been considered, for example, in [161] for the multi-server case, and in [54] for the easier single-server case.

Alternatively (or if no approximations are available), one may opt to estimate related quantities of interest using computer-simulated data. The simulation of the infinite server queue is particularly easy; for example, the number of customers in service is simply given by the difference between arrivals and departures. In case one is interested in the probability of rare events, however, naive Monte Carlo simulation, although conceptually simple, comes with the drawbacks we explained in Section 1.1.2. Importance sampling methods have, for example, been proposed in [127, 145] for infinite-server queues, and in [133] for finite-server queues. In all of these papers the distributions of the interarrival and service times are assumed to be of renewal type and servers are homogeneous. We essentially relax these assumptions in our paper [82], which is presented in the next chapter.

1.2 Organisation and contributions

In the remainder of Part I of this thesis we study the following problems related to the static control of stochastic systems. In Chapter 2 we consider a multi-server queue with Markov-modulated Poisson input and server-dependent phase-type distributed service times. We develop a rare-event simulation technique to estimate the probability that the number of customers in this system reaches a high value. Relying on explicit bounds on the probability under consideration as well as the associated likelihood ratio, we succeed in proving that the proposed estimator is of bounded relative error. Simulation experiments illustrate the significant speed-up that can be achieved by the proposed algorithm.

In Chapter 3 we study another overdispersed arrival process (meaning that the variance of the number of arrivals in an interval of given length is significantly larger than the corresponding mean value). Here, arrivals are assumed to be Poissonian but the arrival rate is resampled every Δ time units. To derive distributional properties of this process, we focus on the evaluation, for n large and $\alpha > 0$, of

$$P_n(a) := \mathbb{P}\left(\text{Pois}\left(n\overline{X}_{n^\alpha}\right) \geqslant na\right), \quad \text{with } \overline{X}_m := \frac{1}{m}\sum_{i=1}^{m} X_i$$

for i.i.d. random variables $X_1, \ldots, X_m$. Relying on elementary techniques, we derive for $\alpha < \frac{1}{3}$ and $\alpha > 3$ the exact asymptotics of $P_n(a)$; for $\alpha \in [\frac{1}{3}, \frac{1}{2})$ and $\alpha \in [2, 3)$ we find a partial solution in terms of an asymptotic lower bound. For the special case that the X_i are gamma distributed we establish the exact asymptotics across all $\alpha > 0$. In addition, we set up an asymptotically efficient importance sampling procedure that produces reliable estimates at low computational cost. We then focus on the infinite-server queue with this overdispersed arrival process. Using a scaling similar to the one featuring in the definition

of $P_n(a)$, we focus on the asymptotics and simulation of the probability $Q_n(a)$ that the number of clients present exceeds na. We then propose a number of staffing algorithms based on the proposed approximation methods, and compare them numerically. Our experiments show the striking feature that the staffing level may *decrease* in the service times' variability.

In Chapter 4, relying only on the classical Bahadur-Rao approximation for large deviations of univariate sample means, we derive strong large deviation approximations for probabilities involving two sets of sample means. The main result concerns the exact asymptotics (as $n \to \infty$) of

$$\mathbb{P}\left(\max_{i \in \{1,\ldots,d_X\}} \overline{X}_{i,n} \leq \min_{i \in \{1,\ldots,d_Y\}} \overline{Y}_{i,n} \right),$$

with the $\overline{X}_{i,n}$s ($\overline{Y}_{i,n}$s, respectively) denoting d_X (d_Y) independent copies of sample means associated with the random variable X (Y). Assuming $\mathbb{E}\,X > \mathbb{E}\,Y$, this is a rare event probability that vanishes essentially exponentially, but with an additional polynomial term. We point out how the probability of interest can be estimated using importance sampling in an asymptotically efficient way. To demonstrate the usefulness of the result, we show how it can be used to compare the order statistics of the sample means of the two populations. This has applications, for instance, in queueing or packing problems.

CHAPTER 2

Efficient simulation of tail probabilities in a queueing model with heterogeneous servers

The multi-server queue is a well-studied object in operations research with widespread applications, for example in the modelling of call centres [71] and healthcare systems [57]. In many situations, the system needs to be designed in such a way that the service level offered is sufficiently high. This is usually translated into the requirement that the probability of the backlog exceeding some critical value should be below a given threshold value.

For the case of homogeneous servers (meaning that the service times at the various servers have a common distribution), a strand of research focuses on evaluating the probability that the number of customers waiting exceeds some high level K. A key result in this area concerns the situation in which the service-time distribution has a finite moment generating function around zero (implying that all moments exist): it was proven by Sadowsky [133] that for such GI/GI/m queues the tail of the probability of interest decays effectively exponentially, cf. also the earlier paper by Takahashi [146] for the setting with phase-type interarrival times and phase-type service times. In addition, [133] provides a fast (importance-sampling based) simulation procedure to estimate this probability with provable optimality properties. More specifically, it was shown that the estimator is logarithmically efficient; this entails that the number of runs needed to obtain an estimate with a given precision grows sub-exponentially in the level K.

In the above literature it was assumed that the servers are homogeneous; this implies, for example, that each service entity serves customers at the same average speed. In many practical situations, however, this assumption is overly restrictive as has been recognised in the work of e.g. [8, 74] (as well as in other references which deal with the problem of routing in systems with heterogeneous servers). Not much is known, however, about the tail distribution of such heterogeneous multi-server systems.

Another aspect that is hardly covered in the importance-sampling literature concerns the incorporation of overdispersion, while for call centre arrival data the variance of the number of arrivals in an interval of given length often turns out to be significantly larger than the corresponding mean value; for references see Section 1.1.3. This phenomenon is better captured by a Cox process, the perhaps most well-known example of which is the Markov-modulated Poisson (MMP) process. For an MMP process the arrival rate is

λ_i when an independently evolving continuous-time, finite-state Markov chain (typically referred to as the *background process*) is in state i. For results on queues with Markov-modulated input we refer to e.g. [9, Chapter XI].

Motivated by the above considerations, the object of study in this chapter, which is based on Kuhn and Mandjes [82], is the multi-server queue with *MMP input* and *server-dependent* phase-type service times. The main contribution is that we devise efficient simulation techniques for the purpose of estimating the tail distribution of the stationary number of customers in the system. In more detail, our work extends the existing literature on importance sampling for multi-server queues are as follows.

- In our set-up we allow the servers to be heterogeneous, whereas [133, 146] assume server-homogeneity. We remark that [133] considers light-tailed service-time distributions, whereas we focus on the subclass of phase-type distributions. It is known, however, that general non-negative distributions can be approximated arbitrarily closely by phase-type distributions so that in practical terms hardly any generality is lost; see e.g. [20, 154] and [9, Theorem III.4.2]. (The focus is still on light-tailed distributions as for heavy-tailed distributions the number of phases needed to adequately model the tails may be excessively large.)

- In addition, we allow for the arrival process to be overdispersed. We focus on the case of MMP arrivals, but, as we will point out, other types of arrival processes can be treated with similar techniques (such as the renewal processes that were studied in [133]).

- We show that our proposed importance-sampling estimator is *strongly efficient*, (or, equivalently, has *bounded relative error*). This means that the number of runs needed to obtain an estimate with given precision remains bounded (i.e., is smaller than some constant that does not depend on K). Recall that in [133] just logarithmic efficiency was proven (implying that the number of runs needed grows sub-exponentially).

In summary, our model can be viewed as a generalization of that of [133] in that we allow for heterogeneous servers as well as overdispersed arrival processes; the (minor) sacrifice that we make is that we assume the service times to be of phase-type, rather than just light-tailed. In more detail, the results obtained are the following.

(i) In the first place, for the queue under study we propose efficient simulation algorithms for the estimation of the probability that the backlog (that is, the number of customers or jobs waiting in the queue) during a busy cycle (during which the system is non-empty) exceeds a given level K. The procedure can be modified for the estimation of related quantities such as the fraction of customers or jobs entering the system while the backlog is larger than K, or the fraction of customers lost in the corresponding model with a waiting room of finite size K.

The algorithms are based on importance sampling, that is, the model is simulated under an alternative measure, under which the event under consideration is *not* rare. We identify an efficient change of measure by solving a particular eigensystem. As it turns out, this change of measure provides us with upper and lower bounds on the probability of interest which are both exponential in the threshold K (and which match up to a multiplicative constant). This property implies that our importance-sampling estimator is strongly efficient.

(ii) As the eigensystem to be solved can become prohibitively large when the dimension of the background process and/or the dimensions of the phase-type distributed service times grow large, we show how the eigensystem can be decoupled in order to identify the change of measure in a computationally more efficient way.

(iii) Finally, we point out how the change of measure can be found for various variants of the arrival and service processes.

The organization of this chapter is as follows. In Section 2.1 we introduce the model and formulate our objectives in greater detail. In Section 2.2 we propose the change of measure that is to be used in the importance-sampling based procedure. We then establish bounds on the probability of interest, which we use to prove that the importance-sampling algorithm has bounded relative error. In Section 2.3 we show that the same change of measure can be obtained when considering the arrival and service processes separately, thus drastically reducing the computational effort needed to compute the change of measure. Section 2.4 contains illustrative numerical experiments that give an impression of the typically achievable speed-up. We conclude in Section 2.5 by discussing how the importance sampling algorithm can be adapted to estimate related quantities, and how it can be useful in static control.

2.1 Framework

2.1.1 Model

In this chapter we primarily focus on the following MMP/Ph/m queue with heterogeneous servers. We now introduce the arrival process and service processes used.

Arrival process. Consider the following MMP arrival process. The transition rate matrix of the (finite-state) background process $(I_t)_{t\geq 0}$ is $Q = (q_{ij})_{i,j=1}^{d}$; define $q_i := -q_{ii} = \sum_{j\neq i} q_{ij}$. When the background process (assumed to be irreducible) is in state i arrivals occur according to a Poisson process with rate $\lambda_i \geq 0$. Let the mean arrival rate be $\lambda := \boldsymbol{\pi}'\boldsymbol{\lambda}$, with $\boldsymbol{\pi}$ the invariant probability measure of the background process and $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_d)'$.

Service processes. There are m heterogeneous servers. Service times at server $\ell \in \{1, \ldots, m\}$ are i.i.d. samples distributed as the non-negative random variable $B^{(\ell)}$. We let $B^{(\ell)}$ be of phase-type [9, Chapter III] with initial distribution $\boldsymbol{\alpha}^{(\ell)}$ concentrated on the transient states, and transition rate matrix

$$T^{(\ell)} = \left(t_{ij}^{(\ell)}\right)_{i,j=1}^{D^{(\ell)}+1} = \begin{pmatrix} S^{(\ell)} & \boldsymbol{s}^{(\ell)} \\ \mathbf{0} & 0 \end{pmatrix}, \tag{2.1}$$

for some $D^{(\ell)} \in \mathbb{N}$. We impose the usual requirement that $t_{ij}^{(\ell)} \geq 0$ for $i \in \{1, \ldots D^{(\ell)}\}$ and $j \in \{1, \ldots D^{(\ell)}+1\}$ with $i \neq j$, and $t_{D^{(\ell)}+1,j} = 0$ for $j \in \{1, \ldots, D^{(\ell)}+1\}$; in addition we define

$$t_i^{(\ell)} := -t_{ii}^{(\ell)} = \sum_{j=1, j\neq i}^{D^{(\ell)}+1} t_{ij}^{(\ell)},$$

so that the row sums are zero. In words, the above means that the service time at server

ℓ stays in phase i for an exponentially distributed amount of time with mean $(t_i^{(\ell)})^{-1}$, and then jumps to state $j \neq i$ with probability $t_{ij}^{(\ell)}/t_i^{(\ell)}$.

Thus, the evolution of the system is recorded by the following trivariate process:

(i) The state of the background process $(I_t)_{t\geq 0}$ taking values in $\mathscr{I} := \{1, \ldots, d\}$.

(ii) The state vector $(\boldsymbol{J}_t)_{t\geq 0}$ of the phase-type distributions of the customers in service; with † indicating that the corresponding server is idle, this takes values in

$$\mathscr{D} := \{1, \ldots, D^{(1)}, \dagger\} \times \cdots \times \{1, \ldots, D^{(m)}, \dagger\}.$$

We will sometimes use the suggestive notation $t_{i,\dagger}^{(\ell)} := t_{i,D^{(\ell)}+1}^{(\ell)}$.

(iii) The number of customers in the system, $(N_t)_{t\geq 0}$, taking values in $\mathbb{N} = \{0, 1, \ldots\}$. We stress that this number includes the customers in service: when $N_t = m + n$, then all servers are occupied, and n customers are waiting.

Observe that $(I_t, \boldsymbol{J}_t, N_t)_{t\geq 0}$ is a continuous-time Markov chain on the state space $\mathscr{I} \times \mathscr{D} \times \mathbb{N}$. Throughout the queue is assumed to be stable, i.e., we impose the condition

$$\lambda < \sum_{\ell=1}^{m} \frac{1}{\mathbb{E}\, B^{(\ell)}},$$

where $\mathbb{E}\, B^{(\ell)}$ can be evaluated in terms of $\boldsymbol{\alpha}^{(\ell)}$ and $T^{(\ell)}$ as in [9, Prop. III.4.1]. This stability criterion can be interpreted as: the average number of clients arriving to the multi-server queue per unit of time should be strictly majorised by the average number of clients that can be served (by the m queues together) per unit of time.

Since servers are heterogeneous, we shall assume that the free server with the lowest index serves the next customer arriving or waiting in the queue. In practice, one may wish to prioritise faster servers; a service policy of this type can be achieved by labelling servers in increasing order according to their average service times.

2.1.2 Objective and methodology

Our first objective is to estimate the probability that the backlog, $\max\{N_t - m, 0\}$, exceeds a given level $K \in \mathbb{N}$ during a *busy cycle*, which in this chapter we define – somewhat unconventionally – as an uninterrupted period during which the system has been *non-empty*. Such a period is initiated by the arrival of a customer to an empty system, and ends by the departure of the last customer (leaving all servers idle). Notice that in our model busy cycles are *not* i.i.d., as the state of the background process at the beginning of the busy cycle has impact on its evolution (as opposed to for instance the situation with renewal-type arrivals that was studied in [133]). We denote by $\mathscr{F}_i$ the event that a busy cycle started when the background process was in state i. We focus on estimating the probability $\varrho_i(K)$ that in a busy cycle the number of customers in the queue exceeds the value K conditional on $\mathscr{F}_i$.

In practice, the probability $\varrho_i(K)$ is usually required to be small, which makes estimating it by crude Monte Carlo simulation inefficient [10, Chapter VI]. We are therefore interested in an estimation procedure that relies on importance sampling [10, Section V.1] in order to limit the required simulation effort. Importance sampling is based on

imposing a 'change of measure' with respect to the original measure $\mathbb{P}$. More concretely, the simulation is performed under a different probability measure $\mathbb{Q}$, and the simulation output is corrected by the 'likelihood ratios' $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ evaluated at the observed outcome in order to retain an unbiased estimation procedure. The challenge is to find an alternative measure $\mathbb{Q}$ that effectively reduces the variance of the estimator. This typically means that $\mathbb{Q}$ should be such that the event under consideration becomes more likely to occur, but in addition it is required that the likelihood ratio $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}$ on the event of interest should have a low variance. This is made explicit in [10, Section VI.1], where various efficiency measures for importance-sampling estimators are discussed.

Compared to the efficient simulation of tail probabilities in an M/M/m queue, a number of complications arises in our set-up. In the first place, as mentioned above, busy cycles are not independent. Furthermore, since servers are heterogeneous, one needs to keep track not only of the number of busy servers but also of their indices and current phases. In addition, the arrival rate is not fixed but depends on the state of the background processes.

Observe, however, that during periods in which N_t is larger than m, the service rate of the system does not change with N_t; one could say it is 'level-homogeneous'. This motivates that we split each busy cycle into subintervals in which $N_t \in \{m+1, m+2, \ldots\}$ (i.e., the queue is not empty; we refer to these intervals as *fully* busy periods), and periods in which $N_t \in \{1, \ldots, m\}$ (i.e., the queue is empty; we call these intervals *partially* busy periods). Thus, during a busy cycle the system alternates between partially and fully busy periods until the system becomes idle again.

Based on the above, we can decompose $\varrho_i(K)$ as follows. With $\varrho_i(K, n)$ the probability that the number of customers attains $m+K$ for the first time in the n-th fully busy period (conditional on $\mathscr{F}_i$), we can write

$$\varrho_i(K) = \sum_{n=1}^{\infty} \varrho_i(K, n).$$

With this decomposition in mind, we first consider the following approach to estimate $\varrho_i(K)$, which will be detailed in Section 2.2. During the fully busy periods, in which the system is level-homogeneous, we use an alternative probability measure $\mathbb{Q}$ under which the queueing system is unstable (so that the rare event under study will occur frequently). During partially busy periods, in which the system is not level-homogeneous, we use the original measure $\mathbb{P}$. To establish particular efficiency properties, the number of fully busy periods (per busy cycle) in which $\mathbb{Q}$ is applied should be bounded by an arbitrary constant $C \in \mathbb{N}$; we return to this subtlety in Section 2.2. Based on the insights gained in Section 2.2 we then show in Section 2.3 how to obtain a change of measure that can be applied throughout the entire busy cycle.

2.2 Importance sampling procedure and its efficiency properties

In this section we describe an importance sampling routine for estimating the quantity $\varrho_i(K)$. As this probability relates to the event that a given level is reached before the number of customers returns to 0, it suffices to track the evolution of the *embedded* discrete-time Markov process, i.e., of the continuous-time Markov chain $(I_t, \boldsymbol{J}_t, N_t)_{t\geq 0}$

observed at its transition epochs. With a mild abuse of notation we refer to the embedded process as

$$(I_n, \boldsymbol{J}_n, N_n)_{n\in\mathbb{N}} \in \mathscr{I} \times \mathscr{D} \times \mathbb{N}, \tag{2.2}$$

where n enumerates the epochs at which any of the three processes makes a transition. Note that at each transition epoch n of this embedded process typically only one of the state vector components changes, the exception being the occurrence of a departure (in which case $\boldsymbol{J}_n$ may change, and N_n decreases by one).

Define σ_k to be the first time that $(N_n)_{n\in\mathbb{N}}$ reaches level k. Assuming that a busy cycle starts with the arrival of a first customer, the backlog exceeds K within that cycle if and only if $\sigma^K := \sigma_{m+K+1} < \sigma_0$. The objective of this section is to find an efficient algorithm for estimating the probability $\varrho_i(K)$ that in a busy cycle the number of customers in the queue exceeds the value K given that the background process is in $i \in \mathscr{I}$ at the start of the busy cycle; that is,

$$\varrho_i(K) = \mathbb{P}(\sigma^K < \sigma_0 \,|\, \mathscr{F}_i)$$

where $\mathscr{F}_i$ corresponds to the event that $I_0 = i$, $J_0^{(1)}$ is sampled according to $\boldsymbol{\alpha}^{(1)}$, $J_0^{(2)} = \cdots = J_0^{(m)} = \dagger$, and $N_0 = 1$ (recall from Section 2.1.1 that the first customer is attended to by the server with the lowest index).

The remainder of this section is organised as follows.

- First, in Section 2.2.1, we focus on a fully busy period; we conveniently shift time, such that the start of the busy period corresponds to time 0. We fix a state $(i, \boldsymbol{j}) \in \mathscr{I} \times \mathscr{D}$, and consider the probability

$$\begin{aligned} q_{i,\boldsymbol{j}}(K) &:= \mathbb{P}_{i,\boldsymbol{j}}(\sigma^K < \sigma_m \,\big|\, N_0 = m+1) \\ &:= \mathbb{P}(\sigma^K < \sigma_m \,\big|\, I_0 = i, \boldsymbol{J}_0 = \boldsymbol{j}, N_0 = m+1). \end{aligned} \tag{2.3}$$

  Observe that $q_{i,\boldsymbol{j}}(K)$ can be interpreted as the probability that the backlog exceeds K within a fully busy period given that such a period has started when the background process and the service times were in state $(i, \boldsymbol{j})$. Relying on the fact that during the fully busy period the system is level-homogeneous, we define a change of measure for estimating $q_{i,\boldsymbol{j}}(K)$. We then propose an importance sampling algorithm for estimating $\varrho_i(K)$ which applies this change of measure during the first $C \in \mathbb{N}$ fully busy periods. (In Section 2.3 we will see how the rates can be twisted in general, without the restriction of changing the measure only during the fully busy periods.)

- In Section 2.2.3 we investigate efficiency properties of the proposed estimators. In the first place, we show that the importance sampling procedure for estimating $q_{i,\boldsymbol{j}}(K)$ has bounded relative error. In addition, the probabilities $q_{i,\boldsymbol{j}}(K)$ and $\varrho_i(K)$ are proven to be 'sufficiently similar' that the procedure for estimating $\varrho_i(K)$ has bounded relative error as well.

- Section 2.2.4 presents a numerical example, in which the new measure $\mathbb{Q}$ is computed. It gives rise to a decomposition property, formalised in Section 2.2.5, which drastically reduces the computational efforts required to evaluate the measure $\mathbb{Q}$.

2.2.1 Change of measure

In this subsection we focus on the estimation of $q_{i,\boldsymbol{j}}(K)$ as defined in (2.3), with fixed i and $\boldsymbol{j}$. Observe that in order to decide whether or not $\sigma^K < \sigma_m$, we consider a time interval during which the value of N_n has not dropped below $m+1$. In other words, the transition matrix of $(I_n, \boldsymbol{J}_n, N_n)$ does not depend on N_n during that interval. It is essentially this property that enables the following construction of the alternative measure, which mimics the construction in [105] for the easier case of the Markov fluid queue.

Consider a discrete-time Markov chain $(\overline{I}_n, \overline{\boldsymbol{J}}_n, \overline{N}_n) \in \mathscr{I} \times \mathscr{D} \times \mathbb{Z}$ that is characterised by the following transition probabilities. Define

$$\varphi_{i,\boldsymbol{j}} := \lambda_i + \sum_{\ell=1}^{m} t_{j_\ell}^{(\ell)} + q_i.$$

Let $\boldsymbol{e}_\ell$ be a vector of dimension m with a one on position ℓ and zeros otherwise. Then the probability of moving from $(i, \boldsymbol{j}, n)$ to $(i, \boldsymbol{j}, n+1)$ is $\lambda_i/\varphi_{i,\boldsymbol{j}}$ (this corresponds to an arrival); the probability of moving from $(i, \boldsymbol{j}, n)$ to $(i', \boldsymbol{j}, n)$ is $q_{ii'}/\varphi_{i,\boldsymbol{j}}$ (this corresponds to a transition of the background process); the probability of moving from $(i, \boldsymbol{j}, n)$ to $(i, \boldsymbol{j} + (k - j_\ell)\boldsymbol{e}_\ell, n)$ is $t_{j_\ell,k}^{(\ell)}/\varphi_{i,\boldsymbol{j}}$ (this corresponds to a transition in the phase of one of the service times, without a departure); and the probability of moving from $(i, \boldsymbol{j}, n)$ to $(i, \boldsymbol{j}+(k-j_\ell)\boldsymbol{e}_\ell, n-1)$ is $\overline{t}_{j_\ell,k}^{(\ell)}/\varphi_{i,\boldsymbol{j}}$ with $\overline{t}_{j_\ell,k}^{(\ell)} := t_{j_\ell,\dagger}^{(\ell)} \alpha_k^{(\ell)}$ (this corresponds to a transition in the phase of one of the service times, but now with a departure). The crucial observation is that during the fully busy period, $(I_n, \boldsymbol{J}_n, N_n)$ behaves as $(\overline{I}_n, \overline{\boldsymbol{J}}_n, \overline{N}_n)$.

We now point out how the alternative measure $\mathbb{Q}$ is constructed. Let $\xi_{i,\boldsymbol{j}}$ denote the net increase of the number of customers $\overline{N}_n$ from an epoch that $(\overline{I}_n, \overline{\boldsymbol{J}}_n)$ is in $(i, \boldsymbol{j})$ until $(\overline{I}_n, \overline{\boldsymbol{J}}_n)$ arrives at a given reference state $(i^*, \boldsymbol{j}^*)$ (we show below that the choice of the reference state does not affect the resulting new measure $\mathbb{Q}$). Let $x_{i,\boldsymbol{j}} \equiv x_{i,\boldsymbol{j}}(\theta) := \mathbb{E}\mathrm{e}^{\theta\xi_{i,\boldsymbol{j}}}$ denote the moment generating function (MGF) of $\xi_{i,\boldsymbol{j}}$. Relying on the usual 'Markovian reasoning', the MGFs satisfy

$$\begin{aligned} x_{i,\boldsymbol{j}} = \frac{\lambda_i}{\varphi_{i,\boldsymbol{j}}} x_{i,\boldsymbol{j}} \mathrm{e}^{\theta} + \sum_{i'=1, i' \neq i}^{d} \frac{q_{ii'}}{\varphi_{i,\boldsymbol{j}}} x_{i',\boldsymbol{j}} + \sum_{\ell=1}^{m} \sum_{k=1, k \neq j_\ell}^{D^{(\ell)}} \frac{t_{j_\ell,k}^{(\ell)}}{\varphi_{i,\boldsymbol{j}}} x_{i,\boldsymbol{j}+(k-j_\ell)\boldsymbol{e}_\ell} \\ + \sum_{\ell=1}^{m} \sum_{k=1}^{D^{(\ell)}} \frac{\overline{t}_{j_\ell,k}^{(\ell)}}{\varphi_{i,\boldsymbol{j}}} x_{i,\boldsymbol{j}+(k-j_\ell)\boldsymbol{e}_\ell} \mathrm{e}^{-\theta}; \end{aligned} \tag{2.4}$$

the first term on the right hand side corresponds to an arrival (hence the factor e^{θ}), the second to a jump of the background process, the third to a phase-transition of one of the service times (but not a departure), and the fourth to a departure and simultaneously the start of a new service (hence the factor $\mathrm{e}^{-\theta}$). The system of equations (2.4) can be regarded as an eigensystem of the form $A\boldsymbol{x} = \boldsymbol{x}$ with eigenvalue 1; the matrix $A \equiv A(\theta)$ is irreducible and non-negative. By Perron-Frobenius all positive eigenvectors of such a matrix lie in the eigenspace corresponding to the largest eigenvalue. The vector $\boldsymbol{x}$ of MGFs is one such eigenvector. In the sequel we denote by θ^* the value of θ such that the largest eigenvalue of A is equal to 1.

We now explain why θ^* can be interpreted as the *Cramér root* [10, Section VI.2a] related to the random variable $\xi \equiv \xi_{i^*,\boldsymbol{j}^*}$. Recall that ξ is the net increase of the number

of customers between two subsequent visits of $(\overline{I}_n, \overline{\boldsymbol{J}}_n)$ to the reference state $(i^*, \boldsymbol{j}^*)$. The alternative measure obtained by an exponential twist of the original measure $\mathbb{P}$ should be such that the MGF of ξ evaluated in θ under $\mathbb{Q}$ matches the MGF of ξ evaluated in $\theta+\theta^*$ under $\mathbb{P}$: in self-evident notation,

$$\mathbb{E}^{\mathbb{Q}}\,\mathrm{e}^{\theta\xi} = \mathbb{E}\,\mathrm{e}^{(\theta+\theta^*)\xi},$$

with θ^* such that $\mathbb{E}\,\mathrm{e}^{\theta^*\xi} = 1$; see again [10, Section VI.2a]. Thus, the vector of MGFs $\boldsymbol{x}$ and θ^* are found from (2.4) by equating $x_{i^*,\boldsymbol{j}^*}$ to 1. It is readily verified, however, that the choice of the reference state has no impact, as Eq. (2.4) is linear in $\boldsymbol{x}$.

Inspired by the above eigensystem, we now propose the following new measure $\mathbb{Q}$ corresponding to an exponential twist of the distribution of ξ, to be used to estimate $q_{i,\boldsymbol{j}}(K)$: when $(\overline{I}_n, \overline{\boldsymbol{J}}_n) = (i, \boldsymbol{j})$,

$$\begin{aligned} \lambda_i^\circ = \lambda_i\,\mathrm{e}^{\theta^*}, \quad q_{ii'}^\circ = q_{ii'}\frac{x_{i',\boldsymbol{j}}}{x_{i,\boldsymbol{j}}}, \quad \left(t_{j_\ell,k}^{(\ell)}\right)^\circ = t_{j_\ell,k}^{(\ell)}\frac{x_{i,\boldsymbol{j}+(k-j_\ell)\boldsymbol{e}_\ell}}{x_{i,\boldsymbol{j}}}, \\ \left(\bar{t}_{j_\ell,k}^{(\ell)}\right)^\circ = \bar{t}_{j_\ell,k}^{(\ell)}\frac{x_{i,\boldsymbol{j}+(k-j_\ell)\boldsymbol{e}_\ell}}{x_{i,\boldsymbol{j}}}\,\mathrm{e}^{-\theta^*}. \end{aligned} \tag{2.5}$$

In the remainder of this subsection we evaluate the likelihood ratio that results from this change of measure when estimating $q_{i,\boldsymbol{j}}(K)$; as it turns out, this has a surprisingly simple form. To this end, we consider an arbitrary path of the process $(\overline{I}_n, \overline{\boldsymbol{J}}_n)$ starting when the fully busy period commences (that is, we have $\overline{I}_0 = i$, $\overline{\boldsymbol{J}}_0 = \boldsymbol{j}$, and $\overline{N}_0 = m+1$), and ending at time $\tau = \min\{\sigma^K, \sigma_m\}$, visiting states $(i_n, \boldsymbol{j}_n)$, where n denotes the n-th transition epoch of the process (2.2). Let $\mathscr{N}_+$ denote the $n \in \mathscr{S} := \{1, \ldots, \tau\}$ corresponding to arrivals, $\mathscr{N}_{\rightsquigarrow}$ the $n \in \mathscr{S}$ corresponding to transitions of the background process, $\mathscr{N}_{\circlearrowleft}^{(\ell)}$ the $n \in \mathscr{S}$ corresponding to a phase-transition of the service time at server ℓ (not being a service completion), and $\mathscr{N}_-^{(\ell)}$ corresponding to a service completion at server ℓ. The likelihood (under $\mathbb{P}$) of such a path is thus given by, with $i = i_0$ and $\boldsymbol{j} = \boldsymbol{j}_0$,

$$\prod_{n\in\mathscr{N}_+}\frac{\lambda_{i_n}}{\varphi_{i_n,\boldsymbol{j}_n}}\prod_{n\in\mathscr{N}_{\rightsquigarrow}}\frac{q_{i_n,i_{n+1}}}{\varphi_{i_n,\boldsymbol{j}_n}}\prod_{\ell=1}^{m}\prod_{n\in\mathscr{N}_{\circlearrowleft}^{(\ell)}}\frac{t_{j_n,j_{n+1}}^{(\ell)}}{\varphi_{i_n,\boldsymbol{j}_n}}\prod_{\ell=1}^{m}\prod_{n\in\mathscr{N}_-^{(\ell)}}\frac{\bar{t}_{j_n,j_{n+1}}^{(\ell)}}{\varphi_{i_n,\boldsymbol{j}_n}}. \tag{2.6}$$

The likelihood of the same path under the new measure $\mathbb{Q}$ has the same form, except that all probabilities in (2.6) are replaced by their counterparts under $\mathbb{Q}$, where, due to (2.4) and the definition of the new rates,

$$\varphi_{i,\boldsymbol{j}}^\circ := \lambda_i^\circ + \sum_{\ell=1}^{m}\left(t_{j_\ell}^{(\ell)}\right)^\circ + q_i^\circ = \lambda_i^\circ + \sum_{\ell=1}^{m}\sum_{k=1,k\neq j_\ell}^{D^{(\ell)}+1}\left(t_{j_\ell,k}^{(\ell)}\right)^\circ + \sum_{i'=1,i'\neq i}^{d} q_{ii'}^\circ = \varphi_{i,\boldsymbol{j}}.$$

It follows that the likelihood ratio over the path takes the form

$$\begin{aligned} L = \prod_{n\in\mathscr{N}_+}\frac{\lambda_{i_n}/\varphi_{i_n,\boldsymbol{j}_n}}{\lambda_{i_n}^\circ/\varphi_{i_n,\boldsymbol{j}_n}^\circ}\prod_{n\in\mathscr{N}_{\rightsquigarrow}}\frac{q_{i_n,i_{n+1}}/\varphi_{i_n,\boldsymbol{j}_n}}{q_{i_n,i_{n+1}}^\circ/\varphi_{i_n,\boldsymbol{j}_n}^\circ}\times \\ \times\prod_{\ell=1}^{m}\prod_{n\in\mathscr{N}_{\circlearrowleft}^{(\ell)}}\frac{t_{j_n,j_{n+1}}^{(\ell)}/\varphi_{i_n,\boldsymbol{j}_n}}{\left(t_{j_n,j_{n+1}}^{(\ell)}\right)^\circ/\varphi_{i_n,\boldsymbol{j}_n}^\circ}\prod_{\ell=1}^{m}\prod_{n\in\mathscr{N}_-^{(\ell)}}\frac{\bar{t}_{j_n,j_{n+1}}^{(\ell)}/\varphi_{i_n,\boldsymbol{j}_n}}{\left(\bar{t}_{j_n,j_{n+1}}^{(\ell)}\right)^\circ/\varphi_{i_n,\boldsymbol{j}_n}^\circ}. \end{aligned}$$

Because $\varphi_{i,\boldsymbol{j}} = \varphi^{\circ}_{i,\boldsymbol{j}}$, by (2.5) this reduces to the 'telescopic product'

$$L = \prod_{n\in\mathscr{N}_+} \mathrm{e}^{-\theta^*} \prod_{n\in\mathscr{N}_{\rightsquigarrow}} \frac{x_{i_n,\boldsymbol{j}_n}}{x_{i_{n+1},\boldsymbol{j}_{n+1}}} \prod_{\ell=1}^{m} \prod_{n\in\mathscr{N}_{\circlearrowleft}^{(\ell)}} \frac{x_{i_n,\boldsymbol{j}_n}}{x_{i_{n+1},\boldsymbol{j}_{n+1}}} \prod_{\ell=1}^{m} \prod_{n\in\mathscr{N}_{-}^{(\ell)}} \frac{x_{i_n,\boldsymbol{j}_n}}{x_{i_{n+1},\boldsymbol{j}_{n+1}}} \mathrm{e}^{\theta^*}$$
$$= \mathrm{e}^{-\theta^*\Sigma_+} \frac{x_{i_0,\boldsymbol{j}_0}}{x_{i_\tau,\boldsymbol{j}_\tau}} \mathrm{e}^{\theta^*\Sigma_-}, \tag{2.7}$$

where Σ_+ is the number of arrivals in $\mathscr{S}$, and Σ_- the number of departures. Observe that at the end of each fully busy period we either have $\Sigma_- - \Sigma_+ = -K$ if $\tau = \sigma^K$, or $\Sigma_- - \Sigma_+ = 1$ if $\tau = \sigma_m$. We find the following identities.

Corollary 2.1. *Let $\overline{I}_0 = i$, $\overline{\boldsymbol{J}}_0 = \boldsymbol{j}$, and $\overline{N}_0 = m+1$. For any $K \in \mathbb{N}$,*

$$L\,\mathbb{1}\{\tau = \sigma^K\} = \mathrm{e}^{-\theta^* K} \frac{x_{i,\boldsymbol{j}}}{x_{i_\tau,\boldsymbol{j}_\tau}}, \quad L\,\mathbb{1}\{\tau = \sigma_m\} = \mathrm{e}^{\theta^*} \frac{x_{i,\boldsymbol{j}}}{x_{i_\tau,\boldsymbol{j}_\tau}}.$$

Remark 2.2.1. It is reassuring to note that the proposed change of measure satisfies Juneja's 'equi-probable cycle' condition, which should hold for an asymptotically optimal change of measure [72]. Namely, if $(\overline{I}_n, \overline{\boldsymbol{J}}_n, \overline{N}_n)_{n\in\mathbb{N}}$ visits a specific state multiple times, the contribution to the likelihood ratio of the interval between two such subsequent visits is equal to one.

2.2.2 Importance sampling algorithm

In the previous subsection we found an alternative measure $\mathbb{Q}$ to be applied during the first $C \in \mathbb{N}$ fully busy periods, whereas during the partially busy periods $\mathbb{P}$ should be used. In Algorithm 1 we provide pseudo code for a single run of the resulting importance sampling procedure. The truncation at C is needed in Section 2.2.3 to prove that the procedure has bounded relative error. In practice, however, this truncation can be neglected since C can be chosen arbitrarily large without compromising the estimator's performance; see Section 2.4 for a numerical comparison of the performance of the estimation algorithm for various values of C.

2.2.3 Bounds and relative error

In this subsection we derive bounds on $\varrho_i(K)$ in terms of the probability $q_{i,\boldsymbol{j}}(K)$, which we then use to prove that the proposed importance sampling estimation procedure leads to bounded relative error.

The partially busy period before each fully busy period commences either when 1 customer is in the system (at the beginning of the busy cycle), or when m customers are in the system (when we just exited a fully busy period). Accordingly, we define, in self-evident notation,

$$\overline{p}_i := \max_i \mathbb{P}\left(\sigma_{m+1} < \sigma_0 \,|\, \mathscr{F}_i\right), \quad p_{i,\boldsymbol{j}} := \mathbb{P}_{i,\boldsymbol{j}}(\sigma_{m+1} < \sigma_0 \,|\, N_0 = m).$$

Algorithm 1 One run of the importance sampling algorithm that applies the change of measure only during the first C fully busy periods.

Set $N = 1$, $L = 1$, $c = 0$. Set i as the initial state of the background process. Generate $j_1 \sim \boldsymbol{\alpha}^{(1)}$, and set $j_\ell = \dagger$ for $\ell = 2, \ldots, m$.
while $N \in \{1, \ldots, m + K\}$ **do**
  if $N \leq m$ **then**
    Set all rates to the original rates. Let $\varphi_{i,\boldsymbol{j}} = \lambda_i + \sum_{\ell:\, j_\ell > 0} t^{(\ell)}_{j_\ell} + q_i$. Set $\boldsymbol{p}$ to be the vector with entries λ_i, $q_{ii'}$ for all $i' \neq i$, and $t^{(\ell)}_{j_\ell,k}$ for all ℓ such that $j_\ell \neq \dagger$ and $k \in \{1, \ldots, D^{(\ell)}, \dagger\}$. Generate the next event from the discrete distribution $\boldsymbol{p}/\varphi_{i,\boldsymbol{j}}$.
    if Arrival **then**
      $N \leftarrow N + 1$
      if $N > m$ **then**
        $c \leftarrow c + 1$
        if $c \leq C$ **then**
          $L \leftarrow L\, x_{i,\boldsymbol{j}}$
        end if
      end if
    else if Transition of the background process **then**
      $i \leftarrow i'$, where i' corresponds to entry $q_{ii'}$ of $\boldsymbol{p}$
    else if Phase transition at server ℓ to $k \leq D^{(\ell)}$ **then**
      $j_\ell \leftarrow k$
    else if Phase transition at server ℓ to $\dagger$ **then**
      $j_\ell \leftarrow \dagger$, and $N \leftarrow N - 1$
    end if
  else if $N > m$ **then**
    if $c \leq C$ **then**
      Set all rates as in (2.5).
    end if
    Update $\varphi_{i,\boldsymbol{j}}$, and set $\boldsymbol{p}$ as for the partially busy period but including rates $\bar{t}^{(\ell)}_{j_\ell,k}$. Generate the next event from the discrete distribution $\boldsymbol{p}/\varphi_{i,\boldsymbol{j}}$.
    if Phase transition at server ℓ with departure **then**
      $j_\ell \leftarrow k$, where k corresponds to entry $\bar{t}^{(\ell)}_{j_\ell,k}$ of $\boldsymbol{p}$, and $N \leftarrow N - 1$
      if $N = m$ and $c \leq C$ **then**
        $L \leftarrow L\, \mathrm{e}^{\theta^*}/x_{i,\boldsymbol{j}}$
      end if
    else if Other transition **then**
      Proceed as for the partially busy period.
    end if
  end if
end while
if $N > m + K$ and $c \leq C$ **then**
  $L \leftarrow L\, \mathrm{e}^{-K\theta^*}/x_{i,\boldsymbol{j}}$
end if
return $L\, \mathbb{1}\{N > m + K\}$

Observe that the number of fully busy periods in a busy cycle is bounded from above by a geometric random variable G with success probability

$$p_+ := \max\left\{\max_i \overline{p}_i,\, \max_{i,\boldsymbol{j}} p_{i,\boldsymbol{j}}\right\} = \max_{i,\boldsymbol{j}} p_{i,\boldsymbol{j}} < 1. \tag{2.8}$$

In every one of these fully busy periods, level $m+K+1$ is reached with a probability that is bounded above by

$$q_+(K) := \max_{i,\boldsymbol{j}} q_{i,\boldsymbol{j}}(K).$$

Supposing that $G = k$, in each of the k attempts the level $m+K+1$ can be reached. The union bound then yields the following upper bound: uniformly in $i \in \mathscr{I}$,

$$\varrho_i(K) \le \sum_{k=1}^{\infty} p_+^k (1-p_+) k\, q_+(K) = \frac{q_+(K)}{p_+}.$$

Now focus on establishing a lower bound based on the probability of reaching $m+K+1$ in the first fully busy period. To this end, we define

$$p_- := \min_i \mathbb{P}(\sigma_{m+1} < \sigma_0 \,|\, \mathscr{F}_i), \quad q_-(K) := \min_{i,\boldsymbol{j}} q_{i,\boldsymbol{j}}(K).$$

Then it follows directly that, uniformly in $i \in \mathscr{I}$,

$$\varrho_i(K) \ge p_- q_-(K).$$

In order to make the bounds on $\varrho_i(K)$ more explicit, we now show how $q_{i,\boldsymbol{j}}(K)$ as defined in (2.3) can be bounded. These bounds are derived by using the change of measure $\mathbb{Q}$ that we identified above. Denoting, as before, the likelihood ratio in the fully busy period by L, we have

$$\begin{aligned} q_{i,\boldsymbol{j}}(K) &= \mathbb{E}^{\mathbb{Q}}_{i,\boldsymbol{j}}\left[L\, \mathbb{1}\{\sigma^K < \sigma_m\} \,|\, N_0 = m+1\right] \\ &= \mathbb{E}^{\mathbb{Q}}_{i,\boldsymbol{j}}\left[L \,|\, \mathscr{E}(K), N_0 = m+1\right] \mathbb{Q}_{i,\boldsymbol{j}}\left(\mathscr{E}(K) | \, N_0 = m+1\right), \end{aligned}$$

with $\mathscr{E}(K) := \{\sigma^K < \sigma_m\}$, and where the subscript $i,\boldsymbol{j}$ denotes conditioning on the initial states $I_0 = i$ and $\boldsymbol{J}_0 = \boldsymbol{j}$ as before. Using Eq. (2.7) (or Corollary 2.1), we thus conclude

$$\eta_- \mathrm{e}^{-\theta^* K} \le \mathbb{E}^{\mathbb{Q}}_{i,\boldsymbol{j}}\left[L \,|\, \mathscr{E}(K), N_0 = m+1\right] \le \eta_+ \mathrm{e}^{-\theta^* K}, \tag{2.9}$$

with the constants η_- and η_+ defined by

$$\eta_- := \min_{i,i' \in \mathscr{I}, \boldsymbol{j},\boldsymbol{j}' \in \mathscr{D}} \frac{x_{i,\boldsymbol{j}}}{x_{i',\boldsymbol{j}'}}, \quad \eta_+ := \max_{i,i' \in \mathscr{I}, \boldsymbol{j},\boldsymbol{j}' \in \mathscr{D}} \frac{x_{i,\boldsymbol{j}}}{x_{i',\boldsymbol{j}'}}.$$

Since $\mathscr{I}$ and $\mathscr{D}$ are finite, and recalling that $\boldsymbol{x}$ is componentwise positive, η_- and η_+ are positive and finite.

Due to the fact that under $\mathbb{Q}$ the queueing system is unstable, we have that, as $K \to \infty$, we have

$$\mathbb{Q}_{i,\boldsymbol{j}}\left(\mathscr{E}(K) | N_0 = m+1\right) \downarrow \mathbb{Q}_{i,\boldsymbol{j}}\left(\mathscr{E}(\infty) | N_0 = m+1\right) > 0.$$

Furthermore, note that (2.9) holds for any fixed $(i, \boldsymbol{j})$, thus, in particular, we can take the minimum or the maximum over such states. We have thus shown that there exist positive and finite numbers κ_- and κ_+ such that

$$\kappa_- \mathrm{e}^{-\theta^* K} \leq q_-(K) \leq q_+(K) \leq \kappa_+ \mathrm{e}^{-\theta^* K}.$$

Combining this with the bounds on $\varrho_i(K)$ established above, we have established the following result.

Proposition 2.2. *For any $K \in \mathbb{N}$, $i \in \mathscr{I}$, and $\boldsymbol{j} \in \mathscr{D}$, uniformly in $i \in \mathscr{I}$,*

$$p_- \kappa_- \mathrm{e}^{-\theta^* K} \leq \varrho_i(K) \leq \frac{\kappa_+}{p_+} \mathrm{e}^{-\theta^* K}.$$

Proposition 2.2 provides us with a lower and upper bound on $\varrho_i(K)$, which are valid across all $K \in \mathbb{N}$, and which differ just by a multiplicative constant. We now use these bounds to assess the estimator's efficiency properties. The probability $\varrho_i(K)$ is estimated by using $\mathbb{Q}$ during the first C fully busy periods, and the original measure $\mathbb{P}$ otherwise. Denoting this 'composite measure' by $\mathbb{Q}_C$, we rely on the identity

$$\varrho_i(K) = \mathbb{E}^{\mathbb{Q}_C}[L \mathbb{1}\{\sigma^K < \sigma_0\} \,|\, \mathscr{F}_i],$$

with the event $\mathscr{F}_i$ as defined above; cf. [10, Section V.1a]. The *relative error* of an estimator is defined by the ratio of the estimator's standard deviation to its mean. Noting that the estimator is unbiased, we obtain that its per-sample standard deviation can be written as

$$\nu^\circ = \sqrt{\mathbb{E}^{\mathbb{Q}_C}\left[L^2 \mathbb{1}\left\{\sigma^K < \sigma_0\right\} \,|\, \mathscr{F}_i\right] - \varrho_i(K)^2}.$$

Thus, the relative error based on n simulation runs is

$$\frac{\nu^\circ}{\sqrt{n}\, \varrho_i(K)} = \sqrt{\frac{\mathbb{E}^{\mathbb{Q}_C}\left[L^2 \mathbb{1}\left\{\sigma^K < \sigma_0\right\} \,|\, \mathscr{F}_i\right]}{n \varrho_i(K)^2} - \frac{1}{n}} \leq \sqrt{\frac{\mathbb{E}^{\mathbb{Q}_C}\left[L^2 \mathbb{1}\left\{\sigma^K < \sigma_0\right\} \,|\, \mathscr{F}_i\right]}{n \varrho_i(K)^2}}. \quad (2.10)$$

Recall that the number of times a fully busy period is reached is bounded from above by a geometric random variable with success probability p_+. Invoking Corollary 2.1, we thus

obtain, for any $i \in \mathscr{I}$,

$$\mathbb{E}^{\mathbb{Q}_C}\left[L^2 \mathbb{1}\left\{\sigma^K < \sigma_0\right\} \mid \mathscr{F}_i\right] \leq \mathrm{e}^{-2\theta^* K} \sum_{k=1}^{C} \mathrm{e}^{2\theta^* k} p_+^k (1-p_+) \eta_+^{2k}. \tag{2.11}$$

For any finite C the right-hand side in (2.11) is finite. (As an aside, observe that C may be chosen as ∞ if $\mathrm{e}^{2\theta^*} p_+ \eta_+^2 < 1$, as then the geometric series converges.) Combining this with the lower bound $\varrho_i(K) \geq p_- \kappa_- \mathrm{e}^{-\theta^* K}$, the bounded relative error follows: the expression in the left-hand side of (2.10) is bounded above by a finite expression that does not depend on K.

Theorem 2.3. *For arbitrary $C \in \mathbb{N}$, the estimator for $\varrho_i(K)$ based on $\mathbb{Q}_C$ has bounded relative error.*

Note that the upper bound given in (2.11) is smallest when $C = 1$, but this obviously does *not* imply that the left-hand side of (2.11) is minimised for $C = 1$. In Section 2.4 we empirically study the impact of the choice of C.

2.2.4 Numerical example

In this subsection we present a small numerical illustration. Consider a two-server system, with $d = 2$ and $D^{(1)} = D^{(2)} = 3$. The arrival rates, initial probabilities and transition rate matrices are

$$Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.1 & -0.1 \end{pmatrix}, \quad \boldsymbol{\alpha}^{(1)} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.2 \\ 0 \end{pmatrix}, \quad T^{(1)} = \begin{pmatrix} -0.9 & 0.2 & 0.1 & 0.6 \\ 0.5 & -1.5 & 0.5 & 0.5 \\ 0.2 & 0.2 & -0.8 & 0.4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{\lambda} = \begin{pmatrix} 0.1 \\ 0.5 \end{pmatrix}, \quad \boldsymbol{\alpha}^{(2)} = \begin{pmatrix} 0.5 \\ 0.2 \\ 0.3 \\ 0 \end{pmatrix}, \quad T^{(2)} = \begin{pmatrix} -1 & 0.2 & 0.2 & 0.6 \\ 1 & -2 & 0.5 & 0.5 \\ 0.2 & 0.2 & -1 & 0.6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We order the states lexicographically and solve the eigensystem (2.4) by first using bisection to find a value of θ such that the largest eigenvalue of the matrix A defining the eigensystem equals one. One thus obtains $\theta^* = 0.86$. We normalise the eigenvector corresponding to this eigenvalue such that its first entry is one, and call the resulting vector $\boldsymbol{x}$.

For example, for states $i = 1$, $i' = 2$, $\boldsymbol{j} = (1,1)$, $\boldsymbol{j}' = (2,3)$, we then have $x_{i,\boldsymbol{j}} = 1$, $x_{i,\boldsymbol{j}'} \approx 1.09$, $x_{i',\boldsymbol{j}} \approx 1.99$, $x_{i',\boldsymbol{j}'} \approx 2.17$ (rounded to two decimal digits). We observe the remarkable property that (up to the rounding error) $x_{i',\boldsymbol{j}'}/x_{i',\boldsymbol{j}} = x_{i,\boldsymbol{j}'}/x_{i,\boldsymbol{j}}$. Generally, it turns out that the obtained MGFs $\boldsymbol{x}$ are such that

$$\frac{x_{i,\boldsymbol{j}}}{x_{i,\boldsymbol{j}'}} = \frac{x_{i',\boldsymbol{j}}}{x_{i',\boldsymbol{j}'}} \quad \text{and} \quad \frac{x_{i,\boldsymbol{j}}}{x_{i',\boldsymbol{j}}} = \frac{x_{i,\boldsymbol{j}'}}{x_{i',\boldsymbol{j}'}}, \quad \text{for any} \quad i \neq i', \boldsymbol{j} \neq \boldsymbol{j}';$$

indicating that there is a certain decoupling among the servers as well as between servers and arrivals. The decoupling means that under $\mathbb{Q}$ (as was the case under the original measure $\mathbb{P}$), (i) the transition rates of the background process do not depend on the phases the customers in service are in, (ii) the service-time distribution at any particular server does not depend on the state of the background process nor the phases the other customers are in.

2.2.5 The structure of A

The observed decoupling can be formally established as follows. Note that if the states $(i, j_1, \ldots, j_m)$ are ordered lexicographically, then the matrix A defining the eigensystem given in (2.4) can be decomposed as

$$A \circ \boldsymbol{\varphi}\mathbf{1} = \Lambda\, \mathrm{e}^{\theta} \otimes I_D + \underline{Q} \otimes I_D + I_d \otimes R = \left[\Lambda\, \mathrm{e}^{\theta} + \underline{Q}\right] \oplus R, \tag{2.12}$$

where $\circ$ denotes the Hadamard product, $\boldsymbol{\varphi}$ is the column vector with entries $\varphi_{i,\boldsymbol{j}}$, $\mathbf{1}$ is a row vector of ones, $\Lambda := \mathrm{diag}\{\boldsymbol{\lambda}\}$; $D := \prod_{\ell=1}^{m} D^{(\ell)}$; I_D is the $D \times D$-identity matrix; $\oplus$ and $\otimes$ denote the Kronecker sum and product, respectively; $\underline{Q}$ denotes the matrix $Q - Q \circ I_d$; and the 'remainder term' R is of dimension $D \times D$ and depends on $T^{(\ell)}$ and $\overline{T}^{(\ell)}$ but not on $\boldsymbol{\lambda}$ or Q. Because the eigenvalues of a Kronecker sum are given by the sums of the eigenvalues of each Kronecker summand [95, Theorem 13.16], this decomposition shows that the eigensystem can be split up into a part corresponding to arrivals and a part corresponding to services.

Let us now consider the remainder term R, which corresponds to the service processes. We note that R contains $D^{-} := \prod_{\ell=1}^{m-1} D^{(\ell)}$ block matrices of size $D^{(m)} \times D^{(m)}$ which have the following structure.

(i) The block matrices on the diagonal are of the form

$$\underline{T}^{(m)} + \overline{T}^{(m)}\, \mathrm{e}^{-\theta} + I_{D^{(m)}} \mathrm{e}^{-\theta} \sum_{\ell=1}^{m-1} \bar{t}^{(\ell)}_{j_\ell, j_\ell}.$$

(ii) Off-diagonal block matrices are either of the form $\left[t^{(\ell)}_{j_\ell, j'_\ell} + \bar{t}^{(\ell)}_{j_\ell, j'_\ell} \mathrm{e}^{-\theta}\right] I_{D^{(m)}}$ with $\ell < m$, or they are zero.

As it turns out, R can thus be decomposed as

$$R = \bigoplus_{\ell=1}^{m} \left(\underline{T}^{(\ell)} + \overline{T}^{(\ell)}\, \mathrm{e}^{-\theta}\right).$$

Inserting this expression in (2.12) we see (e.g. from [95, Theorem 13.16]) that the MGF $\boldsymbol{x}$ that we found as an eigenvector of A can be computed as the Kronecker product of eigenvectors of $m + 1$ decoupled eigensystems, corresponding with the arrivals and the services for each of the m servers. In this way, while the dimension of A is $d\,D$, the measure $\mathbb{Q}$ can be found by solving a system of dimension just $d + \sum_{\ell=1}^{m} D^{(\ell)}$; in the

above example this would yield a reduction of dimension 18 to dimension 8. We detail such an alternative approach in the next section.

2.3 Efficient computation of change of measure

As mentioned above, an intrinsic problem of the change of measure defined in Section 2.2 is that the underlying eigensystem may become prohibitively large, and as a result the computation of $\mathbb{Q}$ becomes problematic. Already for the small example of Section 2.2.4 the length of the vector $\boldsymbol{x}$ is 18; if one has 10 servers with 3-dimensional phase-type distributions, and if $d = 4$, the dimension of the matrix A is as high as $4 \cdot 3^{10} = 2.36 \cdot 10^5$. This explains why we explore an alternative approach to compute $\mathbb{Q}$, in which the twist of the arrival processes and the service times do not interrelate. In the above example with 10 servers, this means that the alternative measure can be found by solving a system of dimension 34. The 'catch' is that the decoupling-based approach requires function evaluations that are more involved, and thus for low-dimensional problems the approach of Section 2.2 may be preferred; see also Section 2.4, where we discuss simulation examples.

2.3.1 Preliminaries

We first recall a few auxiliary results. Consider a sequence of i.i.d. positive random variables $(R_n)_{n\in\mathbb{N}}$ (with a bounded MGF around 0), and its associated counting process

$$R(t) := \sup\left\{n \in \mathbb{N} : \sum_{i=1}^{n} R_i \leq t\right\}.$$

Define the associated limiting logarithmic MGF (ll-MGF):

$$\mathscr{R}(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}e^{\theta R(t)}.$$

Let c be some number larger than $\mathbb{E}R$ (with R being distributed as R_1). Consider a (stable) queue that drains at a constant rate c, where unit-sized jobs arrive with interarrival times $(R_n)_{n\in\mathbb{N}}$. From e.g. [41] we have the logarithmic decay rate of the probability $\mathbb{P}(W > u)$ that the stationary workload W exceeds u,

$$\lim_{u\to\infty} \frac{1}{u} \log \mathbb{P}(W > u) = -\theta,$$

obeys $\mathscr{R}(\theta) - c\theta = 0$. There is however a second way to compute the decay rate, viz. as the solution of $r(-c\theta)\mathrm{e}^{\theta} = 1$, with $r(\theta) := \mathbb{E}\mathrm{e}^{\theta R}$; see e.g. [54]. Note that both lead to the same (c, θ)-pairs. A minor computation yields that

$$\mathscr{R}(\theta) = -r^{-1}(e^{-\theta}). \tag{2.13}$$

We have thus expressed the ll-MGF $\mathscr{R}(\cdot)$ in terms of the MGF of R. For instance for R having an exponential distribution with mean μ^{-1}, this yields $\mathscr{R}(\theta) = \mu(e^{\theta} - 1)$, as it should (recall that in this case $R(t)$ is Poisson with mean μt). See [42] for more background on this type of inversion result.

Let $\mathscr{A}(\theta)$ be the ll-MGF corresponding with the interarrival times $(A_n)_{n\in\mathbb{N}}$, and $\mathscr{B}^{(\ell)}(\theta)$ the ll-MGF corresponding with the service times $(B_n^{(\ell)})_{n\in\mathbb{N}}$ (in case there would always be jobs to serve). Then by [54, Proposition 2] the decay rate of the probability $\mathbb{P}(Q > K)$ that the stationary number of customers Q exceeds K, i.e.,

$$\lim_{K\to\infty} \frac{1}{K} \log \mathbb{P}(Q > K) = -\theta^*,$$

is the solution θ^* of

$$\mathscr{A}(\theta) + \sum_{\ell=1}^{m} \mathscr{B}^{(\ell)}(-\theta) = 0$$

where, with $A(t)$ and $B^{(\ell)}(t)$ defined analogously to $R(t)$,

$$\mathscr{A}(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}e^{\theta A(t)}, \;\; \mathscr{B}^{(\ell)}(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}e^{\theta B^{(\ell)}(t)}.$$

Invoking (2.13) we conclude that θ^* solves

$$\mathscr{A}(\theta) = \sum_{\ell=1}^{m} \left(b^{(\ell)}\right)^{-1} (e^{\theta}), \tag{2.14}$$

where, with $S^{(\ell)}$ and $\boldsymbol{s}^{(\ell)}$ as in Eq. (2.1),

$$b^{(\ell)}(\theta) = \boldsymbol{\alpha}^{(\ell)} \left(-\theta I_{D^{(\ell)}} - S^{(\ell)}\right)^{-1} \boldsymbol{s}^{(\ell)} \tag{2.15}$$

is the MGF associated with server ℓ (see, e.g., [20, Theorem 4.3]).

Regarding the arrival times this means that we have to find an MMP process such that its ll-MGF is $\mathscr{A}^{\circ}(\theta^*) = \mathscr{A}(\theta + \theta^*) - \mathscr{A}(\theta^*)$. As can be found in e.g. Kesidis, Walrand, and Chang [73], $\mathscr{A}(\theta)$ equals $\Xi(Q + (e^{\theta} - 1)\,\mathrm{diag}\{\boldsymbol{\lambda}\})$, where $\Xi(M)$ denotes the largest eigenvalue of M.

Regarding the service times, (2.14) implies that we should construct $\mathbb{Q}$ such that under this new measure the ll-MGF of the service times at server ℓ equals

$$\left(\mathscr{B}^{(\ell)}\right)^{\circ} (\theta^*) = -\left(b^{(\ell)}\right)^{-1} (e^{\theta+\theta^*}) + \left(b^{(\ell)}\right)^{-1} (e^{\theta^*}). \tag{2.16}$$

We now point out how the corresponding changes of measure can be performed.

2.3.2 Twist of the arrival process

As mentioned, $\mathscr{A}(\theta)$ equals $\Xi(Q+\Lambda(e^{\theta}-1))$, and hence we wish to find $\Lambda^{\circ} := \mathrm{diag}\{\boldsymbol{\lambda}^{\circ}\}$ and Q° such that

$$\Xi\big(Q^{\circ}+\Lambda^{\circ}(e^{\theta}-1)\big) = \Xi\big(Q+\Lambda(\mathrm{e}^{\theta+\theta^*}-1)\big) - \mathscr{A}(\theta^*),$$

where it is noted that the right hand side of the previous expression can alternatively be written as, with I_d denoting the $d\times d$-identity matrix, $\Xi\big(Q+\Lambda(\mathrm{e}^{\theta+\theta^*}-1)-I_d\mathscr{A}(\theta^*)\big)$.

Suppose we observe an auxiliary system with the Markov-modulated Poisson process being the input, but served at a constant rate c (larger than the mean input rate of the MMP process). As we argued in the previous section, the decay rate θ^* of the auxiliary system can be evaluated by solving $\mathscr{A}(\theta)=c\theta$. Alternatively, we can find a system of equations that θ^* should satisfy, similar to the approach in Section 2.2.1. Let z_i denote the MGF of the net increase in the number of customers in the auxiliary system during a period in which the background process transitions from i to an arbitrary reference state. Then (θ^*, z_i) should satisfy

$$z_i = \sum_{j\neq i}\frac{q_{i,j}}{q_i}z_j\int_0^{\infty} q_i\,\mathrm{e}^{-q_i t}\mathrm{e}^{-\theta c t}\mathrm{e}^{\lambda_i(\mathrm{e}^{\theta}-1)}\mathrm{d}t = \sum_{j\neq i}\frac{q_{ij}}{q_i-\lambda_i(\mathrm{e}^{\theta}-1)+c\theta}z_j,$$

which can be rewritten as

$$\big(-\lambda_i(e^{\theta}-1)+c\theta\big)z_i = \sum_{j=1}^{d} q_{ij}z_j.$$

Inserting $c\theta^* = \mathscr{A}(\theta^*)$, we conclude that for $\theta=\theta^*$ there exists a componentwise positive vector $\boldsymbol{z}$ such that

$$\big(-(\mathrm{e}^{\theta^*}-1)\Lambda + I_d\mathscr{A}(\theta^*)\big)\boldsymbol{z} = Q\boldsymbol{z}. \tag{2.17}$$

Now let Z denote $\mathrm{diag}\{\boldsymbol{z}\}$. Observe that any eigenvalue of $Q+(e^{\theta+\theta^*}-1)\,\Lambda - I_d\mathscr{A}(\theta^*)$, is eigenvalue of

$$\begin{aligned} &Z^{-1}\big(Q+(\mathrm{e}^{\theta+\theta^*}-1)\,\Lambda - I_d\mathscr{A}(\theta^*)\big)Z \\ &\quad= Z^{-1}QZ + (\mathrm{e}^{\theta+\theta^*}-1)\,\Lambda - I_d\mathscr{A}(\theta^*) \\ &\quad= Z^{-1}QZ + (\mathrm{e}^{\theta^*}-1)\,\Lambda - I_d\mathscr{A}(\theta^*) + (\mathrm{e}^{\theta}-1)\Lambda^{\circ} \end{aligned}$$

(with $\Lambda^{\circ} := \Lambda\mathrm{e}^{\theta^*}$) as well. Now note that, by virtue of (2.17),

$$Q^{\circ} := Z^{-1}QZ + (\mathrm{e}^{\theta^*}-1)\,\Lambda - I_d\mathscr{A}(\theta^*)$$

is a generator matrix. We have thus found that the desired change of measure is

$$\lambda_i^{\circ} := \lambda_i\mathrm{e}^{\theta^*}, \quad q_{ij}^{\circ} := q_{ij}\frac{z_j}{z_i}, \quad q_i^{\circ} := q_i - \lambda_i(\mathrm{e}^{\theta^*}-1) + \mathscr{A}(\theta^*). \tag{2.18}$$

2.3.3 Twist of the service times

We start by pointing out that realizing the desired change of measure such that the ll-MGF becomes (2.16) amounts to exponentially twisting the service times at server ℓ by some ζ_ℓ^* that we specify below. We wish to find service times (with MGF $\overline{b}^{(\ell)}(\cdot)$) such that (2.16) equals $-(\overline{b}^{(\ell)})^{-1}(\mathrm{e}^{\theta})$. Observe that (under the usual regularity conditions) $f^{-1}(yu) = g^{-1}(y)+v$ (for all y) is equivalent to $g(x) = f(x+v)/u$ (for all x). This means that we have to identify a $\overline{b}(\cdot)$ such that

$$\overline{b}^{(\ell)}(\zeta) = \frac{b^{(\ell)}\big(\zeta + (b^{(\ell)})^{-1}(\mathrm{e}^{\theta^*})\big)}{\mathrm{e}^{\theta^*}},$$

which corresponds to exponentially twisting the service times at the ℓ-th server with twist $\zeta_\ell^* := (b^{(\ell)})^{-1}(\mathrm{e}^{\theta^*})$.

We proceed by explaining how the change of measure can be found for each server. Consider a generic server with phase-type distributed service times B, parametrised by the initial distribution $\boldsymbol{\alpha}$, the transition matrix T, and the dimension $D+1$. The twisted measure should satisfy

$$\mathbb{E}_{\mathbb{Q}}\mathrm{e}^{\zeta B} = \frac{\mathbb{E}\mathrm{e}^{(\zeta+\zeta^*)B}}{\mathbb{E}\mathrm{e}^{\zeta^* B}}, \tag{2.19}$$

where $\zeta^* = b^{-1}\left(\mathrm{e}^{\theta^*}\right)$ as argued above. Consider all paths $(i_0, i_1, \ldots, i_{\tau+1})$ of the underlying Markov process, starting from i_0 (sampled according to $\boldsymbol{\alpha}$) and ending at $i_{\tau+1} = D+1$. Let h_j be the time spent between states i_j and i_{j+1}. Then the right hand side of (2.19) can be written as, with $\mathbb{E}\mathrm{e}^{\zeta^* B} = b(\zeta^*) = \mathrm{e}^{\theta^*}$,

$$\begin{aligned}\mathrm{e}^{-\theta^*} \sum_{\text{all paths}} \int_0^\infty \int_{h_j:\sum_{j=0}^{\tau} h_j = h} \alpha_{i_0} \frac{t_{i_0 i_1}}{t_0} t_0 \\ \times e^{-t_{i_0} h_0} \cdots \frac{t_{i_\tau i_{\tau+1}}}{t_{i_\tau}} t_{i_\tau} e^{-t_{i_\tau} h_\tau} \mathrm{e}^{(\zeta+\zeta^*)h} \mathrm{d}h_0 \cdots \mathrm{d}h_\tau \, \mathrm{d}h,\end{aligned}$$

whereas the left hand side reads

$$\sum_{\text{all paths}} \int_0^\infty \int_{h_j:\sum_{j=0}^{\tau} h_j = h} \alpha^\circ_{i_0} t^\circ_{i_0 i_1} e^{-t^\circ_{i_0} h_0} \cdots t^\circ_{i_\tau i_{\tau+1}} e^{-t^\circ_{i_\tau} h_\tau} \mathrm{e}^{\zeta h} \mathrm{d}h_0 \cdots \mathrm{d}h_\tau \, \mathrm{d}h.$$

We wish to identify $\boldsymbol{\alpha}^\circ$ and T° such that both expressions match. To this end, solve the following eigensystem:

$$-\zeta^* y_i = \sum_{j=1}^{D+1} t_{ij} y_j \quad \text{for } i = 1, \ldots, D, \quad \mathrm{e}^{\theta^*} y_{D+1} = \sum_{i=1}^{D} \alpha_i y_i.$$

Then define

$$\alpha_i^\circ := \frac{\alpha_i}{\mathrm{e}^{\theta^*}} \frac{y_i}{y_{D+1}}, \quad t_{ij}^\circ := t_{ij} \frac{y_j}{y_i}, \quad t_i^\circ := t_i - \zeta^*. \tag{2.20}$$

The following two observations are crucial:

- $(\boldsymbol{\alpha}^\circ, T^\circ)$ corresponds to a phase-type distribution. To this end, note that, by definition of the vector $\boldsymbol{y}$, the new initial distribution $\boldsymbol{\alpha}^\circ$ is non-negative and sums to 1. In addition,
$$\sum_{j\neq i} t^\circ_{ij} = \sum_{j\neq i} t_{ij}\frac{y_j}{y_i} = t_i + \zeta^* = t^\circ_i.$$
- It is an easy verification that for the above defined $(\boldsymbol{\alpha}^\circ, T^\circ)$ both MGFs match, as desired.

Thus, the proposed twist of the service times corresponds to a valid change of measure.

Note that the twisted rates we found in this and the previous subsection take the same form as those in Section 2.2.1, the only difference being the MGFs involved. The counterpart to the likelihood ratio given in Eq. (2.7) is

$$L = \left(\mathrm{e}^{-\theta^*\Sigma_+}\frac{z_{i_0}}{z_{i_\tau}}\right) \times \left(\mathrm{e}^{\theta^*\Sigma_-}\prod_{\ell=1}^{m}\frac{y^{(\ell)}_{j_0}}{y^{(\ell)}_{j_\tau}}\right),$$

where, with a slight abuse of notation, z_{i_0} and z_{i_τ} correspond to the first and last state of the background process, and $y^{(\ell)}_{j_0}$ and $y^{(\ell)}_{j_\tau}$ correspond to the first and last phase of server ℓ, respectively, given that we observe a path of length τ as defined in Section 2.2.1. In line with the Kronecker decomposition found in Section 2.2.5, we thus see that $x_{i,\boldsymbol{j}} = z_i\prod_{\ell=1}^m y^{(\ell)}_{j_\ell}$.

Remark 2.3.1. Recall that in the set-up of Section 2.2 we did not quite find the change of measure of the individual service times, as we only identified the twisted version of $\bar{t}^{(\ell)}_{j_\ell,k} := t^{(\ell)}_{j_\ell,\dagger}\alpha^{(\ell)}_k$ rather than the twisted version of $t^{(\ell)}_{j_\ell,\dagger}$ and $\alpha^{(\ell)}_k$ individually. In this section we showed how to twist the rates for each server independently. In other words, we can now use the twisted rates found in this section *throughout* the entire busy cycle, i.e., also outside of fully busy periods. Note that outside of the fully busy period the (total) rates of leaving state $(i,\boldsymbol{j})$, that is, the counterparts to $\varphi_{i,\boldsymbol{j}}$ and $\varphi^\circ_{i,\boldsymbol{j}}$ defined in Section 2.2.1, are *not* equal so that the corresponding terms in the likelihood ratio do not cancel. This means that to evaluate the likelihood ratio, it needs to be continuously updated when the process is not in a fully busy period as it does not have a clean form of the type (2.7).

Remark 2.3.2. There is one important situation in which all expressions simplify considerably: that of no modulation and identical servers. In fact, in this case the interarrival times do not need to be exponential, but any renewal sequence works. To see this, suppose the interarrival times have MGF $a(\cdot)$ and the service times (at each of the servers) have MGF $b(\cdot)$. Then (2.14) reeds

$$-a^{-1}(e^{-\theta}) = mb^{-1}(e^{\theta}),$$

which is solved by $\theta^* := -\log a(-m\alpha)$, where α is such that

$$\log a(-m\alpha) + \log b(\alpha) = 0,$$

which coincides with Eq. (1.7) in [133].

2.4 Simulation examples

In this section we investigate a number of numerical examples, to assess the efficacy of the proposed procedure, and to learn about various aspects of the rare-event behaviour of the multi-server queue under study.

We start by evaluating the impact of heterogeneity among servers on the speed of decay and the the relative error of the estimation procedure. We consider a two-server system with Erlang distributed service times. The service times of Server 1 are distributed with shape parameter 3 and rate parameter μ_1 while the service times of Server 2 are distributed with shape parameter 3 and rate parameter μ_2; the initial phase is distributed as $\boldsymbol{\alpha} = (0.5, 0.2, 0.3)$ in both cases. The arrival process is Poisson with rate 0.1 (i.e., not modulated). We estimate $\rho_1(K)$ using the change of measure proposed in Section 2.2.1 during practically all fully busy periods; that is, we employ the algorithm stated in the appendix with C chosen very large.

Figure 2.1 shows that the convergence of each scaled logarithmic importance sampling estimator (calculated from 10^7 samples) to its corresponding limit $-\theta^*$ appears to be faster when servers are more heterogeneous. For μ_1 and μ_2 fixed, it turns out that the corresponding relative error values are roughly independent of K (in line with our theoretical findings). More precisely, we obtained that for $\mu_1 = \mu_2 = 1$ the relative error of a generic sample is approximately 2 (across a wide range of K-values, independently of n), for $\mu_1 = 1$, $\mu_2 = 2$ it is 2.82, for $\mu_1 = 1$, $\mu_2 = 3$ it is 3.96, and for $\mu_1 = \mu_2 = 2$ it is 7.7. Thus, as one would expect, the deviation from the mean is larger when servers are more heterogeneous. The comparison of the two homogeneous examples suggests that faster service tends to have a negative impact on the relative error performance.

Furthermore, we can check numerically that, as it should be, the twisted rates are the same as those obtained in the way we described in Section 2.3. We now detail how we computed the change of measure using that approach. The underlying idea is that we determine θ^* by solving (2.14); once θ^* has been determined, we can twist the arrival and service processes as in (2.18) and (2.20), respectively. In order to solve (2.14), the following steps need to be performed:

- In the first place it requires the evaluation of $\mathscr{A}(\theta)$ and $(b^{(\ell)})^{-1}(\mathrm{e}^\theta)$ (for $\ell = 1, \ldots, m$) which typically cannot be done in closed-form so that we have to resort to numerics. Here, we used a bisection procedure. To determine $\mathscr{A}(\theta)$ for every θ the eigenvalues of a d-dimensional matrix need to be found; to determine $(b^{(\ell)})^{-1}(\mathrm{e}^\theta)$ the inverse of the function $b^{(\ell)}(\zeta)$ as defined in (2.15) is to be evaluated.

- In the second place a numerical solver needs to be used to solve (2.14). We again used bisection to perform this step.

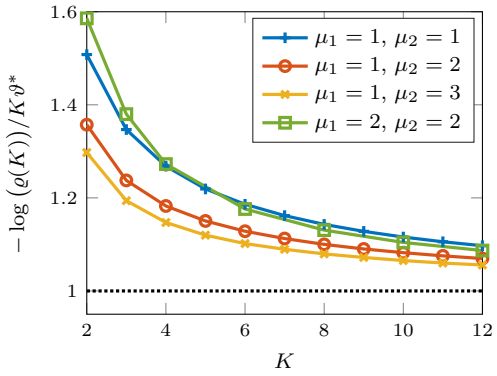


Figure 2.1: Ratio of $\log(\widehat{\rho}_1(K))/K$ and its limit $-\theta^*$ for a two-server system with service times that are Erlang-distributed with shape parameter 3 and rate parameters μ_1 and μ_2, respectively. The horizontal line indicates a ratio of one.

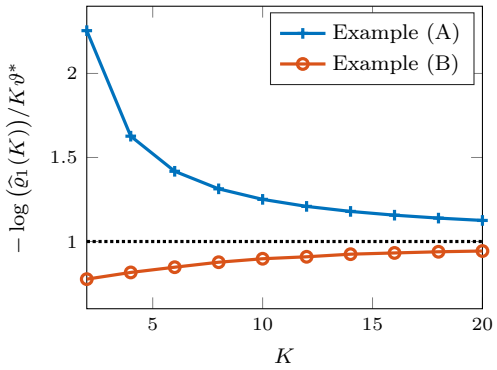


Figure 2.2: Ratio of $\log(\widehat{\rho}_1(K))/K$ and its limit $-\theta^*$ for (A) the example from Section 2.2.4, and (B) a large-scale example with 10 servers. The horizontal line indicates a ratio of one.

For a small example as the ones just discussed, the method of Section 2.2 may be preferred for its conceptual simplicity. For examples of a larger dimension on the other hand, the method of Section 2.2 for computing the change of measure quickly becomes slow or even infeasible due to memory constraints, and the method of Section 2.3 is to be preferred.

We now discuss such a large-scale example, which is too large to be efficiently solved using the method of Section 2.2. We again assume that service times have an Erlang distribution with shape parameter 3. In a system with 10 servers, we set $\boldsymbol{\alpha}^{(\ell)} = (0.5, 0.2, 0.3)$, and choose the Erlang rate parameter as $\ell/3$, for $\ell = 1, \ldots, m$. We set $d = 4$, $\boldsymbol{\lambda} = (1, 2, 3, 4)$, and let Q have off-diagonal entries 0.1 (and diagonal entries -0.3). Recall that the dimension of the matrix A defining the eigensystem (2.4) is as large as $2.36 \cdot 10^5$. Despite this dimension, it turns out that with the methodology of Section 2.3 the change of measure can be computed in less than a second. In Figure 2.2 we show the ratio of the scaled logarithmic importance sampling estimator and its limit $-\theta^*$ obtained in 10^5 simulation runs for (A) the small example from Section 2.2.4, and (B) the large-scale example with

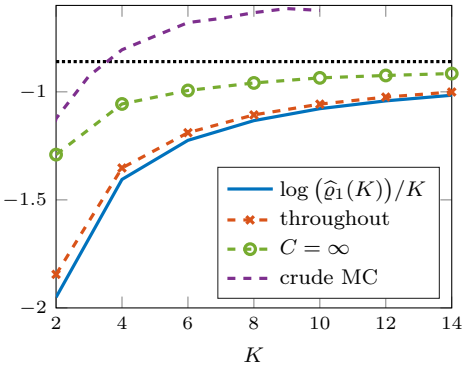


Figure 2.3: Depicted is $\log(\widehat{\rho}_1(K))/K$ obtained for the example of Section 2.2.4. The horizontal dotted line indicates the limit value $-\theta^*$. The change of measure is evaluated using the methodology developed in Section 2.3, and applied either throughout the entire busy cycle, only during fully busy periods ($C = \infty$), or never (crude MC), yielding the (scaled logarithmic) upper 95% confidence bounds that are indicated by the dashed lines.

10 servers; in both cases the change of measure is evaluated as described in Section 2.3, and applied during all fully busy periods.

In the approach we have developed, we use the alternative measure $\mathbb{Q}$ only during (a number of) fully busy periods. As mentioned in Remark 2.3.1, thanks to the decoupling of servers that we described in Section 2.3, the twisted rates can also be applied throughout – during the entire busy cycle – rather than only during fully busy periods. Considering again the example of Section 2.2.4, we compare the sample confidence interval obtained under crude Monte Carlo estimation to that achieved when the change of measure is applied either throughout, or only during fully busy periods (with $C = \infty$). Figure 2.3 shows the scaled logarithmic unbiased estimate of $\varrho_1(K)$ averaged over 10^5 runs. The limit $-\theta^*$ is indicated by the horizontal line. The dashed lines indicate the scaled logarithmic upper bounds of the 95% standard normal confidence intervals. As one would expect, the change of measure significantly improves the accuracy of the estimation procedure for a fixed number of runs. In addition, we observe that when the change of measure is applied throughout (rather than only during fully busy periods) the confidence is noticeably more narrow.

We now investigate the impact of the choice of C for the same example. In Figure 2.4 we compare the relative errors obtained for various values of C in 10^7 runs, where C denotes the number of fully busy periods during which the change of measure is applied. The values obtained when the twisted rates are used either never (crude Monte Carlo) or throughout the entire busy cycle are also shown. The relative error obtained for the crude Monte Carlo estimator (corresponding to $C = 0$) increases exponentially as K grows large. For large C indeed we see that relative error is independent of K. For smaller C instead, the relative error does increase with K until it drops sharply for K large enough. It appears that for small K if the event did not occur in the first period, then it may still occur afterwards even though the original measure is used, which causes a large variance.

Note that for large K the relative error corresponding to $C = 1$ appears to be the smallest, in alignment with the upper bound we found in Section 2.2.3. For an explanation, recall that each fully busy period during which the change of measure is applied

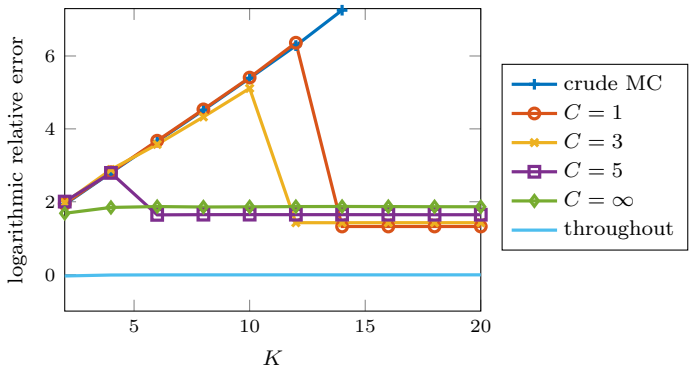


Figure 2.4: Logarithmic relative error values obtained for crude Monte Carlo (MC) estimation of $\rho_1(K)$ compared to the values obtained under importance sampling when the change of measure is applied during the first $C > 0$ fully busy periods or throughout; the rates are chosen as in Section 2.2.4.

contributes to the likelihood ratio by a factor between $\mathrm{e}^{\theta^*}\zeta_-$ and $\mathrm{e}^{\theta^*}\zeta_+$ (cf. Corollary 2.1), and thus potentially increases the variance of the estimator (where it should be kept in mind that $\zeta_- < 1$, $\zeta_+ > 1$, and $\theta^* > 0$). In this sense, each additional fully busy period may have a negative impact on the quality of the estimator. Choosing a good value for C amounts to finding a proper balance between increasing the likelihood of the event of interest and minimizing the possible additional contribution to the variance of the likelihood ratio.

In the experiments that we performed, if the change of measure is applied throughout, then the relative error is remarkably low at about 0.99, substantially lower than when it is applied only during fully busy periods. We see a similar improvement in terms of estimation accuracy for the other examples discussed in this section when $\mathbb{Q}$ is applied throughout.

2.5 Discussion and concluding remarks

In this chapter we developed an algorithm for estimating the probability that the number of customers in a multi-server queueing system reaches a high value. The input is Markov-modulated Poisson, whereas the service-times have server-dependent phase-type distributions. We have identified explicit bounds on the probability under consideration as well as the associated likelihood ratio, which help quantifying the relative error. In particular we have proven that the relative error of our estimator is bounded. We also develop a technique to efficiently compute the alternative measure to be used in our importance-sampling based algorithm, which remains tractable even when the dimension of the system (in terms of the number of servers and the dimensions of the phase-type distributions) is large.

A couple of experiments provide us with indications of the significant speed-up that can be achieved by the proposed algorithm relative to naïve simulation. The focus is on estimating $\varrho_i(K)$, i.e., the probability that the backlog (that is, the number of customers or jobs waiting in the queue) during a busy cycle exceeds a given level K (with the

background process being in state i at the beginning of the busy cycle). The method, however, directly extends to a procedure for estimating the fraction of customers or jobs entering the system while the backlog is larger than K. To this end, first note that this quantity can be written as the ratio of the mean number of customers that entered the system while the backlog is larger than K during a busy cycle, and the mean total number of customers that entered during a busy cycle. Then the idea is to estimate the numerator and denominator of the ratio separately. The denominator does not contain a rare event, and hence can be estimated using the original measure. The numerator *does* involve a rare event, but simulating under $\mathbb{Q}$ (corresponding to a positive drift) would mean that *terminating* the busy cycle would become a rare event. Following e.g. [56], this issue can be remedied by applying a *measure-specific dynamic importance sampling* approach, where $\mathbb{Q}$ is switched off as soon as K has been reached. Along the same lines, one could set up a procedure to estimate the fraction of customers lost in the corresponding model with a waiting room of finite size K, as was done for a similar system in [103].

The proposed procedures could be used as an aid in devising staffing rules. The probability of delay is commonly used as a performance metric in the literature on staffing; see the recent survey [37]. We remark, however, that the fraction of lost customers – which can be estimated as described above – may be a better criterion for staffing in, for instance, call centers, where a moderate delay can be acceptable as long as the call is not lost. The most simple approach would be to use bisection to estimate the required number of servers based on simulation such that the performance metric of choice is fixed to a desired level. If instead the number of servers is given, a similar procedure could be used to determine the size of the waiting room.

In this chapter we considered specific arrival and service processes, but we anticipate that importance sampling procedures for related processes can be developed with the same techniques. As we argued, the MMP is suitable for modelling overdispersion, but there are various other processes that could be used to this end, such as the Cox processes advocated in [79].

CHAPTER 3

Infinite server queue with mixed Poisson arrivals

In the previous chapter we analyzed a queueing model with an overdispersed arrival process, focussing on the MMP process, for which the arrival rate parameter changes at random time instances according to a Markov process. While the MMP model is tractable and often used, it seems reasonable to assume that in practice arrival data is collected in discrete time instances and thus, arrival rates are re-estimated after a fixed number of such instances. In line with this assumption, in this chapter we consider a different type of Cox process that was advocated in [64], for which the arrival rate is *resampled* (in an i.i.d. manner), say every Δ units of time, whereas during the resulting time intervals the arrival rate is assumed constant. We denote these i.i.d. arrival rates by $(X_i)_{i\in\mathbb{N}}$. The content of this chapter, with the exception of Section 3.2.2, is taken from Heemskerk, Kuhn and Mandjes [63].

Number of arrivals. We start by studying the tail asymptotics of the total number of arrivals in a time interval of given length. We do so in a scaling regime that was proposed in [64], in which the arrival rates and sampling frequency are jointly inflated as follows. In the first place, it is natural to assume that arrival rates are large, as these represent the contributions of many potential clients; this can be achieved by letting these arrival rates be $nX_1, nX_2, \ldots$ for i.i.d. $(X_i)_{i\in\mathbb{N}}$ and some large n. In addition, the sampling frequency is set to n^α (assumed to be integer) and hence the size of each time slot is assumed to be $\Delta = n^{-\alpha}$. Here the parameter α can be tuned; evidently, the larger α, the more frequently the arrival rate is resampled.

The focus is thus on the probabilities $P_n(a)$ and $p_n(a)$, where

$$P_n(a) := \mathbb{P}\left(\text{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right), \quad \text{with} \quad \overline{X}_m := \frac{1}{m}\sum_{i=1}^{m} X_i,$$

and $p_n(a)$ denotes the corresponding probability that the mixed Poisson random variable $\text{Pois}\left(n\overline{X}_{n^\alpha}\right)$ equals na (assumed to be integer). We consider the situation that a is larger than $\nu := \mathbb{E}X_i$, which entails that the event under consideration is rare and that we are in the framework of large deviations theory.

It is observed that an important role is played by the time-scale parameter $\alpha > 0$. If

α is large, since the arrival rate is resampled relatively frequently, it is anticipated that the mixed Poisson random variable behaves as a Poisson random variable with parameter $n\nu$. If on the contrary α is small, one would expect that detailed characteristics of the distribution of the X_i s matter. For $\alpha = 1$ both effects play a role. This intuition underlies nearly all results presented in this chapter.

Number of customers in an infinite-server queue. In the second part of this chapter we focus on the infinite-server queue, where the arrival process is the overdispersed process we introduced above, and the service times are i.i.d. samples from a (non-negative) distribution with distribution function $G(\cdot)$. The number of clients in this infinite-server queue, under the arrival process described above, is studied in [64]. As it turns out, one can prove the (plausible) property that the number of clients in the system at time t (which we, for simplicity, assume to be a multiple of Δ), has a *mixed Poisson* distribution, i.e., a Poisson distribution with random parameter. This parameter is given by

$$\sum_{i=1}^{t/\Delta} X_i \, \Delta \, f_i(t, \Delta),$$

where $f_i(t, \Delta)$ denotes the probability that a call arriving at a uniformly distributed epoch in the interval $[(i-1)\Delta, i\Delta)$ is still in the system at time t. Evidently, for small Δ this probability essentially behaves as $\overline{F}(t - i\Delta)$, with $\overline{F}(\cdot) := 1 - F(\cdot)$ denoting the complementary distribution function.

We renormalize time such that $t \equiv 1$ (which can be done without loss of generality), and again impose the scaling along the lines of [64]: the arrival rates are nX_i and the interval width $n^{-\alpha}$. Then the number of clients in the system at time 1 is Poisson with random parameter

$$\sum_{i=1}^{n^\alpha} (nX_i) \, n^{-\alpha} \, f_i(1, n^{-\alpha}) = n^{1-\alpha} \sum_{i=1}^{n^\alpha} X_i \, \omega_i(n^\alpha), \tag{3.1}$$

where $\omega_i(n) := f_i(1, n^{-1}) \approx \overline{F}(1 - i/n)$. A clearly relevant object of study concerns the probability that the number of clients in the system exceeds some threshold na:

$$Q_n(a) := \mathbb{P}\left(\mathrm{Pois}\left(n^{1-\alpha} \sum_{i=1}^{n^\alpha} X_i \, \omega_i(n^\alpha) \right) \geq na \right);$$

$q_n(a)$ denotes the corresponding probability that the mixed Poisson random variable equals na. To ensure that the event under consideration is rare, a is assumed to be larger than

$$\frac{\nu}{n^\alpha} \sum_{i=1}^{n^\alpha} \omega_i(n^\alpha) \approx \frac{\nu}{n^\alpha} \sum_{i=1}^{n^\alpha} \overline{G}(1 - i/n^\alpha) \approx \nu \int_0^1 \overline{G}(x) \mathrm{d}x.$$

A related question of practical interest concerns *staffing*: how many servers should be allocated to ensure a given service level. Approximating the many-server model by its infinite-server counterpart, a staffing rule could be: find the smallest a such that $Q_n(a)$ (or $q_n(a)$) is below some desired (typically small) ε. The resulting procedure has evident applications in the context of call centres, but also has the potential to be applied in cloud computing or in the design of data centres. As we focus on the LD setting, the technique

we develop is specifically useful in the regime in which the performance requirements are strict (in the sense that the probability of service degradation should be kept low).

The contributions and organisation of this chapter are as follows. In Section 3.1 we focus on the evaluation of the probabilities $P_n(a)$ and $p_n(a)$. We first briefly present the logarithmic asymptotics. In Section 3.1.1 we use elementary techniques to derive the exact asymptotics, however, as it turns out, these only apply when the time scales of the arrival process and the resampling are sufficiently separated: we address the cases $\alpha < \frac{1}{3}$ and $\alpha > 3$ (with a partial solution for $\alpha \in [\frac{1}{3}, \frac{1}{2})$ and $\alpha \in [2, 3)$ in terms of an asymptotic lower bound). In Section 3.1.2 it becomes clear why such elementary techniques do not work across all values of α: for the important special case of the X_is corresponding to i.i.d. gamma distributed random variables [71] we find the exact asymptotics for all $\alpha > 0$, and in the range $(\frac{1}{2}, 2) \setminus \{1\}$ these turn out to have a rather intricate shape.

Section 3.1.3 focuses on rare-event simulation as a means to find an accurate approximation at relatively low computational cost: we propose an importance-sampling based technique, which we prove to be asymptotically efficient.

In Section 3.2 we shift our attention to the probabilities $Q_n(a)$ and $q_n(a)$. Again, logarithmic asymptotics and efficient simulation procedures can be found, and in addition we manage to identify the exact asymptotics for the case $\alpha = 1$. By a series of numerical examples it is illustrated how the resulting approximation can be used for staffing purposes. We performed extensive experiments, and make the striking observation that increasing the variability of the service times (e.g. Pareto service times rather than exponential ones) often leads to less conservative staffing rules.

3.1 Tail probabilities of the arrival process

We start by introducing the framework that we consider throughout the chapter. In our set-up we let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed as a generic random variable X, where $\nu := \mathbb{E}X_i$. Assume that Assumptions I.1 and I.2 apply, and that X is non-lattice. Then by the Bahadur-Rao result presented in Section 1.1.1 we have the following exact asymptotics for $\overline{X}_n$: when $a > \nu$,

$$\lim_{n\to\infty} \mathbb{P}\left(\overline{X}_n \geq a\right) \mathrm{e}^{nI_X(a)}\sqrt{n} = C_X(a)$$

with $C_X(a)$ as given in (1.7).

In our analysis the tail asymptotics of Poisson random variables play a crucial role. We note that the Bahadur-Rao asymptotics entail that for the probabilities

$$\psi_n(a \,|\, x) := \mathbb{P}\left(\mathrm{Pois}\,(nx) \geq na\right), \tag{3.2}$$

it holds that

$$\lim_{n\to\infty} \psi_n(a \,|\, x)\mathrm{e}^{n\, I(a\,|\,x)}\sqrt{n} = C(a \,|\, x), \tag{3.3}$$

for $a > x$, where $C(a \,|\, x)$ is given by (1.8), that is,

$$C(a \,|\, x) := \frac{1}{1-\exp\left(\theta^\star\right)} \frac{1}{\sqrt{2\pi\Lambda''(\theta^\star)}} = \frac{1}{1-a/x} \frac{1}{\sqrt{2\pi a}}.$$

As before, $I(\cdot \,|\, x)$ denotes the rate function associated with a Poisson random variable with parameter x, that is, recalling (1.1), we have $I(a \,|\, x) = a\log(a/x) - a + x$.

Let us first present the logarithmic asymptotics of $P_n(a)$ (the same logarithmic asymptotics hold for $p_n(a)$). Here we merely state the results as the proof is exactly as in [64, Section 4.1]. We distinguish between the cases $\alpha > 1$ and $\alpha < 1$; the former case we refer to as the *fast* regime as the X_i's are sampled relatively frequently, whereas the latter case is the *slow* regime. For completeness, the logarithmic asymptotics for the intermediate case $\alpha = 1$, though standard, are included as well.

- In the fast regime n^α is substantially larger than n, and hence the rare event will be essentially due to $\overline{X}_{n^\alpha}$ being close to ν, and the Poisson random variable with parameter (roughly) $n\nu$ exceeding na. Accordingly, following the argumentation in [64], one obtains
$$\lim_{n\to\infty} \frac{1}{n} \log P_n(a) = -I(a \,|\, \nu).$$
This result entails that $P_n(a)$ decays essentially exponentially.

- In the slow regime, assuming the support of X_i is unbounded, the rare event will be a consequence of the joint effect of (i) $\overline{X}_{n^\alpha}$ being close to a, and (ii) the Poisson variable with parameter (roughly) na attaining a typical value; the first event is rare, but the second is not. In this regime, we thus have
$$\lim_{n\to\infty} \frac{1}{n^\alpha} \log P_n(a) = -I_X(a);$$
observe that this corresponds to subexponential decay.

- For $\alpha = 1$, the random variable $\text{Pois}\left(n\overline{X}_{n^\alpha}\right)$ can be written as the sum of n i.i.d. contributions, each of them distributed as $Z := \text{Pois}(X)$. Noting that
$$\log \mathbb{E} \exp\left(\theta Z\right) = \Lambda_X(\mathrm{e}^\theta - 1),$$
a straightforward application of Cramér's theorem [38] yields that the decay is exponential:
$$\lim_{n\to\infty} \frac{1}{n} \log P_n(a) = -\sup_\theta \left(\theta a - \Lambda_X(\mathrm{e}^\theta - 1)\right) =: I_Z(a). \tag{3.4}$$

We derive the exact asymptotics of $P_n(a)$ and $p_n(a)$ in the next section.

3.1.1 Exact asymptotics

In this section we show that for a range of values of α the exact asymptotics of $P_n(a)$ and $p_n(a)$ can be found by relying on elementary probabilistic techniques. We focus on the fast regime in Section 3.1.1.1, and on the slow regime in Section 3.1.1.2. We conclude with the exact asymptotics for the intermediate case $\alpha = 1$, which follow directly from the Bahadur-Rao result.

3.1.1.1 Fast regime

In this section we assume that $\alpha > 1$. We start by proving an upper bound for $P_n(a)$. In self-evident notation, we have

$$P_n(a) = \int_0^\infty \psi_n(a \,|\, x)\, \mathbb{P}\left(\overline{X}_{n^\alpha} \in \mathrm{d}x\right), \tag{3.5}$$

with $\psi_n(a \,|\, x)$ as defined in (3.2). For any δ, Eq. (3.5) is majorised by

$$\int_{\nu-n^\delta}^{\nu+n^\delta} \psi_n(a \,|\, x)\, \mathbb{P}\left(\overline{X}_{n^\alpha} \in \mathrm{d}x\right) + \mathbb{P}\left(\left|\, \overline{X}_{n^\alpha} - \nu \,\right| \geq n^\delta\right); \tag{3.6}$$

we determine an appropriate value for δ later-on. The first term in (3.6) is evidently bounded from above by $\psi_n(a \,|\, \nu + n^\delta)$. Motivated by (3.3), we will show that, as $n \to \infty$,

$$\psi_n(a \,|\, \nu + n^\delta)\, \mathrm{e}^{n\, I(a \,|\, \nu)} \sqrt{n} \to C(a \,|\, \nu), \tag{3.7}$$

whereas the second term in (3.6) turns out to be asymptotically negligible.

To verify that (3.7) holds, note that $C(a \,|\, \nu)/C(a \,|\, \nu + n^\delta) \to 1$ when $\delta < 0$, which follows by a standard continuity argument. We therefore proceed by considering $n\, I(a \,|\, \nu) - n\, I(a \,|\, \nu + n^\delta)$, which behaves as

$$\begin{aligned} & n\left(a \log \frac{a}{\nu} + a - \nu\right) - n\left(a \log \frac{a}{\nu + n^\delta} + a - (\nu + n^\delta)\right) \\ = \;& na \log\left(1 + \frac{n^\delta}{\nu}\right) + n^{1+\delta} = \left(\frac{a}{\nu} + 1\right) n^{1+\delta} + O(n^{1+2\delta}) \to 0 \end{aligned}$$

if $\delta < -1$. Thus, for such δ we have established (3.7).

Now consider the second term of (3.6), and, more specifically,

$$\mathbb{P}\left(\left|\, \overline{X}_{n^\alpha} - \nu \,\right| \geq n^\delta\right)\, \mathrm{e}^{n\, I(a \,|\, \nu)} \sqrt{n}, \tag{3.8}$$

for $n \to \infty$. Due to a Chernoff bound, we have

$$\mathbb{P}\left(\overline{X}_{n^\alpha} \geq \nu + n^\delta\right) \leq \exp\left(-n^\alpha \sup_\theta \left(\theta(\nu + n^\delta) - \log \mathbb{E}\, \mathrm{e}^{\theta X_i}\right)\right) = \mathrm{e}^{-n^\alpha I_X(\nu + n^\delta)},$$

and hence (3.8) is majorised by

$$\mathrm{e}^{-n^\alpha I_X(\nu+n^\delta)}\, \mathrm{e}^{n\, I(a \,|\, \nu)} \sqrt{n} + \mathrm{e}^{-n^\alpha I_X(\nu-n^\delta)}\, \mathrm{e}^{n\, I(a \,|\, \nu)} \sqrt{n}.$$

Now realize that $I_X(\nu + n^\delta) = \frac{1}{2} I_X''(\nu) n^{2\delta} + O(n^{3\delta})$ and similarly for $I_X(\nu - n^\delta)$. Thus, the expression from the previous display vanishes when $\alpha + 2\delta > 1$, or, equivalently, $\delta > (1 - \alpha)/2$, where $(1 - \alpha)/2 < 0$ since $\alpha > 1$.

We note that the requirements $\delta < -1$ (corresponding to the first term) and $\delta > (1 - \alpha)/2$ (corresponding to the second term) are both fulfilled when $\alpha > 3$. Thus, we have shown that for $\alpha > 3$ an asymptotic upper bound for $P_n(a)$ is given by (3.7).

Let us now turn to the corresponding lower bound. The probability of interest ma-

jorises

$$\psi_n(a \,|\, \nu - n^\delta) \int_{\nu-n^\delta}^{\nu+n^\delta} \mathbb{P}\left(\overline{X}_{n^\alpha} \in \mathrm{d}x\right).$$

As above, we can check that for $\delta < -1$,

$$\psi_n(a \,|\, \nu - n^\delta)\, \mathrm{e}^{n\, I(a\,|\,\nu)} \sqrt{n} \to C(a \,|\, \nu),$$

and, by the Bahadur-Rao result (1.6),

$$\int_{\nu-n^\delta}^{\nu+n^\delta} \mathbb{P}\left(\overline{X}_{n^\alpha} \in \mathrm{d}x\right) \sim 1 - 2\exp\left(-\frac{1}{2} I_X''(\nu) n^\alpha n^{2\delta}\right) \to 1,$$

when $\delta > -\alpha/2$. This can be realized when $\alpha > 2$ (and is hence fulfilled when $\alpha > 3$ as well). This proves the lower bound.

Combining the upper and lower bounds, we thus find the following result.

Proposition 3.1. *For $\alpha > 3$, as $n \to \infty$,*

$$P_n(a) \sim \mathrm{e}^{-n\, I(a\,|\,\nu)} \frac{C(a \,|\, \nu)}{\sqrt{n}}.$$

For $\alpha \in (2, 3]$,

$$\liminf_{n\to\infty} P_n(a)\, \mathrm{e}^{n\, I(a\,|\,\nu)} \sqrt{n} \geq C(a \,|\, \nu).$$

Remark 3.2. This result is in accordance with the intuition we gave at the beginning of the section – in the fast regime the asymptotics of $P_n(a)$ depend on the distribution of the X_i only through their mean ν. This also gives an indication as to why the asymptotics for α closer to 1 may be more delicate to deal with. One can imagine that for more moderate values of α the result may not be precise enough, and that also large deviations coming from $\overline{X}_{n^\alpha}$ may play a role in that regime. This is confirmed in Section 3.1.2, where we consider an example with $X_i \sim \mathrm{Exp}(\lambda)$. It turns out that the exact asymptotic expression for $\alpha \in (1, 2)$ is indeed more intricate than the expression provided in Theorem 3.1. ♢

Remark 3.3. Along the same lines the asymptotics for $p_n(a)$ can be found. They turn out to be, for $\alpha > 3$, as $n \to \infty$,

$$p_n(a) \sim \mathrm{e}^{-n\, I(a\,|\,\nu)} \frac{C(a \,|\, \nu)}{\sqrt{n}} \left(1 - \mathrm{e}^{-I'(a\,|\,\nu)}\right).$$

This is in line with the result of Proposition 3.1: informally,

$$\begin{aligned} p_n(a) &= P_n(a) - P_n(a + 1/n) \\ &\approx \frac{C(a \,|\, \nu)}{\sqrt{n}} \mathrm{e}^{-n\, I(a\,|\,\nu)} - \frac{C(a + 1/n \,|\, \nu)}{\sqrt{n}} \mathrm{e}^{-n\, I(a+1/n\,|\,\nu)} \\ &\approx \frac{C(a \,|\, \nu)}{\sqrt{n}} \mathrm{e}^{-n\, I(a\,|\,\nu)} \left(1 - \mathrm{e}^{-I'(a\,|\,\nu)}\right), \end{aligned}$$

for large n, based on elementary Taylor arguments. ♢

3.1.1.2 Slow regime

We now consider the slow regime, i.e., $\alpha < 1$. We have to distinguish between two cases.

- In Case I we assume that X_i may have outcomes larger than a with positive probability:
$$b_+ := \sup\{b : \mathbb{P}(X_i > b) > 0\} > a;$$
as a consequence $I_X(a) < \infty$. Recall that in this case, for n^α substantially smaller than n, it can be argued that $P_n(a)$ essentially behaves as $\mathbb{P}(\overline{X}_{n^\alpha} \geq a)$.

- In Case II we consider the opposite situation: $b_+ < a$. Then the intuition is that the rare event under consideration is the consequence of large deviations of both random components: of (i) $\overline{X}_{n^\alpha}$ being close to b_+, and (ii) the Poisson variable with parameter (roughly) nb_+ attaining the atypical value na.

We present here only the analysis for Case I; the derivation for Case II relies on the same ideas but is more intricate, details can be found in [63]. As in the fast regime, we start by establishing an upper bound. Note that $P_n(a)$ is majorised by

$$\mathbb{P}\left(\overline{X}_{n^\alpha} \geq a - n^\delta\right) + \psi_n(a \,|\, a - n^\delta).$$

Due to the Bahadur-Rao result, the first term is asymptotically equivalent to

$$n^{-\alpha/2} C_X(a - n^\delta) \mathrm{e}^{-n^\alpha I_X(a - n^\delta)},$$

which behaves as $n^{-\alpha/2} C_X(a) \mathrm{e}^{-n^\alpha I_X(a)}$ when $\delta < -\alpha$ (as a direct consequence of the standard expansion $I_X(a - n^\delta) = I_X(a) - n^\delta I'_X(a) + O(n^{2\delta})$). In addition, again using the Chernoff bound, we have

$$\mathrm{e}^{n^\alpha I_X(a)}\, \psi_n(a \,|\, a - n^\delta) \leq \mathrm{e}^{n^\alpha I_X(a)} \exp\left(-n\left(a \log \frac{a}{a - n^\delta} + n^\delta\right)\right). \tag{3.9}$$

Observe that the exponent in the second factor of the right hand side of (3.9) behaves as $n^{2\delta+1}$. We conclude that (3.9) vanishes if $2\delta + 1 > \alpha$, or, equivalently, $\delta > (\alpha - 1)/2$ (note that $(\alpha - 1)/2 < 0$). In order to simultaneously meet $\delta < -\alpha$ and $\delta > (\alpha - 1)/2$, we need to have $\alpha < \frac{1}{3}$.

We now turn to the lower bound. The probability of interest is bounded from below by

$$\psi_n(a \,|\, a + n^\delta)\, \mathbb{P}\left(\overline{X}_{n^\alpha} \geq a + n^\delta\right).$$

The first factor is bounded from below by 1 minus a term that decays as $\exp(-n^{1+2\delta})$ (which goes to 1 when $\delta > -\frac{1}{2}$), whereas the second behaves as $n^{-\alpha/2} C_X(a) \mathrm{e}^{-n^\alpha I_X(a)}$ when $\delta < -\alpha$. In other words, there is an appropriate δ for all $\alpha < \frac{1}{2}$. We have thus arrived at the following result.

Proposition 3.4. *Assume $b_+ > a$. For $\alpha < \frac{1}{3}$, as $n \to \infty$,*

$$P_n(a) \sim \mathrm{e}^{-n^\alpha I_X(a)} \frac{C_X(a)}{n^{\alpha/2}}.$$

For $\alpha \in [\frac{1}{3}, \frac{1}{2})$,

$$\liminf_{n\to\infty} P_n(a)\, \mathrm{e}^{n^\alpha I_X(a)} n^{\alpha/2} \geq C_X(a).$$

Remark 3.5. Note that here, in contrast with Proposition 3.1, the rate function is that of X rather than the Poisson random variable. As expected, when α is small, the rare event is typically a result of a large deviation of $\overline{X}_{n^\alpha}$. However, for values of α closer to 1 the same reasoning as in Remark 3.2 applies, and we do not expect a simple asymptotic expression as given in Proposition 3.4 to hold for all $\alpha \in (\frac{1}{3}, 1)$ (as will be confirmed in Section 3.1.2, which covers the special case in which the X_i are exponentially distributed). ♢

Remark 3.6. As in Remark 3.3, the asymptotics for $p_n(a)$ can be found as well. As it turns out, as $n \to \infty$,

$$p_n(a) \sim \mathrm{e}^{-n^\alpha I_X(a)} \frac{C_X(a) I'_X(a)}{n^{1-\alpha/2}}.$$

This is consistent with the result stated in Proposition 3.4:

$$\begin{aligned} p_n(a) &= P_n(a) - P_n(a + 1/n) \\ &\approx \frac{C_X(a)}{n^{\alpha/2}} \mathrm{e}^{-n^\alpha I_X(a)} - \frac{C_X(a + 1/n)}{n^{\alpha/2}} \mathrm{e}^{-n^\alpha I_X(a+1/n)} \\ &\approx \frac{C_X(a)}{n^{\alpha/2}} \mathrm{e}^{-n^\alpha I_X(a)} \left(1 - \mathrm{e}^{-n^{\alpha-1} I'_X(a)}\right) \approx C_X(a) I'_X(a) \mathrm{e}^{-n^\alpha I_X(a)} n^{\alpha/2-1}, \end{aligned}$$

for large n. Note that the asymptotic expansion of $P_n(a)$ has a polynomial factor $n^{-\alpha/2}$, whereas $p_n(a)$ has a polynomial factor $n^{\alpha/2-1}$. So in this case $P_n(a)$ and $p_n(a)$ are not (asymptotically) off by a constant, but by a constant multiplied by $n^{\alpha-1}$. ♢

3.1.1.3 Intermediate range

We finally consider the case $\alpha = 1$. The random variable $\mathrm{Pois}(n\overline{X}_{n^\alpha})$ is distributed as the sum of n i.i.d. contributions, each of them distributed as $Z := \mathrm{Pois}(X)$. Assuming that maximum in the definition (3.4) of $I_Z(a)$ is attained at θ^*_Z, the Bahadur-Rao result yields, as $n \to \infty$,

$$P_n(a) \sim \mathrm{e}^{-nI_Z(a)} \frac{C_Z(a)}{\sqrt{n}},$$

where now

$$\begin{aligned} C_Z(a) :=& \frac{1}{1 - \mathrm{e}^{\theta^\star_Z}} \frac{1}{\sqrt{2\pi \Lambda''_Z(\theta^\star_Z)}} \\ =& \frac{1}{1 - \mathrm{e}^{\theta^\star_Z}} \frac{1}{\sqrt{2\pi \left(\mathrm{e}^{\theta^\star_Z} \Lambda'_X\left(\mathrm{e}^{\theta^\star_Z} - 1\right) + \mathrm{e}^{2\theta^\star_Z} \Lambda''_X\left(\mathrm{e}^{\theta^\star_Z} - 1\right)\right)}} \\ =& \frac{1}{1 - \mathrm{e}^{\theta^\star_Z}} \frac{1}{\sqrt{2\pi \left(a + \mathrm{e}^{2\theta^\star_Z} \Lambda''_X\left(\mathrm{e}^{\theta^\star_Z} - 1\right)\right)}}. \end{aligned}$$

Based on the same arguments as in Remark 3.3 we infer that

$$p_n(a) \sim \frac{1}{\sqrt{2\pi n \left(a + \mathrm{e}^{2\theta^\star_Z} \Lambda''_X\left(\mathrm{e}^{\theta^\star_Z} - 1\right)\right)}} \mathrm{e}^{-nI_Z(a)}.$$

3.1.2 Example with gamma arrival rates

In this section we consider the special case that the X_i s are i.i.d. samples from the gamma distribution. The use of this specific mixed Poisson distribution for call center staffing purposes is advocated in e.g. [71]. In the analysis, this can be reduced to the case where the X_i s are exponentially distributed with parameter λ (i.e., mean λ^{-1}), see Remark 3.8.

To start the exposition, we note that if the X_i s are exponential with parameter λ, then $\sum_{i=1}^{n^\alpha} X_i$ has a gamma distribution with parameters n^α and λ. The objective of this section is to evaluate the asymptotics of $p_n(a)$ across all values of α; later we comment on what the corresponding $P_n(a)$ looks like. We assume throughout that a is larger than λ^{-1}. The computations are facilitated by the fact that an exact expression for $p_n(a)$ is available. It takes a routine calculation, which we include for completeness, to compute $p_n(a)$:

$$\begin{aligned}
p_n(a) &= \int_0^\infty \frac{(n^{1-\alpha}x)^{na}}{(na)!}\mathrm{e}^{-(\lambda+n^{1-\alpha})x}\frac{\lambda^{n^\alpha}}{(n^\alpha-1)!}x^{n^\alpha-1}\,\mathrm{d}x \\
&= \frac{(n^{1-\alpha})^{na}}{(na)!}\frac{\lambda^{n^\alpha}}{(n^\alpha-1)!}\int_0^\infty \mathrm{e}^{-(\lambda+n^{1-\alpha})x}x^{na+n^\alpha-1}\,\mathrm{d}x \\
&= \frac{(na+n^\alpha-1)!}{(na)!(n^\alpha-1)!}\frac{(n^{1-\alpha})^{na}\lambda^{n^\alpha}}{(\lambda+n^{1-\alpha})^{na+n^\alpha}} \\
&\qquad\qquad \int_0^\infty \frac{(\lambda+n^{1-\alpha})^{n^\alpha}}{(na+n^\alpha-1)!}\mathrm{e}^{-(\lambda+n^{1-\alpha})x}x^{na+n^\alpha-1}\,\mathrm{d}x \\
&= \binom{na+n^\alpha-1}{na}\left(\frac{n^{1-\alpha}}{\lambda+n^{1-\alpha}}\right)^{na}\left(\frac{\lambda}{\lambda+n^{1-\alpha}}\right)^{n^\alpha}.
\end{aligned}$$

Remark 3.7. We recognize here the probability that a *negative binomially distributed* random variable with success probability $p := n^{1-\alpha}/(\lambda+n^{1-\alpha})$ attains na successes before n^α failures have occurred. This can be understood as follows. Note that a Poisson random variable with parameter xT represents the number of $\mathrm{Exp}(x)$ "success clocks" expiring within a period of length T. In our case the rate of the success clocks is $x = n^{1-\alpha}$ and the length of the period corresponds to the time it takes for n exponential "failure clocks" of rate λ to expire, that is, we have $T = \sum_{i=1}^{n^\alpha} X_i$. Thus, $p_n(a)$ is the probability that na success clocks expire before the n^αth failure clock expires and the period ends. The success probability is indeed given by p as it is the probability that the next $\mathrm{Exp}(n^{1-\alpha})$ success clock expires before a $\mathrm{Exp}(\lambda)$ failure clock. ♢

Remark 3.8. In the above setup we considered exponentially distributed X_i s. Note, however, that our analysis only relies on $\sum_{i=1}^{n^\alpha} X_i$ having a gamma distribution, and thus can easily be extended to the practically relevant case [71] that the X_i s are i.i.d. samples from a gamma distribution. It is noted that the gamma distribution has two parameters (as opposed to the exponential distribution), and therefore allows for more modelling flexibility (e.g., the mean and variance can be fitted). ♢

As a first step in deriving the exact asymptotics of $p_n(a)$, we approximate the binomial coefficients by applying Stirling's formula, which says that $n! \sim \sqrt{2\pi n}\, n^n \mathrm{e}^{-n}$. As a

consequence we find that

$$\binom{na+n^\alpha-1}{na} \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{na+n^\alpha-1}}{\sqrt{na}\sqrt{n^\alpha-1}} \frac{(na+n^\alpha-1)^{na+n^\alpha-1}}{(na)^{na}(n^\alpha-1)^{n^\alpha-1}}$$

Applying this in the expression for $p_n(a)$ then yields

$$\begin{aligned} p_n(a) &= \binom{na+n^\alpha-1}{na}\left(\frac{n^{1-\alpha}}{\lambda+n^{1-\alpha}}\right)^{na}\left(\frac{\lambda}{\lambda+n^{1-\alpha}}\right)^{n^\alpha} \\ &\sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{na+n^\alpha-1}}{\sqrt{na}\sqrt{n^\alpha-1}} \frac{(na+n^\alpha-1)^{na+n^\alpha-1}}{(na)^{na}(n^\alpha-1)^{n^\alpha-1}} \\ &\qquad \times \left(\frac{n^{1-\alpha}}{\lambda+n^{1-\alpha}}\right)^{na}\left(\frac{\lambda}{\lambda+n^{1-\alpha}}\right)^{n^\alpha} \\ &= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{n^\alpha-1}}{\sqrt{na}\sqrt{na+n^\alpha-1}} \left(\frac{na+n^\alpha-1}{a\lambda(n^\alpha+\frac{n}{\lambda})}\right)^{na} \\ &\qquad \times \left(\frac{\lambda(na+n^\alpha-1)}{(\lambda+n^{1-\alpha})(n^\alpha-1)}\right)^{n^\alpha}. \end{aligned} \tag{3.10}$$

In order to determine the asymptotic behaviour of this expression for large n, we again consider the three regimes separately. We do so by evaluating the three factors in (3.10).

3.1.2.1 Fast regime

We start by examining the case $\alpha > 1$. For the first factor we have

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{n^\alpha-1}}{\sqrt{na}\sqrt{na+n^\alpha-1}} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{na}}\,.$$

The middle factor can be addressed as follows. For ease we analyse its logarithm:

$$\begin{aligned} na\log\left(\frac{na+n^\alpha-1}{a\lambda(n^\alpha+n/\lambda)}\right) &= -na\log(a\lambda) + na\log\left(1+n^{1-\alpha}a-n^{-\alpha}\right) \\ &\quad - na\log\left(1+n^{1-\alpha}/\lambda\right) \end{aligned} \tag{3.11}$$

For the last factor we similarly obtain

$$\begin{aligned} n^\alpha\log\left(\frac{\lambda(na+n^\alpha-1)}{(\lambda+n^{1-\alpha})(n^\alpha-1)}\right) &= n^\alpha\log(1+n^{1-\alpha}a-n^{-\alpha}) - \\ &\quad n^\alpha\log\left(1+\frac{1}{\lambda}n^{1-\alpha}-\frac{1}{\lambda}n^{1-2\alpha}-n^{-\alpha}\right) \end{aligned} \tag{3.12}$$

Define $\overline{k} := (\alpha-1)^{-1}$ and $k_+ := \lfloor \overline{k} \rfloor$. The logarithms can be expanded relying on their standard Taylor series form, but it can be argued that the resulting infinite series

can be truncated. For instance,

$$\begin{aligned} na\log\left(1+n^{1-\alpha}a-n^{-\alpha}\right) &= na\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(n^{(1-\alpha)}a-n^{-\alpha})^k \\ &\sim na\sum_{k=1}^{k_+}\frac{(-1)^{k+1}a^k}{k}n^{(1-\alpha)k}. \end{aligned}$$

Likewise,

$$na\log\left(1+n^{1-\alpha}/\lambda\right) \sim na\sum_{k=1}^{k_+}\frac{(-1)^{k+1}\lambda^{-k}}{k}n^{(1-\alpha)k}.$$

We thus find that (3.11) asymptotically equals

$$-na\log(a\lambda)+na\sum_{k=1}^{k_+}\frac{(-1)^k(\lambda^{-k}-a^k)}{k}n^{(1-\alpha)k}.$$

For the last factor, note that from $k_+ + 1$ on all terms vanish, leaving us with

$$\begin{aligned} n^{\alpha}\log(1+n^{1-\alpha}a-n^{-\alpha}) &\sim n^{\alpha}\sum_{k=1}^{k_++1}\frac{(-1)^{k+1}a^k}{k}n^{(1-\alpha)k}-1, \\ n^{\alpha}\log(1+n^{1-\alpha}/\lambda-n^{1-2\alpha}/\lambda-n^{-\alpha}) &\sim n^{\alpha}\sum_{k=1}^{k_++1}\frac{(-1)^{k+1}\lambda^{-k}}{k}n^{(1-\alpha)k}-1. \end{aligned}$$

After a bit of rewriting, we conclude that (3.12) equals

$$n\sum_{k=0}^{k_+}\frac{(-1)^k(a^{k+1}-\lambda^{-(k+1)})}{k+1}n^{(1-\alpha)k}.$$

Defining

$$\xi_0 := -a\log(\lambda a)+a-\frac{1}{\lambda}, \quad \xi_k := (-1)^k\left(\lambda^{-k}\left(\frac{a}{k}-\frac{1/\lambda}{k+1}\right)-a^{k+1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right),$$

we conclude that in case $\alpha > 1$,

$$p_n(a) \sim \frac{1}{\sqrt{2\pi an}}\,\mathrm{e}^{\xi_0 n}\exp\left(\sum_{k=1}^{k_+}\xi_k n^{(1-\alpha)k+1}\right).$$

In particular, if $\alpha > 2$, then the last factor equals 1 (the empty sum being defined as 0). It is not hard to check that this result agrees with what has been found for $\alpha > 3$ in Section 3.1.1.

3.1.2.2 Slow regime

If $\alpha < 1$, then the first factor behaves as

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{n^\alpha - 1}}{\sqrt{na}\sqrt{na + n^\alpha - 1}} \sim \frac{1}{\sqrt{2\pi}} \frac{1}{a} n^{\alpha/2-1} .$$

For the logarithm of the middle factor we now have

$$\begin{aligned} & na \log \left(\frac{na + n^\alpha - 1}{a\lambda(n^\alpha + n/\lambda)} \right) \\ & \quad = na \log \left(1 + \frac{1}{a}(n^{\alpha-1} - n^{-1}) \right) - na \log \left(1 + \lambda n^{\alpha-1} \right) . \end{aligned} \tag{3.13}$$

With $\widetilde{k} := \alpha(1-\alpha)^{-1}$ and $k_- := \lfloor \widetilde{k} \rfloor$, we obtain that this factor asymptotically equals

$$\begin{aligned} & na \sum_{k=1}^{k_-+1} \frac{(-1)^{k+1}(a^{-k} - \lambda^k)}{k} n^{(\alpha-1)k} - 1 \\ & \quad = na \sum_{k=0}^{k_-} \frac{(-1)^k (a^{-(k+1)} - \lambda^{k+1})}{k+1} n^{(\alpha-1)(k+1)} - 1 . \end{aligned}$$

For the last factor we find

$$\begin{aligned} n^\alpha \log \left(\frac{\lambda(na + n^\alpha - 1)}{(\lambda + n^{1-\alpha})(n^\alpha - 1)} \right) = {} & n^\alpha \log (\lambda a) + n^\alpha \log \left(1 + \frac{1}{a} n^{\alpha-1} - \frac{1}{a} n^{-1}) \right) \\ & - n^\alpha \log \left(1 + \lambda n^{\alpha-1} - \lambda n^{-1} - n^{-\alpha} \right) \end{aligned}$$

where

$$\begin{aligned} n^\alpha \log \left(1 + \frac{1}{a} n^{\alpha-1} - \frac{1}{a} n^{-1} \right) & \sim n^\alpha \sum_{k=1}^{k_-} \frac{(-1)^{k+1}}{k} a^{-k} n^{(\alpha-1)k} , \\ n^\alpha \log \left(1 + \lambda n^{\alpha-1} - \lambda n^{-1} - n^{-\alpha} \right) & \sim n^\alpha \sum_{k=1}^{k_-} \frac{(-1)^{k+1}}{k} \lambda^k n^{(\alpha-1)k} - 1 . \end{aligned}$$

Combining the above we conclude

$$p_n(a) \sim \frac{1}{\sqrt{2\pi a}} n^{\frac{\alpha}{2}-1} \, \mathrm{e}^{\zeta_0 n^\alpha} \exp \left(\sum_{k=1}^{k_-} \zeta_k n^{(\alpha-1)k+\alpha} \right) , \tag{3.14}$$

where

$$\zeta_0 := \log(\lambda a) + 1 - \lambda a , \quad \zeta_k := (-1)^k \left(\lambda^k \left(\frac{1}{k} - \frac{a\lambda}{k+1} \right) - a^{-k} \left(\frac{1}{k} - \frac{1}{k+1} \right) \right) .$$

It can again be verified that this result coincides for $\alpha < \frac{1}{3}$ with the one derived in Section 3.1.1.

3.1.2.3 Intermediate regime

For completeness, we also include the result for the case $\alpha = 1$. We find

$$p_n(a) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{na(a+1)}} \exp\left(-n\left(a\log\left(a\,\frac{1+\lambda}{1+a}\right) + \log\left(\frac{1}{\lambda}\,\frac{1+\lambda}{1+a}\right)\right)\right). \tag{3.15}$$

It is noted that the asymptotics of $P_n(a)$ and $p_n(a)$ could have been found by applying the Bahadur-Rao result directly:

$$P_n(a) \sim \frac{1}{1-\mathrm{e}^{\theta^\star}} \frac{1}{\sqrt{2\pi n \Lambda_Z''(\theta_Z^\star)}} \mathrm{e}^{-nI_Z(a)} = \frac{1}{1-a\,\frac{1+\lambda}{1+a}} \frac{1}{\sqrt{2\pi na(a+1)}} \mathrm{e}^{-nI_Z(a)}.$$

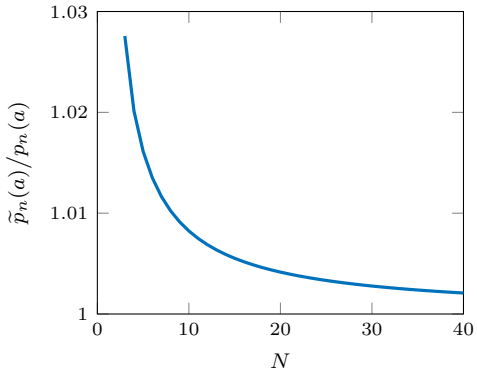
and

$$p_n(a) = P_n(a) - P_n\left(a+\frac{1}{n}\right) \sim \frac{1}{\sqrt{2\pi na(a+1)}} \mathrm{e}^{-nI_Z(a)},$$

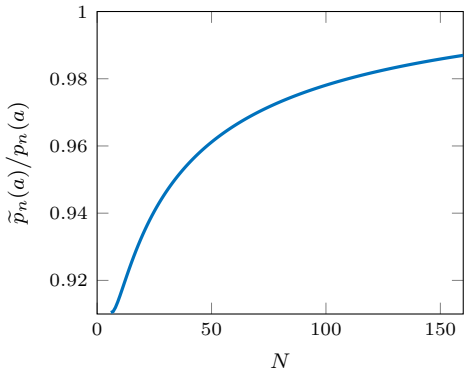
where it can be verified that $I_Z(a)$ coincides with the exponent found in (3.15).

3.1.2.4 Example

In Figure 3.1 we illustrate the accuracy of the approximation, by displaying the ratio of the approximation $\widetilde{p}_n(a)$ and the exact expression for $p_n(a)$. As expected, we observe that this ratio tends to 1 as n grows.



(a) Fast regime, $\alpha = 5$.



(b) Slow regime, $\alpha = \frac{1}{5}$.

Figure 3.1: Ratio of approximation $\widetilde{p}_n(a)$ and exact value $p_n(a)$, where X_i is exponentially distributed with parameter $\lambda = 2.5$ and $a = 1$.

3.1.3 Importance sampling

In the previous sections we found exact asymptotics for the rare-event probabilities $p_n(a)$ and $P_n(a)$ for (i) a specific range of α, and (ii) for the specific case that the X_i are exponentially distributed. To facilitate numerical evaluation (which we need, for example, if (i) and (ii) do not apply), we propose in this section importance sampling estimators for $p_n(a)$ and $P_n(a)$. We establish asymptotic efficiency properties, thus guaranteeing fast

computation even for large n. As before, we distinguish the cases $\alpha < 1$ and $\alpha > 1$; the case $\alpha = 1$ can be addressed by using a classical importance sampling procedure.

3.1.3.1 Fast regime

Recall that in this regime a rare event is typically the result of a large deviation of the Poisson random variable, while the sample mean $X_1, \ldots, X_{n^\alpha}$ will typically be close to ν (under their true distribution, which we shall indicate by a subscript ν). In view of this, we propose a somewhat unconventional importance sampling estimator (*cf.* the more classical estimator (3.19) that we will come across in the slow regime). Based on $r \in \mathbb{N}$ runs, $P_n(a)$ can be unbiasedly estimated by

$$\widehat{P}_n^{\mathrm{IS}}(a) = \frac{1}{r} \sum_{i=1}^{r} \frac{\mathbb{P}\big(\mathrm{Pois}\left(n\overline{X}_{N^\alpha}^{(i)}\right) = Z^{(i)}\big)}{\mathbb{P}\left(\mathrm{Pois}(na) = Z^{(i)}\right)} \mathbb{1}\left\{Z^{(i)} \geq na\right\}, \tag{3.16}$$

where $Z^{(1)}, \ldots, Z^{(r)}$ are independent samples from $\mathrm{Pois}(na)$, and $\overline{X}_{n^\alpha}^{(1)}, \ldots, \overline{X}_{n^\alpha}^{(r)}$ are independently sampled under the original measure.

Observe that the contribution from the ith run depends on $\overline{X}_{n^\alpha}^{(i)}$ as well as $Z^{(i)}$. It is therefore easier to analyse the corresponding estimator for $p_n(a)$,

$$\widehat{p}_n^{\mathrm{IS}}(a) := \frac{1}{r} \sum_{i=1}^{r} \frac{\mathbb{P}\left(\mathrm{Pois}\left(n\overline{X}_{n^\alpha}^{(i)}\right) = Z^{(i)}\right)}{\mathbb{P}\left(\mathrm{Pois}(na) = Z^{(i)}\right)} \mathbb{1}\left\{Z^{(i)} = na\right\},$$

which does not depend on values of $Z^{(i)}$ that are larger than na (as $Z^{(i)}$ equals na almost surely). We later comment on efficient estimation of $P_n(a)$.

The contribution due to the likelihood ratio of the ith run is

$$L\left(\overline{X}_{n^\alpha}^{(i)}\right) := \left(\frac{\overline{X}_{n^\alpha}^{(i)}}{a}\right)^{na} \mathrm{e}^{n\left(a - \overline{X}_{n^\alpha}^{(i)}\right)}.$$

The variance of the estimator (with respect to the joint distribution of $Z \sim \mathrm{Pois}(na)$ and $\overline{X}_{n^\alpha}$) can be evaluated to be

$$\frac{1}{n}\,\mathbb{E}\left[\left(L\big(\overline{X}_{n^\alpha}\big)\mathbb{1}\left\{Z = na\right\}\right)^2\right] - p_n(a)^2 = \frac{1}{n}\,\mathbb{E}\left[L^2\big(\overline{X}_{n^\alpha}\big)\mathbb{1}\left\{Z = na\right\}\right] - p_n(a)^2, \tag{3.17}$$

with Z distributed as each of the $Z^{(i)}$, and $\overline{X}_{n^\alpha}$ as each of the $\overline{X}_{n^\alpha}^{(i)}$.

Proposition 3.9. *The estimator $\widehat{p}_n^{\mathrm{IS}}(a)$ is asymptotically efficient for estimating $p_n(a)$; that is*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[L^2\left(\overline{X}_{n^\alpha}\right)\mathbb{1}\left\{Z = na\right\}\right] \leq -2I(a \,|\, \nu).$$

Proof. First, note that

$$\begin{aligned}\mathbb{E}\left[L^2\big(\overline{X}_{n^\alpha}\big)\mathbb{1}\left\{Z=na\right\}\right] &= \mathbb{E}_\nu\left[\left(\frac{\overline{X}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n(a-\overline{X}_{n^\alpha})}\right]\mathbb{P}(Z=na)\\ &\leq \mathbb{E}_\nu\left[\left(\frac{\overline{X}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n(a-\overline{X}_{n^\alpha})}\right].\end{aligned}$$

Define $\mathcal{F}_\varepsilon^{(n)} := \left\{\overline{X}_{n^\alpha} \in (\nu-\varepsilon, \nu+\varepsilon)\right\}$, where $\varepsilon > 0$. Then

$$\mathbb{E}_\nu\left[\left(\frac{\overline{X}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n\,(a-\overline{X}_{n^\alpha})}\mathbb{1}\left\{\mathcal{F}_\varepsilon^{(n)}\right\}\right] \leq \left(\frac{\nu+\varepsilon}{a}\right)^{2na} \mathrm{e}^{2n(a-\nu+\varepsilon)}. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}_\nu\left[\left(\frac{\overline{X}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n\,(a-\overline{X}_{n^\alpha})}\mathbb{1}\left\{\left(\mathcal{F}_\varepsilon^{(n)}\right)^c\right\}\right] &= \mathbb{E}_\nu\left[\mathrm{e}^{-2nI\left(a\,\middle|\,\overline{X}_{n^\alpha}\right)}\mathbb{1}\left\{\left(\mathcal{F}_\varepsilon^{(n)}\right)^c\right\}\right]\\ &\leq \mathbb{P}\left(\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right).\end{aligned}$$

where the last inequality is due to $I(a\,|\,x) \geq 0$ for any x. Invoking Chernoff's bound, we note that

$$\mathbb{P}\left(\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right) \leq 2\exp\left(-n^\alpha j_\varepsilon\right), \text{ where } j_\varepsilon := \inf_{x\notin(\nu-\varepsilon,\nu+\varepsilon)} I_X(x) > 0.$$

We conclude that for $\alpha > 1$,

$$\limsup_{n\to\infty} \frac{n^\alpha}{n}\frac{1}{n^\alpha}\log\mathbb{P}\left(\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right) \leq \limsup_{n\to\infty} -\frac{n^\alpha}{n} j_\varepsilon = -\infty.$$

Combining this with (3.18), we conclude that

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[\left(L\left(\overline{X}_{n^\alpha}\right)\mathbb{1}\left\{Z=na\right\}\right)^2\right] \leq 2a\log\left(\frac{\nu+\varepsilon}{a}\right) + 2(a-\nu+\varepsilon).$$

The desired result follows when taking $\varepsilon \downarrow 0$. □

Formally, this result on asymptotic efficiency for $\widehat{p}_n^{\mathrm{IS}}(a)$ does not imply asymptotic efficiency for $\widehat{P}_n^{\mathrm{IS}}(a)$. In practice, however, we can use

$$\widehat{P}_n^{\mathrm{IS}}(a) = \sum_{k=na}^{K} \widehat{p}_n(k/n),$$

with K sufficiently large, to estimate $P_n(a)$.

3.1.3.2 Slow regime

In the slow regime, assuming that $b_+ \geq a$, the rare event is typically caused by a large deviation of $\overline{X}_{n^\alpha}$. Suppose that $\overline{X}_{n^\alpha}^{(1)}, \ldots, \overline{X}_{n^\alpha}^{(n)}$ are independently sampled according to

the original measure $\mathbb{P}_\nu$ (where the subscript indicates that the expectation of each of the sample means $\overline{X}^{(i)}_{n^\alpha}$ involved is ν). In this case we suggest the estimator

$$\widehat{P}^{\mathrm{IS}}_n(a) = \frac{1}{r}\sum_{i=1}^{r} \frac{\mathbb{P}_\nu\left(\overline{X}^{(i)}_{N^\alpha} \in \mathrm{d}Y^{(i)}\right)}{\mathbb{P}_a\left(\overline{X}^{(i)}_{N^\alpha} \in \mathrm{d}Y^{(i)}\right)} \mathbb{1}\left\{\mathrm{Pois}\left(NY^{(i)}\right) \geq Na\right\}, \tag{3.19}$$

where $Y^{(1)}, \ldots, Y^{(r)} \sim \mathbb{P}_a$. The measure $\mathbb{P}_a$ corresponds to the exponentially twisted version such that the mean becomes a (rather than ν).

For each run we have the likelihood ratio, with $\boldsymbol{x} = (x_1, \ldots, x_{n^\alpha})$,

$$L(\boldsymbol{x}) = \prod_{i=1}^{n^\alpha} M_X(\theta_a)\, \mathrm{e}^{-\theta_a x_i},$$

where we recall that $M_X(\cdot)$ is the moment-generating function of X and θ_a is the unique solution to

$$\mathbb{E}_a[X] = \mathbb{E}_\nu\left[X\frac{\mathrm{e}^{\theta X}}{M_X(\theta)}\right] = \frac{M'_X(\theta)}{M_X(\theta)} = a\,.$$

In this case we have seen before that $n^{-\alpha}\log P_n(a) \to -I_X(a)$ as $n \to \infty$.

Proposition 3.10. *The estimator $\widehat{P}^{\mathrm{IS}}_n(a)$ is asymptotically efficient for estimating $P_n(a)$; that is*

$$\limsup_{n\to\infty} \frac{1}{n^\alpha}\log \mathbb{E}_a\left[\left(L(\boldsymbol{X})\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right\}\right)^2\right] \leq -2I_X(a)\,.$$

Proof. Note that

$$\begin{aligned}&\mathbb{E}_a\left[\left(L(\boldsymbol{X})\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right\}\right)^2\right]\\ &\quad = M(\theta_a)^{2n^\alpha}\,\mathbb{E}_a\left[\mathrm{e}^{-2\theta_a n^\alpha \overline{X}_{n^\alpha}}\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right\}\right].\end{aligned}$$

On $\mathcal{F}^{(n)}_\varepsilon := \left\{\overline{X}_{n^\alpha} \in (a-\varepsilon, \infty)\right\}$ we have

$$\mathbb{E}_a\left[\mathrm{e}^{-2\theta_a n^\alpha \overline{X}_{n^\alpha}}\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right\}\mathbb{1}\left\{\mathcal{F}^{(n)}_\varepsilon\right\}\right] \leq \mathrm{e}^{-2\theta_a n^\alpha(a-\varepsilon)},$$

while outside of $\mathcal{F}^{(n)}_\varepsilon$ we have

$$\begin{aligned}&\mathbb{E}_a\left[\mathrm{e}^{-2\theta_a n^\alpha \overline{X}_{n^\alpha}}\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right\}\mathbb{1}\left\{\left[\mathcal{F}^{(n)}_\varepsilon\right]^c\right\}\right]\\ &\quad \leq \mathbb{P}_a\left(\mathrm{Pois}\left(n(a-\varepsilon)\right) \geq na\right),\end{aligned}$$

where we used that $\theta_a > 0$ because $a > \nu$ [38, Lemma 2.2.5]. By virtue of the Chernoff bound,

$$\mathbb{P}_a\left(\mathrm{Pois}\left(n(a-\varepsilon)\right) \geq na\right) \leq \mathrm{e}^{-nI(a\,|\,a-\varepsilon)}, \text{ where } I(a\,|\,a-\varepsilon) > 0.$$

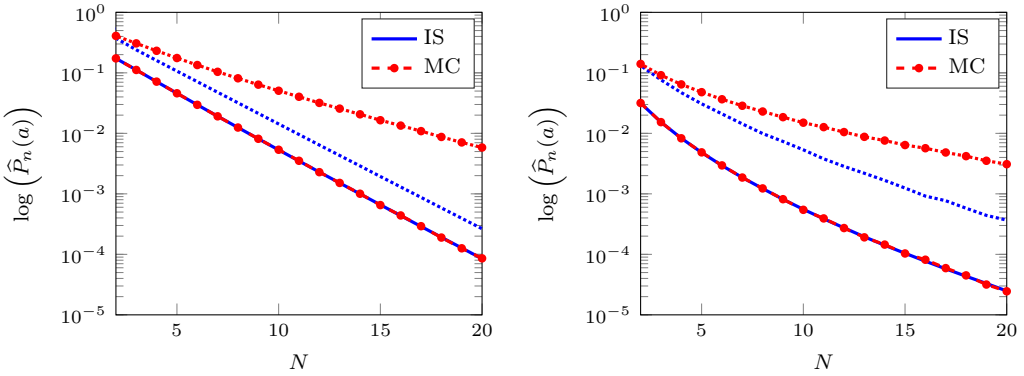
This implies that

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{E}_a \left[\mathrm{e}^{-2\theta_a n^\alpha \overline{X}_{n^\alpha}} \mathbb{1}\left\{ \mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na \right\} \mathbb{1}\left\{ \left[\mathcal{F}_\varepsilon^{(n)} \right]^c \right\} \right]$$
$$\leq \limsup_{n\to\infty} -\frac{n}{n^\alpha} I(a \,|\, a-\varepsilon) = -\infty.$$

We let first $n \to \infty$ and then $\varepsilon \downarrow 0$, to conclude that

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{E}_a \left[\left(L(\boldsymbol{X}) \mathbb{1}\left\{ \mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na \right\} \right)^2 \right]$$
$$\leq 2 \log M_X(\theta_a) - 2\theta_a a = -2 I_X(a)\,,$$

as claimed. □



(a) Fast regime, $\alpha = 2$.

(b) Slow regime, $\alpha = 0.5$.

Figure 3.2: Logarithmic importance sampling (IS) and crude Monte Carlo (MC) estimators for $P_n(a)$, where X_i is exponentially distributed with parameter λ_α (where $\lambda_2 = 1$, $\lambda_{0.5} = 2.5$) and $a = 2$, averaged over $n = 10^7$ samples. The upper bounds of the sample confidence intervals are indicated by dashed lines; the width of the intervals is inflated by a factor 10^3 for better visibility.

3.1.3.3 Numerical example

We provide a numerical example with exponentially distributed X_i. Specifically, we consider $X_i \sim \mathrm{Exp}(1)$, $a = 2$, and $\alpha \in \{0.5, 2\}$. Figure 3.2 shows the logarithm of $\widehat{P}_n^{\mathrm{IS}}(a)$ as well as the corresponding crude Monte Carlo estimators, as a function of n. We generated $\sum_{i=1}^{n^\alpha} X_i$ by drawing from the gamma distribution with parameters n^α and $1/\lambda$. This allowed us to include values of n for which $n^\alpha \notin \mathbb{N}$ in Fig. 3.2b. The dotted lines in the figures indicate the upper bounds of the standard normal 95% confidence intervals evaluated using sample standard deviations (multiplied by a factor 10^3 to make them visible). It can be seen that for the importance sampling estimator the width of the confidence interval hardly depends on n. In contrast, for the Monte Carlo estimator the width of the confidence interval increases significantly.

3.2 Tail probabilities of the system size process

In this section we investigate the asymptotic behaviour of $Q_n(a)$ ($q_n(a)$), the probability that the number of clients in the system exceeds (equals) some threshold na. We consider the scaled system previously studied in [64].

Define

$$S_n := \sum_{i=1}^{n} Z_i, \quad \text{with} \quad Z_i \stackrel{\mathrm{d}}{=} \mathrm{Pois}\left(X_i\, \omega_i(n)\right),$$

where $\omega_i(n)$ is the probability that a call arriving at a uniform epoch in the interval $[(i-1)/n, i/n)$ is still present at time 1. Suppose the arrival rates X_i are i.i.d. with distribution $\mathbb{P}_\nu$ such that $\mathbb{E}X_i = \nu$, and the service times are i.i.d. with distribution function G. Then it can be verified that

$$\omega_i(n) = n \int_{(i-1)/n}^{i/n} \overline{G}(1-x)\mathrm{d}x;$$

because the X_i are i.i.d., we can reverse time, and hence replace $\overline{G}(1-x)$ in the previous display by $\overline{G}(x)$. As mentioned in the introduction (*viz.* Eqn. (3.1)), under the scaling of [64], the probability that the number of clients in the system exceeds some threshold na is given by

$$Q_n(a) = \mathbb{P}\left(\mathrm{Pois}(n\overline{X\omega}_{n^\alpha}) \geq na\right), \tag{3.20}$$

where

$$\overline{X\omega}_n := n^{-1} \sum_{i=1}^{n} X_i\, \omega_i(n);$$

for details see [64]. The corresponding probability mass function is denoted by

$$q_n(a) := \mathbb{P}\left(\mathrm{Pois}(n\overline{X\omega}_{n^\alpha}) = na\right).$$

Note that if $\alpha = 1$, we have $Q_n(a) = \mathbb{P}(S_n \geq na)$, and $q_n(a) = \mathbb{P}(S_n = na)$. We shall also use the notation

$$\overline{\omega}_n := n^{-1} \sum_{i=1}^{n} \omega_i(n).$$

Assume that the distribution function G associated with the distribution of the service times is twice differentiable. Then using exactly the same techniques as in [64, Section 4.1] we can show that that $Q_n(a)$ satisfies a large deviations principle,

$$\lim_{n\to\infty} n^{-\min\{1,\alpha\}} \log\left(Q_n(a)\right) = -J_\alpha(a),$$

where the rate function is given by

$$J_\alpha(a) := \begin{cases} I\left(a \,\middle|\, \nu \int_0^1 \overline{G}(x)\mathrm{d}x\right) & \text{if } \alpha > 1, \\ \sup_\theta \left\{\theta a - \int_0^1 \Lambda_X\left(\left(\mathrm{e}^\theta - 1\right)\overline{G}(x)\right)\mathrm{d}x\right\} & \text{if } \alpha = 1, \\ \sup_\theta \left\{\theta a - \int_0^1 \Lambda_X\left(\theta\overline{G}(x)\right)\mathrm{d}x\right\} & \text{if } \alpha < 1. \end{cases} \tag{3.21}$$

We will assume that the suprema in the definition of J_α for $\alpha < 1$ and $\alpha = 1$, respectively, are attained at $\theta^* \equiv \theta^*_\alpha$.

In the remainder of this section, we first determine the exact asymptotics for the special case $\alpha = 1$. That is, we assume that the arrival rates are resampled every $1/n$ time units, and we are interested in the number of customers present at time 1 (that is, after n time periods of length $1/n$). As it turns out, the case $\alpha \neq 1$ is considerably harder to deal with, and therefore left for future research. We conclude this section by a set of numerical experiments.

3.2.1 Exact asymptotics

Let in this section $\alpha = 1$, and

$$\theta^\star = \arg\sup_\theta \left\{ \theta a - \int_0^1 \Lambda_X \left(\overline{G}(x)(\mathrm{e}^\theta - 1) \right) \mathrm{d}x \right\}.$$

It is immediately verified that

$$\mathbb{E}\left[\mathrm{e}^{\theta S_n}\right] = \prod_{i=1}^n M_X \left(\omega_i(n)(\mathrm{e}^\theta - 1) \right). \tag{3.22}$$

The idea is now to construct a measure $\mathbb{Q}$ under which the event of interest is not rare so that a central limit theorem applies. Concretely, we choose $\mathbb{Q}$ to be an $\theta^\star$-twisted version of the original measure such that S_n has moment generating function (*cf.* (3.22))

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{\theta S_n}\right] = \prod_{i=1}^n M_X \left(\omega_i(n)(\mathrm{e}^{\theta+\theta^\star} - 1) \right) \Big/ \prod_{i=1}^n M_X \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1) \right). \tag{3.23}$$

As a consequence, $q_n(a) = \mathbb{E}_{\mathbb{Q}} L \mathbb{1}\{S_n = na\}$, with the likelihood ratio

$$L := \mathrm{e}^{-\theta^\star S_n} \prod_{i=1}^n M_X \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1) \right).$$

It thus follows that

$$q_n(a) = \mathrm{e}^{-\theta^\star na} \left(\prod_{i=1}^n M_X \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1) \right) \right) \mathbb{Q}(S_n = na).$$

We now point out how to evaluate the middle factor in the previous display (i.e., the product), namely, we check that asymptotically this middle factor behaves as

$$\exp\left(n \int_0^1 \Lambda_X\big(\tau \overline{G}(x)\big) \mathrm{d}x \right), \tag{3.24}$$

with $\tau := \mathrm{e}^{\theta^\star} - 1$. The logarithm of the middle factor is

$$\sum_{i=1}^n \Lambda_X\big(\tau \omega_i(n)\big) = \sum_{i=1}^n \Lambda_X \left(\tau n \int_{(i-1)/n}^{i/n} \overline{G}(x) \mathrm{d}x \right),$$

where, by a Taylor expansion of $\overline{G}$,

$$n\int_{(i-1)/n}^{i/n}\overline{G}(x)\mathrm{d}x=\overline{G}\left(\frac{i-1}{n}\right)+\frac{1}{2n}\,\overline{G}\,'\left(\frac{i-1}{n}\right)+O\left(\frac{1}{n^2}\right).$$

As a consequence, from a Taylor expansion of $\Lambda_X(\cdot)$ we have

$$\begin{aligned}&\sum_{i=1}^{n}\Lambda_X\big(\tau\omega_i(n)\big)\\&\quad=\sum_{i=1}^{n}\Lambda_X\left(\tau\overline{G}\left(\frac{i-1}{n}\right)\right)+\frac{\tau}{2n}\sum_{i=1}^{n}\overline{G}\,'\left(\frac{i-1}{n}\right)\Lambda_X'\left(\tau\overline{G}\left(\frac{i-1}{n}\right)\right)+O\left(\frac{1}{n}\right),\end{aligned}$$

where, as $n\to\infty$,

$$\begin{aligned}\frac{\tau}{2n}\sum_{i=1}^{n}\overline{G}\,'\left(\frac{i-1}{n}\right)\Lambda_X'\left(\tau\overline{G}\left(\frac{i-1}{n}\right)\right)&\to\frac{\tau}{2}\int_0^1\overline{G}\,'(x)\,\Lambda_X'\left(\tau\overline{G}(x)\right)\mathrm{d}x\\&=\frac{1}{2}\left(\Lambda_X\big(\tau\overline{G}(1)\big)-\Lambda_X\big(\tau\overline{G}(0)\big)\right),\end{aligned}$$

provided that $\overline{G}(\cdot)$ is twice differentiable on $[0,1]$ (recognize the *left Riemann sum* approximation). Now recall the *trapezoidal rule* version of the Riemann sum approximation, that holds for any Riemann-integrable $f(\cdot)$:

$$\frac{1}{n}\sum_{i=1}^{n}f(i/n)=\int_0^1 f(x)\mathrm{d}x+\frac{1}{2n}(f(1)-f(0))+O\left(\frac{1}{n^2}\right).$$

Since Λ_X is Riemann integrable on $[0,1]$, this can be applied to yield

$$\begin{aligned}&n\int_0^1\Lambda_X\big(\tau\overline{G}(x)\big)\mathrm{d}x\\&\quad=\sum_{i=1}^{n}\Lambda_X\left(\tau\overline{G}\left(\frac{i}{n}\right)\right)-\frac{1}{2}\left(\Lambda_X\big(\tau\overline{G}(1)\big)-\Lambda_X\big(\tau\overline{G}(0)\big)\right)+O\left(\frac{1}{n}\right)\\&\quad=\sum_{i=1}^{n}\Lambda_X\left(\tau\overline{G}\left(\frac{i-1}{n}\right)\right)+\frac{1}{2}\left(\Lambda_X\big(\tau\overline{G}(1)\big)-\Lambda_X\big(\tau\overline{G}(0)\big)\right)+O\left(\frac{1}{n}\right).\end{aligned}$$

We have thus arrived at

$$q_n(a)\sim\mathrm{e}^{-\theta^\star na}\exp\left(n\int_0^1\Lambda_X\big(\overline{G}(x)(\mathrm{e}^{\theta^\star}-1)\big)\mathrm{d}x\right)\mathbb{Q}(S_n=na).$$

We are left to evaluate $\mathbb{Q}(S_n=na)$. We do so by first proving the claim that, under $\mathbb{Q}$, S_n obeys a central limit theorem: as $n\to\infty$,

$$\frac{S_n-na}{\sqrt{n}}$$

converges to a zero-mean normal random variable. Recall from (3.23) that we have

$$\log \mathbb{E}_{\mathbb{Q}}\, \mathrm{e}^{\theta S_n} = \sum_{i=1}^{n} \Lambda_X \left(\omega_i(n)(\mathrm{e}^{\theta+\theta^\star} - 1)\right) - \sum_{i=1}^{n} \Lambda_X \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1)\right).$$

In order to establish that S_n satisfies the anticipated central limit theorem, we prove that $\Psi_n(\theta) := \log \mathbb{E}_{\mathbb{Q}}\, \mathrm{e}^{\theta S_n/\sqrt{n}} - \theta a\sqrt{n} \to \frac{1}{2}\sigma^2\theta^2$, for some $\sigma^2 > 0$. This is done as follows. Observe that we can write the logarithmic moment generating function $\Psi_n(\theta)$ as

$$\sum_{i=1}^{n} \Lambda_X \left(\omega_i(n) \left(\mathrm{e}^{\theta^\star} - 1 + \left(\mathrm{e}^{\theta^\star}(\mathrm{e}^{\theta/\sqrt{n}} - 1)\right)\right)\right) - \sum_{i=1}^{n} \Lambda_X \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1)\right) - \theta a\sqrt{n}.$$

By applying a Taylor expansion to $\mathrm{e}^{\theta/\sqrt{n}} - 1$, this can be written as (neglecting higher order terms)

$$\sum_{i=1}^{n} \Lambda_X \left(\omega_i(n) \left(\mathrm{e}^{\theta^\star} - 1 + \left(\mathrm{e}^{\theta^\star} \left(\frac{\theta}{\sqrt{n}} + \frac{\theta^2}{2n}\right)\right)\right)\right) - \sum_{i=1}^{n} \Lambda_X \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1)\right) - \theta a\sqrt{n}.$$

This can be expanded to, up to terms that are $o(1)$ as $n \to \infty$,

$$\sum_{i=1}^{n} \left[\Lambda_X' \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1)\right) \omega_i(n) \mathrm{e}^{\theta^\star} \left(\frac{\theta}{\sqrt{n}} + \frac{\theta^2}{2n}\right) + \frac{1}{2}\Lambda_X'' \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1)\right) \omega_i(n)^2 \mathrm{e}^{2\theta^\star} \frac{\theta^2}{n}\right] - \theta a\sqrt{n}. \tag{3.25}$$

Now note that, similar to what we have seen before,

$$\frac{1}{n} \sum_{i=1}^{n} \Lambda_X' \left(\omega_i(n)(\mathrm{e}^{\theta^\star} - 1)\right) \omega_i(n)\, \mathrm{e}^{\theta^\star} = \int_0^1 \Lambda_X' \big(\overline{G}\,(x)\,(\mathrm{e}^{\theta^\star} - 1)\big) \overline{G}(x)\, \mathrm{e}^{\theta^\star} \mathrm{d}x + O\left(\frac{1}{n}\right),$$

where the integral equals a by the definition of $\theta^\star$. We conclude that (3.25) converges to $\frac{1}{2}\sigma^2\theta^2$ as $n \to \infty$, where the corresponding variance is given by

$$\begin{aligned}\sigma^2 :=& \int_0^1 \Lambda_X' \big(\overline{G}(x)(\mathrm{e}^{\theta^\star} - 1)\big) \overline{G}(x)\, \mathrm{e}^{\theta^\star} \mathrm{d}x + \int_0^1 \Lambda_X'' \big(\overline{G}(x)(\mathrm{e}^{\theta^\star} - 1)\big) \overline{G}^{\,2}(x)\, \mathrm{e}^{2\theta^\star} \mathrm{d}x \\ =& a + \int_0^1 \Lambda_X'' \big(\overline{G}(x)(\mathrm{e}^{\theta^\star} - 1)\big) \overline{G}^{\,2}(x)\, \mathrm{e}^{2\theta^\star} \mathrm{d}x.\end{aligned}$$

We have thus established that, under $\mathbb{Q}$, S_n satisfies the claimed central limit theorem.

It directly implies that, by applying the usual continuity correction idea, $\mathbb{Q}(S_n = na)$ behaves inversely proportionally to $\sqrt{n}$ in the sense that

$$\sqrt{n}\,\mathbb{Q}(S_n = na) \sim \sqrt{n}\,\mathbb{P}\left((0,\sigma^2) \in \left(-\frac{1}{2\sqrt{n}}, \frac{1}{2\sqrt{n}}\right)\right) \to \frac{1}{\sqrt{2\pi}\sigma}.$$

Upon combining the above, we conclude that the following asymptotic relationship holds.

Proposition 3.11. *As $n \to \infty$, if $\overline{G}(\cdot)$ is twice differentiable on $[0,1]$,*

$$q_n(a) \sim \widetilde{q}_n(a) := \mathrm{e}^{-\theta^\star na} \exp\left(n \int_0^1 \Lambda_X\big(\overline{G}(x)(\mathrm{e}^{\theta^\star} - 1)\big)\mathrm{d}x\right) \frac{1}{\sqrt{2\pi n}\sigma}.$$

Similar to Remark 3.3, we can convert the asymptotics of $q_n(a)$ into those of $Q_n(a)$. More precisely, it can be argued that $Q_n(a)$ has the same asymptotics as $q_n(a)$, except that the expansion for $q_n(a)$ should be divided by $1 - \mathrm{e}^{-\theta^\star}$ (which is smaller than 1). Note also that for the case $\overline{G}(\cdot) \equiv 1$ we indeed recover the expression that we provided in Section 3.1.1.3. Furthermore, it is easily verified that if $\mathbb{P}(X_i = \lambda) = 1$ (so the arrival rates are deterministic), the approximation we obtained in Proposition 3.11 coincides with that of the transient distribution of an M/G/∞ queue. With $\varrho(1) := \lambda \int_0^1 \overline{G}(x)\mathrm{d}x$, recall that the number of customers present at time 1 is Poisson with mean $\varrho(1)$. By applying Stirling's approximation, and using that $\theta^* = \log(a/\varrho(1))$,

$$q_n(a) = (n\varrho(1))^{na}\,\mathrm{e}^{-n\varrho(1)}\frac{1}{(na)!} \sim \left(\frac{\varrho(1)}{a}\right)^{na} \mathrm{e}^{n(a-\varrho(1))}\frac{1}{\sqrt{2\pi na}} = \widetilde{q}_n(a).$$

3.2.2 Importance sampling

To complement the asymptotic results, in this section we consider importance sampling algorithms for efficient estimation of $Q_n(a)$. We proceed similarly as in Section 3.1.3.

3.2.2.1 Fast regime

For $\alpha > 1$, similar to the estimator proposed in Section 3.1.3.1 we now consider

$$\begin{aligned}\widehat{Q}_n^{\mathrm{IS}}(a) &= \frac{1}{r}\sum_{i=1}^{r} \frac{\mathbb{P}\big(\mathrm{Pois}\big(n^{1-\alpha}\sum_{j=1}^{n^\alpha} X_j^{(i)}\,\omega_j(n^\alpha)\big) = Z^{(i)}\big)}{\mathbb{P}\left(\mathrm{Pois}(na) = Z^{(i)}\right)} \mathbb{1}\left\{Z^{(i)} \geq na\right\} \\ &= \frac{1}{r}\sum_{i=1}^{r} \frac{\mathbb{P}\big(\mathrm{Pois}\big(n\overline{X^{(i)}\omega}_{n^\alpha}\big) = Z^{(i)}\big)}{\mathbb{P}\left(\mathrm{Pois}(na) = Z^{(i)}\right)} \mathbb{1}\left\{Z^{(i)} \geq na\right\},\end{aligned}$$

where $Z^{(i)}$ are samples from $\mathrm{Pois}(na)$, and $X_j^{(i)}$ are i.i.d. random variables distributed as $\mathbb{P}_\nu$.

In parallel to the procedure for estimating $P_n(a)$ in Section 3.1.3.1, we prove asymptotic efficiency for

$$\widehat{q}_n^{\mathrm{IS}}(a) = \frac{1}{r}\sum_{i=1}^{r} \frac{\mathbb{P}\big(\mathrm{Pois}\big(n\overline{X^{(i)}\omega}_{n^\alpha}\big) = Z^{(i)}\big)}{\mathbb{P}\left(\mathrm{Pois}(na) = Z^{(i)}\right)} \mathbb{1}\left\{Z^{(i)} = na\right\}. \tag{3.26}$$

Proposition 3.12. *For $\alpha > 1$, the estimator $\widehat{q}_n^{\mathrm{IS}}(a)$ defined in (3.26) is asymptotically efficient for estimating $q_n(a)$; that is*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[L^2\left(\overline{X\omega}_{n^\alpha}\right) \mathbb{1}\left\{Z = na\right\}\right] \leq -2\, I\left(a \,\middle|\, \nu \int_0^1 \overline{G}(x)\mathrm{d}x\right).$$

Proof. The proof follows closely that of 3.9. First, note that

$$\mathbb{E}\left[L^2\big(\overline{X\omega}_{n^\alpha}\big)\mathbb{1}\left\{Z = na\right\}\right] \leq \mathbb{E}_\nu\left[\left(\frac{\overline{X\omega}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n(a-\overline{X\omega}_{n^\alpha})}\right].$$

Define $\mathcal{F}_\varepsilon^{(n)} := \left\{\overline{X\omega}_{n^\alpha} \in (\nu\overline{\omega}_{n^\alpha} - \varepsilon, \nu\overline{\omega}_{n^\alpha} + \varepsilon)\right\}$, where $\varepsilon > 0$. Then

$$\mathbb{E}_\nu\left[\left(\frac{\overline{X\omega}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n\,(a-\overline{X\omega}_{n^\alpha})}\mathbb{1}\left\{\mathcal{F}_\varepsilon^{(n)}\right\}\right] \leq \left(\frac{\nu\overline{\omega}_{n^\alpha} + \varepsilon}{a}\right)^{2na} \mathrm{e}^{2n(a-\nu\overline{\omega}_{n^\alpha}+\varepsilon)}. \tag{3.27}$$

Since G is monotone and thus Riemann-integrable on $[0,1]$, we have

$$\lim_{n\to\infty} \overline{\omega}_{n^\alpha} = \int_0^1 \overline{G}(x)\mathrm{d}x.$$

Hence, from (3.27) we obtain that for n large enough

$$\begin{aligned}\mathbb{E}_\nu\left[\left(\frac{\overline{X\omega}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n\,(a-\overline{X\omega}_{n^\alpha})}\mathbb{1}\left\{\mathcal{F}_\varepsilon^{(n)}\right\}\right] \\ \leq \left(\frac{\nu\int_0^1 \overline{G}(x)\mathrm{d}x + \varepsilon}{a}\right)^{2na} \mathrm{e}^{2n\left(a-\nu\int_0^1 \overline{G}(x)\mathrm{d}x+\varepsilon\right)}.\end{aligned} \tag{3.28}$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}_\nu\left[\left(\frac{\overline{X\omega}_{n^\alpha}}{a}\right)^{2na} \mathrm{e}^{2n\,(a-\overline{X\omega}_{n^\alpha})}\mathbb{1}\left\{\left(\mathcal{F}_\varepsilon^{(n)}\right)^c\right\}\right] &= \mathbb{E}_\nu\left[\mathrm{e}^{-2nI\left(a\,\middle|\,\overline{X\omega}_{n^\alpha}\right)}\mathbb{1}\left\{\left(\mathcal{F}_\varepsilon^{(n)}\right)^c\right\}\right] \\ &\leq \mathbb{P}\left(\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right),\end{aligned}$$

where the last inequality holds because $I(a|x) \geq 0$ for any x. Invoking Chernoff's bound, we note that

$$\mathbb{P}\left(\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right) \leq 2\exp\left(-\sum_{i=1}^{n^\alpha} J_{i,\varepsilon}\right), \text{ where } J_{i,\varepsilon} := \inf_{x\notin(\nu\overline{\omega}_{n^\alpha}-\varepsilon,\, \nu\overline{\omega}_{n^\alpha}+\varepsilon)} I_{X\omega_i}(x) > 0.$$

We conclude that for $\alpha > 1$,

$$\limsup_{n\to\infty} \frac{n^\alpha}{n}\frac{1}{n^\alpha}\log\mathbb{P}\left(\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right) \leq \limsup_{n\to\infty} -\frac{n^\alpha}{n}\overline{J_{\varepsilon}}_{n^\alpha} = -\infty,$$

where we used that $\overline{J_{\varepsilon}}_{n^\alpha} := n^{-\alpha}\sum_{i=1}^{n^\alpha} J_{i,\varepsilon}$ is bounded since $\omega_i(n^\alpha) \in [0,1]$. Combining

this with (3.28), we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[\left(L\left(\overline{X\omega}_{n^\alpha}\right) \mathbb{1}\left\{Z = na\right\}\right)^2\right] \leq 2a \log\left(\frac{\nu \int_0^1 \overline{G}(x)\mathrm{d}x + \varepsilon}{a}\right) + 2\left(a - \nu \int_0^1 \overline{G}(x)\mathrm{d}x + \varepsilon\right).$$

The desired result follows when taking $\varepsilon \downarrow 0$. □

3.2.2.2 Slow regime

In this section we assume that X_i takes values greater than a with positive probability, and we focus on the case $\alpha < 1$. In this regime we expect $Q_n(a)$ to behave as

$$\mathbb{P}\left(n\overline{X\omega}_{n^\alpha} \geq na\right) = \mathbb{P}\left(\overline{X\omega}_{n^\alpha} \geq a\right).$$

We therefore propose to twist the distribution of $X_i\omega_i$ with twisting parameter

$$\theta^* := \arg\max_\theta \left\{\theta a - \int_0^1 \Lambda_X\left(\theta\overline{G}(x)\right)\mathrm{d}x\right\},$$

that is, under the twisted measure X_i has probability density

$$\frac{\mathrm{e}^{\theta^*\omega_i}}{M_X(\omega_i\theta^*)} f_X(x).$$

The corresponding importance sampling estimator is

$$\widehat{Q}_n^{\mathrm{IS}}(a) = \frac{1}{r}\sum_{i=1}^r L(\boldsymbol{X}^i)\,\mathbb{1}\left\{\overline{X_i\omega}_{n^\alpha} \geq na\right\}, \tag{3.29}$$

where $X_j^{(i)}$ are drawn from the twisted distribution of X_j with twisting parameter $\theta^*\omega_j$, whence the likelihood ratio is given by

$$L(\boldsymbol{X}) = \prod_{j=1}^{n^\alpha} \mathrm{e}^{-\theta^*\omega_i X_j} M_X(\theta^*\omega_j).$$

Proposition 3.13. *For $\alpha < 1$, the estimator $\widehat{Q}_n^{\mathrm{IS}}(a)$ as defined in (3.29) is asymptotically efficient for estimating $Q_n(a)$; that is*

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{E}_{\mathbb{Q}}\left[\left(L(\boldsymbol{X})\,\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X\omega}_{n^\alpha}\right) \geq na\right\}\right)^2\right] \leq -2J_\alpha(a).$$

Proof. The proof follows closely that of Proposition 3.10. Note that

$$\mathbb{E}_{\mathbb{Q}}\left[\left(L(\boldsymbol{X})\,\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X\omega}_{n^\alpha}\right) \geq na\right\}\right)^2\right] = \prod_{i=1}^{n^\alpha} M_X\left(\omega_i\theta\right)\,\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\theta^* n^\alpha \overline{X\omega}_{n^\alpha}}\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X\omega}_{n^\alpha}\right) \geq na\right\}\right].$$

On $\mathcal{F}_\varepsilon^{(n)} := \left\{\overline{X\omega}_{n^\alpha} \in (a-\varepsilon, \infty)\right\}$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\theta^* n^\alpha \overline{X\omega}_{n^\alpha}} \mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X\omega}_{n^\alpha}\right) \geq na\right\} \mathbb{1}\left\{\mathcal{F}_\varepsilon^{(n)}\right\}\right] \leq \mathrm{e}^{-2\theta^* n^\alpha (a-\varepsilon)}$$

while outside of $\mathcal{F}_\varepsilon^{(n)}$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\theta_a n^\alpha \overline{X}_{n^\alpha}} \mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X\omega}_{n^\alpha}\right) \geq na\right\} \mathbb{1}\left\{\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right\}\right] \leq \mathbb{Q}\left(\mathrm{Pois}\left(n(a-\varepsilon)\right) \geq na\right),$$

where we used that $\theta^* > 0$ by [38, Lemma 2.2.5]. By virtue of the Chernoff bound,

$$\mathbb{Q}\left(\mathrm{Pois}\left(n(a-\varepsilon)\right) \geq na\right) \leq \mathrm{e}^{-nI(a\,|\,a-\varepsilon)}, \text{ where } I(a\,|\,a-\varepsilon) > 0.$$

This implies that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\theta_a n^\alpha \overline{X\omega}_{n^\alpha}} \mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X\omega}_{n^\alpha}\right) \geq na\right\} \mathbb{1}\left\{\left[\mathcal{F}_\varepsilon^{(n)}\right]^c\right\}\right] \\ \leq \limsup_{n\to\infty} -\frac{n}{n^\alpha} I(a\,|\,a-\varepsilon) = -\infty.\end{aligned}$$

We let first $n \to \infty$ and then $\varepsilon \downarrow 0$, to conclude that

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \mathbb{E}_{\mathbb{Q}}\left[\left(L(\boldsymbol{X})\mathbb{1}\left\{\mathrm{Pois}\left(n\overline{X}_{n^\alpha}\right) \geq na\right\}\right)^2\right] \leq \frac{2}{n^\alpha} \log \prod_{i=1}^{n^\alpha} M_X(\theta^*\omega_i) - 2\theta^* a,$$

The claim then follows because as in Section 3.2.1 we can check that

$$J_\alpha(a) \sim \sup_\theta \left\{\theta a - \frac{1}{n^\alpha} \log \prod_{i=1}^{n^\alpha} M_X\left(\omega_i \theta\right)\right\}.$$

□

3.2.2.3 Intermediate regime

We now focus on the intermediate case with $\alpha = 1$. Under the original distribution, the arrival rate parameters X_i have probability density f_X; given $X_i = x_i$, the number of arrivals, S_i, is Poisson distributed with parameter x_i; and given $S_i = s_i$, the number of customers in the system at time 1, Z_i, has a binomial distribution with parameters s_i and ω_i. This can be seen from the components of the moment-generating function of $Z_i \overset{d}{=} \mathrm{Pois}(X\omega_i)$, which is given by

$$\begin{aligned}M_{Z_i}(\theta) &= \mathbb{E}\left[\mathrm{e}^{\theta Z_i}\right] = M_X\left(\omega_i\left(\mathrm{e}^\theta - 1\right)\right) \\ &= \int_0^\infty f_X(x)\mathrm{e}^{-x}\,\mathrm{e}^{x\left(\mathrm{e}^\theta \omega_i + 1 - \omega_i\right)}\mathrm{d}x \\ &= \int_0^\infty f_X(x) \sum_{n=0}^\infty \frac{\mathrm{e}^{-x}x^n}{n!}\left(\omega_i \mathrm{e}^\theta + 1 - \omega_i\right)^n \mathrm{d}x.\end{aligned} \tag{3.30}$$

Now we consider importance sampling with exponential twisting. Let θ^* be the value of θ that optimizes $J_1(a)$. The twisted moment generating function for the random variable

Z_i is

$$\frac{\mathbb{E}\left[\mathrm{e}^{(\theta+\theta^*)Z_i}\right]}{\mathbb{E}\left[\mathrm{e}^{\theta^* Z_i}\right]}.$$

The numerator can be evaluated as

$$\begin{aligned}
&\int_0^\infty f_X(x) \sum_{n=0}^\infty \mathrm{e}^{-x} \frac{x^n}{n!} \left(\mathrm{e}^{\theta^*+\theta}\omega_i + 1 - \omega_i\right)^n \mathrm{d}x \\
&= \int_0^\infty f_X(x) \sum_{n=0}^\infty \mathrm{e}^{-x} \frac{x^n}{n!} q^n \left(\frac{1}{q}\left(\mathrm{e}^{\theta^*+\theta}\omega_i + 1 - \omega_i\right)\right)^n \mathrm{d}x \\
&= \int_0^\infty f_X(x) \mathrm{e}^{(q-1)x} \sum_{n=0}^\infty \frac{\mathrm{e}^{-qx}(qx)^n}{n!} \left(\frac{\mathrm{e}^{\theta^*}\omega_i}{q}\mathrm{e}^{\theta} + 1 - \frac{\mathrm{e}^{\theta^*}\omega_i}{q}\right)^n \mathrm{d}x
\end{aligned}$$

where $q := \mathrm{e}^{\theta^*}\omega_i + 1 - \omega_i$. This suggests that the rate parameter X_i should be twisted with twisting parameter $q - 1 = \left(\mathrm{e}^{\theta^*} - 1\right)\omega_i$; given $X = x$, the number of arrivals S should be simulated as a Poisson random variable with parameter qx; and finally, given $S = s$ we have that Z_i should be sampled from a Binomial distribution with parameters s and $\left(\mathrm{e}^{\theta^*}\omega_i\right)/q$. By a similar computation as in (3.30) we can check that under the twisted measure Z_i is distributed as $\mathrm{Pois}(\widetilde{X}\omega_i\mathrm{e}^{\theta^*})$, where $\widetilde{X}$ has probability density $\mathrm{e}^{(q-1)x} f_X(x)/M_X(q-1)$. We thus propose the following algorithm.

1. Find θ^* that maximises $J_1(a)$.

2. Set $L = 1$. For each $i = 1, \ldots, n$:

   (a) Generate X_i from the twisted distribution with density

   $$\frac{\mathrm{e}^{\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right)x}}{M_X\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right)} f_X(x).$$

   (b) Given $X_i = x_i$, generate $Z_i \sim \mathrm{Pois}(x_i\omega_i\mathrm{e}^{\theta^*})$.

   (c) Update the likelihood ratio $L \to L \cdot \ell_i(Z_i)$, where

   $$\ell_i(Z_i) = \frac{\mathbb{P}\left(\mathrm{Pois}(X_i\omega_i) = Z_i\right)}{\mathbb{P}\left(\mathrm{Pois}(\widetilde{X}\omega_i\mathrm{e}^{\theta^*}) = Z_i\right)}. \tag{3.31}$$

   This can be evaluated in closed form, see the computation below.

3. Evaluate $S_n = \sum_{i=1}^n Z_i$.

4. Carry out steps $(1) - (3)$ r times and evaluate the estimator for $\mathbb{P}(S_n \geq a)$ as the sample average over $\mathbb{1}\{S_n \geq a\} L$:

$$\widehat{Q}_n^{\mathrm{IS}}(a) = \frac{1}{r}\sum_{j=1}^r \mathbb{1}\left\{S_n^{(j)} \geq a\right\} L^{(j)}.$$

The likelihood ratio increments (3.31) can be evaluated as

$$\begin{aligned}
\ell_i(z) &= \frac{\int_0^\infty (x\omega_i)^z \mathrm{e}^{-x\omega_i} f_X(x)\,\mathrm{d}x}{\int_0^\infty (y\omega_i \mathrm{e}^{\theta^*})^z \mathrm{e}^{-y\omega_i \mathrm{e}^{\theta^*}} \frac{\exp\left(\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right)y\right)}{M_X\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right)} f_X(y)\,\mathrm{d}y} \\
&= \frac{\int_0^\infty (x\omega_i)^z \mathrm{e}^{-x\omega_i} f_X(x)\,\mathrm{d}x}{\frac{1}{M_X\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right)} \int_0^\infty \left(\mathrm{e}^{\theta^*}\omega_i y\right)^z \mathrm{e}^{-\omega_i y} f_X(y)\mathrm{d}y} \\
&= \mathrm{e}^{-\theta^* z} M_X\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right) \frac{\int_0^\infty (\omega_i x)^z \mathrm{e}^{-\omega_i x} f_X(x)\,\mathrm{d}x}{\int_0^\infty (\omega_i y)^z \mathrm{e}^{-\omega_i y} f_X(y)\mathrm{d}y} \\
&= \mathrm{e}^{-\theta^* z} M_X\left(\left(\mathrm{e}^{\theta^*}-1\right)\omega_i\right).
\end{aligned}$$

Let $\mathbb{E}^*$ denote the θ^*-twisted expectation. Note from the above computation that $\ell_i(z)$ is the same likelihood ratio as obtained from the one-step algorithm where we simply twist Z_i by θ^*, and thus, the proposed importance sampling estimator is asymptotically optimal:

Proposition 3.14. *The estimator $\widehat{Q}_n(a)$ is asymptotically efficient for estimating $\mathbb{P}(S_n \geq na)$; that is*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}^* \left[\left(L(\boldsymbol{Z})\, \mathbb{1}\left\{S_n \geq na\right\}\right)^2 \right] \leq -2J_1(a),$$

where

$$L(\boldsymbol{Z}) = \prod_{i=1}^n \ell_i(Z_i),$$

with $\boldsymbol{Z}$ denoting the vector $(Z_1, \ldots, Z_n)$.

Proof. Recall first that

$$J_1(a) \sim \sup_\theta \left\{ \theta a - \frac{1}{n} \log \prod_{i=1}^n M_X\left(\omega_i\left(\mathrm{e}^{\theta^*}-1\right)\right)\right\}; \tag{3.32}$$

this follows by applying the Taylor series approximation as in Section 3.2.1. Now it is readily checked that

$$\begin{aligned}
&\frac{1}{n} \log \mathbb{E}_{\mathbb{Q}} \left[\left(L(\boldsymbol{Z})\, \mathbb{1}\left\{S_n \geq na\right\}\right)^2 \right] \\
&\quad = \frac{1}{n} \log \mathbb{E}_{\mathbb{Q}} \left[\mathrm{e}^{-2\theta^* S_n} \prod_{i=1}^n M_X^2\left(\omega_i\left(\mathrm{e}^{\theta^*}-1\right)\right) \mathbb{1}\left\{S_n \geq na\right\}\right] \\
&\quad \leq \frac{1}{n} \log \left[\mathrm{e}^{-2\theta^* na} \prod_{i=1}^n M_X^2\left(\omega_i\left(\mathrm{e}^{\theta^*}-1\right)\right)\right].
\end{aligned}$$

Invoking (3.32), we see that the upper bound is asymptotically equivalent to $-2J_1(a)$ by definition of θ^*. □

3.2.3 Application example

We consider the following numerical example, which illustrates how Proposition 3.11 can be useful in devising staffing rules for instance for call centres. Per time slot of length 5 min. (which we refer to as Δ) a new arrival rate is sampled from a given distribution with a mean such that on average λ clients arrive in the time slot of length Δ. The service times have a fixed mean E.

Let us assume the system starts empty, say at 9 AM. Suppose we wish to determine an appropriate staffing rule for slot 100, i.e., between 10:39 AM and 10:40 AM (evidently, any other slot can be dealt with analogously). Then we choose $n = 100$ (recall the way we normalized time), and after scaling we have $\mathbb{E}[nX_i\Delta] = \lambda$ (as $n\Delta = 1$). Suppose the service facility wishes to maintain a rather strict quality level; its objective is to choose the number of servers in slot 100 to be $\lfloor na \rfloor$ (or, alternatively, $\lceil na \rceil$), where a is the smallest number such that $Q_n(a)$ drops below ε.

For the service times we consider the following three distributions:

- In the first place, we assume that the service times are exponential with mean service time E, that is, $\overline{G}(x) = \mathrm{e}^{-x/E}$.
- A second choice is to assume that the service times are deterministically equal to E, that is we define $\overline{G}(x) = \mathbb{1}\{x < E\}$.
- A third choice is to assume that the service times have a Pareto(2) distribution with mean E, that is, $\overline{G}(x) = (1 + x/E)^{-2}$.

As indicated in the introduction, in practice arrival rates for modeling call centers are typically not constant over time, but may be fluctuating around some mean value [71]. We assume that arrival rates follow a Poisson distribution in Section 3.2.3.1. In Section 3.2.3.2 we consider discrete arrival rates alternating between two values (corresponding to busy and quiet periods), motivated by applications in cloud computing, where the workload of virtual machines exhibits such bursty behavior [168].

3.2.3.1 Poisson arrival rates

In this example we take $X_i \sim \mathrm{Pois}(\lambda)$. We then have

$$\Lambda_X(\theta) = \lambda\left(\mathrm{e}^{\theta} - 1\right); \quad \Lambda'_X(\theta) = \Lambda''_X(\theta) = \lambda\,\mathrm{e}^{\theta}.$$

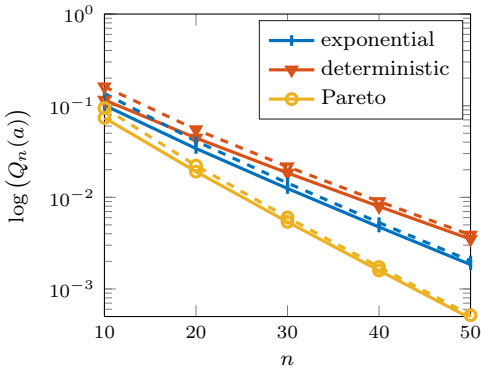
To compute θ^* and σ^2, we evaluate

$$\int_0^1 \Lambda_X\left(\overline{G}(x)(\mathrm{e}^{\theta} - 1)\right)\mathrm{d}x = \int_0^1 \lambda\left(\exp\left(\overline{G}(x)(\mathrm{e}^{\theta} - 1)\right) - 1\right)\mathrm{d}x$$

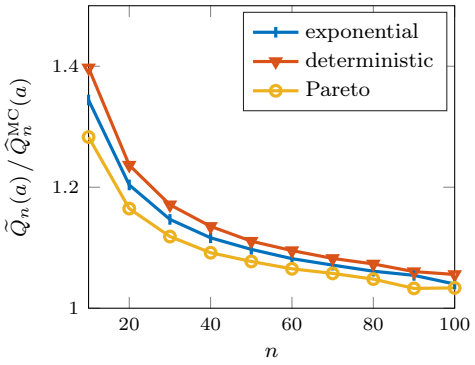
and

$$\int_0^1 \lambda\,\exp\left(\overline{G}(x)(\mathrm{e}^{\theta^\star} - 1)\right)\overline{G}^{\,2}(x)\,\mathrm{e}^{2\theta^\star}\,\mathrm{d}x$$

by numerical integration. Inserting the resulting quantities into the formula provided in Proposition 3.11, we can compute the approximation $\widetilde{Q}_n(a)$ as $\widetilde{q}_n(a)(1-\mathrm{e}^{-\theta^*})^{-1}$ for various a. Consider Figure 3.3 for a comparison of $\widetilde{Q}_n(a)$ with the corresponding estimators $\widehat{Q}_n^{\mathrm{MC}}(a)$ that are obtained by crude Monte Carlo estimation of the probability $Q_n(a)$ as defined in (3.20).



(a) $\widehat{Q}_n(a)$ (solid line) vs. $\widetilde{Q}_n(a)$ (dashed line).



(b) Ratio of $\widetilde{Q}_n(a)$ divided by $\widehat{Q}_n(a)$.

Figure 3.3: Comparison of crude Monte Carlo estimators $\widehat{Q}_n^{\mathrm{MC}}(a)$ and the approximation $\widetilde{Q}_n(a)$ as provided in Proposition 3.11. Parameters are chosen as $a = 0.2$, $\lambda = 0.1$, $E = 1$.

We then proceed to find the value of a, denoted by $a(\varepsilon)$, for which we have $|\widetilde{Q}_n(a) - \varepsilon| < 10^{-9}$ using a bisection method. The results are displayed in Table 3.1; together with M_1, the expected number of customers present at time 1; the Monte Carlo estimates $\widehat{Q}_n^{\mathrm{MC}}\big(a(\varepsilon)\big)$; and the values of $\widetilde{Q}_n(\underline{a})$ and $\widetilde{Q}_n(\overline{a})$, where $\underline{a}$ and $\overline{a}$ are such that the number of servers is integer-valued: $n\underline{a} = \lfloor na(\varepsilon) \rfloor$ and $n\overline{a} = \lceil na(\varepsilon) \rceil$. Surprisingly, the results we obtain for $a(\varepsilon)$ and M_1 suggest that the number of servers required decreases as the variability of the service distribution increases: a relatively small number of servers suffices when service times are Pareto(2), whereas a large number of servers is required for deterministic service times.

At first sight, this outcome may seem counter-intuitive: one would perhaps have expected that unsteady service times would imply that more servers are needed. It is, however, easy to see that this conclusion is not necessarily valid (and in fact false for the example at hand). While it is true that customers arriving at an early slot can be served in time by the 'deterministic servers' with probability 1, customers arriving in later slots can never complete their service in time. For 'random servers' instead, customers arriving early may not finish their service in time but on the other hand customers arriving late still have a chance of completing their service.

In our example, this is reflected in the values of $\omega_i(n)$: bearing in mind that we fixed the value of the mean service time E, the arrival rates in the system with Pareto service times are thinned less in early slots but more in later slots, compared to deterministic service times (see Figure 3.4). That Pareto service times turn out to be better is a result of the fact that the Pareto service times are smaller than E with large probability, and hence the regime in which the Pareto servers outperforms the deterministic servers matters more than the regime in which the deterministic servers are better. Formally, we have that the sum of $\omega_i(n)$ is smallest in the case of Pareto servers, and hence, $S_n = \sum_{i=1}^{n} \mathrm{Pois}\big(X_i\omega_i(n)\big)$ has the smallest exceedance probability in that case.

To further investigate this issue, it is instructive to compute the variance of the steady-state number of clients in the system for the three models for the infinite-server queue. To this end, we can use the formulae that were provided in [64, Eq. (2.31)] for the special case of exponential service times, noting that they can analogously be derived for more

Table 3.1: Values of $a(\varepsilon)$ needed to achieve $|\widetilde{Q}_n\big(a(\varepsilon)\big)-\varepsilon| < 10^{-9}$ with $n = 100$, expected arrival rate $\lambda = 2$, and mean service time E. The Monte Carlo estimates $\widehat{Q}_n\big(a(\varepsilon)\big)$ are also provided (based on 10^9 runs) together with CI, the width of the standard normal 95% confidence interval, as well as the values of the approximation $\widetilde{Q}_n(a)$ with $\underline{a}$ ($\overline{a}$, respectively) such that $n\underline{a} = \lfloor na \rfloor$ ($n\overline{a} = \lceil na \rceil$, respectively). The inferred number of servers is $n\overline{a}$, which should be larger than the expected number of customers M_1 at time 1.

G	ε	E	$a(\varepsilon)$	$n\overline{a}$	$\lceil M_1 \rceil$	$\frac{1}{\varepsilon}\left[\widehat{Q}_n^{\mathrm{MC}}(a(\varepsilon)) \pm \frac{\mathrm{CI}}{2}\right]$	$\frac{1}{\varepsilon}\left(\widetilde{Q}_n(\underline{a}), \widetilde{Q}_n(\overline{a})\right)$
Exponential	10^{-3}	0.05	0.2516	26	10	0.5568 ± 0.0015	$(1.1009, 0.6033)$
		0.5	1.2602	127	87	0.7215 ± 0.0017	$(1.0053, 0.7802)$
		1	1.7537	176	127	0.8099 ± 0.0018	$(1.0784, 0.8780)$
	10^{-4}	0.05	0.2885	29	10	0.8436 ± 0.0057	$(1.7277, 0.9039)$
		0.5	1.3460	135	87	0.8380 ± 0.0057	$(1.1858, 0.8921)$
		1	1.8587	186	127	0.9122 ± 0.0059	$(1.2238, 0.9702)$
Deterministic	10^{-3}	0.05	0.2782	28	10	0.8382 ± 0.0018	$(1.4983, 0.9133)$
		0.5	1.4809	149	100	0.7645 ± 0.0017	$(1.0185, 0.8279)$
		1	2.6636	267	200	0.8353 ± 0.0018	$(1.0565, 0.9070)$
	10^{-4}	0.05	0.3223	33	10	0.6146 ± 0.0049	$(1.1319, 0.6547)$
		0.5	1.5857	159	100	0.8463 ± 0.0057	$(1.1407, 0.9036)$
		1	2.8048	281	200	0.8590 ± 0.0057	$(1.0869, 0.9136)$
Pareto(2)	10^{-3}	0.05	0.2350	24	10	0.6630 ± 0.0016	$(1.3845, 0.7229)$
		0.5	1.0074	101	67	0.8559 ± 0.0018	$(1.2375, 0.9268)$
		1	1.4250	143	100	0.8224 ± 0.0018	$(1.1252, 0.8894)$
	10^{-4}	0.05	0.2688	27	10	0.5721 ± 0.0057	$(1.8616, 0.9194)$
		0.5	1.0818	109	67	0.7223 ± 0.0053	$(1.0613, 0.7633)$
		1	1.5167	152	100	0.8642 ± 0.0058	$(1.1959, 0.9164)$

general service time distributions. We obtain

$$\operatorname{Var}\left(\sum_{i=1}^{n} Z_i\right) = \operatorname{Var} X \sum_{i=1}^{n} \omega_i^2(n) + \mathbb{E}X \sum_{i=1}^{n} \omega_i(n).$$

In case the service times are typically considerably smaller than 1, this behaves as

$$\begin{aligned} n \operatorname{Var} X \int_0^1 \overline{G}^{\,2}(x)\mathrm{d}x + n\,\mathbb{E}X \int_0^1 \overline{G}(x)\mathrm{d}x \\ \approx n \operatorname{Var} X \int_0^\infty \overline{G}^{\,2}(x)\mathrm{d}x + n\,\mathbb{E}X \int_0^\infty \overline{G}(x)\mathrm{d}x. \end{aligned} \tag{3.33}$$

In this decomposition the second part can be interpreted as the variance that one would obtain if the arrival process were Poisson with a constant (non-random) rate $\mathbb{E}X$, whereas the first part is the contribution due to overdispersion. In our example, because X has a Poisson distribution, $\mathbb{E}X = \lambda = \operatorname{Var} X$.

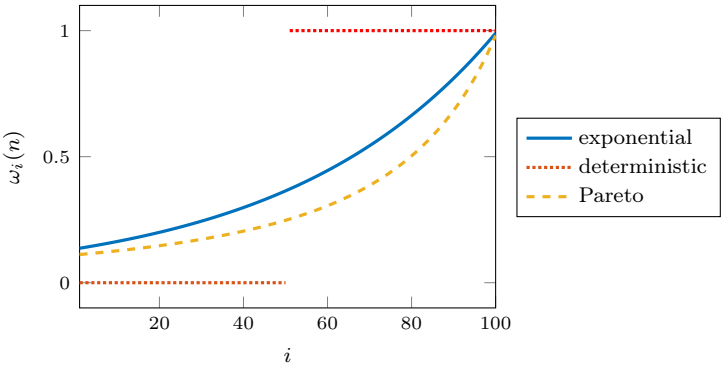


Figure 3.4: Values of $\omega_i(n)$, the probability that a customer arriving in the i-th time slot is still in the system at time 1, where $n = 100$, $E = 0.5$.

The mean number in the system in stationarity is

$$M_\infty := n\,\mathbb{E}X \int_0^\infty \overline{G}(x)\mathrm{d}x = n\,\lambda\,E, \tag{3.34}$$

which shows that this term depends on the service-time distribution only through its mean E.

It thus follows that the second term in the right-hand side of (3.33) equals $n\,\lambda\,E$. We now consider the first (overdispersion-related) term. In the exponential case,

$$\int_0^\infty \overline{G}^{\,2}(x)\mathrm{d}x = \int_0^\infty \mathrm{e}^{-2x/E}\mathrm{d}x = \frac{E}{2};$$

in the deterministic case,

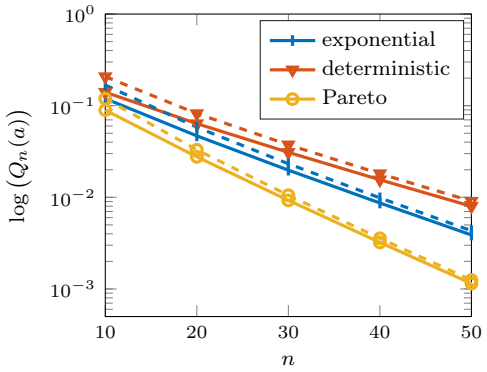
$$\int_0^\infty \overline{G}^{\,2}(x)\mathrm{d}x = \int_0^E \mathrm{d}x = E;$$

and in the Pareto(2) case,

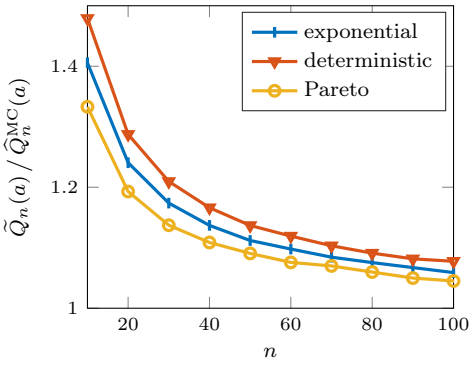
$$\int_0^\infty \overline{G}^{\,2}(x)\mathrm{d}x = \int_0^\infty (1 + x/E)^{-4}\mathrm{d}x = \frac{E}{3}.$$

These computations confirm that the variability in the number of clients in the system is highest when the service times are deterministic, and lowest when they are Pareto(2). This entails that – as we saw from the results in Table 3.1 – if there is overdispersion (i.e., $\mathrm{Var}\,X > 0$), the Pareto(2) case allows for a relatively conservative staffing policy, whereas in the deterministic case comparatively many servers are required.

The table also shows that the required number of servers given by $n\overline{a}$ is, for obvious reasons, larger than M_1, the expected number of customers at time 1. At the same time, $n\overline{a}$ can be substantially *lower* than the expected number of customers in the system in stationarity (i.e., M_∞, as defined in (3.34)), due to the fact that the system has not necessarily reached stationarity at time $t = 1$ (recall that the system starts empty at time 0).



(a) $\widehat{Q}_n(a)$ (solid line) vs. $\widetilde{Q}_n(a)$ (dashed line).



(b) Ratio of $\widetilde{Q}_n(a)$ divided by $\widehat{Q}_n(a)$.

Figure 3.5: Comparison of crude Monte Carlo estimators $\widehat{Q}_n^{\mathrm{MC}}(a)$ and the approximation $\widetilde{Q}_n(a)$ as provided in Proposition 3.11. Parameters are chosen as $E = 0.5$, $p = 0.75$, $\lambda_1 = 1$ and $\lambda_2 = 5$, with $a = 1.6$ for deterministic, $a = 1.4$ for exponential and $a = 1.2$ for Pareto service times.

3.2.3.2 Bursty arrival rate parameters

In a second example we assume that the arrivals are Poisson and usually occur with a certain rate λ_1, but occasionally occur with some larger rate λ_2 (corresponding to peak times in the network). Queueing networks with such "bursty" arrival behavior are of interest in the context of cloud computing, see for example [121, 168].

Specifically, we assume that $\mathbb{P}(X_i = \lambda_1) = p$ and $\mathbb{P}(X_i = \lambda_2) = 1 - p =: \overline{p}$, where p is typically substantially larger than $\frac{1}{2}$. A routine calculation shows that the cumulant-generating function is

$$\Lambda_X(\theta) = \log\left(p\mathrm{e}^{\theta\lambda_1} + \overline{p}\mathrm{e}^{\theta\lambda_2}\right),$$

with derivatives

$$\Lambda_X'(\theta) = \frac{\lambda_1 p\mathrm{e}^{\theta\lambda_1} + \lambda_2\overline{p}\mathrm{e}^{\theta\lambda_2}}{p\mathrm{e}^{\theta\lambda_1} + \overline{p}\mathrm{e}^{\theta\lambda_2}}, \qquad \Lambda_X''(\theta) = \frac{p\overline{p}(\lambda_1 - \lambda_2)^2\mathrm{e}^{\theta(\lambda_1+\lambda_2)}}{\left(p\mathrm{e}^{\theta\lambda_1} + \overline{p}\mathrm{e}^{\theta\lambda_2}\right)^2}.$$

As before, we evaluate the approximation provided in Proposition 3.11 numerically. The obtained approximations and the corresponding Monte Carlo estimates are depicted in Figure 3.5. The counterpart to Table 3.1 is Table 3.2, where the parameters are chosen as in Section 3.2.3.1 (we put $\lambda_1 = 1$, $\lambda_2 = 5$ and $p = 0.75$ so that the mean arrival rate is 2 as before). Compared to the previous example, it seems that here the required number of servers is overall somewhat larger due to the greater variance of the X_i. The ordering of the service time distributions in terms of the required number of servers remains the same as before: the queueing system with deterministic service times requires the largest number of servers.

Table 3.2: Parameters are chosen as in Table 3.1, with arrival rate parameters $p = 0.75$, $\lambda_1 = 1$ and $\lambda_2 = 5$ (so that the expected arrival rate is 2).

G	ε	E	$a(\varepsilon)$	$n\overline{a}$	$\lceil M_1 \rceil$	$\frac{1}{\varepsilon}\left[\widehat{Q}_n^{\mathrm{MC}}(a(\varepsilon)) \pm \frac{\mathrm{CI}}{2}\right]$	$\frac{1}{\varepsilon}\left(\widetilde{Q}_n(\underline{a}), \widetilde{Q}_n(\overline{a})\right)$
Exponential	10^{-3}	0.05	0.2662	27	10	0.7501 ± 0.0017	$(1.4061, 0.8115)$
		0.5	1.2991	130	87	0.9002 ± 0.0019	$(1.2266, 0.9787)$
		1	1.8061	181	127	0.8576 ± 0.0018	$(1.1182, 0.9307)$
	10^{-4}	0.05	0.3056	31	10	0.7199 ± 0.0053	$(1.4107, 0.7615)$
		0.5	1.3942	140	87	0.8089 ± 0.0056	$(1.1124, 0.8601)$
		1	1.9234	193	127	0.8230 ± 0.0056	$(1.0742, 0.8717)$
Deterministic	10^{-3}	0.05	0.3012	31	10	0.6173 ± 0.0015	$(1.0539, 0.6640)$
		0.5	1.5438	155	100	0.8215 ± 0.0018	$(1.0708, 0.8934)$
		1	2.7487	275	200	0.9035 ± 0.0019	$(1.1232, 0.9827)$
	10^{-4}	0.05	0.3484	35	10	0.8783 ± 0.0058	$(1.5388, 0.9209)$
		0.5	1.6632	167	100	0.8187 ± 0.0056	$(1.0669, 0.8690)$
		1	2.9094	291	200	0.9316 ± 0.0060	$(1.1532, 0.9905)$
Pareto(2)	10^{-3}	0.05	0.2461	25	10	0.7264 ± 0.0017	$(1.4490, 0.7888)$
		0.5	1.0381	104	67	0.8755 ± 0.0018	$(1.2856, 0.7069)$
		1	1.4671	147	100	0.8651 ± 0.0018	$(1.1606, 0.9393)$
	10^{-4}	0.05	0.2817	29	10	0.5315 ± 0.0045	$(1.1255, 0.5649)$
		0.5	1.1200	113	67	0.6948 ± 0.0052	$(1.0002, 0.7408)$
		1	1.5688	157	100	0.9138 ± 0.0059	$(1.2335, 0.9709)$

3.3 Conclusion

In this chapter we considered an infinite-server queue with doubly stochastic Poisson arrivals, where the arrival rate is resampled every $n^{-\alpha}$ time units. Among the main contributions of the paper are exact (non-logarithmic, that is) asymptotic expressions for $P_n(a)$, the tail distribution of the number of arrivals at a given time (for $\alpha > 3$ or $\alpha < \frac{1}{3}$); as well as for $Q_n(a)$, the tail probability that more than na customers are present in the system (for the case $\alpha = 1$).

As we saw for the specific example of exponentially distributed arrival rates, the asymptotic expression for $P_n(a)$ can have a rather intricate shape for $\alpha \in \left[\frac{1}{2}, 2\right]$. We do, however, believe that it is possible to derive the asymptotics for the cases $\alpha \in \left[\frac{1}{3}, \frac{1}{2}\right)$ and $\alpha \in (2, 3]$ by using more precise bounds based on the Berry-Esseen inequality.

In numerical examples we showed how the approximation for $Q_n(a)$ can be useful when determining the required number of servers such that at a specific time t (e.g. a certain time of the day) a specific performance target is met. In the future, we aim to extend this staffing rule to one that achieves the desired performance level during an extended period of time, rather than at a single time point. We expect that more refined techniques are needed then, since the staffing level at a certain point in time has impact on the number of customers that are present in the subsequent time interval (although we feel that the

procedure we have developed in this paper may serve as a reasonably accurate proxy).

Finally, we believe that it is possible to extend the results of the chapter by relaxing the assumption that the arrival rates be independent and identically distributed. Instead, one could consider the situation in which the arrival rates in subsequent time intervals depend on each other in a Markovian way. Another interesting topic relates to the infinite-server model in which the random rate of the arrival process changes continuously; in this context we could for instance consider a Coxian arrival process with a shot-noise rate.

CHAPTER 4

Sample-mean related rare-event probabilities

The problem we consider in this chapter is more generic compared to the setting of Chapters 2 and 3; we will point out possible applications later-on. The content of this chapter has appeared in Kuhn, Mandjes and Taimre [88].

Let the sequence $(X_i)_{i=1}^n$ ($(Y_i)_{i=1}^n$, respectively) consist of i.i.d. samples, all of them distributed as a random variable X (Y, respectively); in addition, the sequences are assumed to be mutually independent. In a broad range of applications, it is relevant to quantify the behaviour of the probability, for $n \in \mathbb{N}$,

$$\alpha_1(n) := \mathbb{P}\left(\overline{X}_n \leq \overline{Y}_n\right),$$

with $\overline{X}_n$ and $\overline{Y}_n$ denoting the sample averages

$$\overline{X}_n := \frac{1}{n}\sum_{j=1}^n X_j \quad , \quad \overline{Y}_n := \frac{1}{n}\sum_{j=1}^n Y_j.$$

We throughout assume that $\mathbb{E}\,X > \mathbb{E}\,Y$, which entails that $\alpha_1(n)$ corresponds to a rare event, and therefore vanishes as n grows large. The logarithmic asymptotics of $\alpha_1(n)$ easily follow from Cramér's theorem. Furthermore, recall from Section 1.1.1 that the Bahadur-Rao result states that

$$\alpha_1(n) \sim \frac{C}{\sqrt{n}}\mathrm{e}^{-nI},$$

for positive constants C and I.

A natural next question concerns the context in which there are d independent copies of each of the sample means. More specifically, with $\overline{X}_{1,n}$ up to $\overline{X}_{d,n}$ ($\overline{Y}_{1,n}$ up to $\overline{Y}_{d,n}$, respectively) being i.i.d. copies of $\overline{X}_n$ ($\overline{Y}_n$, respectively), we wish to identify the exact asymptotics of

$$\alpha_d(n) := \mathbb{P}\left(\mathscr{E}_n\right), \quad \text{with } \mathscr{E}_n := \left\{\max_{i\in\{1,\ldots,d\}} \overline{X}_{i,n} \leq \min_{i\in\{1,\ldots,d\}} \overline{Y}_{i,n}\right\}.$$

Some straightforward bounds on $\alpha_d(n)$ can easily be found. It is for instance clear that a necessary condition for $\mathscr{E}_n$ is that $\overline{X}_{i,n} \leq \overline{Y}_{i,n}$ for all $i \in \{1,\ldots,d\}$, and hence the independence of the individual sample means implies the following obvious asymptotic upper bound (in self-evident notation):

$$\alpha_d(n) \lesssim \frac{C^d}{n^{d/2}} \mathrm{e}^{-ndI}, \tag{4.1}$$

as $n \to \infty$ (with C and I as above). The main result of this chapter is that we show that (4.1) is not tight: we prove that, for some $\widetilde{C}_d > 0$, as $n \to \infty$,

$$\alpha_d(n) \sim \frac{\widetilde{C}_d}{n^{d-\frac{1}{2}}} \mathrm{e}^{-ndI} \tag{4.2}$$

(where obviously $\widetilde{C}_1 = C$). The proof relies on careful use of the Bahadur-Rao approximation for all sample means involved.

The exact asymptotics of $\alpha_d(n)$ do not follow from results that have appeared in the literature before, as we point out now. We first observe that the setting introduced above can be cast in a more general framework, involving d^2 sample means. Indeed, with

$$\overline{\boldsymbol{Z}}_n = (\overline{X}_{1,n},\ldots,\overline{X}_{d,n},\overline{Y}_{1,n},\ldots,\overline{Y}_{d,n})^{\mathrm{T}} \in \mathbb{R}^{2d},$$

we can write $\alpha_d(n) = \mathbb{P}(A\overline{\boldsymbol{Z}}_n \geq \boldsymbol{0})$, for an appropriately chosen $d^2 \times 2d$ matrix A with entries in $\{0,\pm 1\}$. Asymptotics of probabilities of the type $\mathbb{P}(A\overline{\boldsymbol{Z}}_n \geq \boldsymbol{b})$ are derived (for $\boldsymbol{b} \in \mathbb{R}^{d^2}$), under specific conditions, by Chaganty and Sethuraman in [29]; they typically have the form of a product of a constant, the polynomial function $n^{-d^2/2}$, and a function that decays exponentially in n. Later on in this chapter, however, we will verify that for the event of our interest the conditions imposed in [29] are *not* met. (Indeed, the polynomial decay term in our asymptotic form (4.2) is $1/n^{d-1/2}$, rather than the $1/n^{d^2/2}$ that one would obtain in the setting of [29].)

We extend the asymptotics of $\alpha_d(n)$ in several ways. In the first place, in Theorem 4.1 we actually establish a slightly more general version of the above asymptotic equivalence, in which the number of sample means $\overline{X}_{i,n}$, say d_X, does not necessarily coincide with the number of sample means $\overline{Y}_{i,n}$, say d_Y. This result is then easily extended to the case where we consider sample means $\overline{X}_{i,p_i n}$ and $\overline{Y}_{j,q_j n}$ where $p_i\, n,\ q_j\, n \in \mathbb{N}$, see Eq. (4.19). We also provide an importance sampling procedure for estimating such probabilities fast and accurately, and we prove that the underlying algorithm is optimal in the sense that it is asymptotically efficient.

In addition, we apply our main result to derive probabilities of practical relevance. More concretely, we obtain an asymptotic expression for the false rejection probability in log-likelihood ratio testing, as well as for the probability of observing at least $k \in \{1,\ldots,d\}$ unordered pairs (where the pair (i,j) is said to be unordered if $\overline{X}_{i,n} < \overline{Y}_{j,n}$). The latter can be formulated in terms of a comparison of order statistics, and may, for example, be understood as the probability that at least k jobs cannot be served, or that at least k items cannot be packed; see Section 4.3.2.

This chapter is organised as follows. In Section 4.1 we define the problem we wish to consider in more detail. Section 4.2 provides the decay rate of $\alpha_d(n)$, and we explain

why this decay rate cannot be obtained from [29]. The result is illustrated by numerical examples, and in this context we also devise an efficient simulation procedure. In Section 4.3 we apply our main result to compare the order statistics of the sample means, again illustrated by an example. We conclude in Section 4.4.

4.1 Problem formulation

We define the set-up considered in our main result (stated and proved in the next section). We let $(X_{i,j})_{j=1}^n$ (with $i \in \{1,\dots,d_X\}$) be independent sequences of i.i.d. random variables $X_{i,j}$, all of them distributed as the generic random variable X. Similarly, for $i \in \{1,\dots,d_Y\}$ we define the i.i.d. sequences $(Y_{i,j})_{j=1}^n$ with $Y_{i,j} \sim Y$. All sequences are assumed to be mutually independent.

Define the sample averages

$$\overline{X}_{i,n} := \frac{1}{n}\sum_{j=1}^{n} X_{i,j},\ \ i \in \{1,\dots,d_X\}, \qquad \overline{Y}_{i,n} := \frac{1}{n}\sum_{j=1}^{n} Y_{i,j},\ \ j \in \{1,\dots,d_Y\}\,,$$

where we assume $\mathbb{E}X > \mathbb{E}Y$. In this chapter we focus on the non-lattice case. The lattice case is more delicate to deal with, and therefore ruled out (as in e.g. [28]). Throughout the chapter we assume that Assumptions I.1 and I.2 are fulfilled for X and Y. We further impose the following assumption.

Assumption I.3. The distributions of the random variables X and Y are continuous.

We now introduce a number of functions and quantities that are useful in Section 4.2. In the first place it turns out to be convenient to define

$$a_{d_X,d_Y} := \arg\min_{a\in\mathbb{R}} J_{d_X,d_Y}(a) =: a^*, \quad J_{d_X,d_Y}(a) := d_X\, I_X(a) + d_Y\, I_Y(a) =: J(a)\,. \tag{4.3}$$

For better readability, we will use the abbreviated notation a^* and $J(a)$ respectively where no confusion can arise. Note that a^* is guaranteed to exist due to the strong convexity of the Legendre transforms [38, Exercise 2.2.24], and can be seen to lie between $\mathbb{E}Y$ and $\mathbb{E}X$. Note that since a^* minimises $J(a)$, it satisfies

$$d_Y\theta_Y(a^*) = d_Y I_Y'(a^*) = -d_X I_X(a^*) = -d_X\theta_X(a^*)\,, \tag{4.4}$$

where $I_X'(a^*) < 0$ and $I_Y'(a^*) > 0$, as a consequence of $\mathbb{E}Y < a^* < \mathbb{E}X$; this 'symmetry' will be useful, particularly in Section 4.2.2. In addition we will need the function

$$K_{d_X,d_Y}(a) := \big(-C_X(a)\big)^{d_X} C_Y(a)^{d_Y}\, d_Y I_Y'(a) := K(a)\,,$$

with $C_X(a)$ and $C_Y(a)$ as defined in (1.7). Note here that $C_X(a) < 0$ and $C_Y(a) > 0$; to see this, bear in mind that $\theta_{-X}(-a) = -\theta_X(a)$.

For our exact asymptotics to hold, we further impose the following regularity condition.

Assumption I.4. $K_{d_X,d_Y}(a)$ is continuous at a^*, and C_X or C_Y are differentiable in a neighbourhood of a^*.

4.2 Exact asymptotics

In this section we provide in Theorem 4.1 the strong large deviations approximation of

$$\alpha_{d_X,d_Y}(n) := \mathbb{P}\left(\mathscr{E}_n\right), \quad \text{with } \mathscr{E}_n := \left\{ \max_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \leq \min_{i\in\{1,\dots,d_Y\}} \overline{Y}_{i,n} \right\}.$$

The result and its proof are presented in Section 4.2.1. In Section 4.2.2 we explain why this result cannot be obtained using the seeming sufficiently general result [29, Theorem 3.4]. In Section 4.2.3 we provide two numerical examples featuring normal and exponential random variables, and point out how these could be estimated efficiently relying on the importance sampling simulation methodology.

4.2.1 Main result

We first state the main result in Theorem 4.1. It says that $\alpha_{d_X,d_Y}(n)$ decays (roughly) exponentially, where the decay rate is given by $J(a^*)$ (with a^* as defined in (4.3)). The polynomial term is of the power $-(d_X+d_Y)/2+1/2$.

Theorem 4.1. *Suppose that X and Y satisfy Assumptions I.1–I.3, and in addition Assumption I.4 applies. Then,*

$$\lim_{n\to\infty} \alpha_{d_X,d_Y}(n)\, \mathrm{e}^{nJ(a^*)}\, n^{(d_X+d_Y)/2-1/2} = K(a^*)\sqrt{\frac{2\pi}{J''(a^*)}}\,. \tag{4.5}$$

Proof. Assume first that C_Y is differentiable (which we can do, due to Assumption I.4). Then our starting point is the obvious identity (that is due to conditional independence)

$$\alpha_{d_X,d_Y}(n) = \int_{-\infty}^{\infty} \left(\mathbb{P}\left(\overline{Y}_{1,n} \geq a\right)\right)^{d_Y} \mathbb{P}\left(\max_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \in \mathrm{d}a\right). \tag{4.6}$$

If instead C_X is differentiable, we can start from

$$\alpha_{d_X,d_Y}(n) = \int_{-\infty}^{\infty} \left(\mathbb{P}\left(\overline{X}_{1,n} \leq a\right)\right)^{d_X} \mathbb{P}\left(\min_{i\in\{1,\dots,d_Y\}} \overline{Y}_{i,n} \in \mathrm{d}a\right),$$

then proceed analogously. We prove a lower and an upper bound of (4.6), which asymptotically coincide.

Lower bound: The first step is to just consider the contribution of $a \in (a^*-\varepsilon, a^*+\varepsilon)$ in (4.6), where we choose ε such that $(a^*-\varepsilon, a^*+\varepsilon)$ is fully covered in the interval $(\mathbb{E}\,Y, \mathbb{E}\,X)$. The Bahadur-Rao result [13], which holds due to Assumptions I.1 and I.2, entails that for any $\delta > 0$ there is an n_0 such that $\alpha_{d_X,d_Y}(n)$ majorises for any $n \geq n_0$,

$$(1-\delta)\int_{a^*-\varepsilon}^{a^*+\varepsilon} \left(\frac{C_Y(a)}{\sqrt{n}}\mathrm{e}^{-nI_Y(a)}\right)^{d_Y} \mathbb{P}\left(\max_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \in \mathrm{d}a\right); \tag{4.7}$$

recall that the convergence in the Bahadur-Rao result holds uniformly [65, 123]. We

proceed by applying integration by parts. To this end, first define

$$\begin{aligned} g(a,n) &:= (1-\delta)\left(\frac{C_Y(a)}{\sqrt{n}}\mathrm{e}^{-nI_Y(a)}\right)^{d_Y}\mathbb{P}\left(\max_{i\in\{1,\ldots,d_X\}}\overline{X}_{i,n}\leq a\right) \\ &\sim (1-\delta)\left(\frac{C_Y(a)}{\sqrt{n}}\mathrm{e}^{-nI_Y(a)}\right)^{d_Y}\left(-\frac{C_X(a)}{\sqrt{n}}\mathrm{e}^{-nI_X(a)}\right)^{d_X}; \end{aligned}$$

where the asymptotic equality '$\sim$' again follows from the Bahadur-Rao result.

Applying integration by parts, we find that Expression (4.7) asymptotically equals the sum of three terms:

$$\begin{aligned} &g(a^*+\varepsilon,n)-g(a^*-\varepsilon,n)-(1-\delta)\int_{a^*-\varepsilon}^{a^*+\varepsilon}\left(-\frac{C_X(a)}{\sqrt{n}}\mathrm{e}^{-nI_X(a)}\right)^{d_X} \\ &\quad d_Y\left(\frac{C_Y(a)^{d_Y-1}C_Y'(a)-nC_Y(a)^{d_Y}I_Y'(a)}{n^{d_Y/2}}\right)\mathrm{e}^{-nd_YI_Y(a)}\mathrm{d}a. \end{aligned} \tag{4.8}$$

Recall that by e.g. [38, Lemma 1.2.15] the decay rate of the sum of three terms equals the largest of the decay rates that correspond to the individual terms. By definition of a^* and the function $J(\cdot)$, for any $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\log g(a^*\pm\varepsilon,n)<-J(a^*); \tag{4.9}$$

later-on we will see that the last term in (4.8) has decay rate $-J(a^*)$, and hence this means that these terms can be asymptotically neglected.

We therefore focus on the last term in (4.8), which can be checked to asymptotically equal

$$(1-\delta)\int_{a^*-\varepsilon}^{a^*+\varepsilon}\frac{\big(-C_X(a)\big)^{d_X}C_Y(a)^{d_Y}}{n^{(d_X+d_Y)/2-1}}\,d_YI_Y'(a)\,\mathrm{e}^{-nJ(a)}\mathrm{d}a.$$

Now define the convex functions $h_X(a):=I_X(a)-I_X(a^*)$ and $h_Y(a):=I_Y(a)-I_Y(a^*)$, which both equal 0 at a^*. We thus find, for n sufficiently large,

$$\begin{aligned} &\alpha_{d_X,d_Y}(n)\,\mathrm{e}^{nJ(a^\star)}\geq \\ &\quad(1-\delta)\int_{a^*-\varepsilon}^{a^*+\varepsilon}\frac{\big(-C_X(a)\big)^{d_X}C_Y(a)^{d_Y}}{n^{(d_X+d_Y)/2-1}}\,d_YI_Y'(a)\,\mathrm{e}^{-n\left[d_Xh_X(a)+d_Yh_Y(a)\right]}\mathrm{d}a. \end{aligned} \tag{4.10}$$

We now study $d_Xh_X(a)+d_Yh_Y(a)$ around $a=a^*$. Setting up a Taylor expansion of $J(a)$ around a^*, we can find a positive function $\psi(a)=o(a^2)$ such that

$$d_Xh_X(a)+d_Yh_Y(a)\leq\frac{1}{2}J''(a^*)(a-a^*)^2+\psi(a-a^*),J''(a^*):=\frac{\mathrm{d}^2}{\mathrm{d}a^2}J(a)\bigg|_{a=a^*}>0\,, \tag{4.11}$$

where we used that $J(a)$ is convex and minimal in a^*. Defining

$$\kappa(a^*,\varepsilon):=\inf_{a\in(a^*-\varepsilon,a^*+\varepsilon)}K_{d_X,d_Y}(a)=\inf_{a\in(a^*-\varepsilon,a^*+\varepsilon)}\big(-C_X(a)\big)^{d_X}C_Y(a)^{d_Y}\,d_YI_Y'(a).$$

and applying the above upper bound (4.11) on $d_Xh_X(a)+d_Yh_Y(a)$, it follows that the

right-hand side of (4.10) majorises

$$\frac{1-\delta}{n^{(d_X+d_Y)/2-1}}\kappa(a^*,\varepsilon)\int_{-\varepsilon}^{\varepsilon}\mathrm{e}^{-n[\frac{1}{2}J''(a^*)a^2+\psi(a)]}\mathrm{d}a. \tag{4.12}$$

To further evaluate the integral in (4.12), we now apply the transformation $b=\sqrt{nJ''(a^*)}\,a$ (such that $\mathrm{d}b=\sqrt{nJ''(a^*)}\,\mathrm{d}a$), so that the expression given in (4.12) reads

$$\frac{1-\delta}{n^{(d_X+d_Y)/2-1/2}}\frac{\kappa(a^*,\varepsilon)}{\sqrt{J''(a^*)}}\int_{-\varepsilon\sqrt{nJ''(a^*)}}^{\varepsilon\sqrt{nJ''(a^*)}}\mathrm{e}^{-b^2/2-n\psi(b/\sqrt{nJ''(a^*)})}\mathrm{d}b.$$

As $n\to\infty$, relying on 'dominated convergence', and recalling that $\psi(a)=o(a^2)$, the integral in the previous display converges to a constant:

$$\int_{-\varepsilon\sqrt{nJ''(a^*)}}^{\varepsilon\sqrt{nJ''(a^*)}}\mathrm{e}^{-b^2/2-n\psi(b/\sqrt{nJ''(a^*)})}\mathrm{d}b\to\int_{-\infty}^{\infty}\mathrm{e}^{-b^2/2}\mathrm{d}b=\sqrt{2\pi}.$$

Combining this with (4.10), we have thus found the asymptotic lower bound, as $n\to\infty$,

$$\liminf_{n\to\infty}\alpha_{d_X,d_Y}(n)\,\mathrm{e}^{nJ(a^*)}\,n^{(d_X+d_Y-1)/2}\geq(1-\delta)\kappa(a^*,\varepsilon)\sqrt{\frac{2\pi}{J''(a^*)}}.$$

Recall that $\delta>0$ and $\varepsilon>0$ were chosen arbitrarily. We thus obtain the lower bound by letting $\delta\downarrow 0$ and $\varepsilon\downarrow 0$:

$$\liminf_{n\to\infty}\alpha_{d_X,d_Y}(n)\,\mathrm{e}^{nJ(a^*)}\,n^{(d_X+d_Y-1)/2}\geq K(a^*)\sqrt{\frac{2\pi}{J''(a^*)}}\,, \tag{4.13}$$

where $K(a^*):=\lim_{\varepsilon\to 0}\kappa(a^*,\varepsilon)$ (where we use I.4).

Upper bound: The upper bound follows by showing that in (4.6) the contributions corresponding to $a\leq a^*-\varepsilon$ (say $\alpha^-_{d_X,d_Y}(n)$) and $a\geq a^*+\varepsilon$ (say $\alpha^+_{d_X,d_Y}(n)$) are asymptotically negligible; the contribution corresponding to the interval $(a^*-\varepsilon,a^*+\varepsilon)$ (say $\alpha^\circ_{d_X,d_Y}(n)$) can be analysed as in the lower bound, in that it can be verified that, under the assumptions imposed,

$$\limsup_{n\to\infty}\alpha^\circ_{d_X,d_Y}(n)\,\mathrm{e}^{nJ(a^*)}\,n^{(d_X+d_Y-1)/2}\leq K(a^*)\sqrt{\frac{2\pi}{J''(a^*)}}.$$

Let us focus on $\alpha^-_{d_X,d_Y}(n)$, i.e., the contribution corresponding to $(-\infty,a^*-\varepsilon]$ (as the contribution due to the interval $[a^*+\varepsilon,\infty)$ can be dealt with precisely analogously); our objective is to prove that its exponential decay rate is strictly smaller than $-J(a^*)$. For

all $\delta > 0$ we can find an n_0 such that for $n \geq n_0$, $\alpha^-_{d_X,d_Y}(n)$ is majorised by

$$(1+\delta)\int_{-\infty}^{\mathbb{E}Y} \mathbb{P}\left(\max_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \in \mathrm{d}a\right) \tag{4.14}$$
$$+ (1+\delta)\int_{\mathbb{E}Y}^{a^*-\varepsilon} \mathrm{e}^{-nd_Y I_Y(a)}\, \mathbb{P}\left(\max_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \in \mathrm{d}a\right);$$

here a Chernoff bound argument is used in the second probability.

We start by considering the first term in (4.14). Suppressing the factor $(1+\delta)$ for the moment, it can be written as

$$\mathbb{P}\left(\max_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \leq \mathbb{E}\,Y\right) = \left(\mathbb{P}\left(\overline{X}_{i,n} \leq \mathbb{E}\,Y\right)\right)^{d_X} \leq \mathrm{e}^{-nd_X I_X(\mathbb{E}Y)}.$$

Now observe that

$$d_X I_X(\mathbb{E}Y) = d_X I_X(\mathbb{E}Y) + d_Y I_Y(\mathbb{E}Y) > d_X I_X(a^*) + d_Y I_Y(a^*) = J(a^*).$$

We conclude that the decay rate of the first term of (4.14) is strictly smaller than $-J(a^*)$.

We now focus on the second term in (4.14). Using integration by parts, we obtain that this is smaller than

$$(1+\delta)\Bigg[\mathrm{e}^{-nd_Y I_Y(a^*-\varepsilon)}\mathbb{P}\left(\overline{X}_{1,n} \leq a^*-\varepsilon\right)^{d_X} \tag{4.15}$$
$$+ \int_{\mathbb{E}Y}^{a^*-\varepsilon} nd_Y I_Y'(a)\mathrm{e}^{-nd_Y I_Y(a)}\mathbb{P}\left(\overline{X}_{1,n} \leq a\right)^{d_X} \mathrm{d}a\Bigg]. \tag{4.16}$$

Since the event $\{\overline{X}_{1,n} \leq a\}$ is rare for $a \leq a^* - \varepsilon < \mathbb{E}X$, we can apply the Bahadur-Rao result to $\mathbb{P}\left(\overline{X}_{1,n} \leq a^*-\varepsilon\right)^{d_X}$. Then, for large n, the first term in (4.15) behaves as

$$\mathrm{e}^{-nd_Y I_Y(a^*-\varepsilon)}\left(\frac{-C_X(a^*-\varepsilon)}{\sqrt{n}}\mathrm{e}^{-nI_X(a^*-\varepsilon)}\right)^{d_X} = \mathrm{e}^{-nJ(a^*-\varepsilon)} n^{d_X/2}(-C_X(a^*-\varepsilon))^{d_X}.$$

Taking the logarithm and dividing by n we see that for large n the decay rate is $-J(a^*-\varepsilon)$, which is smaller than $-J(a^*)$.

Now consider the second term in (4.15), which is asymptotically equal to

$$\int_{\mathbb{E}Y}^{a^*-\varepsilon} n^{1-d_X/2} d_Y I_Y'(a)\big(-C_X(a)\big)^{d_X}\mathrm{e}^{-nJ(a)}\mathrm{d}a. \tag{4.17}$$

Since the Legendre transform $J(\cdot)$ is convex, it follows that $J(a) \geq J'(a^*-\varepsilon)(a-a^*+\varepsilon) + J(a^*-\varepsilon)$ for any a, and thus (4.17) is at most

$$\mathrm{e}^{-nJ(a^*-\varepsilon)}\int_{\mathbb{E}Y}^{a^*-\varepsilon} n^{1-d_X/2} d_Y I_Y'(a)\big(-C_X(a)\big)^{d_X}\mathrm{e}^{-nJ'(a^*-\varepsilon)(a-a^*+\varepsilon)}\mathrm{d}a.$$

Taking the logarithm and dividing by n, we obtain that the decay rate of the second term

in (4.15) is majorised by

$$-J(a^*-\varepsilon)+\limsup_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{E}Y}^{a^*-\varepsilon} d_Y I'_Y(a)\big(-C_X(a)\big)^{d_X}\mathrm{e}^{-nJ'(a^*-\varepsilon)(a-a^*+\varepsilon)}\mathrm{d}a\,.$$

Since J is convex and takes its minimum in a^*, the derivate at $a^*-\varepsilon$ is negative: $J'(a^*-\varepsilon)<0$. On $(-\infty,a^*-\varepsilon]$ we also have $a-a^*+\varepsilon\le 0$, and hence the exponential is at most 1. Hence,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{E}Y}^{a^*-\varepsilon} d_Y I'_Y(a)\big(-C_X(a)\big)^{d_X}\mathrm{e}^{-nJ'(a^*-\varepsilon)(a-a^*+\varepsilon)}\mathrm{d}a\\ \le\limsup_{n\to\infty}\frac{1}{n}\log\int_{\mathbb{E}Y}^{a^*-\varepsilon} d_Y I'_Y(a)\big(-C_X(a)\big)^{d_X}\mathrm{d}a=0.\end{aligned}$$

We conclude that the decay rate of the second term in (4.15) is smaller than $-J(a^*-\varepsilon)$.

Combining the above findings, we have established that the asymptotic exponential decay rate of $\alpha^-_{d_X,d_Y}(n)$ is strictly smaller than $-J(a^*)$ (i.e., the decay rate of $\alpha^\circ_{d_X,d_Y}(n)$).

As we mentioned above, an analogous procedure can be followed for the probability $\alpha^+_{d_X,d_Y}(n)$. Combining all the above elements, it now follows that an asymptotic upper bound on $\alpha_{d_X,d_Y}(n)$ is given by

$$\limsup_{n\to\infty}\alpha_{d_X,d_Y}(n)\,\mathrm{e}^{nJ(a^\star)}\,n^{(d_X+d_Y-1)/2}\le K(a^*)\sqrt{\frac{2\pi}{J''(a^*)}}. \tag{4.18}$$

The lower bound (4.13) and the upper bound (4.18) together yield the desired result (4.5). □

The result can be easily adapted to the situation in which the individual sample means correspond to different numbers of samples. We find that, for p_in, $q_in\in\mathbb{N}$, as $n\to\infty$,

$$\begin{aligned}&\mathbb{P}\left(\max_{i\in\{1,\dots,d_X\}}\overline{X}_{i,np_i}\le\min_{i\in\{1,\dots,d_Y\}}\overline{Y}_{i,nq_i}\right) \\ &\quad\sim\frac{(-C_X(a^*))^{d_X}\,C_Y(a^*)^{d_Y}}{n^{(d_X+d_Y-1)/2}}\,\frac{\overline{q}}{\prod_{i=1}^{d_X}\sqrt{p_i}\,\prod_{j=1}^{d_Y}\sqrt{q_j}}\,I'_Y(a^*)\sqrt{\frac{\pi}{J''_{\overline{p},\overline{q}}(a^*)}}\,\mathrm{e}^{-nJ_{\overline{p},\overline{q}}(a^*)},\end{aligned} \tag{4.19}$$

where now $a^*=\arg\min_a J_{\overline{p},\overline{q}}(a)$ with $\overline{p}:=\sum_{i=1}^{d_X}p_i$, and $\overline{q}:=\sum_{i=1}^{d_Y}q_i$. This more general asymptotic relation may be useful in applications, for example those we mention in Section 4.3.

4.2.2 Comparison with earlier results

In this subsection we compare the main result, as derived in the previous section, with related results from the literature. If $d_X=d_Y=1$, then the asymptotics of (4.2) could also be obtained by applying the Bahadur-Rao result from [13] directly. Therefore, we first verify that indeed our expression coincides with that of Bahadur and Rao in this case.

As mentioned earlier, the event of interest can be written in terms of d_Xd_Y inequalities involving the sample means $\overline{X}_{i,n}$ and $\overline{Y}_{j,n}$, which suggests that we can analyse the

probability $\alpha_{d_X,d_Y}(n)$ using the asymptotic equivalence (1.10) derived in [29]. In case $d_X > 1$ and $d_Y > 1$, however, we show that one of the conditions imposed in [29] is not fulfilled, entailing that our result is thus new for this case. (If either $d_X = 1$ or $d_Y = 1$, then the result from [29] *does* apply.)

The case $d_X = d_Y = 1$. Consider first the case where $d := d_X = d_Y = 1$. Define the sample mean $\overline{Z}_n := n^{-1}\sum_{j=1}^n (Y_j - X_j)$, and note that from the Bahadur-Rao approximation stated in (1.6) we have

$$\alpha_{1,1}(n) = \mathbb{P}\left(\overline{Z}_n \geq 0\right) \sim \frac{C_Z(0)}{\sqrt{n}} \mathrm{e}^{-nI_Z(0)}\,. \tag{4.20}$$

In order to compare this with (4.5), we first check that $\theta_Z(0) = \theta_Y(a^*) = -\theta_X(a^*)$, where the latter equality holds by (4.4). We thus have that $\theta_Y(a^*)$ solves $a^* - \Lambda'_Y(\theta) = 0$ as well as $-a^* + \Lambda'_X(-\theta) = 0$. In conclusion, $\theta_Y(a^*)$ is the unique solution to $\Lambda'_Z(\theta) = \Lambda'_Y(\theta) - \Lambda'_X(-\theta) = 0$, and hence $\theta_Y(a^*) = \theta_Z(0)$. With this relationship it is now readily checked that $J(a^*) = I_Z(0)$. Note that

$$\begin{aligned} J''(a) = a\left[\theta''_X(a) + \theta''_Y(a)\right] + 2\left[\theta'_X(a) + \theta'_Y(a)\right] \\ - \left[\theta''_X(a)\Lambda''_X\big(\theta_X(a)\big) + \theta''_Y(a)\Lambda''_Y\big(\theta_Y(a)\big)\right. \\ \left. + \theta'_X(a)^2\Lambda''_X\big(\theta_X(a)\big) + \theta'_Y(a)^2\Lambda_Y\big(\theta_Y(a)\big)\right]. \end{aligned}$$

Because $\theta'(a) = 1/\Lambda''\big(\theta(a)\big)$ and $\Lambda'\big(\theta(a)\big) = a$, this reduces to

$$J''(a) = \frac{1}{\Lambda''_X\big(\theta_X(a)\big)} + \frac{1}{\Lambda''_Y\big(\theta_Y(a)\big)}\,.$$

We then obtain

$$\begin{aligned} &K(a^*)\sqrt{\frac{2\pi}{J''(a^*)}} \\ &\quad = -\frac{1}{\theta_X(a^*)\sqrt{2\pi\Lambda''_X\big(\theta_X(a^*)\big)}}\,\frac{1}{\theta_Y(a^*)\sqrt{2\pi\Lambda''_Y\big(\theta_Y(a^*)\big)}}\,\theta_Y(a^*)\sqrt{\frac{2\pi}{J''(a^*)}} \\ &\quad = \frac{1}{\theta_Y(a^*)\sqrt{2\pi\left[\Lambda''_X\big(\theta_X(a^*)\big) + \Lambda''_Y\big(\theta_Y(a^*)\big)\right]}} = C_Z(0)\,. \end{aligned}$$

Thus, we conclude that (4.5) reduces to (4.20) if $d = 1$.

The case $d_X > 1$ and $d_Y > 1$. Now we consider the case that both $d_X > 1$ and $d_Y > 1$ and show that our result does not fall in the framework of [29]. As was already briefly pointed out in the introduction, we can rewrite $\alpha_{d_X,d_Y}(n)$ as $\mathbb{P}(A\overline{\boldsymbol{Z}}_n \geq \mathbf{0})$, where

$$\overline{\boldsymbol{Z}}_n = (\overline{X}_{1,n}, \ldots, \overline{X}_{d_X,n}, \overline{Y}_{1,n}, \ldots, \overline{Y}_{d_Y,n})^{\mathrm{T}},$$

and A an appropriately chosen matrix of dimension $d_X d_Y \times (d_X + d_Y)$. In [29, Theorem 3.4] it is proved that, conditional on certain assumptions being satisfied, for positive

constants C and I,

$$\mathbb{P}(A\overline{\boldsymbol{Z}}_n \geq \mathbf{0}) \sim \frac{C}{n^{(d_X\, d_Y)/2}} \mathrm{e}^{-nI}.$$

In Theorem 4.1 we showed that the polynomial factor in the asymptotics is of the form $n^{-(d_X+d_Y)/2+1/2}$ rather than $n^{-(d_X\, d_Y)/2}$; in this section we show that this seeming inconsistency is due to the fact that [29, Condition (B)] is not met. Observe that if $d_X = 1$ or $d_Y = 1$ the powers match; we therefore consider the situation that both d_X and d_Y are strictly larger than 1.

Let us first define the multivariate cumulant-generating function. To this end, we write $\overline{W}_{ij,n} = \overline{Y}_{j,n} - \overline{X}_{i,n}$, with $i \in \{1, \ldots, d_X\}$ and $j \in \{1, \ldots, d_Y\}$; observe that the probability of our interest equals $\mathbb{P}(\overline{\boldsymbol{W}}_n \geq \mathbf{0})$, where $\overline{\boldsymbol{W}}_n$ is the $d_X d_Y$-vector with entries $W_{ij,n}$. Then the corresponding multivariate moment generating function is given by

$$M(\boldsymbol{\theta}) := \prod_{j=1}^{d_Y} \mathbb{E}\left[\mathrm{e}^{Y_j \sum_{i=1}^{d_X} \theta_{i,j}}\right] \prod_{i=1}^{d_X} \mathbb{E}\left[\mathrm{e}^{-X_i \sum_{j=1}^{d_Y} \theta_{i,j}}\right],$$

and hence the multivariate cumulant function equals

$$\Lambda(\boldsymbol{\theta}) := \log M(\boldsymbol{\theta}) = \sum_{j=1}^{d_Y} \Lambda_Y\left(\sum_{i=1}^{d_X} \theta_{i,j}\right) + \sum_{i=1}^{d_X} \Lambda_X\left(-\sum_{i=1}^{d_Y} \theta_{i,j}\right).$$

Let $\boldsymbol{\theta}^*$ solve $\Lambda'(\boldsymbol{\theta}) = \mathbf{0}$; it is readily checked that all $d_X\, d_Y$ entries of $\boldsymbol{\theta}^*$ are equal (say, have value τ), and solve the equation $\Lambda_Y'\big(d_X\tau\big) = \Lambda_X'\big(-d_Y\tau\big)$. Then [29, Condition (B)] states that the determinant of the Hessian of $\Lambda(\boldsymbol{\theta}^*)$ should be different from 0. An elementary computation yields that the elements of this Hessian are given by, with $k, \overline{k} \in \{1, \ldots, d_X\}$ and $\ell, \overline{\ell} \in \{1, \ldots, d_Y\}$,

$$\frac{\partial^2 \Lambda(\boldsymbol{\theta})}{\partial \theta_{k,\ell} \partial \theta_{\overline{k},\overline{\ell}}} = r_\ell \mathbb{1}\{\ell = \overline{\ell}\} + s_k \mathbb{1}\{k = \overline{k}\},$$

where

$$r_\ell := \Lambda_Y''\left(\sum_{i=1}^{d_X} \theta_{i,\ell}\right), \quad s_k := \Lambda_X''\left(-\sum_{j=1}^{d_Y} \theta_{k,j}\right).$$

Let $R(\boldsymbol{\theta}) := \mathrm{diag}\{\boldsymbol{r}\}$ and $S(\boldsymbol{\theta}) := \mathrm{diag}\{\boldsymbol{s}\}$; in addition, E is a $d_X \times d_X$ all-ones matrix, and F a $d_Y \times d_Y$ all-ones matrix. Then we can write the Hessian compactly by

$$H(\boldsymbol{\theta}) = R(\boldsymbol{\theta}) \otimes E + F \otimes S(\boldsymbol{\theta}),$$

where $\otimes$ denotes the Kronecker product. Let $\boldsymbol{e}_k$ be the k-th d_X-dimensional unit row vector (i.e., $\boldsymbol{e} \in \mathbb{R}^{d_X}$ such that the k-th entry is 1 and all other entries 0). Likewise, $\boldsymbol{f}_\ell$ denotes the ℓ-th d_Y-dimensional unit row vector. Then define, for arbitrary $k \neq \overline{k}$ and $\ell \neq \overline{\ell}$ (which is possible as $d_X \geq 2$ and $d_Y \geq 2$),

$$\boldsymbol{v} := (\boldsymbol{e}_k \otimes \boldsymbol{f}_\ell) - (\boldsymbol{e}_k \otimes \boldsymbol{f}_{\overline{\ell}}) - (\boldsymbol{e}_{\overline{k}} \otimes \boldsymbol{f}_\ell) + (\boldsymbol{e}_{\overline{k}} \otimes \boldsymbol{f}_{\overline{\ell}}).$$

It is then an elementary computation to conclude that $\boldsymbol{v} H(\boldsymbol{\theta}^*) = \mathbf{0}$, and hence $H(\boldsymbol{\theta}^*)$ is

singular. We conclude that [29, Condition (B)] does not apply.

The intuitive reason for the violation of the condition is that some of the $d_X\, d_Y$ restrictions are essentially redundant. For example, if $d_X = d_Y = 2$, then $\overline{Y}_{1,n} - \overline{X}_{1,n} > 0$ will usually occur by a realisation in which $\overline{Y}_{1,n} \approx \overline{X}_{1,n}$, and similarly for $\overline{Y}_{1,n} - \overline{X}_{2,n} > 0$ and $\overline{Y}_{2,n} - \overline{X}_{1,n} > 0$. Thus, informally speaking, these three conditions boil down to requiring that $\overline{Y}_{1,n} \approx \overline{X}_{1,n} \approx \overline{Y}_{2,n} \approx \overline{X}_{2,n}$. As a consequence, the fourth constraint, i.e., $\overline{Y}_{2,n} - \overline{X}_{2,n} > 0$, is already ensured to hold by the first three conditions with high likelihood. With this line of reasoning it also becomes intuitively clear that we should have $n^{-(d_X+d_Y)/2+1/2}$ as a pre-factor, as we obtained in (4.5). Informally, [29, Condition (B)] ensures that none of the restrictions imposed by $\overline{\boldsymbol{W}}_n \geq \mathbf{0}$ is redundant.

The case $d_X = 1$ or $d_Y = 1$. We finally show that in case $d_X = 1$ or $d_Y = 1$ the result from [29] does apply. This can be seen as follows. Let us assume that $d_X = d \geq 1$ and $d_Y = 1$ (the opposite case works analogously). Then by Sylvester's theorem it follows that

$$|H(\boldsymbol{\theta})| = |S(\boldsymbol{\theta})| \left|I + S(\boldsymbol{\theta})^{-1} r E\right| .$$

Note that $S(\boldsymbol{\theta})^{-1} r E$ is a matrix with rows $(r/s_k, \ldots, r/s_k)$. Furthermore, $|S(\boldsymbol{\theta})| = \prod_{k=1}^{d} s_k$. It can then be checked that

$$|H(\boldsymbol{\theta})| = \sum_{x \in \chi} \prod_{i=1}^{d} x_i \,,$$

where χ denotes the set of all combinations of length d from $\{r, s_1, \ldots, s_d\}$ (hence, $|\chi| = d+1$). Now, inserting $r = \Lambda_Y''\big(d\tau\big)$ and $s_k = \Lambda_X''\big(-\tau\big)$, we obtain that the determinant of $H(\boldsymbol{\theta}^*)$ is non-zero:

$$|H(\boldsymbol{\theta}^*)| = d\Lambda_X''\big(-\tau\big)^{d-1} \Lambda_Y''\big(d\tau\big) + \Lambda_X''\big(-\tau\big)^{d} .$$

Invoking (4.4) we note that $d\tau = \theta_Y(a^*) = -d\theta_X(a^*)$. Thus, the result from [29] states that

$$\alpha_{d,1}(n) \sim \frac{1}{(2\pi n)^{d/2}} \left(d\Lambda_X''\big(\theta_X(a^*)\big)^{d-1} \Lambda_Y''\big(\theta_Y(a^*)\big) + \Lambda_X''\big(\theta_X(a^*)\big)^{d}\right)^{-1/2} \mathrm{e}^{n\left[d\Lambda_X\left(\theta_X(a^*)\right) + \Lambda_Y\left(\theta_Y(a^*)\right)\right]} .$$

This can be checked to be equivalent to the expression given in Theorem 4.1, using that

$$J_{d,1}''(a^*) = \frac{d}{\Lambda_X''(\theta_X(a^*))} + \frac{1}{\Lambda_Y''(\theta_Y(a^*))} .$$

4.2.3 Examples and importance sampling

In this subsection we compute the asymptotic expressions we derived for two examples with Gaussian and exponentially distributed random variables, respectively. For the purpose of comparing these asymptotically accurate approximations to the value of $\alpha_{d_X,d_Y}(n)$ as obtained by simulation, we point out how to set up a provably asymptotically efficient

importance sampling procedure for general random variables satisfying Assumptions I.1 and I.2.

1. *Gaussian.* In this example we let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ and fix n. We have
$$\Lambda_{-X}(\theta) = -\theta\mu_X + \frac{1}{2}\sigma_X^2\theta^2\,,$$
so that $\theta_{-X}(-a) = -(a-\mu_X)/\sigma_X^2 = -\theta_X(a)$. It follows directly that
$$I_{-X}(-a) = \frac{1}{2}\left(\frac{a-\mu_X}{\sigma_X}\right)^2 = I_X(a)\,.$$
A similar procedure can be followed for Y. Furthermore, note that $J(a) = d_X I_X(a) + d_Y I_Y(a)$ is minimised by
$$a^* = \frac{d_X\mu_X\sigma_Y^2 + d_Y\mu_Y\sigma_X^2}{d_X\sigma_Y^2 + d_Y\sigma_X^2}\,;$$
indeed, as we remarked earlier, this quantity lies in the interval (μ_Y, μ_X)), We thus arrive at the following expression for the decay rate of $\alpha_{d_X,d_Y}(n)$:
$$J(a^*) = \frac{d_X d_Y}{2}\frac{(\mu_Y-\mu_X)^2}{d_Y\sigma_X^2 + d_X\sigma_Y^2}\,.$$
For $d_X = d_Y = 1$ and $\sigma_X = \sigma_Y$ this is just the Kullback-Leibler divergence between X and Y. Moreover, note that
$$\theta_{-X}(-a^*) = -\frac{d_Y(\mu_Y-\mu_X)}{d_X\sigma_Y^2 + d_Y\sigma_X^2}\,, \quad \theta_Y(a^*) = \frac{d_X(\mu_X-\mu_Y)}{d_X\sigma_Y^2 + d_Y\sigma_X^2}\,,$$
and hence
$$-C_X(a^*) = -\frac{d_X\sigma_Y^2 + d_Y\sigma_X^2}{d_Y(\mu_Y-\mu_X)\sqrt{2\pi\sigma_X^2}}\,, \quad C_Y(a^*) = \frac{d_X\sigma_Y^2 + d_Y\sigma_X^2}{d_X(\mu_X-\mu_Y)\sqrt{2\pi\sigma_Y^2}}$$
(which can both be checked to be positive). With $I_Y'(a^*) = \theta_Y(a^*)$, we thus obtain
$$K(a^*) = \big(-C_X(a^*)\big)^{d_X} C_Y(a^*)^{d_Y}\, d_Y I_Y'(a^*)\,.$$

2. *Exponential.* The logarithmic MGF of an exponential random variable with parameter λ is, for $\theta < \lambda$, given by $\Lambda(\theta) = \log\lambda - \log(\lambda-\theta)$, so that (with $\theta(a) = \lambda - 1/a$, assuming $a \neq 0$), $I(a) = \lambda a - 1 - \log(\lambda a)$. For exponential X and Y with $\lambda_X < \lambda_Y$ we thus have
$$J(a) = a\,(d_X\lambda_X + d_Y\lambda_Y) - (d_X+d_Y)\big(\log(a)+1\big) - d_X\log(\lambda_X) - d_Y\log(\lambda_Y)\,,$$
which is minimal at
$$a^* = \frac{d_X+d_Y}{d_X\lambda_X + d_Y\lambda_Y}\,.$$

We obtain

$$\theta_{-X}(-a^*) = -\frac{d_Y(\lambda_X - \lambda_Y)}{d_X + d_Y}, \quad \theta_Y(a^*) = \frac{d_X(\lambda_Y - \lambda_X)}{d_X + d_Y}$$

(thus, indeed $\theta_{-X}(-a^*) < \lambda_X$ and $\theta_Y(a^*) < \lambda_Y$, and hence the MGFs are defined at these points). We have

$$C_X(a^*) = \frac{d_X\lambda_X + d_Y\lambda_Y}{\sqrt{2\pi d_Y(\lambda_X - \lambda_Y)}}, \quad C_Y(a^*) = \frac{d_X\lambda_X + d_Y\lambda_Y}{\sqrt{2\pi d_X(\lambda_Y - \lambda_X)}}$$

and $J(a^*) = -(d_X + d_Y)\log(a^*) - d_X\log(\lambda_X) - d_Y\log(\lambda_Y)$.

Our asymptotic results describe how $\alpha_{d_X,d_Y}(n)$ behaves as $n \to \infty$, but do not provide any error bound for a given $n_0 \in \mathbb{N}$. We therefore now describe an importance sampling algorithm that can be used to efficiently estimate $\alpha_{d_X,d_Y}(n)$ for moderate n by means of simulation.

Let $f_X(\cdot)$ be the density of X, and $f_Y(\cdot)$ the density of Y. Now associate the exponentially twisted measure $\mathbb{Q}$ with the system in which the $X_{i,k}$ and $Y_{j,\ell}$ are sampled according to the densities

$$g_X(x) = \frac{\mathrm{e}^{\theta_X(a^*)x}}{M_X(\theta_X(a^*))} f_X(x), \quad g_Y(x) = \frac{\mathrm{e}^{\theta_Y(a^*)y}}{M_Y(\theta_Y(a^*))} f_Y(x)\,.$$

Recall that a^* minimises $J(a)$, and therefore solves $-d_X\theta_X(a) = d_Y\theta_Y(a)$ (where it is used that $I_X'(a) = \theta_X(a)$ and $I_Y'(a) = \theta_Y(a)$). It is readily checked that $\mathbb{E}\,X > \mathbb{E}\,Y$ implies that $\theta_X(a) < 0$ and $\theta_Y(a) > 0$.

The usual change-of-measure argument entails that, in self-evident notation,

$$\alpha_{d_X,d_Y}(n) = \mathbb{E}_{\mathbb{Q}}\left[L\,\mathbb{1}_{\{\mathscr{E}_n\}}\right], \quad \text{where} \quad L := \left(\prod_{i=1}^{d_X}\prod_{k=1}^{n} L_X(X_{i,k})\right)\left(\prod_{j=1}^{d_Y}\prod_{\ell=1}^{n} L_Y(Y_{j,\ell})\right),$$

with the per-sample likelihood ratios defined by

$$L_X(x) = M_X(\theta_X(a^*))\mathrm{e}^{-\theta_X(a^*)x}, \quad L_Y(y) = M_Y(\theta_Y(a^*))\mathrm{e}^{-\theta_Y(a^*)y}\,.$$

To prove asymptotic efficiency of the resulting estimator, we need to show that

$$\limsup_{n\to\infty} \frac{1}{n}\log \mathbb{E}_{\mathbb{Q}}(L^2\,\mathbb{1}_{\{\mathscr{E}_n\}}) \le \limsup_{n\to\infty} \frac{2}{n}\log \mathbb{E}_{\mathbb{Q}}(L\,\mathbb{1}_{\{\mathscr{E}_n\}}) = -2J(a^*)\,.$$

To this end, we first rewrite $\mathbb{E}_{\mathbb{Q}}(L^2\,\mathbb{1}_{\{\mathscr{E}_n\}})$ as

$$\begin{aligned}&\left(M_X(\theta(a^*))\right)^{2nd_X}\left(M_Y(\theta(a^*))\right)^{2nd_Y}\\ &\qquad \mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\theta_X(a^*)\sum_{i=1}^{d_X}\sum_{k=1}^{n}X_{i,k}}\mathrm{e}^{-2\theta_Y(a^*)\sum_{j=1}^{d_Y}\sum_{\ell=1}^{n}Y_{j,\ell}}\mathbb{1}_{\{\mathscr{E}_n\}}\right].\end{aligned}$$

The next step is to bound, on the event $\mathscr{E}_n$, the exponential term. To this end, note that, on $\mathscr{E}_n$, for all $i \in \{1,\dots,d_X\}$ and $j \in \{1,\dots,d_Y\}$, we have that $\sum_{k=1}^{n} Y_{i,k} \ge \sum_{\ell=1}^{n} X_{j,\ell}$.

Summing this inequality over all i and j and dividing by $d_X d_Y$ we obtain, on $\mathscr{E}_n$,

$$\frac{1}{d_X}\sum_{i=1}^{d_X}\sum_{k=1}^{n} X_{i,k} \le \frac{1}{d_Y}\sum_{j=1}^{d_Y}\sum_{\ell=1}^{n} Y_{j,\ell}\,.$$

It now follows that, recalling that $-d_X\theta_X(a^*) = d_Y\theta_Y(a^*)$,

$$\begin{aligned}
-\theta_Y(a^*)\sum_{i=1}^{d_Y}\sum_{k=1}^{n} Y_{i,k} &= -d_Y\theta_Y(a^*)\frac{1}{d_Y}\sum_{i=1}^{d_Y}\sum_{k=1}^{n} Y_{i,k} \\
&\le -d_Y\theta_Y(a^*)\frac{1}{d_X}\sum_{j=1}^{d_X}\sum_{\ell=1}^{n} X_{j,\ell} \\
&= d_X\theta_X(a^*)\frac{1}{d_X}\sum_{j=1}^{d_X}\sum_{\ell=1}^{n} X_{j,\ell} = \theta_X(a^*)\sum_{j=1}^{d_X}\sum_{\ell=1}^{n} X_{j,\ell}\,,
\end{aligned}$$

from which we conclude that, for any $n \in \mathbb{N}$,

$$\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-2\theta_X(a^*)\sum_{i=1}^{d_X}\sum_{k=1}^{n} X_{i,k}}\mathrm{e}^{-2\theta_Y(a^*)\sum_{j=1}^{d_Y}\sum_{\ell=1}^{n} Y_{j,\ell}}\mathbb{1}_{\{\mathscr{E}_n\}}\right] \le 1\,.$$

This yields the desired inequality:

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}_{\mathbb{Q}}\left[L^2\,\mathbb{1}_{\{\mathscr{E}_n\}}\right] &\le 2d_X\,\Lambda_X(\theta_X(a^*)) + 2d_Y\,\Lambda_Y(\theta_Y(a^*)) \\
&= -2a^*\left[d_X\theta_X(a^*) + d_Y\theta_Y(a^*)\right] \\
&\quad + 2d_X\,\Lambda_X(\theta_X(a^*)) + 2d_Y\,\Lambda_Y(\theta_Y(a^*)) \\
&= -2J(a^*)\,.
\end{aligned}$$

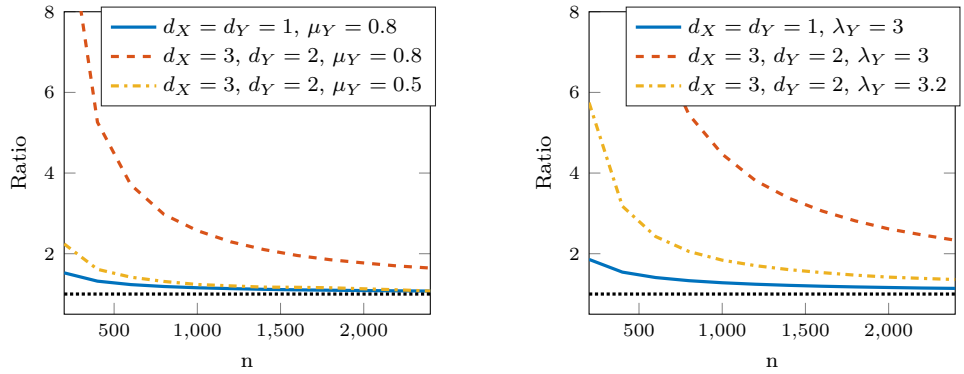
We have thus found the following result.

Proposition 4.2. *The measure $\mathbb{Q}$ yields an asymptotically efficient procedure for estimating $\alpha_{d_X,d_Y}(n)$.*

In the remainder of this section we examine the accuracy of approximation by the exact asymptotics of $\alpha_{d_X,d_Y}(n)$. With the proposed importance sampling procedure, and inserting the explicit expressions we found for Gaussian and Exponential random variables, we can compare the asymptotic formula given by Theorem 4.1 to the probabilities as estimated by simulation. Some examples are provided in Fig. 4.1. The two examples indicate that the approximation tends to be more accurate if (i) d_X and d_Y are smaller or (ii) if the means of X and Y differ more. if the means of X and Y differ more. The former could be a consequence of the fact that we used a number of additional approximation steps compared to Bahadur and Rao in order to extend their result. The latter may be due to the fact that in this case the event is more rare in which case the applied large deviations approximations are more accurate.

4.3 Further refinements and applications

Motivated by specific practical applications, we now study two variants of our main result.



(a) Gaussian random variables, $\mu_Y = 1$, $\sigma = 2$. (b) Exponential random variables, $\lambda_X = 2.8$.

Figure 4.1: Ratio of the asymptotic expression (4.5) and simulated probabilities $\alpha_{d_X,d_Y}(n)$. The dotted horizontal line indicates a ratio of 1.

4.3.1 At least one unordered sample mean pair

It is directly seen that Theorem 4.1 allows us to conclude that

$$\mathbb{P}\left(\min_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \leq \max_{i\in\{1,\dots,d_Y\}} \overline{Y}_{i,n}\right) \sim \sum_{i=1}^{d_X}\sum_{j=1}^{d_Y} \mathbb{P}\left(\overline{Y}_{j,n} - \overline{X}_{i,n} \geq 0\right) \tag{4.21}$$

because the decay rate corresponding to events $\left\{\overline{Y}_{j,n} - \overline{X}_{i,n} \geq 0\right\}$ is $-\inf_a J_{1,1}(a)$, which is larger than the rate functions corresponding to any number of intersections of such events given that those correspond to $\inf_a J_{i,j}(a)$ for $i + j > 2$. Then the asymptotic relation (4.21) follows from the inclusion-exclusion principle. It is thus evident that

$$\begin{aligned}
&\mathbb{P}\left(\min_{i\in\{1,\dots,d_X\}} \overline{X}_{i,n} \leq \max_{i\in\{1,\dots,d_Y\}} \overline{Y}_{i,n}\right) \\
&\quad\sim d_X d_Y \mathbb{P}\left(\overline{X}_{1,n} \leq \overline{Y}_{1,n}\right) \\
&\quad\sim \mathrm{e}^{-nJ_{1,1}(a_{1,1})} \frac{1}{\sqrt{n}} d_X d_Y K_{1,1}(a_{1,1}) \sqrt{\frac{2\pi}{J''_{1,1}(a_{1,1})}}\,.
\end{aligned}$$

This probability has applications in LLR testing. Specifically, it allows to derive the exact asymptotics of the false selection probability we consider in the anomaly identification problem in Chapter 9.

For a more specific example, suppose d_X signals are sent from an echo sounding system, and in return $d_X + d_Y$ echoes are received, d_Y of which have to thus to be identified as noise. If this echo sounding experiment is carried out n times, the probability of wrongly discarding a signal as noise can be evaluated as a probability of the form (4.21).

If we relax the assumption that the distributions of X and Y are known (for example, replace the MGFs of X and Y by their maximum likelihood estimators), one may also think of applications in ordinal optimisation problems such as stochastic bandit problems, see e.g. [51, 52].

4.3.2 At least k unordered sample mean pairs

Denote the order statistics of the sample means of X and Y by $\overline{X}_{(i),n}$ and $\overline{Y}_{(j),n}$; we assume that these order statistics have been put in decreasing order. We here focus on the evaluation of the probability $1 - \mathbb{P}\left(\overline{X}_{(1),n} > \overline{Y}_{(1),n}, \ldots, \overline{X}_{(d),n} > \overline{Y}_{(d),n}\right)$, or, more generally (as we can put $k = 1$)

$$\beta_{d,k}(n) := \mathbb{P}\left(\exists i \in \{1, \ldots, d-k+1\} : \overline{X}_{(i),n} \leq \overline{Y}_{(k+i-1),n}\right), \tag{4.22}$$

which is the probability that for every bijection mapping the set of indices of $\overline{X}_n$ to the set of indices of $\overline{Y}_n$ there exist at least k unordered pairs (the pair (i, j) is unordered if $\overline{X}_{i,n} < \overline{Y}_{j,n}$).

For a potential application of this type of probability, think of the following static control problem. We have d ships with n containers, and dnc items that need to be packed onto these ships. We assume that the items are separated into d loads (for example, they came from d trucks) of n batches of c items. The expected capacity of each container is μ_X – the actual capacity is random (for example, it might be that the containers arrive more or less empty than expected). The total observed capacity of ship i is $n\overline{X}_{i,n}$. The items have an expected size of μ_Y/c, so that each batch of c items has an expected size of μ_Y. The total size of load j is $n\overline{Y}_{j,n}$. After observing $n\overline{X}_{i,n}$ and $n\overline{Y}_{j,n}$ each of the d loads needs to be brought to a ship and packed into the containers. In this case the question of whether the full load can be packed (if the batches are assigned carefully) boils down to whether or not there exists a perfect matching of order statistics. More generally we can ask for the probability that at least k loads cannot be packed, which is given by (4.22). This could, for example, be used as a performance criterion which can be set to a desired level in order to estimate the required number of ships assuming all other parameters are fixed.

For another application, suppose that we want to assign memory space/server capacities to serve d batches/queues of jobs in an open-loop (static) manner. Suppose there are $np \in \mathbb{N}$ jobs in each batch. (As we remarked in (4.19) it is easy to adapt our results for the case where one of the populations has sample size pn instead of n.) The expected job size/duration is μ_Y. The size of the jobs in batch i amounts to $np\overline{Y}_{i,np}$. Each batch has to be assigned to one of d server pools, each with n servers with expected capacity $\mathbb{E}X$. The actual service capacity of server pool j amounts to $n\overline{X}_{j,n}$. Clearly a quantity of interest is of the form (4.22), which can be interpreted as the probability that at least k job batches cannot be served. This may be a useful performance criterion in the context of staffing; see also the discussion in Section 2.5.

The main result of this subsection is as follows. It states that the asymptotics of $\beta_{d,k}(n)$ are essentially determined by those of $\alpha_{d_X,d_Y}(n)$.

Proposition 4.3. *Assume that Assumptions I.1–I.4 hold, and in addition that i^* defined by*

$$i^* := \operatorname*{arg\,min}_{i \in \{1,\ldots,d-k+1\}} J_{d-i+1,k+i-1}(A_i), \tag{4.23}$$

with $A_i := a_{d-i+1,k+i-1}$, is unique. Then,

$$\beta_{d,k}(n) \sim \binom{d}{k+i^*-1} \binom{d}{d-i^*+1} \alpha_{d-i^*+1,k+i^*-1}(n)\,. \tag{4.24}$$

Proof. First, we note that we can write

$$\begin{aligned}
&\mathbb{P}\left(\overline{X}_{(i),n} \leq \overline{Y}_{(i+k-1),n}\right) \\
&\quad = \begin{pmatrix} d \\ k+i-1 \end{pmatrix} \begin{pmatrix} d \\ d-i+1 \end{pmatrix} \mathbb{P}\Big(\min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} > \max_{j\in\{i,\dots,d\}} \overline{X}_{j,n}, \\
&\qquad\qquad \min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} \geq \max_{j\in\{k+i,\dots,d\}} \overline{Y}_{j,n}, \\
&\qquad\qquad \max_{j\in\{i,\dots,d\}} \overline{X}_{j,n} \leq \min_{j\in\{1,\dots,i-1\}} \overline{X}_{j,n} \Big).
\end{aligned}$$

The probability on the right-hand side can be computed as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{a}^{\infty} \mathbb{P}\left(\max_{j\in\{k+i,\dots,d\}} \overline{Y}_{j,n} \leq b \right) \mathbb{P}\left(\min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} \in \mathrm{db} \right) \\
&\qquad\qquad \mathbb{P}\left(\max_{j\in\{1,\dots,i-1\}} \overline{X}_{j,n} \geq a \right) \mathbb{P}\left(\max_{j\in\{i,\dots,d\}} \overline{X}_{j,n} \in \mathrm{da} \right).
\end{aligned} \tag{4.25}$$

We again prove a lower and an upper bound which asymptotically coincide.

Lower bound: A lower bound for (4.25) is given by

$$\begin{aligned}
&\int_{A_{i^*}-\varepsilon}^{A_{i^*}+\varepsilon} \int_{a}^{A_{i^*}+\varepsilon} \mathbb{P}\left(\max_{j\in\{k+i,\dots,d\}} \overline{Y}_{j,n} \leq A_{i^*} - \varepsilon \right) \mathbb{P}\left(\min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} \in \mathrm{db} \right) \\
&\qquad\qquad \mathbb{P}\left(\max_{j\in\{1,\dots,i-1\}} \overline{X}_{j,n} \geq A_{i^*} + \varepsilon \right) \mathbb{P}\left(\max_{j\in\{i,\dots,d\}} \overline{X}_{j,n} \in \mathrm{da} \right),
\end{aligned}$$

which asymptotically equals

$$\int_{A_{i^*}-\varepsilon}^{A_{i^*}+\varepsilon} \int_{a}^{A_{i^*}+\varepsilon} \mathbb{P}\left(\min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} \in \mathrm{db} \right) \mathbb{P}\left(\max_{j\in\{i,\dots,d\}} \overline{X}_{j,n} \in \mathrm{da} \right).$$

This can be rewritten as

$$\begin{aligned}
&\int_{A_{i^*}-\varepsilon}^{A_{i^*}+\varepsilon} \mathbb{P}\left(\min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} \geq a \right) \mathbb{P}\left(\max_{j\in\{i,\dots,d\}} \overline{X}_{j,n} \in \mathrm{da} \right) \\
&\qquad - \int_{A_{i^*}-\varepsilon}^{A_{i^*}+\varepsilon} \mathbb{P}\left(\min_{j\in\{1,\dots,k+i-1\}} \overline{Y}_{j,n} \geq A_{i^*} + \varepsilon \right) \mathbb{P}\left(\max_{j\in\{i,\dots,d\}} \overline{X}_{j,n} \in \mathrm{da} \right).
\end{aligned}$$

This lower bound holds for any i; we pick $i = i^*$. The above expression is asymptotically

equal to, with $\overline{d} := d - i^* + 1$ and $\overline{k} := k + i^* - 1$,

$$\begin{aligned}\alpha_{\overline{d},\overline{k}}(n) &- \frac{C_Y(A_{i^*}+\varepsilon)^{\overline{k}}}{n^{\overline{k}/2}}\mathrm{e}^{-n\overline{k}I_Y(A_{i^*}+\varepsilon)} \\ &\times\left[\frac{C_X(A_{i^*}-\varepsilon)^{\overline{d}}}{n^{\overline{d}/2}}\mathrm{e}^{-n\overline{d}I_X(A_{i^*}-\varepsilon)} - \frac{C_X(A_{i^*}+\varepsilon)^{\overline{d}}}{n^{\overline{d}/2}}\mathrm{e}^{-n\overline{d}I_X(A_{i^*}+\varepsilon)}\right] \\ = \alpha_{\overline{d},\overline{k}}(n) &- \frac{C_Y(A_{i^*}+\varepsilon)^{\overline{k}}}{n^{\overline{k}/2}}\mathrm{e}^{-n\,J_{\overline{d},\overline{k}}(A_{i^*}+\varepsilon)} \\ &\times\left[\frac{C_X(A_{i^*}-\varepsilon)^{\overline{d}}}{n^{\overline{d}/2}}\mathrm{e}^{n\overline{d}\left[I_X(A_{i^*}+\varepsilon)-I_X(A_{i^*}-\varepsilon)\right]} - \frac{C_X(A_{i^*}+\varepsilon)^{\overline{d}}}{n^{\overline{d}/2}}\right]\end{aligned}$$

where $\alpha_{\overline{d},\overline{k}}(n)$ is as in Section 4.2.

Recall that the exponential term in $\alpha_{\overline{d},\overline{k}}(n)$ is $J_{\overline{d},\overline{k}}(A_{i^*})$. Since A_{i^*} minimises $J_{\overline{d},\overline{k}}(a)$, we have that $\exp(-nJ_{\overline{d},\overline{k}}(A_{i^*}))$ asymptotically dominates $\exp(-nJ_{\overline{d},\overline{k}}(A_{i^*}+\varepsilon))$. Furthermore, recall that $A_{i^*} < \mathbb{E}X$, and therefore $I_X(A_{i^*}+\varepsilon) - I_X(A_{i^*}-\varepsilon) < 0$. We thus conclude that the lower bound is asymptotically equal to $\alpha_{\overline{d},\overline{k}}(n)$.

Upper bound: We can simply replace probabilities by one to obtain that

$$\int_{-\infty}^{\infty}\int_{a}^{\infty}\mathbb{P}\left(\min_{j\in\{1,\ldots,k+i-1\}}\overline{Y}_{j,n}\in\mathrm{db}\right)\mathbb{P}\left(\max_{j\in\{i,\ldots,d\}}\overline{X}_{j,n}\in\mathrm{da}\right)$$

is an upper bound for (4.25). Since this is equal to

$$\int_{-\infty}^{\infty}\mathbb{P}\left(\min_{j\in\{1,\ldots,k+i-1\}}\overline{Y}_{j,n}\geq a\right)\mathbb{P}\left(\max_{j\in\{i,\ldots,d\}}\overline{X}_{j,n}\in\mathrm{da}\right),$$

the results of Section 4.2 state that an upper bound is given by $\alpha_{d-i+1,k+i-1}(n)$, which thus coincides with the lower bound. We thus find, asymptotically,

$$\mathbb{P}\left(X_{(i),n}\leq Y_{(k+i-1),n}\right)\sim\binom{d}{k+i-1}\binom{d}{d-i+1}\alpha_{d-i+1,k+i-1}(n)\,. \tag{4.26}$$

It now follows that

$$\beta_{d,k}(n)\lesssim\sum_{i=1}^{d-k+1}\binom{d}{k+i-1}\binom{d}{d-i+1}\alpha_{d-i+1,k+i-1}(n)\,.$$

Asymptotically what matters is the dominating summand given by i^* as defined in (4.23); as $n\to\infty$ the other summands are asymptotically negligible (under the uniqueness assumption that we imposed). Since every single summand is a lower bound for $\beta_{d,k}(n)$, we then have the asymptotic relation (4.24). □

If there is no unique optimiser i^*, we have proven the asymptotic upper bound

$$\beta_{d,k}(n)\leq\sum_{i\in\mathscr{I}}\binom{d}{k+i-1}\binom{d}{d-i+1}\alpha_{d-i+1,k+i-1}(n)\,, \tag{4.27}$$

where $\mathscr{I}$ denotes the set of optimising $i\in\{1,\ldots,d-k+1\}$. Furthermore, every summand

of the right-hand side is an asymptotic lower bound.

One may now wonder whether the upper bound in (4.27) is asymptotically tight. Observe that the inequality in (4.27) is essentially a Bonferroni inequality, and one might expect that probabilities of intersections of the corresponding events are asymptotically negligible, in which case by the inclusion-exclusion principle the upper bound would be asymptotically tight (similar to the argument we gave in Section 4.3.1). The following heuristic argument indicates, however, that this reasoning is not valid in this case, and this is confirmed numerically in the example provided below.

In our example we consider the simplest case possible: we suppose that $d = 2$ and $k = 1$. We define the events

$$E_{i,j} := \left\{\overline{X}_{i,n} \leq \overline{Y}_{j,n}\right\}, \quad F_{i,j} := \left\{\overline{X}_{(i),n} \leq \overline{Y}_{(j),n}\right\},$$

where $i, j \in \{1, 2\}$. We have

$$\beta_{2,1}(n) = \mathbb{P}\left(F_{1,1} \cup F_{2,2}\right) = \mathbb{P}\left(F_{1,1}\right) + \mathbb{P}\left(F_{2,2}\right) - \mathbb{P}\left(F_{1,1} \cap F_{2,2}\right).$$

It is directly verified that

$$\begin{aligned}
\mathbb{P}(E_{1,1} \cap E_{1,2}) &= \mathbb{P}(E_{2,1} \cap E_{2,2}),\\
\mathbb{P}(E_{1,1} \cap E_{2,1}) &= \mathbb{P}(E_{1,2} \cap E_{2,2}),\\
\mathbb{P}(E_{1,1} \cap E_{2,2}) &= \mathbb{P}(E_{1,2} \cap E_{2,1}).
\end{aligned}$$

Furthermore, relying on arguments similar to those used in Section 4.2.2, it follows that some events essentially imply each other, in that

$$\mathbb{P}(E_{1,1} \cap E_{2,1} \cap E_{2,2}) \approx \mathbb{P}(E_{1,1} \cap E_{2,2}), \quad \mathbb{P}(E_{1,2} \cap E_{2,1} \cap E_{2,2}) \approx \mathbb{P}(E_{1,2} \cap E_{2,1}), \ldots$$

and analogously for other probabilities of this form. Based on these findings, and applying elementary set theory, we have that $\mathbb{P}\left(F_{1,1}\right) + \mathbb{P}\left(F_{2,2}\right)$ behaves as

$$\left[2\,\mathbb{P}(E_{1,1} \cap E_{1,2}) - \mathbb{P}(E_{1,1} \cap E_{2,2})\right] + \left[2\,\mathbb{P}(E_{1,1} \cap E_{2,1}) - \mathbb{P}(E_{1,1} \cap E_{2,2})\right], \tag{4.28}$$

whereas $\mathbb{P}\left(F_{1,1} \cap F_{2,2}\right) \approx \mathbb{P}(E_{1,1} \cap E_{2,2})$. We conclude that this probability is thus not negligible compared to (4.28), and as a consequence (4.27) is not asymptotically tight.

Gaussian example. We consider again the example with X and Y both being normally distributed, as introduced in Section 4.2.3. First, assume that $\sigma_X \neq \sigma_Y$. Define the differentiable function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) := \frac{1}{2}\frac{(d - x + 1)(k + x - 1)(\mu_Y - \mu_X)^2}{(k + x - 1)\sigma_X^2 + (d - x + 1)\sigma_Y^2}.$$

As can be checked by an explicit calculation, we have that $h''(x) < 0$, and hence $h(\cdot)$ is concave. Note that for $i \in \{1, \ldots, d - k + 1\}$ we have $J_{d-i+1,k+i-1}(A_i) = h(i)$. We conclude that $J_{d-i+1,k+i-1}(A_i)$ is concave as a function of $i \in \{1, \ldots, d - k + 1\}$, and thus takes its minima at the boundaries, that is, for $i \in \{1, d - k + 1\}$. A straightforward calculation reveals that $J_{d-i+1,k+i-1}(A_i)$ is minimised at $i^* = 1$ if $\sigma_Y > \sigma_X$, and at $i^* = d - k + 1$ otherwise.

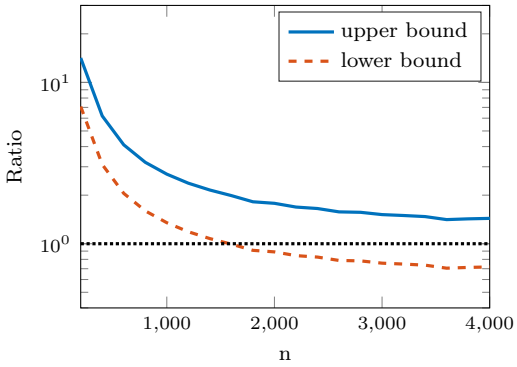


Figure 4.2: Ratio of the asymptotic bounds (4.29) and simulated probabilities $\beta_{d,k}(n)$, where $d = 3$, $k = 2$, for Gaussian random variables with $\mu_X = 1$, $\mu_Y = 0.8$, $\sigma = 2$. The dotted horizontal line indicates a ratio of 1.

Now consider the case $\sigma_X = \sigma_Y$. Then the function $h(\cdot)$ simplifies:

$$J_{d-i+1,k+i-1}(A_i) = \frac{(d-i+1)(k+i-1)}{2\sigma^2} \frac{(\mu_Y - \mu_X)^2}{d+k} .$$

As before, this concave function can attain its minimum value only at the boundary points $i \in \{1, d-k+1\}$, but note that at these points the function value is the same. Hence, from (4.27) we have

$$\left(\begin{array}{c} d \\ k \end{array} \right) \alpha_{d,k}(n) \leq \beta_{d,k}(n) \leq \left(\begin{array}{c} d \\ k \end{array} \right) \left[\alpha_{d,k}(n) + \alpha_{k,d}(n) \right] . \tag{4.29}$$

Numerical experiments such as Fig. 4.2 indicate that these bounds are not tight.

4.4 Conclusion

We have derived exact asymptotics for the rare event probability that all sample means of a population Y exceed all sample means of an independent population X while $\mathbb{E}X > \mathbb{E}Y$. The proof heavily relies on Bahadur-Rao type asymptotics that describe the tail distribution of the sample mean of i.i.d. random variables. Our result is new: it seemingly fits in the framework of [29], but careful inspection shows that the conditions imposed in [29] are not met in our situation (and we do obtain another asymptotic form than suggested in [29], with the polynomial factor being $1/n^{d-1/2}$, rather than $1/n^{d^2/2}$). We also provide an asymptotically efficient importance sampling procedure for estimating the probability of our interest.

We then showed that this result yields an expression for the exact asymptotics of the probability that there exists a sample mean from Y that exceeds a sample mean from X, which is relevant in log-likelihood ratio testing. We also used our result to derive the probability that there are at least k unordered sample means in every possible matching of sample means between X and Y; we explained that this probability may be of practical interest for example in particular queueing or packing problems.

Part II

Dynamic Control Problems

CHAPTER 5

Introduction

The static control strategies we considered in Part I of this thesis did not take into account updated information about the current state of the system. In this part we turn to *dynamic* control policies that are designed to exploit such updated state information.

An important class of control problems typically solved by dynamic control policies is that of the multiarmed bandits. The term refers to a sequential decision model in which, in each time slot, a decision maker decides to activate k out of d competing projects. In this thesis we focus on the Markovian formulation (for a survey on alternative bandit models we refer to [23]): it is assumed that the state processes of these projects can be modelled as discrete time Markov chains, and only if a project is activated, it transits from one state to the next according to some known transition kernel. The decision maker then receives a certain reward (possibly negative), which depends on the new state of the project. His objective is to make decisions in such a way that he maximises the average or discounted reward that he can expect to accumulate over an infinite time-horizon.

Obviously, we do not need to apply the bandit model for selecting 'projects' in the most literal sense of the word: we can think of many alternative application possibilities such as the problem of selecting a location where to drill for oil, or that of deciding which drug to give to test patients in a clinical trial. The name 'bandit problem' refers to the interpretation of the 'projects' as arms of a slot machine (a so-called 'bandit'), and the decision maker is then a gambler who can play k arms at a time, yielding a certain reward depending on the state of these arms.

In the classical Markovian multiarmed bandit problem the states of unplayed ('passive') arms remain frozen, and the current states of all arms are known to the gambler. An important generalisation has been proposed by Whittle [163], who assumed that *all* arms evolve to a new state, irrespective of whether they are being played or not, yet the action of playing the arm may affect the transition kernels of the underlying Markov state processes. This generalisation is referred to as the *restless multiarmed bandit* (RMAB) problem.

In this thesis we will only consider one particular variant of the RMAB where – in the spirit of partially observable Markov decision processes (POMDPs) [143] – it is assumed that the gambler can only observe the state of a certain arm when he chooses to activate it, in which case he receives a state-dependent reward. We will refer to this kind of

partially observable RMAB as *reward-observing* RMAB (RORMAB).

A clever gambler will certainly try to use the information gained from each play to make better decisions in the future. If the problem is reward-observing, the gambler then faces a trade-off between exploiting those arms that he believes will yield a high reward at the next decision time, and exploring other arms in order to collect information about their current state. This is incorporated in the concept of a *policy*, that is, a strategy that specifies the decision to be made at time t given the information that is available to the decision maker at that time. In the RORMAB setting the available information consists of the history of observed states (or functionals thereof), and we shall assume that we have knowledge of a stochastic model describing the state evolution of the arms.

An *index policy* maps the information available for each arm to some real-valued priority index, where the index value of a certain arm does not depend on the characteristics of any other arm but on the history of this arm only. At every decision time the policy then activates those k arms that correspond to the k largest indices. A policy is *optimal* if it maximises the expected total reward accumulated over time. No generally optimal solution is known to the RMAB but in many cases index policies have been found to perform well, they are computationally simple due to the decoupling of arms, and often turn out to be optimal (at least in some asymptotic sense) under certain conditions.

We more formally define the aforementioned bandit problems in Section 5.1.1. Index policies are discussed in Section 5.1.2. Both sections closely follow our exposition in [89]. The further organisation of this part of the thesis is provided in Section 5.2.

5.1 Background

5.1.1 Markovian bandit models

In this section we formulate the RMAB and the RORMAB problem. Consider d independent processes $\big(X_i(t)\big)_t$, where $X_i(t)$ denotes the state of arm i at time $t \in \mathbb{N}_0$, taking values in a state space S_i. We assume that for each arm i we have fitted a stochastic model describing its evolution. At every point in time the decision maker may choose whether or not to play arm i. We denote the action of playing arm i by $a_i(t) = 1$ (active), while $a_i(t) = 0$ (passive) refers to the action of not playing. We require that exactly k arms have to be activated at each decision time, i.e., the action vector $\boldsymbol{a}(t) := \big(a_1(t), \dots, a_d(t)\big)$ satisfies $\sum_{i=1}^{d} a_i(t) = k$. Based on the decisions $\boldsymbol{a}(t)$, each of the processes evolves as

$$X_i(t+1) = \begin{cases} \mathcal{A}_i\big(X_i(t), U_i(t)\big), & \text{if} \quad a_i(t) = 1, \\ \mathcal{P}_i\big(X_i(t), U_i(t)\big), & \text{if} \quad a_i(t) = 0, \end{cases} \tag{5.1}$$

where $\{U_i(t),\, i = 1, \dots, d\}$ are independent i.i.d. (driving) sequences of uniform $[0,1]$ random variables, and $\mathcal{A}_i$ and $\mathcal{P}_i$ map $S_i \times [0,1]$ to S_i. This means that the state processes evolve according to different Markovian transition kernels, depending on whether the action applied to the arm is active or passive. We assume that the state-dependent reward is given by $R_i^{a_i(t)}\big(X_i(t)\big)$, where $R_i^a(\cdot)$ is a known deterministic function from the underlying state space to $\mathbb{R}$, which may be dependent on the arm i and the action $a \in \{0,1\}$. In the upcoming chapters $R_i^0(\cdot)$ is taken to be zero everywhere; that is, an arm that is not played does not yield a reward.

A policy π is a mapping from the observation space $O \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ to the action space $\{0,1\}^d$ and may be parametrised by time, that is, $\pi_t\big(\boldsymbol{Y}(0), \dots, \boldsymbol{Y}(t-1)\big) \to \boldsymbol{a}(t)$

Table 5.1: Bandit models defined based on the state updating rule (5.1), and the mode of observation. The state update given $a_i(t) = 1$ is in all cases given by $\mathcal{A}_i\big(X_i(t), U_i(t)\big)$.

	$X_i^0(t+1)$ if $a_i(t) = 0$	$Y_i(t)$ if $a_i(t) = 1$	$Y_i(t)$ if $a_i(t) = 0$
RMAB	$\mathcal{P}_i\big(X_i(t), U_i(t)\big)$	$X_i(t)$	$X_i(t)$
MAB	$X_i(t)$	$X_i(t)$	$X_i(t)$
RORMAB	$\mathcal{P}_i\big(X_i(t), U_i(t)\big)$	$R_i^1\big(X_i(t)\big)$	$f_i\big(X_i\big(t - \eta_i(t)\big), \eta_i(t)\big)$

where $Y_i(s)$ denotes the i-th entry of $\boldsymbol{Y}(s)$ and corresponds to the observation of arm i obtained at time s (typically, it is assumed that $Y_i(s)$ is determined by $a_i(s)$ and $X_i(s)$, see below). The aim is to find a policy that maximises the accumulated rewards over an infinite time horizon as evaluated by the *total expected discounted reward criterion*

$$V^\pi(\boldsymbol{x}) := \lim_{T\to\infty} \mathbb{E}_{\boldsymbol{x}}^\pi \left[\sum_{t=0}^{T} \beta^t \sum_{i=1}^{d} R_i^{a_i(t)}\big(X_i(t)\big) \right], \tag{5.2}$$

where $\beta \in (0,1)$, and the subscript $\boldsymbol{x} := (x_1, \ldots, x_d)$ indicates conditioning on $X_i(0) = x_i$; or the *average expected reward criterion*

$$G^\pi(\boldsymbol{x}) := \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}_{\boldsymbol{x}}^\pi \left[\sum_{t=0}^{T-1} \sum_{i=1}^{d} R_i^{a_i(t)}\big(X_i(t)\big) \right]. \tag{5.3}$$

Assuming that the suprema exist, the corresponding optimal value functions are defined as

$$V(\boldsymbol{x}) := \sup_\pi V^\pi(\boldsymbol{x}), \quad G(\boldsymbol{x}) := \sup_\pi G^\pi(\boldsymbol{x}). \tag{5.4}$$

A policy is optimal if it achieves the supremum. If every set in the state space can be reached from any other state (that is, if the MDP is unichain, see [126]), the optimal *average* reward is the same for all initial states [17, Prop. 5.2.3]. Under fairly general conditions [126], the optimal policy turns out to be stationary and Markovian in the sense that the optimal decision only depends on the set of observations collected last: $\pi_t^*\big(\boldsymbol{Y}(0), \ldots, \boldsymbol{Y}(t)\big) \equiv \pi^*\big(\boldsymbol{Y}(t)\big)$. In the following we will only consider such policies.

Take $f_i(\cdot)$ to be a deterministic function from $S_i \times \mathbb{N}$ to S_i. Further define

$$\eta_i(t) := \min\{s \in \mathbb{N} \,|\, a_i(t-s) = 1\}$$

to be the number of time steps ago arm i was last observed. (We assume that we are aware of the actions we took in the past so that at time t $\eta_i(t)$ is deterministic.) Depending on the specific form of the observations, reward functions, and the state updating mappings given in (5.1), we distinguish bandit models as summarised in Table 5.1.

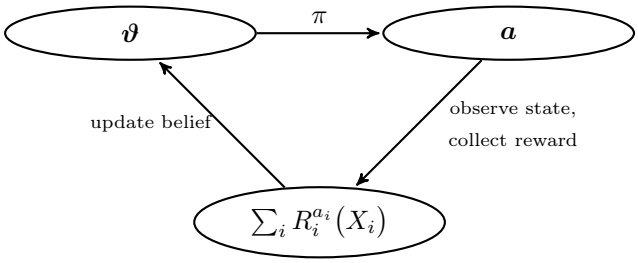
RORMAB. In Chapters 6 and 7 we will focus on the RORMAB model, for which state information of an arm is only obtained whenever that arm is played. Assuming that the states $X_i(t)$ are sequentially correlated, the decision maker can use the previous observation $X_i\big(t - \eta_i(t)\big)$ to make predictions about the current state of arm i. Thus, the relevant information at time t (prior to making the decision) can be summarised and

represented by $\mathcal{I}(t) := \big(\mathcal{I}_1(t), \ldots, \mathcal{I}_d(t)\big)$, where for $i = 1, \ldots, d$,

$$\mathcal{I}_i(t) = \Big(\eta_i(t), X_i\big(t - \eta_i(t)\big)\Big). \tag{5.5}$$

Based on $\mathcal{I}_i(t)$, the controller's *belief* about the state of arm i at time t is summarised by $F_i(x) := \mathbb{P}\big(X_i(t) \leq x \,|\, \mathcal{I}_i(t)\big)$, the conditional distribution of arm i given the information collected up to that time. For the state processes we consider in this thesis, this probability distribution is characterised by scalar- or vector-valued sufficient statistics. That is, for arm i there exists a parameter $\vartheta_i(t)$ that fully specifies the probability distribution of $X_i(t)$ given the information $\mathcal{I}_i(t)$. For example, in the 'Gilbert-Elliott' model we consider in Chapter 6, $F_i(\cdot)$ is a Bernoulli distribution with success probability $\vartheta_i(t)$. In the second model we study in Chapter 6, $F_i(\cdot)$ is a normal distribution, hence, $\vartheta_i(t)$ is a two-dimensional vector specifying the conditional mean and conditional variance. Using the terminology common in literature on POMDPs [126], we refer to $\vartheta_i(t)$ as the *belief state* of arm i at time t — indeed $\vartheta_i(t)$ represents our belief concerning the state of the arm.

In summary, as time evolves from t to $t + 1$, given the current belief state $\boldsymbol{\vartheta} := (\vartheta_1, \ldots, \vartheta_d)$ and a channel selection policy π, the following chain of actions takes place:



5.1.2 Index policies

For the classical multiarmed bandit problem it is known that if $k = 1$, then an optimal policy is given by the so-called *Gittins index* policy [50, 49]. If $k > 1$, then optimality holds under additional conditions as was proven in [119]. In contrast, finding an optimal policy for the RMAB in its full generality is typically intractable. However, for a relaxed version of the problem where we only require that k arms are activated *on average*, Whittle [163] found that an index policy similar to the Gittins index describes an optimal solution. This *Whittle index* policy reduces to the Gittins index under the additional assumption that passive arms do not change states.

Under the constraint that *exactly* k arms are played at each time slot, the Whittle index policy is usually not optimal. However, in a number of application examples the Whittle index was shown to be optimal [3, 99] or asymptotically optimal [116, 157]. Weber and Weiss [160] proved, that in the general RMAB setting, the Whittle index policy is asymptotically optimal as k and d tend to infinity, under the assumptions that the ratio of k and d remains fixed, and that the differential equation describing the fluid approximation to the index policy has a globally stable equilibrium point.

In this section we explain the idea behind the use of index policies and specifically the Whittle index, a generalisation of the well-studied Gittins index [48, 163]. Whittle proposed this type of index as a heuristic solution to RMAB problems. We first de-

scribe index policies in general before we turn to Whittle's optimisation problem and the associated Whittle index policy.

Index policies are defined in terms of functions $\iota_1, \dots, \iota_d$ such that ι_i maps the current state of arm i to a certain priority index, irrespective of the current state of any other arm.

Definition 5.1. Let $\mathbf{x} := (x_1, \dots, x_d)$ denote the vector of states in a system with d arms. An index policy π_ι activates those k arms that correspond to the k largest indices,

$$\pi_\iota\big(\mathbf{x}\big) = \operatorname*{arg\,max}_{\boldsymbol{a}:\, \sum_{i=1}^d a_i = k} \sum_{i=1}^{d} \iota_i\big(x_i\big)\, a_i \,.$$

Ties are broken uniformly at random, but in compliance with the requirement that k arms have to be selected.

For an intuitive justification as to why index policies may work well in large systems, consider the following: Pick an arbitrary arm and suppose we want to decide whether to activate it or leave it passive, based on the current state. Generally, we would make our decision dependent on the states of the remaining arms: we care about the reward we can expect from a certain arm *relative to* the reward that other arms would yield. In this way, our decision policy is highly influenced by the proportion of arms that are in a certain state. In a large system with many arms, however, this proportion can be expected to remain relatively stable over time (assuming the underlying Markovian state processes are stationary). In this sense, the larger the system, the less important it is for us to consider other arms when making a decision as we always find ourselves in roughly the same situation for decision making. In conclusion, in a system with many arms, little is lost if we make decisions for each arm solely based on its current state, disregarding the current state of any other arm in the system. We will return to this idea in Section 7.3.1.

The question remains how to define the index functions ι_i. A simple example is the myopic index. In the context of the RORMAB, where states are actually belief states, it is defined by $\iota_i^M(\vartheta_i) = \mathbb{E}_{\vartheta_i}\big[R_i^{a_i}(X_i)\big]$. Thus, under the myopic policy the decision maker exploits those arms that promise the largest immediate rewards. However, as one may expect, it turns out that such a 'greedy' policy is usually not optimal (see our numerical examples in Chapters 6–7). It may be favourable to give some priority to exploring other arms in order to decrease the decision maker's uncertainty with respect to their current state.

Going back to the more general RMAB, this motivates us to consider the more sophisticated Whittle index, which takes the possible need for exploration into account. To derive his heuristic, Whittle relaxed the constraint that exactly k arms have to be selected at each time point, and replaced it by the requirement that k arms are selected *on average*. Since the latter constraint is weaker, the optimal total expected (discounted or average) value of the MDP under this constraint is an upper bound for the optimal total expected value that can be achieved in the original problem. We shall see that this relaxation allows to formulate the decision making problem as a Lagrange optimisation problem, from which Whittle obtained a rule for determining $\iota_i\big(x_i\big)$.

Whittle's optimisation problem. For ease of exposition we consider an MDP with a finite state space $S = S_1 \times \cdots \times S_d$. Recall that for the models considered in this thesis it will be assumed that $r_i^0(\cdot) \equiv 0$. For brevity we focus on this setting when stating

Whittle's optimisation problem below. Furthermore, we focus now on the average reward case. The discounted reward case can be dealt with similarly, see [111]. Under suitable regularity conditions, the optimal average value is independent of the initial state of the system (see e.g. [18]); here we assume that we are in such a setting.

Recall the definition of the average reward criterion, Eq. (5.3). For a time horizon T (where we let $T \to \infty$) we sum up the rewards that are obtained from the selected arms at each time point. Equivalently, we could group selected arms according to their states, and keep track of how many arms were selected while being in a specific state, over the whole time horizon. That is, rather than considering each time step t separately and adding up rewards as obtained at each time step, we can consider how many arms were in a certain state when selected, and multiply this proportion with the reward that is obtained from an arm in that state. If we do so for all states, then the total is equivalent to the value of the average reward as $T \to \infty$.

This is the viewpoint we are taking in this subsection; it is inspired by the exposition in [111]. Define $p_i(x)$ as the expected long-run fraction of time that arm i is selected when it currently is in state $x \in S_i$; that is,

$$p_i(x) := \lim_{T\to\infty} \frac{1}{T} \mathbb{E}^\pi \left[\sum_{t=0}^{T-1} \mathbb{1}\{i \in a(t),\, X_i(t) = x\} \right] .$$

Subject to Whittle's relaxation, we can then formulate the optimisation problem as the linear programming (LP) problem:

$$G^W = \max_p \sum_{i=1}^{d} \sum_{x\in S_i} r_i(x)\, p_i(x), \quad \text{subject to} \quad \sum_{i=1}^{d} \sum_{x\in S_i} p_i(x) = k\,, \tag{5.6}$$

where $r_i(x)$ denotes the reward that is obtained from selecting arm i when its state is x (as before $r_i(\cdot)$ is a known, deterministic function). Formulating this as an equivalent Lagrangian optimisation problem we obtain:

$$\begin{aligned} \mathcal{L}(\lambda) :=& \max_p \sum_{i=1}^{d} \sum_{x\in S_i} r_i(x)\, p_i(x) - \lambda \left(\sum_{i=1}^{d} \sum_{x\in S_i} p_i(x) \; - k \right) \\ =& \max_p \sum_{i=1}^{d} \sum_{x\in S_i} \big(r_i(x) - \lambda\big) p_i(x) + \lambda k \\ =& \sum_{i=1}^{d} \mathcal{L}_i(\lambda) + \lambda k\,, \end{aligned} \tag{5.7}$$

with

$$\mathcal{L}_i(\lambda) := \max_{p_i} \sum_{x\in S_i} \big(r_i(x) - \lambda\big) p_i(x)\,. \tag{5.8}$$

Since in (5.7) there is no longer a common constraint for the arm, each arm can be optimised separately through (5.8). By strong LP duality we know that there exists a Lagrange multiplier λ^* that yields $\mathcal{L}(\lambda^*) = G^W$. LP complementary slackness ensures that (assuming $\lambda^* \neq 0$) any optimal solution to (5.7) must satisfy the relaxed constraint, and is therefore also optimal for Whittle's relaxed problem (5.6). It was observed by

Whittle [163] that we can interpret the Lagrange multiplier as a cost for selecting an arm (or equivalently, as a subsidy for not selecting an arm). That is, imposing a cost of λ^* on the selection of an arm causes the controller to select k arms on average under a policy that optimises G^W.

Indexability and the Whittle index. In accordance with [163], we make the following reasonable regularity assumption.

Assumption II.1. Arm i is *indexable*, that is, the set of states for which it is optimal to select arm i, $S_i^*(\lambda)$, say, decreases monotonically from S_i to $\emptyset$ as the cost λ increases from $-\infty$ to ∞. This property holds for every arm in the system.

While this assumption is intuitively appealing and does seem to hold in many cases, one can construct examples for which it is not valid [157, 163]. Proving that a particular bandit problem is indexable can be surprisingly difficult [48]. Indexability implies that for each arm i there exists a function of the current state, $\lambda_i(x)$, such that it is optimal to select the arm whenever $\lambda_i(x) > \lambda$ and to leave it passive otherwise (the decision maker is indifferent when $\lambda = \lambda_i(x)$). In this sense, $\lambda_i(x)$ measures the "value" of arm i when it is in state x. Furthermore, applying this policy to all arms in the case where $\lambda = \lambda^*$ (that is, selecting arm i whenever $\lambda_i(x) > \lambda^*$) results in a policy that is optimal for Whittle's relaxed problem (5.6)[1]. This motivates choosing the index function $\iota_i(\cdot)$ as $\iota_i^W(x) := \lambda_i(x)$ (as was proposed in [163]).

Now, how do we find $\lambda_i(\cdot)$? Recall that the decision maker is indifferent when $\lambda = \lambda_i(x)$, and that we are interested in the case where the cost λ is chosen to be the optimal cost λ^* that causes the decision maker to select k arms on average. Furthermore, we saw that the Lagrangian (5.7) can be solved by considering arms one by one (in accordance with the intuition described at the beginning of this section, where it is argued that not much is lost by decoupling arms provided the system is large enough). In fact, (5.8) is the Lagrangian corresponding to a *one-arm sub-problem* in which there is only a single arm which can be selected or not, and where playing the arm yields the state-dependent reward as before but also costs a certain amount λ. Now the optimal $\lambda_i(x)$ is the one that makes us indifferent between playing or not playing the arm when it is in state x. In summary, we define the Whittle index as follows (cf. [163]).

Definition 5.2. The Whittle index of arm i when in state x is the largest cost λ in (5.8) such that it is still optimal to play the arm in the one-arm sub-problem:

$$\iota_i^W(x) = \sup\{\lambda : x \in S_i^*(\lambda)\}.$$

Intuitively, the Whittle index can perhaps best be thought of as an opportunity cost, to be paid for losing the opportunity to select one of the other arms in the constrained system with multiple arms. Naturally, we then prioritise arms with higher opportunity cost.

Computing the Whittle index. As stated in Definition 5.2, the Whittle index is derived from the optimal policy for the one-arm sub-problem. Thus, the computational

[1] When $\lambda_i(x) = \lambda^*$, one needs to decide for the action to be taken in state x in an appropriately randomised fashion that ensures that the relaxed constraint is satisfied [160, 163].

complexity of the Whittle index increases only linearly with the number of arms: we need the optimal policy for at most d non-identical one-arm sub-problems. In contrast, the complexity of computing the optimal policy for the full system increases exponentially (the latter problem is in fact PSPACE hard [120]).

It is well-known [126] that in great generality the optimal average reward G is constant (independent of the initial state), and satisfies Bellman's optimality equation. For the one-arm sub-problem associated with arm i and applying the average reward criterion, this optimality equation reads as,

$$G + h(x) = \max\left\{ r_i(x) - \lambda + \mathbb{E}\Big[h\big(\mathcal{A}_i(x,\, U)\big)\Big],\;\; \mathbb{E}\Big[h\big(\mathcal{P}_i(x,\, U)\big)\Big] \right\}, \tag{5.9}$$

with $x \in S_i$ and U a uniform $(0, 1)$ random variable. Here, $r_i(x) - \lambda$ is the expected immediate reward obtained from deciding to use the arm, corrected by the opportunity cost λ. The bias function h accounts for the transient effect caused by starting at initial state x rather than at equilibrium.

In contrast, the optimality equation under the total expected β-discounted reward criterion is

$$V(x) = \max\left\{ r_i(x) - \lambda + \beta\, \mathbb{E}\Big[V\big(\mathcal{A}_i(x,\, U)\big)\Big],\;\; \beta\, \mathbb{E}\Big[V\big(\mathcal{P}_i(x,\, U)\big)\Big] \right\}. \tag{5.10}$$

The optimal policies for these one-arm sub-problems are then to choose the action that maximises the right-hand side of (5.9) and (5.10), respectively. They can be found from dynamic programming algorithms such as (relative) value or policy iteration. Then the Whittle index for state x can be effectively computed by solving (5.9) or (5.10) for an increasing sequence of λ and finding the maximal λ (namely $\lambda_i(x)$) for which playing the arm is still optimal. Assuming continuity, this means that the Whittle index is the value of λ for which both arguments of the maximum are equal.

We remark that Whittle originally defined λ as a subsidy for not playing an arm and consequently added λ on the right-hand side of the optimality equation rather than subtracting it on the left. These two interpretations are, of course, equivalent, and we merely chose to define λ as opportunity cost in this section because this interpretation seemed more natural to us. In the upcoming chapters we will write the optimality equations in the traditional way.

5.2 Organisation and contributions

As mentioned in the introduction, a possible application of the RORMAB is as a model for the problem of selecting transmission channels in a wireless network. We will deal with bandit models for this application more closely in Chapters 6 and 7.

The literature on RORMAB modelling of the channel selection problem has so far mostly focussed on the Gilbert-Elliot (GE) model, in which channels are assumed to evolve as a two state Markov chain, and the Gaussian autoregressive (AR) channels of order 1, where the state of the channel is given by the logarithmic signal-to-noise ratio. In Chapter 6 we unify the presentation of both types of models under the umbrella of our newly defined RORMAB. We investigate a mixed-channel model in a number of numerical examples, and provide a literature survey on the subject of wireless channel selection with RORMABs. The content of Chapter 6 has appeared in [89].

In Chapter 7, which is based on our work [83, 84], we focus on the AR channel model in more detail. We establish structural properties for the Whittle index policy, which motivate a parametric index policy that is computationally much simpler than the Whittle index but can still outperform the myopic policy. Furthermore, we examine the many-arm behaviour of the system under the parametric policy, identifying equations describing its asymptotic dynamics. Based on these insights we provide a simple heuristic algorithm to evaluate the performance of index policies; the latter is used to optimise the parametric index.

CHAPTER 6

Reward-observing bandits for channel selection

Communication devices are often configured to transmit on several alternative channels, which may differ in their type (e.g. WiFi versus cellular) or in their physical frequencies. Further, due to physical transmitter limitations, a device can only use a limited number of channels at any given time. Thus, the question arises which channels to select for transmission so as to maximise the throughput that is achieved over time.

To illustrate the problem, we consider the scenario depicted in Figure 6.1. At every discrete time instance, the transmitter has the choice to use either channel 1 or channel 2. The channels cannot be used in parallel, for instance, due to hardware limitations and/or energy constraints. The selected channel then yields an immediate reward that depends on the condition of the channel (e.g. the reward may be measured as the number of bits successfully transmitted). Consequently, an observation of the state of that channel is also obtained. The unselected channel on the other hand is not observed in this time instance.

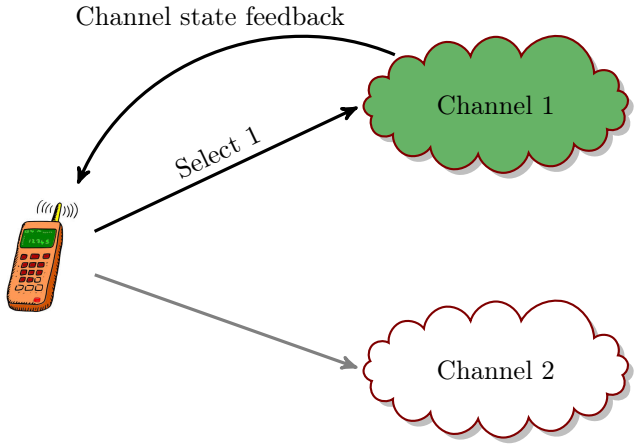


Figure 6.1: A transmitting device needs to choose whether to transmit on Channel 1 or Channel 2. Transmitting on a channel results in immediate channel state feedback.

Ideally, the transmitter would choose channels in a way that achieves the largest throughput over time. However, the nature of communication channels is often stochastic and thus the transmitter does not know the current state of each channel with certainty. A good prediction of the channel state can be obtained when there is strong dependence between the current state of a channel and its state in the (recent) past. Such *channel memory* can for example be caused by other users interfering on the same channel, multipath of physical signals, or other persistent disturbances.

In utilising channel memory, to make wise channel selection decisions, the transmitter needs to balance a trade-off between exploitation and exploration: On the one hand, based on the controller's belief regarding the current state of each channel, it may make sense to choose the channel expected to transmit the highest number of bits over the next time slot. On the other hand, it may be sensible to check the condition of the other channel so as to decrease uncertainty regarding its current state. How should channels be selected, based on the information available, so as to maximise the long-run expected throughput?

A first step towards answering this control problem is to devise suitable *channel models*; that is, models that capture channel behaviour reasonably well and at the same time are simple enough to be mathematically tractable. To capture such a dependence, channel states are often modelled as Markov processes. One such very simple process is the so-called *Gilbert-Elliot channel* (GE), where there are two possible states, 0 ("bad") and 1 ("good"), and transitions between states occur in a Markovian manner. As mentioned in Section 5.1.1, in this case the belief state of arm i is summarised by the success probability of a Bernoulli distribution. The application of the GE model in channel selection and specifically opportunistic spectrum access is motivated by its ability to capture bursty traffic of primary users [40]. Due to its simplicity it has been very popular in modelling channel selection problems (refer to the literature review in Section 6.3).

Another class of models, which has only recently come to attention in the context of wireless channel selection [11, 83], are *Gaussian autoregressive processes of order* 1 (which we denote by AR). It has been found that Gaussian autoregressions model the logarithmic signal-to-noise ratio of a channel reasonably well; for details see [2]. Since states are normally distributed, the objectives of exploitation and exploration naturally correspond to the conditional mean and the conditional variance of a channel, which at the same time contain all relevant information concerning its state and thus fully describe the belief state of the arm. The evolution of this vector-valued belief state can be described by a simple linear recursion perturbed by Gaussian noise.

The virtue of both the GE and the AR model is that they are quite tractable, yet allow the capture of the exploration–exploitation trade-off that the controller faces. The models are simple in the sense that the *belief* which the controller maintains about the state of the channel is neatly summarised by sufficient statistics. In the GE case, this sufficient statistic is given by the conditional probability of being in the good state, given the information that is available to the controller at the time. In the AR case, it is sufficient to keep track of the conditional mean and variance of the state, which quantify the expected gain from exploitation and the need for exploration, respectively.

An optimal policy for the channel selection problem needs to balance this exploration–exploitation trade-off. Note that the above described channel selection problem can be regarded as a RORMAB. Recall from Chapter 5 that the Whittle index is a computationally more tractable alternative to the dynamic programming solution to the problem, and that one may hope that in a system with many channels its performance is near-optimal.

While the chapter does not contain new results, it is unique in that it provides a unified

treatment of both the GE and the AR approaches for channel models, and considers also the channel selection problem in the mixed case where some of the channels are modelled as GE while the others are AR. This is of interest in networks where some but not all of the channels may be subject to user interference.

The remainder of this chapter is structured as follows. In Section 6.1 we formulate the RORMAB problem, and present the GE and AR models in a unified manner. In Section 6.2 we show how to use channel models to evaluate the Whittle index numerically, and use it as a solution strategy for an example channel selection problem. In the latter section we also provide a number of performance comparisons. Section 6.3 contains a literature survey, and points out some open problems.

6.1 Models and framework

In this section we formulate the RORMAB problem within the context of wireless channel selection. We consider channels $X_1(t), \dots, X_d(t)$, operating as independent Markov processes in discrete time $t \in \mathbb{N}_0$. We assume that the models and their parameters are known but do not have to be the same for each channel. At every decision time exactly k channels are selected; channels that are not selected are not observed and do not yield a reward. The reward obtained from channel i is $a_i(t)\, R_i(X_i(t))$, where $R_i(\cdot)$ is a known deterministic function. Throughout this chapter we focus on the average reward criterion (5.3), which we feel is more natural in the context of channel selection.

Because the channel states are sequentially dependent due to the channel memory, the controller can use information about the previous state of a channel to make predictions about the current state. The accuracy of the prediction depends on the *age* of the information, i.e. the number of time steps ago that a channel was last observed. Recall that this number is denoted by $\eta_i(t)$, so that the available information about channel i at time t is $\mathcal{I}_i(t)$ as defined in (5.5). We denote the belief states by $\vartheta_i(t)$ as before.

Using the tower property of conditional expectations it can be proven formally [18, 143] that a POMDP with partially observable states $X_i(t)$ and rewards $R_i\big(X_i(t)\big)$ is equivalent to a fully observable MDP with states $\vartheta_i(t)$ and rewards $r_i\big(\vartheta_i(t)\big) := \mathbb{E}_{\vartheta_i(t)}\big[R_i\big(X_i(t)\big)\big]$ in the sense that the best throughput that can be achieved is the same for both, and it is achieved by the same optimal policy. This justifies that we consider the MDP with states $\vartheta_i(t)$ in the remainder of this chapter.

Belief state evolution. For RORMABs, the decisions determined by a policy π affect the updating of the belief state based on the *observation update* mapping $\mathcal{O}_i(\cdot)$ and the *belief propagation* operator $\mathcal{T}_i(\cdot)$ as follows:

$$\vartheta_i(t+1) = \begin{cases} \mathcal{O}_i\big(X_i(t)\big), & \text{if } a_i(t) = 1, \\ \mathcal{T}_i\big(\vartheta_i(t)\big), & \text{if } a_i(t) = 0. \end{cases} \tag{6.1}$$

The *observation update* mapping $\mathcal{O}_i(\cdot)$ determines how the belief state of channel i is updated when that channel is selected for transmission. In this case we observe $X_i(t)$, and hence its realisation can be used by the observation update rule when implementing the controller. Further, for analytical, modelling and simulation purposes, when $a_i(t) = 1$, the distribution of $X_i(t)$ is determined by the known value $\vartheta_i(t)$, so that $X_i(t)$ can be replaced by a generic random variable stemming from this distribution.

The *belief propagation* operator $\mathcal{T}_i(\cdot)$ defines the update of the belief state of a channel when it is not selected for transmission. Because in this case no new observation is obtained, the update is deterministic.

Since a channel may remain unobserved for several consecutive time slots, it is useful to also consider $\mathcal{T}_i^k(\cdot)$ (the k-step operator obtained by applying $\mathcal{T}_i(\cdot)$ k times) as well as attracting fixed points of the operator $\mathcal{T}_i(\cdot)$. As we describe below, in both the GE and the AR model, the k-step operator has an explicit form and converges to a unique attracting fixed point; this is useful for understanding the dynamics of the model.

We now specify the observation update and belief propagation operations in the context of each of the two channel models.

6.1.1 Gilbert-Elliot channels

In this case $X_i(t)$ is a two state Markov chain on the state space $\{0, 1\}$, where 0 represents a "bad" state and 1 is a "good" state of the channel. The transition matrix can be parametrised as

$$P_i = \begin{bmatrix} p_i^{00} & p_i^{01} \\ p_i^{10} & p_i^{11} \end{bmatrix} = \begin{bmatrix} 1-\gamma_i\,\overline{\rho_i} & \gamma_i\,\overline{\rho_i} \\ \overline{\gamma_i}\,\overline{\rho_i} & 1-\overline{\gamma_i}\,\overline{\rho_i} \end{bmatrix},$$

where we denote $\overline{x} := 1 - x$. One standard parametrisation of this Markov chain uses transition probabilities $p_i^{01}, p_i^{10} \in [0, 1]$ (and sets $p_i^{00} = \overline{p_i^{01}}$, $p_i^{11} = \overline{p_i^{10}}$). Alternatively we may specify the stationary probability of being in state 1, denoted by $\gamma_i \in [0, 1]$, together with the second eigenvalue of P_i, denoted by $\rho_i \in \big[1-\min(\gamma_i^{-1}, \overline{\gamma_i}^{-1}),\ 1\big]$. Then ρ_i quantifies the time-dependence of the chain. If $\rho_i = 0$, the chain is i.i.d., otherwise there is memory. Specifically, when $\rho_i > 0$ there is positive correlation between successive channel states and when $\rho_i < 0$ that correlation is negative. The relationship between these parametrisations is given by $\gamma_i = p_i^{01}/(p_i^{01} + p_i^{10})$, and $\rho_i = 1 - p_i^{01} - p_i^{10}$. The parametrisation with transition probabilities $p_i^{\ell k} \in [0, 1]$ is standard. As opposed to that, our alternative parametrisation in terms of γ_i and ρ_i has not been used much in the literature. Nevertheless, we find it captures the behaviour of the model in a useful manner, especially when carrying out numerical comparisons.

As the Bernoulli distribution is fully specified by the success probability, it suffices to keep track of this parameter. That is, we have

$$\vartheta_i(t) = \omega_i(t) := \mathbb{P}\big(X_i(t) = 1 \,|\, \mathcal{I}_i(t)\big),$$

and hence the belief state space of channel i, denoted by S_i, is given by the interval $[0, 1]$. Now the *observation update* operation is defined by:

$$\mathcal{O}_i(x) = \begin{cases} p_i^{01}, & \text{if } x = 0\,, \\ p_i^{11}, & \text{if } x = 1\,. \end{cases}$$

That is, if the observed channel was "bad" ($x = 0$), then the chance of a good channel is given by the entry p_i^{01}, and otherwise ($x = 1$) by p_i^{11}. The *belief propagation* operation is

$$\mathcal{T}_i(\omega) = \omega p_i^{11} + \overline{\omega} p_i^{01} = \rho_i\,\omega + \gamma_i\,\overline{\rho_i}\,.$$

This follows by evaluating the probability of $\{X_i(t+1) = 1\}$ based on $\omega_i(t)$ and the probability transition matrix P_i. It is a standard exercise for two state Markov chains (recurrence relations) to show that the k-step transition probability, and thus the k-step

belief propagation operator, takes the form

$$\mathcal{T}_i^k(\omega) = \gamma_i + \rho_i^k(\omega - \gamma_i).$$

Note that γ_i is a fixed point of this operator, and the sequence $\mathcal{T}_i^k(\omega)$ converges to this fixed point. Further note that if $\rho_i > 0$, this sequence is monotonic whereas if $\rho_i < 0$, it oscillates about γ_i as it converges to it. The case of $\rho_i = 0$ is not of interest in terms of decision making because in that case there is no channel memory.

6.1.2 Gaussian autoregressive channels

In this case the channel states follow an AR process of order 1, that is,

$$X_i(t) = \varphi_i\, X_i(t-1) + \varepsilon_i(t)\,,$$

with $\{\varepsilon_i(t) : t \in \mathbb{N}_0\}$ denoting an i.i.d. sequence of $\mathcal{N}(0, \sigma_i^2)$ random variables. We assume $|\varphi_i| < 1$, in which case the processes are stable in the sense that as time evolves they converge to a stationary version. Note that if $\varphi_i \in (0,1)$, the states are positively correlated over time; for $\varphi_i \in (-1,0)$ the correlation is negative. The case $\varphi_i = 0$ may be neglected as it corresponds to observations being independent. Linear combinations of independent Gaussian random variables are still Gaussian, and hence, their conditional distribution at time t is fully described by the conditional mean $\mu_i(t)$ and the conditional variance $\nu_i(t)$. That is, the sufficient statistic (vector) for the state of channel i is:

$$\vartheta_i(t) = \big(\mu_i(t),\, \nu_i(t)\big),$$

where $\mu_i(t)$ and $\nu_i(t)$ denote the conditional mean and variance at time t defined as

$$\begin{aligned}
\mu_i(t) &:= \mathbb{E}\Big[X_i(t) \,\Big|\, X_i\big(t - \eta_i(t)\big), \eta_i(t)\Big] && (6.2)\\
&= \varphi_i^{\eta_i(t)} X_i\big(t - \eta_i(t)\big),\\
\nu_i(t) &:= \mathrm{Var}\left(X_i(t) \,\middle|\, X_i\left(t - \eta_i(t)\right), \eta_i(t)\right) && (6.3)\\
&= \sigma_i^2 \sum_{h=0}^{\eta_i(t)-1} \varphi_i^{2h} = \sigma_i^2 \frac{1 - \varphi_i^{2\eta_i(t)}}{1 - \varphi_i^2}.
\end{aligned}$$

Hence, the *observation update* operation is:

$$\mathcal{O}_i(x) = \big(\varphi_i\, x,\, \sigma_i^2\big).$$

while, the *belief propagation* operation is given by

$$\mathcal{T}_i(\mu_i,\, \nu_i) = \big(\varphi_i\, \mu_i,\, \varphi_i^2\, \nu_i + \sigma_i^2\big)\,. \tag{6.4}$$

It is easy to show by recursion of the mean and the variance that the k-step belief propagation is:

$$\mathcal{T}_i^k(\mu_i,\, \nu_i) = \left(\varphi_i^k\, \mu_i,\, \varphi_i^{2k}\, \nu_i + \frac{1 - \varphi_i^{2k}}{1 - \varphi_i^2}\, \sigma_i^2\right)\,. \tag{6.5}$$

The belief state space in this case is $S_i = \mathbb{R} \times [\nu_i^{\min}, \nu_i^{\max})$ where $\nu_i^{\min} = \sigma_i^2$ and $\nu_i^{\max} = \sigma_i^2/(1-\varphi_i^2)$. The attracting fixed point of $\mathcal{T}_i(\cdot)$ is the mean-variance pair $(0, \nu_i^{\max})$. It is further interesting to note that the second coordinate of the belief state can only attain values in a countable subset of $[\nu_i^{\min}, \nu_i^{\max})$ (see Eq. (6.3)). This is because when the channel is selected, the conditional variance decreases to the value $\nu_i^{\min}$, and thus, ν_i in (6.5) is always proportional to σ_i^2, where the factor is given by a geometric series in φ_i^2. Observe further that, since $\nu_i < \nu_i^{\max}$ and because $|\varphi_i| < 1$, it always holds that the variance increases when updated with $\mathcal{T}_i(\cdot)$, that is, the decision maker's uncertainty regarding the state of the channel indeed grows as long as no new observation is obtained.

6.1.3 Mixed model example

Having specified the GE and AR channel models, we now consider a mixed model example, which is also used for numerical illustration in Section 6.2. Research in this field to date seems to have focussed on problems with channels of the same type (mostly GE, some AR); it is therefore interesting to investigate a mixed channel model example, where a proportion $q \in [0,1]$ of the channels is GE and the others are AR. This can occur in examples where the dominating phenomena of some of the channels is user interference (GE channels), while for other channels the key feature is slow-fading behaviour (AR channels).

Our model parameters are α_i, γ_i for $i = 1, \ldots, q\,d$ (GE channels) and φ_j, σ_j^2 for $j = q\,d+1, \ldots, d$ (AR channels); we assume that $q\,d$ is an integer.

In order to be able to compare GE and AR channels, we consider the following reward functions:

$$R_i(x_i) = \frac{x_i - \gamma_i}{\sqrt{\gamma_i\,(1-\gamma_i)}}\,, \quad \text{and} \quad R_j(x_j) = \frac{\sqrt{1-\varphi_j^2}}{\sigma_j} x_j\,, \tag{6.6}$$

where x_i is a value observed in GE channel i and x_j is the value observed in AR channel j. Note that these functions are chosen in such a way that the steady state values of rewards from both channels have zero-mean and unit-variance, hence making the channels equivalent in this sense.

The state space of the MDP with (joint belief) states $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_d)$, with scalars ϑ_i, $i = 1, \ldots, q\,d$, and 2-dimensional vectors ϑ_j, $j = q\,d+1, \ldots, d$, is given by:

$$\Theta := [0,1]^{q\,d} \times \prod_{j=qd+1}^{d} \mathbb{R} \times [\nu_j^{\min}, \nu_j^{\max}).$$

An optimal policy π for this MDP is not known in closed form. It can be computed approximately with the aid of dynamic programming algorithms, on a discretised and truncated state space. This is feasible with sufficient accuracy only if d is very small (and indeed this is carried out as part of the numerical examples provided in Section 6.2 for $d = 2$).

6.2 Numerical comparison

In this section we compare the performance of the Whittle index policy to that of the myopic policy and, for small d, to the optimal policy. To evaluate the Whittle indices, we usually need the optimal policy associated with Whittle's one-armed sub-problem.

We obtain the latter from relative value iteration (on a discretised state space) using the optimality equation (5.9). This can be written more explicitly using the reward functions from (6.6) as

$$G + h(\omega) = \max\left\{ R_i(\omega) - \lambda + \omega\, h\big(p_i^{11}\big) + \overline{\omega}\, h(p_i^{01})\,,\; h\big(\omega p_i^{11} + \overline{\omega} p_i^{01}\big) \right\} \tag{6.7}$$

when the channel is GE (so that $\vartheta = \omega$, and $r_i(\vartheta) = R_i(\omega)$ since $R_i(\cdot)$ as defined in (6.6) is affine), and

$$G + h(\mu,\, \nu) = \max\left\{ R_j(\mu) - \lambda + \int_{-\infty}^{\infty} h(\varphi\, y,\, \sigma^2)\, \phi_{\mu,\nu}(y)\, dy,\; h\big(\varphi\mu, \varphi^2\nu + \sigma^2\big) \right\} \tag{6.8}$$

when the channel is AR (in which case $\vartheta = (\mu, \nu)$, and $r_j(\vartheta) = R_j(\mu)$ since $R_j(\cdot)$ is linear). Here $\phi_{\mu,\nu}$ denotes the normal density with mean μ and variance ν. Note that in the case of GE channels, the Whittle indices are in fact available in closed form, [99]. Still, for the purposes of exposition, we carry out relative value iteration in this section numerically.

Figure 6.2 shows the optimal switching curve for a small mixed system with one AR and one GE channel. To the left of the curve, where ω is large in comparison to μ, the optimal policy is to select the GE channel. To the right of the curve selecting the AR channel is optimal. The curve shifts with the age of the AR channel: the more time has passed since the AR channel has last been observed, the more inclined the transmitter should be to select that channel in order to update the available information regarding its state. In other words, it is indeed optimal to give some priority to exploration if AR channels are present in the system. Note, however, that for 'older' channels this effect is less pronounced because in that case the resulting change in the conditional variance ν is smaller (recall the belief propagation of ν defined by (6.4)).

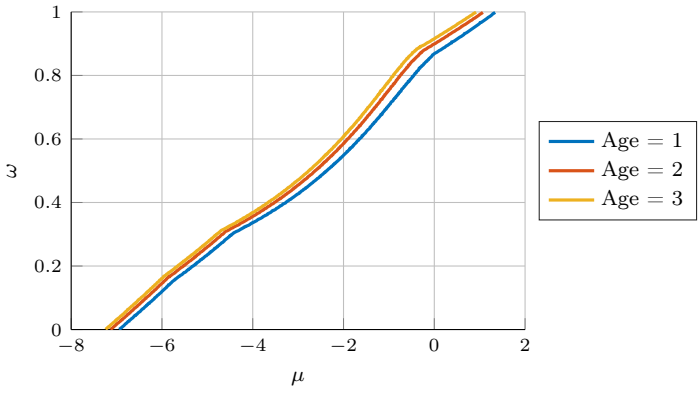


Figure 6.2: Optimal switching curves for a system with $d = 2$ channels: an AR channel with $\varphi = 0.8$ and $\sigma = 2$, and a GE channel with $\rho = 0.5$ and $\gamma = 0.8$. This figure shows the switching curves on the ω, μ plane, one curve per age $\eta \in \{1, 2, 3\}$.

Figure 6.3 shows a comparison of the rewards that are obtained per channel on average under different policies. Here, $k = d/2$ channels are selected at a time in a system with d channels, where half of the channels are GE and the other half are AR. All of the AR channels are with $\varphi = 0.8$ and $\sigma = 2$ (as in Figure 6.6). The GE channels on the

other hand are heterogeneous, with $\gamma = 0.8$ and $\rho_i \in [0.2, 0.8]$ evenly spaced such that $0.8 = \rho_1 > \cdots > \rho_{d/2} = 0.2$. Depicted are the average rewards per arm obtained under the Whittle and the myopic index policy, and, as an upper bound, we also computed the average rewards that could be obtained in a fully observable system under the myopic policy. Due to the high computational complexity, the optimal policy is only evaluated for $d = 2$.

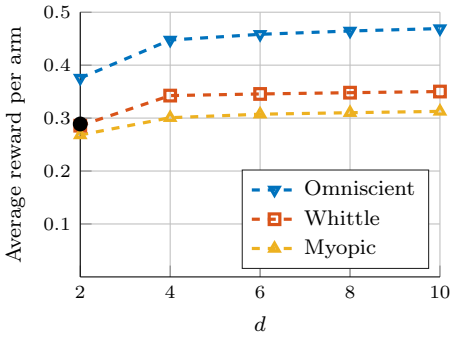


Figure 6.3: Comparison of Whittle and myopic index policies for increasing number of channels d when half of the channels are GE and the other half is AR. For $d = 2$, the average reward obtained under the optimal policy is indicated by a black dot. We compare to the average reward that could be obtained if both arms were observed at each time point (that is in the fully observable or 'omniscient' setting).

All policies seem to approach a certain steady performance in terms of average reward per arm rather quickly as the number of channels grows large while the ratio k/d remains fixed. The achieved average reward level demonstrates the significant improvement in throughput that can be achieved by utilising the channel memory: the average reward is increased by more than 30% compared to the zero average reward that is obtained when channel memory is not used.

Figure 6.3 also confirms that some degree of exploration is favourable: In this example the Whittle index policy improves the average reward per arm by about 5% with respect to the myopic policy. This is in contrast to scenarios where all channels are GE and stochastically identical. In the latter case it can be shown that the Whittle and the myopic index policy are equivalent. We give further details in (i) below.

From a practical perspective an increase of 5% may appear small, however, it can be crucial in systems that are nearing fundamental limits. For example, for wireless devices with limited battery life such an increase may effectively correspond to a decrease of a few percent in power consumption, which may be significant in increasing the operational time of the device.

Next, we investigate the Whittle indices $\iota^W(\omega)$ obtained for GE channels with various parameter combinations (Figure 6.4). We observe the following properties of $\iota^W(\omega)$:

(i) The index function $\iota^W(\omega)$ increases monotonically; the larger the conditional probability that the channel is in a good state, the more priority should be given to that channel. This implies that the Whittle index is equivalent to the myopic policy in systems with *identical* channels, as we mentioned above.

(ii) $\iota^W(\omega)$ is affine within the ranges $\big[0, \min\{p^{01}, p^{11}\}\big]$ and $\big[\max\{p^{01}, p^{11}\}, 1\big]$. Further, it changes slope at γ.

These properties have been proven in [99] for GE channels with reward function $r(\omega)$ given by the identity function.

We further note that the Whittle indices are overall smaller if γ is larger because in this case the rewards are smaller (as R_j defined by (6.6) is decreasing in γ).

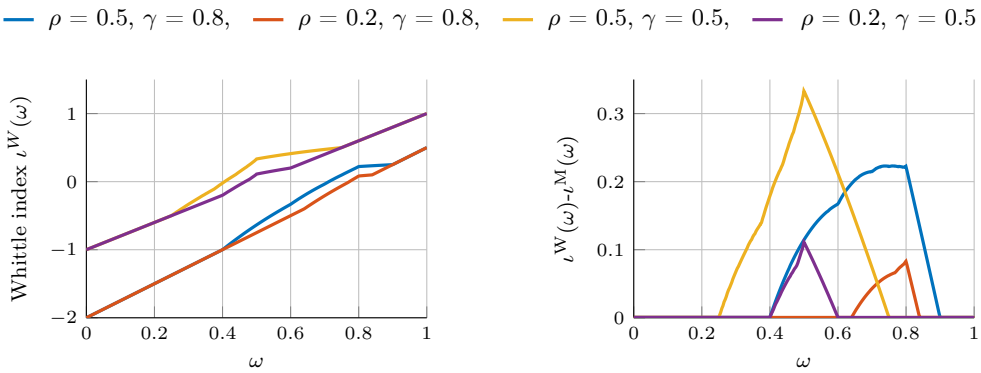


Figure 6.4: Whittle indices for GE channels parametrised by α and γ.

Figure 6.5: Difference between Whittle and myopic index function.

In Figure 6.5 we show the difference between the Whittle and the myopic index function. It can be seen that the index functions are basically identical on $\left[0, \min\{p^{01}, p^{11}\}\right]$ and $\left[\max\{p^{01}, p^{11}\}, 1\right]$: In these regions exploration is not essential as it is rather certain that the state will evolve towards γ. Accordingly, we see that ι^W and ι^M do differ around γ, and on a larger interval to the left of γ.

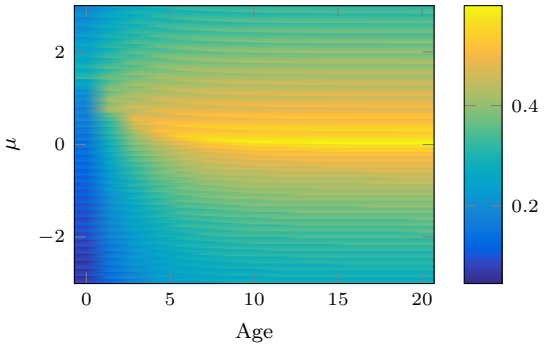


Figure 6.6: Contour plot of $\iota^W(\mu, \nu) - \iota^M(\mu)$ (the difference of Whittle and myopic indices), for an AR channel with $\varphi = 0.8$, $\sigma = 2$.

Figure 6.6 depicts the difference between Whittle and myopic indices as obtained for an AR channel. Both indices increase with μ because the expected immediate reward is larger. Note that for increasing age the Whittle indices increase relative to the myopic indices. Again this suggests that exploration pays off. Furthermore, for high ages the difference between the Whittle and the myopic indices is largest around zero, which corresponds to the unconditional mean reward of the channel. Similarly to the GE case, this may be explained by noting that exploration is more important if μ is close to the unconditional mean as it is less clear in which direction the belief state will evolve. If

μ is far away from the unconditional mean on the other hand, then it is likely that the updated conditional mean will be closer to the unconditional mean. However, when the age is close to zero, then due to the positive correlation of the channel it is also important that μ was large just an instance ago. Thus, while for small ages the Whittle indices are generally close to the myopic indices, the largest difference can be seen for positive μ (however not too far away from the unconditional mean of the channel).

6.3 Literature survey

There is a vast body of literature on MDP as well as topics related to (restless) multiarmed bandits. Here, we focus on the RORMAB formulation of the basic channel selection problem as formulated in this chapter, with GE or AR channels. Other (approximate) solutions to this MDP problem have been put forward [67], but are not considered here.

GE Channels. The GE model was proposed in [47] for the purpose of modelling burst-noise telephone circuits. It was the first non-trivial channel model with memory. Since the 1990's, the model and its generalisations have been used for modelling flat-fading channels in wireless communication networks. Its application in the context of Opportunistic Spectrum Access (OSA) is motivated by the bursty traffic of primary users [70, 173]. For an account on the history of the GE model we refer to [132].

Due to its simplicity, the GE model is mathematically tractable and has been analysed extensively in the context of channel selection in wireless networks. We survey a number of papers that model the problem as a RORMAB with GE channels. Unless otherwise stated, channels are assumed to be independent and stochastically identical.

One of the first papers in this context appears to be [78]. The paper is motivated by the problem of allocating bandwidth of a shared wireless channel between a base station and multiple homogeneous mobile users. Thus, from an engineering perspective, the set-up slightly differs from the problem considered in this chapter; the model and the mathematical analysis, however, apply directly to the channel selection scenario (where simply "users" are replaced by "channels").

In [78], the noisiness of the link for the users is modelled using the GE model. At any point in time a user may either be connected to the base station or not. The current state of a user is only observed when a packet is transmitted to that user. Rewards are given by the number of successful transmissions. The analysis is with respect to the *discounted* reward criterion over an infinite time horizon. The authors show that the myopic policy is generally optimal for the case of $d = 2$ users. For the case $d > 2$ and positively correlated channels it is proven that the myopic policy is optimal if the discount factor is small enough (Condition (A) in [78]). Furthermore, in the positively correlated scenario the myopic policy is seen to be equivalent to a "persistent round robin" policy where the link is dedicated to each user in a cyclic fashion according to their initial probability of being in a good state, and packets are transmitted to the same user until a packet fails to be transmitted correctly.

Following this work, the GE channel model has been analysed extensively in a surge of research on OSA, which goes back to [77]. The aim of this branch of research is to find secondary user policies that efficiently exploit transmission opportunities created by the bursty usage patterns of licensed primary users in wireless networks.

One of the first to formulate the RORMAB with GE channels in the context of OSA were Zhao et al. in 2005 [171]. The authors compare the transmission rate achieved by

the myopic policy to the optimal policy using numerical examples.

This work was the starting point of a sequence of papers analysing the performance of the myopic policy. In [169], optimality is proven for the case of choosing one out of two channels, with respect to expected total discounted rewards over finite as well as infinite time horizon.

The scenarios in which the myopic policy is optimal are then generalised in a sequence of papers. Javidi et al. [68] consider the case of selecting 1 out of d channels and prove optimality of the myopic policy under the discounted reward criterion for positively correlated channels provided the discounted factor satisfies a certain inequality with respect to the transition probabilities. Under the additional ergodicity criterion

$$|p^{11} - p^{00}| < 1, \tag{6.9}$$

the myopic policy is further shown to be optimal under the average reward criterion (cf. (5.3)). The work of [68] is extended in [98] to the case of selecting $d-1$ out of d channels.

In [170] for the case of choosing 1 out of d channels the result of [78] is confirmed that the myopic policy is a persistent round robin scheme if channels are positively correlated. It is further shown that if correlation is negative, then the myopic policy is a round robin scheme, where the circular order is reversed in every time slot (and as for the positively correlated case, the user switches to the next channel as soon as the currently used channel signals has transitioned to the bad state). For the case $d = 2$, the myopic policy is shown to be optimal in general, as had already been established in [78]. Furthermore, it is shown that the performance of the myopic policy is determined by the stationary distributions of a higher-order countable-state Markov chain. The stationary distribution is known in closed form for the case $d = 2$. For the case $d > 2$, lower and upper bounds are established.

For negatively correlated channels and the case of selecting 1 out of d channels, the finite and infinite horizon discount-reward optimality of the myopic policy is proven in [7], provided that either $d \in \{2, 3\}$ or the discount factor is less than half. These results also hold under average rewards under the additional ergodicity condition (6.9). For the finite-horizon discounted reward criterion, the results of [7] are generalised in [4] to the case of selecting k channels.

In 2014, Liu et al. [101] provide a unifying framework of the optimality conditions for the myopic policy that resulted from the OSA-motivated research of the channel selection problem with GE channels. The problem formulation in [101] is more general as it allows one to sense k out of d (identically distributed) channels but select only $l \leq k$ of those channels for transmission based on the outcome of the sensing. The authors provide a set of unifying sufficient conditions under which the myopic policy is optimal. It is shown that the optimal policy is not generally myopic if $l < k$. (This is intuitive because the user is allowed to explore channels without having to use them.)

The Whittle index policy has also been studied both for the bandwidth allocation problem that was put forward in [78], and also in the context of OSA. As opposed to [78], a paper by Niño-Mora [112] handles the problem of bandwidth allocation when users are *heterogeneous*. The author proves that the problem is indexable and provides closed-form expressions for the index function.

For the basic RORMAB with GE channels, Liu and Zhao [99] prove that the Whittle index and the myopic policy are equivalent for positively correlated identical channels, thus, yielding the optimality of the Whittle index in this case. In [117], the indexability and closed-form expression for the Whittle index in the case of discounted rewards are

derived for a more general model where the achievable transmission rate (the reward) for a channel in the bad state is, in general, non-zero and any rate above this achievable rate leads to outage.

Apart from the index policies proposed in this line of research, algorithms for approximating an optimal policy have also been investigated. See, for example, [58, 60], where algorithms for the more general model with correlated channels are proposed and investigated regarding their performance.

In the context of GE channels a number of generalisations of the basic model considered in this chapter have been considered. For example, a paper by Niño-Mora [113] allows for non-identical channels with sensing errors/measurement noise. Imperfect sensing was also considered in [100, 159]. In [117] the authors consider a problem where in both states, good and bad, transmission may fail with a certain non-zero probability, and it is only observed whether transmission was successful or not. Another recent paper with imperfect sensing is [110]. In this paper the focus is on stability issues of queues associated with channel (server) selection in the context of imperfect sensing.

The paper [109] deals with random delay of feedback arrivals. Correlated channels were considered in [58, 59, 60]. Action-dependency of channel model parameters is taken into account in [153]. A very substantial paper is [157], which considers an RMAB in continuous time, and allows for non-identical channels, a time-dependent number of channels, and multiple actions. In this paper, a more general class of index policies is considered, which includes the Whittle index if the bandit problem is indexable. Asymptotic optimality for this class is proven for systems with many channels.

AR Channels. The AR channel model has only recently come to attention in the context of channel selection, and consequently the mathematical analysis is still at its infancy. The first to propose the application of this model for channel selection were Avrachenkov *et al.* [11] in 2012. This is motivated by empirical studies [2], showing that the AR model captures the signal-to-noise ratio (SNR) behaviour of the channels reasonably well.

In [11], the authors compare the performance of the myopic and an ad-hoc randomised policy to the optimal policy by means of numerical examples. It is concluded that the myopic policy is "nearly optimal" when all channels are similarly correlated, with respect to the long-run average reward criterion. In contrast, the randomised policy appears to perform better when there is a significant difference in the magnitude of the correlation of the channels.

Subsequently, the authors show how to model the problem when two transmitters are present that can possibly interfere with each other. In this case the SNR is replaced by the signal-plus-interference-to-noise-ratio (SINR) to model the states of the channels. The scenario is formalised as a competitive MDP (also called a stochastic game) – an MDP in which the instantaneous rewards for each player and the transition probabilities among the states are controlled by the joint actions of the players in each state. Then, similar to the single user case, a randomised and a myopic policy are suggested (now based on the SINR).

We also note that a related body of literature deals with the problem of optimal sensing of Kalman filters. A key paper in this line of research is [115]. A related paper is [114] as well as the recent [34] which appears to provide an indexability proof using a new novel method. It is possible that ideas put forward in these papers dealing with the Whittle index and simple Gaussian processes may be fruitful for the RORMAB problem

with AR channels. This avenue of research remains to be explored.

In the next chapter we present present the results of our own research on the AR channel model as published in [83, 84].

CHAPTER 7

Reward-observing bandits with Gaussian autoregressive arms

In this chapter we focus on the RORMAB for channel selection with Gaussian AR channels as introduced in Chapter 6. The content of this chapter has appeared in Kuhn, Mandjes and Nazarathy [83, 84].

We first briefly describe the type of results presented in this chapter. Considering the discounted reward case, we find structural properties of the one armed sub-problem associated with the Whittle index. We establish convexity and monotonicity properties and show numerically the existence of a switching curve and the monotonicity of the related Whittle index. These properties motivate a simple parametric index which quantifies the virtue of exploration compared to exploitation in terms of variance and mean. For this index we investigate the mean-field behaviour of the system in the average reward case. In particular, we put forward a deterministic measure-valued recursion that approximately describes the distribution of belief states when the number of arms is large. We merge these ideas into a performance evaluation and optimisation procedure.

The organisation of this chapter is as follows. In Section 7.1 we formulate the decision problem. Sections 7.2 and 7.3 present our contributions with respect to the structural and asymptotic analysis of the problem. We conclude in Section 7.4.

7.1 Model and framework

The AR model features independent state processes $\boldsymbol{X}(t) := \big(X_1(t), \ldots, X_d(t)\big)$, which satisfy the AR(1) recursion

$$X_i(t) = \varphi\, X_i(t-1) + \varepsilon_i(t),$$

with $\big\{\varepsilon_i(t)\big\}_{t\in\mathbb{N}_0}$ denoting an i.i.d. sequence of $\mathcal{N}\big(0,\sigma^2\big)$ random variables. The parameters φ, σ are assumed to be known. We restrict our exposition to the case $\varphi \in (0,1)$, whence the processes are stable and observations are positively correlated over time. (The case $\varphi < 0$ appears to be more difficult and is excluded from this discussion.)

We are in the partially observable setting, that is, the state of an arm is only observed

when that arm is activated, while at every time slot all arms evolve to the next state. Recall that the belief state is fully characterised by the conditional mean and variance defined in (6.2) and (6.3). We denote the joint (belief) state space of (μ_i, ν_i) by $\Theta := \Theta_1 \times \Theta_2$ so that $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \Theta^d$.

The aim is to find a policy π so as to maximise the accumulated rewards over an infinite time horizon as evaluated by the total expected discounted reward criterion defined in (5.2), which we repeat here for convenience:

$$V^\pi(\boldsymbol{\mu}, \boldsymbol{\nu}) := \lim_{T\to\infty} \mathbb{E}^\pi_{\boldsymbol{\mu},\boldsymbol{\nu}} \left[\sum_{t=0}^{T} \beta^t \sum_{i=1}^{d} X_i(t)\, a_i(t) \right]. \tag{7.1}$$

(As before, $\beta \in (0,1)$ is the discount factor, and the subscript indicates conditioning on $\boldsymbol{X}(0) \sim \mathcal{N}\big(\boldsymbol{\mu}, \operatorname{diag}(\boldsymbol{\nu})\big)$.) Similarly, the average expected reward criterion now reads

$$G^\pi(\boldsymbol{\mu}, \boldsymbol{\nu}) := \liminf_{T\to\infty} \frac{1}{T} \mathbb{E}^\pi_{\boldsymbol{\mu},\boldsymbol{\nu}} \left[\sum_{t=0}^{T-1} \sum_{i=1}^{d} X_i(t)\, a_i(t) \right]. \tag{7.2}$$

Note that $X_i(t)$ in (7.1) and (7.2) can be replaced by $\mu_i(t)$ by the tower property of conditional expectations.

Lemma 7.1. *The function V^π is well-defined in the sense that the limit in (5.2) exists and is finite. Furthermore, the optimal value function $\sup_\pi V^\pi$ is finite.*

Proof. For each arm i we have

$$\begin{aligned} \mathbb{E}_{\mu_i,\nu_i} |X_i(t)| \leq & \mathbb{E}_{\mu_i,\nu_i} \left[\varphi^t |X_i(0)| + \sum_{j=0}^{t-1} \varphi_i^j |\varepsilon_i(t-j)| \right] \\ \leq & \sqrt{\frac{2\nu_i}{\pi}} + |\mu_i| + \sqrt{\frac{2}{\pi}} \frac{1}{1-\varphi} =: B(\mu_i, \nu_i), \end{aligned}$$

which is finite, and thus,

$$\sup_\pi \sum_{t=0}^{\infty} \beta^t \mathbb{E}^\pi_{\mu_i,\nu_i} \left[\sum_{i=1}^{d} |X_i(t) a_i(t)| \right] \leq \frac{d \max_i B(\mu_i, \nu_i)}{1-\beta} < \infty,$$

whence $\sup_\pi V^\pi(\boldsymbol{\mu}, \boldsymbol{\nu})$ is finite, and $\sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{d} |X_i(t) a_i(t)|$ converges almost surely to a finite limit. The variables $Z_T := \sum_{t=0}^{T} \beta^t \sum_{i=1}^{d} X_i(t) a_i(t)$ thus converge almost surely as $T \to \infty$ and are dominated by the absolute sum which has finite mean. Hence, by dominated convergence

$$\mathbb{E}^\pi_{\boldsymbol{\mu},\boldsymbol{\nu}} Z_T \to \mathbb{E}^\pi_{\boldsymbol{\mu},\boldsymbol{\nu}} \left[\sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{d} X_i(t) a_i(t) \right].$$

□

In view of computational tractability we restrict our exposition to policies from the class of index policies.

7.2 Index policies

Recall that an index policy is a policy of the form

$$\pi_\iota(\boldsymbol{\mu}, \boldsymbol{\nu}) = \underset{\boldsymbol{a}:\sum_{i=1}^d a_i = k}{\arg\max} \left\{ \sum_{i=1}^{d} \iota\left(\mu_i, \nu_i\right) a_i \right\}.$$

Without loss of generality the index function can be written as

$$\iota(\mu, \nu) = \mu + q(\mu, \nu) \tag{7.3}$$

for some known function $q : \Theta \to \mathbb{R}$. For the myopic index we have $q \equiv 0$. As the resulting policy does not account for information growing obsolete (giving full priority to exploitation), the performance of the myopic policy deteriorates as $\beta \uparrow 1$. We therefore focus on the Whittle index that was introduced in Section 5.1.2.

7.2.1 Whittle index

In this section we show how to obtain the Whittle index when the underlying states are AR, and provide structural results. Our focus is on the discounted reward criterion (7.1).

In Definition 5.2 we defined the Whittle index for arm i as the largest cost such that it is still optimal to play the arm in the one-arm sub-problem. Equivalently (and this is the more classical interpretation), we may define it to be the smallest subsidy for which it is optimal to leave the arm passive:

Definition 7.2. Let $\mathcal{P}_\lambda$ denote the *passive set* associated with the one-armed problem with subsidy λ, that is,

$$\mathcal{P}_\lambda := \{(\mu, \nu) \,|\, a = 0 \text{ is optimal action}\}.$$

The Whittle index associated with this arm and state (μ, ν) is given by

$$\omega(\mu, \nu) = \inf\left\{\lambda \,|\, (\mu, \nu) \in \mathcal{P}_\lambda\right\}.$$

As before, we shall assume that indexability holds, as confirmed through extensive numerical experimentation (see, for example, Figure 7.1). The discount-optimal value function $V^\lambda := \sup_\pi V^{\lambda,\pi}$ for the one-armed sub-problem with subsidy can be obtained using value iteration (see Proposition 7.3).

First we introduce the operator $Tv := \max_{a\in\{0,1\}} T_a v$, where

$$T_a v\,(\mu, \nu) := \begin{cases} \lambda + \beta\, v(\varphi\,\mu,\, \varphi^2\nu + \sigma^2), & a = 0, \\ \mu + \beta \int_{-\infty}^{\infty} v\left(\varphi\, y,\, \sigma^2\right) \phi_{\mu,\nu}(y)\, dy, & a = 1, \end{cases}$$

with $\phi_{\mu,\nu}$ denoting the normal density with mean μ and variance ν.

Proposition 7.3. *For $V_0^\lambda \equiv 0$ the iteration*

$$V_n^\lambda = TV_{n-1}^\lambda \tag{7.4}$$

converges to a unique function $V^\lambda : \Theta \to \mathbb{R}$ *as* $n \to \infty$ *that satisfies the Bellman equation,*

$$V^\lambda = TV^\lambda.$$

This V^λ *is the discount-optimal value function for the one-arm bandit problem with subsidy* λ*. An optimal policy for this problem maps* (μ, ν) *to action* a *if* $V^\lambda(\mu,\nu) = T_a V^\lambda(\mu,\nu)$*.*

Proof. Define $b(\mu) := 1 + \mu^2$. Then, because $|\max\{m, \mu\}| \le \big(1 + |m|\big)b(\mu)$, the absolute value of the expected immediate reward under both actions (passive, active) is bounded by $\big(1 + |m|\big)b(\mu)$ for any belief state in Θ. Furthermore,

$$\int_{-\infty}^{\infty} b(\varphi y)\phi_{\mu,\nu}(y)dy \le \big(1 + \varphi^2\nu_{\max}\big)\, b(\mu)$$

and hence, b is an upper bounding function in the sense of [16, Definition 7.1.2]. The implication of this is that the space $\mathcal{V}$ of measurable functions $v : \Theta \to \mathbb{R}$ with finite weighted supremum norm defined by

$$\|v\|_b := \sup_{\mu,\nu} \frac{|v(\mu,\nu)|}{b(\mu)} < \infty$$

contains the optimal value function V. We apply [16, Theorem 7.2.1]. To verify the main condition of the latter, define the operator Q by

$$Qv(\mu,\nu) := \max\left\{\beta v(\varphi\mu, \varphi^2\nu + \sigma^2), \beta \int_{-\infty}^{\infty} v(\varphi y, \sigma^2)\phi_{\mu,\nu}(y)dy\right\}.$$

Take $b(\mu,\nu) = b(\mu)$ as defined above and observe that

$$Q^n b(\mu,\nu) \le \beta^n\Big(1 + \varphi^2\nu_{\max} + \varphi^{2n}\left(\nu + \mu^2\right)\Big),$$

whence $Q^n b \to 0$ as $n \to \infty$. Noting that the further regularity conditions of [16, Theorem 7.2.1] are satisfied, we obtain that with initial choice $V_0 \equiv 0$ the value iteration converges to an optimal value function, and an optimal policy exists; namely, it is optimal to take the action that maximises the right-hand side of the Bellman equation. The uniqueness of the value function can be seen as follows. Let v and w be two fixed points of T. Then

$$T^n v(\mu,\nu) = T\left(T^{n-1}v(\mu,\nu)\right) = \cdots = v(\mu,\nu),$$

for every $n \in \mathbb{N}$. Because $Q^n \to 0$ we know that for every (μ,ν) there exists n_μ such that

$$\frac{|T^{n_\mu}v(\mu,\nu) - T^{n_\mu}w(\mu,\nu)|}{b(\mu,\nu)} \le \alpha \sup_{\mu,\nu} \frac{|v(\mu,\nu) - w(\mu,\nu)|}{b(\mu,\nu)}$$

for some $\alpha \in (0,1)$. Hence,

$$\|v - w\|_b = \sup_{\mu,\nu} \frac{|T^{n_\mu}v(\mu,\nu) - T^{n_\mu}w(\mu,\nu)|}{b(\mu,\nu)} \le \alpha\|v - w\|_b,$$

which implies that $v \equiv w$. □

Structural properties. Let us first consider monotonicity properties of the optimal value function V^λ.

Lemma 7.4. *Let $\varphi \in (0,1)$. Then $V^\lambda(\cdot,\nu)$ is convex, continuous on open intervals, non-decreasing, and not constant; and $V^\lambda(\mu,\cdot)$ is non-decreasing.*

Proof. The proof is by induction on (7.4), and consists of three parts. Part (a) refers to the convexity assertion, which implies continuity on open intervals. In Part (b) we prove the monotonicity properties of $V^\lambda(\cdot,\nu)$ for fixed ν, whereas in Part (c) we show monotonicity of $V^\lambda(\mu,\cdot)$ with μ fixed.

(a) Suprema, expectations, compositions of convex and increasing functions as well as linear combinations with non-negative weights of convex functions are convex. Then the result follows from (7.4) by induction.

(b) For $V_0^\lambda \equiv 0$, we have that $V_1^\lambda(\cdot,\nu)$ is non-decreasing and thus we may assume that $V_n^\lambda(\cdot,\nu)$ is non-decreasing for some n. If $\mu_1 \le \mu_2$, then by a stochastic ordering of $Y_{\mu_i,\nu} \sim \mathcal{N}(\mu_i,\nu)$ it holds that

$$\mathbb{E}\left[V_n^\lambda\left(\varphi\, Y_{\mu_1,\nu},\, \sigma^2\right)\right] \le \mathbb{E}\left[V_n^\lambda\left(\varphi\, Y_{\mu_2,\nu},\, \sigma^2\right)\right].$$

It follows by induction that V_n^λ is non-decreasing in μ for all $n \in \mathbb{N}$, and thus their limit V^λ is non-decreasing in μ. Furthermore, since a lower bound for V^λ is given by the value obtained when always playing active, $\mu/(1-\varphi\beta)$, which is strictly increasing in μ, it is evident that V^λ cannot be constant in μ.

(c) Let $V_0^\lambda \equiv 0$. Then $V_1^\lambda(\mu,\cdot)$ is constant and thus non-decreasing. Assume that $V_n^\lambda(\mu,\cdot)$ is non-decreasing. We prove below that $\mathbb{E}\left[V_n^\lambda\left(\varphi\, Y_{\mu,\nu},\, \sigma^2\right)\right]$ is non-decreasing in ν. Then it follows by induction that V_n^λ is non-decreasing in ν for every n, and thus, so is V^λ.

For brevity we assume that $V^\lambda(\cdot,\nu)$ is differentiable (otherwise the arguments need to be formulated in terms of subdifferentials). Define $g(y) := V_n^\lambda(\varphi\sqrt{\nu}y + \mu, \sigma^2)$; this is increasing and convex as it is a composition of a convex and a monotone increasing function. Applying Jensen's inequality we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\nu}\mathbb{E}\left[V_n^\lambda\left(\varphi\, Y_{\mu,\nu},\, \sigma^2\right)\right] \\
&\quad= \frac{\varphi}{2\sqrt{\nu}}\int_{-\infty}^{\infty} y\, \frac{\partial}{\partial x} V_n^\lambda\left(x,\, \sigma^2\right)\Big|_{x=\varphi\sqrt{\nu}y+\mu} \phi_{0,1}(y)\, dy \\
&\quad= \frac{1}{2\nu}\int_{-\infty}^{\infty} y\, g'(y)\, \phi_{0,1}(y)\, dy \\
&\quad\ge \frac{1}{2\nu}\int_{-\infty}^{\infty} \big(g(y) - g(0)\big)\, \phi_{0,1}(y)\, dy \ge 0
\end{aligned}$$

because g is convex (and thus $g(b) - g(a) \le g'(b)(b-a)$ for all a, b in $\mathbb{R}$, the domain of g). □

Figure 7.1 and similar numerical experiments[1] suggest that the passive set $\mathcal{P}_\lambda$ and the active set $\mathcal{P}_\lambda^c$ are separated by a *switching curve* (defined on the countable space Θ_2).

[1] To execute the value iteration we truncate Θ_1 to $[-6\sigma, 6\sigma]$, and consider $\lambda \in [-2\sigma, 2\sigma]$. Discretisation is done in steps of size 0.01, which is preserved when truncating Θ_2.

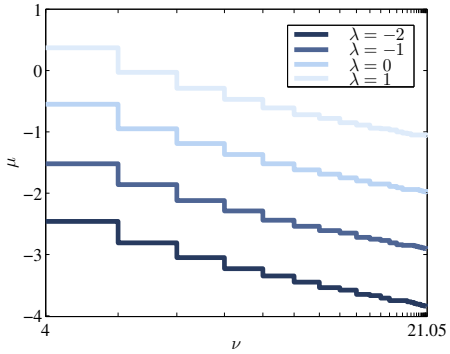


Figure 7.1: Switching curves: below the curve the optimal action is passive, above it is active. Parameters are chosen as $\beta = 0.8$, $\varphi = 0.9$, $\sigma = 2$. Note that the state space of ν is countable within $[\sigma^2, \sigma^2/(1-\varphi^2))$.

Conjecture 7.5. A policy that achieves the optimal value function V^λ is a threshold policy: There exists a switching curve (sequence) $\zeta_\lambda : \Theta_2 \to \Theta_1$ such that $\mathcal{P}_\lambda = \{(\mu, \nu) \,|\, \mu \leq \zeta_\lambda(\nu)\}$.

Let $\mu_1 \leq \mu_2$, then Conjecture 7.5 implies that $(\mu_1, \nu) \in \mathcal{P}_{\omega(\mu_2,\nu)}$. Hence, we have $\omega(\mu_2, \nu) \in \{m \,|\, (\mu_1, \nu) \in \mathcal{P}_m\}$ and thus, $\omega(\mu_1, \nu) \leq \omega(\mu_2, \nu)$ by definition of the Whittle index as an infimum.

Numerical evidence such as provided in Figure 7.1 suggests that the switching curve is in fact strictly decreasing, i.e. it is optimal to give some priority to exploration. Assuming this is the case, one would expect that $\omega(\mu, \cdot)$ is monotone non-decreasing, which would imply that the Whittle index policy assigns comparatively larger indices to arms that have not been activated for a longer time. In accordance with this observation, Figure 6.6 in Chapter 6 showed that the correction term $q(\mu, \nu)$ is positive and increases in ν. Furthermore, we confirmed numerically that the slope of $\zeta_\lambda(\nu)$ increases as β increases as in this case exploration becomes more beneficial.

7.2.2 Parametric index

As no closed form for the Whittle index is available, the Whittle indices have to be computed and stored for every belief state in Θ, while the evaluation of the optimal value function for the one-armed problem with subsidy is computationally expensive. Therefore, instead of finding the index for the one-armed problem with subsidy that is optimal (the Whittle index), we propose to find the index that is optimal when restricting to a family of parametric functions. A simple example is obtained by picking a function $q(\mu, \nu)$ that is proportional to ν, the most obvious measure for the decision maker's uncertainty. This yields the *parametric index*

$$\iota(\mu, \nu) = \mu + \theta\nu, \tag{7.5}$$

where $\theta \geq 0$ because the Whittle index function is non-decreasing in ν. The correction term $\theta\nu$ allows to adjust the priority the decision maker wants to give to exploration. We denote the associated policy by π_θ.

The parametric index can be related to the Whittle index as follows. Numerical experiments (such as Figure 7.1 for discounted, and related experiments for average rewards)

suggest that the optimal switching curve may be well approximated by a linear function, the slope of which is negative but does not depend on λ. The position of the curve on the other hand does depend on λ. Such an approximation is given by $\zeta_\lambda(\nu) \approx -\theta\nu + \lambda + c$ with $\theta \geq 0$, $c \in \mathbb{R}$. As $\zeta_\lambda(\nu)$ takes some value $\mu \in \mathbb{R}$, solving for λ (which may correspond to the Whittle index) suggests using an index of the form (7.5), where without loss of generality we take $c = 0$.

In the next section we show that we can explicitly describe the asymptotic dynamics of the system induced by π_θ.

7.3 System with many arms

We investigate the behaviour of the system with many arms as $d \to \infty$ and $k_d/d \to r$. In Section 7.3.1 we outline the main idea that is based on the intuition that the limiting proportion of belief states remains stable in an equilibrium system with infinitely many arms. In Section 7.3.2 we relate the equilibrium system to a single arm process, and use this connection to propose an algorithm for performance evaluation. This algorithm is used to optimise π_θ in Section 7.3.3.

7.3.1 Limiting empirical distribution

Consider the system with d arms as before, and to simplify the exposition suppose for now that the system is stationary. Let $I_i(t)$ denote the process of indices associated with arm i, that is, $I_i(t) := \iota\big(\mu_i(t), \nu_i(t)\big)$. Note that the index processes $I_i(t)$ and $I_j(t)$, $i, j = 1, \ldots, d$, are generally dependent because the belief states of both arms depend on the action that was chosen, which in turn depends on the index of all arms in the system (as they are coupled by the requirement that those k arms with the largest indices are activated). Let us now focus on a single arm i in this system and suppose that its belief state evolves from ϑ at time t to another belief state $\widetilde{\vartheta}$ at time $t+1$. While d is small, this certainly changes the proportion of arms with current belief state ϑ considerably. However, if d is very large, we should be able to find another arm j whose new belief state at time t is (in close proximity to) ϑ. It thus seems reasonable to expect that, as we add more arms to the system, it approaches a mean-field limit in which the proportion of arms associated with a certain belief state remains fixed. Thus, in the limit, the action chosen for a certain arm is independent of the current belief state of any other arm, as there is always the same proportion of arms associated with a certain belief state in the system.

Let us now more formally investigate the proportion of arms that are associated with a certain belief state at time t. We focus on parametric index functions as defined in (7.5). The empirical measure

$$M^d(C, t) := \frac{1}{d} \sum_{i=1}^{d} \mathbb{1}\Big\{ \big(\mu_i(t),\, \nu_i(t)\big) \in C \Big\} \tag{7.6}$$

quantifies the proportion of arms in the d-dimensional system whose belief state falls into $C \in \mathcal{B}(\Theta)$ at time t, where $\mathcal{B}(\Theta)$ denotes the Borel σ-algebra on Θ. It is related to the

measure on indices,

$$\widetilde{M}^d(B,t) := \frac{1}{d}\sum_{i=1}^{d} \mathbb{1}\{I_i(t) \in B\} \tag{7.7}$$

with $B \in \mathcal{B}(\mathbb{R})$, through

$$\widetilde{M}^d(B,t) = M^d\Big(\Big\{(\mu,\nu) \in \Theta \,|\, \mu + \theta\nu \in B\Big\},\, t\Big). \tag{7.8}$$

We examine the dynamics of $M^d(C,t)$. To this end we enumerate the elements in Θ_2, that is, $\nu^{(h)} = \sigma^2(1-\varphi^{2(h+1)})/(1-\varphi^2)$, $h = 0,1,2\ldots$, so that $h+1$ is the number of time steps since an arm was played last. As before, we refer to h as the *age* of an arm. Then (7.6) can be written as, with $B \in \mathcal{B}(\mathbb{R})$,

$$\sum_{h=0}^{\infty} M_h^d(B,t) := \sum_{h=0}^{\infty} \frac{1}{d} \sum_{i:\, \nu_i(t)=\nu^{(h)}} \mathbb{1}\{\mu_i(t) \in B\}. \tag{7.9}$$

Many-arms asymptotics. As motivated at the beginning of this section, it is reasonable to believe that the limiting proportion of arms associated with a certain belief state evolves deterministically, and thus, that the dynamics of the limiting system can be described by non-random measures $m_h(\cdot,t)$. For brevity we write $m_h(x,t)$ for $m_h\big((-\infty,x],t\big)$, and denote by $\Phi_{\mu,\nu}$ the normal distribution function with mean μ and variance ν. We define $m_h(\cdot,t)$ by the recursion

$$m_h\big(x,t+1\big) \tag{7.10}$$
$$= \begin{cases} \sum_{h=0}^{\infty} \int_{\ell_h(t)}^{\infty} \Phi_{z,\nu^{(h)}}\left(\frac{x}{\varphi}\right) m_h(dz,\, t), & h = 0, \\ m_{h-1}\Big(\min\big\{\frac{x}{\varphi},\, \ell_{h-1}(t)\big\},\, t\Big), & h \geq 1, \end{cases}$$

where $\ell_h(t) := \ell(t) - \theta\nu^{(h)}$ with $\ell(t)$ defined as the quantile function of $\iota\big(\vartheta(t)\big)$ evaluated at $1-r$, that is,

$$\ell(t) = \inf\left\{\ell \,\Big|\, \sum_{h=0}^{\infty} \widetilde{m}_h\big([\ell,\infty),\, t\big) \leq r\right\}. \tag{7.11}$$

Here, $\widetilde{m}_h$ denotes the measure on indices, i.e.

$$\widetilde{m}_h(B,\, t) = m_h\Big(\big\{\mu \in \Theta_1 \,|\, \mu + \theta\nu^{(h)} \in B\big\},\, t\Big), \tag{7.12}$$

cf. Eq. (7.8). Note that $\ell_h(t)$ is a threshold such that at time t the policy π_θ activates all arms that are of age h and have conditional mean $\mu(t) \geq \ell_h(t)$, $h \geq 0$. Obviously, if the policy is myopic, then $\ell_h(t) = \ell(t)$ does not depend on the age of an arm and the above expressions can be simplified. Recursion (7.10) is obtained based on the dynamics of the belief states. The evolution of $m_0(\cdot,t)$ is determined by the evolution of the belief state of all arms that have been played in the previous time slot. If $h > 0$ on the other hand, we use that arms of age h must have been of age $h-1$ at the previous decision time; and since they have not been activated, their mean must have been below the threshold $\ell_h(t-1)$.

For the (pre-limit) empirical processes M^d it obviously holds that $M^d(\Theta, t) = 1$, as well as (with $h = 0$) $M_0^d(\Theta_1, t) = k_d/d$. These properties carry over to the conjectured limiting measure.

Lemma 7.6. *If the sequence* $\{m_h(B,0)\}_h$ *satisfies*

$$m_0\big(\Theta_1, 0\big) = r, \quad \textit{and} \quad \sum_{h=0}^{\infty} m_h\big(\Theta_1, 0\big) = 1, \tag{7.13}$$

then the same holds for $m_h(B,t)$ *for all* $t > 0$.

This is easily proven by induction using (7.10)–(7.12). We believe that (7.10) indeed describes the mean-field behaviour of the dynamical system: Assuming that $M_h^d(B,0)$ converges weakly to $m_h(B,0)$ for all $h \geq 0$,

$$M_h^d(B,0) \xrightarrow{w} m_h(B,0),$$

as $d \to \infty$ while $\lim_{d\to\infty} k_d/d = r$, then, for all $t, h \geq 0$, we expect to have

$$M_h^d(B,t) \xrightarrow{w} m_h(B,t).$$

Long-run equilibrium. Note from (7.10) that for $h \leq t$ we can express $m_h(B,t)$ in terms of $m_0(B,t)$,

$$m_h\big(x,\, t\big) = m_0\left(\min_{j=1,\dots,h}\left\{\frac{x}{\varphi^h},\, \frac{\ell_{h-j}(t-j)}{\varphi^{h-j}}\right\},\, t-h\right).$$

Then the fixed-point equation corresponding to (7.10) is given by

$$\begin{aligned} m_0^*\big(x\big) &= \sum_{h=0}^{\infty} \int_{\ell_h^*}^{\infty} \Phi_{z,\nu^{(h)}}\left(\frac{x}{\varphi}\right) m_h^*(dz) \\ &= \sum_{h=0}^{\infty} \int_{\frac{\ell_h^*}{\varphi^h}}^{\min_j \frac{\ell_{h-j}^*}{\varphi^{h-j}}} \Phi_{\varphi^h z,\nu^{(h)}}\left(\frac{x}{\varphi^h}\right) m_0^*(dz) \end{aligned}$$

where $j = 1, \dots, h$. Here, $\ell_h^* = \ell^* - \theta\nu^{(h)}$ where the steady state ℓ^* is defined by

$$\ell^* = \inf\left\{\ell \,\Big|\, \sum_{h=0}^{\infty} \widetilde{m}_h^*\Big([\ell, \infty)\Big) \leq r\right\}, \tag{7.14}$$

and $\widetilde{m}^*$ again denotes the measure on indices, cf. (7.12).

The above system of equations describes possible equilibrium points of the measure valued dynamical system. It is intricate due to the coupling of ℓ^*, $\widetilde{m}^*$ and the measures m_h, $h \geq 0$. The virtue of this representation is that its solution can potentially be described through a single measure, namely m_0^*.

For the special case of $\theta = 0$ (myopic) we verified numerically that with arbitrary initial choice $\{m_h(\cdot, 0)\}$ satisfying (7.13) an equilibrium point satisfying (7.14) is indeed attracting. Furthermore, when d and t are large enough, the proportion of arms associated

with a certain belief state in a simulated system with d arms is indeed fixed and well approximated by the solution to (7.14) when operated under the myopic policy.

7.3.2 The equilibrium index process

We now relate the system with many arms operated under π_θ to a special *one-armed process with threshold*. For this process the arm is activated whenever the index exceeds a specified threshold ℓ, i.e. $a(t) = \mathbb{1}\{\mu(t) + \theta\nu(t) \geq \ell\}$. Because the evolution of the belief state and thus the evolution of the index depends on ℓ, we denote the associated stochastic process of indices by $I^\ell(t) := \mu(t) + \theta\nu(t)$.

Suppose that ℓ is picked in such a way that we activate with probability r; denote it by $\overline{\ell}$. Then a policy $\pi_\theta^{\overline{\ell}}$ that chooses action $a_i(t) = \mathbb{1}\left\{\mu_i(t) + \theta\nu_i(t) \geq \overline{\ell}\right\}$ for every arm i in an unconstrained system with d arms is a policy which activates $r\,d$ arms *on average* (this is essentially the idea behind Whittle's relaxation [163]). Thus, as $d \to \infty$, the policy $\pi_\theta^{\overline{\ell}}$ activates approximately a proportion r of arms at every decision time.

We believe that in steady state (as $t \to \infty$ or under stationarity) the equilibrium of the measure-valued dynamical system is directly related to the one-armed process with this particular threshold $\overline{\ell}$, and further $\overline{\ell}$ equals ℓ^* of (7.14). Assume that the index is parametric, and that $I^\ell(t)$ is stationary. Then we believe that the equation

$$\mathbb{P}\left(I^\ell(t) \geq \ell\right) = r \tag{7.15}$$

has a unique solution ℓ^*, which satisfies Eq. (7.14), and

$$\mathbb{P}\left(I^{\ell^*}(t) \in B\right) = \sum_{h=0}^{\infty} \widetilde{m}_h^*(B),\ \forall\, B \in \mathcal{B}(\mathbb{R}).$$

A practical implication is that in the limit, as $d \to \infty$ and $t \to \infty$, a parametric index policy π_θ is equivalent to the policy that activates arm i in an unconstrained system whenever $I_i(t) \geq \ell^*$, where ℓ^* is defined by (7.15). This motivates the following simple algorithm for performance evaluation.

Algorithm 2 Performance evaluation.

1: For large T determine $\widehat{\ell^*}$ (e.g. using a bisection method) such that $T^{-1}\sum_{t=0}^{T} a_i(t) = r$ is achieved for a policy $\pi_\theta^{\widehat{\ell^*}}$.
2: Use the sample path of Step 1 to obtain an estimate $\overline{G}$ for the expected average reward of the one-armed system.
3: Output $\overline{G}_d := d\,\overline{G}$ as an approximation of the expected average reward of the multi-armed system with d arms operated under π_θ.

The virtue of this algorithm is that the behaviour of the many-armed system is approximated by simulating a much simpler one-armed problem.

7.3.3 Optimised parametric index

The algorithm can be used to approximate the best parameter values for a parametrised index policy. We approximate $\theta^* := \arg\max_\theta G_d(\theta)$ by $\overline{\theta}^* := \arg\max_\theta \overline{G}(\theta)$, where $G_d(\theta)$ is the average reward obtained under π_θ for the problem with d arms, and $\overline{G}(\theta)$ is the

estimator for $G(\theta)$ as obtained from Step 2 of the algorithm. Figure 7.2 depicts the estimated expected average reward $\overline{G}(\theta)$ as a function of θ. The figure suggests that for large φ, the myopic policy (which corresponds to $\theta = 0$) can be improved significantly.

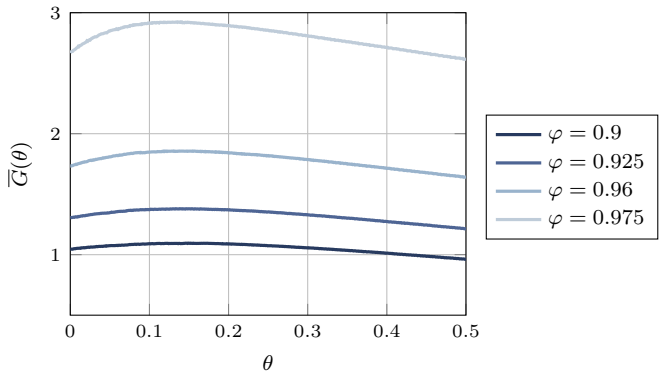


Figure 7.2: Expected average reward $\overline{G}(\theta)$ computed by the algorithm as a function of θ. $\sigma = 2$, $\varphi \in \{0.9, 0.925, 0.95, 0.975\}$, $r = 0.4$, $T = 2 \times 10^6$.

We now examine the performance of π_θ when the parameter is chosen to be $\overline{\theta}^*$. In contrast to the approximation $\overline{G}_d(\theta)$ that is obtained from the algorithm, we denote the estimated average reward obtained by Monte Carlo simulation of the d-armed system by $\widehat{G}_d(\theta)$. We define $\widehat{\theta}_d^* := \arg\max_\theta \widehat{G}_d(\theta)$. Accordingly, $\widehat{G}_d(\overline{\theta}^*)$ and $\widehat{G}_d(\widehat{\theta}_d^*)$ are the average rewards obtained when simulating the system under π_θ, where θ is chosen as $\overline{\theta}^*$ and $\widehat{\theta}_d^*$ respectively. In Figure 7.3 we compare these quantities to the average rewards obtained when simulating the system under the Whittle index and the myopic policy. Unsurprisingly, the Whittle index policy outperforms the other index policies – in fact, we believe it to be asymptotically optimal. However, the parametrised index does considerably better than the myopic.

Importantly, we note from Figure 7.3 that $\widehat{\theta}_d^*$ is indeed well approximated by $\overline{\theta}^*$. Thus, instead of optimising the parameter by simulating the multidimensional d-armed system, we can approximate the best θ-value directly from the one-armed process with threshold for any value of d (such that $k_d = \lfloor r\, d \rfloor$).

7.4 Conclusion

This chapter provides a starting point for a rigorous investigation of the structural properties and performance of index policies in partially observable restless bandit problems with AR arms. This incorporates (i) the analysis of the Whittle index as a likely candidate for an asymptotically optimal policy as $d \to \infty$ while $k_d/d \to r$, and (ii) insights into the behavior of the system in this asymptotic regime. In addition to our conjectures in Section 7.3, we also believe that some form of asymptotic independence holds for the index processes as the number of arms grows large. In this context we mention that I_i, $i = 1, \ldots, d$, are exchangeable [5]. This may yield a path for proving asymptotic independence. The recursions on measures defining the limiting dynamical system can perhaps be treated along the lines of [106].

Furthermore, it can be explored whether the ideas in this chapter can be generalised.

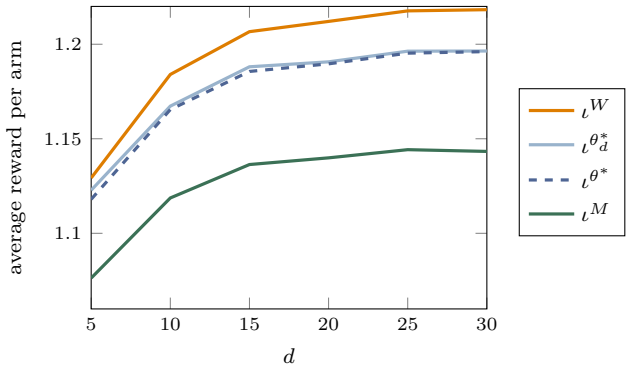


Figure 7.3: Comparison of average rewards achieved per arm under the Whittle, the parametric index (7.5) and the myopic policy. The parameter θ is found by optimising (i) the problem with d arms, and (ii) the one-armed problem (dashed). $\varphi = 0.9$, $\sigma = 2$, $r = 0.4$, $T = 10^6$.

For example, the results obtained in Section 7.2.1 for discounted rewards can be expected to hold in the average reward case as well. Beyond that, we can extend the treatment to AR processes of higher order, heterogeneous arms and bandit problems with correlated arms.

Part III

Testing and Monitoring

CHAPTER 8

Introduction

In Parts I and II we discussed problems related to the control of stochastic systems arising in communication networks and other application areas. The performance of the devised control policies, however, crucially depends on the validity of one's modelling assumptions. Therefore, in this part of the thesis we focus on methods for testing and monitoring stochastic systems.

We distinguish between off-line and on-line testing as follows: Off-line testing problems are posed in such a way that the question of interest is to be answered as accurately as possible given a set of available data; time does not play a role in this case. If instead we assume that new data becomes available as we go, the problem is sequential, and the objective is to arrive at a conclusion as quickly as possible subject to a constraint on the number of false positives. This we regard as an on-line (or sequential) testing problem.

In Chapter 9 we consider a network of data streams, and the problem is to identify the process that stems from a given target distribution, subject to a constraint on the number of observations. Due to this constraint, the main question to be answered is how to allocate the samples in such a way that the accuracy of the test result is optimised, and consequently we treat the problem as an off-line testing problem. In contrast, in Chapters 10 and 11 we focus on sequential testing problems related to the timely detection of changes in the parameters of the assumed stochastic model.

In each of these cases we focus on testing procedures which feature a log-likelihood ratio (LLR) type test statistic. LLR test statistics are widely applied in all areas of statistical testing because they often turn out to be optimal in the sense that they maximise the power of the test subject to a constraint on the type I error. We discuss this and further preliminaries on LLR testing in Section 8.1.1. In Section 8.1.2 we proceed to discuss some relevant background on change point detection. The further organisation of this part of the thesis is detailed in Section 8.2.

8.1 Background

We briefly introduce LLR testing in Section 8.1.1, and review some basics on change point detection with a focus on the cumulative sum (CUSUM) method in Section 8.1.2.

8.1.1 Sequential LLR testing

In statistics, an important technique to decide whether a certain assumption should be accepted as true, is the method of *hypothesis testing*, often with a LLR test statistic (see, for instance, [97]). We briefly recall the basics of simple (off-line) LLR testing, and give a quick introduction to sequential analysis based on Chapter 2 in [141].

Let X denote a (discrete or continuous) random variable (possibly multivariate) with density function f_X, and suppose we wish to test $H_0 : f_X = p$ against $H_1 : f_X = q$, where p and q are two different probability density functions defined on the same domain. A *false alarm* (also referred to as *false positive* or *type I error*) occurs whenever we falsely reject H_0. The *power* of the test is defined as the probability of a true positive (rejecting H_0 when it is indeed false). A simple LLR test features the following decision rule:

Reject H_0 *if* $\mathscr{L}(X) \geq b$,

Accept H_0 *if* $\mathscr{L}(X) < b$,

where $\mathscr{L}(X) = \log\big(p(X)/q(X)\big)$ denotes the LLR. Then it follows from the Neyman-Pearson lemma that this LLR test is optimal in the sense that no other test of H_0 vs H_1 based on X and with false alarm probability no larger than $\delta := \mathbb{P}_0\big(\mathscr{L}(X) \geq b\big)$ can achieve a power larger than $\mathbb{P}_1\big(\mathscr{L}(X) \geq b\big)$ (where $\mathbb{P}_0$ and $\mathbb{P}_1$ denote the probability measure under H_0 and H_1).

In sequential testing we do not have to decide based on a fixed data set. Instead, we can also opt to collect more data, and update the test statistic sequentially whenever a new observation is obtained. A decision is made as soon as the test statistic exits a certain interval, that is, at a random time T which is a stopping time with respect to the natural filtration generated by the observations. As before, a natural choice for the test statistic is the LLR, which is now sequentially updated to include new observations. This leads to the following *sequential probability ratio test* due to Wald [158].

Let (X_t), $t \in \mathbb{N}$, denote a sequence of random vectors with joint density functions p and q under H_0 and H_1, respectively, and denote their joint LLR by

$$\mathscr{L}_{1:n} := \log\left(\frac{q(X_1, \ldots, X_n)}{p(X_1, \ldots, X_n)}\right) = \sum_{t=1}^{n} \ell(X_t),$$

where

$$\ell(x_t) := \frac{q(x_t \,|\, x_1^{t-1})}{p(x_t \,|\, x_1^{t-1})} \tag{8.1}$$

denotes the LLR increments, and $q(x_t \,|\, x_1^{t-1})$ denotes the conditional density evaluated at x_t given $x_1^{t-1} := \{x_1, \ldots, x_{t-1}\}$. Then the sequential LLR test features the decision rule

Reject H_0 *if* $\mathscr{L}_{1:T} \geq b$,

Accept H_0 *if* $\mathscr{L}_{1:T} \leq a$,

where $0 < a < b < \infty$ (usually $a < 1 < b$), and

$$T := \min\{n \in \mathbb{N} : \mathscr{L}_{1:n} \notin (a, b)\}$$

is a stopping time, which we assume to be finite almost surely under both $\mathbb{P}_0$ and $\mathbb{P}_1$. Similarly to its off-line counterpart, the sequential LLR test has strong optimality properties:

If the observations (X_t) are i.i.d., it minimises $\mathbb{E}_i T$ for $i \in \{0,1\}$ among all sequential tests having no larger error probabilities. A proof of this fact is given in [45, p. 365]; for an informal, more intuitive, argumentation see [141, Chapter II.4].

In the next section we focus on a particular class of sequential testing problems referred to as change point detection. We shall see that these can be solved with the aid of LLR-type test statistics.

8.1.2 Change point detection

The problem of detecting changes in the probability distribution of certain data streams is an important problem in many application domains, including quality control, target detection, signal processing, and computer network surveillance; see [151, Section 1.3]. A comprehensive treatment of the available methodology related to change point detection is provided in [15] and the more recent book [151]; for a more concise overview we refer to the surveys [93, 142, 149, 156].

Consider again a sequence of observations (X_t) in discrete time taking values in $\mathbb{R}^d$, $d \geq 1$.

Definition 8.1. Let $k \in \{1, \ldots, n\}$ be a point in time such that $(X_s)_{s=1}^{k-1}$ and $(X_t)_{t=k}^{n}$ are independent, and distributed with conditional probability density $p\left(x_s | x_1^{s-1}\right)$ and $q\left(x_t | x_k^{t-1}\right)$, respectively. Then k is called a *change point*.

We are interested in statistical procedures capable of recognising such a change in an on-line manner. Note that in this case we are not concerned with a simple binary hypothesis problem as those of Section 8.1.1. Rather, at any time point n the hypotheses of interest are

H_0 : *No change has occurred.*

H_1 : *There is a change point k with $k \in \{1, \ldots, n\}$.*

Thus, the alternative hypothesis is the union of hypotheses

$H_1(k)$: *A change occurred exactly at time k, for a specific $k \in \{1, \ldots, n\}$.*

There are two standard formulations of the problem: the Bayesian formulation as developed by Shiryaev [137, 138, 139], where the change point is assumed to be random with a known prior distribution; and the minimax formulation due to Lorden [102], where the change point is regarded as a deterministic parameter. In both cases the objective is to minimise a certain measure of the detection delay subject to a constraint on the number of false alarms. The focus in the upcoming chapters is on the minimax formulation.

Let T be a stopping time with respect to the natural filtration $\mathcal{F}$ associated with the observations, that is, $\mathcal{F}_t := \sigma(X_1, \ldots, X_t)$. We write $\mathbb{P}_i$ and $\mathbb{E}_i$, $i \in \{0,1\}$ for the probability measure and expectation under H_i. Furthermore, we define $\mathbb{P}_1^k$ and $\mathbb{E}_1^k$ to be the probability measure and expectation under $H_1(k)$. The following delay criteria have been considered in the literature: the worst-case expected delay due to [102]

$$\sup_{k \geq 1} \operatorname*{ess\,sup}_{\mathcal{F}_{k-1}} \mathbb{E}_1^k \left[(T - k + 1)^+ \,\middle|\, \mathcal{F}_{k-1} \right], \tag{8.2}$$

and the less pessimistic delay criterion

$$\sup_{k \geq 1} \mathbb{E}_1^k [T - k \,|\, T \geq k] \tag{8.3}$$

due to [124].

The classic criterion for the number of false alarms is the average run length (ARL) $\mathbb{E}_0 T$ — the expected time until the first false alarm is raised. We propose two alternative false alarm criteria in Chapter 10.

A common choice for the decision rule defining T is the CUSUM rule.

The CUSUM method. The CUSUM method first proposed by Page [118] is essentially a sequential LLR test where the unknown change point k is replaced by its maximum likelihood estimator (interestingly Page apparently did not recognise this when he suggested it [142]). Let us consider the LLR test for testing H_0 against $H_1(k)$. Evidently, the test statistic to be considered is

$$\mathscr{L}_{k:n} := \sum_{t=k}^{n} \ell(X_t).$$

To handle the fact that H_1 equals the union of the $H_1(k)$, we have to verify whether there is a $k \in \{1, \ldots, n\}$ such that $\mathscr{L}_{k:n}$ exceeds a certain critical value. As a result, the statistic for the *composite* test (that is, H_0 versus H_1) is

$$S_n := \max_{k \in \{1,\ldots,n\}} \mathscr{L}_{k:n}. \tag{8.4}$$

Then, for a given threshold $b > 0$, the CUSUM method raises an alarm at time τ, with

$$\tau := \inf \{n \geq 1 : \, S_n > b\}. \tag{8.5}$$

The name of the test is explained by noting that the test statistic S_n can be rewritten in terms of the cumulative sums $C_k := \sum_{t=1}^{k} \ell(X_t)$ as follows,

$$S_n = C_n - \min_{k \in \{1,\ldots,n-1\}} C_k.$$

This is convenient with respect to computational efficiency.

Lorden [102] proved that if observations are i.i.d. before and after the change point, then the CUSUM method is asymptotically optimal in the sense that with b chosen such that $\mathbb{E}_0 \tau = \kappa$ it minimises the worst-case expected delay (8.2) among all procedures satisfying $\mathbb{E}_0 T \geq \kappa$ where $\kappa \to \infty$. Later Moustakides [108] strengthened this result to the non-asymptotic case. Asymptotic optimality results for dependent random variables have also been established under certain conditions starting with an influential paper by Lai [92]; for details see, for example, [151, Chapter 8]. In Chapter 10 we verify that Lai's proofs imply asymptotic optimality for dependent observations with respect to two alternative false alarm criteria.

Threshold selection. Existing literature on minimax change point detection typically considers the ARL $\mathbb{E}_0 T$ as a false alarm criterion. Then a threshold is chosen such that the ARL exceeds a desired (large) constant. To this end, because in general closed form expressions for the ARL are not available, the latter is in practice often evaluated based on simulation. As Page already has noticed [118] for the case of independent observations, it is also possible to express the ARL in terms of iterated Fredholm integral equations (see [151, Section 8.2.6.1]). Unfortunately, these equations can typically not be solved

analytically, and while solving them numerically can lead to very accurate results, it is computationally expensive.

A method that is easier to apply is to select the threshold based on closed-form approximations to the ARL. Previous results on how to select the threshold usually require the data points to be *independent*. Firstly, in this case Wald's approximations can be applied; see [141, Chapter II] and [15, Section 5.2.2.2]. These approximations are in many cases not very accurate since they ignore the excess with which the LLR crosses the threshold; a corrected approximation method has been developed in [140, 141]. Furthermore, under the independence assumption the conceivable fact is proven that (under an appropriate scaling) a functional central limit theorem (CLT) holds, meaning that the cumulative random walk process converges to a Brownian motion. This result enables us to approximate the test's false alarm probability (for details see [25, 141]). Apart from the CLT regime, asymptotic expressions for the false alarm probability have been derived under a LD scaling as well, see e.g. [24, Chapter VI.E] and [39] (for useful background on LD theory recall Chapter 1.1.1).

Considerably less work has focused on the case of *dependent* observations. If the stochastic process to be tested is linear (in the sense that Wold's decomposition applies [22, Section 5.7.1]), then the above methods can still be applied to detect changes in the i.i.d. sequence of *innovations* that is obtained by passing the original observations through a whitening filter. This approach has been advocated, for example, in [15, 128].

However, not every type of change in the sequence of observations results in an equivalent change in the sequence of innovations [15, Chapter 8]. In such cases, if p and q stem from a parametric family of locally asymptotically normal distributions, one can apply the following so called *local approach* (for details see, e.g., [15]). To this end, it needs to be assumed that H_0 and H_1 become increasingly similar as the number of observations grows (with a speed of convergence of order $n^{-1/2}$, where n denotes the number of observations. Then a CLT approximation to the Taylor expansion of the LLR with respect to the (decreasing) magnitude of the change is exploited in order to translate the original problem to an asymptotically equivalent problem of testing an i.i.d. sequence of Gaussians against a change in mean.

For further details on asymptotic methods for threshold selection see also Chapter 10.

8.2 Organisation and contributions

In Chapter 9 we consider a network of data streams from which an anomalous process with known target distribution is to be identified. Motivated by the realisation that in practice obtaining observations may be expensive, we assume that there is a constraint on the total number of observations based on which the decision has to be made. Due to this constraint, the main question to be answered is how to allocate the samples, and we treat the problem as an off-line testing problem. Using LD asymptotics to approximate the probability of false identification, we derive a sufficient condition on the sampling budget such that the error probability is kept below some desired level. Furthermore, we show how to obtain a sampling allocation that can improve upon equal sampling allocation and achieves the desired accuracy.

In Chapters 10 and 11 we focus on sequential testing problems related to the timely detection of changes in the parameters of the assumed stochastic model based on the CUSUM method. As we mentioned in Section 8.1.2, in the literature on minimax change point detection the number of false alarms is usually measured by the ARL – the expected

duration until the first false alarm. However, this is does not in general allow one to control the number of false alarms at every particular time instance. This motivates the investigation of two stronger false alarm criteria in Chapter 10. We propose asymptotic procedures for evaluating these criteria, considering extreme value (EV), CLT, as well as LD regimes. In numerical comparisons the LD approximations seem to yield the best performance.

In Chapter 11 we provide a number of examples in which we design change point detection procedures for correlated data points using the LD approach. We consider the detection of proportional changes in the mean and variance of *autoregressive moving average* (ARMA) processes, as well as detecting changes in the mean value of state space models.

CHAPTER 9

Anomaly identification with limited sampling budget

In this chapter we consider an off-line testing problem, for which a simple LLR testing procedure is developed. The content of this chapter is based on Kuhn, Mandjes and Taimre [87].

Consider a network of d processes which are to be monitored with the objective to identify the anomalous process (we focus on the case where exactly one such process is present), that is, the process that stems from a given target distribution G rather than the reference distribution F. We may, for example, be interested in identifying an idle channel in a network of communication channels [31], the presence of a certain animal species in one of a number of monitored habitats, or the drug that is efficient in curing a certain disease.

It is known that if only a single process is to be monitored and the problem is to decide as quickly as possible whether or not it stems from a target distribution subject to a constraint on the error probability, Wald's sequential probability ratio test, which we briefly introduced in Section 8.1.1, is optimal [141]. In such a sequential setting, target identification in networks of multiple processes has been considered in [91, 150] under the assumption that all processes are observed at every time point.

In practice, it is often the case that obtaining an observation is expensive. For example, in decentralised sensor networks there is usually a cost associated with the communication between sensors and the fusion center [14]. In many other application areas observations can only be obtained from experiments that involve human intervention, in which case an observation is particularly costly. In medical applications it occurs that only a limited number of samples are available, e.g. for diagnostic testing. Such considerations motivate the investigation of how an anomalous process or target can be identified efficiently based on only a limited number of observations.

In a sequential setting, [14] investigated the question of efficient sampling allocation in the context of change point detection, where the anomaly is not present initially but may appear at some unknown time point. In [172] a sequential procedure for sampling allocation was proposed subject to the constraint that only $k < d$ of all data streams can be observed at every time point. In contrast, in this chapter we do not consider a sequential testing problem but instead assume that the decision has to be made as accurately as possible with a given sampling budget.

It seems plausible that a good sampling allocation should explicitly take into account the specific characteristics of each process. For example, processes with a high variance should be sampled more often; and more samples should be taken from processes that are more similar to the anomalous process. Motivated by such considerations we are interested in determining an allocation $\boldsymbol{\rho} := (\rho_1, \ldots, \rho_d) \in (0,1)^d$ with $\sum_{i=1}^d \rho_i = 1$ such that the decision can be made with the desired accuracy based on $\rho_i n$ samples from process $i \in \{1, \ldots, d\}$. (We neglect the minor technical issues arising when $\rho_i n$ is not integer-valued.)

We assume no prior knowledge as to which of the data streams in the network is anomalous. In this framework, initially all processes have to be observed for a certain amount of time so as to collect information about their nature. We therefore first derive a bound on the number of samples needed to obtain the desired minimal security about which process is the anomalous one. To do this, we use a LD approximation to the probability of wrongly identifying a process as anomalous. Then, in a second step, we show how the remaining sampling budget can be allocated so as to optimise the accuracy of the identification. This boils down to solving a convex optimisation problem that we formulate in terms of the LD rate functions.

As is common in the literature on anomaly identification [91, 150] we assume that we know which behaviour should be characterised as anomalous and which behaviour is normal, in that we know the distribution of the data streams in both cases. This is a reasonable assumption for example for the problem of searching the idle channel in a communication network [172].

In line with [150] we assume that we know that exactly one anomalous process is present in the network. If $k > 1$ processes are anomalous (k known), then the algorithm we propose is still valid, but deriving the LD approximation as we propose in Section 9.2 requires more care (see the analysis in Chapter 4). We remark that if the number of anomalous processes is unknown, the algorithm cannot be applied in its current form; we briefly comment on this issue in Section 9.4.

The methodology we apply follows closely the work of [51, 52] on ordinal optimisation. The authors consider the problem of finding the process with the largest sample mean from a given set of i.i.d. observations, subject to a constraint on the available sampling budget.

This chapter is organised as follows. In Section 9.1 we explain the problem and propose an algorithm for sampling allocation and target identification that provably does better than a pre-specified accuracy. In Sections 9.2 and 9.3 we explain how the steps of the algorithms can be carried out. We conclude in Section 9.4.

9.1 Problem formulation and sampling algorithm

Denote the observation of process i at time n by $X_i(n)$. We assume that the sequences $\big(X_i(n)\big)_n$ and $\big(X_j(n)\big)_n$ are independent for $i \neq j$. Unless otherwise stated, the observations need not be independent over time. Without loss of generality, we assume that the anomalous process is the process labelled by 1. Suppose that $X_2(n), \ldots, X_d(n)$ have a distribution with density f, while $X_1(n)$ has a distribution with density g. Using an off-line LLR test, we declare process i to be anomalous based on a total of N observations

sampled according to $\boldsymbol{\rho}$ if for all $j \neq i$, $j \in \{1, \dots, d\}$ we have

$$\mathscr{L}_i(N\rho_i) := \frac{1}{N\rho_i} \sum_{n=1}^{N\rho_i} \ell_i(n) \geq \mathscr{L}_j(N\rho_j)\,,$$

where the LLR increments are now given by

$$\ell_i(n) := \log \frac{g\big(x_i(n)\big|x_i(1), \dots, x_i(n-1)\big)}{f\big(x_i(n)\big|x_i(1), \dots, x_i(n-1)\big)}.$$

We assume that $\mathbb{E}\big[\ell_1(1)\big] > \mathbb{E}\big[\ell_j(1)\big]$, $j \in \{2, \dots, d\}$, so that the anomalous process is indeed distinguishable. This assumption was also imposed in [51].

Suppose the total sampling budget is N. Then, since at the beginning of the testing it is not known which process is from the target distribution g, the naive approach is to allocate an equal number of samples to each process. Later we will see, however, that we can easily improve upon this simple approach.

The event of a false selection (FS), i.e. of declaring the wrong process to be anomalous, based on n samples is given by

$$\mathrm{FS}(n, \boldsymbol{\rho}) := \left\{ \mathscr{L}_1(n\rho_1) < \max_{i \in \{2,\dots,d\}} \mathscr{L}_i(n\rho_i) \right\}.$$

Note that this corresponds to the type I error of the testing procedure. We propose the following "algorithm" for determining the sampling allocation $\boldsymbol{\rho}$ such that $\mathrm{FS}(N, \boldsymbol{\rho})$ is small and guaranteed to be below a chosen level α.

(A) Fix α. Observe all processes until time $\min\{\overline{n}/d, N/d\}$, where $\overline{n}$ is such that we have $\mathbb{P}(\mathrm{FS}(\overline{n}, \boldsymbol{\rho}_d)) \leq \alpha$ for $\boldsymbol{\rho}_d := (1/d, \dots, 1/d)$.

(B) If $\overline{n} \leq N$, distribute the remaining sampling budget according to the allocation $\boldsymbol{\rho}^*$ that solves

$$\begin{aligned} &\min_{\boldsymbol{\rho}} \mathbb{P}\big(\mathrm{FS}(N, \boldsymbol{\rho})\big) \\ \text{s. t.} \quad &\sum_{i=1}^{d} \rho_i = 1\,, \quad \rho_i N \geq \overline{n}/d \;\forall\, i\,. \end{aligned} \tag{9.1}$$

Step (A) ensures that we explore all process sufficiently well so that the desired false selection probability can be achieved. In Step (B) we exploit the information collected in Step (A) to find the allocation that minimises the false selection probability subject to the constraint that the total budget spent on process i exceeds the budget we already allocated in Step (A).

Provided that the LLR increments satisfy the conditions of the law of large numbers, it is clear that $\overline{n}$ from Step (A) of the algorithm indeed exists. If the false selection probability decreases monotonically in n for all $n \geq \overline{n}$, then the false selection probability achieved using the above algorithm based on N samples is guaranteed to be below α. This motivates us to investigate the monotonicity of the false selection probability in the remainder of this section.

If $\mathbb{P}\big(\mathrm{FS}(n, \boldsymbol{\rho})\big)$ is not everywhere decreasing in n, we can still ensure that $\mathbb{P}\big(\mathrm{FS}(N, \boldsymbol{\rho})\big)$ is no larger than α by deriving $\overline{n}$ in Step (A) from an upper bound on the false selection

probability that does decrease monotonically in n; we provide such a bound in Section 9.2.

In the remainder of this section we focus on the case of i.i.d. observations. Denote the mean and variance of $\ell_i(1)$ by μ_i and σ_i^2, respectively. We introduce the random process

$$Z_j(n) := \mathscr{L}_1(n\rho_1) - \mathscr{L}_j(n\rho_j)\,, \quad j \in \{2, \dots, d\}\,.$$

Then the false selection probability is everywhere monotonically decreasing in n if

$$\mathbb{P}\big(Z_j(n+1) \leq 0\big) \leq \mathbb{P}\big(Z_j(n) \leq 0\big) \tag{9.2}$$

holds for all $n \in \mathbb{N}$, $j \in \{2, \dots, d\}$. The mean of $Z_j(n)$ is $a_j := \mu_1 - \mu_j > 0$ and the variance is $n^{-1}v_j^2(\boldsymbol{\rho})$, where $v_j^2(\boldsymbol{\rho}) := \sigma_1^2/\rho_1 + \sigma_j^2/\rho_j$. We further define $v_j^2 := \sigma_1^2 + \sigma_j^2$.

We can check that (9.2) holds for all $n \in \mathbb{N}$ in the case of Gaussian random variables (Lemma 9.1). The distribution function of the standard normal distribution is denoted by Φ.

Lemma 9.1. *For each i assume that $X_i(n)$ are i.i.d. Gaussian random variables. Then the false selection probability decreases monotonically as a function of n.*

Proof. By assumption the processes $Z_j(n)$ are Gaussian as a convolution of independent Gaussian random variables. Then (9.2) is equivalent to

$$\Phi\left(-\frac{\sqrt{n}a_j}{v_j(\boldsymbol{\rho})}\right) \geq \Phi\left(-\frac{\sqrt{n+1}a_j}{v_j(\boldsymbol{\rho})}\right)\,,$$

which clearly holds for any $n \in \mathbb{N}$ since $a_j > 0$. □

In general, the false selection probability need not be decreasing. We can show, however, that $\mathbb{P}(Z_j(n) \leq 0)$ lies within an interval the center of which is decreasing in n, and with bounds that become increasingly tight as n grows.

To this end, note that $Z_j(n)$ is a sample mean of n independent random variables

$$Y_j(t) := \frac{\ell_1(t)}{\rho_1}\mathbb{1}\{t \leq n\rho_1\} - \frac{\ell_j(t)}{\rho_j}\mathbb{1}\{t \leq n\rho_j\}\,,$$

with mean $a_j(t) > 0$ and variance $v_j(t)^2$. Then the following lemma readily follows from the Berry-Esséen theorem [136].

Lemma 9.2. *Let $X_i(t)$ be i.i.d. random variables such that $a_j(t) > 0$, $v_j^2(t) < \infty$ for $t \in \mathbb{N}$, and*

$$\sup_{t\in\mathbb{N}} \mathbb{E}\left[|Y_j(t) - a_j(t)|^3\right] < B\,,$$

for some $B < \infty$. Then $\mathbb{P}\left(Z_j(n) \leq 0\right)$ is bounded by

$$\Phi\left(-\sqrt{n}\,\frac{a_j}{v_j}\right) \pm \frac{3\,B}{5v_j^3\sqrt{n}}\,. \tag{9.3}$$

The condition on the absolute third moment is very mild; for example, for i.i.d. Gaussian observations the absolute third moments are bounded by $2v_j^3\sqrt{2/\pi}$.

In Sections 9.2 and 9.3 we discuss how to carry out Steps (A) and (B) of the algorithm, respectively.

9.2 Sufficient sampling budget

In this section, for a given allocation $\boldsymbol{\rho}$, we derive a bound on the total number of observations that ensures that the false selection probability is kept below a desired level α. That is, we want to determine $n_{\boldsymbol{\rho}}$ such that

$$\mathbb{P}\big(\mathrm{FS}(n_{\boldsymbol{\rho}}, \boldsymbol{\rho})\big) \leq \alpha\,.$$

(Such an $n_{\boldsymbol{\rho}}$ exists because we assumed that $\mu_1 > \mu_j$ for $j \in \{2, \dots, d\}$.)

In special cases, it may possible to compute the false selection probability explicitly by inducing independence via conditioning on the value of $\mathscr{L}_1(n\rho_1)$. In general, however, it is difficult to evaluate $\mathbb{P}\big(\mathrm{FS}(n, \boldsymbol{\rho})\big)$, and therefore we now show how it can be approximated using LD results from Section 1.1.1 when the given sampling budget n is large. In this chapter we write

$$\Lambda_i(\theta) := \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{\theta\, \mathscr{L}_i(n)}\right]$$

for the limiting cumulant-generating function Λ_i of the LLR of process i (which we assume to exist), and accordingly denote its Legendre transform by $I_i(x)$.

Proposition 9.3. *The false selection probability can be approximated as*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\mathrm{FS}(n, \boldsymbol{\rho})\big) = -\min_{j\in\{2,\dots,d\}} G_j(\boldsymbol{\rho})\,, \tag{9.4}$$

where

$$G_j(\boldsymbol{\rho}) := \inf_x \left\{\rho_1 I_1(x) + \rho_j I_j(x)\right\}\,. \tag{9.5}$$

Proof. Using that $\lim_{n\to\infty} n^{-1}\log(d-1) = 0$, similar to [38, Lemma 1.2.15] it is readily obtained that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\mathrm{FS}(n, \boldsymbol{\rho})\big) = \\ \max_{j\in\{2,\dots,d\}} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathscr{L}_j(n\rho_j) > \mathscr{L}_1(n\rho_1)\right)\,.\end{aligned} \tag{9.6}$$

From the Gärtner-Ellis theorem (see Theorem 1.8), we have that for $x > \mathbb{E}\ell_i(1)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\mathscr{L}_i(n\rho_i) > x\big) = -\rho_i I_i(x)\,.$$

Let $\mathcal{I}_j(\boldsymbol{x}) := \rho_1 I_1(x_1) + \rho_j I_j(x_j)$. Because $\mathscr{L}_j$ and $\mathscr{L}_1$ are independent, it follows that for $B \subset \mathbb{R}^2$ such that $\inf_{x\in B^\circ} \mathcal{I}_i(x) = \inf_{x\in\overline{B}} \mathcal{I}_i(x) =: \mathcal{I}_i(B)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\big(\mathscr{L}_1(n\rho_1), \mathscr{L}_j(n\rho_j)\big)' \in B\right) = -I_j(B)\,.$$

Using the properties of the rate functions I_i, we can then argue as in [51, Section 2.2] that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathscr{L}_j(n\rho_j) > \mathscr{L}_1(n\rho_1)\right) = -G_j(\boldsymbol{\rho})\,.$$

This, together with (9.6) proves (9.4). □

Let $D_{I_i} := \{x \in \mathbb{R} : I_i(x) < \infty\}$ denote the essential domain of $I_i(\cdot)$. Recall from Section 1.1.1 that for each $x \in D_{I_i}^o$ the Fenchel-Legendre transform is strictly convex and differentiable, and satisfies $I_i(x) \geq 0$ with $I_i(\mu_i) = 0$.

The following lemma can be proven analogously to [51, Lemma 3]. Loosely speaking, assumption (9.7) of the lemma states that the LLRs can take any value in the interval $[\mu_d, \mu_1]$; in particular, it implies that $\mathbb{P}\big(Z_j(n\rho_j) \leq 0\big) > 0$. It holds for example if the distribution of $\ell_i(t)$ stems from the normal or the gamma family.

Lemma 9.4 (Glynn and Juneja [51])**.** *Assume that all observations are i.i.d., and that*

$$[\mu_d, \mu_1] \subset \bigcap_{i=1}^{d} D_{I_i}^o. \tag{9.7}$$

Then for a given allocation $\boldsymbol{\rho}$ and sampling budget n we have

$$\mathbb{P}\big(\mathrm{FS}(n, \boldsymbol{\rho})\big) \leq (d-1) \exp\left(-n \min_{j \in \{2,\ldots,d\}} G_j(\boldsymbol{\rho})\right). \tag{9.8}$$

Thus, a lower bound for the minimal value of n can be achieved by putting the upper bound given in (9.8) equal to α, and solve for n. Note that this will yield a function of n that depends on the allocation $\boldsymbol{\rho}$.

In Step (A) of the algorithm proposed in the previous section we assume $\rho_i = 1/d$ for all i. We provide an example below.

Example. Suppose the observations of each stream are i.i.d. and normally distributed with $X_i(t) \sim \mathcal{N}(m_i, s^2)$ for $i \in \{1, \ldots, d\}$, where $m_1 = \widetilde{m}$, whereas $m_j = m$ for $j \in \{2, \ldots, d\}$. It is easy to see that the LLR increments are

$$\ell_i(t) = \frac{\widetilde{m} - m}{s^2} \left(X_i(t) - \frac{m + \widetilde{m}}{2}\right),$$

and hence $\ell_i(t) \sim \mathcal{N}\left(\mu_i, \sigma^2\right)$, where

$$\mu_i := \frac{\widetilde{m} - m}{s^2} \left(m_i - \frac{m + \widetilde{m}}{2}\right), \quad \sigma := \frac{\widetilde{m} - m}{s}.$$

Then the LLRs at time n have distribution $\mathscr{L}_i(\rho_i n) \sim \mathcal{N}\big(\mu_i, \sigma^2/(\rho_i n)\big)$. The cumulant generating function is

$$\Lambda_i(\theta) = \theta\, \mu_i + \frac{1}{2}\sigma^2\, \theta^2,$$

so that

$$I_i(x) = \frac{(x - \mu_i)^2}{2\sigma^2}.$$

Therefore, we obtain that

$$G_i(\boldsymbol{\rho}) = \frac{(\mu_i - \mu_1)^2}{2\sigma^2(1/\rho_1 + 1/\rho_j)}.$$

Consider the allocation $\boldsymbol{\rho} = (1/d, \ldots, 1/d)$. From (9.8) we know that if n satisfies

$$\alpha = (d-1)\exp\left(-nG_j(\boldsymbol{\rho})\right)$$

for arbitrary $j \in \{2, \ldots, d\}$, then $\mathbb{P}\big(\mathrm{FS}(n, \boldsymbol{\rho})\big) \leq \alpha$. Solving for n we obtain that under allocation $\boldsymbol{\rho}$ we can make a decision with the desired accuracy if

$$n \geq \overline{n} := \frac{2\sigma^2\left(1/\rho_1 + 1/\rho_j\right)}{(\mu_j - \mu_1)^2} \log\left(\frac{d-1}{\alpha}\right).$$

Figure 9.1 shows the values of $\overline{n}$ obtained with equal allocation. Naturally, the value of $\overline{n}$ decreases as the difference between the anomalous process and the other processes increases. The achieved false selection probabilities are very conservative. The jumps are due to the rounding of $\overline{n}/d$.

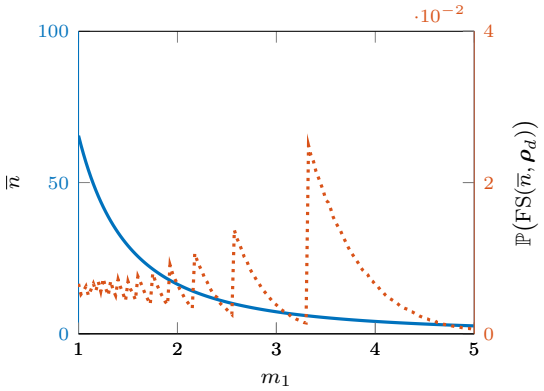


Figure 9.1: We plot the obtained values of $\overline{n}$ as a function of $m_1 = \mathbb{E}X_1(t)$ for a network with 4 processes under equal allocation (solid line). Other parameters are set as $m_j = 0$ for $j = 2, 3, 4$, $s = 1$, $\alpha = 0.01$. The dotted line shows the simulated values of $\mathbb{P}\big(\mathrm{FS}(\overline{n}, \boldsymbol{\rho}_d)\big)$, with values on the right vertical axis.

9.3 Asymptotically optimal allocation

An asymptotically optimal allocation $\boldsymbol{\rho}^*$ that (approximately) solves (9.1) if N is large can be found by minimising the rate function given in (9.4). Note that the rate function is concave in $\boldsymbol{\rho}$ as a minimum over affine functions. We can formulate the optimisation problem as follows:

$$\begin{aligned} \max \quad z \quad \text{s. t.} \quad & G_j(\boldsymbol{\rho}) - z \geq 0 \\ & \sum_{i=1}^{d} \rho_i - 1 = 0 \qquad (9.9) \\ & \rho_i N - \overline{n}/d \geq 0\,. \qquad (9.10) \end{aligned}$$

From the Karush-Kuhn-Tucker conditions [12] we know that there exists multipliers μ_j, η_i and λ such that

$$\sum_{j=2}^{d} \mu_j \frac{\partial G_j(\boldsymbol{\rho}^*)}{\partial \rho_1} + \eta_1 N = \lambda \tag{9.11}$$

$$\mu_j \frac{\partial G_j(\boldsymbol{\rho}^*)}{\partial \rho_j} + \eta_j N = \lambda\,, \quad j \in \{2, \dots, d\} \tag{9.12}$$

$$1 - \sum_{j=2}^{d} \mu_j = 0 \tag{9.13}$$

$$\mu_j \big(z - G_j(\boldsymbol{\rho}^*)\big) = 0\,, \quad j \in \{2, \dots, d\} \tag{9.14}$$

$$\eta_i \left(\overline{n}/d - \rho_i^* N\right) = 0\,, \quad i \in \{1, \dots, d\}\,. \tag{9.15}$$

This yields conditions on the asymptotically optimal allocation, which we formalise in the following proposition.

Proposition 9.5. *Let $N > \overline{n}$. If an allocation $\boldsymbol{\rho}^*$ with $\sum_{i=1}^{d} \rho_i^* = 1$ and $\rho_i^* \geq \overline{n}/(dN)\,\forall\, i$ minimises $\mathbb{P}\big(\mathrm{FS}(N, \boldsymbol{\rho})\big)$, then*

$$G_j(\boldsymbol{\rho}^*) = G_k(\boldsymbol{\rho}^*) \tag{9.16}$$

for $k, j \in \{2, \dots, d\}$ such that $\rho_k^*, \rho_j^* \neq \overline{n}/(dN)$. If $\rho_i^* \neq \overline{n}/(dN)\,\forall\, i$, we also have

$$\sum_{j=2}^{d} \frac{\partial G_j(\boldsymbol{\rho}^*)/\partial \rho_1}{\partial G_j(\boldsymbol{\rho}^*)/\partial \rho_j} = 1\,. \tag{9.17}$$

Proof. From (9.13) we have that there exists $j \in \{2, \dots, d\}$ such that $\mu_j > 0$. Because $\partial G_j(\boldsymbol{\rho}^*)/\partial \rho_j > 0$, together with (9.12) this implies that that $\lambda > 0$.This means that for $j \geq 2$, if $\eta_j = 0$, then $\mu_j > 0$, in which case by (9.14) we have (9.16). By (9.15), $\eta_j = 0$ holds if $\rho_j^* \neq \overline{n}/(dN)$.

The latter also implies that if $\rho_j^* \neq \overline{n}/(dN)$ for all i, then $\eta_i = 0$ for all i, in which case (9.12) implies that $\mu_j = \lambda/\big[\partial G_j(\boldsymbol{\rho}^*)/\partial \rho_j\big]$. Substituting this into (9.11) then gives (9.17). □

Note that since we assumed that the processes $2, \dots, d$ are stochastically identical, (9.16) implies that $\rho_2 = \cdots = \rho_d =: \widetilde{\rho}$ (as one would expect). If (9.16) and (9.17) yield a feasible solution (i.e. an allocation that satisfies (9.9) and (9.10)), then this solution is optimal because the optimisation problem is concave and thus the Karush-Kuhn-Tucker conditions are sufficient. Hence, in this case we assign $\rho_1^* N$ samples to the process that had the largest LLR after Step (A) of the algorithm, and $\rho_j^* N$ samples to all others. Otherwise, the optimal solution is $\rho_i^* = \overline{n}/(dN)$ for all i except the process that yielded the largest LLR with the first $\overline{n}/(dN)$ observations, which then has allocation $1 - (d-1)\overline{n}/(dN)$.

Example. We assume that the observations are i.i.d. Gaussian so that the LLR increments have distribution $\mathcal{N}\big(\mu_i, \sigma_i^2\big)$, where processes $2, \dots, d$ are assumed to be stochastically identical. We compute the allocation $\boldsymbol{\rho}^*$ obtained based on LD approximations as suggested above. For illustration purposes, we assume that $\overline{n} = 0$. Note that $\rho_i^* > 0$

because otherwise $\min_i G_i(\boldsymbol{\rho}^*) = 0$ while we know that $G_i(\boldsymbol{\rho}_d) > 0$ for every i. As in Example 1 in [51] we obtain from (9.16) and (9.17) that $\boldsymbol{\rho}^*$ satisfies the conditions

$$\rho_1^* = \sigma_1 \sqrt{\sum_{j=2}^{d} \frac{\rho_j^{*2}}{\sigma_j^2}}\,, \quad \sum_{i=1}^{d} \rho_i = 1\,,$$

and

$$(\mu_j - \mu_1)^2 \left(\frac{\sigma_1^2}{\rho_1^*} + \frac{\sigma_k^2}{\rho_k^*}\right) = (\mu_k - \mu_1)^2 \left(\frac{\sigma_1^2}{\rho_1^*} + \frac{\sigma_j^2}{\rho_j^*}\right)$$

for $j, k \in \{2, \dots, d\}$. Thus, the asymptotically optimal allocation $\boldsymbol{\rho}^*$ can be obtained by solving the resulting system of equations.

Since we assume that processes $j \in \{2, \dots, d\}$ are identically distributed, we readily obtain

$$\rho_j = \frac{\sigma_j}{\sigma_1\sqrt{d-1} + \sigma_j(d-1)}\,, \quad j \in \{2, \dots, d\}$$
$$\rho_1 = \frac{\sigma_1}{\sigma_j}\sqrt{d-1}\,\rho_j\,.$$

Note that the budget allocated to the anomalous process increases as σ_1 increases; as one would expect, one needs to take more samples if the variance is larger.

We now compare the false selection probabilities obtained numerically under $\boldsymbol{\rho}^*$ with those achieved under equal allocation, see Fig. 9.2. Naturally, the false selection probabilities overall decrease as m_1 and thus the difference between the means of the processes increases. We further note that $\mathbb{P}\big(\mathrm{FS}(10, \boldsymbol{\rho}_d)\big)$ is generally greater than $\mathbb{P}\big(\mathrm{FS}(10, \boldsymbol{\rho}^*)\big)$; the performance gain is more than 10%.

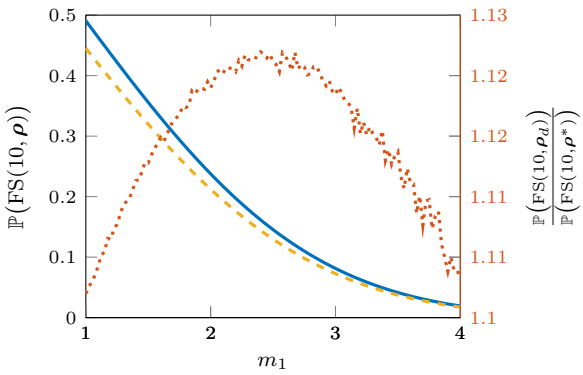


Figure 9.2: False selection probabilities achieved under equal allocation $\boldsymbol{\rho}_d$ (solid line) and under asymptotically optimal allocation $\boldsymbol{\rho}^*$ (dashed line) as a function of $m_1 := \mathbb{E}X_i(t)$ for a network with 4 processes under equal allocation. Their ratio is also depicted (dotted line, values on the right vertical axis). Other parameters are set as $m_j = 0$ for $j = 2, 3, 4$, $s = 2$, $N = 10$.

9.4 Conclusion

We proposed an algorithm for identifying a target process in a network of independent data streams subject to a constraint on the total number of observations. We showed that the probability of false selection can be decreased substantially compared to equal allocation of samples.

In future research asymptotically exact approximations such as those developed in Chapter 4 should be applied instead of the logarithmic asymptotics we used here in order to improve the accuracy of the results. Note that using Theorem 4.2.1 it is possible to consider the case where more than one anomalous process is present in the network (assuming that their exact number is known). Apart from this, the impact of the specific characteristics of each process on the optimal allocation should be investigated.

The case where the the number of anomalous processes is unknown requires more substantial modifications of the proposed algorithm and is thus left for future research. One could think of a procedure in which a process is declared anomalous when the LLR test statistic exceeds an appropriately specified threshold.

CHAPTER 10

False alarm control beyond average run length

In this chapter, which is based on [85], we turn to sequential testing procedures for change point detection. More specifically, we consider threshold selection methods for the CUSUM procedure. As mentioned in Section 8.1.2, the typical false alarm criterion is the ARL. The ARL criterion is, however, not always informative: in [107], examples are provided where the ARL is infinite even though the detection delay is finite (see also the discussion in [148]). More importantly, we stress that a large ARL does not necessarily ensure that the false alarm probability is low at every particular time instance (see [92] and [151, Section 6.3.5]). Generally, the ARL does not fully determine the distribution of the stopping time – although in the case of i.i.d. random variables the CUSUM stopping time turns out to be approximately exponential for large thresholds [125], this does not have to hold in general. This means that even if the ARL is controlled to exceed a certain level, the variance of the stopping time may still be large. In practice, this can be crucial. For example, in a reliability context, imagine one is monitoring the status of many network elements in parallel. Then if the false alarm rate is highly variable, it is not unlikely that false alarms are raised for a large number of elements at the same time, which could cause the capacity of the technicians to attend to all (including true) alarms to be exceeded. More generally, this argumentation applies to any scenario where multiple independent data streams are to be monitored simultaneously.

Consequently, more stringent false alarm criteria are desirable. Perhaps the best available candidate is a criterion coined *maximal local false alarm probability* (MLFA) in [151], which seems to have first appeared in [92]. It is defined as

$$\sup_{n\geq 1} \mathbb{P}_0(T \leq n + N - 1 \,|\, T > n - 1) \leq \alpha\,, \tag{10.1}$$

where T is a stopping time with respect to the filtration generated by the observations, and $\mathbb{P}_0$ indicates that the probability is evaluated under the null hypothesis that no change has occurred, α is the desired level of false alarms, and $N \geq 1$ is a design parameter. Asymptotic optimality (for small α/large b) of the CUSUM method under this criterion has been proven in [147]. For the case of i.i.d. observations, the MLFA can be evaluated numerically using [151, (8.110)]. The MLFA is, however, difficult to evaluate in closed form; in their recent book [151] Tartakovsky *et al.* note that even an upper bound is lacking

for the general non-i.i.d. case. Similar arguments apply to the approach developed in [107] (see [148]). The difficulty arises from the fact that the distribution of the stopping time is hard to evaluate in closed form, even if the distribution of the test statistic is known.

In view of the above, one wishes for further understanding of the distribution of the stopping time as well as simple but effective methods for selecting the threshold such that the probability of raising a false alarm is kept low in a stronger sense than allowed by the ARL criterion.

Another issue is that in practice testing the full history of observations may be computationally expensive, and data points will typically not be stationary over a long period of time. This motivates one to consider window-limited change point detection where data is tested in windows of fixed size; for every new observation arriving the oldest observation is dropped. Window-limited testing has for example been considered in [43, 92, 151, 167], and, for problems with unknown change size, in [164].

In this chapter we consider two simpler false alarm criteria, which are based on (10.1), and are consequently stronger than the ARL criterion. These simplifications are motivated by known results on the behaviour of (10.1). We believe that considering those simplified criteria is worthwhile from a practical point of view because approximations are available and thus the selection of the threshold is facilitated.

We now describe our contributions in more detail. As a simplification of (10.1), Lai [92] considered

$$\sup_{n\geq 1} \mathbb{P}_0(n \leq T < n+N) \leq \alpha \tag{10.2}$$

as a performance criterion. It has, however, since been shown that the probability in (10.2) is approximately exponential for small α, if observations are independent or weakly dependent [125]. Therefore, by the memoryless property of the exponential distribution, we have

$$\sup_{n\geq 1} \mathbb{P}_0(n \leq T < n+N) \approx \mathbb{P}_0(T \leq N)\,.$$

In the first part of the chapter we therefore focus on the simple criterion

$$\mathbb{P}_0(T \leq N) \leq \alpha\,. \tag{10.3}$$

We check that CUSUM (with or without windows) is still asymptotically optimal under this modified false alarm criterion, and investigate methods for selecting the threshold such that it is satisfied. To do this exactly, one would need closed-form expressions for the distribution of the stopping time. Since such expressions are not known in general, we first show how the distribution of the window-limited stopping time can be described in terms of recursive integral equations. For detection procedures without windows, integral equations have been derived based on renewal theory [118, 125, 141]; here we follow a different approach, using results on the maximum of autoregressive processes [165, 166]. We remark that the obtained recursions are not restricted to CUSUM but hold for a broad class of testing procedures including exponentially weighted moving average schemes (EWMA, see [129]).

However, while we thus in principle know the exact distribution of the window-limited stopping time, in general these expressions cannot be solved for the threshold other than numerically, and the latter is only feasible for the left tail of the distribution (see Sec-

tion 10.2.2.1 for discussion). We therefore provide non-asymptotic bounds for the distribution of the CUSUM stopping time when windows are used as well as for testing without windows. These bounds relate the distribution of the CUSUM stopping time to the crossing probability of a sum with random increments, and are therefore easier to evaluate. For example, we can then apply available approximation methods to find a threshold (function) that ensures the proposed false alarm criterion is satisfied. We compare the use of central limit theorem (CLT), large deviations (LD), as well as extreme value (EV) approximations. The latter two methods allow one to obtain a threshold *function* rather than a constant threshold; this increased flexibility can yield an improved delay performance (see Section 10.3 as well as the example in [43]).

In the second part of the chapter, we focus on the criterion

$$\sup_{n\geq 1} \mathbb{P}_0(T = n \,|\, T \geq n-1) \leq \alpha\,. \tag{10.4}$$

Note that this implies that the false alarm probability is limited at any given time n. This criterion is the most conservative case of (10.1), where we chose $N = 1$ because (10.1) increases monotonically as a function of N. We motivate its application and show how the aforementioned approximations can be used to select the threshold. We believe that (10.4) serves better than (10.2) as a simplification of (10.1) because it does not depend on the design parameter N, and it is stronger than (10.3) (for details see Section 10.3).

The remainder of this chapter is organised as follows. In Section 10.1 we define the change point detection problem and the CUSUM method. In Section 10.2 we discuss the first criterion, (10.3). In Section 10.2.1 we check that asymptotic optimality of CUSUM still holds, before we turn to analyse the distribution of the stopping time with applications for threshold selection in Sections 10.2.2 (when testing windows of fixed size) and 10.2.3 (when testing the full history of observations). The alternative of limiting the false alarm probability at any given time via (10.4) is discussed in Section 10.3. We illustrate the procedures with a simple example with independent Gaussian observations. For two more detailed examples with dependent multivariate observations we refer to Chapter 11.

10.1 Problem and procedures

We are concerned with testing a sequence of observations $(V_t) \in \mathbb{R}^{d_v}$ against a change in the underlying probability distribution. At every point in discrete time a new observation arrives and is to be included in the test sample. That is, at time $t \in \mathbb{N}$ we want to test H_0, the null hypothesis of no change before time t, against H_1, the alternative that there was a change point $k \in \{1, \ldots, t\}$. Note that the alternative is essentially a union of hypotheses $H_1(k)$ that a change occurred at a specific time k. In practice, in view of computational expense, it may often be necessary to test data in windows of fixed size n, rather than keeping the whole history of observations. In this case we restrict k to the set $\{t-n+1, \ldots, t\}$. Note that testing the full history of observations can equivalently be regarded as testing with expanding windows: at time t the size of the window to be tested is t. We take this viewpoint in the remainder of the chapter as it allows to treat both cases in a more unified manner.

Throughout the chapter we assume that the observations are stationary under H_0 and H_1, where observations after the change point are assumed to be independent from observations before the change point. We further assume that the observations are distributed with common density p under H_0, and with common density q under H_1 after the change

point. We focus on the case of independent observations in Sections 10.2.2.2 and 10.2.3.2 to facilitate the comparison of different approximation methods as a tool for threshold selection.

We focus on the CUSUM method. Recall from Section 8.1.2 that the CUSUM essentially a sequential LLR test, where the unknown change point is replaced by its maximum likelihood estimator. A sequence of LLRs can be regarded as a sequence of partial sums with random increments $\ell(V_t)$ given by (8.1). We can identify the LLRs corresponding to $H_1(1), \ldots, H_1(n)$ with a stochastic process

$$\boldsymbol{Y}_m := \big(\mathscr{L}_{1:n}(m), \ldots, \mathscr{L}_{n:n}(m)\big)', \tag{10.5}$$

where

$$\mathscr{L}_{k:n}(m) := \sum_{i=k+m-1}^{n+m-1} \ell(V_i)\,, \tag{10.6}$$

so that $m \geq 1$ corresponds to the number of the first observation within the window that is to be tested. If no windows are to be considered (i.e., in the case of expanding windows), then $m = 1$ is fixed, and the size of the window n increases with time. In this case, we write $\mathscr{L}_{k:n} := \mathscr{L}_{k:n}(1)$ to simplify the notation. To consider windows of fixed size n, let m increase with time instead.

We recall that the standard CUSUM testing procedure with expanding windows features the stopping time

$$\tau = \inf\left\{n \geq 1 : \max_{k\in\{1,\ldots,n\}} \mathscr{L}_{k:n} > b\right\}. \tag{10.7}$$

If window-limited testing (with windows of fixed size) is desired, we define the stopping time to be

$$\omega := \inf\left\{m \geq 1 : \max_{k\in\{1,\ldots,n\}} \mathscr{L}_{k:n}(m) > b\right\}, \tag{10.8}$$

The threshold is to be selected in such a way that the number of false alarms is kept at a desired level. The available methods for threshold selection are usually focussed on evaluation of the ARL, and often it is assumed that observations are independent (for details see Section 8.1.2). Moreover, these methods are only practical if one is aiming for a *constant* threshold. In this chapter we also provide methods for selecting the threshold as a *function* $b_n(k)$ of k (corresponding to $H_1(k)$) and n (the latter is only needed in Section 10.3). This greater flexibility can yield performance improvements as discussed in Section 10.3.

In the next sections we consider the performance criteria as briefly introduced in the introduction, where we have to distinguish between testing with windows of fixed vs. expanding size.

Criterion:	Window size:	
	Fixed	Expanding
(10.3) (Section 10.2)	$\mathbb{P}_0(\omega \leq N) \leq \alpha$ (Section 10.2.2)	$\mathbb{P}_0(\tau \leq N) \leq \alpha$ (Section 10.2.3)
(10.4) (Section 10.3)	$\mathbb{P}_0(\omega = n \mid \tau \geq n) \leq \alpha$ for all n	$\mathbb{P}_0(\tau = n \mid \tau \geq n) \leq \alpha$ for all n

10.2 False alarm before time N

In this section we focus on the criterion

$$\mathbb{P}_0(T \leq N) \leq \alpha\,, \tag{10.9}$$

where $T = \omega$ or $T = \tau$, depending on whether or not window sizes are fixed, and for a fixed quantity N. Note that (10.9) is less conservative than the MLFA defined in (10.1), but indeed stronger than the traditional ARL criterion. The latter requires that $\mathbb{E}_0 T \geq \kappa$, for some given (large) constant κ. Given (10.9), for $1 \leq h \leq N$ we have $\mathbb{P}_0\big(T > h\big) \geq 1 - \alpha$, which implies that

$$\mathbb{E}_0 T \geq \sum_{h=1}^{N} (1-\alpha) = N(1-\alpha)\,. \tag{10.10}$$

Note that N plays the role of the length of the time interval considered in (10.2). In practice, if the maximum testing period is known to be bounded, then the length of the testing period seems a natural choice for N. Otherwise, motivated by (10.10), one could specify κ and α as desired (that is, according to one's practical requirements on the ARL and the false alarm probability), and choose $N = \kappa/(1-\alpha)$.

10.2.1 Asymptotic optimality of CUSUM

For independent observations it is known that if the CUSUM procedure with stopping time τ satisfies $\mathbb{E}_0\tau = \kappa$, then it is optimal with respect to certain delay criteria among all procedures that satisfy $\mathbb{E}_0\tau \geq \kappa$. However, in practice it is usually not possible to achieve $\mathbb{E}_0\tau = \kappa$ because $\mathbb{E}_0\tau$ is not known in closed form. Therefore, asymptotic optimality results are of interest which establish optimality of CUSUM with τ satisfying $\mathbb{E}_0\tau \geq \kappa$ asymptotically for large κ. (For details see, for example, [151], Chapter 8.) Similarly, we can prove asymptotic optimality of $T \in \{\tau, \omega\}$ under (10.9). This amounts to carefully checking that the proofs given in [92] for the case of (10.2) still go through; we detail the steps below.

Let T be a stopping time with respect to the natural filtration $\mathcal{F}$ associated with the observations, that is, $\mathcal{F}_t := \sigma(V_1, \ldots, V_t)$. As before we write $\mathbb{P}_i$ and $\mathbb{E}_i$, $i \in \{0, 1\}$ for the probability measure and expectation under H_i. Furthermore, we define $\mathbb{P}_1^k$ and $\mathbb{E}_1^k$ to be the probability measure and expectation under $H_1(k)$.

We now verify that CUSUM is asymptotically optimal among all procedures satisfying (10.9) with respect to the two delay criteria defined in Section 8.1.2. To this end, we check that the usual asymptotic lower bound on the detection delay still holds (see Prop. 10.1), and that this lower bound is attained for small α in combination with large N (Prop. 10.2 below).

Proposition 10.1. *Suppose that for some finite positive constant I we have*

$$\lim_{m\to\infty} \mathbb{P}_1^1\left(\max_{1\leq t\leq m} \mathscr{L}_{1:t} \geq I(1+\delta)m\right) = 0 \quad \forall \delta > 0\,. \tag{10.11}$$

Assume that $\alpha := \alpha_N \leq (\log N)/N$. *Then*

$$\begin{aligned}\inf\left\{\sup_{k\geq 1}\operatorname*{ess\,sup}_{\mathcal{F}_{k-1}}\mathbb{E}_1^k\big[(T-k+1)^+\,|\,\mathcal{F}_{k-1}\big]:\,\mathbb{P}_0(T\leq N)\leq\alpha\right\}\\ \geq\left\{\sup_{k\geq 1}\mathbb{E}_1^k[T-k\,|\,T\geq k]:\,\mathbb{P}_0(T\leq N)\leq\alpha\right\}\\ \geq\big(I^{-1}+o(1)\big)\log\big(N(1-\alpha)\big)\,.\end{aligned}$$

as $N\to\infty$.

Proof. The proof is a small modification of Lai's proof [92, Theorem 1]. Suppose $\mathbb{P}_0(T\leq N)\leq\alpha$. To simplify notation, define $\gamma_N := \log\big(N(1-\alpha)\big)$. We show that for any $\delta>0$,

$$\mathbb{P}_1^1\left(T-1\geq(1-\delta)I^{-1}\gamma_N\right)\to 1 \tag{10.12}$$

as $N\to\infty$. This then implies that

$$\sup_{k\geq 1}\mathbb{E}_1^k[T-k\,|\,T\geq k]\geq\mathbb{E}_1^1[T-1]\geq\big(I^{-1}+o(1)\big)\gamma_N\,.$$

Since $\{T\geq k\}\in\mathcal{F}_{k-1}$, we have

$$\operatorname*{ess\,sup}\mathbb{E}_1^k\big[(T-k+1)^+\,|\,\mathcal{F}_{k-1}\big]\geq\mathbb{E}_1^k[T-k\,|\,T\geq k]\,,$$

and we obtain the statement of the theorem.

To show (10.12), we consider the sets

$$\begin{aligned}C_\delta &:= \left\{T<(1-\delta)I^{-1}\gamma_N,\,\mathscr{L}_{1:T}\leq(1-\delta^2)\gamma_N\right\}\,,\\ \overline{C_\delta} &:= \left\{T<(1-\delta)I^{-1}\gamma_N,\,\mathscr{L}_{1:T}>(1-\delta^2)\gamma_N\right\}\,.\end{aligned}$$

(i) Show that $\mathbb{P}_1^1(C_\delta)\to 0$ for every $0<\delta<1$. First, we note that

$$\mathbb{P}_1^1(C_\delta)=\int_{C_\delta}\frac{\mathrm{d}\mathbb{P}_1^1}{\mathrm{d}\mathbb{P}_0}\mathrm{d}\mathbb{P}_0=\int_{C_\delta}e^{\mathscr{L}_{1:T}}\mathrm{d}\mathbb{P}_0\leq e^{(1-\delta^2)\gamma_N}\,\mathbb{P}_0(C_\delta)\,.$$

Therefore, for N large enough such that $\alpha\leq(\log N)/N\leq I$, we have:

$$\begin{aligned}\mathbb{P}_1^1(C_\delta)&\leq e^{(1-\delta^2)\gamma_N}\mathbb{P}_0(T<(1-\delta)I^{-1}\gamma_N)\\ &\leq\big(N(1-\alpha)\big)^{1-\delta^2}\mathbb{P}_0\left(T\leq N\right)\\ &\leq(1-\alpha)^{1-\delta^2}N^{-\delta^2}\log N\,,\end{aligned}$$

which tends to zero as $N\to\infty$.

(ii) To prove that $\mathbb{P}_1^1(\overline{C_\delta})\to 0$, we note that

$$\mathbb{P}_1^1(\overline{C_\delta})\leq\mathbb{P}_1^1\left(\max_{t\leq(1-\delta)I^{-1}\gamma_N}\mathscr{L}_{1:1+t}\geq I(1+\delta)\,(1-\delta)I^{-1}\gamma_N\right)\,.$$

The upper bound tends to zero by (10.11).

□

It now follows with the same arguments as used to prove Theorem 4.(ii) in [92] that the lower bound is attained by ω (see Prop. 10.2). Since $\omega \geq \tau$ almost surely, this implies that the bound is also attained by τ, and thus, both are asymptotically optimal.

Proposition 10.2 (Lai (1998) [92])**.** *Assume that the threshold $b = b_N$ and the window size $n = n_N$ are chosen such that $\mathbb{P}_0(\omega \leq N) \leq \alpha$, where $\alpha = \alpha_N \to 0$ as $N \to \infty$. Further assume that for some positive constant I and $m \in \mathbb{N}$ we have*

$$\liminf_{N\to\infty} n_N I / b_N > 1\,, \tag{10.13}$$

$$\lim_{R\to\infty} \sup_{k\in\{1,\dots,m\}} \operatorname{ess\,sup} \mathbb{P}_1^k \left(\frac{1}{R} \sum_{i=m}^{m+R} X_i < I \,\middle|\, \mathcal{F}_{m-1} \right) = 0\,, \tag{10.14}$$

Then we have:

$$\sup_{k\in\mathbb{N}} \operatorname{ess\,sup} \mathbb{E}_1^k \big[(\omega - k + 1)^+ \,|\, \mathcal{F}_{k-1} \big] \leq \big(I^{-1} + o(1) \big) b_N \tag{10.15}$$

where $o(1) \to 0$ as $N \to \infty$.

Proof. We can essentially follow the steps in [92]. Let $u \in \mathbb{N}$, $k \in \{1, \dots, n\}$, and define $d_N := \lfloor b_N / I \rfloor$. By (10.13), we have that for large N

$$\begin{aligned} &\operatorname{ess\,sup} \mathbb{P}_1^k \left(\omega - k + 1 > (u-1)\, d_N \,\middle|\, \mathcal{F}_{k-1} \right) \\ &\quad \leq \operatorname{ess\,sup} \mathbb{P}_1^k \left(\max_{m\in\{0,\dots,(u-1)\, d_N + k - 1\}} \max_{l\in\{1,\dots,d_N\}} \mathscr{L}_{l:d_N}(m) \leq b \,\middle|\, \mathcal{F}_{k-1} \right). \end{aligned}$$

The right-hand side is upper bounded by

$$\begin{aligned} &\operatorname{ess\,sup} \mathbb{P}_1^k \left(\sum_{i=(j-1)d_N+k}^{jd_N+k-1} X_i < b \quad \forall\, 1 \leq j \leq u \,\middle|\, \mathcal{F}_{k-1} \right) \\ &\quad = \operatorname{ess\,sup} \prod_{j=1}^{u} \mathbb{P}_1^k \left(\sum_{i=(j-1)d_N+k}^{jd_N+k-1} X_i < b \,\middle|\, \mathcal{F}_{(j-1)d_N+k-1} \right). \end{aligned}$$

By (10.14) we have that, for $m \in \mathbb{N}$,

$$\sup_{k\in\{1,\dots,m\}} \operatorname{ess\,sup} \mathbb{P}_1^k \left(\sum_{i=m}^{\lfloor b_N/I \rfloor + m - 1} X_i < b_N \,\middle|\, \mathcal{F}_{m-1} \right) \to 0\,,$$

as $N \to \infty$. Hence, for any $\delta > 0$ we can find N sufficiently large such that

$$\mathbb{P}_1^k \left(\sum_{i=(j-1)d_N+k}^{jd_N+k-1} X_i < b \,\middle|\, \mathcal{F}_{(j-1)d_N+k-1} \right) \leq \delta\,.$$

Thus, we conclude that, for large N,

$$\operatorname{ess\,sup} \mathbb{P}_1^k \left(\omega - k + 1 > (u-1)\, d_N\right) \leq \delta^u \,,$$

in which case we have

$$\sup_{k\geq 1} \operatorname{ess\,sup} \mathbb{E}_1^k \left[(\omega - k + 1)^+ / d_N \,\middle|\, \mathcal{F}_{k-1}\right] \leq \sum_{u=0}^{\infty} \delta^u = \frac{1}{1-\delta} \,.$$

Since we can do this for all $\delta > 0$, this implies that

$$\sup_{k\geq 1} \operatorname{ess\,sup} \mathbb{E}_1^k \left[(\omega - k + 1)^+ \,\middle|\, \mathcal{F}_{k-1}\right] \leq \big(I^{-1} + o(1)\big) b_N \,,$$

as $N \to \infty$. □

For example, if observations are i.i.d., the conditions (10.11) and (10.14) are satisfied with I the Kullback–Leibler information number, $I = \mathbb{E}_1^1 \ell(V_1)$, assuming the later is finite.

In summary, we have the following corollary.

Corollary 10.3. *If $b \sim \log\big(N(1-\alpha)\big)$, $\alpha \leq \log(N)/N$, and (10.11), (10.13), and (10.14) are satisfied with $I > 0$, then ω is asymptotically optimal as $N \to \infty$ in the sense that it minimises the detection delay among all stopping times T satisfying $\mathbb{P}_0(T \leq N) \leq \alpha$.*

In order to select a threshold that ensures (10.9), we need to be able to evaluate the distribution of the stopping time. We focus on ω in Section 10.2.2 and turn to τ in Section 10.2.3. In both sections we first provide results on the distribution of the stopping time, and then show how the threshold function can be chosen based on approximations to $\mathbb{P}_0(T \leq N)$.

10.2.2 Window-limited testing

First, we show an exact expression for the distribution of the stopping time ω in terms of iterated integrals. Since these are hard to evaluate in practice, we then propose an EV approximation that can be used to select the threshold in order to ensure (10.9).

10.2.2.1 Exact expression in terms of iterated integrals

We show that the test statistic of a large class of change point detection procedures (including the window-limited CUSUM procedure) can be expressed in form of a first order vector autoregressive process (VAR(1)). We can then obtain the distribution of the corresponding stopping time using results on the distribution of the maximum of autoregressive processes [166].

We are interested in finding an expression for

$$\mathbb{P}_0(\omega \leq m) = \mathbb{P}_0\big(\exists j \in \{1, \ldots, n\} : \boldsymbol{H}_{m,j} > b(j)\big) \,, \tag{10.16}$$

where $\boldsymbol{H}_m$ is the n-vector with $j-$th element

$$\begin{aligned} H_{m,j} &:= \max\left\{\mathscr{L}_{j:n}(1), \ldots, \mathscr{L}_{j:n}(m)\right\} \\ &= \max\left\{\boldsymbol{Y}_{1,j}, \ldots, \boldsymbol{Y}_{m,j}\right\}. \end{aligned} \tag{10.17}$$

Note that the process $\boldsymbol{Y}_m$ follows the recursion

$$\boldsymbol{Y}_m = \Psi(\boldsymbol{Y}_{m-1}) + \vartheta \mathbf{1} \ell(V_{m+n-1}) \,, \tag{10.18}$$

where $\mathbf{1}$ denotes an n-vector of ones. To obtain the window-limited CUSUM procedure, ϑ is set equal to one, and Ψ is defined as $\Psi(\boldsymbol{y}) = C\boldsymbol{y}$, where $C = (c_{i,j})_{i,j=1,\dots,n}$ with $c_{i,i+1} = 1$ for $i = 1, \dots, n-1$ and $c_{i,j} = 0$ otherwise. Interestingly, other popular change point detection methods can also be expressed in this way: for example, to obtain an exponentially weighted moving average (EWMA, see [129]) procedure based on LLRs, define $\Psi(\boldsymbol{y}) = (1-\vartheta)C\boldsymbol{y}$ for $\vartheta \in (0,1)$. Thus, while in this chapter we are focussed on the CUSUM procedure, the result in Prop. 10.4 would allow one to compute the stopping time for the window-limited case more generally.

Note that (10.18) is a VAR(1) process, albeit with a degenerate noise process. A paper that gives exact expressions (in terms of iterated Fredholm integrals) for the distribution of $\boldsymbol{H}_m$ for a VAR(1) process is [166]. We adapt their results to our setting.

Let, for fixed $\boldsymbol{x} \in \mathbb{R}^n$,

$$Q_m(\boldsymbol{x}, \boldsymbol{y}) := \mathbb{P}\left(\boldsymbol{H}_m \leq \boldsymbol{x},\, \boldsymbol{Y}_m \leq \boldsymbol{y}\right) \tag{10.19}$$

for $m \geq 0$. Denote by $\boldsymbol{x}_j$ the j-th entry of the vector $\boldsymbol{x}$. Let $\min\{\boldsymbol{x}, \boldsymbol{y}\}$ be the componentwise minimum of $\boldsymbol{x}$ and $\boldsymbol{y}$. Let F be the distribution function of $\ell(V_i)$.

Proposition 10.4. *We have $Q_m(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{K}\, Q_{m-1}(\boldsymbol{x}, \boldsymbol{y})$ for $m \geq 0$ and fixed $\boldsymbol{x}$, where $\mathcal{K}$ is defined by*

$$\mathcal{K}\, h(\boldsymbol{y}) = \int_{\mathbb{R}} F\left(\frac{1}{\vartheta} \min_{i=1,\dots,n} \left\{ \left(\min\{\boldsymbol{x}, \boldsymbol{y}\} - \Psi(\boldsymbol{z}) \right)_i \right\} \right) \mathrm{d}h(\boldsymbol{z}) \,. \tag{10.20}$$

Proof. We adapt the steps in the proof of [166], Theorem 3.1). Note that $\boldsymbol{H}_m \leq \boldsymbol{x}$ implies that $\boldsymbol{Y}_l \leq \boldsymbol{x}$ for $l \leq m$, so $Q_m(\boldsymbol{x}, \boldsymbol{y}) = Q_m\big(\boldsymbol{x}, \min\{\boldsymbol{x}, \boldsymbol{y}\}\big)$. Then for $m \geq 1$:

$$\begin{aligned}
&Q_m(\boldsymbol{x}, \boldsymbol{y}) \\
&\quad = \mathbb{P}_0\big(\boldsymbol{H}_{m-1} \leq \boldsymbol{x},\, \boldsymbol{Y}_m \leq \min\{\boldsymbol{x}, \boldsymbol{y}\}\big) \\
&\quad = \mathbb{P}_0\big(\boldsymbol{H}_{m-1} \leq \boldsymbol{x},\, \vartheta \mathbf{1} \ell(V_{m+n}) \leq \min\{\boldsymbol{x}, \boldsymbol{y}\} - \Psi(\boldsymbol{Y}_{m-1})\big) \\
&\quad = \mathbb{E}_F\left[\mathbb{P}_0\left(\boldsymbol{H}_{m-1} \leq \boldsymbol{x},\, \ell(V_{m+n}) \leq \frac{1}{\vartheta} \min_{i=1,\dots,n} \left\{ \left(\min\{\boldsymbol{x}, \boldsymbol{y}\} - \Psi(\boldsymbol{Y}_{m-1}) \right)_i \right\} \middle|\, \ell(V_{m+n}) \right) \right],
\end{aligned}$$

where the latter equation follows by invoking the law of total probability and Bayes' rule, writing $\mathbb{E}_F$ for the expectation under F. With $\mathcal{K}$ as defined in (10.20) we can write this as $\mathcal{K}\, Q_{m-1}(\boldsymbol{x}, \boldsymbol{y})$ as claimed. □

Thus, for $\mathbb{P}_0\left(\boldsymbol{H}_m \leq \boldsymbol{x}\right) = Q_m(\boldsymbol{x}, \boldsymbol{\infty})$ we obtain

$$\mathcal{K}\, h(\boldsymbol{\infty}) = \int_{\mathbb{R}} F\left(\frac{1}{\vartheta} \min_{i=1,\dots,n} \left\{ \left(\boldsymbol{x} - \Psi(\boldsymbol{z}) \right)_i \right\} \right) \mathrm{d}h(\boldsymbol{z}) \,. \tag{10.21}$$

In principle, this allows us to compute (10.16) as

$$\mathbb{P}_0(\omega \leq m) = 1 - \mathbb{P}_0\left(\boldsymbol{H}_{m,i} \leq b(i),\, i = 1, \dots, n\right) \,.$$

To evaluate this in practice, at least for small m one can use approximations based on the eigenvalues of the Fredholm kernel $\mathcal{K}$ (see [166]). However, we are aiming for expressions that can be solved for the threshold function $b(\cdot)$. Therefore, even though $\mathbb{P}_0(\omega \leq m)$ is known exactly, we are interested in approximate expressions for the latter that are easier to evaluate.

10.2.2.2 Approximation for threshold selection

When testing is window-limited, we can apply EV theory to approximate the false alarm probability (10.9). This provides an easily applicable method to select b, which we outline in this section for the example of independent Gaussian observations. We remark that EV results have been applied very recently in [69] in the context of non-parametric change point detection.

Define

$$\gamma_{i,j}(h) := \mathrm{Cov}_0(\boldsymbol{Y}_{m,i}, \boldsymbol{Y}_{m+h,j}),.$$

By application of the result of [6], we obtain the following corollary.

Corollary 10.5. *Assume that observations are i.i.d. Gaussian, and that $\sigma > 1$. Then the process $(\boldsymbol{H}_m)$ defined by (10.17) satisfies*

$$\mathbb{P}_0\left(\frac{\boldsymbol{H}_{m,i} - (n-i+1)\mu}{\sqrt{n-i+1}\sigma} \leq a_m\, \boldsymbol{x}_i + c_m,\ i = 1,\dots,n\right) \to \prod_{i=1}^{n} \exp\big(-\exp(-\boldsymbol{x}_i)\big)\,, \quad \text{as } m \to \infty\,,$$

where

$$a_m = (2\log m)^{-1/2}\,,$$
$$c_m = (2\log m)^{1/2} - \frac{1}{2}(2\log m)^{-1/2}(\log\log m + \log 4\pi)\,.$$

Proof. It has been shown in [6] that as $m \to \infty$ the limiting distribution of the process of component-wise maxima of any standard Gaussian process coincides with that of n independent Gumbel variables, provided that the following conditions hold:

$$|\gamma_{i,j}(0)| < r \text{ for } i,j = 1,\dots,n,\ i \neq j\,, \tag{10.22}$$

$$\sum_{h=1}^{\infty} |\gamma_{i,j}(h)|^r < \infty \text{ for all } i,j = 1,\dots,n\,. \tag{10.23}$$

(The former condition was overlooked in [6] as has been noted in [66].)

We apply this theorem to the n-dimensional process $\widetilde{\boldsymbol{M}}_m$ with $i-$th component

$$\frac{\boldsymbol{H}_{m,i} - (n-i+1)\mu}{\sigma\sqrt{n-i+1}}\,.$$

Note that

$$\frac{\mathscr{L}_{k:n}(m) - (n-k+1)\mu}{\sigma\sqrt{n-k+1}}$$

has a standard normal distribution, so that $\widetilde{\boldsymbol{M}}_m$ is indeed the process of component-wise maxima of a standard Gaussian process.

To verify (10.22), we note that, for $l, k \in \{1, \ldots, n\}$ with $l > k$,

$$\operatorname{Cov}\left(\frac{\mathscr{L}_{k:n}}{\sigma\sqrt{n-k+1}}, \frac{\mathscr{L}_{l:n}}{\sigma\sqrt{n-l+1}}\right) = \frac{1}{\sigma\sqrt{n-k+1}}\,, \tag{10.24}$$

which is smaller than 1 by assumption.

Finally, (10.23) is satisfied because for $k, l \in \{1, \ldots, n\}$, $h \in \mathbb{N}$, we have

$$\operatorname{Cov}\left(\sum_{i=k}^{n} \ell(V_i), \sum_{j=l+h}^{n+h} \ell(V_j)\right) = \big(n - \max\{k, l+h\} + 1\big)^+\,,$$

which is zero for h large enough. □

Recall that we wish to choose a threshold function that yields

$$\mathbb{P}_0(\omega \le N) = 1 - \mathbb{P}_0(\boldsymbol{H}_{m,i} \le b(i),\, i = 1, \ldots, n) \le \alpha\,.$$

Thus, for fixed N and n, Corollary 10.2.3.2 suggests we choose

$$\begin{aligned} b(\beta) = \Bigg[&- a_N \log\left(-\frac{1}{n}\log(1-\alpha)\right) + c_N \Bigg] \\ &\times \sqrt{n(1-\beta)}\,\sigma + n(1-\beta)\,\mu + \delta\,, \end{aligned} \tag{10.25}$$

where the change point k is written as $n\beta + 1$, $\beta \in \mathcal{B}_n := \{0/n, 1/n, \ldots, (n-1)/n\}$ (this notation will turn out to be useful particularly in Section 10.2.3.2). The parameter δ is a design parameter to be chosen based on simulation. Because $c_N \to \infty$, adding a constant δ (constant with respect to N) is negligible for large N. Numerical experiments suggest that for small n the choice $\delta = 0$ seems to work well, but that for larger n a negative δ should be chosen (possibly as a function of the other parameters). We suggest a choice for δ in Section 10.2.3.2, for the case of expanding windows.

10.2.3 Testing with expanding windows

We first derive non-asymptotic bounds on the distribution of τ, which can then be used to apply the CLT, LD, and EV approximations to select the threshold. The latter two approaches yield a threshold function rather than a fixed threshold, and the achieved false alarm performance is overall closer to the desired level.

10.2.3.1 Non-asymptotic bounds

The complication in evaluating $\mathbb{P}_0(\tau \le N)$ arises from the fact that this involves a double maximum of a sum with random increments:

$$\mathbb{P}_0\left(\tau \le N\right) = \mathbb{P}_0\left(\max_{1 \le m \le N} \max_{1 \le k \le m} \mathscr{L}_{k:m} > b\right)\,.$$

In this section we provide bounds that circumvent this problem. The upper bounds we provide below in (10.26) and (10.27) turn out to be very tight, particularly if the size of the

change is large (see Fig. 10.1). We use these in Section 10.2.3.2. We remark that similar bounds could be obtained for $\mathbb{P}_0(\omega \leq N)$; the adaptation to this case is straightforward and therefore omitted.

First, note that we have

$$\begin{aligned}
\mathbb{P}_0\left(\tau \leq N\right) &= \mathbb{P}_0\left(\max_{1\leq m\leq N}\max_{1\leq k\leq m} \mathscr{L}_{k:m} > b\right) \\
&= \mathbb{P}_0\left(\max_{1\leq m\leq N}\max_{m\leq n\leq N} \mathscr{L}_{m:n} > b\right) \\
&= \mathbb{P}_0\left(\exists\, m \in \{1,\ldots,N\} : \max_{m\leq n\leq N} \mathscr{L}_{m:n} > b\right) \\
&= \mathbb{P}_0\left(\min_{1\leq m\leq N} \tau_m \leq N\right) ,
\end{aligned}$$

where

$$\tau_m =: \inf\{n \geq m : \mathscr{L}_{m:n} > b\} .$$

Therefore, the CUSUM stopping time can be written as

$$\tau = \min_{m\geq 1} \tau_m .$$

Hence, we have

$$\begin{aligned}
\mathbb{P}_0\left(\tau \leq N\right) &= \mathbb{P}_0\left(\min_{1\leq m\leq N} \tau_m \leq N\right) \\
&= 1 - \mathbb{P}_0\left(\min_{1\leq m\leq N} \tau_m > N\right) \\
&= 1 - \mathbb{P}_0(\tau_N > N) \prod_{i=2}^{N} \mathbb{P}_0\left(\tau_{i-1} > N \,|\, \tau_i > N\right) ,
\end{aligned}$$

which yields the bounds

$$1 - \mathbb{P}_0(\tau_N > N) \leq \mathbb{P}_0\left(\tau \leq N\right) \leq 1 - \prod_{i=1}^{N} \mathbb{P}_0(\tau_i > N) . \tag{10.26}$$

We furthermore note that the right-hand side is smaller than

$$\begin{aligned}
&1 - \left(\min_{h\in\{1,\ldots,N\}} \mathbb{P}_0(\tau_h > N)\right)^N \\
&\qquad = 1 - \mathbb{P}_0\left(\max_{h\in\{1,\ldots,N\}} \mathscr{L}_{1:h} \leq b\right)^N .
\end{aligned} \tag{10.27}$$

Approximations to (10.27) are available, based on which we can devise simple yet effective procedures, see Section 10.2.3.2.

As Fig. 10.1 shows, the upper bounds turn out to be very tight. The lower bounds are closer when the size of the change is smaller. To see why this should be true, consider the following heuristic argument. Since the mean μ of the LLR increments is negative,

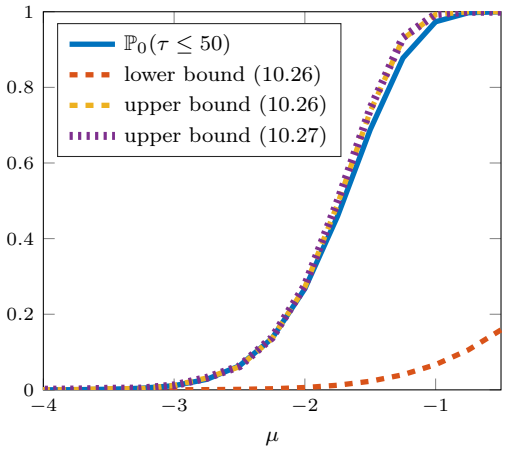


Figure 10.1: Comparison of $\mathbb{P}_0(\tau \leq N)$ and the bounds provided in (10.26) and (10.27), with $N = 50$, $\sigma = 1$ and threshold $b = 0.5$.

let us suppose that all increments were negative. In this case $\mathscr{L}_{i:n} < b$ would imply that $\mathscr{L}_{i-1:n} < b$, and hence $\tau_i > N$ would imply that $\tau_{i-1} > N$. Thus, when μ is small compared to σ^2, then $\mathbb{P}_0(\tau \leq N) \approx \mathbb{P}_0(\tau_N \leq N) = \mathbb{P}_0(X_N > b)$. One would thus expect that an alarm is typically raised at the end of the current window, as is confirmed in numerical experiments (see Fig. 10.4).

We now discuss how the bound (10.27) can be used for threshold selection.

10.2.3.2 Approximations for threshold selection

From the upper bound (10.27) we obtain that a sufficient condition for $\mathbb{P}_0(\tau \leq N) \leq \alpha$ is

$$1 - \mathbb{P}_0\left(\max_{h\in\{1,\dots,N\}} \mathscr{L}_{1:h} \leq b\right)^N \leq \alpha\,,$$

or, equivalently,

$$\mathbb{P}_0\left(\max_{h\in\{1,\dots,N\}} \mathscr{L}_{h:N} > b\right) \leq 1 - (1-\alpha)^{1/N}\,. \tag{10.28}$$

Below we discuss different limiting regimes that yield approximations to (10.28). In this section we assume that observations are independent to facilitate the comparison of the different methods. We remark, however, that the LD approximations suggested below can be extended to the case of correlated observations (as was done in [43] for observations following a Gaussian autoregressive process). We restrict ourselves to a Gaussian example only for the EV approximation (so that Corollary holds); the LD and the CLT approximation apply more generally.

EV approximation As opposed to the approach in Section 10.2.2.2, we now consider the univariate process of partial sums. That is, in this case we are interested in the

maximum of $\mathscr{L}_{k:N}$ over $k \in \{1, \ldots, N\}$. Therefore, to achieve (10.28), in the case of i.i.d. Gaussian random variables the threshold function can be chosen as

$$\begin{aligned} b(\beta) = &\sqrt{N(1-\beta)}\,\sigma\,\left[-a_N \log\left(-\frac{1}{N}\log(1-\alpha)\right) + c_N\right] \\ &+ N(1-\beta)\,\mu + \delta \end{aligned} \tag{10.29}$$

where $\beta \in \mathcal{B}_N$. For choosing δ we recall our remark from the previous section that one may expect that – at least for large changes – a change tends to be detected at the end of the window, where a single increment is considered. Thus, it seems intuitive to choose δ such that $b\big((N-1)/N\big)$ equals the $1-\big(1-(1-\alpha)^{1/N}\big)$-quantile of the distribution of the LLR increments. It is confirmed in numerical experiments that this choice indeed yields good performance of the resulting testing procedure; see the independent data example provided at the end of this section as well as the example in Section 11.2.

LD approximation Since we wish the false alarm probability α to be *small*, we may regard this as a rare event scenario; this motivates us to invoke LD theory. Change point detection procedures based on LD approximations have been considered in [24, 43, 81] for i.i.d. and VARMA models, yielding a threshold function $b(\cdot)$ that depends on the assumed position of the change point under the alternative hypothesis. We now explain how to obtain a threshold function from LD approximations. We express the change point k in terms of N, that is, we write $k = N\beta + 1$, where $\beta \in \mathcal{B}_N$. First, note that

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_0\left(\max_{\beta\in\mathcal{B}_N} \frac{1}{N}\mathscr{L}_{N\beta+1:N} > b\right) = \max_{\beta\in\mathcal{B}_N} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_0\left(\frac{1}{N}\mathscr{L}_{N\beta+1:N} > b\right)$$

(for details see [43], Section 2). In this case the logarithmic LD asymptotics suggest that for fixed β the false alarm probability can be approximated by

$$\mathbb{P}_0\left(N^{-1}\mathscr{L}_{N\beta+1:N} > b\right) \approx \exp\left(-N\mathcal{I}(b)\right), \tag{10.30}$$

where $\mathcal{I}$ denotes a Legendre transform we specify below. There is, however, no need to pick a constant b, as in the case of the EV approximation, we can pick a function $b(\beta)$ instead, such that (10.30) holds with b replaced by $b(\beta)$ for all $\beta \in \mathcal{B}_N$. That is, we propose to pick the threshold function $b(\cdot)$ such that it satisfies

$$1-(1-\alpha)^{1/N} = \exp\Big(-N\mathcal{I}\big(b(\beta)\big)\Big) \quad \forall \beta \in \mathcal{B}_N. \tag{10.31}$$

This choice entails that raising a false alarm is essentially equally likely irrespective of the supposed location of the change point within the window, and it is therefore optimal in terms of type II error performance; see [24], Chapter VI.E.

Now let us make the above more rigorous. The limiting cumulant-generating function $\Lambda(\theta)$ associated with the distribution of the LLR is:

$$\begin{aligned} \Lambda(\theta) &= \lim_{N\to\infty} \frac{1}{N(1-\beta)} \log M_{N\beta}(\theta) \\ &:= \lim_{N\to\infty} \frac{1}{N(1-\beta)} \log \mathbb{E}_0\left(e^{\theta\mathscr{L}_{N\beta+1:N}}\right); \end{aligned} \tag{10.32}$$

we assume for now that this function exists and is finite for every $\theta \in \mathbb{R}$. Define $\mathcal{I}$ as the

Legendre transform of $\Lambda(\theta)$. Provided that $\Lambda(\theta)$ exists for all $\theta \in \mathbb{R}$, noting that we can rescale as written out in (10.33), the Gärtner–Ellis theorem (Theorem 1.8) yields

$$\lim_{N\to\infty} \frac{1-\beta}{N(1-\beta)} \log \mathbb{P}_0 \left(\frac{1}{N(1-\beta)} \mathscr{L}_{N\beta+1:N} - \frac{b(\beta)}{1-\beta} > 0 \right) = -\mathcal{I}\big(b(\beta)\big). \tag{10.33}$$

In accordance with the idea expressed in (10.31), we choose the threshold function $b(\cdot)$ such that it satisfies

$$-\mathcal{I}(b(\beta)) = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_0 \left(\frac{1}{N} \mathscr{L}_{N\beta+1:N} - b(\beta) > 0 \right) = -\gamma \tag{10.34}$$

for some positive $\gamma = -N^{-1} \log\big(1-(1-\alpha)^{1/N}\big)$, across all $\beta \in \mathcal{B}_N$. Then asymptotically for large N we have that (10.28) is satisfied.

CLT approximation As a third alternative, we consider the approximation of the false alarm probability based on CLT arguments. Applying a CLT approximation has been considered in [86] for parametric change point detection (the setting of this chapter), and in [122] in the context of non-parametric change point detection. Motivated by Donsker's theorem, we can approximate the probability in (10.28) by [141], Eq. (3.15),

$$\mathbb{P}_0 \left(\max_{t\in[0,N]} \sigma B_t + \mu\, t > b \right) = 1 - \Phi\left(\frac{b-\mu\, N}{\sigma\sqrt{N}} \right) + e^{\frac{2b\mu}{\sigma^2}}\, \Phi\left(\frac{-b-\mu\, N}{\sigma\sqrt{N}} \right), \tag{10.35}$$

where B_t is a standard Brownian motion (Wiener process). Then a fixed threshold b (rather than a function as before) can be obtained numerically from setting (10.35) equal to $1-(1-\alpha)^{1/N}$.

Remark 10.6. It is known [125] that for independent observations the limiting distribution of the stopping time is exponential under H_0 as $b \to \infty$. This is not contradicting the use of a CLT approximation because the latter applies in a different limiting regime. One could also obtain a constant threshold assuming an exponential distribution of the stopping time. However, we shall see in the independent data example provided below that a constant threshold cannot be expected to work better than a threshold function. This is consistent with an example provided in [43], where the constant threshold is optimised based on simulation and compared to a threshold function obtained from an LD approximation. For this reason, we refrain from including a second method for selecting a constant threshold in this discussion.

Independent-data example For illustration we provide an example with independent data, see Figs. 10.2–10.3. Note that when testing an independent sequence of $\mathcal{N}(0,\nu)$ observations against a shift in mean of size θ, then the LLR $\mathscr{L}_{k:n}(m)$ corresponding to testing against $H_1(k)$ is given by

$$\mathscr{L}_{k:n}(m) = \sum_{i=k+m-1}^{n+m-1} \ell(V_i) = \sum_{i=k+m-1}^{n+m-1} \frac{\widetilde{m}}{\nu^2} \left(V_i - \frac{\widetilde{m}}{2} \right).$$

Hence, under H_0 the LLR increments are normally distributed with mean value $\mu = -(\widetilde{m}/\nu)^2/2$ and variance $\sigma^2 = (\widetilde{m}/\nu)^2$.

Application of the EV and CLT approximations is then straightforward. To apply the LD approximation, we need to compute the limiting log-moment-generating function $\Lambda(\theta)$ in more explicit terms (this way we also check that it indeed exists and is finite for all θ). Because the sequence of observations is independent, with $k = N\beta + 1$, we can write the associated moment-generating function as

$$M_{N\beta}(\theta) = \mathbb{E}_0 \left[\exp\left(\theta \sum_{t=k}^{N} \log \frac{q(V_t)}{p(V_t)} \right) \right] = \prod_{t=k}^{N} \exp\left[\frac{\theta}{2}(\theta - 1) \left(\frac{\widetilde{m}}{\nu} \right)^2 \right] .$$

With this expression we can compute a threshold function $b(\beta)$ from

$$\gamma = \sup_{\theta} \left\{ \theta b(\beta) + (1-\beta)\frac{\theta}{2}(1-\theta) \left(\frac{\widetilde{m}}{\nu} \right)^2 \right\} = \mathcal{I}\big(b(\beta)\big) . \tag{10.36}$$

The optimising θ is $1/2 + b(\beta)/\big[(1-\beta)\,(\widetilde{m}/\nu)^2\big]$, so that from (10.36) we obtain the desired closed-form expression for $b(\cdot)$:

$$b(\beta) = -\frac{1-\beta}{2} \left(\frac{\widetilde{m}}{\nu} \right)^2 + \sqrt{2\gamma\,(1-\beta)}\,\frac{\widetilde{m}}{\nu} . \tag{10.37}$$

It is interesting to compare this to the EV threshold function (10.29): we note that in both cases (up to scaling by N because the LD test statistic is divided by N) the threshold function is of the form

$$b(\beta) = N(1-\beta)\mu + \sqrt{N(1-\beta)}\,\sigma\,\zeta(\cdot),$$

where $\zeta(\cdot)$ is some function of the parameters. This form is intuitively appealing: it makes sense to select a threshold that exceeds the expected value of $\mathscr{L}_{N\beta+1:N}$ by some function of the standard deviation.

Using the three different thresholds, we can evaluate $\mathbb{P}_0(\tau \leq N)$ by Monte Carlo simulation. We evaluate the false alarm rate as the relative frequency with which a false alarm is raised. Figure 10.2 shows that the performance in terms of false alarms is conservative, as was to be expected because we approximate the upper bound (10.28) rather than $\mathbb{P}_0(\tau \leq N)$ itself. Nevertheless, the false alarm rates are close to the desired level α when the EV approximation is applied, while the LD approximation is more conservative. The CLT approximation does not seem to adjust enough for different α. This may be related to the fact that we have to solve for b numerically in this case while in absolute terms $1 - (1-\alpha)^{1/N}$ in (10.28) does not change much with α. Moreover, it has also been found in Chapter III of [141] that the CLT approximation typically underestimates the probability of interest. An explanation for this is that in (10.35) it is assumed that the maximum is taken over a continuous (and thus larger) interval.

Figure 10.3 displays the obtained delay values for various values of α. Here, the delay is evaluated as the sample average of the difference between the first detection time and the actual change point. Note the trade-off between the false alarm probability and the resulting delay for the LD and CLT approximation. Interestingly, the EV approximation yields a higher delay even though the false alarm probability is higher, suggesting that the shape of the threshold function does not match the shape of the LLRs $\mathscr{L}_{N\beta+1:N}$. (Note that this is not generally the case: for the state space model example discussed in Section

11.2 the EV approximation achieves the better delay performance.)

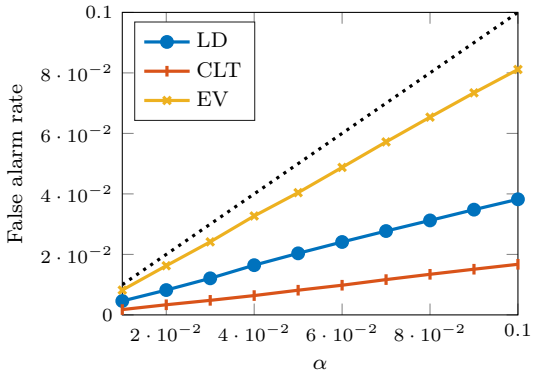


Figure 10.2: False alarm rates under criterion $\mathbb{P}_0(\tau \leq N) \leq \alpha$, with $\widetilde{m} = 2$, $N = 150$, $\nu = 1$. Comparison for various α (indicated by the dotted line).

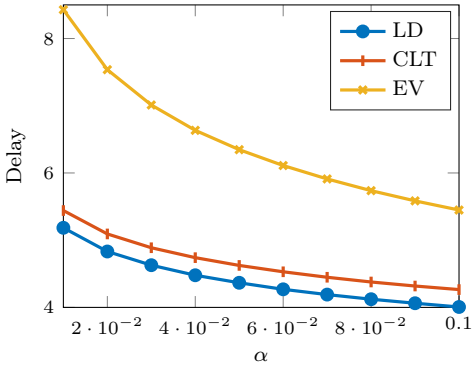


Figure 10.3: Delay values under criterion $\mathbb{P}_0(\tau \leq N) \leq \alpha$, with $\widetilde{m} = 2$, $N = 150$, $\nu = 1$.

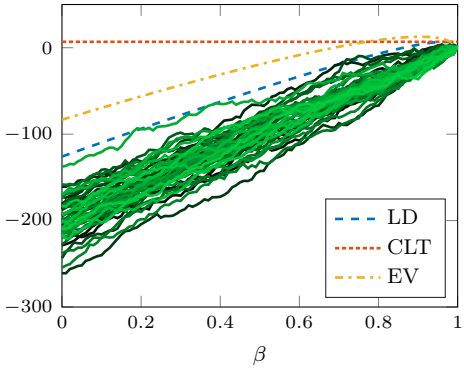


Figure 10.4: Comparison of LLRs $\mathscr{L}_{N\beta+1:N}$ and the threshold functions devised in Section 10.2.3.2 (the LD threshold is multiplied by N for comparability).

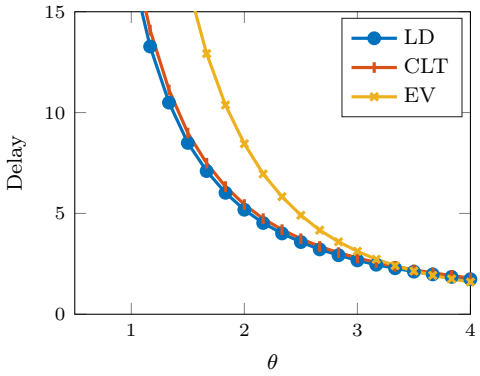


Figure 10.5: Delay values under criterion $\mathbb{P}_0(\tau \leq N) \leq \alpha$, with $\alpha = 0.01$, $N = 150$, $\nu = 1$. Comparison for various values of the shift size $\widetilde{m}$.

To further investigate this issue, we plot a graph of the threshold function as well as the LLRs, both as a function of $\beta \in \mathcal{B}_N$, see Fig. 10.4. Indeed, the distance between the EV threshold and the LLR is not uniform across β. The shape of the LD threshold, however, matches the LLRs very well. The figure also suggests that particularly when using the EV threshold, false alarms usually occur at the end of the window. One may thus wonder whether one could simply choose a constant threshold equal to the $1 - \big(1 - (1-\alpha)^{1/N}\big)$-quantile of F, the distribution of the LLR increments. This choice, however, does not work well, the obtained false alarm rate is usually considerably higher than the desired level (in this example it is close to 1). The figure shows clearly why a threshold function is to be preferred with respect to a constant threshold: the CLT threshold is far away from the actual LLRs, except when β is close to 1. Choosing a function is favourable particularly in view of the detection delay, provided that it closely mimics the behaviour of the LLRs.

Figure 10.5 shows a comparison of the delay for various choices of the shift size $\widetilde{m}$. As expected, the delay performance improves as the shift size increases. We remark that, reassuringly, for different choices of $\widetilde{m}$ the resulting false alarm performance is similar to Fig. 10.2.

10.3 More control over false alarms

The criterion considered in the previous section may not always be restrictive enough, as is illustrated in Fig. 10.6. This figure shows the alarm rate obtained when testing a sequence of independent Gaussian observations with expanding windows. We compute the alarm rate as the relative frequency of the alarms raised – thus, the alarm rate before the change point corresponds to the false alarm rate we discussed in Section 10.2.3.2, whereas the alarm rate after the change point is to be interpreted as the rate of detection. The position of the change point is indicated by the vertical line. The threshold is chosen such that (10.9) is achieved. It can be seen that at the beginning of the period, where only a small number of data points are tested, the false alarm rate is too high but because it then decreases below the desired level, the criterion is still satisfied. This also confirms once more that one should choose b to be a function, rather than a constant threshold as

is often assumed. With a constant threshold, as the example shows, $\mathbb{P}_0(\tau \leq N) \leq \alpha$ can only be achieved if $\mathbb{P}_0(\tau = 1) \leq \alpha$. This is true more generally; recall that for independent or weakly dependent observations, it has been shown, for example, in [125, 147] that the distribution of τ is approximately exponential when the threshold b is large but constant.

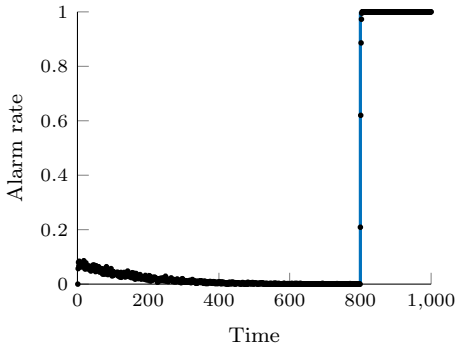


Figure 10.6: Alarm ratios obtained when testing an i.i.d. Gaussian sequence without windows. A constant threshold is chosen such that $\mathbb{E}_0\tau \geq N(1-\alpha)$, where $N = 1,000$ and $\alpha = 0.01$.

In view of the above, we propose to choose a threshold function that limits the false alarm rate for the current window to be $\widetilde{\alpha}$ (which can be related to α from before as outlined below). That is, we require

$$\mathbb{P}_0(T = n \,|\, T > n-1) = \mathbb{P}_0\left(\max_{k\in\{1,\ldots,n\}} \mathscr{L}_{k:n} > b_n\right) = \widetilde{\alpha} \tag{10.38}$$

to hold, uniformly across all n, where $T \in \{\tau, \omega\}$. If $T = \omega$, the above can be simplified because

$$\mathbb{P}_0(\omega = 1) = \mathbb{P}_0(\omega = n \,|\, \omega > n-1)$$

for any n. As mentioned in the introduction, in the definition of the MLFA (10.1) we can choose $N = 1$ to recover (10.38). In view of Fig. 10.6 this seems a good choice as one would like to control the false alarm rate at every time instance; it is, however, the least conservative because the MLFA increases monotonically in N.

To relate (10.38) to (10.3), note that

$$\mathbb{P}_0(T \leq N) = \sum_{n=1}^{N} \mathbb{P}_0(T = n)\,, \tag{10.39}$$

and

$$\begin{aligned}\mathbb{P}_0(T = n) &= \mathbb{P}_0(T = n \,|\, T > n-1)\,\mathbb{P}_0(T > n-1) \\ &= \mathbb{P}_0(T = n \,|\, T > n-1)\left(1 - \sum_{t=1}^{n-1} \mathbb{P}_0(T = t)\right).\end{aligned}$$

Using this recursive equation, it is possible to express each $\mathbb{P}_0(T = n)$ in (10.39) in terms of conditional probabilities of the form $\mathbb{P}_0(T = n \,|\, T > n-1)$. One obtains that $\mathbb{P}_0(T = n)$

can be written as

$$\mathbb{P}_0(T = n \,|\, T > n-1) \prod_{t=1}^{n-1} \big(1 - \mathbb{P}_0(T = t \,|\, T > t-1)\big) .$$

Thus, in principle one can allow for $\widetilde{\alpha}$ to depend on the current window size n as well, and choose a sequence of $\widetilde{\alpha}_n$ such that $\mathbb{P}_0(\tau \leq N) \leq \alpha$ is achieved. For example, we can set

$$\widetilde{\alpha}_1 = \frac{\alpha}{N}, \quad \widetilde{\alpha}_n = \frac{\alpha}{N} \left[\prod_{t=1}^{n-1} (1 - \widetilde{\alpha}_t) \right]^{-1} . \tag{10.40}$$

Therefore, the condition (10.38) indeed allows a better control over the false alarm performance as desired.

Approximations for (10.38) are readily available. For example, we can apply EV, LD, and CLT approximations as in Section 10.2.3.2 with N replaced by n, and $1-(1-\alpha)^{1/N}$ replaced by $\widetilde{\alpha}$ (we give more details in Section 11.2.2 for the state space model). In order to ensure (10.38), we now need the threshold function to depend on the current window size n. Thus, if the window size is fixed, the threshold function is the same for every window. If windows are expanding, we obtain an adaptive threshold function. In the latter case, it is all the more important that evaluation of the threshold function is simple so that this can be carried out on-line as a new observation arrives.

Independent-data example For illustration we consider again the independent data example from Section 10.2.2.2, yet now false alarm rates are evaluated according to (10.38). See Fig. 10.7 for an example with stopping time ω, which displays the probability $\mathbb{P}_0(\omega = 1)$ that is achieved on average, for various choices of $\widetilde{\alpha}$. (A comparison of different shift sizes is not depicted because the false alarm behaviour remains very stable, as desired.) In comparison to the example in Section 10.2.3.2 we note that the LD and EV approximations are closer but slightly above the desired false alarm rate; whereas in Section 10.2.3.2 they were rather conservative. This difference may be explained by the fact that in Section 10.2.3.2 we approximated an upper bound to $\mathbb{P}_0(\tau \leq N)$ rather than the probability itself. When windows are expanding (and the stopping time is τ and thresholds are adaptive), a very similar false alarm performance is obtained.

In Fig. 10.8 we check that when the threshold is obtained as suggested in the current section, with the sequence $\widetilde{\alpha}_n$ defined by (10.40), we indeed obtain a false alarm performance similar to Fig. 10.2, where the threshold was chosen with the aim of achieving $\mathbb{P}_0(\tau \leq N) \leq \alpha$. The performance in terms of delay is then as in Fig. 10.3, as a consequence of the similar false alarm performance.

In summary, Figs. 10.7 and 10.8 together confirm that (10.38) is a stronger false alarm criterion that allows better control over the false alarms at any given time point.

We provide more involved examples featuring dependent Gaussian processes in Chapter 11.

10.4 Conclusion

In this chapter we considered two false alarm criteria derived from the MLFA (10.1). Both criteria are more stringent than the traditional ARL, however, the former is less stringent than the MLFA, whereas the second is a special case.

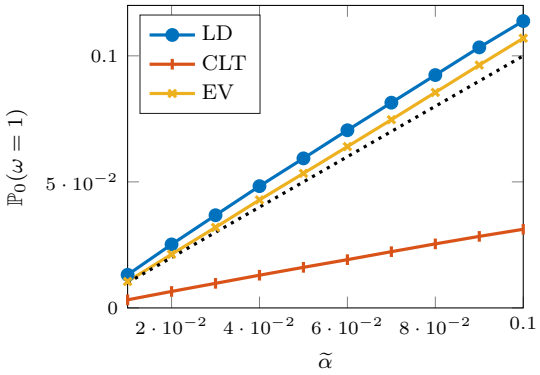


Figure 10.7: Comparison of probabilities $\mathbb{P}_0(\omega = n|\omega > n-1)$ obtained with adaptive thresholds chosen such that (10.38) is achieved with $n = 50$, $\widetilde{m} = 1$, $\nu = 1$, for various $\widetilde{\alpha}$ (indicated by the dotted line).

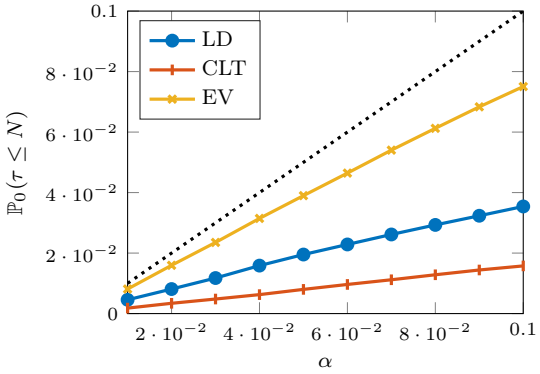


Figure 10.8: Comparison of probabilities $\mathbb{P}_0(\tau \leq N)$ obtained with adaptive thresholds chosen such that (10.38) is achieved, where the sequence of $\widetilde{\alpha}_n$ is chosen according to (10.40) such that $\mathbb{P}_0(\tau \leq N) \leq \alpha$ holds as in (10.9), with $N = 150$, $\widetilde{m} = 1$, $\nu = 1$, for various α (indicated by the dotted line).

We then provided methods for the selection of the threshold such that the false alarm criteria under consideration hold at least approximately. With respect to numerical methods for threshold selection these are more easily applicable, and moreover allow the selection of a threshold *function* rather than a constant threshold. We investigated the performance of the resulting detection procedures in numerical examples. In terms of false alarm performance, the EV approximation was usually closest to the desired level. However, the LD threshold function typically mimicked the shape of the LLRs more closely, and thus yielded the best trade-off between false alarm and delay performance. We also saw that a threshold function generally seems to be preferable in comparison to a constant threshold (and accordingly the EV and the LD threshold functions outperformed the constant CLT threshold).

A topic for future research is the improvement of the EV approximation: we saw that a shift of the resulting threshold function yields a good false alarm performance; however, it should be determined what the optimal size of that shift is, depending on the

available parameters. Furthermore, the LD approximation requires the evaluation of the limiting logarithmic moment generating function of the LLR. In this chapter, we only provided these computations for the case of Gaussian observations. Similarly, for the EV approximation we assumed Gaussian observations. In future research other distributions could also be considered in more detail.

CHAPTER 11

Change point detection for Gaussian processes

To illustrate the methodologies we proposed in the previous chapter, we apply the obtained procedures to detect changes in Gaussian processes that are serially correlated (correlated over time).

First, in Section 11.1, we note that in the case of linear Gaussian processes it is usually possible to express the LLR of the observations as a function of a sequence of independent *innovations* that are obtained by passing the observations through a whitening filter (this is a common approach, see also, for example, [15]). We present the testing procedure for detecting a proportional change in the mean and variance of dependent Gaussian sequences with an LD-based threshold. By this example, we verify that the innovations-based approach is computationally easier, and achieves similar performance compared to that of using the observations directly. The content of Section 11.1 is taken from Kuhn, Ellens and Mandjes [81].

In Section 11.2 we consider a state space model, for which we compare the performance of the thresholds obtained from LD, EV and CLT approximations using the innovations-based approach. This model is of interest because it is a special type of hidden Markov model; for this class of models it has been found in [107] that the ARL can be infinite. Moreover this example is not straightforward because the size of the change is not fixed, but rather a function of $t-k$, where t is the current time and k denotes the change point. The content of Section 11.2 is based on work by Kuhn, Mandjes and Taimre [85, 86].

11.1 Change in mean and variance

Traditionally, much of the literature on change point detection has focussed on detecting a change in the mean value. However, in the context of communication networks, an increase in the number of active users tends to be not reflected by a change in the mean only, but rather as a *change in scale* [43] – a change in both the mean and (proportionally) the corresponding variance. Therefore, to only focus on the detection of mean shifts neglects an additional indicator that a change has taken place [15, Ex. 4.1.9]. While by now there exists work on detecting more general types of change (see e.g. [151] and the references therein); none besides the unpublished work report [43] appear to have focussed on the change in scale in particular.

In this section, which is based on [81], we apply the LD based testing technique from Chapter 10 for the detection of such a change in multidimensional and serially correlated sequences. We focus on Gaussian sequences (X_t), where the observations X_t have a multivariate normal distribution. While Gaussian models are not generally appropriate for modelling network traffic, it can be argued that, as networks grow, the behaviour of the traffic streams should approach that of a Gaussian process [1, 104]. This view has been confirmed in [155] and more recently in [35] in empirical studies based on network traffic measurement data.

Let us consider the illustrative example of a link of a communication network. If the bandwidth consumed by different users is i.i.d., then the mean and variance of the total bandwidth consumption are both proportional to the number of users. As a consequence, a change in the number of users can be considered as a change of scale in the sense that the mean and variance exhibit the same relative (i.e., percentage-wise) change. When measuring not only at a single link but at various points in the network, then more information is available, potentially facilitating earlier detection or a lower risk of false alarms. In this case, apart from serial correlation also cross-correlation between data sequences generated by different sensors has to be taken into account, because the same traffic may be captured by several sensors (and hence we explicitly allow for multidimensional observations). For large traffic aggregates, using a Gaussian model is justified by central-limit type of arguments. Based on the above, we conclude that the set-up considered in this paper can be used to detect changes in load, caused by, for instance, a (legal) increase in the number of users, or a DDoS (distributed denial of service) attack.

A number of procedures have been proposed that allow for data streams to be either serially correlated [128] or multidimensional [30]. In [152] a detection method for testing serially correlated and multidimensional data streams, however, the components of each multidimensional observation are assumed to be mutually independent. All these papers focus on detecting a change in mean only.

The more general setting of cross-dependent multidimensional data streams is covered in Basseville and Nikoforov [15], where testing against a change in mean or variance is considered separately. For the purpose of testing against a change in mean only it is shown how to reduce the problem of testing serially correlated Gaussian sequences to that of testing independent Gaussian sequences using the aforementioned whitening transformation to obtain the i.i.d. sequence of innovations. For testing against more general types of changes Basseville and Nikoforov consider the local approach. For threshold selection the authors suggest the known approximation methods for i.i.d. sequences we listed in Section 8.1.2.

In this section, we first note that the innovations-based approach can be applied for testing against a change in scale, and there is therefore no need to turn to the local approach, which is less easy to apply and requires additional assumptions. We compare the innovations-based approach, which we sometimes refer to by the letter (A), with the method of testing the sequence of observations directly (in this we can follow [80] by merely replacing the scaling factor), denoted as (B). For the former we impose the weak assumption [22, Section 5.7.1] that the process be linear and invertible, while for the observations-based approach we need additional assumptions on the underlying correlation structure. We validate the proposed tests in a series of numerical experiments, which (i) study the trade-off between the detection ratio and the corresponding delay, (ii) assess the gain of multidimensional testing procedures (over multiple one-dimensional tests), and (iii) provide a systematic comparison between (A) and (B).

This section is organised as follows. In the next subsection we explain the change

in scale and the set-up of our LD-based hypothesis test in greater detail. In Section 11.1.2 we compute the LLR test statistics for (A) and (B), respectively, before we derive the threshold functions in Section 11.1.3. The results of the numerical evaluation are presented in Section 11.1.4. We conclude in Section 11.1.5.

11.1.1 Observations- vs. innovations-based testing

In this section, we introduce the setting, assumptions and notation, and provide a short comparison between approaches (A) and (B) in Table 11.1.

We are concerned with testing a stationary multidimensional Gaussian sequence (X_t) against a change in scale, where after the change both the mean and variance are multiplied by some constant c. Each X_t is a d-dimensional column vector consisting of the measurements of d different 'sensors' at (discrete) time t. The sequence of *innovations* (ε_t) is defined by

$$\varepsilon_t := X_t - \mathbb{E}\left(X_t \,|\, X_{t-1}, \ldots, X_1\right). \tag{11.1}$$

We focus window-limited testing with LD threshold. Recall that the change point k corresponds to $n\beta + 1$, where (throughout the section) $\beta \in \mathcal{B}_n$. We write $\mathscr{L}_{n\beta}$ for the LLR $\mathscr{L}_{n\beta+1:n}$. For clarity, whenever it matters, we will explicitly spell out whether the LLR is a function of observations or innovations: we write (A) $\mathscr{L}^X_{n\beta}(\cdot)$, or (B) $\mathscr{L}^\varepsilon_{n\beta}(\cdot)$. The threshold function is denoted by (A) $b_X(\beta)$ or (B) $b_\varepsilon(\beta)$. The alarm is raised at the stopping time ω defined in (10.8).

We now state the hypotheses to be tested, and the assumptions required when using (A) or (B).

(A) For the observations based approach, to compute the LLR, we consider the n observations within each window jointly. The joint distribution of X under H_0 is $\mathcal{N}_{nd}(\mathbf{0}, \Sigma)$, a Gaussian distribution of dimension nd. We write the covariance matrix Σ of the joint observations as a block Toeplitz matrix of the individual autocovariance matrices $\Gamma_h = \mathrm{Cov}\,(X_t, X_{t-h})$.

Now we can formulate H_0 and H_1 more specifically. For all $\beta \in \mathcal{B}_n$ we want to test

$$H_0 : X \sim \mathcal{N}_{dn}(\mathbf{0}, \Sigma) \quad \text{vs.} \quad H_1(n\beta+1) : X \sim \mathcal{N}_{dn}(\boldsymbol{\nu}, T),$$

where

$$\boldsymbol{\nu} = (\mathbf{0}' \ldots, \mathbf{0}', \overline{\boldsymbol{\nu}}', \ldots, \overline{\boldsymbol{\nu}}')', \quad T = \left(\begin{array}{c|c} \Sigma^{(dn\beta)} & 0 \\ \hline 0 & c \cdot \Sigma^{(dn(1-\beta))} \end{array}\right),$$

with $\overline{\boldsymbol{\nu}} = c\boldsymbol{\mu} - \boldsymbol{\mu}$, $\boldsymbol{\mu}$ denoting the mean vector before centering, and where m in $\Sigma^{(m)}$ denotes the dimension of the matrix. For method (A), we assume that the sequence before $n\beta + 1$ is independent of the sequence afterwards. This assumption enables computations, and is reasonable if a change has taken place, and the cause of the change is 'external' (as in the examples mentioned in the introduction).

(B) For the innovations based approach we need to impose further assumptions (see also Table 11.1). We focus on *causal linear* processes, i.e., we assume that X_t satisfies $\sup_t \mathbb{E}|X_t| < \infty$ and can be modelled as

$$X_t = \sum_{j=0}^{\infty} \Psi_j Z_{t-j}, \tag{11.2}$$

Table 11.1: Characteristics of (A) the observations and (B) the innovations based approach

(A)	(B)
Suitable test statistic for changes in mean and variance but also in coefficients	Suitable for detecting changes in mean or variance
Computationally expensive	Recursive computation of LLR and reduced dimensionality
How to define the threshold function in the multidimensional case is unclear, unless there is no shift in mean or data streams are independent	We can compute the threshold function for the change in scale explicitly
The process does not need to be invertible	Requires invertibility
The observations are well-defined test statistics	Since innovations are defined in terms of past observations, initial conditions are required

with uncorrelated error terms $Z_t \sim \mathcal{N}_d(\mathbf{0}, \Omega)$, and where the Ψ_j form an absolutely summable sequence of coefficient matrices [22, Proposition 3.1.1]. Using the lag operator L defined by $LZ_t := Z_{t-1}$, $L^j Z_t = Z_{t-j}$, and defining $\Psi(z) := I + \sum_{j=1}^{\infty} \Psi_j z^j$, Eq. (11.2) can be written as $X_t = \Psi(L)Z_t$.

We further need to assume that the process be *invertible*, i.e., that the i.i.d. sequence of innovations (ε_t) can be extracted as a well-defined function of present and past observations (lie in their closed linear span). If X_t is given by a VARMA(p,q) process

$$X_t = \sum_{i=1}^{p} A_i X_{t-i} + \sum_{j=1}^{q} B_j Z_{t-j} + Z_t,$$

then a well-known sufficient condition for invertibility is that $|B(z)|$ has no roots on the unit disk, where $B(z) = I + \sum_{j=1}^{q} B_j z^j$ denotes the MA-polynomial [22, Theorem 3.1.2].

In Sections 11.1.2 and 11.1.3, we provide explicit computations for the LLR and the threshold function, respectively.

11.1.2 Evaluation of the test statisic

We now evaluate the LLR when using approach (A) and (B), respectively. In both cases we may assume, without loss of generality, that the pre-change process has mean vector $\mathbf{0}$ (we may subtract the original mean vector to achieve this).

(A) The LLR for testing $X \sim \mathcal{N}_{nd}(\mathbf{0}, \Sigma)$ against the simple alternative hypothesis $X \sim \mathcal{N}_{nd}(\boldsymbol{\nu}, T)$ can be computed as

$$\mathscr{L}_n^X(X) = \frac{1}{2}\log|\Sigma| - \frac{1}{2}\log|T| + \frac{1}{2}X^T\Sigma^{-1}X - \frac{1}{2}(X-\boldsymbol{\nu})'T^{-1}(X-\boldsymbol{\nu}).$$

Filling in $\boldsymbol{\nu}$, Σ, T, the LLR for testing against a change in scale at a specific point $n\beta + 1$ becomes

$$\mathscr{L}^X_{n\beta}(X) = -\frac{1}{2}dn(1-\beta)\log c + \frac{1}{2}\widetilde{X}'\left(\Sigma^{(dn(1-\beta))}\right)^{-1}\widetilde{X} - \frac{1}{2c}\left(\widetilde{X} - \boldsymbol{\nu}^{(dn(1-\beta))}\right)'\left(\Sigma^{(dn(1-\beta))}\right)^{-1}\left(\widetilde{X} - \boldsymbol{\nu}^{(dn(1-\beta))}\right), \tag{11.3}$$

where $\widetilde{X} := (X'_{n\beta+1}, \ldots, X'_n)'$.

(B) Given the invertibility assumption holds, a proportional change in the covariance matrix of the observations (i.e. covariances are inflated by c) can be detected as a proportional change in the covariance matrix of the innovations, as it is known [22, Eq. (11.1.13)] that under H_0 the autocovariances of X_t are given by $\Gamma_h = \sum_j \Psi_j \Omega \Psi'_{j-h}$. It has been shown in [15] that (for VARMA processes) the sequence of innovations is a sufficient statistic for detecting a change in the mean value.

Then, defining $\xi = \Psi(L)^{-1}\overline{\boldsymbol{\nu}}$, the above hypotheses can equivalently be formulated as

$$H_0 : \varepsilon_t \sim \mathcal{N}_d(\mathbf{0}, \Omega),\ t = 1, \ldots, n \quad \text{vs.} \quad H_1(n\beta+1) : \begin{cases} \varepsilon_t \sim \mathcal{N}_d(\mathbf{0}, \Omega), & t \le n\beta, \\ \varepsilon_t \sim \mathcal{N}_d(\boldsymbol{\xi}, c\Omega), & t > n\beta. \end{cases}$$

Since the innovations are independent, the LLR $\mathscr{L}^{\varepsilon}_{n\beta}(\varepsilon)$ for testing H_0 against $H_1(n\beta+1)$ can be expressed as the sum of the LLRs at time $t > n\beta$ (since the LLR is zero for $t \le n\beta$). Therefore, $\mathscr{L}^{\varepsilon}_{n\beta}$ becomes

$$\mathscr{L}^{\varepsilon}_{n\beta}(\varepsilon) = \sum_{t=n\beta+1}^{n} \frac{1}{2}\log\frac{1}{c^d} + \frac{1}{2}\varepsilon'_t\Omega^{-1}\varepsilon_t - \frac{1}{2c}(\varepsilon_t - \xi)'\Omega^{-1}(\varepsilon_t - \xi). \tag{11.4}$$

Note that in this case we can compute the LLR for each new window recursively. On the other hand, in practice the true innovations after the change points can only be estimated as the recursion (11.1) requires initial conditions. The effect is minor if the order of the process is small (see Section 11.1.4).

The LLR test statistics obtained for (A) and (B) are compared with the associated threshold functions as evaluated in the next section.

11.1.3 Threshold selection

In this section we show how to obtain the threshold function as $b_X(\beta)$ for the observations based or $b_\varepsilon(\beta)$ for the innovations-based approach. To that end, we need to compute the limiting cumulant-generating function $\Lambda(\theta)$ in more explicit terms (this way we also check that it indeed exists and is finite for all θ).

(A) In Section 3 of [43] it is outlined how to compute the MGF $M_n(\theta)$ for testing $X \sim$

$\mathcal{N}_{nd}(\mathbf{0}, \Sigma)$ against $X \sim \mathcal{N}_{nd}(\boldsymbol{\nu}, T)$ (for arbitrary $\boldsymbol{\nu}, \Sigma, T$):

$$M_n(\theta) = \left(\frac{|\,\Sigma\,|}{|\,T\,|}\right)^{\theta/2} \frac{1}{|\,\theta T^{-1}\Sigma + (1-\theta) I_{dn}\,|^{1/2}} \times \exp\left(-\frac{\theta}{2}\boldsymbol{\nu}' T^{-1}\boldsymbol{\nu} + \frac{\theta^2}{2}\boldsymbol{\nu}' T^{-1}\left(\theta T^{-1} + (1-\theta)\Sigma^{-1}\right)^{-1} T^{-1}\boldsymbol{\nu}\right).$$

Filling in the specific $\boldsymbol{\nu}, \Sigma, T$ for testing against a change in scale, this expression reduces to

$$M_{n\beta}(\theta) = c^{-\theta dn(1-\beta)/2}\left(\frac{\theta}{c} + 1 - \theta\right)^{-dn(1-\beta)/2} \times \exp\left(\overline{\boldsymbol{\nu}}' s_{n\beta}\overline{\boldsymbol{\nu}} \frac{\theta^2 - \theta}{2(\theta + c - \theta c)}\right),$$

where $s_{n\beta}$ denotes the sum of all d dimensional covariance matrices within the lower right $dn(1-\beta) \times dn(1-\beta)$ dimensional block matrix in Σ^{-1}.

Using the expression we obtained for $M_n(\theta)$, the limiting log-moment generating function as defined in (10.32) becomes

$$\Lambda(\theta) = -\frac{1}{2}\theta d(1-\beta)\log(c) - \frac{1}{2}d(1-\beta)\log\left(\frac{\theta}{c} + 1 - \theta\right) + \lim_{n\to\infty}\frac{1}{n}\overline{\boldsymbol{\nu}}' s_{n\beta}\overline{\boldsymbol{\nu}}\frac{\theta^2 - \theta}{2(\theta + c - \theta c)}.$$

We can evaluate the limit in the specific cases (i) X_t can be modelled as d independent ARMA processes

$$X_{it} = Z_{it} + \sum_{j=1}^{p} a_{ij} X_{i,t-j} + \sum_{j=1}^{q} b_{ij} Z_{i,t-j},$$

(i.e., the d monitored traffic streams are independent), or (ii) there is no shift in mean, i.e. $\overline{\boldsymbol{\nu}} = \mathbf{0}$. The latter may happen, for example, if the number of users stays constant while the variance of their load changes (e.g. due to application changes).

(i) In the first case, the autocovariance matrices Γ_h are diagonal, and thus the expression $\overline{\boldsymbol{\nu}}' s_{n\beta}\overline{\boldsymbol{\nu}}$ reduces to $\sum_{i=1}^{d}\overline{\nu}_i^2 t_{i,n\beta}$, where $\overline{\nu}_i$ is the size of the mean shift of X_{it}, and $t_{i,n\beta}$ denotes the sum of the entries of the lower right $n(1-\beta) \times n(1-\beta)$-dimensional block matrix of Σ_i^{-1}, the inverse covariance matrix of X_{it}. From [43, Lemma 1] we have

$$\lim_{n\to\infty}\frac{t_{i,n\beta}}{n(1-\beta)} = \left(\frac{1 - \sum_{j=1}^{p} a_{ij}}{\sigma_i\left(1 + \sum_{j=1}^{q} b_{ij}\right)}\right)^2 =: \tau_i,$$

and hence, the limiting cumulant-generating function exists and is finite. The threshold $b_X(\beta)$ can then be evaluated by putting the resulting rate function

$$\sup_{\theta}\left\{\theta b_X(\beta) + \frac{1}{2}(1-\beta)\left[\theta d\log c + d\log\left(\frac{\theta}{c} + 1 - \theta\right) - \frac{\theta^2 - \theta}{\theta + c - \theta c}\sum_{i=1}^{d}\overline{\nu}_i^2\tau_i\right]\right\}$$

equal to γ. Defining $\eta = -d\,(1-c)^2\,/2\sum_{i=1}^{d}\overline{\nu}_i^2\tau_i$, we compute the optimising

θ to be

$$\frac{c}{1-c}\left[\left(\eta+\sqrt{\eta^2+c-d+1+\frac{4c\eta}{1-c}\left(\frac{b(\beta)}{1-\beta}+\frac{1}{2}\log c\right)}\right)^{-1}-1\right]. \quad (11.5)$$

The threshold function $b_X(\beta)$ can be evaluated using standard numerical procedures.

(ii) If there is no shift in mean, then $M_n(\theta)$ does not depend on $s_{n\beta}$. Hence the limiting cumulant-generating function always exists, and $b_X(\beta)$ follows from

$$\gamma = \mathscr{I}(b_X(\beta)) = \sup_\theta\left(\theta b_X(\beta)+\frac{1}{2}d(1-\beta)\left[\theta\log c+\log\left(\frac{\theta}{c}+1-\theta\right)\right]\right).$$

The optimising θ is

$$-\left(\frac{d(1-\beta)}{2b_X(\beta)+d(1-\beta)\log c}+\frac{c}{1-c}\right).$$

(B) When using the innovations-based approach, we may make use of the fact that innovations are independent, in which case the LLR can be written as a sum of the form $\sum_{t=n\beta+1}^{n}\ell(\varepsilon_t)$ as given in (11.4). It follows that $\Lambda(\theta)$ exists as a finite number:

$$\Lambda(\theta)=\lim_{n\to\infty}\frac{1}{n}\log\left[\mathbb{E}_0\exp\left(\theta\ell(\varepsilon_1)\right)\right]^{n(1-\beta)}=(1-\beta)\log\mathbb{E}_0\exp\left(\theta\ell(\varepsilon_1)\right).$$

The threshold can be found from putting

$$\sup_\theta\left[\theta b_\varepsilon(\beta)+\frac{1}{2}(1-\beta)\left(\theta d\log c+d\log\left(\frac{\theta}{c}+1-\theta\right)-\frac{\theta^2-\theta}{\theta+c-\theta c}\xi'\Omega^{-1}\xi\right)\right] \quad (11.6)$$

equal to γ.

The optimising θ is similar to (11.5) (replace η by $-d\left(1-c\right)^2/2\xi'\Omega^{-1}\xi$).

As expected, both approaches yield the same threshold function in case there is no shift in mean. We now know how to compute the LLR and the threshold function either using the observations- or the innovations-based approach. In the next section we evaluate the performance of the resulting detection procedures.

11.1.4 Numerical results

In this section we summarise the results of our numerical experimentation. We investigate the performance of detection methods (A) and (B) with respect to the false alarm rate and the detection delay, when testing vector autoregressive (VAR) processes against a change in scale.

We begin in Section 11.1.4.1 with an illustrative example which outlines how the testing methods (A) and (B) could be applied in practice. Then, in Section 11.1.4.2, we explain how the performance measures, false alarm rate and detection delay, are evaluated. Finally, in Section 11.1.4.3, we demonstrate the potential gain from using

multidimensional detection procedures by comparing the multidimensional procedure to the corresponding one-dimensional procedure that tests each data stream individually.

11.1.4.1 On-line detection

Let us first explain how to apply the detection methods set up in this paper for on-line detection of changes in scale in multidimensional Gaussian processes. We assume that one new observation arrives at a time, and the n most recent observations are being tested against a change with scaling factor c. As an illustrative example, we run the following procedure.

- We simulate a VAR(1) process of length N according to

$$X_t = AX_{t-1} + Z_t, \tag{11.7}$$

  where Z_t is Gaussian white noise with $Z_t \sim \mathcal{N}(\mathbf{0}, \Omega)$ for $t = 1, \ldots, k-1$, $0 < k < N$, and $Z_t \sim \mathcal{N}(\boldsymbol{\xi}, c\Omega)$ afterwards.

- We consider windows of size $n < k$, adding one new observation at a time while deleting the oldest.

- In order to test whether a change in scale with scaling factor c has occurred in a particular window, we determine whether we have arrived at the stopping time ω. If approach (B) is used, the innovations are extracted as $X_t - AX_{t-1}$ *for all* t, and thus, the assumed independence between pre- and post-change observations is neglected. We do so to account for the fact that in practice the true value of ε_k is not known as it depends on unknown initial values.

- We repeat the above steps $15,000$ times, and divide the total number of alarms raised for each window by $15,000$ so as to obtain the alarm ratio for each window.

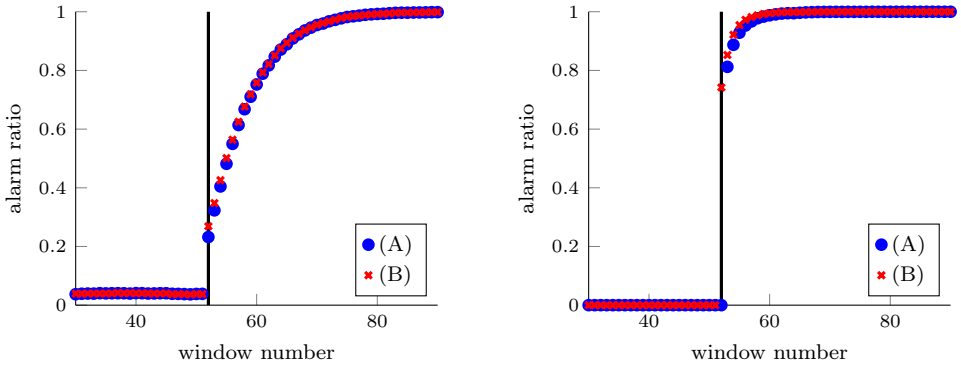
Two examples are presented in Fig. 11.1. It can be seen that the false alarm rate (the ratio of alarms before the change point as indicated by the vertical line) is indeed kept at a low level, whereas the alarm rate increases gradually to 1 after the change has occurred. It is not surprising that the detection ratio depends on the position of the change point within the window – the more observations have been affected by the change, the easier the change can be detected.

The figure shows that method (B) results in a slightly higher detection rate than method (A). This may be due to the fact that in the test set-up for approach (A) we neglected the dependence between $X_1, \ldots, X_{k-1}$ and $X_k, \ldots, X_N$ under H_0.

As expected, we also see that if $\overline{\boldsymbol{\nu}} \neq \mathbf{0}$, i.e., if there is a change in the mean value also, then both false alarm rate and detection rate improve; the shift in mean is an additional indicator that a change has occurred (for a formal proof of this intuitive result, see [15, Ex. 4.1.9]). We focus on the "worst case setting" with $\overline{\boldsymbol{\nu}} = \mathbf{0}$ when numerically evaluating the performance measures, false alarm ratio and the detection delay, in Section 11.1.4.2.

11.1.4.2 Performance measures

To evaluate the *false alarm rate*, we perform the above experiment; however, instead of shifting windows along a series of length $N > n$, we now consider a single window of observations that all correspond to H_0. Then every alarm that is raised in $15,000$ runs



(a) Mean shift $\overline{\boldsymbol{\nu}} = (0, 0, 0)'$ (b) Mean shift $\overline{\boldsymbol{\nu}} = (2, 2, 2)'$

Figure 11.1: Alarm ratios obtained when testing a three-dimensional AR(1) sequence of observations, simulated according to (11.7) with diagonal coefficient matrix A with diagonal entries 0.5 and diagonal input variance matrix Ω with diagonal entries 1, against a change in scale with $c = 2$, $\alpha = 0.01$, window size $n = 50$. The first window containing the change is indicated by a vertical line.

is a false alarm, and hence, the number of change points detected on average gives an estimate for the false alarm rate. The significance level is set to $\alpha \in \{0.01, 0.05\}$, and we pick $c = 2$ (as no change is simulated, the choice of c has little impact on the test results).

In order to evaluate the *detection delay*, we simulate a VAR(1) sequence where the first 49 observations correspond to H_0 while all later observations have been affected by the change. We test windows of size 50, at each point in time adding one new observation and dropping the oldest (thus, in window i only i out of 50 observations have been affected by the change). The procedure is stopped as soon as the change has been recognised, i.e., when the first alarm was raised. We then take the number of the first window for which this happened, averaged over 30,000 runs (to obtain an estimate for the ARL under H_1), and subtract one to obtain the detection delay.

The results of these experiments, where data streams are tested *jointly*, are presented in Table 11.2 for a number of two-dimensional examples (next to the results from testing the streams separately as explained in Section 11.1.4.3). It can be seen that – as expected – the outcome of the experiments is similar for methods (A) and (B), and the false alarm rate is generally close to the significance level α as desired. Table 11.2 also shows that the detection delay is small, and provides quantitative insight into the the trade-off between the false alarm rate and the detection delay: It suffices if 22% of the observations have been affected by the change when $\alpha = 0.01$ while less than 12% need to be affected when $\alpha = 0.05$.

These and similar examples suggest that the test performance is affected neither by the sign (positive or negative) nor by the magnitude of the correlation induced by Ω because the change size is relative to the size of the covariances if $\overline{\boldsymbol{\nu}} = 0$. (The effect of the shift size $\overline{\boldsymbol{\nu}}$ has been investigated in [43] for the case of a change in mean only.) A higher correlation via A on the other hand seems to have a positive effect on the delay – the effect of a change is enhanced due to the cross correlation.

11.1.4.3 A case for multidimensional testing procedures

In this section we demonstrate the merits of multidimensional detection procedures. In general, the signature of a change in scale is stronger when it affects $d > 1$ data streams simultaneously. In fact, in case the d tested data streams are independent, and the detection probability for each of them is p, then the detection probability when testing the d streams simultaneously is $1-(1-p)^d$. For example, if the detection probability for one data stream is 0.8, then the detection probability for testing three i.i.d. data streams simultaneously is 0.992. As a consequence, the multidimensional procedure outperforms a procedure that tests one of the individual data streams.

The more interesting question is whether the multidimensional procedure (testing data streams jointly) performs better than a one-dimensional approach where each of the d data streams is tested *separately* but an alarm is raised as soon as a change has been detected in *any* of the streams. In the latter case the significance level is corrected using the (conservative) Bonferroni method [27], that is, it is put to α/d for each one-dimensional testing procedure.

The main conclusion we draw from the results presented in Table 11.2 is that indeed the multidimensional detection procedure outperforms the method of separate testing of data streams in terms of false alarm rate and detection delay, even if the sequences are independent. However, it should be noted that this benefit comes at the cost of a longer computation time.

Furthermore, it can be seen that testing the data streams separately results in a considerably larger false alarm rate as soon as the data streams are mutually dependent via the coefficient matrix A; due to the increased correlation, the process X_t makes larger jumps, but the separate testing does not account for this. It is surprising that the performance in terms of detection delay is good when streams are tested separately, but this may be explained by the high false alarm rate.

Cross-correlations in the covariance matrix of the innovations process on the other hand have a negative impact on the detection delay when testing the streams separately, whereas the false alarm rate remains low. This is because the fluctuations of the process X_t are of smaller magnitude if the error terms Z_{it} are cross-correlated. (In the example given in the table, Z_t is generated as $Z_t = \Omega^{1/2} Y_t$, where the two components of Y_t are independent standard normals. Therefore, $Z_{1t} = Y_{1t}$ and $Z_{2t} = 0.5Y_{1t} + 0.866Y_{2t}$. This way it can be seen that jumps of Z_t are more moderate than when there is no cross-correlation in Ω.)

11.1.5 Conclusion

In this section we explained how to set up a testing procedure for detecting a change in scale in multidimensional serially correlated Gaussian processes, and found appropriate threshold functions. In the networking context, this type of change may occur for instance as a change in scale in correlated traffic streams due to an increase in the number of users, or due to an attack on the network.

We applied the testing procedure to (A) the sequence of observations and (B) the sequence of innovations. We listed benefits and drawbacks of each approach, and saw that both performed well in numerical experiments. We also demonstrated the supremacy of multidimensional detection procedures – compared to one-dimensional testing methods – for detecting changes that affect multiple data streams simultaneously, even if the data streams are independent.

Both approaches (A) and (B) were seen to be equivalent in some sense, and indeed they yielded a comparable performance. Thus, we focus on the innovations-based approach in the next section, where we consider a specific multidimensional Gaussian model that we test against a change in mean.

11.2 Change in mean in state space models

In this section we apply the innovations-based approach for testing against a change in mean in state space models. State space models are popular for modelling stochastic networks such as communication or road networks, as they allow to take into account that observations of the true state of a system may be corrupted by measurement noise (usually, a Gaussian noise process is assumed). Although they are very general, state space models are still relatively tractable in that the true system state can be estimated efficiently by a recursive procedure known as Kalman filtering. For a gentle introduction to state space models and Kalman filter estimation we refer to [61] and [22, Ch. 8].

State space models can be regarded as a special type of hidden Markov model. As such, they are a flexible modelling tool that has been found useful, for example, for modelling economic time series [62], urban traffic flow [144], and ozone data [26]. For the application to communication networks we assume that the current *state* of a channel (e.g. measured by the probability of package loss) is not observed directly, but has to be inferred from the received package flow. A change in the mean value of the hidden state sequence or a change in the mean value of the received package flow can deteriorate the performance of the network if it remains unrecognised. State space models can also be used to model road networks to account for uncertainty in the measurement of travel times. For example, in the latter case we may suppose that travel times have to be estimated from flow and occupancy data. An increase in the unobserved mean travel time can be caused by traffic congestion; a shift in the mean value of the observations on the other hand could indicate a bias of the sensors.

This motivates us to investigate procedures for testing against a shift in mean in the observations and hidden state sequence of state space models with Gaussian noise. Since the observations are generally not independent, we employ the approach of testing the innovations, which are in this case conveniently obtained as a by-product from Kalman filter estimation of the hidden states. Change point detection for state space models has also been considered in [26, 94] for the case where the size of the mean shift is unknown, in which case a generalised LLR test can be applied (the latter is due to [164]).

We then show how to apply the EV, LD and CLT approximations for the selection of the threshold. An interesting complication that arises is that a persistent change in the mean value of the observations results in a dynamic change in the mean value of the innovations, which are therefore not identically distributed after the change point. However, it follows from the stability properties of the Kalman filter that under weak conditions the magnitude of the shift converges to a constant.

This section is organised as follows. In Section 11.2.1 we define the state space model with a change in the mean value of both the hidden and the observed sequence. In Sections 11.2.2 and 11.2.3 the testing procedure is explained; that is, we determine the LLR test statistic and show how the threshold can be approximated. Section 11.2.4 provides a comparison of the numerical performance of the tests under both types of limiting regimes with respect to the false alarm probability and the detection delay. We conclude in Section 11.2.5.

11.2.1 Model and framework

We consider an example from Kuhn, Mandjes and Taimre [86] that features the following state space model of a sequence of observations (V_t), with a shift in mean at the *change*

point k:

$$X_{t+1} = AX_t + Y_t + \Gamma\, \mathbb{1}_{\{t \geq k\}}\,, \quad V_t = BX_t + Z_t + \Upsilon\, \mathbb{1}_{\{t \geq k\}}\,.$$

The d_x-dimensional process (X_t) represents the unobserved state of the system, with state transition matrix $A \in \mathbb{R}^{d_x \times d_x}$ that has eigenvalues within the unit circle, in which case the system is stable [61]. The vectors Γ and Υ model the shift in mean. We assume that A, B, Q, R, Γ, and Υ, are known, and that the Gaussian white noise processes $Y_t \sim \mathcal{N}(0, Q)$ and $Z_t \sim \mathcal{N}(0, R)$ are independent.

Denote $V_s^t := \{V_s, \ldots, V_t\}$ for $s, t \in \mathbb{N}$. The minimum variance estimator $\widehat{X}_t = \mathbb{E}_0\left[X_t \,|\, V_1^{t-1}\right]$ for the hidden state X_t can be computed efficiently using the well-known Kalman filter [for details see e.g. [55]] as

$$\widehat{X}_t = A\widehat{X}_{t-1} + K_{t-1}(V_{t-1} - B\widehat{X}_{t-1})\,, \quad \widehat{X}_0 = x_0\,.$$

where $K_t := A\Sigma_t B'(B\Sigma_t B' + R)^{-1}$ is the *Kalman gain*, and $\Sigma_t = A\Sigma_{t-1}A' + Q - K_{t-1}(B\Sigma_{t-1}B' + R)K_{t-1}'$ is the *state error covariance matrix*. As a by-product the sequence of *innovations* is obtained,

$$\varepsilon_t := V_t - B\,\widehat{X}_t\,,$$

which represent the new information which is not contained in V_1^{t-1}. They are independent and normally distributed with mean 0 and covariance matrix

$$\Omega_t := \mathrm{Cov}(\varepsilon_t) = B\Sigma_t B' + R\,.$$

The persistent change in mean in X_t and V_t results in a shift in mean on ε_t that can be described by (see [15, Eq. (7.2.110)])

$$\rho(t,k) = B\left[\psi(t,k) - A\,\zeta(t-1,k)\right] + \Upsilon,$$

where $\psi(t,k) = A\,\psi(t-1,k) + M$, $\zeta(t,k) = A\,\zeta(t-1,k) + K_t\,\rho(t,k)$, with initial conditions $\psi(k,k) = 0$, $\zeta(k-1,k) = 0$. Therefore, in order to detect a shift on (X_t) or (V_t), we test

$$H_0 : \varepsilon_t \sim \mathcal{N}(0,\, \Omega_{t|t-1}) \quad \text{versus} \quad H_1 : \bigcup_{k=1}^{n} \left[H_1(k) : \varepsilon_t \sim \mathcal{N}\big(\rho(t,k),\, \Omega_{t|t-1}\big)\right]$$

with $t \geq k$. That is, we have to test whether any of the hypotheses $H_1(k)$ holds.

Note that $\rho(t,k)$ depends on both k and t during the transient phase of the Kalman filter. Provided that Σ_t converges to some matrix Σ as t grows large, it can be shown that the Kalman gain K_t converges to $K = \Sigma B'(B\Sigma B' + R)^{-1}$ ([15], Section 3.2.3.2). Conditions under which this holds are provided in [55], Section 7.3.1.2. The limit Σ (if it exists) can then be obtained as the solution to the algebraic Riccati equation

$$\Sigma - A\Sigma A' + A\Sigma B'(B\Sigma B' + R)^{-1}B\Sigma A' - Q = 0\,.$$

In this case, as noted in ([15], Eq. (7.2.112)), we have that asymptotically

$$\rho(t,k) \to B\big(I - A(I-KB)\big)^{-1}\Gamma + (I - B\big(I - A(I-KB)\big)^{-1}AK)\Upsilon =: \rho\,.$$

Then it also holds that $\Omega_t \to B\Sigma B' + R =: \Omega$.

These limiting expressions can be used to obtain approximations for the false alarm probability as outlined in Section 10.3. Moreover, they yield an approximation to the LLR test statistic that can be computed in a recursive manner – in Section 11.2.4 we numerically evaluate the test performance when the approximate LLR is used rather than the actual LLR.

11.2.2 Evaluation of the test statistic

In this section we evaluate the LLR test statistic $\mathscr{L}_{k:n}$, for $k \in \{1,\dots,n\}$, $n \in \{1,\dots,N\}$.

The joint likelihood of V_k^n is given by

$$\begin{aligned} p\big(V_k^n\big) &= \prod_{t=k}^{n} p\big(V_t \,|\, V_1^{t-1}\big) \\ &= \prod_{t=k}^{n} \frac{\exp\Big[-\frac{1}{2}\big(V_t - B\widehat{X}_t\big)\Omega_t^{-1}\big(V_t - B\widehat{X}_t\big)'\Big]}{\sqrt{(2\pi)^{d_v}\big|\Omega_t\big|}}\,. \end{aligned}$$

Thus we have that $p(V_k^n) = \prod_{t=k}^{n} p(\varepsilon_t)$, where (abusing notation) $p(\cdot)$ denotes the density function corresponding to its argument. Hence, we can write the LLR as

$$\mathscr{L}_{k:n} = \sum_{t=k}^{n} \rho(t,k)'\Omega_t^{-1}\varepsilon_t - \frac{1}{2}\rho(t,k)'\Omega_t^{-1}\rho(t,k)\,. \tag{11.8}$$

Note that this is not a backward recursion over k because the recursive computation of $\rho(t,k)$ proceeds forward. However, for large $n-k$ we have

$$\mathscr{L}_{k:n} \approx \sum_{t=k}^{n} \ell(\varepsilon_t) := \sum_{t=k}^{n} \rho'\Omega^{-1}\varepsilon_t - \frac{1}{2}\rho'\Omega^{-1}\rho\,. \tag{11.9}$$

We show numerically in Section 11.2.4 that the test performance remains good if the LLR (11.8) is replaced by the approximate LLR (11.9). The mean and variance of the asymptotic likelihood increments (under H_0) are

$$\mu = \mathbb{E}\big[\ell(\varepsilon_t)\big] = -\frac{1}{2}\rho'\Omega^{-1}\rho\,, \quad \sigma^2 = \mathrm{Var}\,\big(\ell(\varepsilon_t)\big) = \rho'\Omega^{-1}\rho\,. \tag{11.10}$$

Then the threshold function b can be chosen as outlined in Section 10.3; we provide explicit computations in the next section.

11.2.3 Threshold selection

Using the explicit expressions obtained for μ and σ^2, defining the threshold function according to (10.29) is straightforward. We obtain the EV threshold

$$b(\beta) = \left[-a_n \log\left(-\frac{1}{n}\log(1-\alpha)\right) + c_n\right]\sqrt{n(1-\beta)\rho'\Omega^{-1}\rho} - n(1-\beta)\frac{1}{2}\rho'\Omega^{-1}\rho + \delta\,, \tag{11.11}$$

for $\beta \in \mathcal{B}_n$, with a_n and c_n as defined in Corollary 10.2.3.2, and δ chosen to be the $(1-\alpha)$-quantile of the $\mathcal{N}(\mu, \sigma^2)$-distribution. For the CLT approximation we apply (10.35), replacing N by n and again using μ and σ as defined in (11.10). To obtain the LD based threshold, we can again proceed as in Section 10.2.3.2. Because the sequence of innovations is independent, with $k = n\beta + 1$, we can write the associated moment-generating function as

$$M_{n\beta}(\theta) = \mathbb{E}_0\left[\exp\left(\theta\sum_{t=k}^{n}\log\frac{q(V_t \mid V_k^{t-1})}{p(V_t \mid V_k^{t-1})}\right)\right] = \prod_{t=k}^{n}\mathbb{E}_{0,t}\left[\left(\frac{q(\varepsilon_t)}{p(\varepsilon_t)}\right)^{\theta}\right],$$

where, abusing notation, p and q refer to the distribution of their argument under H_0 and $H_1(k)$ respectively, and $\mathbb{E}_{0,t}$ indicates that the expectation is taken with respect to $p(\varepsilon_t)$. As in [43, Section 3], we can evaluate this as

$$\prod_{t=k}^{n}\exp\left[\frac{\theta}{2}(\theta-1)\rho(t,k)'\Omega_{t|t-1}^{-1}\rho(t,k)\right].$$

Combining the above, we may take $\Lambda(\theta) \approx \frac{\theta}{2}(\theta-1)\rho'\Omega^{-1}\rho$ as an approximation for the cumulant-generating function. This can be used to compute a threshold function $b(\beta)$ as

$$b(\beta) = -\frac{1-\beta}{2}\,\rho'\,\Omega^{-1}\rho \;+\; \sqrt{2(1-\beta)\rho'\,\Omega^{-1}\rho\,\gamma}\,, \tag{11.12}$$

where $\gamma = -n^{-1}\log\widetilde{\alpha}$.

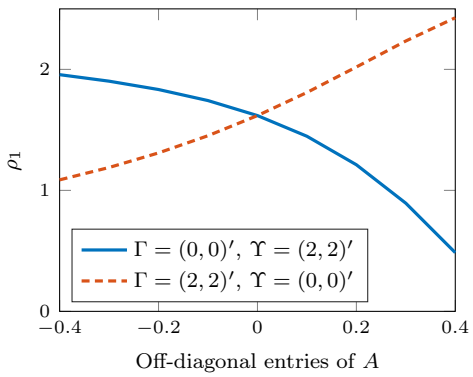


Figure 11.2: Values for the shift size ρ (here $\rho_1 = \rho_2$)

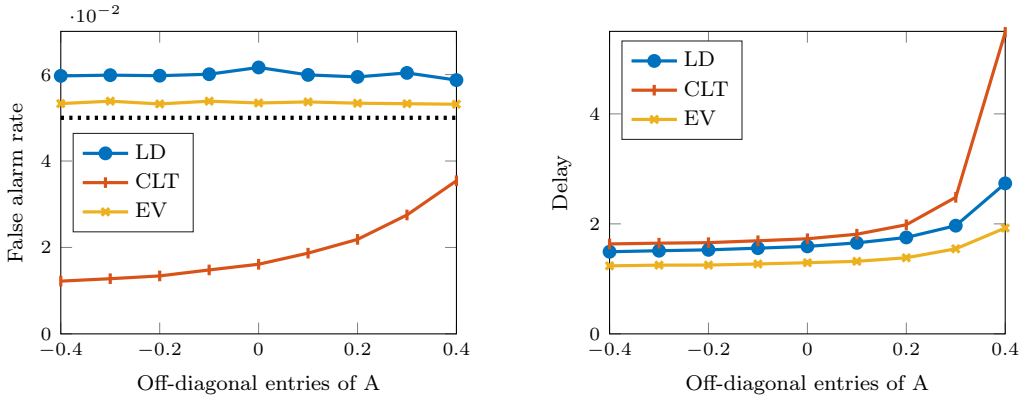


Figure 11.3: False alarm rate per window and delay values with $\Gamma = (0,0)'$, $\Upsilon = (2,2)'$ and $\widetilde{\alpha} = 0.05$ (dotted line).

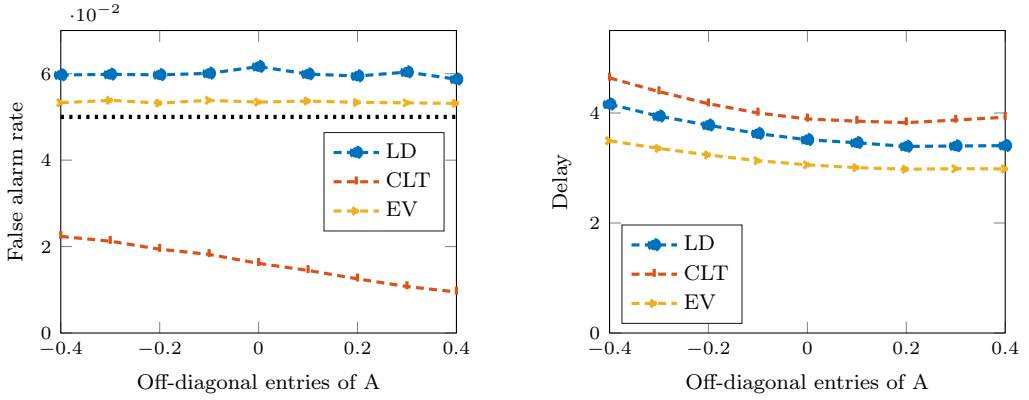


Figure 11.4: False alarm rate per window and delay values with $\Gamma = (2,2)'$, $\Upsilon = (0,0)'$ and $\widetilde{\alpha} = 0.05$ (dotted line).

11.2.4 Numerical results

We now investigate the performance of the procedures defined in Section 11.2.2. In order to gain insight regarding the impact of cross-correlation, we fix the diagonal entries of A to be $A_{11} = A_{22} = 0.5$, and vary the off-diagonal entries (both are taken to be equal, $A_{12} = A_{21}$). For various shift sizes, we provide the achieved false alarm and detection rates when using thresholds obtained based on EV, CLT, or LD approximations. Further, we fix $B = 0.5\, I_2$, $Q = R = I_2$, and $\widetilde{\alpha}$ and put either Γ or Υ equal to $\mathbf{0}$. The resulting shift sizes are depicted in Fig. 11.2. The values plotted in Figs. 11.3–11.4 were obtained by averaging the relative frequencies of false and true alarms obtained over 10,000 runs. The significance level $\widetilde{\alpha}$ is indicated by the horizontal dotted black line.

The LD threshold yields false alarm rates that are consistently close to but slightly above the specified level α, while the CLT threshold is conservative overall. The best false alarm performance is achieved by the EV threshold. The delay values depend on the size of ρ: a larger change is easier to detect (compare to Fig. 11.2). The accuracy of the CLT approximations seems to improve when ρ is small. In this case $\rho(t,k)$ is closer

to ρ, even when t is small; this may explain why the CLT approximation works better in this case. Interestingly, the EV approximation works better than the LD approximation in this example: both the false alarm rates as well as the delay values are better.

11.2.5 Conclusion

Numerical experiments (some of which were presented in this thesis) indicate that both the EV and the LD approximations work reasonably well while the CLT threshold yields an overly conservative performance in terms of false alarms (and no significant gain for the delay).

The fact that the tests performed well in our numerical examples also suggests that the test performance is rather robust with respect to the use of the LLR approximation (11.9) rather than the actual LLR (11.8). This is a great advantage with respect to computation time, and thus important for applying the proposed procedures for on-line testing.

Table 11.2: False alarm rates and detection delays obtained from testing two-dimensional VAR(1) sequences, using (A) the observations-based approach and (B) the innovations-based approach, with $c = 2$, window size $n = 50$, mean zero. Streams are tested jointly with significance level α, and separately (ignoring interdependence) with significance level $\alpha/2$. In the latter case an alarm is raised as soon as a change point is found in any of the d streams. The standard error is given in parentheses.

Example	α	Testing	False alarm rate		Delay	
			(A)	(B)	(A)	(B)
$A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$	0.01	separately	0.007 (0.0006)	0.007 (0.0006)	14.278 (0.075)	14.139 (0.075)
		jointly	0.008 (0.0007)	0.007 (0.0007)	10.510 (0.058)	10.289 (0.058)
$\Omega = \begin{pmatrix} 1.0 & 0 \\ 0 & 1.0 \end{pmatrix}$	0.05	separately	0.031 (0.0014)	0.032 (0.0015)	7.998 (0.050)	7.818 (0.050)
		jointly	0.038 (0.0016)	0.038 (0.0016)	5.992 (0.040)	5.802 (0.040)
$A = \begin{pmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \end{pmatrix}$	0.01	separately	0.397 (0.0040)	0.374 (0.0040)	3.264 (0.036)	3.438 (0.037)
		jointly	0.008 (0.0007)	0.007 (0.0007)	7.384 (0.055)	6.970 (0.054)
$\Omega = \begin{pmatrix} 1.0 & 0 \\ 0 & 1.0 \end{pmatrix}$	0.05	separately	0.552 (0.0041)	0.529 (0.0041)	1.527 (0.022)	1.625 (0.023)
		jointly	0.038 (0.0016)	0.038 (0.0016)	4.105 (0.037)	3.768 (0.036)
$A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$	0.01	separately	0.006 (0.0006)	0.006 (0.0006)	15.502 (0.082)	15.340 (0.082)
		jointly	0.008 (0.0007)	0.007 (0.0007)	10.509 (0.058)	10.289 (0.058)
$\Omega = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$	0.05	separately	0.031 (0.0014)	0.031 (0.0014)	8.782 (0.055)	8.634 (0.055)
		jointly	0.038 (0.0016)	0.038 (0.0016)	5.992 (0.040)	5.802 (0.040)
$A = \begin{pmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \end{pmatrix}$	0.01	separately	0.515 (0.0041)	0.485 (0.0041)	2.674 (0.035)	2.919 (0.037)
		jointly	0.008 (0.0007)	0.007 (0.0007)	7.458 (0.055)	7.023 (0.055)
$\Omega = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$	0.05	separately	0.640 (0.0039)	0.610 (0.0040)	1.295 (0.023)	1.428 (0.022)
		jointly	0.038 (0.0016)	0.038 (0.0016)	4.146 (0.037)	3.796 (0.036)

Summary

The purpose of the research comprised in this thesis entitled 'Monitoring and Control of Stochastic Systems' was

I to contribute new methods for evaluating performance criteria for stochastic models as applied in telecommunications, manufacturing or health care;

II to investigate what performance and behaviour of such stochastic systems can be expected when operating under certain control policies, or subject to certain design choices; and

III to devise procedures for monitoring and testing in order to verify the underlying modelling assumptions.

In agreement with these three aspects related to the monitoring and controlling of stochastic systems behaviour, the thesis is divided into three parts, the contents of which are briefly summarised below.

Part I: Static Control Problems. In Part I, we focus on the evaluation of certain performance characteristics relevant with regard to the static (open-loop) control and design choices concerning the system. Specifically, we derive large deviations asymptotics as well as efficient simulation procedures for the evaluation of probabilities related to rare events such as network congestion or failure. Control measures derived from such an analysis are *static* in the sense that they can be decided on in advance, in an off-line manner, and will then remain in place as long as the underlying model is considered valid.

First, we consider two particular queuing models which are suitable for instance for modelling call centres. A common phenomenon occurring in call centres appears to be that call arrival data is often more volatile than that of a homogeneous Poisson process, which is the traditional model used for such data. This effect is known as *overdispersion*. The models we focus on aim to solve this issue by introducing a 'second layer' of randomness: instead of assuming that the Poisson process has a fixed arrival rate, we allow for the arrival rate itself to evolve randomly over time. The resulting process is also referred to as 'doubly stochastic' Poisson process.

Performance criteria of interest include, for example, the probability that a certain (large) number of customers ends up in the queue at a particular time, or the probability that a certain fraction of customers is lost. For the queuing models we consider, with overdispersed arrival rates, no exact analytical expressions are known for these performance characteristics. Therefore, in this thesis we develop (i) efficient algorithms for the

estimation of relevant rare-event probabilities based on importance sampling; and (ii) expressions that are asymptotically accurate, or, as common in the field of large deviations theory, at least exact on a particular logarithmic scale.

Finally, in Part I we derive asymptotically exact expressions and importance sampling procedures for more generic performance measures involving the comparison of order statistics. These are of interest in the context of queuing and packing problems, but also describe the false selection (type II error) probability for hypothesis testing problems such as those considered in Part III.

Part II: Dynamic Control Problems. Part II of the thesis is about control problems that require a *dynamic* control policy, where decision rules are updated at regular time instances based on the current state of the network at those times. In particular, we focus on a class of Markovian decision problems known as *restless multi-armed bandits* (RMAB). As an application we consider the problem of selecting transmission channels in a wireless network so as to maximise the average throughput. We compare the two models that have been proposed as RMAB models of the channel selection problem. The first model is simpler and hence better understood; here, each channel is assumed to alternate randomly between two states ('good' and 'bad') in a Markovian manner. In the second model the state of the channel (e.g. the logarithmic signal-to-noise ratio) is assumed to behave as an autoregressive Gaussian process.

The objective in Part II is to investigate the performance and structural properties of dynamic control policies, in particular for the latter model, which is less well-studied so far. In view of computational tractability we focus on the class of index policies. We investigate the behaviour of some example policies in simulation experiments, provide a number of structural results, and perform an extensive literature survey regarding the use of the two models as an RMAB model for channel selection.

Part III: Testing and Monitoring. The performance of any control policy derived from the analysis of a model crucially depends on the validity of the underlying modelling assumptions. Part III of the thesis is therefore concerned with statistical testing procedures that can identify a certain type of process or detect persistent changes in the process behaviour. Motivated by the good optimality properties of such tests, the focus is on testing procedures that feature a likelihood ratio test statistic.

We first propose a testing procedure for the purpose of deciding which process in a network of stochastic processes is the one that follows a given target distribution. Because in practice collecting measurements is often costly, we assume that the decision has to be made based on a limited number of samples. We exploit large deviations theory to control the probability of making a false selection.

The remainder of Part III is focussed on the detection of *change points*, points in time at which the underlying probability distribution of a process changes. We consider two novel false alarm (type II error) criteria for such sequential testing procedures. These criteria are more stringent than the traditional average run length criterion, with which the average number of false alarms can be controlled but not their variance. We show how the false alarm performance of the test can be regulated with respect to these criteria by using large deviations and other asymptotic approximations. As examples we present and carry out the change point detection procedures for different Gaussian models for the purpose of detecting changes in the first and second moment of the distribution.

Samenvatting *(in Dutch)*

De doelen van het onderzoek omvat in dit proefschrift met de titel 'Monitoring and Control of Stochastic Systems' (Monitoring en Besturing van Stochastische Systemen) zijn:

I het ontwikkelen van methoden ter evaluatie van het gedrag van verschillende stochastische modellen, met toepassingen in met name telecommunicatienetwerken, productiesytemen en de gezondheidszorg;

II het onderzoeken van de prestatie en het gedrag van zulke stochastische systemen opererend onder een gegeven besturingsmechanisme; en

III het opzetten van toets- en monitoring-procedures voor de verificatie van onderliggende modelaannames.

In overeenstemming hiermee is dit proefschrift ingedeeld in drie delen. Deze worden hieronder kort samengevat.

Deel I: Statische Regelingsproblemen. In Deel I richten we ons op de evaluatie van bepaalde prestatie-indicatoren die relevant zijn met het oog op statische ('open-loop') besturing van het system. In het bijzonder leiden we asymptotisch accurate uitdrukkingen af voor kansen op bepaalde 'grote afwijkingen' (*large deviations*), zoals een hoog congestieniveau of het falen van netwerk-componenten. Besturingsmechanismes die worden afgeleid via een dergelijke analyse zijn 'statisch' in die zin dat ze vooraf gespecificeerd worden, en dan van toepassing blijven zolang het onderliggende model van kracht blijft.

Eerst bekijken we twee specifieke wachtrijmodellen die geschikt zijn voor (bijvoorbeeld) het modelleren van call-centers. Een belangrijk fenomeen in call-centers is dat de tijden waarop gesprekken binnenkomen veel onregelmatiger verdeeld zijn dan wat men zou verwachten op basis van een homogeen Poisson proces (het model dat traditioneel wordt toegepast voor dit type datareeksen). Dit effect wordt *overdispersie* genoemd. Hier concentreren wij ons op modellen die dit gedrag beter beschrijven door een tweede vorm van toevalligheid te introduceren: de vaste parameter van het Poisson proces wordt vervangen door een tweede stochastisch proces. Het resulterende proces wordt dan ook wel een 'tweevoudig stochastisch Poisson proces' (*doubly stochastic Poisson process*) genoemd.

Interessante prestatie-indicatoren zijn, bijvoorbeeld, de kans dat een bepaald (groot) aantal klanten op een gegeven tijdstip in de wachtrij belandt. Voor de wachtrijmodellen die wij hier bekijken zijn geen analytische uitdrukkingen bekend voor deze prestatiemaat. Derhalve ontwikkelen we in dit proefschrift (i) efficiënte *importance-sampling* algoritmes voor het schatten van zulke kansen (in het bijzonder gericht op regimes waarin deze kansen klein zijn), en (ii) uitdrukkingen die of asymptotisch exact zijn, of, zoals gebruikelijk op

het gebied van de large deviations theorie, tenminste asymptotisch exact zijn op een bepaalde logaritmische schaal.

Ten slotte leiden we in Deel I asymptotisch exacte uitdrukkingen en importance sampling procedures af voor een aantal meer generieke prestatiematen die gerelateerd zijn aan het vergelijken van geordende steekproefelementen. Deze zijn relevant in specifieke industriële toepassingen, maar houden ook verband met de kans op een zgn. 'fout-positief' ('fout van de eerste soort') in bepaalde hypothesetoetsen zoals diegene die geanalyseerd worden in Deel III.

Deel II: Dynamische Regelingsproblemen. Deel II van dit proefschrift behandelt besturingsproblemen die vragen om *dynamische* aanpassingen. Hierbij worden de beslissingsregels bijgewerkt op periodieke tijdstippen op basis van de toestand van het system van dat moment. In het bijzonder bekijken we een type Markov-beslissingsprobleem dat bekend staat als *restless multi-armed bandits* (RMAB). Als toepassing beschouwen we het probleem waarbij communicatiekanalen in een draadloos netwerk zo gekozen moeten worden dat de gemiddelde doorvoer maximaal is.

Twee modellen die zijn voorgesteld als RMAB modellen voor dit 'kanaalselectieprobleem' worden vergeleken. Het eerste model is eenvoudiger en beter bestudeerd; in dit model wordt aangenomen dat een kanaal tussen twee toestanden wisselt ('goed' en 'slecht') volgens een Markov proces. In het tweede model wordt aangenomen dat de toestand van het kanaal (bijvoorbeeld de logaritme van de 'signaal-versus-ruis ratio') zich gedraagt als een autoregressief Gaussisch proces.

Het doel in Deel II is om de prestatie en de structurele eigenschappen van dynamische regelingen te onderzoeken, in het bijzonder voor het tweede, minder onderzochte, model. Met het oog op de computationele complexiteit richten we ons op zgn. 'index-regels'. Het gedrag van een aantal regels wordt onderzocht aan de hand van simulatie-experimenten. Daarnaast bespreken we structurele eigenschappen, en presenteren we een uitgebreid literatuuroverzicht ten aanzien van de toepassing van deze twee modellen als RMAB modellen voor kanaalselectie.

Part III: Toetsen en Surveillance. Cruciaal voor de prestatie van elke besturingsstrategie voor een specifiek model is de geldigheid van de onderliggende aannames van dit model. Deel III van dit proefschrift behandelt statistische methoden om bepaalde type processen te kunnen identificeren, en om blijvende veranderingen in het gedrag van het proces te detecteren. Op basis van de bekende optimaliteits-eigenschappen van dit soort toetsen, concentreren we ons op toetsen die gebaseerd zijn op de 'likelihood ratio'.

We ontwikkelen een testprocedure om te bepalen welk proces in een netwerk van stochastische processen hetgene is die een gegeven 'doelverdeling' volgt. Omdat het in de praktijk vaak duur is om metingen te verzamelen, wordt aangenomen dat een beslissing moet worden gemaakt op basis van een beperkt aantal metingen. Large deviations theorie wordt gebruikt om de kans van een verkeerde keuze te kunnen beteugelen.

Daarnaast behandelt Deel III de detectie van zgn. 'veranderpunten' (*change points*), punten waarop de onderliggende kansverdeling van een proces verandert. We bekijken twee nieuwe vals-alarm ('fouten van de tweede soort') criteria voor zulke sequentiële toetsprocedures. Deze criteria zijn strikter en breder toepasbaar dan het gemiddelde-loopduur (*average run length*) criterium, waarmee alleen het gemiddelde maar niet de variantie van het aantal valse alarmen kan worden gecontroleerd. We laten zien hoe prestatie van de toets (in termen van het aantal valse alarms) afhangt van de onderliggende parameters,

door gebruik te maken van large deviations en andere asymptotische benaderingen. Als voorbeelden bekijken we procedures voor de detectie van change points in de context van verschillende Gaussische modellen, waarbij we ons met name richten op veranderingen in het eerste of tweede moment van de verdeling.

List of authors' contributions

Ref.	Included in	Author	Statement of Contribution
[63]	Chapter 3	M. Heemskerk	Mainly in Sections 3.1.1 and 3.1.2
		J. Kuhn	Mainly Sections 3.1.2, 3.1.3, 3.2.3, as well as Section 3.2.2, which is new with respect to the cited work
		M. Mandjes	Idea and Sections 3.1.1, 3.1.3, 3.2.1
[81]	Section 11.1	J. Kuhn	Conception, computations, writing, editing, simulations
		W. Ellens	Conception, editing
		M. Mandjes	Idea, conception, computations, editing
[82]	Chapter 2	J. Kuhn	Conception, writing, editing, simulations, computations
		M. Mandjes	Idea, conception, computations, proofs, writing, editing
[83] [84]	Chapter 7	J. Kuhn	Idea, conception, writing, editing, proofs, simulations
		M. Mandjes	Editing
		Y. Nazarathy	Idea, conception, editing
[85]	Chapter 10	J. Kuhn	Idea, conception, writing, editing, proofs, simulations
		M. Mandjes	Idea, editing
		T. Taimre	Editing
[86]	Section 11.2	J. Kuhn	Idea, conception, computations, simulations
		M. Mandjes	Idea, editing
		T. Taimre	Editing
[87]	Chapter 9	J. Kuhn	Idea, conception, writing, editing, computations, simulations
		M. Mandjes	Idea, editing
		T. Taimre	Computations, editing
[88]	Chapter 4	J. Kuhn	Idea, conception, writing, editing, proofs, simulations
		M. Mandjes	Idea, conception, proofs, writing, editing
		T. Taimre	Editing
[89]	Chapter 6	J. Kuhn	Conception, writing, editing, simulations, extensive literature survey
		Y. Nazarathy	Idea, conception

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