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Publication date

2017

Document Version

Final published version

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Citation for published version (APA):

Fan, Z. (2017). *Essays on international portfolio choice and asset pricing under financial contagion*. [Thesis, fully internal, Universiteit van Amsterdam].

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Zhenzhen Fan

Essays on International Portfolio Choice and Asset Pricing under Financial Contagion

The 2008 financial crisis has witnessed prices of assets traded on different exchange markets, of various asset classes, from different geographical locations plunge simultaneously or in close succession, causing serious problems for banks, insurance companies, and other financial institutions. It calls for models that account for the unconventional dependence structure of asset prices beyond the classical paradigm. This doctoral dissertation contributes to the modeling of the financial contagion phenomenon and the analysis of the impact of contagion on financial investment decisions, hedging strategies, and asset prices.

Zhenzhen obtained her MSc in Tilburg University (field: quantitative finance and actuarial sciences) in 2011. After taking selected courses from the Tinbergen Institute and the University of Amsterdam, she started her Ph.D. in 2012 at the Department of Quantitative Economics of the University of Amsterdam. Zhenzhen has been an Assistant Professor at the School of Finance of Nankai University (Tianjin, China) since 2016.

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**ESSAYS ON INTERNATIONAL
PORTFOLIO CHOICE AND ASSET
PRICING UNDER FINANCIAL
CONTAGION**

Zhenzhen Fan

ISBN: 978 90 3610 485 2

Cover design: Crasborn Graphic Designers bno, Valkenburg a.d. Geul

This book is No. **695** of the Tinbergen Institute Research Series, established through cooperation between Rozenberg Publishers and the Tinbergen Institute. A list of books which already appeared in the series can be found in the back.

Essays on International Portfolio Choice and Asset Pricing under Financial Contagion

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Universiteit van Amsterdam
op gezag van de Rector Magnificus
Prof. dr. ir. K.I.J. Maex
ten overstaan van een door het College voor Promoties ingestelde
commissie, in het openbaar te verdedigen in
de Agnietenkapel
op woensdag 21 Juni 2017, te 10.00 uur

door

Zhenzhen Fan

geboren te Jiangsu, China

Promotiecommissie

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Faculteit Economie en Bedrijfskunde



The research was partially supported by APG Asset Management.

Acknowledgements

I'd love to express my sincere gratitude to my University supervisor, Prof. dr. Roger Laeven, who, not only guided me through my entire journey of Ph.D. research, but also set an example of a professional scholar through his perfectionism towards every aspect of the research, never-ending questions on the details, and perseverance until every possible improvement is exhausted.

I am heartily thankful to my company supervisor, Dr. Rob van den Goorbergh, who has supported me and helped me with his abundant industry experience and expertise. I have been working with Prof. Laeven and Dr. van den Goorbergh since my master thesis writing. I never thought of pursuing a Ph.D. when I first came to the Netherlands for a master degree. It was through working with them did I become interested in research and decide to do a Ph.D. eventually.

This thesis would not have been possible without the sufficient financial and technical support provided by APG Asset Management. I would like to thank the Solutions team: Louis Chaillet, Bart Kuipers, Eduard Ponds, and Martin Prins. I benefitted a lot from working one day a week with them. Their enthusiasm towards econometrics, economics, and finance, and strong work ethics greatly influenced the way I work.

My sincere thanks go to the committee members, Prof. dr. H. Peter Boswijk, Dr. E. Eiling, Prof. dr. Frank C. J. M. de Jong, Prof. dr. ir. Cornelis W. Oosterlee and Prof. dr. ir. Michel H. Vellekoop. Thank you for your willingness to be my committee.

I would also like to show my gratitude to the participants in APG Quant Round Table Seminars, Brown Bag seminars at the University of Amsterdam and Tinbergen Institute, who provided useful comments during presentations. I appreciate useful comments from seminar participants at Nanjing University, Nankai University of China, and conference participants at the Winter School of Mathematical Finance 2013, International Congress on Insurance: Mathematics and Economics 2014, Quantitative Methods in Finance 2014, Annual Meetings of the European Financial Management Association 2015, European Meeting of Statisticians 2015.

I will always appreciate the suggestions and support offered during my job search period. I was greatly supported by my supervisor Roger Laeven. He gave me useful comments on my job market paper and organized an internal seminar for me to present my job market paper. I am grateful to my other two references, Rob Kaas and Rob van den Goorbergh. I am thankful for the information from Jose Luis Moraga during the Tinbergen Institute Ph.D. job market session. I appreciate the useful advice from the mock interviewers, Jose Luis Moraga, Katarzyna Lasak, Andre Lucas, Jan Schnitzler and Arjen Siegmann. I am also grateful for the help and encouragement from my friends Zhiling Wang, Xiao Xiao and Simin He, who accompanied me through the entire job market process and made it less stressful for me.

Many thanks to my Ph.D. colleagues for providing a stimulating and fun environment in which to learn and grow. I'd like to give special thanks to my office mate Merrick Li, Yuan Yue, Jitze Hooijsma, Frank van Berkum, and Jan de Kort. Sharing an office with you during these years was one of the pleasures of doing a Ph.D. In particular, I'd love to thank Rob Sperna Weiland, Servaas van Bilsen, and Andrei Lalu, who have been very good friends of mine as well as colleagues. They have helped me solve many questions that I cannot have dealt on my own. I am especially grateful to Rob Sperna Weiland for his contribution to the Dutch summary in this dissertation.

I am indebted to my roommates, Yuan Yue and Fang Wang. You have made my life in Amsterdam colorful and "delicious". The apartment was home to me thanks to you. I'd also

like to thank two most important guests of the apartment, Hao Fang and Lin Zhao. Thank you for your great company. I will never forget the enjoyable times when we cook together, played card games, went out hunting for Pokemons – the list goes on and on.

The badminton group was an important part of my Ph.D. life. Many thanks to our captain, Lintao Li, and Mrs caption, Yang Qin. Special thanks to Junwei Wang, Zhiling Wang, Junze Sun, Hubert, Haochen Jiang, etc. It was great fun to play badminton and to hang out with you.

Being surrounded by many friends made my Ph.D. life enjoyable in Tilburg and Amsterdam. Hailong Bao, Te Bao, Yun Dai, Zichen Deng, Lex Dielen, Yufeng Huang, Zhenxing Huang, Lingwei Kong, Zina Lekniute, Hao Li, Jinghua Lei, Bowen Luo, Miao Nie, Geng Niu, Luqing Shi, Lei Shu, Ruixin Wang, Yun Wang, Yusi Wang, Ran Xing, Ruoyu Xing, Yan Xu, Yilong Xu, Jingni Yang, Diandian Yi, Yifan Yu, Yuejuan Yu, Huashan Zhang, Jun Zhang, Kun Zheng, Sudan Zheng, and Bo Zhou, thank you all so much for your friendship.

I owe a lot to my parents and grandparents. Without you I would not have had the courage to embark on this journey in the first place. Thank you for your understanding and support.

I would like to express my sincere gratitude to my husband, Hong Li, who has always been encouraging and inspiring me throughout my Ph.D.

Lastly, I offer my regards and blessings to all of those who supported me in any respect during the completion of the Ph.D.

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Chapter 1

Introduction

The 2008 financial crisis has witnessed prices of assets traded on different exchange markets, of various asset classes, from different geographical locations plunge simultaneously or in close succession, causing serious problems for banks, insurance companies, and other financial institutions. It calls for models that account for the unconventional dependence structure of asset prices beyond the classical paradigm. This doctoral dissertation contributes to the modeling of the financial contagion phenomenon and the analysis of the impact of contagion on financial investment decisions, hedging strategies, and asset prices.

Financial contagion loosely refers to the phenomenon that asset prices exhibit excess cross-market linkages (especially on the downside) during economic downturns.¹ Cross-sectionally, the stochastic dependence among international equities in a financial crisis is typically asymmetric and therefore cannot be captured by standard (linear, symmetric) correlations. The class of mutually exciting jump-diffusion processes is a promising workhorse for modeling financial contagion in continuous-time finance. Different from Lévy type models that are widely applied in the literature, mutually exciting jumps are both cross-sectionally and serially dependent. The class provides a parsimonious model of jump propagation, allowing for cross-sectional asymmetry and serial dependence through time: a jump that takes place in one asset market today leads to a higher probability of experiencing future jumps in the same market as well as in other markets around the world.

Many investment and risk management implications derived from classical models are no longer valid in the context of financial contagion. This raises many important questions. For example, when financial risks are contagious, how are investors compensated for the systemic risks they are taking, to what extent should an investor diversify his/her equity portfolio internationally, and how to deal with unhedgable risks, etc.? This thesis tries to address these questions by reconsidering some of the classical problems in finance, most noticeably asset pricing, portfolio choice, hedging, and valuation, in the presence of contagion.

Chapter 3 answers the question of how to optimally diversify the equity portfolio when equity jump risk is contagious across geographical markets. Based on Fan [53], this chapter analyzes the optimal equity portfolio choice problem in the classical Merton context [96]. We propose to model the contagious financial market using mutually exciting jumps to account for excess comovement during economic downturns led by the US market. The mutually exciting jumps generate asymmetric jump excitation, thus allowing for some equity markets to be more capable of spreading their risks than others. A typical example is that crashes in the US get reflected quickly in Europe, while the reverse transmission may not be as pronounced.

¹ See Forbes and Rigobon [62] for a discussion of various definitions of contagion.

Modeling asymmetric equity excitation is of significant empirical relevance. Literature has found that the US equity market plays a unique role in the international financial market. One example is that lagged US equity returns significantly predict returns in numerous non-US countries, while lagged non-US returns display limited predictive ability for the US returns. In our model, the leading role of the US equity market is characterized by having a large cross-section excitor as the “source jump component”, the jump component that transmits risks, and a small cross-section excitor as the “target jump component”, the jump component that receives the transmission. The asymmetric excitation structure of the US equity market indicates, on the one hand, its capability of spreading domestic jump risks worldwide and on the other its resistance to foreign equity risk spillover.

We employ the martingale method to solve the optimal asset allocation problem and we are able to derive closed form solutions. The optimal portfolio has two important features: (1) It is sufficiently diversified, in the sense that it consists of a large number of individual assets to diversify away idiosyncratic risks; (2) Instead of an optimally diversified equity portfolio, as suggested in the classical asset allocation literature, the optimal portfolio for an expected CRRA utility investor is biased towards the US, which is more capable of transmitting equity jump risks worldwide, a phenomenon that we term “the US bias”. Intuitively, the US bias arises because price jumps in the US are likely to get reflected in the world economy by raising jump intensities of the other markets. Since the US equity drives jump intensities more than other equity markets, the investor demands more US equity exposure to hedge against the uncertainty in the jump intensities of global equity markets.

We apply the model to historical return data and find that excitation asymmetry can explain the observed US bias in the market portfolio. Neither Poisson jumps nor self exciting jumps can produce the pattern of the US bias in the market portfolio. Only when jumps are mutually exciting with an asymmetric excitation structure does the optimal portfolio exhibit the US bias.

To focus on the international equity contagion, Chapter 3 assumes that investors hedge away all currency risk when investing internationally. While this is feasible in theory, it may not be the optimal way of handling exchange rate risk. Chapter 4, therefore, investigates the currency hedging problem when investors invest in equities denominated in different currencies. Here, based on Fan [54], we consider the optimal and equilibrium currency hedging strategies in the context of equity and currency contagion.

Different from Chapter 3, one cannot simply add mutually exciting jumps to the classic reduced-form model to produce the contagion between the equity market and the FX market. As Backus et al. [11] point out, the change in exchange rate is effectively the ratio of the change in pricing kernel processes of the two countries. Therefore the exchange rate dynamics should be consistent with the pricing kernel specification. We propose a structural model that prices equity risks and models currency dynamics consistently while generating equity-currency contagion.

While complying with the foreign exchange literature findings, we allow the equity jump component and currency jump components to be mutually exciting. An equity price jump today increases the probability of experiencing further price jumps in the equity market as well as the probability of the occurrence of currency jumps, and vice versa. The normal dependence is captured by instantaneous covariance and the dependence during market turmoil by jump excitation.

Besides being empirically relevant, this framework allows us to intuitively distinguish a safe haven currency and an investment currency. The existing literature invariably refers to a “safe-haven” currency as the currency with low covariance with equities. Nevertheless, the

covariance between currency returns and equity returns can be time varying, and can even change signs over time. Also, the 2008 crisis has seen many currencies that were not at the center of the turmoil depreciated, even those which were regarded as safe-haven currencies preceding the crisis. By contrast, we characterize the “safe-haven” currencies by a small equity-currency excitor, indicating that a price plunge in the equity market is not likely to trigger a depreciation of that currency.

We first solve the optimal asset allocation problem over the asset universe of equities, equity derivatives, and currencies. Unlike Chapter 3 where the equity market is asymptotically complete, the market in this model is incomplete in the sense that there are more risk factors than assets available. Solving the portfolio choice problem analytically in this incomplete market is a more challenging task than that in Chapter 3. With careful specification, we are able to solve for the optimal portfolio choice in closed form. We revisit Black’s equilibrium currency hedging problem [15] by further imposing security market clearing conditions to derive the equilibrium currency hedging strategies. We find that all else equal, investors have a larger hedging ratio for investment currencies, those that are relatively more prone to equity market turmoil. The preference for the safe haven currencies cannot be readily replicated using symmetric dependence measures, such as correlation.

In both cases (that is, equity portfolio optimization and currency hedging), ignoring financial contagion leads to substantial utility loss and results in under-estimated risk exposure that may have devastating consequences for institutional as well as individual investors, especially during financial crises.

While Chapter 3 and Chapter 4 look into the contagion phenomenon, either among geographical markets or asset classes, Chapter 5 focuses on another aspect of the financial crisis – the emergence of arbitrage opportunities during the market turmoil. One of the unique phenomena in the financial crisis is the breakdown of the Covered Interest rate Parity (CIP) that is used to be taken for granted. Chapter 5 is based on my job market paper Fan [55]. The industry has been puzzled by the deviations from CIP in 2007-2009, the magnitude of which was unprecedented. Traditionally, a deviation from the parity of over 20 bps is regarded as exploitable arbitrage profits. Let alone the 400 bps spike observed during the crisis.

We first establish that the currency basis (deviations from CIP) during the crisis has a convex, downward sloping term structure curve for USD/EUR, USD/AUD, and USD/CAD currency pairs. As a first theoretical attempt (to my knowledge) to price the currency basis, we propose an asset pricing model that explicitly accounts for market liquidity risk, the risk factor found to be relevant in the empirical literature.

In particular, we consider the position unwinding risk, the risk that investors may need to exit the position to free the capital for liquidity preference. To exploit the CIP violation, investors need to form a currency basis trade, which is a capital-intensive portfolio. It may happen that before the maturity date, investors may need to exit the position in order to free the capital in the cash market. Unfortunately, premature deposits and off-the-run bonds (bonds which are not the newest issue) are usually sold at a lower price than on-the-run ones (bonds of the latest issue) of the same kind. The arbitrageur, therefore, bears a liquidity cost when the market liquidity is low (during which time the on-the-run/off-the-run spread widens). We show that when we consider the position unwinding risk, the forward currency rate given by the asset pricing model exceeds the CIP implied forward rate by the risk neutral expectation of future liquidity cost weighted by the position unwinding probability, where the future liquidity cost is measured by the on-the-run/off-the-run spread.

Our model nests as a special case the CIP, in the absence of liquidity risk. We show numerically that the model is consistent with the empirical observation: (1) (on the time

series dimension) consistent with the empirical literature, liquidity risk contributes to a non-negative currency basis; (2) (on the cross-section dimension) our model is able to produce a convex, downward sloping term structure that closely resembles the reality.

This dissertation is organized as follows. Chapter 2 gives an overview of the existing literature on the topics covered in Chapter 3, 4 and 5. Chapter 3 studies the optimal equity portfolio choice problem under equity contagion across geographical markets, assuming that all currency risks are hedged away. Chapter 4 investigates how investors can optimally hedge currency risk when there is contagion between the equity market and the foreign exchange market. Chapter 5 focuses on explaining the deviation from the covered interest rate parity during the financial crisis.

Chapter 2

Overview of the literature

We define financial contagion according to Forbes and Rigobon [62]. The mutually exciting jump process is used extensively as the workhorse to model asset prices in this dissertation. The mutually exciting jump process is a multivariate version of the Hawkes process, which was originally developed by Hawkes [76], Hawkes [77], and introduced in finance to model the dynamics of asset returns by Aït-Sahalia, Cacho-Diaz, and Laeven [6] and to model credit default by Aït-Sahalia, Laeven, and Pelizzon [5].

Chapter 3 and 4 belong to the international portfolio choice literature. Chapter 3 deals with the optimal equity portfolio problem without exchange rate risks, assuming that investors fully hedge currency risks. Chapter 4 relaxes this assumption and takes into account currency risks in international asset allocation. Chapter 5 is a study on the exchange rate behavior, in particular, how it is related to limits to arbitrage in the capital market.

The modern portfolio theory can be traced back to the efficient frontier theory of Markowitz [95]. On the basis of the single-period model of Markowitz [95], Merton [96] introduces the intertemporal optimal portfolio choice in a continuous time framework. Since Solnik [111], the potential benefits of investing internationally have been known to equity investors. The majority of the literature on international equity portfolio optimization assumes that investors will fully hedge currency risks when investing in the international market; thus currency risks can be taken out of the story. The past two decades have seen many efforts to address the portfolio optimization problem in richer stochastic environments of equity dynamics. To name a few, Wachter [117] and Chacko and Viceira [32] solve in closed-form the consumption and portfolio problem in a diffusion market with mean-reverting state variables; Liu [90] solves the asset allocation problem for general diffusion return processes; Das and Upal [42] and Aït-Sahalia et al. [4] study the portfolio implications of systemic jumps under constant investment opportunities; Liu et al. [92] look at the portfolio optimization problem when both price and volatility can jump; Jin and Zhang [81] consider the asset allocation for general Lévy processes.

Sophisticated as the models get, the extant literature on the equity portfolio optimization has not been able to address the US bias, that is, investors tend to invest more in the US equity market than in the peripheral markets compared to classic portfolio predictions. Chapter 3 contributes to the international portfolio choice literature by (1) deriving the equity portfolio implication analytically under the context of financial contagion, and (2) theoretically generating the US bias, that is, all else equal, investors hold more US equities than predictions made by classic portfolio choice models.

To our knowledge, the existing models in the extant literature cannot produce a US bias in portfolio choice. Sophisticated portfolio choice models that admit closed-form solutions

often times focus on univariate settings with a single stock in the market. For instance, see the stochastic volatility model solved by Liu [90], stochastic volatility with jumps model proposed in Liu and Pan [91], the double jump model of Liu et al. [92], and Branger et al. [21], who extend Liu and Pan [91] to allow for multiple jumps in volatility but stay within the single stock framework. Since the dependence structure of international equities can be nonlinear and asymmetric (see Ang and Chen [9], Christoffersen et al. [36]), the asymmetric excitation feature cannot be replaced by stochastic volatility (see Buraschi et al. [27]) or regime-switching models (see Ang and Bekaert [8]) even in a truly multivariate framework.

When it comes to international portfolio choice, the modeling of currency returns and their interplay with equity risks is inevitable. The studies above make the simplified assumption that all currency risks are hedged away. A more realistic approach would be to allow investors to choose their optimal currency exposure. So far there has been no consensus on how much currency risks to hedge or even whether to hedge currency risks at all. Empirical work has been carried out to answer this question. On the one hand, many studies have found that hedging currency risks reduces portfolio risks. For example, Glen and Jorion [71] investigate the benefits from currency hedging with forward contracts and find that currency hedging significantly improves the performance of portfolios. Campbell et al. [28] consider an investor with an exogenous portfolio of equities or bonds and ask how the investor can use foreign currencies to manage the risk of the portfolio. They find that the correlations between exchange rates and equity returns vary a lot across different currency pairs. On the other hand, papers like De Roon et al. [45] conclude that currency hedging reduces the volatility of portfolio returns at the cost of lower expected return and fatter tails of international equity return distribution. On the extreme, Froot [67] claims that currency exposure should be left unhedged for long-term investors based on the assumption that purchasing power parity holds in the long run and exchange rates display mean reversion.

While there is an extensive literature on the exchange rate dynamics¹, theoretical studies that account for the interdependence between the capital market and the exchange rate market are relatively scarce. One example is Bakshi et al. [12], who decompose the stochastic discount factor (hence exchange rates) into interest rate risk, equity risk, and an orthogonal component. A similar factor structure model can be found in Brusa et al. [26], who include an equity factor, a carry factor, and a Dollar factor in modeling exchange rate dynamics. Another attempt is Lettau et al. [89], who propose to explain the currency return in a downside risk capital asset pricing model by including a downside equity beta.

The interdependent structure between equity and currency has important implications on international portfolio choice and optimal currency hedging strategies. The study on the theoretical multi-currency hedging in an equilibrium framework starts with Solnik [112], which is expanded by Sercu [109], Stulz [113], Adler and Dumas [2], etc. While the literature on modeling equity and exchange rate dynamics has grown fast in the past decade, relatively little is known on the international portfolio choice with currency risks in more realistic scenarios. Some exceptions include Brown et al. [23], who study the optimal currency hedging problem in the context of stochastic volatility, and Torres [115], who explores the optimal portfolio choice problem in a Poissonian jump diffusion model.

One of the first attempts on the equilibrium currency hedging is made by Black [15], who

¹Examples are the factor models proposed by Backus et al. [11], later extended by Lustig et al. [93] to account for the cross section of carry trade returns. Bates [13] is one of the pioneers that include jumps in stochastic volatility models to model exchange rate dynamics. Since then, FX models with jumps to capture crash risk in currency returns can be found in Chernov et al. [35], Farhi and Gabaix [56], Farhi et al. [57], Carr and Wu [30], Jurek [82]. Jumps in exchange rates have also been documented and studied using high frequency data by Lahaye et al. [87], Chatrath et al. [34] and Lee and Wang [88], etc.

assumes that equity returns and currency returns follow i.i.d. multivariate Gaussian distribution. The paper derives a striking result: in equilibrium, every investor hedges the same amount of any risky currency regardless of the investor's home currency. This universal currency hedging ratio depends only on the average risk tolerance and on total wealth and total assets held by investors in each country. Surprisingly, Black's universal hedging ratio remains the prevailing opinion on currency hedging both in the industry and in academia even 27 years after the paper was published. Among existing literature that studies the international portfolio choice problem with currency risks, (conditional) covariance between exchange rates and equity (bond) risks is used exclusively as the measure of interdependence between currencies and other asset classes, despite the sophistication of the equity and the foreign exchange market.

Chapter 4 aims to bridge this gap. We revisit Black's equilibrium currency hedging problem under the context of equity-currency contagion. We propose a realistic model that generates equity-currency contagion, which enables a theoretical characterization of the "safe haven" properties of a risky currency. We derive the equilibrium currency hedging strategies under this context.

Having studied the portfolio implication with and without currency risks, Chapter 5 takes a closer look at exchange rates behavior in the spot and derivatives market. In particular, Chapter 5 studies the validity of the Covered Interest rate Parity. Early studies tend to agree that the market is efficient in the sense that after taking into account data imperfections, brokerage fees, and other transaction costs, the Covered Interest rate Parity holds.² In a landmark study, Taylor [114] documents small but potentially exploitable profitable arbitrage opportunities during periods of turbulence. The recent financial crisis starting in late 2007 has again witnessed an enormous breakdown of the CIP condition, leaving "arbitrage opportunities" unexploited. Many papers have documented persistent and significant deviations from CIP in various currency pairs using different interest rate instrument with different sampling frequencies. Such studies include Baba et al. [10], Coffey et al. [37], Fong et al. [60], Genberg et al. [69], Sarkar [107], Hui et al. [79], and Mancini-Griffoli and Rinaldo [94]. So far the extant literature has been focusing on the time series behavior of the currency basis. The cross section patterns of the currency basis remain relatively unexplored.

Chapter 5 tries to empirically study the cross section as well as time series behavior of the deviations from CIP. Inspired by the stylized facts, this chapter proposes an asset pricing model and shows how position unwinding risk and the on-the-run/off-the-run spread can contribute to a nontrivial currency basis that resembles the reality.

²See Frenkel and Levich [66], Deardorff [46], Rhee and Chang [105], and the survey paper by Officer and Willett [99].

Chapter 3

Asymmetric Excitation and the US Bias in Portfolio Choice¹

3.1 Introduction

The potential benefits of international diversification have been known to equity investors for long (see, for example, Solnik [111]). Nevertheless, the actual equity portfolios held by investors appear to be far from optimally diversified as measured by classic models. The equity home bias, for instance, is a well-recognized pattern of under-diversification. It refers to the empirical finding that investors over-invest in domestic equities relative to the theoretically optimal investment portfolio. Since the seminal paper by French and Poterba [65], there has been extensive research on the measurement and explanation of home bias. Information asymmetry and familiarity are commonly offered as potential explanations for the equity home bias.²

However, the equity home bias is only part of the under-diversification puzzle. It is a well-documented fact that investors hold biased equity portfolios not only towards home equities but also towards some other equities. Kang et al. [83] study the foreign ownership in Japanese firms and find that investors hold foreign portfolios tilted towards large firms with good accounting performance rather than those with better Sharpe ratios. Chan et al. [33] show that markets that are more developed and larger in market capitalization attract more foreign investors. Ferreira and Matos [58] study the preference of institutional investors worldwide and conclude that institutional investors prefer firms that are cross-listed in the US and constituents of the Morgan Stanley Capital International World Index. Bekaert and Wang [14] compare actual country equity holdings to a theoretical optimal allocation given by the CAPM framework and find that investors significantly over-invest in the US and under-invest in Japan. Forbes [61] states that both size and liquidity contribute to the attractiveness of US financial markets on top of the risk-return tradeoff. Diyarbakirlioglu [47] studies mutual fund holdings and finds that investors' foreign portfolios tend to be concentrated in large stock markets and well-developed economies.

A related strand of the literature suggests that the US plays a special role in the international financial market. For instance, King and Wadhvani [84] investigate high-frequency returns for the US, Japan, and UK, and find that when New York opens, there is a jump in the London price reflecting the information contained in the New York opening price. Eun and Shim [52] employ a vector autoregression system and find that innovations in the US

¹Professor Roger J. A. Laeven and Dr. Rob van den Goorbergh made helpful comments and suggestions.

²See, e.g., Epstein and Miao [51], Uppal and Wang [116], Bekaert and Wang [14], Boyle et al. [19].

are rapidly transmitted to other markets whereas no single foreign market can significantly explain US market movements. Similarly, Hamao, Masulis, and Ng [75] find significant volatility spillover effects from New York to London and Tokyo but no price volatility spillover effects to New York are observed. In a more recent paper, Rapach et al. [104] show that lagged US equity returns significantly predict returns in numerous non-US countries, while lagged non-US returns display limited predictive ability with respect to US returns. They state that

“the lead-lag relationships are an important feature of international stock return predictability, with the United States generally playing a leading role. ... our results call for an international asset pricing model that explicitly incorporates the leading role of the United States.” (p. 1636)

3.1.1 Three-region market portfolio

Taking the perspective of a world investor, free from home bias, the actual international market portfolio is not optimally diversified according to the classic asset allocation theory. Figure 3.1 plots the dynamics of the Merton mean-variance portfolio (see Merton [96]) on the left and of the (three-region) market portfolio on the right for US, Japanese, and European equities over the period June 1996 to May 2015, with expected returns estimated over the full sample and covariances estimated from an expanding window. We see that the market portfolio is consistently over-weighting the US equity and under-weighting the other two.

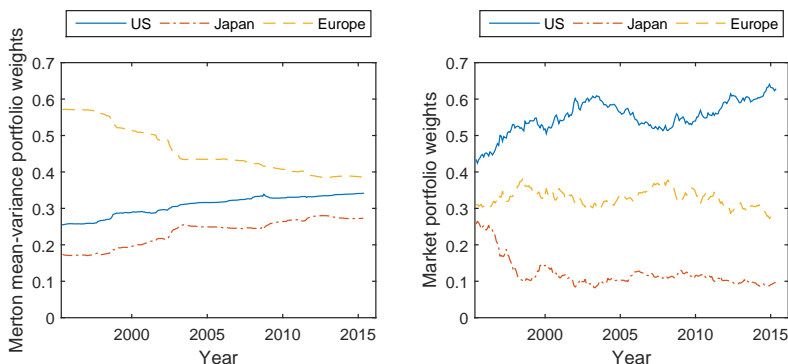


Figure 3.1: Market equity portfolio weights (right panel) and the Merton mean-variance equity portfolio weights (left panel) on US, Japanese, and European equity markets from June 1996 to May 2015. The market portfolio weights are calculated by dividing the market values (US dollar denominated) of MSCI US, Japan, and Europe by their sum at each time point. The Merton mean-variance portfolio is computed using excess log returns of MSCI indices over local risk-free rates. US 3-month treasury bill rates, Japan base discount rates, and 3-month Euribor rates are used as proxies for the local risk-free rates. Expected excess returns are fixed as the mean excess returns of the monthly *total return* from January 1970 to May 2015. The variance matrices are estimated using daily MSCI *price index* with an expanding window. Weights are normalized to add up to 1.

Acknowledging the fact that expected returns cannot be estimated statistically significantly in samples of finite length (see Merton [98]), we can equivalently ask the question of

what expected returns would explain the market weights, assuming that investors are mean-variance optimizers and allocate their wealth among the equity markets of the US, Japan, and Europe. Denote the portfolio weights on the (currency-hedged) risky assets by w and the weights *within* the equity portfolio by \bar{w} . Given the coefficient of relative risk aversion, γ , the covariance matrix, Σ , and the expected excess log returns adjusted for half the variance, μ , a Merton mean-variance investor has optimal portfolio weight $w = \frac{1}{\gamma} \Sigma^{-1} \mu$. The equity portfolio composition, however, is independent of the risk aversion coefficient γ , and given by

$$\bar{w} = \frac{w}{w' \iota} = \frac{\Sigma^{-1} \mu}{\iota' \Sigma^{-1} \mu},$$

with ι a vector of ones. Following French and Poterba [65], we calculate the implied expected excess returns which make a mean-variance investor hold the market equity portfolio. We take Σ to be the estimated covariance matrix of excess log returns of MSCI indices in the corresponding sample and calculate the implied expected excess log return that delivers the observed market equity portfolio weights \bar{w} . Table 3.1 compares the empirical mean excess log returns with the implied expected excess log returns for different sample periods. We normalize the implied expected excess log returns such that either the implied Japanese return (JA as reference) or the implied European return (EU as reference) is the same as its historical estimate. In the full sample estimate (Panel A), the implied expected excess log return for US is over 9 percentage points higher than its empirical value when the Japanese equity is used as reference, and is about 3 percentage points higher when the European equity is used as reference. If we exclude the turbulent period of the global financial crisis and terminate the sample at the end of 2006, we find the US implied return to be around 8 percentage points higher than the empirical counterpart when using the Japanese equity as reference, and 3 percentage points higher when using the European equity as reference. It is unlikely that the US equity can deliver such a high expected excess return consistently. In other words, according to classic portfolio choice models, the risk return trade off of the US equity is not good enough to attract as much investment as it does.

Empirical vs. implied expected excess log returns (% per annum)			
Panel A: Full sample			
	Empirical	Implied (JA as reference)	Implied (EU as reference)
US	4.6	13.8	7.4
JA	3.7	3.7	1.4
EU	4.8	9.0	4.8
Panel B: Until the end of 2006			
	Empirical	Implied (JA as reference)	Implied (EU as reference)
US	4.5	12.2	7.7
JA	4.5	4.5	2.5
EU	5.3	8.4	5.3

Table 3.1: Empirical and implied expected excess returns denoted in % per annum. Implied expected excess returns are computed based on the market values of MSCI US, Japan, and Europe, and are solutions of $\bar{w} = \frac{\Sigma^{-1}\mu}{\iota'\Sigma^{-1}\mu}$, with \bar{w} the observed market equity portfolio weights, ι a vector of ones, Σ the estimated covariance matrix of excess log returns of MSCI indices using daily returns. Empirical mean excess log returns are estimated using monthly MSCI total return indices over local risk-free rates, for which US 3 month treasury bill rates, Japan base discount rates, and 3 month Euribor rates are used as proxies. Both samples start in January 1972. Panel A reports the estimates using the full sample and Panel B reports the estimates using the sample truncated at the end of 2006.

Figure 3.1 and Table 3.1 show that the question of why an equity market remains larger (smaller) and more (less) attractive than others cannot be answered by differences in Sharpe ratios alone. Forbes [61] also finds that the amount of foreign investment in the US cannot be explained by standard portfolio allocation models and diversification motives, and puts forward the question:

“Why are foreigners willing to invest an average of well over 5 billion every day in the United States – especially given low returns relative to comparable investments in other countries...?” (p. 3)

It brings us to a phenomenon that is equally interesting as the home bias, and which we refer to as the “US bias”, i.e., the extent to which a global investment portfolio over-weights US equity compared to classic asset allocation models.

3.1.2 Contribution

Inspired by the empirical results from these strands of the literature, we propose a global asset return model that explicitly takes into account the leading feature of the US equity market. We model the lead-lag relation using asymmetric jump excitation, which allows a price plunge in the US to get reflected in future prices of foreign equities but much less the other way around.

Specifically, we model a contagious financial market with mutually exciting jumps to account for excess comovement during economic downturns led by the US market. Different from Lévy type models that are widely applied in the literature, mutually exciting jumps are both cross sectionally and serially dependent, meaning that a large price movement that

happens in the domestic market today increases the probability of experiencing further price jumps in the same market in the future as well as the probability of experiencing price jumps in other markets. There are two important indicators that measure the cross-sectional excitation capability of an equity market – how much a domestic price crash can get reflected in future foreign equity prices and how a foreign price crash can affect future domestic equity prices. The empirical evidence mentioned earlier suggests that the excitation is typically not symmetric. Consistent with the empirical findings, we allow for an asymmetric jump excitation structure. The leading role of the US equity market is characterized by having a large cross section excitor as the “source jump component” – the jump component that transmits risks – and a small cross section excitor as the “target jump component” – the jump component that receives transmission. The asymmetric excitation structure of the US equity market indicates on the one hand its pronounced transmission of domestic jump risks worldwide and on the other its limited susceptibility to foreign equity risk spillovers.

Apart from allowing for jump propagation, we deviate from the standard asset allocation literature in two respects. First, instead of using representative assets of every regional market, we assume that each local market is made up of a large number of individual assets, which are exposed to regional risk factors as well as idiosyncratic risks. While the regional risk factors are systematic and cannot be diversified away, idiosyncratic risks can be eliminated by holding a well-diversified portfolio. Second we adopt the factor investing perspective and focus on allocation to risk factors rather than assets. Inspired by Ang [7], who remarks that “factors are to assets what nutrients are to food; factor risks are the driving force behind risk premiums”, we derive optimal portfolio exposure to risk factors instead of optimal portfolio weights of each individual asset. In this way, thousands of assets reduce to only a few manageable risk factors.

These specifications allow us to solve in closed form the portfolio optimization problem with multiple regions and a large number of assets that are exposed to mutually exciting jump risks, systematic Brownian risks, and idiosyncratic Brownian risks. The optimal portfolio exploits the diversification benefits among independent risk factors, and at the same time exploits the hedging potential within the dependence structure among risk factors and state variables. As a result, the optimal portfolio in this high-dimensional contagious market: (1) is sufficiently diversified, in the sense that it consists of a large number of individual assets to diversify away idiosyncratic risks; and (2) is biased towards the US equity market as compared to classic portfolio predictions. Intuitively, the US bias arises because price jumps in the US are likely to get reflected in the world economy by raising jump intensities of the other markets. Since the US equity drives jump intensities more than other equity markets, the investor demands more US equity exposure in order to hedge against the uncertainty in the jump intensities of global equity markets.

Generally speaking, incorporating jump risks brings three effects to traditional asset allocation, where equities are assumed to be driven by Brownian motions alone. First, as discovered by Das and Uppal [42] and Ait-Sahalia et al. [4], the investment in risky assets is smaller for an investor who accounts for jumps. This is due to the fact that, when a jump occurs, wealth can drop significantly before the investor has a chance to adjust the portfolio as he/she would when faced with Brownian risks. As a result, the investor prefers a smaller leverage to stay on the safe side. Second, compared with the constant jump intensity case, jump excitation increases the demand for risky assets. When a jump occurs, the state variables that drive the world economy (jump intensities in this case) and the equity prices move in opposite directions. To reduce the uncertainty in the state variables, the investor should increase the exposure to risky assets in order to exploit the hedging potential in the

jump components. This effect is first seen in Liu, Longstaff, and Pan [92], who show in a univariate model, that jumps in volatility increase the optimal portfolio weight on the risky asset. However, the implications of stochastic jump intensities in a multivariate setting are not well explored, possibly due to the difficulty in formulating a flexible yet tractable model, which yields analytical solutions for the optimal asset allocation. We extend the existing non-Poissonian jump diffusion asset allocation literature to a multivariate setting using mutually exciting jumps. The multivariate model gives rise to the third effect, the US bias, meaning that, compared to the Merton mean-variance portfolio, an investor over-invests in the US market whose jump component is more capable of exciting the jump components in other markets but is less prone to be excited by the other jump components. We find that the US bias arises when jump excitation is asymmetric, in which case jump components have heterogeneous hedging potential against state variables. The investor thus tilts the portfolio towards the US equity market for a more effective hedging. Ignoring price discontinuities or the excitation nature of jumps results in substantial welfare losses. The first two effects are known in the literature, while the third effect is the main finding of this paper.

We apply our model to historical prices of MSCI US, Japan, and Europe. We estimate the parameters of our model using daily price data. We show that neither Poisson jumps nor self exciting jumps are able to reproduce the pattern of the US bias in the market portfolio. Only when jumps are mutually exciting with an asymmetric excitation matrix does the optimal portfolio exhibit the US bias. The portfolio prediction generated by the mutually exciting jump diffusion model closely resembles the risk profile of the market portfolio.

To our knowledge, an analytical characterization of the US bias using the lead-lag relationships in international returns cannot be easily replicated by other existing portfolio choice models in the literature. Sophisticated portfolio choice models that admit closed form solutions often times focus on univariate settings with a single stock in the market. Moreover, even in a truly multivariate framework, the asymmetric feature cannot be replicated by stochastic volatility nor regime-switching models (see Buraschi et al. [27]). The linear correlation, as a measure of dependence used in such models, is a symmetric and contemporaneous relation, which does not allow for a lead-lag or asymmetric relation between two equity markets. Ang and Bekaert [8] propose a regime-switching model to account for the fact that correlations between international equity returns are higher during bear markets than during bull markets. While regime-switching models are able to account for excess linear dependence during economic downturns, the dependence structure of international equities can be nonlinear and asymmetric. More importantly, Rapach et al. [104] show that US shocks are only fully reflected in non-US equity prices with a lag. Therefore the dependence structure of international equities goes beyond conditional linear correlations. The nonlinearity, asymmetry and lead-lag properties distinguish our asymmetric excitation model from stochastic volatility and regime-switching models. We believe that the mutually exciting jump diffusion model provides a natural, realistic and parsimonious way that gives rise to this effect.

The remainder of this chapter is organized as follows. Chapter 3.2 postulates a model of asset prices that generates lead-lag relations in international equity returns featuring mutually exciting jumps. We solve for the optimal portfolio using the martingale method in a market with a large number of individual assets. We show that the market is asymptotically complete, in the sense that the optimal portfolio path can be closely tracked by investing in a large basket of individual stocks to diversify away idiosyncratic risks. In Chapter 3.3, we study the property of the optimal exposure to jump risks using comparative statics analysis and show that the optimal portfolio exhibits the US bias. In Chapter 3.4, we quantify the

certainty equivalent loss in terms of annualized returns if the investor were to ignore jump excitation. Chapter 3.5 reports the calibration results and numerical findings. Chapter 3.6 concludes.

3.2 Optimal asset allocation in a contagious financial market

In this section, we propose a model of asset prices that generates lead-lag relations in international returns. This is achieved by extending a pure diffusion process of asset returns to include both cross sectionally and serially dependent jump components, namely, mutually exciting jumps. We specify a general contagious financial market with mutually exciting jumps in Chapter 3.2.1. In Chapter 3.2.2, we discuss the general features of the optimal portfolio weights in this market. We impose additional structure on equity risk premiums in Chapter 3.2.3, which enables us to solve the portfolio optimization problem in closed form using the martingale approach in Chapter 3.2.4. In Chapter 3.2.5, we discuss the completeness of the financial market.

3.2.1 A general model

We work in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ that satisfies the usual conditions. Let $N_t = (N_{1,t}, \dots, N_{n,t})'$ be mutually exciting jump processes with intensities $\lambda_{i,t}, i = 1, \dots, n$:

$$\begin{cases} \mathbb{P}[N_{i,t+s} - N_{i,t} = 0 | \mathcal{F}_t] = 1 - \lambda_{i,t}s + o(s); \\ \mathbb{P}[N_{i,t+s} - N_{i,t} = 1 | \mathcal{F}_t] = \lambda_{i,t}s + o(s); \end{cases} \quad (3.1)$$

and $\mathbb{P}[N_{i,t+s} - N_{i,t} > 1 | \mathcal{F}_t] = o(s), s > 0$, where the intensities follow the exponentially decaying dynamics

$$d\lambda_{i,t} = \alpha_i(\lambda_{i,\infty} - \lambda_{i,t})dt + \sum_{j=1}^n \beta_{i,j} dN_{j,t}, \quad \alpha_i, \beta_{i,j}, \lambda_{i,\infty} \geq 0, \quad i, j = 1, \dots, n. \quad (3.2)$$

The occurrence of a jump in component j at time t , i.e., $dN_{j,t} = 1$, not only raises the intensity of jump component j , $\lambda_{j,t}$, by a non-negative amount $\beta_{j,j}$, but also increases the intensities of other jump components, $\lambda_{i,t}, i \neq j$, by a non-negative amount $\beta_{i,j}$. After being excited, the intensity of each jump component $\lambda_{i,t}$ mean reverts to the steady state $\lambda_{i,\infty}$ at an exponentially decaying rate α_i , until it gets excited by a next jump occurrence.

In the remainder, we call β , defined as

$$\beta := (\beta_1, \dots, \beta_n) = \begin{pmatrix} \beta_{1,1} & \dots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \dots & \beta_{n,n} \end{pmatrix},$$

the *excitation matrix*; $\beta_{i,i}$ is called the *self-excitor* of jump component i ; $\beta_{i,j}$ is called the *cross section excitor* of jump component j , in which jump component j is called the *source jump component*, and the jump component i is called the *target jump component*.

The unconditional expectation of the jump intensity is given by $\mathbb{E}[\lambda_t] = (I - \beta / (\alpha'))^{-1} \lambda_\infty$. The intensity processes can be made stationary by imposing $(I - \beta / (\alpha'))^{-1} > 0$ [6]. Here,

I is an n by n identity matrix; α, λ_∞ are vectors of $\alpha_i, \lambda_{i,\infty}, i = 1, \dots, n$, respectively; $\mathbf{1}$ is a column vector of ones. We adopt the convention of denoting vectors and matrices using boldface characters to distinguish them from scalars. We use \circ to denote element-wise multiplication of matrices and \cdot to denote element-wise division. We use “;” for column breaks and “;” for row breaks in a matrix. The pair (N, λ) is jointly Markov.

Let there be a risk-free asset S_t^0 , generating an instantaneous risk free return r_t ,

$$S_t^0 = S_0^0 \exp\left(\int_0^t r_s ds\right), \quad S_0^0 > 0, \quad t \in [0, T].$$

We assume that each S_t^0 -deflated security price process is in the space \mathcal{H}^2 of adapted càdlàg (hence, progressively measurable) and square integrable semi-martingale processes $S_{i,t}, i = 1, \dots, m$, following

$$\begin{cases} \frac{dS_{i,t}}{S_{i,t-}} = \mu_{i,t} dt + \sum_{j=1}^m \sigma_{i,j,t} dW_{j,t} + \sum_{l=1}^n d_{i,l} z_{l,t} dN_{l,t}, \\ d\lambda_{i,t} = \alpha_i(\lambda_{i,\infty} - \lambda_{i,t}) dt + \sum_{j=1}^n \beta_{i,j} dN_{j,t}. \end{cases} \quad (3.3)$$

Here, $\mu_{i,t} > 0$ is the (state-dependent) excess return of asset i ; $\sigma_{i,j,t}$ is the (state-dependent) exposure of asset i to the Brownian risk $W_{j,t}$. $\mathbf{W}_t = (W_{1,t}, \dots, W_{m,t})'$ is a vector of standard and independent Brownian motions. We use $S_{i,t-}$ to denote the left-limit of $S_{i,t}$. The $(i, j)^{\text{th}}$ entry of the instantaneous covariance matrix Σ_t is given by

$$\Sigma_t[i, j] = \sum_{k=1}^m \sigma_{i,k,t} \sigma_{j,k,t}. \quad (3.4)$$

The exposure of asset i to jump component l is denoted by a constant $d_{i,l}$. The amplitude of jump component l is denoted by $z_{l,t}$, supposed to be i.i.d. random variables which determine the percentage change in the asset price caused by an occurrence in jump component N_l at time t . The jump amplitudes are assumed to be independent of all risk factors.

The mutually exciting jump diffusion model postulated in (3.3) is able to reproduce important stylized facts of asset returns. For example, the asset returns exhibit jump clustering as a result of the time series excitation, and systemic jumps as a result of the cross section excitation. The model generates lead-lag and asymmetric relations in international equity returns. Unlike dependence generated by (stochastic) covariance, which is simultaneous and symmetric in the sense that $\text{Cov}_t(X_1, X_2) = \text{Cov}_t(X_2, X_1)$, contagion allows for lagged dependence. The dependence structure can further be made asymmetric by setting $\beta_{i,j} \neq \beta_{j,i}$ to accommodate that some jump components have a larger potential to excite other jump components. Equities with these jump components tend to lead international equity returns, since a price plunge there can get reflected in future prices of other equities.

The model also generates excess comovement during market turmoil. During tranquil periods, international asset returns are correlated through the instantaneous covariance Σ_t . In periods of financial crises, initiated by the first few downside jumps, jump intensities build up and give rise to clustered subsequent jumps in the initial market as well as in other markets across the world, creating nonlinear excess tail dependence in economic downturns.

3.2.2 Optimal asset allocation

We consider an expected utility investor with power utility $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}, \gamma > 0$. The investor is given a non-stochastic initial endowment $x_0 > 0$ to invest in the risk-free and risky

assets. The investor neither consumes nor receives any intermediate income. Assume that the investor can rebalance the portfolio in continuous time without incurring any transaction costs. The objective is to maximize the expected utility over terminal wealth X_T through optimal continuous time trading. Denote the portfolio weights (percentage of wealth) of the risky assets at time t by $\mathbf{w}_t = (w_{1,t}; \dots; w_{m,t})$, $0 \leq t \leq T$, assumed to be adapted càdlàg processes, bounded in \mathcal{L}^2 .³ We do not impose leverage restrictions, so the position on the risk-free asset at time t , given by $w_{0,t} = 1 - \sum_{i=1}^m w_{i,t}$, can be a negative amount. The S_t^0 -deflated wealth process X_t is self-financing:

$$\begin{aligned} \frac{dX_t}{X_{t-}} &= \sum_{i=1}^m w_{i,t} \frac{dS_{i,t}}{S_{i,t-}} \\ &= \mathbf{w}_t' \boldsymbol{\mu}_t dt + \mathbf{w}_t' \boldsymbol{\Sigma}_t^{1/2} d\mathbf{W}_t + \sum_{l=1}^n \mathbf{w}_t' \mathbf{d}_l z_{l,t} dN_{l,t}. \end{aligned} \quad (3.5)$$

Here, $\boldsymbol{\mu}_t$, \mathbf{d}_l are vectors containing $\mu_{i,t}$ and $d_{i,l}$ introduced in Equation (3.3).

The asset allocation problem is formulated as

$$\sup_{\{\mathbf{w}_t, 0 \leq t \leq T\}} \mathbb{E}[u(X_T) | \mathcal{F}_0]. \quad (3.6)$$

To solve the asset allocation problem, we initially employ stochastic control theory, following Merton. Define the indirect utility function J at time t as

$$J(t, x, \boldsymbol{\lambda}) = \sup_{\{\mathbf{w}_s, t \leq s \leq T\}} \mathbb{E}_t \left[\frac{X_T^{1-\gamma}}{1-\gamma} \right],$$

where the expectation is conditional on the information available at time t .⁴ Then, employing dynamic programming and the appropriate version of Itô's Lemma, the Hamilton-Jacobi-Bellman (HJB) equation reads (we omit the arguments $t, x, \boldsymbol{\lambda}$ of the function J when no confusion is caused)

$$\begin{aligned} 0 = \sup_{\{\mathbf{w}_s, t \leq s \leq T\}} & \left\{ J_t + \mathbf{w}_t' \boldsymbol{\mu}_t J_x x + \frac{1}{2} \mathbf{w}_t' \boldsymbol{\Sigma}_t \mathbf{w}_t J_{xx} x^2 + \sum_{l=1}^n \alpha_l (\lambda_{l,\infty} - \lambda_l) J_{\lambda_l} \right. \\ & \left. + \sum_{l=1}^n \lambda_l \mathbb{E}[J(t, x(1 + \mathbf{w}_t' \mathbf{d}_l z_{l,t}), \boldsymbol{\lambda} + \boldsymbol{\beta}_l) - J] \right\}, \end{aligned} \quad (3.7)$$

where we use J_t, J_x, J_{λ_l} to denote the partial derivatives of $J(t, x, \boldsymbol{\lambda})$ with respect to t, x, λ_l and similarly for the higher order derivatives. The expectation is taken over the jump amplitude distribution of $z_{l,t}$. $\boldsymbol{\beta}_l$ denotes the excitor vector when N_l is the source jump component, which is the l^{th} column of the excitation matrix, $\boldsymbol{\beta}_l = (\beta_{1l}; \dots; \beta_{nl})$.

It is known that the indirect utility function for a power utility investor can be written as

$$J(t, x, \boldsymbol{\lambda}) = \frac{x^{1-\gamma}}{1-\gamma} f(t, \boldsymbol{\lambda}),$$

³Since portfolio weights cannot anticipate jumps, they are \mathcal{F}_{t-} measurable and left continuous (cf. Aït-Sahalia and Hurd [3]).

⁴We sometimes denote the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_t]$ as $\mathbb{E}_t[\cdot]$. We use the two notations interchangeably.

where $f(t, \lambda)$ is a deterministic function of time t and the value of the state variables λ .⁵ We substitute for this functional form in the HJB Equation (3.7), and solve the first order condition with respect to w_t to get the following implicit function that characterizes the optimal portfolio weights w_t^* . For $0 \leq t \leq T$, the optimal portfolio weights w_t^* solve

$$\mu_t - \gamma \Sigma_t w_t^* + \sum_{l=1}^n \frac{f(t, \lambda + \beta_l)}{f} \lambda_l \mathbb{E}[(1 + w_t^{*'} d_l z_{t,l})^{-\gamma} d_l z_{t,l}] = 0, \quad (3.8)$$

with $f(t, \lambda)$ satisfying

$$\begin{aligned} 0 = & f_t + (1 - \gamma) w_t^{*'} \mu_t f - \frac{1}{2} \gamma (1 - \gamma) w_t^{*'} \Sigma_t w_t^* f + \sum_{l=1}^n \alpha_l (\lambda_{l,\infty} - \lambda_l) f_{\lambda_l} \\ & + \sum_{l=1}^n \lambda_l \mathbb{E}[(1 + w_t^{*'} d_l z_{t,l})^{1-\gamma} f(t, \lambda + \beta_l) - f]. \end{aligned} \quad (3.9)$$

One can easily verify that the pair $(w_t^*, f(t, \lambda))$ jointly determined by Equations (3.8) and (3.9) satisfies the HJB Equation (3.7).

The optimal portfolio weights derived from Equation (3.8) can be decomposed into the following components:

$$w_t^* = \underbrace{\frac{1}{\gamma} \Sigma_t^{-1} \mu_t}_{(I)} + \frac{1}{\gamma} \Sigma_t^{-1} \left(\underbrace{\sum_{l=1}^n \lambda_l M_{l,t}}_{(II)} + \underbrace{\sum_{l=1}^n \lambda_l \frac{f(t, \lambda + \beta_l) - f}{f} M_{l,t}}_{(III)} \right), \quad (3.10)$$

where

$$M_{l,t} := \mathbb{E}[(1 + w_t^{*'} d_l z_{t,l})^{-\gamma} d_l z_{t,l}].$$

The optimal portfolio weights consist of a mean-variance demand (I), a myopic buy-and-hold demand (II), and an intertemporal hedging demand (III). The mean-variance demand (I) is given by the mean-variance weights, exploiting diversification benefits of the instantaneous covariance structure.

The myopic buy-and-hold demand (II) arises because the asset prices have discontinuities. As explained by Liu et al. [92], unlike continuous fluctuations, jumps may occur before the investor has the opportunity to adjust the portfolio. Jump risks, therefore, are similar to “illiquidity risk”: the investor has to hold the asset until the jump has occurred. Observe that

$$M_{l,t} \propto \nabla_{w_t} \mathbb{E}[u(X_t) - u(X_{t-}) | N_{l,t} - N_{l,t-} = 1].$$

⁵Since

$$J(t, cx, \lambda) = \sup \mathbb{E}_{t,cx,\lambda} \left[\frac{(cX_T)^{1-\gamma}}{1-\gamma} \right] = c^{1-\gamma} \sup \mathbb{E}_{t,x,\lambda} \left[\frac{X_T^{1-\gamma}}{1-\gamma} \right] = c^{1-\gamma} J(t, x, \lambda),$$

we conclude that the value function is homogeneous of degree $1 - \gamma$ in the wealth level. Let $c = \frac{1}{x}$. It holds that

$$J(t, 1, \lambda) = x^{-(1-\gamma)} J(t, x, \lambda).$$

Rearrange and get

$$J(t, x, \lambda) = \frac{x^{1-\gamma}}{1-\gamma} f(t, \lambda),$$

where

$$f(t, \lambda) = (1 - \gamma) J(t, 1, \lambda),$$

with terminal condition $f(T, \lambda) = 1$.

$\mathbb{E}[u(X_t) - u(X_{t-}) | N_{l,t} - N_{l,t-} = 1]$ is the expected utility gain at time t conditional on an occurrence in jump component l at time t . Therefore (II) is the expected marginal utility increase induced by jump component l from investing in one unit of risky assets at time t . The buy-and-hold demand is “myopic” in the sense that it does not take into account the uncertainties of the future jump intensities.

The last term (III) is tailored to account for the fact that the jumps are mutually exciting. Since the asset prices S_t and the state variables λ_t are both driven by jumps N_t , the risky assets can be used to hedge future realizations of the state variables. Intuitively, the mean-variance demand and the myopic buy-and-hold demand exploit the risk-return trade-off of the risky assets, whereas the intertemporal hedging demand is only concerned with state variable uncertainties.

All three components of the portfolio weights can be time-varying, but for different reasons. The mean-variance demand (I) and myopic buy-and-hold demand (II) depend on the spot values of the asset return parameters. Hence they change with the spot values instantaneously. The intertemporal hedging demand, on the other hand, depends not only on the spot values of the asset return parameters, but also on how the returns and the state variables evolve over the investment horizon. The information of future outcomes is contained in $f(\cdot)$, which is horizon dependent.

Remark 3.1. *In principle, one can allow the state variable λ_t , similar to equities, to follow a jump diffusion process*

$$d\lambda_t = y_0(\lambda_t) dt + y_1(\lambda_t) dW_t + y_1^\circ(\lambda_t) dW_t^\circ + y_2 dN_t, \quad (3.11)$$

with appropriate regularity conditions on y_0 , y_1 , y_1° , and y_2 . Here, W_t° is a $m \times 1$ vector of Brownian motions that are independent of W_t ; $y_0(\lambda_t)$ is an $n \times 1$ vector of drift terms; $y_1(\lambda_t)$ and $y_1^\circ(\lambda_t)$ are both $n \times m$ matrices; y_2 is an $n \times n$ matrix of constants. The optimal portfolio weights in (3.10) would then include a fourth volatility hedging component (IV), $\frac{1}{\gamma} y_1' \frac{\nabla \lambda_t^f}{f}$, in order to use the risky assets to hedge the common Brownian risks W_t in the state variables. In this case, the model nests the stochastic volatility model of Liu [90], the Poisson jump diffusion model in Das and Uppal [42] and Ait-Sahalia et al. [4], the contagion model in Ait-Sahalia and Hurd [3], and the univariate double jump model in Liu et al. [92] and Branger et al. [21]. Although jump-diffusion-driven state variables will in principle not create additional difficulties in the technical analysis, we focus on the more parsimonious mutually exciting jump diffusion model for simplicity. In case of mutually exciting jumps, it holds that

$$y_0(\lambda_t) = \alpha \circ (\lambda_\infty - \lambda_t), \quad y_1(\lambda_t) = y_1^\circ(\lambda_t) = 0, \quad y_2 = \beta.$$

The purpose of this paper is to evaluate the impact of excitation asymmetry on the optimal portfolio choice, rather than to develop a general multivariate jump diffusion model to nest existing models in the literature.

3.2.3 Proportional risk premium

In general, the function $f(\cdot)$ in Equation (3.8) does not admit an analytical expression. In order to fully solve the asset allocation problem, we impose some additional structure, in particular on the equity risk premium. Inspired by Ait-Sahalia, Cacho-Diaz, and Laeven [6], we propose a parsimonious model to focus on jump propagation through time and across different geographic markets. Let there be n regions with m_i assets in region i . Let N_1, \dots, N_n represent regional jump components to capture large price drops in equity indices.

Assume that each asset is only exposed to the jump risk of its own region but not to those of the other regions. Equivalently, the jump exposure $d_{i,l}$ in Equation (3.3) takes the form

$$d_{i,l} = \begin{cases} 1, & \text{if } i = l, \\ 0, & \text{if } i \neq l. \end{cases}$$

Even though jump components in “peripheral” markets do not influence “domestic” asset prices directly, jump risks are systemic in the sense that they mutually excite. The jump intensities λ_t follow

$$d\lambda_{i,t} = \alpha(\lambda_{i,\infty} - \lambda_{i,t}) dt + \sum_{j=1}^n \beta_{i,j} dN_{j,t}, \quad (3.12)$$

where for simplicity we assume that all jump intensities share the same mean-reversion rate α , as in Aït-Sahalia et al. [6].

We model the “normal” (day-to-day) covariances among regions by *correlated* Brownian motions $\hat{\mathbf{W}}_t = (\hat{W}_{1,t}, \dots, \hat{W}_{n,t})'$ given by

$$\hat{\mathbf{W}}_t = \mathbf{L}\mathbf{W}_t.$$

Here, $\mathbf{L}\mathbf{L}'$ is a correlation matrix with ones on the diagonal and correlation coefficients off-diagonal. Besides systematic Brownian risks, assets are also subject to idiosyncratic fluctuations, captured by standard and independent Brownian motions $\mathbf{Z}_t = (Z_{i,t}^k)$, $i = 1, \dots, n$, $k = 1, \dots, m_i$, which are independent of the “regional” Brownian risks $\hat{W}_{i,t}$, $i = 1, \dots, n$.

Following, among others, French and Poterba [65], we further assume that the representative investor hedges 100% of the exchange rate risk using, say, forward exchange rate contracts. The hedged return is given by

$$R_t^{\text{hedged}} = r_t + (R_t^{\text{local}} - r_t^{\text{local}}),$$

where r_t is the risk-free rate of a reference country and r_t^{local} is the local risk-free rate. In other words, the hedged excess log returns are computed as local returns denominated in local currency over local risk-free rates. Taking as numeraire the risk-free asset $S_t^0 = S_0^0 \exp\left(\int_0^t r_s ds\right)$ from a reference currency, we normalize all price processes as S_t^0 -deflated prices henceforth. According to the Numeraire Invariance Theorem (see, for example, Duffie [48]), such normalization places essentially no economic effects.

To focus on the effect of jump propagation on the optimal asset allocation, we assume, for simplicity, that the expected return and volatility of equity prices are state independent. Denote individual asset identities by superscripts and denote region identities by subscripts. We suppose that the *currency-hedged* deflated price of a risky asset k from region i is in the space $\mathcal{S}^2 \in \mathcal{H}^2$ containing $S_{i,t}^k$, $i = 1, \dots, n$, $k = 1, \dots, m_i$, $t \in [0, T]$, which follows

$$\frac{dS_{i,t}^k}{S_{i,t}^k} = \nu_i^k dZ_{i,t}^k + \sigma_i^k (d\hat{W}_{i,t} + \eta_i dt) + z_i^k (dN_{i,t} - (1 + \kappa_i)\lambda_{i,t} dt), \quad (3.13)$$

with constant $\sigma_i^k, \eta_i, \nu_i^k, \kappa_i \geq 0$ for all i, k . Within a given region, the price of any individual asset k is driven by both region specific systematic risks, $\hat{W}_{i,t}, N_{i,t}$, as well as idiosyncratic risks, $Z_{i,t}^k$.

Following Cox and Ross [40] and Liu and Pan [91], we assume that the jump amplitudes z_i^k , $i = 1, \dots, n$, $k = 1, \dots, m_i$, are constant. In addition, we restrict that $-1 < z_i^k \leq 0$ to rule out probability of ruin and to indicate that jumps are unfavorable events. Conditioning on an occurrence in jump component N_i , each asset k in region i drops by a deterministic amount. This assumption simplifies the analysis, allowing us to focus on the impact of adverse rare events and the contagious nature of such events.

Comparing Equation (3.13) with Equation (3.3), we see that the drift term of asset k from region i is a linear function of the state variable $\lambda_{i,t}$,

$$\mu_{i,t}^k = \sigma_i^k \eta_i - z_i^k (1 + \kappa_i) \lambda_{i,t}.$$

The covariance matrix Σ is constant over time and has the structure

$$\Sigma_t[p, p] = (\sigma_i^p)^2 + (\nu_i^p)^2, \quad \Sigma_t[p, q] = \rho_{i,j} \sigma_i^p \sigma_j^p,$$

where i, j are the regional markets to which asset p and asset q belongs, respectively, and $\rho_{i,j}$ is the $[i, j]^{\text{th}}$ entry of the correlation matrix $\mathbf{L}\mathbf{L}'$.

Equation (3.13) specifies dynamics under the physical measure P . If the market is free of arbitrage, there exists an equivalent martingale measure Q , under which the expected excess return of any S_t^0 -deflated risky asset is zero, i.e., $\mathbb{E}^Q \left[\frac{dS_{i,t}^k}{S_{i,t}^k} \right] = 0$. We start with specifying a pricing kernel process that uniquely prices the three sources of risks: the idiosyncratic Brownian risks, the systematic Brownian risks, and the jump risks. Next we show in Chapter 3.2.5 that the market is complete in the sense that any random payoff that is consistent with this pricing kernel can be replicated by investing in the available assets in the market.

Define the systematic Brownian risk premium vector $\boldsymbol{\eta} = (\eta_1; \dots; \eta_n)$ and the jump risk premium vector $\boldsymbol{\kappa} = (\kappa_1; \dots; \kappa_n)$. Following Liu and Pan [91], consider a pricing kernel process π_t given by

$$\frac{d\pi_t}{\pi_{t-}} = -\boldsymbol{\eta}'(\mathbf{L}\mathbf{L}')^{-1} d\hat{\mathbf{W}}_t + \sum_{i=1}^n \kappa_i (dN_{i,t} - \lambda_{i,t} dt), \quad \pi_0 = 1. \quad (3.14)$$

It is clear from Equation (3.14) that π is a *local* martingale. If π is actually a martingale, one can verify according to the Lenglart-Girsanov Theorem that π can serve as a Radon-Nikodym derivative that changes the measure P to a risk neutral measure Q , under which asset prices evolve according to

$$\frac{dS_{i,t}^k}{S_{i,t}^k} = \nu_i^k dZ_{i,t}^{k,Q} + \sigma_i^k d\hat{W}_{i,t}^Q + z_i^k (dN_{i,t}^Q - (1 + \kappa_i) \lambda_{i,t} dt), \quad (3.15)$$

where $\hat{W}_{i,t}^Q, Z_{i,t}^{k,Q}$ are standard Brownian motions under Q . The jump process $N_{i,t}^Q$ has intensity $(1 + \kappa_i) \lambda_{i,t}$ under Q , while the jump amplitude z_i^k remains unchanged. Consequently, $S_{i,t}^k$ is a local martingale under risk measure Q .

The asset dynamics under P (Equation (3.13)) and those under Q (Equation (3.15)) reveal how the three types of risks are priced. First, similar to Merton [97], the idiosyncratic risk \mathbf{Z}_t is assumed to be perfectly diversifiable. As a result, the market portfolio is free of idiosyncratic Brownian risks and the market price of idiosyncratic Brownian risk \mathbf{Z}_t is zero. Only systematic risks are priced.

Second, note that the Brownian risk $\Delta \hat{W}_{i,t}$ has constant variance Δ , while the jump risk $\Delta N_{i,t}$ has variance $\lambda_{i,t} \Delta$ (approximately). Loosely speaking, we are assuming that the risk

premium is proportional to the “risk” of the risk factors – the Brownian risk is compensated with $\eta_i \Delta$ and the jump risk is compensated with $\kappa_i \lambda_{i,t} \Delta$. A similar jump risk premium specification can be found in Pan [100], Liu et al. [92], and Boswijk et al. [18]. It implies that the expected stock returns are increasing in the jump intensities λ_t . Intuitively, this risk premium is sensible. During recessions, when there is a high probability of experiencing large price drops, the investor is compensated by a better risk-return tradeoff. This jump premia specification is consistent with the empirical estimation results in Bollerslev and Todorov [16], who show that most peaks in the equity jump risk premia are associated with events that mark the market turmoil, and also with Santa-Clara and Yan [106], who find that the equilibrium equity risk premium is a function of the jump intensity. Furthermore, in our model, the jump intensities under the risk neutral measure are larger than the physical jump frequencies, an empirically relevant fact that has been confirmed in the non-parametric estimation by Bollerslev and Todorov [16].

3.2.4 Optimal portfolio exposure to risk factors

We present the optimal portfolio results in this section from the perspective of a world representative investor. In addition, different from the traditional portfolio choice literature that solves for the optimal portfolio weights, we formulate the problem in terms of portfolio exposure to risk factors. Essentially, this places no effects on the optimal wealth path, since assets generate excess returns because of their underlying exposures to systematic risk factors (see Ang [7]). It turns out to be a more convenient approach in this case where there are a few systematic risk factors but numerous assets that are bundles of such factors.

Problem 3.1. *Suppose there are no arbitrage opportunities in the market introduced in Chapter 3.2.3 and all assets are priced according to the pricing kernel given by Equation (3.14). Let θ_t^Z be a $\left(\sum_i^n m_j\right) \times 1$ vector process, and $\theta_t^W, \theta_t^N = (\theta_t^{N_i}), i = 1, \dots, n$, be $n \times 1$ vector processes, which are adapted, càdlàg, and bounded in \mathcal{L}^2 . Define the portfolio optimization problem for an expected utility investor with power utility, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \gamma > 0$ as*

$$\sup_{\{\theta_t^Z, \theta_t^W, \theta_t^N\}} \mathbb{E}_0 \left[\frac{X_T^{1-\gamma}}{1-\gamma} \right], \quad (3.16)$$

subject to the budget constraint:

$$\frac{dX_t}{X_t} = \theta_t^{Z'} dZ_t + \theta_t^{W'} (d\hat{W}_t + \eta dt) + \sum_{i=1}^n (\exp(\theta_t^{N_i}) - 1) (dN_{i,t} - (1 + \kappa_i) \lambda_{i,t} dt). \quad (3.17)$$

We invoke the martingale method developed by Cox and Huang [39] to solve for the optimal portfolio exposure to risk factors. The main results are stated in the following proposition.

Proposition 3.1 (Optimal portfolio exposure to risk factors). *Consider Problem 3.1. The optimal portfolio exposure to risk factors is given by*

$$\begin{cases} \theta_t^{Z*} =: \theta^{Z*} = 0, \\ \theta_t^{W*} =: \theta^{W*} = \frac{1}{\gamma} (LL')^{-1} \eta, \\ \theta_t^{N_i*} = -\frac{1}{\gamma} \log(1 + \kappa_i) + \beta_i' B(t), \end{cases} \quad (3.18)$$

and the indirect utility function at $t = 0$ is given by

$$J(0, x_0, \boldsymbol{\lambda}_0) = \frac{x_0^{1-\gamma}}{1-\gamma} \exp(\gamma(A(0) + \mathbf{B}(0)' \boldsymbol{\lambda}_0)). \quad (3.19)$$

Here, $A(t)$ and $\mathbf{B}(t)$ satisfy

$$\begin{cases} \dot{\mathbf{B}}(t) = \frac{\gamma-1}{\gamma} \boldsymbol{\kappa} + \alpha \mathbf{B}(t) - (\boldsymbol{\kappa} + 1)^{\frac{\gamma-1}{\gamma}} \circ e^{\beta' \mathbf{B}(t)} + 1, \\ \dot{A}(t) = \frac{\gamma-1}{2\gamma} \boldsymbol{\eta}'(\mathbf{L}\mathbf{L}')^{-1} \boldsymbol{\eta} - \alpha \mathbf{B}'(t) \boldsymbol{\lambda}_\infty, \end{cases} \quad (3.20)$$

with $A(T) = 0$ and $\mathbf{B}(T) = \mathbf{0}$.

Notice that the optimal exposure to Brownian risks (both idiosyncratic and systematic) is time-independent and that the optimal exposure to jump risks is a continuous deterministic process. Therefore the optimal portfolio exposure to all risk factors satisfies the càdlàg assumption.

Alternatively, we may also use the stochastic control method outlined in Chapter 3.2.2 with risk exposure $(\boldsymbol{\theta}_t^Z, \boldsymbol{\theta}_t^W, \boldsymbol{\theta}_t^N)$ as control variables. The HJB equation is given by

$$\begin{aligned} 0 = \sup_{\{\boldsymbol{\theta}_t^Z, \boldsymbol{\theta}_t^W, \boldsymbol{\theta}_t^N\}} & \left\{ J_t + \left(\boldsymbol{\theta}_t^{W'} \boldsymbol{\eta} - \sum_{i=1}^n (\exp(\theta_t^{N_i}) - 1)(1 + \kappa_i) \lambda_{i,t} \right) J_x x \right. \\ & + \frac{1}{2} (\boldsymbol{\theta}_t^{W'} \mathbf{L}\mathbf{L}' \boldsymbol{\theta}_t^W + \boldsymbol{\theta}_t^{Z'} \boldsymbol{\theta}_t^Z) J_{xx} x^2 + \sum_{l=1}^n \alpha(\lambda_{l,\infty} - \lambda_l) J_{\lambda_l} \\ & \left. + \sum_{l=1}^n \lambda_l \left(J(t, x \exp(\theta_t^{N_l}), \boldsymbol{\lambda} + \beta_l) - J \right) \right\}. \end{aligned} \quad (3.21)$$

From Proposition 3.1, we already know that the indirect utility is exponentially affine in jump intensities. Let $J(t, x, \boldsymbol{\lambda}) = \frac{x^{1-\gamma}}{1-\gamma} \exp(A(t) + \mathbf{B}(t)' \boldsymbol{\lambda}_t)$. Plugging it into the HJB equation and taking first order conditions with respect to $\boldsymbol{\theta}_t^Z, \boldsymbol{\theta}_t^W, \boldsymbol{\theta}_t^{N_i}$, respectively, one can easily show that the optimal risk exposure coincides with $\boldsymbol{\theta}^{Z*}, \boldsymbol{\theta}^{W*}, \boldsymbol{\theta}_t^{N_i*}$ given by Equation (3.18).

We next show that the optimal exposure to Brownian risks $\boldsymbol{\theta}^{W*}$ given in Proposition 3.1 corresponds to the mean-variance demand. In particular, we show that the Merton mean-variance portfolio has exposure $\boldsymbol{\theta}^{W*}$ to systematic Brownian risks. Lemma 3.1 establishes the connection between the optimal portfolio exposure to Brownian risks $\boldsymbol{\theta}^{W*}$ and the Merton mean-variance portfolio.

Lemma 3.1 (The Merton mean-variance portfolio). *Let there be regional representative assets which are free of idiosyncratic risks and jump risks, i.e., $m_i = 1$, $\nu_i^k = z_i^k = 0, \forall k, i$. The representative assets therefore follow*

$$\frac{dS_{i,t}}{S_{i,t}} = \sigma_i (d\hat{W}_{i,t} + \eta_i dt), \quad i = 1, \dots, n.$$

Then, the optimal portfolio weights $\mathbf{w}_{\text{Merton}}^*$ of the Merton mean-variance portfolio are given by

$$\mathbf{w}_{\text{Merton}}^* = \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu},$$

with $\boldsymbol{\Sigma}$ the covariance matrix given by $\boldsymbol{\Sigma} = \boldsymbol{\sigma} \mathbf{L}\mathbf{L}' \boldsymbol{\sigma}'$, and $\boldsymbol{\mu}$ the expected excess return vector given by $\boldsymbol{\mu} = \boldsymbol{\sigma} \boldsymbol{\eta}$, where $\boldsymbol{\sigma}$ is a diagonal matrix with σ_i on the diagonals. In addition,

the optimal exposure to systematic Brownian risks of the Merton mean-variance portfolio is given by

$$\theta_{Merton}^{W*} := \sigma' w_{Merton}^* = \theta^{W*} = \frac{1}{\gamma} (LL')^{-1} \eta.$$

There are three features of the optimal portfolio exposure we would like to point out. First, the exposure to risk factors at time t , $(\theta^{Z*}, \theta^{W*}, \theta_t^{N*})$, is independent of the wealth level x_t and the realization of the state variables λ_t . Even if mutually exciting jumps give rise to stochastic investment opportunities, under the assumption of proportional risk premia, the optimal portfolio composition does not vary with the investor's wealth or realizations of the state variables that indicate economic cycles. In other words, there is no market timing of the portfolio strategy. The independence of the wealth level is a result of the wealth homogeneity property of the power utility. The optimal risk exposure being independent of the realization of the state variables λ_t stems from the assumption that the jump risk premium $\kappa_i \lambda_{i,t}$ is a multiple of $\lambda_{i,t}$. (Proportional risk premia are a common assumption in the literature.) In general, one may expect that when the current jump intensities λ_t are high, the optimal portfolio exposure to jump risks should be low to stay away from the high probability of a price plunge. In our model, the investor is rewarded proportionally to jump intensities. When the probability of jump occurrences is high, the risk premium is also high to the extent that the demand for the jump risk is independent of the jump intensity.

The second property of the optimal risk exposure is that, although state independent, the optimal jump risk exposure is horizon dependent. In the special case where $\beta = 0$, the jumps are not mutually exciting and the investment opportunities are constant. In this case, there are no hedging incentives, hence no horizon dependence in the jump risk exposure. When $\beta \neq 0$, the investment opportunities are stochastic, giving rise to incentives to hedge against changes in the investment opportunities. Observe that $B(t)$ in Equation (3.19) measures the sensitivity of the log indirect utility function to the values of the state variables, i.e.,

$$B(t) = \frac{1}{\gamma} \nabla_{\lambda_t} \log J(t, x_t, \lambda_t).$$

The longer the investment horizon, the further away $B(t)$ is from zero, implying a larger impact of state variables on indirect utility, which in turn leads to a stronger motivation for the investor to hedge against the changes in the state variables.

The third property of the optimal portfolio is that the optimal portfolio has no exposure to idiosyncratic risks, i.e., $\theta^{Z*} = 0$. Naturally, since the exposure to idiosyncratic risks are not compensated by any risk premium, the investor stays away from these risk factors. In practice, it means that the investor should invest in a large basket of assets in every region to diversify away the idiosyncratic risks as much as possible.

3.2.5 Market completeness

In this section, we discuss the completeness property of the financial market. We show that the market is asymptotically complete in the sense that the investor is able to construct the optimal portfolio given in Proposition 3.1 by investing in a large number of assets in each region. We start with a simplified setting where there are as many assets as risk factors and no idiosyncratic risks.

Suppose for now that the idiosyncratic risks Z_t are absent. Each region introduces two risk factors – a systematic Brownian motion $\tilde{W}_{i,t}$ and a jump component $N_{i,t}$. The following lemma states that as long as there are two investable assets (which are not linearly dependent)

from each region, we have a complete market in the sense that any no-arbitrage payoff path in the space \mathcal{S}^2 defined above can be replicated. Denote the S_t^0 -deflated value of the replicating portfolio at time t by P_t with weight $w_{i,t}^k$ on asset k of region i . It holds that

$$P_t = \sum_{i=1}^n \sum_{k=1}^{m_i} w_{i,t}^k S_{i,t}^k, \quad t \in [0, T].$$

The following lemma states the complete market result.

Lemma 3.2 (Market completeness). *Suppose the market is free of arbitrage opportunities, with the pricing kernel given by Equation (3.14). Let there be two non-redundant assets (not linearly dependent) from each region, consistently priced by the pricing kernel, which follow*

$$\begin{aligned} \frac{dS_{i,t}^k}{S_{i,t}^{k-}} &= \nu_i^k dZ_{i,t}^k + \sigma_i^k (d\hat{W}_{i,t} + \eta_i dt) + z_i^k (dN_{i,t} - (1 + \kappa_i)\lambda_{i,t} dt), \\ k &= 1, 2, i = 1, \dots, n, \end{aligned}$$

with $\nu_i^k \equiv 0, \forall k, i$. For any payoff $\{F_t\} \in \mathcal{S}^2$ which follows

$$\frac{dF_t}{F_{t-}} = \sum_{i=1}^n g_i (d\hat{W}_{i,t} + \eta_i dt) + \sum_{i=1}^n h_i (dN_{i,t} - (1 + \kappa_i)\lambda_{i,t} dt), \quad (3.22)$$

there exists a $2n \times 1$ vector \mathbf{w}_t containing portfolio weights, with $w_{i,t}^k$ being the weight on asset k of region i , such that the resulting portfolio value P_t is equal to F_t almost surely, i.e.,

$$P_t = F_t, \quad \text{a.s. } \forall t.$$

When assets are exposed to idiosyncratic risks, however, we need more than two assets in each region so as to diversify away idiosyncratic risks. Let $\mathbf{m} = (m_1, \dots, m_n)$ be a vector containing m_i as the number of available assets in region i . In fact, a similar result as in Lemma 3.2 holds when the number of assets in each region goes to infinity. The next proposition formalizes this result.

Proposition 3.2 (Asymptotic completeness). *Let there be m_i non-redundant assets (i.e., not linearly dependent) in region i following:*

$$\frac{dS_{i,t}^k}{S_{i,t}^{k-}} = \nu_i^k dZ_{i,t}^k + \sigma_i^k (d\hat{W}_{i,t} + \eta_i dt) + z_i^k (dN_{i,t} - (1 + \kappa_i)\lambda_{i,t} dt),$$

with $\nu_i^k \geq 0, \forall k, i$. For any payoff $\{F_t\} \in \mathcal{S}^2$ which follows (3.22), there exist portfolio weights $w_{i,t}^k, k = 1, \dots, m_i, i = 1, \dots, n$, such that for any $0 \leq t \leq T$, the replicating portfolio $P_t(\mathbf{m})$ satisfies

$$P_t(\mathbf{m}) \rightarrow F_t,$$

with probability one, as the numbers of assets $m_i, i = 1, \dots, n$, all go to infinity. As a result, there exist portfolio weights $w_{i,t}^k, k = 1, \dots, m_i, i = 1, \dots, n$, such that for any $0 \leq t \leq T$,

$$P_t(\mathbf{m}) \rightarrow X_t^*,$$

with probability one, as the numbers of assets $m_i, i = 1, \dots, n$, all go to infinity.

Proposition 3.2 shows that the investor is indeed able to construct the optimal portfolio by investing in a large number of assets in each region. Appendix 3.B gives one explicit example of how this can be done.

3.3 Analysis of the optimal portfolio exposure to jump risks and the effect of excitation asymmetry

In Chapter 3.2.2 we show that the optimal portfolio weights consist of a mean-variance demand, a myopic buy-and-hold demand and an intertemporal hedging demand. In Lemma 3.1, we have seen that the optimal Brownian risk exposure θ^{W*} corresponds to the Merton mean-variance demand. In this section, we analyze the properties of the optimal portfolio exposure to jump risks. In Chapter 3.3.1, we decompose the jump risk exposure θ_t^{N*} into a Poisson jump risk exposure and a contagion risk exposure. In Chapter 3.3.2, we conduct comparative statics analysis of the contagion risk exposure with respect to jump risk parameters. In Chapter 3.3.3, we study the effect of excitation asymmetry on portfolio exposure to jump risks. We show that the optimal portfolio can be biased towards an equity market when the excitation structure is asymmetric.

3.3.1 Decomposition of exposure to jump risks

In this section we show that the jump risk exposure θ_t^{N*} can be decomposed into a Poisson jump risk exposure θ^J which corresponds to the myopic buy-and-hold demand, and a contagion risk exposure θ_t^C which corresponds to the intertemporal hedging demand.

Note that the exposure to a jump component $\theta_t^{N_i}$ is equal to $\log(1 + w_{i,t}z_i)$, where $w_{i,t}$ is the portfolio weight, and $z_i < 0$ is the jump amplitude. If the investor longs the asset, i.e., $w_{i,t} > 0$, then it holds that $\theta_t^{N_i} = \log(1 + w_{i,t}z_i) < 0$. The more negative the exposure is, the more appealing the jump factor is to the investor. The portfolio exposure to the jump factor of region i can be written as

$$\begin{aligned}\theta_t^{N_i*} &= -\frac{1}{\gamma} \log(1 + \kappa_i) + \sum_{j=1}^n \beta_{j,i} B_j(t) \\ &= \underbrace{-\frac{1}{\gamma} \log(1 + \kappa_i)}_{\theta^{J_i}} + \underbrace{\overbrace{\beta_{i,i} B_i(t)}^{\theta_t^{J_i}} + \sum_{j \neq i}^n \overbrace{\beta_{j,i} B_j(t)}^{\theta_t^{C_{i,j}}}}_{\theta_t^{C_i}} \\ &=: \theta^{J_i} + \theta_t^{C_i}.\end{aligned}\tag{3.23}$$

The static component θ^{J_i} is the portfolio exposure to Poisson jump risk. When $\beta = 0$, the jumps are Poissonian with constant intensities, in which case the investor's optimal portfolio exposure to jump risks reduces to θ^{J_i} . The exposure to Poisson jump risks does not take into account the stochastic nature of the jump intensities and therefore plays the role of a myopic buy-and-hold demand.

An interesting comparison is to see what happens when the uncertainties brought by the jump risk are mimicked by Brownian motions that display the same mean and variance. The jump factor $(dN_{i,t} - (1 + \kappa_i)\lambda_{i,t} dt)$ has mean $-\kappa_i\lambda_{i,t} dt$ and variance $\lambda_{i,t} dt$. The instantaneous correlation between the jump components is 0. Consider instead a Brownian motion with the same mean and variance. Then, the investor will have an exposure of

$$\hat{\theta}^{J_i} = -\frac{1}{\gamma} \kappa_i.\tag{3.24}$$

One can show that⁶

$$|\theta^{J_i}| < |\hat{\theta}^{J_i}|.$$

It implies that the exposure to a risk factor is smaller when it is recognized as a Poisson jump than a Brownian motion, given its mean (risk premium) and volatility (risk). In a situation where asset prices move continuously, the investor can rebalance the portfolio after any infinitesimal changes in value to avoid large losses. However, since the investor cannot anticipate jumps, his/her wealth can change substantially before the investor has an opportunity to perform any adjustment. For this reason, the investor is reluctant to take too much risk exposure due to fear of disastrous events.

The horizon-dependent component $\theta_t^{C_i}$ is the portfolio exposure to contagion risks. Because jump risk factors drive the latent state variables as well as the asset prices, risky assets can be used to hedge against the uncertainties in the state variables. The hedging function gives rise to the additional term in the optimal portfolio exposure to jump risks, $\theta_t^{C_i}$, which can be understood as an intertemporal hedging demand for jump risks.

The contagion risk exposure, $\theta_t^{C_i}$, can be decomposed further into exposure to time series contagion risk, $\theta_t^{is_i}$, as a result of self excitation of jump component i , and exposure to cross section contagion risk, $\theta_t^{cs_{i,j}}$, as a result of cross section excitation from jump component i to jump component j , $j \neq i$.

3.3.2 Comparative statics analysis of the optimal portfolio exposure to contagion risks

In this section, we numerically show how the optimal portfolio exposure to contagion risks $\theta_t^{C_i}$ depends on the stochastic characteristics of the jump intensities. Provided risks are compensated properly so that investors long risky assets, then the exposure to Poisson jump risks θ^{J_i} is negative. The exposure to contagion risks increases the overall demand for jump risk in region i , if it has the same sign as θ^{J_i} . For a more insightful interpretation, we plot the negative of the exposure to contagion risks of the jump component N_i , $-\theta^{C_i}$, at the beginning of the investment horizon, understood as the hedging demand of the jump component. We suppress the time subscript to indicate that the exposure to contagion risks is evaluated at the beginning of the investment horizon. Larger hedging demand implies larger stake of the region in the portfolio.

We illustrate the model in a two-region market and calculate the optimal portfolio exposure to contagion risks given in Equation (3.23). We fix a set of base case parameter values and conduct comparative statics analysis of parameters of the intensity processes. Specifically, we set the mean reversion rate at $\alpha = 21$, the investment horizon at $T = 1$. We specify a symmetric excitation matrix with reasonable values, $\beta = (15, 3; 3, 15)$, according to the parameter estimates in Ait-Sahalia, Cacho-Diaz, and Laeven [6]. We also impose identical jump risk premia $\kappa_1 = \kappa_2 = 0.3$, so that the jump components of the two regions are not distinguishable. Then we only need to analyze the contagion exposure to one of the jump components. This allows us to analyze how parameters affect contagion exposure as directly as possible.

In addition, we fix the risk aversion parameter to be $\gamma = 3$. As noted by Liu [90], when $\gamma > 1$, investors are more risk averse than those with a log utility function and choose to

⁶Note that both θ^{J_i} and $\hat{\theta}^{J_i}$ are negative when the investor takes long positions. We have $\theta^{J_i} > \hat{\theta}^{J_i}$, which actually implies that $\hat{\theta}^{J_i}$ leads to larger exposure to jump risks. Therefore we compare absolute values to avoid confusion.

hedge against changes in the state variables. When $\gamma < 1$, investors not only forgo the hedging potential, but seek the high risk premium by betting on the future outcome.⁷ The comparative statics analysis in Figures 3.2 and 3.3 for investors with $\gamma < 1$ has opposite patterns to those with $\gamma > 1$, as is the case in Liu and Pan [91] and Liu et al. [92]. Since it is unlikely that investors have such small risk aversion, we restrict our analysis to the case where $\gamma > 1$.

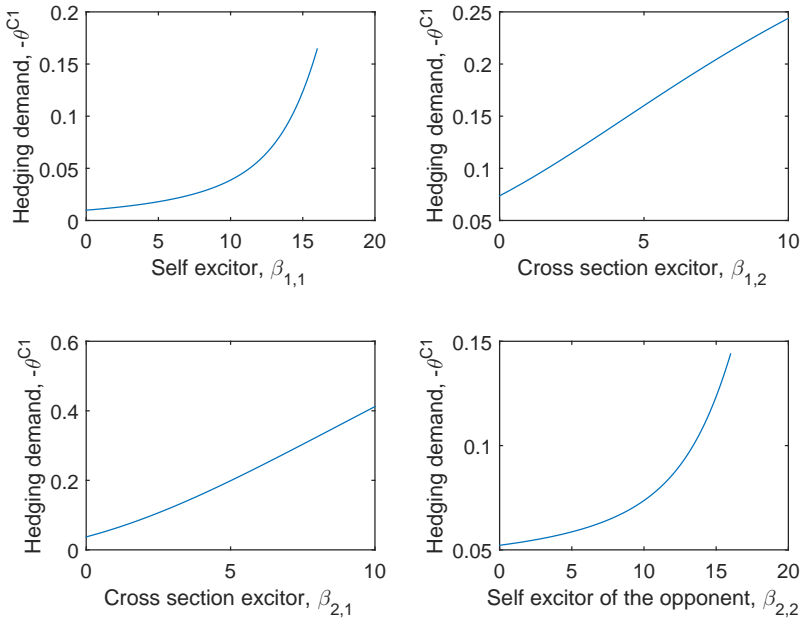


Figure 3.2: Comparative statics analysis of the hedging demand of jump component 1, $-\theta^{C1}$. The market consists of two regions with identical jump risk factors. The hedging demand of jump component 1 (the other will be symmetric) is plotted as functions of elements in the excitation matrix β . The base case parameters are $\alpha = 21$, $\beta = (15, 3; 3, 15)$, $T = 1$, $\kappa_1 = \kappa_2 = 0.3$, $\gamma = 3$.

⁷Note that the investors with $0 < \gamma < 1$ are still risk averse.

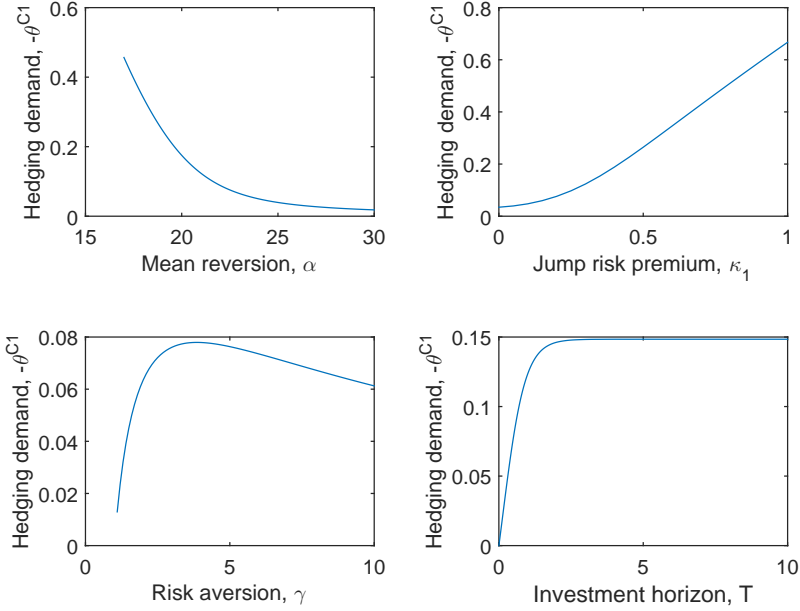


Figure 3.3: Comparative statics analysis of the hedging demand of jump component 1, $-\theta^{C_1}$. The market consists of two regions with identical jump risk factors. The hedging demand of jump component 1 (the other will be symmetric) is plotted as functions of the mean reversion rate α , jump risk premium κ_1 , risk aversion γ and investment horizon T . The base case parameters are $\alpha = 21$, $\beta = (15, 3; 3, 15)$, $T = 1$, $\kappa_1 = \kappa_2 = 0.3$, $\gamma = 3$.

Figure 3.2 plots the hedging demand of jump component 1, $-\theta^{C_1}$, as functions of excitation parameters. The figure shows that increasing any element of the excitation matrix β leads to increasing demand for jump component 1, whether it be the self excitor of jump component 1, $\beta_{1,1}$ (top left), the cross section excitor from jump component 1 to component 2, $\beta_{2,1}$ (bottom left), the cross section excitor from jump component 2 to component 1, $\beta_{1,2}$ (top right), or the self excitor of the opponent jump component $\beta_{2,2}$ (bottom right). The strongest increase in hedging demand occurs when $\beta_{2,1}$ increases.

Figure 3.3 plots the hedging demand of jump component 1, $-\theta^{C_1}$, as functions of the mean reversion rate α (top left), jump risk premium κ_1 (top right), risk aversion γ (bottom left) and investment horizon T (bottom right). Larger risk premium and longer investment horizon result in increasing jump risk demand. On the contrary, faster mean reversion rate decreases the exposure to contagion risks and in turn decreases the demand for jump risk. Interestingly, increasing the risk aversion first increases then decreases the hedging demand.

Since a jump occurrence moves asset prices and state variables in opposite directions, jump excitation enables the risky assets to be used as a static hedge against the effects of jumps in the state variables, as pointed out by Liu et al. [92]. For instance, an occurrence in jump component i decreases the asset price (since $z_i^k < 0$) but increases the state variables λ_i by $\beta_i \geq 0$.⁸ When $\gamma > 1$, investors take extra exposure to jump risks to hedge changes in

⁸The fact that asset prices and jump intensities jump in different directions is essential to generate hedging

the state variables to reduce uncertainties of the indirect utility.

Larger excitation and slower mean reversion imply that the jump intensity process (3.12) is more volatile. As one may expect, the more uncertainty there is in the state variables, the larger hedging incentive investors have. Larger risk premium results in larger exposure to Poisson jump risks θ^J , which leads to larger jump risks in the portfolio to be hedged. Similarly, longer investment horizon leads to increased sensitivity of indirect utility to state variables. In short, hedging demand rises when there are increasing uncertainties in investor's indirect utility.

The effect of increasing the risk aversion, however, is not clear. On the one hand, increasing the risk aversion decreases the exposure to Poisson jump risk θ^J , implying a smaller amount to be hedged, thereby decreasing the hedging demand. On the other hand, a more risk averse investor is more inclined to hedge against the changes in the state variables, and has a larger hedging demand. The final result depends on which effect is larger. Figure 3.3 shows that the effect of increasing risk aversion is not monotone: it first increases the jump risk demand and then reduces it.

3.3.3 The effect of asymmetric excitation

Interesting phenomena arise when the excitation structure becomes asymmetric. The excitor $\beta_{j,i}, j \neq i$, measures the excitation capability of jump component N_i as the source jump component, i.e., how large an occurrence in N_i raises the intensities of another jump component j . The excitor $\beta_{i,j}, j \neq i$, on the other hand, measures the inclination to excitation of jump component N_i as the target jump component, i.e., how large an occurrence in some jump component j raises the intensity of N_i . As we see in reality, equity prices in other geographical markets usually crash in close succession with an equity plunge in the US, whereas the transmission in the reverse order is not as often observed. It implies that $\beta_{j,i} > \beta_{i,j}$ when N_i represents the jump component in the US equity.

Recall from Equation (3.23) that $\theta^{C_i} = \beta_{i,i}B_i + \sum_{j \neq i}^n \beta_{j,i}B_j$. The portfolio exposure to contagion risks of jump component i depends on the potential of jump component i to excite other jump components as well as itself. Observe that $\beta_{j,i}$ plays a different role from $\beta_{i,j}$ in determining θ^{C_i} : a larger cross section excitor $\beta_{i,j}$ leads to a larger increase in the jump intensity of region i , and a larger portfolio exposure to the jump risk factor of region j (instead of region i). Regions with comparable expected jump intensities may be weighted differently in the optimal investment portfolio due to asymmetric excitation.

demand. If the equity jump $z_i^k > 0$, investors take smaller exposure to jump risks as a result of jump excitation. This result is consistent with Liu et al. [92].

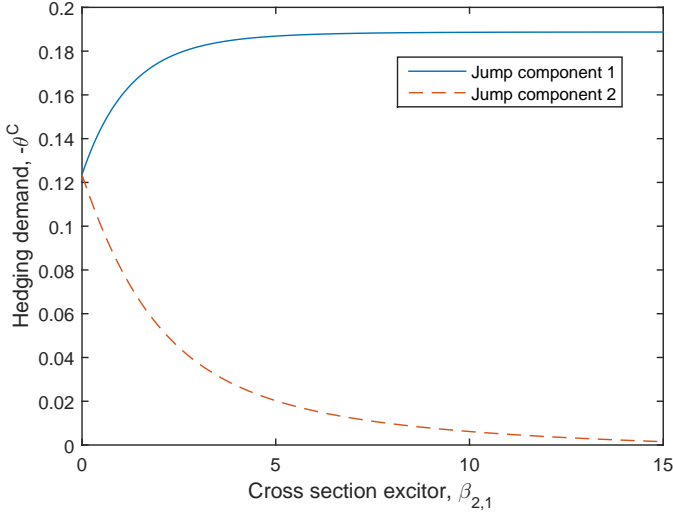


Figure 3.4: The hedging demand of the two jump components $-\theta^{C_1}$ (solid curve), $-\theta^{C_2}$ (dotted curve) as functions of the cross section excitor $\beta_{2,1}$. The excitation matrix is $\beta = (18, 0; \beta_{2,1}, \beta_{2,2})$. We let $\beta_{2,1}$ increase and find the corresponding $\beta_{2,2}$ such that the expected jump intensities do not vary with the excitation matrix. The jump risk premium is set to be equal with $\kappa_1 = \kappa_2 = 0.3$. The other parameters are $\alpha = 21, \gamma = 3, T = 1$. Given the choice of parameter values, the expected jump intensity is $\mathbb{E}[\lambda_{1,t}] = \mathbb{E}[\lambda_{2,t}] = 2.1$.

Figure 3.4 gives a numerical illustration of the effect of excitation asymmetry in a two-region market. The figure plots the hedging demand $-\theta^{C_1}$ (solid curve), $-\theta^{C_2}$ (dotted curve) as functions of $\beta_{2,1}$. It shows how the investor's demand for jump risks changes as the excitation structure becomes more asymmetric. We fix the first row of the excitation matrix to be $\beta_{1,1} = 18, \beta_{1,2} = 0$. We close the excitation channel from region 2 to region 1 by setting $\beta_{1,2} = 0$, so that the jump risk only propagates from region 1 to region 2 but not the other way around. We let $\beta_{2,1}$ increase while finding the corresponding $\beta_{2,2}$ that delivers the same unconditional expected jump intensity $\mathbb{E}[\lambda_{2,t}]$. As $\beta_{2,1}$ increases, the excitation matrix gets more asymmetric due to a larger difference between $\beta_{2,1}$ and $\beta_{1,2}$, strengthening the bias towards region 1, with everything else (e.g., jump risk premia, expected jump intensities) unchanged. We observe from the figure that jump component 1 becomes more appealing to the investor as contagion becomes more asymmetric, while the jump components have the same risk profile in all other aspects.

As mentioned in the previous section, jump excitation increases the demand for jump risks because of their hedging potential. Consistent with this intuition, when jump components have heterogeneous capabilities in raising jump intensities, the jump component with a higher excitation capability is the more favorable risk factor due to its larger hedging potential against jump intensities. The end point on the x -axis in Figure 3.4 stands for $\beta = (18, 0; 15, 3)$, in which case an occurrence in jump component 1 raises $\lambda_{1,t}$ by 18, and $\lambda_{2,t}$ by 15, while an occurrence in jump component 2 only raises $\lambda_{2,t}$ by 3. Apparently, jump component 1 has a larger influence on the state variables (jump intensities) than jump component 2 and consequently jump component 1 has the larger hedging potential. The investor therefore tilts the portfolio towards region 1 for a more effective hedge of the state variables.

One may expect that, everything else equal, the US equity will take a larger share in the investor's portfolio as compared to the classic model predictions due to excitation asymmetry.

3.4 Utility loss of suboptimal trading strategies

In the previous sections we have shown that jump propagation changes the risk profile of the optimal portfolio. In particular, we see that excitation asymmetry leads to larger investment towards the equity market with the highest degree of transmission of jump risks. In this section, we examine the utility loss for an investor who fails to construct the equity portfolio optimally. We only focus on the case when the suboptimal portfolio is fully diversified, i.e., $\theta^Z \equiv 0$. A suboptimal portfolio is defined as the equity portfolio whose risk exposure to the systematic Brownian and jump risk factors is different from the optimal exposure. To quantify the utility loss of implementing suboptimal strategies, we adopt the measure introduced in Liu and Pan [91]. The utility loss L of a suboptimal portfolio \hat{x} is defined as

$$L = \frac{\log x^* - \log \hat{x}}{T}, \quad (3.25)$$

where $x^*(\hat{x})$ is the certainty equivalent wealth of the optimal (suboptimal) portfolio strategy defined as

$$\begin{aligned} \frac{x^{*1-\gamma}}{1-\gamma} &:= \mathbb{E}_{0,x,\lambda} \left[\frac{X_T(\theta^{W*}, \theta^{N*})^{1-\gamma}}{1-\gamma} \right] = J(0, x, \lambda), \\ \frac{\hat{x}^{1-\gamma}}{1-\gamma} &:= \mathbb{E}_{0,x,\lambda} \left[\frac{X_T(\hat{\theta}^W, \hat{\theta}^N)^{1-\gamma}}{1-\gamma} \right], \end{aligned}$$

where $X_T(\cdot, \cdot)$ emphasizes the dependence of the terminal wealth on the portfolio risk exposure.

Effectively, L measures the loss in terms of the annualized, continuously compounded return in certainty equivalent wealth of the suboptimal portfolio strategy. The larger L is, the larger the utility loss of implementing the suboptimal portfolio strategy.

The next proposition computes the utility loss of implementing the portfolio strategy with a suboptimal risk exposure $(\hat{\theta}_t^W, \hat{\theta}_t^N)$.

Proposition 3.3 (Utility loss of suboptimal strategies). *If the expected power utility investor implements a suboptimal portfolio strategy with risk exposure $(\hat{\theta}_t^W, \hat{\theta}_t^N)$, then the associated utility loss is given by*

$$L = \frac{1}{(1-\gamma)T} \left(\gamma A(0) - \hat{A}(0) + (\gamma B(0) - \hat{B}(0))' \lambda_0 \right), \quad (3.26)$$

where $B(0), A(0)$ are given by (3.20), and $\hat{B}(0), \hat{A}(0)$ can be solved from

$$\dot{\hat{B}}(t) = (1-\gamma)(1+\kappa) \circ \hat{\theta}_t^N + \alpha \hat{B}(t) - (\hat{\theta}_t^N + 1)^{1-\gamma} \circ e^{\beta' \hat{B}} + 1, \quad (3.27)$$

$$\dot{\hat{A}}(t) = -(1-\gamma) \left(\hat{\theta}_t^{W'} \eta - \frac{1}{2} \hat{\theta}_t^{W'} \mathbf{L} \mathbf{L}' \hat{\theta}_t^W \right) - \frac{1}{2} (1-\gamma)^2 \hat{\theta}_t^{W'} \mathbf{L} \mathbf{L}' \hat{\theta}_t^W - \alpha \hat{B}' \lambda_\infty, \quad (3.28)$$

with $\hat{B}(T) = 0, \hat{A}(T) = 0$.

Two relevant cases are: (1) when the investor fails to recognize the exciting nature of jump components and implements the portfolio strategy as if the equity price is generated by Poisson jump diffusion, and (2) when the investor simply ignores the discontinuities in equity returns and implements the Merton mean-variance strategy. To calculate the utility loss associated with these suboptimal strategies, according to Proposition 3.3, we simply substitute for $\hat{\theta}_t^W$ in Equation (3.28) θ^{W*} , and for $\hat{\theta}_t^N$ in Equation (3.27) we substitute θ^J , $\hat{\theta}^J$, respectively.

Figure 3.5 plots the utility loss of the aforementioned two cases as functions of the currency jump intensity (top left), jump risk premium κ_1 (top right), risk aversion γ (bottom left) and investment horizon T (bottom right). The utility loss when the investor incorrectly recognizes the return generating model as Poisson jump diffusion (PJD) is plotted as the solid curve, and the case when the investor incorrectly implements the Merton mean-variance strategy is plotted as the dotted curve. Notice that the two curves move together, with the PJD curve above the Merton curve most of the time.

Although neither the optimal portfolio nor the suboptimal portfolio depends on the realization of the state variables, the investor's utility over terminal wealth is dependent on the current jump intensities. As a result, utility loss is sensitive to the current jump intensities, as shown in the upper left panel of Figure 3.5. During a financial crisis, jump intensities may build up to as large as 100 or higher as found by Ait-Sahalia et al. [6]. Ignoring jump excitation then costs a loss of over 40 percentage points in annualized portfolio returns.

The upper right panel plots the utility loss as a function of the jump risk premium, κ_1 . The graph shows that the utility loss increases with the jump risk premium. For instance, when the jump risk premium $\kappa_1 = 0.5$ (which gives an equity jump premium of 5% as estimated by Bollerslev and Todorov [16]), the expected annual return of a portfolio strategy that does not account for jump excitation is 15 percentage points lower than that of the true optimal portfolio. Recall from Figure 3.3 that the jump risk premium increases contagion exposure. A larger jump risk premium leads to larger deviations from the optimal jump exposure, which in turn leads to bigger utility loss. The same effect holds for the investment horizon shown in the lower right panel.

The lower left panel plots the utility loss as a function of the risk aversion γ . When $\gamma = 1$, the investor has log utility. Log utility investors are myopic, in the sense that they only care about the current realization of the state variables, and therefore do not have an incentive to hedge against future changes of the state variables. As a result, ignoring jump excitation generates no utility loss in the PJD strategy for a log utility investor. The utility loss of the Merton strategy, however, is nontrivial even when the investor has log utility. As the investor becomes more risk averse, the utility losses of both suboptimal strategies start to increase. This is because the more risk averse the investor is, the more concerned he/she is about the changes in the state variables as a result of jump excitations. Therefore deviations from the optimal portfolio lead to larger utility loss for more risk averse investors.

A surprising fact is that the utility loss of ignoring jump excitation is even larger than that of ignoring jumps in total (except when γ is close to 1). In other words, if the true return generating model is mutually exciting jump diffusion, then it is better for the investor not to account for jumps at all than to recognize jumps but hold them for the wrong type. In Chapter 3.3.1, we have shown that the exposure to a risk factor is smaller when it is recognized as a Poisson jump than a Brownian motion. However, when jumps are mutually exciting, the investor increases the exposure to the jump risk factor in order to exploit the hedging potential inherited. The Poisson jump diffusion strategy turns out to be too conservative.

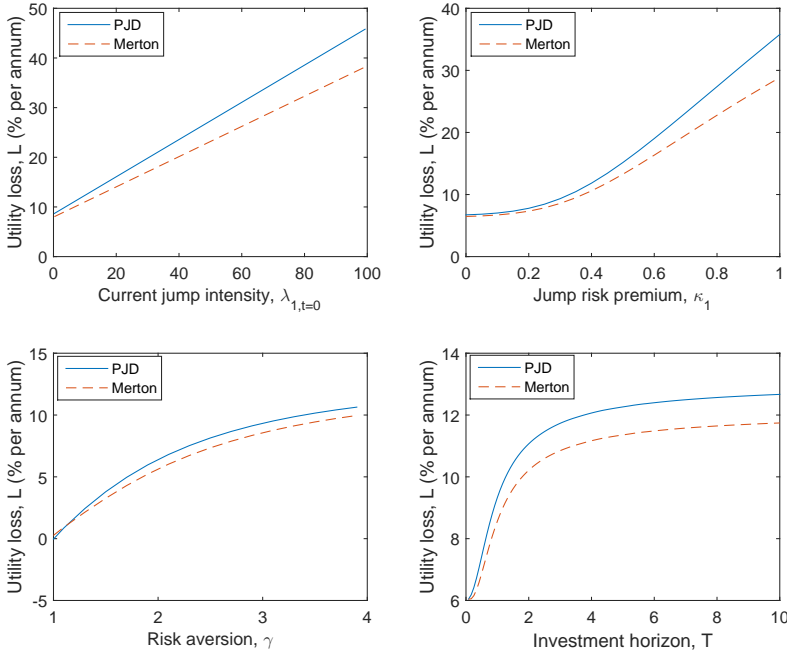


Figure 3.5: Utility loss in terms of the annualized continuously compounded return (% per annum) if the investor incorrectly assumes $\beta = 0$ and implements the portfolio strategy as if the model is Poisson jump diffusion (solid curve), or if the investor implements the Merton mean-variance strategy (dotted curve). The “true” base case parameters are $\eta = (0.3; 0.3)$, $\alpha = 21$, $T = 1$, $\lambda_\infty = (0.3; 0.3)$, $\kappa = (0.3; 0.3)$, $\lambda_{t=0} = (2; 2)$, $\beta = (15, 3; 3, 15)$, $\gamma = 3$. The top left panel plots the utility loss as a function of the current value of the intensity of jump component 1, $\lambda_{1,t=0}$, while all other parameters remain the same. The top right graph plots the utility loss as a function of the risk premium of jump component 1, κ_1 . The bottom left panel plots the utility loss as a function of the risk aversion γ . The bottom right graph plots the utility loss as a function of the investment horizon T .

3.5 Application to international equity returns

In this section, we estimate the mutually exciting jump diffusion model to index returns of US, Japan, and Europe. We show that the under-diversification of the market portfolio, especially the over-weighting of the US equity of the market portfolio compared to classic asset allocation models, can be explained by an asymmetric excitation structure. In Appendix 3.C, we also show how the portfolio tilts towards the home market for country representative investors who are ambiguity averse towards foreign equity markets. In Chapter 3.5.1, we describe the equity index data we use in the empirical analysis. We disentangle jumps from continuous returns and estimate the diffusion and jump parameters in Chapter 3.5.2. Chapter 3.5.3 compares the empirical market portfolio exposure to risk factors with model predictions to show that excitation asymmetry is able to generate the observed US bias in the market portfolio.

3.5.1 Data

We consider three well-developed stock markets: the United States, Japan, and Europe, and make the simplifying assumption that these regions represent the global financial market. We choose these three equity markets because: (1) these equity markets have little barriers to international investment; (2) these three equity markets already represented around 63% of the world equity market capitalization by the end of 2012 according to the World Bank. More markets could be included theoretically, but parameter estimation is likely to become cumbersome. Therefore we do not go beyond three equity markets in the empirical analysis.⁹

We examine excess returns on Morgan Stanley Capital International (MSCI) indices of US, Japan, and Europe, over *local* risk-free rates in *local* currencies. The MSCI indices are value weighted stock indices. There are 642 individual stocks included in MSCI US index, 314 stocks in MSCI Japan index, and 444 stocks in MSCI Europe index in May 2015. Expected returns are estimated as the sample mean of the log returns on the MSCI *total return* index from Jan 1970 to May 2015. The total return index has been adjusted for dividends and other noncash payments to shareholders. We estimate the covariance and jump parameters using the daily *price index* from Jan 3, 1972 to May 29, 2015, a total of 11325 observations.¹⁰ Excitation parameters would not be estimated accurately on less frequent data such as weekly or monthly data, since multiple jumps could happen within adjacent observations. We use US 3-month Treasury bill rates, Japan base discount rates, 3-month Euribor rates as proxies for the local risk-free rates of the three regions.¹¹

Table 3.2 contains the descriptive statistics of the annualized excess log returns on MSCI indices. Japan has the lowest mean return of 3.7% and Europe the highest with 4.8% on an annual basis. Return volatilities vary from around 15% to 19%. Comparing the risk return tradeoffs of these three major equity markets, the European market generates a fairly high mean with the lowest volatility. By contrast, Japanese equity is characterized by the lowest expected return and the highest volatility. In spite of the unfavorable risk-return tradeoff of Japanese equity, the correlation between returns on the Japanese market and those on the US market is as small as 0.11. The correlation between returns of Japan and Europe is also lower than that between US and Europe. The relatively small dependence between the Japanese market and the other two equity markets makes the Japanese equity a better candidate for international diversification. All returns are left-skewed, implying larger extreme losses than gains. For all regions, the excess kurtosis is substantially larger than 0, as would be caused by jumps. Our econometric analysis requires a careful treatment of the time zone differences. We refer to Appendix 3.F for further details and robustness checks with respect to our treatment of the time zones.

⁹Similar assumptions that a few representative markets make up the world economy are also made in, for example, Ang and Bekaert [8], Uppal and Wang [116], Das and Uppal [42]. A more extensive empirical analysis is beyond our scope and left to future research.

¹⁰We do not model dividends separately in our model. Preferably we would like to use daily returns on the *total return* index in all estimations. Unfortunately, the *daily* data of total return of MSCI indices are only available since 2001. We compare the sample covariance of the daily returns of price index and total return index in the overlapping period. They turn out to be almost identical. The same holds for the jump intensities. We conclude that on an index level, most variation in daily total return comes from price moves, not dividends.

¹¹The 3-month Euribor rates are only available since 1999, after the Euro was introduced as an accounting currency. We use 3-month Fibor rates, the German interbank offered rates, before the Euribor is available. The two rates are almost identical during the overlapping period.

Descriptive statistics of annualized excess log returns			
	US	JA	EU
Number of constituents	642	314	444
Mean	4.6%	3.7%	4.8%
Standard deviation	17.2%	18.9%	15.1%
Correlation	1	0.11	0.46
		1	0.30
			1
Skewness	-1.04	-0.31	-0.40
Excess kurtosis	26.3	11.4	9.7

Table 3.2: Descriptive statistics of annualized excess log returns on MSCI indices over local risk-free rates. The sample mean is computed using the monthly total return data from Jan 1970 to May 2015. Higher moments are computed using daily price index data from Jan 3, 1972 to May 29, 2015.

3.5.2 Parameter estimation of nested models

The equity indices, by construction, are well diversified local portfolios of a large number of individual assets. We henceforth assume that the equity indices are representative assets free of idiosyncratic risks and follow the dynamics

$$\frac{dS_{i,t}}{S_{i,t-}} = \sigma_i(d\hat{W}_{i,t} + \eta_i dt) + z_i(dN_{i,t} - (1 + \kappa_i)\lambda_{i,t} dt), \quad i = 1, 2, 3, \quad (3.29)$$

with an instantaneous 3 by 3 covariance matrix of local portfolios $\Sigma = \sigma LL'\sigma$, where σ is a diagonal matrix with σ_i on the diagonals.

We consider the following nested models: the diffusion only model, the Poisson jump diffusion model (PJD), the self exciting jump diffusion model (SEJD) and finally the full-fledged mutually exciting jump diffusion model (MEJD). Table 3.3 lists nested models as special cases of the mutually exciting jump diffusion model with proper parameter restrictions. If we restrict jump amplitudes $\mathbf{z} = (z_1; z_2; z_3) = \mathbf{0}$, i.e., asset prices do not jump at all, we have the classic diffusion only model in which asset prices follow geometric Brownian motions. The model becomes a Poisson jump diffusion model if $\beta = \mathbf{0}$, implying that jumps do not excite and intensities are kept constant at the level λ_∞ . More generally, if we only close the cross section excitation channels and restrict β to be a diagonal matrix, asset prices will follow self exciting jump diffusion processes. In the most general case where no restrictions are imposed, we have a full-fledged mutually exciting jump diffusion model.

Nested models as special cases	
Nested models	Parameter restrictions
Diffusion only	$\mathbf{z} = \mathbf{0}$
PJD	$\alpha = 0, \beta = 0$
SEJD	β diagonal
MEJD	none

Table 3.3: Nested models as special cases of the mutually exciting jump diffusion model with proper parameter restrictions. “PJD” stands for the Poisson jump diffusion model; “SEJD” represents the self exciting jump diffusion model; “MEJD” represents the mutually exciting jump diffusion model.

We are going to estimate each model listed in Table 3.3 using the historical returns on M-SCI indices. Parameter estimates of the diffusion only model can be easily obtained through the first and second moments of the log returns reported as the summary statistics in Table 3.2. For jump diffusion models, similar to Liu et al. [92], we disentangle jumps from the continuous part of log returns in order to estimate diffusion and jump parameters separately.¹² We define truncation thresholds as negative three times the sample standard deviation. Daily log returns that fall below the thresholds are regarded as jump returns. Figure 3.6 plots the resulting jump occurrences in US, Japan, and EU. We observe jump clustering during the Asian crisis (1997-1999), the stock market downturn (2002) and the Subprime mortgage crisis (2007-2012).

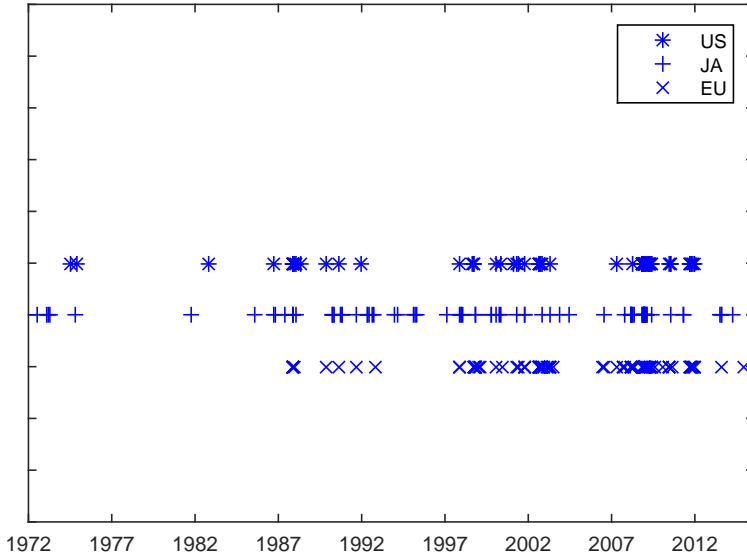


Figure 3.6: Filtered jump occurrences in US, Japan, and Europe. Each mark indicates a jump occurrence. The top row stands for the jump occurrences in the US equity market, the middle row Japan and the bottom row Europe. Daily log returns that fall below negative three times the sample standard deviation are recorded as jump occurrences.

Parameter estimates of the nested jump diffusion models are reported in Table 3.4. Jump amplitudes of log returns, $\log(1 + z)$, are estimated as the differences between average jump returns and non-jump returns. In a Poisson jump diffusion model, the constant Poisson jump intensities are obtained by dividing the sum of total jump occurrences detected in each M-SCI return by the number of years. In case of self excitation and mutual excitation, jump excitation parameters α, β are estimated using maximum likelihood, while λ_∞ is estimated

¹²The parametrization of the mutually exciting jump diffusion model is rich and econometrically challenging. Ideally, we would like to apply the Generalized Method of Moments developed by Aït-Sahalia et al. [6] which minimizes the effects of the “Peso problem” inherited. However, given the 3-dimensional nature of our problem, we use this somewhat crude two-step maximum likelihood approach to identify the excitation structure of equity markets. A more refined econometric treatment is beyond the scope of this paper.

Parameter estimates of nested jump diffusion models						
Parameters	PJD	SEJD			MEJD	
σ		$\begin{pmatrix} 15.9\% & 0 & 0 \\ 0 & 17.5\% & 0 \\ 0 & 0 & 13.6\% \end{pmatrix}$				
LL'		$\begin{pmatrix} 1 & 0.10 & 0.40 \\ 0.10 & 1 & 0.25 \\ 0.40 & 0.25 & 1 \end{pmatrix}$				
z		$\begin{pmatrix} -4.9\% \\ -5.0\% \\ -4.0\% \end{pmatrix}$				
λ_∞	$\begin{pmatrix} 1.7 \\ 2.0 \\ 2.6 \end{pmatrix}$	$\begin{pmatrix} 0.56 \\ 1.00 \\ 0.60 \end{pmatrix}$			$\begin{pmatrix} 0.44 \\ 0.74 \\ 0.32 \end{pmatrix}$	
α	0	17.7***			29.3***	
β	0	$\begin{pmatrix} 11.8^{***} & 0 & 0 \\ 0 & 8.8^{***} & 0 \\ 0 & 0 & 13.6^{***} \end{pmatrix}$			$\begin{pmatrix} 12.6^{***} & 0.0 & 5.8^{***} \\ 9.3^{***} & 7.4^{***} & 2.3 \\ 24.1^{***} & 2.7^* & 8.1^{***} \end{pmatrix}$	

Table 3.4: Parameter estimates of nested jump diffusion models. “PJD” stands for the Poisson jump diffusion model; “SEJD” represents the self exciting jump diffusion model; “MEJD” represents the mutually exciting jump diffusion model. Each column is in the order of “US, Japan, Europe”. *, **, *** indicate significance at 95%, 97.5%, and 99.5% confidence levels, respectively.

iteratively such that the unconditional expected jump intensity $\mathbb{E}[\lambda]$ is equal to the average jump occurrences per year. The algorithm of computing the likelihood functions is discussed in Appendix 3.D. Having identified the jumps, we construct the truncated returns by removing detected jumps from returns. We regard the truncated data as being generated by the continuous part of the model. We estimate the instantaneous volatility σ and the correlation matrix LL' from the truncated data for models with jumps.

We see from Table 3.4 that equities with higher volatility have larger jump amplitudes but not necessarily more frequent jumps. This is not unreasonable since volatility measures the variation in the bulk of the data, while jumps contribute to the very left tails of the return distribution. Among three regions, the Japanese market has the largest jump amplitude, -4.97%, with an average of 2.02 jumps per year. Europe has the smallest jump amplitude, -4.07%, but has the most frequent jumps – average 2.63 jumps occur per year. The US equity market has moderate jump amplitude and jump frequency. Jumps display statistically significant self excitation as well as cross section excitation. The jump propagation from US to the other two regions is significant and large in magnitude. The US, on the other hand, is only excited by itself or the European market; the spillover effect from Japan to the US is almost zero. The existence of cross section excitation between Japan and Europe is barely significant. The estimated excitation structure is in line with the pairwise estimation results in Ait-Sahalia et al. [6], who show that the US always has a larger cross section excitor as the source jump component than as the target jump component when paired with other economies.

3.5.3 Empirical vs. implied portfolio exposure to risk factors

In this section, we calculate the exposure to risk factors as predicted by the model and compare this exposure to the risk exposure induced by the market portfolio.

First, we infer the market exposure to systematic risk factors using market portfolio weights. Let M_t be the market equity portfolio. Denote by $h_{i,t}$ the weight on the local equity index $S_{i,t}$ in the market equity portfolio M_t , with $\sum_i h_{i,t} = 1$. We suppose that the dynamics of $S_{i,t}$ are given by Equation (3.29). Thus, it holds that

$$\frac{dM_t}{M_{t-}} = \sum_{i=1}^n h_{i,t} \left(\frac{dS_{i,t}}{S_{i,t-}} \right) \quad (3.30)$$

$$= \sum_{i=1}^n h_{i,t} \left(\sigma_i (d\hat{W}_{i,t} + \eta_i dt) + z_i (dN_{i,t} - (1 + \kappa_i) \lambda_{i,t} dt) \right). \quad (3.31)$$

In an equilibrium situation (with common beliefs), the dynamics of the market portfolio M_t should reflect the dynamics of the optimal wealth X_t^* of the representative investor. In Chapter 3.2.4, we show that the optimal wealth for an expected power utility investor is given by

$$\frac{dX_t^*}{X_{t-}^*} = \sum_{i=1}^n \left\{ \theta^{W_i^*} (d\hat{W}_{i,t} + \eta_i dt) + (\exp(\theta_t^{N_i^*}) - 1) (dN_{i,t} - (1 + \kappa_i) \lambda_{i,t} dt) \right\}.$$

Thus, we should have that

$$\frac{dM_t}{M_{t-}} \approx \frac{dX_t^*}{X_{t-}^*}. \quad (3.32)$$

Notice that in reality we have an approximation instead of an equality here. As shown in Chapter 3.2.5, the replicating portfolio converges to the optimal wealth process as the number of assets goes to infinity. In reality, unfortunately, the equity indices are made up of a finite number (although many) of individual assets. The market portfolio M_t , therefore, would not be completely free of idiosyncratic risks.

Under this setting, Equation (3.32) implies the following approximation:

$$\begin{cases} \theta^{W_i^*} \approx \sigma_i h_{i,t}, \\ \theta_t^{N_i^*} \approx z_i h_{i,t}. \end{cases} \quad (3.33)$$

Note that Equation (3.33) holds as long as assets follow jump diffusion processes, regardless of whether the jumps are mutually exciting, self exciting or Poissonian.

Table 3.5 reports the normalized portfolio weights and (approximated) exposure to risk factors, $\frac{\theta^{W_i^*}}{\sum \theta^{W_i^*}}$ and $\frac{\theta^{N_i^*}}{\sum \theta^{N_i^*}}$, of the market portfolio. The market values (US dollar denominated) of MSCI US, Japan, and Europe on May 29, 2015, (the last day in our sample) serve as a proxy of the market portfolio. The portfolio weights are calculated by dividing the market values of each region by the sum of the market values, so that the weights on US, Japan, and Europe add up to 1. Risk exposure is calculated using approximation (3.33). Notice that the portfolio exposure to Brownian risks and that to jump risks are similar, since a region with higher volatility has on average a larger jump amplitude (see Table 3.4). Normalization further narrows the differences. Both the US Brownian risks and jump risks comprise the majority of the market portfolio. As we will see shortly, the over-exposure to US Brownian risks stems from the high Brownian risk premium in the US, while the over-exposure to US jump risks is due to the asymmetric excitation structure.

Empirical market portfolio weights and exposure to risk factors			
	US	JA	EU
Portfolio weights	0.63	0.10	0.28
Exposure to systematic Brownian risks	0.65	0.11	0.24
Exposure to jump risks	0.66	0.10	0.24

Table 3.5: Empirical market portfolio weights and exposure to risk factors in May, 2015. The market weights are computed from the MSCI market values. Risk exposures are calculated according to Equation (3.33). Statistics are normalized to add up to 1 on the rows.

Next, we derive the model predicted risk exposure. As discussed in Chapter 3.3.1, the optimal exposure to mutually exciting jumps can be decomposed into three components (see Equation (3.23)): exposure to Poisson jump risks, exposure to time series contagion risks, and exposure to cross section contagion risks. The optimal jump exposure predicted by any nested model, as a result, is a combination of these components, depending on the stochastic features of the jump factors. Table 3.6 lists the portfolio exposure to risk factors predicted by nested models. Notice that although the term θ^{ts} (exposure to time series contagion risks) appears in the self exciting jump model as well as the mutually exciting jump model, they have different values because the excitation matrices of SEJD and MEJD are different not only in off-diagonal elements but also in diagonal elements (c.f. Table 3.4). For comparability, we also include the risk exposure of a pure diffusion model. In a pure diffusion model, the Brownian risk factors remain unchanged. Investors regard the jump factors as if they were generated by Brownian motions. $\hat{\theta}^J$, the exposure to “jump risk factors” when jump components are recognized as (mimicked by) Brownian motions, is given by (3.24).

Optimal risk exposure of nested models		
Nested models	Brownian exposure	Jump exposure
Diffusion only	θ^{W*}	$\hat{\theta}^J$ (Brownian mimicked)
PJD	θ^{W*}	θ^J
SEJD	θ^{W*}	$\theta^J + \theta^{ts}(\beta^D)$
MEJD	θ^{W*}	$\theta^J + \theta^{ts}(\beta) + \theta^{cs}(\beta)$

Table 3.6: Model predicted exposure to risk factors. “PJD” stands for the Poisson jump diffusion model; “SEJD” represents the self exciting jump diffusion model; “MEJD” represents the mutually exciting jump diffusion model.

Because all models predict the same exposure to Brownian risks, θ^{W*} , we focus on the comparison of the model predictions on the exposure to jump risks. We calibrate the Brownian risk premium η such that θ^{W*} is equal to the Brownian risk exposure of the market portfolio. To do that, we first decompose the equity premium into variance premium and jump premium. Let μ be the expected excess return: $\mu = \mathbb{E}[\mathrm{d}S_t/S_t] = \mathbb{E}[\log(S_t)] + \frac{1}{2}\mathbb{E}[(\log(S_t) - \mathbb{E}[\log(S_t)])^2]$.¹³ The total equity premium μ can therefore be divided into the

¹³We are aware of the fact that the first moment of equity returns cannot be consistently estimated using a sample of 41 years alone. The main purpose here is to show that for given risk premium estimates, while acknowledging this fact, the estimated excitation structure gives rise to the US bias observed in the market portfolio. Other papers also use the empirical first moment for asset allocation purposes. See, among others, French and Poterba [65], Liu et al. [92], Liu and Pan [91], Das and Uppal [42], and Jin and Zhang [81].

variance premium and the jump premium:

$$\begin{aligned}\mu &= \mu_{\text{variance}} + \mu_{\text{jump}}, \\ &:= \sigma\eta + (-\kappa \circ z \circ \mathbb{E}[\lambda_t]).\end{aligned}\quad (3.34)$$

Since we consider the *relative* allocations to US, Japan, and Europe, η is only identified up to a positive constant: $\eta \propto LL' \frac{\theta^{W*}}{\nu \theta^{W*}}$.

Using (3.18), (3.33), and (3.34), the risk premium parameters reported in Table 3.7 are calibrated according to

$$\eta = LL' \frac{\theta^{W*}}{\nu \theta^{W*}}, \quad (3.35)$$

$$\kappa = (\sigma\eta - \mu) \cdot z \cdot \mathbb{E}[\lambda_t]. \quad (3.36)$$

As shown in Table 3.7, the market portfolio exposure to Brownian risk imposes a large variance premium in the US and a small variance premium in Japan. The jump premium, calculated as the equity premium less the variance premium, is comparable across regions, with Japan having the largest jump premium and US the lowest.

Calibrated risk premium parameters					
	η	κ	Variance premium	Jump premium	Total equity premium
US	0.12	0.48	1.86	3.99	5.85
JA	0.04	0.47	0.63	4.62	5.25
EU	0.08	0.44	1.12	4.57	5.69

Table 3.7: Calibrated risk premium parameters and the corresponding variance and jump risk premiums. The parameters η, κ are calibrated using Equations (3.35), (3.36). The variance and the jump premium are calculated with (3.34) and are denoted in percentage per annum.

The jump risk premium in US reported in Table 3.7 is consistent with Pan [100], who estimates the S&P 500 average mean excess rate of return demanded for the jump risk to be 3.5% per year. The estimates of the jump risk premium from other papers can vary. For instance, Bollerslev and Todorov [16] non-parametrically estimate the average US jump risk premium to be approximately 5%. Santa-Clara and Yan [106] find that the jump risk premium is on average more than half of the total equity premium. In a self-exciting jump diffusion model using the US equity and option data, Boswijk et al. [18] estimate the jump risk premium to be around 8.82 times the spot jump intensity, implying a κ of the US market of 0.79. The actual choice of risk premiums does not have a qualitative impact on the presence of the US bias.

The property of no market timing of the optimal portfolio discussed in Chapter 3.2.4 allows us to construct unconditional optimal portfolios without having to estimate the latent state variable process – jump intensities in our case. All portfolios are constructed using static parameter estimates in Table 3.4.

Table 3.8 reports the optimal jump risk exposure under the four nested models for various coefficients of relative risk aversion and investment horizons. Observe that when jumps do not excite, the normalized exposure to jump risks does not change with investors' risk aversions or investment horizons. Compared to the prediction generated by the classic diffusion only model, neither the jump itself nor jump self excitation has a noticeable impact on

the relative jump risk allocation of the optimal portfolio. Although Poisson jumps and self exciting jumps reduce or increase the *total* investment in risky assets, they barely affect the composition *within* the equity portfolio. The diffusion only, no excitation as well as the self excitation model predict comparable exposure to jump risks of the three regions.

When jumps are mutually exciting, investors demand more US jump exposure than the cases of self excitation or no excitation. As either risk aversion or investment horizon increases, the bias towards US jump factors becomes more prominent. Self exciting jumps, which in fact have a symmetric excitation structure, are not able to generate the US bias. It is only when an asymmetric excitation matrix β comes into play that we observe a shift from Japanese jump risk factors to US jump risk factors. For example, when $\gamma = 5$, $T = 10$ (bold cells), of the total jump exposure of the portfolio, around 58% comes from US, and only 14% from Japan. Therefore we conclude that the serial dependence of jumps alone does not lead to the US bias, but rather the excitation asymmetry contributes to the over-weighting of the US equity market in the optimal portfolio.

It becomes clear that the over-exposure to the US Brownian risk is due to the large variance premium in the US (as shown in Table 3.7), while the over-exposure to the US jump risk is due to the asymmetric excitation structure. If the over-exposure to the US jump factor were, too, caused by a higher jump premium in the US, then we would observe the US bias in the portfolio predictions of the other models as well. In fact, as we observe in Table 3.7, US has the lowest jump premium but the largest demand. By contrast, Japan has the highest jump premium but the lowest demand. This implies that the investor takes the largest stake in the US jump factors not for its risk-return tradeoff but for its hedging potential.

Model implied exposure to jump risks					
Risk aversion	Horizon	Diffusion only	PJD	SEJD	MEJD
1.5	month	0.35	0.35	0.35	0.38
		0.34	0.34	0.33	0.31
		0.32	0.32	0.32	0.31
	quarter	0.35	0.35	0.35	0.41
		0.34	0.34	0.32	0.28
		0.32	0.32	0.33	0.31
	10-year	0.35	0.35	0.34	0.43
		0.34	0.34	0.30	0.27
		0.32	0.32	0.35	0.30
3	month	0.35	0.35	0.35	0.42
		0.34	0.34	0.32	0.28
		0.32	0.32	0.33	0.30
	quarter	0.35	0.35	0.35	0.49
		0.34	0.34	0.30	0.22
		0.32	0.32	0.35	0.29
	10-year	0.35	0.35	0.34	0.53
		0.34	0.34	0.24	0.18
		0.32	0.32	0.42	0.29
5	month	0.35	0.35	0.35	0.43
		0.34	0.34	0.32	0.27
		0.32	0.32	0.33	0.30
	quarter	0.35	0.35	0.35	0.52
		0.34	0.34	0.29	0.19
		0.32	0.32	0.36	0.29
	10-year	0.35	0.35	0.32	0.58
		0.34	0.34	0.19	0.14
		0.32	0.32	0.49	0.28

Table 3.8: Optimal exposure to jump risks in a jump diffusion market where investors specify the return generating process to be pure diffusion (“Diffusion only”), Poisson jump diffusion (“PJD”), self exciting jump diffusion (“SEJD”) and mutually exciting jump diffusion (“MEJD”). Each column is in the order of “US, Japan, Europe”. Parameter values are contained in Tables 3.4 and 3.7. The figures are normalized so that the exposure to the three regions adds up to 1.

3.6 Conclusion

Inspired by the empirical findings that US equity plays a leading role in international stock returns and that investors tend to over-invest in US equities compared to classic model implications, we postulate a mutually exciting jump diffusion model for equity prices that (1) accounts for the lead-lag relationships of international returns, and (2) theoretically generates a US bias in a representative investor’s equity portfolio.

In particular, we allow for asymmetric jump excitation in international equity prices. We show that the leading role of the US equity can be generated by having larger cross section excitor(s) as the source jump component than the cross section excitor(s) as the target jump component. We solve the asset allocation problem in closed form in this market using the martingale approach and apply the theoretical results to historical returns on MSCI indices.

We show that the optimal portfolio exhibits the US bias, i.e., US equity is over-weighted compared to classic portfolio predictions.

The analytical nature of the solution helps establish the economic intuition of the optimal portfolio risk exposure, which can be summarized into the following properties: (1) The optimal portfolio is sufficiently diversified in the sense that it includes a large number of individual stocks to diversify away idiosyncratic risks. (2) Similar to the Merton mean-variance portfolio, it exploits the covariance structure of the Brownian risks. (3) The exposure to jump risks includes exposure to Poisson jump risk and exposure to contagion risks. The former is a risk-return tradeoff term and the latter is a hedging demand. (4) The portfolio exposure to jump risks is biased towards US equity, which heavily transmits jump risks worldwide and is much less prone to foreign equity markets turmoil. Using parameter estimates on MSCI indices, we show that the US bias observed in the market portfolio can at least in part be explained by excitation asymmetry.

Appendices

3.A Proofs

Proof of Proposition 3.1. Since the market is free of arbitrage opportunities and admits a unique martingale measure Q induced by Equation (3.14), the portfolio optimization problem defined in (3.1) is equivalent to

$$\sup_{X_T \in \mathcal{X}} \mathbb{E}_0 \left[\frac{X_T^{1-\gamma}}{1-\gamma} \right],$$

where \mathcal{X} is the set of admissible square integrable terminal wealths:

$$\mathcal{X} = \{X_T \text{ is } \mathcal{F}_T\text{-measurable} : \mathbb{E}_0[\pi_T X_T] \leq x_0\}.$$

The corresponding Lagrange function is constructed as

$$\mathcal{L} = \frac{X_T^{1-\gamma}}{1-\gamma} + y(x_0 - \pi_T X_T).$$

The optimal terminal wealth X_T^* can be derived by taking the first order condition with respect to X_T :

$$X_T^* = (y\pi_T)^{-\frac{1}{\gamma}},$$

where y is the Lagrange multiplier that satisfies $\mathbb{E}_0[\pi_T X_T^*] = x_0$, which implies that

$$y^{-\frac{1}{\gamma}} = \frac{x_0}{\mathbb{E}_0[\pi_T^{1-1/\gamma}]}.$$

Since at any time $0 \leq t \leq T$ the optimal wealth process satisfies the no arbitrage assumption $X_t^* \pi_t = \mathbb{E}_t[X_T^* \pi_T]$, we have

$$X_t^* = \frac{\mathbb{E}_t[X_T^* \pi_T]}{\pi_t} = \frac{\mathbb{E}_t[y^{-\frac{1}{\gamma}} \pi_T^{1-\frac{1}{\gamma}}]}{\pi_t} = \frac{x_0}{\pi_t} \frac{\mathbb{E}_t[\pi_T^{\frac{\gamma-1}{\gamma}}]}{\mathbb{E}_0[\pi_T^{\frac{\gamma-1}{\gamma}}]}.$$

The solution requires the computation of the expected value of the exponential of a stochastic integral. Let $M_t := \log(\pi_t)$. It holds that

$$\begin{aligned} dM_t &= \frac{d\pi_t}{\pi_t} - \frac{1}{2} \frac{d\langle \pi_t \rangle}{\pi_t^2} \\ &= \left(-\frac{1}{2} \boldsymbol{\eta}'(LL')^{-1} \boldsymbol{\eta} - \sum_{i=1}^n \kappa_i \lambda_{i,t} \right) dt - \boldsymbol{\eta}'(LL')^{-1} d\hat{\mathbf{W}}_t + \sum_{i=1}^n \log(1 + \kappa_i) dN_{i,t}. \end{aligned}$$

Write $\mathbf{Y}_t = (M_t; \boldsymbol{\lambda}_t)$. We have

$$d\mathbf{Y}_t = \boldsymbol{\mu}(\mathbf{Y}_t) dt + \boldsymbol{\sigma}(\mathbf{Y}_t) d\hat{\mathbf{W}}_t + d\mathbf{N}_t.$$

\mathbf{Y}_t admits the affine structure:

$$\begin{cases} \boldsymbol{\mu}(\mathbf{Y}_t) = \mathbf{K}_0 + \mathbf{K}_1 \mathbf{Y}_t, \\ [\boldsymbol{\sigma}(\mathbf{Y}_t) \boldsymbol{\sigma}(\mathbf{Y}_t)']_{ij} = [\mathbf{H}_0]_{ij} + [\mathbf{H}_1]_{ij} \cdot \mathbf{Y}_t, \\ \lambda^i(\mathbf{Y}_t) = l_0^i + l_1^i \cdot \mathbf{Y}_t, \end{cases}$$

where

$$\begin{aligned} K_0 &= \begin{pmatrix} -\frac{1}{2}\eta'(\mathbf{L}\mathbf{L}')^{-1}\eta \\ \alpha\lambda_{1,\infty} \\ \vdots \\ \alpha\lambda_{n,\infty} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & -\kappa_1 & \dots & -\kappa_n \\ 0 & -\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\alpha \end{pmatrix}, \\ H_0 &= \begin{pmatrix} \eta'(\mathbf{L}\mathbf{L}')^{-1}\eta & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad H_1 = 0, \\ l_0^i &= 0, \quad l_1^i = (0; \mathbf{e}_i), \quad i = 1, \dots, n. \end{aligned}$$

Here, \mathbf{e}_i denotes a vector of zeros with 1 at the i^{th} entry. The jump transform $\zeta^i(c)$ that determines the jump size distribution of jump component i is $\zeta^i(c) = (1 + \kappa_i)^{c_1} \exp(\sum_{j=1}^n c_{j+1} \beta_{j,i})$, $i = 1, \dots, n$. According to Duffie et al. [50], the conditional expectation takes the form

$$\mathbb{E}_t \left[\pi_T^{\frac{\gamma-1}{\gamma}} \right] = \pi_t^{\frac{\gamma-1}{\gamma}} \exp(A(t) + \mathbf{B}'(t)\boldsymbol{\lambda}_t),$$

where $A(t)$ and $\mathbf{B}(t)$ satisfy

$$\begin{cases} \dot{\mathbf{B}}(t) = \frac{\gamma-1}{\gamma} \boldsymbol{\kappa} + \alpha \mathbf{B}(t) - (\boldsymbol{\kappa} + 1)^{\frac{\gamma-1}{\gamma}} \circ e^{\beta' \mathbf{B}(t)} + 1, \\ \dot{A}(t) = \frac{\gamma-1}{2\gamma} \eta'(\mathbf{L}\mathbf{L}')^{-1}\eta - \alpha \mathbf{B}'(t)\boldsymbol{\lambda}_\infty, \end{cases}$$

with $A(T) = 0, \mathbf{B}(T) = \mathbf{0}$. Therefore it holds that

$$X_t^* = x_0 \pi_t^{-1/\gamma} \exp(A(t) + \mathbf{B}'(t)\boldsymbol{\lambda}_t - A(0) - \mathbf{B}'(0)\boldsymbol{\lambda}_0),$$

from which we derive the SDE of the optimal wealth path

$$\begin{aligned} \frac{dX_t^*}{X_t^*} &= \frac{1}{\gamma} \eta'(\mathbf{L}\mathbf{L}')^{-1}\eta dt + \frac{1}{\gamma} \eta'(\mathbf{L}\mathbf{L}')^{-1} d\hat{\mathbf{W}}_t \\ &\quad + \sum_{l=1}^n \left((1 + \kappa_l)^{-\frac{1}{\gamma}} \exp(\beta_l' \mathbf{B}(t)) - 1 \right) (dN_{l,t} - (1 + \kappa_l) \lambda_{l,t} dt) \\ &= \boldsymbol{\theta}^{W*'} (d\hat{\mathbf{W}}_t + \boldsymbol{\eta} dt) + \sum_{l=1}^n \left(\exp(\theta_l^{N_l*}) - 1 \right) (dN_{l,t} - (1 + \kappa_l) \lambda_{l,t} dt). \end{aligned}$$

□

Proof of Lemma 3.1. The economy has constant investment opportunities with asset prices generated by geometric Brownian motions. We can write the asset price dynamics in matrix form as

$$\text{diag}(\mathbf{S}_t)^{-1} d\mathbf{S}_t = \boldsymbol{\sigma} \boldsymbol{\eta} dt + \boldsymbol{\sigma} d\hat{\mathbf{W}}_t.$$

Then according to Merton [96], the portfolio weights are given by

$$\mathbf{w}_{\text{Merton}}^* = \frac{1}{\gamma} (\boldsymbol{\sigma} \mathbf{L} \mathbf{L}' \boldsymbol{\sigma}')^{-1} \boldsymbol{\sigma} \boldsymbol{\eta}.$$

The portfolio wealth process follows

$$\frac{dX_t^*}{X_t^*} = \mathbf{w}_{\text{Merton}}^{*\prime} \boldsymbol{\sigma} \boldsymbol{\eta} dt + \mathbf{w}_{\text{Merton}}^{*\prime} \boldsymbol{\sigma} d\hat{\mathbf{W}}_t.$$

It holds that

$$\boldsymbol{\theta}_{\text{Merton}}^{W*} = \boldsymbol{\sigma}' \mathbf{w}_{\text{Merton}}^* = \frac{1}{\gamma} (\mathbf{L} \mathbf{L}')^{-1} \boldsymbol{\eta} = \boldsymbol{\theta}^{W*}.$$

□

Proof of Lemma 3.2. To follow the price dynamics of $\{F_t\} \in \mathcal{S}^2$ given by Equation (3.22), it requires that for every $i = 1, \dots, n$,

$$\begin{cases} w_{i,t}^{k=1} \sigma_i^{k=1} + w_{i,t}^{k=2} \sigma_i^{k=2} = g_i, \\ w_{i,t}^{k=1} z_i^{k=1} + w_{i,t}^{k=2} z_i^{k=2} = h_i. \end{cases} \quad (3.37)$$

Hence at time t , for every region i , we have two unknowns and two equations that are not linearly dependent. We can solve for the weighting vector at any time t and replicate the price dynamics of F_t . Since $\{P_t\}$ and $\{F_t\}$ are solutions to the same stochastic differential equations, they are indistinguishable processes. □

Proof of Proposition 3.2. Inspired by Merton [97], p. 137, the portfolio weights $w_{i,t}^k$ can be restricted such that they satisfy

$$\begin{cases} \sum_{k=1}^{m_i} w_{i,t}^k \sigma_i^k = g_i, \\ \sum_{k=1}^{m_i} w_{i,t}^k z_i^k = h_i, \end{cases} \quad (3.38)$$

and can be represented as

$$w_{i,t}^k =: \frac{\omega_{i,t}^k}{m_i}, \quad (3.39)$$

where $\omega_{i,t}^k$ is a finite constant, independent of the total number of assets. The replicating portfolio P_t thus follows

$$\begin{aligned} \frac{dP_t}{P_t} &= \sum_{i=1}^n g_i (d\hat{W}_{i,t} + \eta_i dt) + \sum_{i=1}^n h_i (dN_{i,t} - (1 + \kappa_i) \lambda_{i,t} dt) + \sum_{i=1}^n \sum_{k=1}^{m_i} w_{i,t}^k \nu_i^k dZ_{i,t}^k \\ &= \frac{dF_t}{F_t} + \sum_{i=1}^n d\zeta_{i,t}(m_i), \end{aligned}$$

where

$$d\zeta_{i,t}(m_i) := \sum_k^{m_i} w_{i,t}^k \nu_i^k dZ_{i,t}^k.$$

Let $d\varepsilon_{i,t}^k = \omega_{i,t}^k \nu_i^k dZ_{i,t}^k$. We have that

$$d\zeta_{i,t}(m_i) = \sum_k^{m_i} w_{i,t}^k \nu_i^k dZ_{i,t}^k = \frac{\sum_{k=1}^{m_i} d\varepsilon_{i,t}^k}{m_i}.$$

Note that $d\varepsilon_{i,t}^k$ are independent, since each $dZ_{i,t}^k$ represents the idiosyncratic risk of asset k in region i . By the Law of Large Numbers, it holds that, for all i , with probability one,

$$d\zeta_{i,t}(m_i) \rightarrow 0,$$

which implies the stated result. □

Proof of Proposition 3.3. For a given well-diversified portfolio (i.e., free of idiosyncratic risks), suppose its exposure to the systematic Brownian risks is $\hat{\theta}_t^W$, and its exposure to the jump risks is $\hat{\theta}_t^N$. The wealth process is given by

$$\begin{aligned} d \log \hat{X}_t = & \left(-\frac{1}{2} \hat{\theta}_t^{W'} L L' \hat{\theta}_t^W + \hat{\theta}_t^{W'} \eta - \sum_{l=1}^n (e^{\hat{\theta}_t^{N_l}} - 1) (1 + \kappa_l) \lambda_{l,t} \right) dt + \hat{\theta}_t^{W'} d\hat{W}_t \\ & + \sum_{l=1}^n \hat{\theta}_t^{N_l} dN_{l,t}, \end{aligned}$$

with $X_0 = x_0$. Similar to the proof of Proposition 3.1, we employ the formula in Duffie, Pan, and Singleton [50] to evaluate the expected utility of terminal wealth and write $\mathbf{Y}_t = (\log \hat{X}_t, \boldsymbol{\lambda}_t)$. We have

$$d\mathbf{Y}_t = \boldsymbol{\mu}(\mathbf{Y}_t) dt + \boldsymbol{\sigma}(\mathbf{Y}_t) d\hat{W}_t + d\mathbf{N}_t.$$

\mathbf{Y}_t admits the affine structure:

$$\begin{cases} \boldsymbol{\mu}(\mathbf{Y}_t) = \mathbf{K}_0 + \mathbf{K}_1 \mathbf{Y}_t, \\ [\boldsymbol{\sigma}(\mathbf{Y}_t) \boldsymbol{\sigma}(\mathbf{Y}_t)']_{ij} = [\mathbf{H}_0]_{ij} + [\mathbf{H}_1]_{ij} \cdot \mathbf{Y}_t, \\ \lambda^i(\mathbf{Y}_t) = l_0^i + \mathbf{l}_1^i \cdot \mathbf{Y}_t, \end{cases}$$

where

$$\begin{aligned} \mathbf{K}_0 &= \begin{pmatrix} -\frac{1}{2} \hat{\theta}_t^{W'} L L' \hat{\theta}_t^W + \hat{\theta}_t^{W'} \eta \\ \alpha \lambda_{1,\infty} \\ \vdots \\ \alpha \lambda_{n,\infty} \end{pmatrix}, \\ \mathbf{K}_1 &= \begin{pmatrix} 0 & -(1 + \kappa_1) (e^{\hat{\theta}_t^{N_1}} - 1) & \dots & -(1 + \kappa_n) (e^{\hat{\theta}_t^{N_n}} - 1) \\ 0 & -\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\alpha \end{pmatrix}, \\ \mathbf{H}_0 &= \begin{pmatrix} \hat{\theta}_t^{W'} L L' \hat{\theta}_t^W & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad \mathbf{H}_1 = 0, \\ l_0^i &= 0, \quad \mathbf{l}_1^i = (0; \mathbf{e}_i), \quad i = 1, \dots, n. \end{aligned}$$

The jump transform $\zeta^i(c)$ is given by $\zeta^i(c) = \exp(c_1 \hat{\theta}_t^{N_i} + \sum_{j=1}^n c_{j+1} \beta_{j,i})$, $i = 1, \dots, n$. The conditional expectation takes the form

$$\frac{1}{1 - \gamma} \mathbb{E}[\hat{X}_T^{1-\gamma}] = \frac{x_0^{1-\gamma}}{1 - \gamma} \exp(\hat{A}(0) + \hat{\mathbf{B}}'(0) \boldsymbol{\lambda}_0), \quad (3.40)$$

where

$$\begin{aligned} \dot{\hat{\mathbf{B}}}(t) &= (1 - \gamma)(\boldsymbol{\kappa} + 1) \circ (e^{\hat{\theta}_t^N} - 1) + \alpha \hat{\mathbf{B}}(t) - \exp((1 - \gamma) \hat{\theta}_t^N + \beta' \hat{\mathbf{B}}) + 1, \\ \dot{\hat{A}}(t) &= -(1 - \gamma) \left(\hat{\theta}_t^{W'} \eta - \frac{1}{2} \hat{\theta}_t^{W'} L L' \hat{\theta}_t^W \right) - \frac{1}{2} (1 - \gamma)^2 \hat{\theta}_t^{W'} L L' \hat{\theta}_t^W - \alpha \hat{\mathbf{B}}' \boldsymbol{\lambda}_\infty, \end{aligned}$$

with $\hat{B}(T) = 0$, $\hat{A}(T) = 0$. The certainty-equivalent wealth is

$$\hat{x}_0 = x_0 \exp \left(\frac{1}{1-\gamma} (\hat{A}(0) + \hat{B}'(0)\lambda_0) \right).$$

□

3.B Portfolio construction with a large basket of assets

In this section we demonstrate how to represent the portfolio weights by Equation (3.39) in a given region, thereby completing the proof of Proposition 3.2.

Let (σ^k, ν^k, z^k) , $k = 1, \dots, m$, characterize the price dynamics of individual asset k in a given region (for which we omit the subscript denoting the region identify). First, we randomly pair the m assets into $m/2$ pairs, denoted by $p, p = 1, \dots, m/2$.¹⁴ For any pair p with asset S^k, S^l , let ω_t^k, ω_t^l be the weights on assets S^k, S^l as if S^k, S^l make up the entire portfolio in that region. Since the optimal regional portfolio produces the risk exposure f, g , it holds that

$$\begin{cases} \omega_t^k \sigma^k + \omega_t^l \sigma^l = f, \\ \omega_t^k z^k + \omega_t^l z^l = g. \end{cases}$$

Since assets are not linearly dependent, the linear equation system has a unique solution for ω_t^k, ω_t^l . Define the paired portfolio P_t^p as

$$dP_t^p := \omega_t^k dS_t^k + \omega_t^l dS_t^l.$$

It holds that the price of any paired portfolio X_t^p follows the dynamic of the local optimal portfolio plus some tracking errors:

$$\frac{dP_t^p}{P_t^p} = f(d\hat{W}_t + \eta dt) + g(dN_t - (1 + \kappa)\lambda_t dt) + d\zeta_t^p,$$

where

$$d\zeta_t^p := \omega_t^k \nu^k dZ_t^k + \omega_t^l \nu^l dZ_t^l =: \nu_t^p dZ_t^p.$$

Here, dZ_t^p is the idiosyncratic Brownian motion of the paired portfolio, independent of all other paired portfolios. The last equality of the previous equation holds in distribution.

Now that we have $m/2$ paired portfolios. Any weighted average of these paired portfolios produces the optimal exposure to systematic risk factors. Denote the weights on the paired portfolios by $\mathbf{a}(m) = (a^1, \dots, a^{m/2})'$. The replicating portfolio $P_t(m)$ is given by

$$dP_t(m) = \mathbf{a}(m)' dP_t^p.$$

For example, in an equal weighted scheme, each paired portfolio is assigned the same weight $a^p = 2/m, p = 1, \dots, m/2$. Then asset k in the m -asset portfolio gets weight

$$w_t^k = \frac{2\omega_t^k}{m},$$

in which $2\omega_t^k$ is independent of the number of asset m . We have therefore showed that the representation proposed in (3.39) is indeed feasible.

¹⁴In case m is odd, we simply create a $m + 1$ asset by, for example, $S^{m+1} = \frac{1}{2}S^m + \frac{1}{2}S^{m-1}$. Therefore we assume m is even without loss of generality.

3.C Portfolios that exhibit home bias

The mutually exciting jump diffusion model proposed in Chapter 3.2.3 allows for ambiguity averse preferences, modeled by, for example, the multiple prior preferences proposed by Gilboa and Schmeidler [70]. An investor from region i does not have the full knowledge of the true probability law of asset prices. Instead of optimizing the expected utility under the physical measure, the investor specifies plausible candidate ambiguity measures $G_i \in \mathcal{G}$ and optimizes the expected utility under the worst case scenario:

$$\sup_{\theta^W, \theta_i^N} \inf_{G_i \in \mathcal{G}} \mathbb{E}_0^G[u(X_T)]. \quad (3.41)$$

In general, Investors from different regions have different ambiguity levels towards other regions. Equation (3.41) describes the optimization problem for an investor from region i with ambiguity measure \mathcal{G} . We omit the region identity i for notation simplicity.

In order to have a tractable solution, we further assume that ambiguity comes from parameter uncertainty. Instead of relying entirely on the point estimates of risk premiums, investors construct confidence intervals of parameter estimates, and optimize under the worst case parameter values. Based on the pure diffusion model in Garlappi, Uppal, and Wang [68], we restrict the Brownian risk premium η to lie within $[\underline{\eta}, \bar{\eta}]$, and similarly, the jump risk premium parameter κ to lie within $[\underline{\kappa}, \bar{\kappa}]$. Then we may characterize the ambiguity measure G by η^G, κ^G and write $G(\eta^G, \kappa^G)$, where

$$\eta^G \in [\underline{\eta}, \bar{\eta}], \quad 0 \leq \underline{\eta} \leq \eta \leq \bar{\eta}, \quad [\eta^G]_i = \eta_i, \quad (3.42)$$

$$\kappa^G \in [\underline{\kappa}, \bar{\kappa}], \quad 0 \leq \underline{\kappa} \leq \kappa \leq \bar{\kappa}, \quad [\kappa^G]_i = \kappa_i. \quad (3.43)$$

All operators are element-wise comparisons. The constraints (3.42) and (3.43) specify the ambiguity level – the larger the sets $[\underline{\eta}, \bar{\eta}]$, $[\underline{\kappa}, \bar{\kappa}]$, the more ambiguity averse the investor is. Nevertheless, however large the investor's ambiguity aversion is, he/she has no ambiguity towards the home equity, which is reflected through $[\eta^G]_i = \eta_i$, $[\kappa^G]_i = \kappa_i$. Since $\eta \in [\underline{\eta}, \bar{\eta}]$, $\kappa \in [\underline{\kappa}, \bar{\kappa}]$, the ambiguity sets contain the true measure P . Notice that the ambiguity set \mathcal{G} is convex and compact.

Under the ambiguity measure $G(\eta^G, \kappa^G)$, asset k from region i follows the dynamics

$$\frac{dS_{i,t}^k}{S_{i,t}^k} = \sigma_i^k (d\hat{W}_{i,t}^G + \eta_i^G dt) + \nu_i^k dZ_{i,t}^{k,G} + z_i (dN_{i,t}^G - (1 + \kappa_i^G) \lambda_{i,t} dt),$$

where $\hat{W}_{i,t}^G, Z_{i,t}^{k,G}$ are martingales under the ambiguity measure G , and $N_{i,t}^G, i = 1, \dots, n$, are mutually exciting jumps with intensities $(1 + (\kappa - \kappa^G)) \circ \lambda_t$ under the ambiguity measure G . Consistent with the asset dynamics under G is the measure change process $\frac{dG}{dP} |_{\mathcal{F}_t} = \xi_t$ that follows

$$\frac{d\xi_t}{\xi_{t-}} = -(\eta - \eta^G)'(LL')^{-1} d\hat{W}_t + \sum_{i=1}^n (\kappa_i - \kappa_i^G) (dN_{i,t} - \lambda_{i,t} dt), \quad \xi_0 = 1. \quad (3.44)$$

Expected utility is nested when the ambiguity set \mathcal{G} is a singleton in which the physical measure P is the only element. The optimal risk exposure $\theta^{W*}, \theta_t^{N*}$ derived in Proposition 3.1 can therefore be regarded as a special case of the more general function $\theta^{W*}(\eta^G), \theta_t^{N*}(\kappa^G)$, when $\eta^G = \eta, \kappa^G = \kappa$. The following proposition is a generalization of Proposition 3.1 to incorporate ambiguity averse preferences.

Proposition 3.4 (Optimal portfolio choice with ambiguity aversion). *In a contagious economy with asset prices given by (3.13), suppose that a representative investor from a certain region is ambiguity averse and aims to solve*

$$\sup_{\theta^W, \theta_t^N} \inf_{G \in \mathcal{G}} \mathbb{E}^G \left[\frac{X_T^{1-\gamma}}{1-\gamma} \middle| X_0 = x_0 \right], \gamma > 1, \quad (3.45)$$

with

$$\mathcal{G} = \left\{ G(\boldsymbol{\eta}^G, \boldsymbol{\kappa}^G) : \frac{dG}{dP} \Big|_{\mathcal{F}_t} = \xi_t \right\},$$

where ξ_t is given by (3.44). Then the optimal portfolio value $X^*(\boldsymbol{\eta}, \boldsymbol{\kappa})$ follows the dynamics

$$\begin{aligned} \frac{dX_t^*}{X_t^*} = & \boldsymbol{\theta}^{W*}(\boldsymbol{\eta})' (d\hat{\mathbf{W}}_t + \boldsymbol{\eta} dt) \\ & + \sum_{i=1}^n \left(\exp(\theta_t^{N_i*}(\boldsymbol{\kappa})) - 1 \right) \left(dN_{i,t} - (1 + \kappa_i) \lambda_{i,t} dt \right), \end{aligned}$$

where the risk exposure is given by

$$\begin{cases} \boldsymbol{\theta}^{W*}(\boldsymbol{\eta}) = \frac{1}{\gamma} (\mathbf{L}\mathbf{L}')^{-1} \boldsymbol{\eta}, \\ \boldsymbol{\theta}_t^{N*}(\boldsymbol{\kappa}) = -\frac{1}{\gamma} \log(1 + \boldsymbol{\kappa}) + \boldsymbol{\beta}' \mathbf{B}(\boldsymbol{\kappa}; t). \end{cases} \quad (3.46)$$

Here, $\mathbf{B}(\boldsymbol{\kappa}; t)$ is given by

$$\dot{\mathbf{B}}(\boldsymbol{\kappa}; t) = \frac{\gamma-1}{\gamma} \boldsymbol{\kappa} + \alpha \mathbf{B}(\boldsymbol{\kappa}; t) - (\boldsymbol{\kappa} + 1)^{\frac{\gamma-1}{\gamma}} \circ e^{\boldsymbol{\beta}' \mathbf{B}(\boldsymbol{\kappa}; t)} + 1, \quad (3.47)$$

with $\mathbf{B}(\boldsymbol{\kappa}; T) = 0$.

Proof of Proposition 3.4. Since any prior $G \in \mathcal{G}$ is equivalent to P , and \mathcal{G} is by construction convex and compact, it holds that¹⁵

$$\sup_{\theta^W, \theta_t^N} \inf_{G \in \mathcal{G}} \mathbb{E}^G \left[\frac{X_T^{1-\gamma}}{1-\gamma} \middle| X_0 = x_0 \right] = \inf_{G \in \mathcal{G}} \sup_{\theta^W, \theta_t^N} \mathbb{E}^G \left[\frac{X_T^{1-\gamma}}{1-\gamma} \middle| X_0 = x_0 \right].$$

For a given measure, we first solve the inner supremum problem. The result in Proposition 3.1 can be directly applied. For any $G(\boldsymbol{\eta}^G, \boldsymbol{\kappa}^G) \in \mathcal{G}$, the optimal portfolio exposure to risk factors is given by

$$\begin{cases} \boldsymbol{\theta}^{W*}(\boldsymbol{\eta}^G) = \frac{1}{\gamma} (\mathbf{L}\mathbf{L}')^{-1} \boldsymbol{\eta}^G, \\ \boldsymbol{\theta}_t^{N*}(\boldsymbol{\kappa}^G) = -\frac{1}{\gamma} \log(1 + \boldsymbol{\kappa}^G) + \boldsymbol{\beta}' \mathbf{B}(\boldsymbol{\kappa}^G; t). \end{cases}$$

where

$$\dot{\mathbf{B}}(\boldsymbol{\kappa}^G; t) = \frac{\gamma-1}{\gamma} \boldsymbol{\kappa}^G + \alpha \mathbf{B}(\boldsymbol{\kappa}^G; t) - (\boldsymbol{\kappa}^G + 1)^{\frac{\gamma-1}{\gamma}} \circ e^{\boldsymbol{\beta}' \mathbf{B}(\boldsymbol{\kappa}^G; t)} + 1,$$

with $\mathbf{B}(\boldsymbol{\kappa}^G; T) = 0$. Having solved the inner supremum problem, one can easily show that the indirect utility function given in (3.19) is strictly decreasing in $\boldsymbol{\eta}, \boldsymbol{\kappa}$ for $\boldsymbol{\eta} \geq 0, \boldsymbol{\kappa} \geq 0$. Therefore it suffices to replace $\boldsymbol{\eta}^G$ by $\boldsymbol{\eta}$, and $\boldsymbol{\kappa}^G$ by $\boldsymbol{\kappa}$ in $\boldsymbol{\theta}^{W*}(\boldsymbol{\eta}^G), \boldsymbol{\theta}_t^{N*}(\boldsymbol{\kappa}^G)$. \square

The proposition confirms that the results of Garlappi et al. [68], who find that ambiguity aversion towards expected return in a pure diffusion market is equivalent to a lower risk premium, can be readily extended to our jump diffusion model. The coexistence of home bias and foreign bias found in investors' equity portfolios can therefore be generated by taking realistic values of the ambiguity parameters.

¹⁵The proof of this equality can be found in, for example, Theorem 2 of Schied and Wu [108].

3.D Transition density of jump arrivals

We first give the algorithm of computing the log transition density function of jump arrivals for the univariate case.

Algorithm 3.1 (Univariate). *The algorithm of computing the transition densities for a univariate self exciting jump process, given the K jump arrival times $\{u_1, \dots, u_K\}$ within a time span $[0, T]$, conditional on $\lambda_0 = \lambda_\infty, N_0 = 0$:*

1. *Set the initial conditions: $u_0 = 0, \lambda_t = \lambda_\infty, t \in [0, u_1^-]$ and $k \in \{1, 2, \dots, K\}$. Define $u_{K+1} := T$.*
2. *Denote the log likelihood of observing a jump occurrence at time u_k conditional on the information available by time u_{k-1} by $f(u_k | \mathcal{F}_{k-1})$. It holds that*

$$f(u_k | \mathcal{F}_{k-1}) = \log \lambda_{u_k^-} - \Lambda(k),$$

where

$$\Lambda(k) := -\frac{1}{\alpha}(\lambda_{u_{k-1}} - \lambda_\infty)(e^{-\alpha(u_k - u_{k-1})} - 1) + \lambda_\infty(u_k - u_{k-1}).$$

3. *Record the jump intensity at u_k to be*

$$\lambda_{u_k} = \lambda_{u_k^-} + \beta.$$

4. *Compute the intensity just before the next jump arrival u_{k+1} :*

$$\lambda_{u_{k+1}^-} = (\lambda_{u_k} - \lambda_\infty)e^{-\alpha(u_{k+1} - u_k)} + \lambda_\infty.$$

5. *Repeat step 2-4 until $k = K$.*
6. *The total log likelihood L is*

$$L = \sum_{k=1}^K f(u_k | \mathcal{F}_{k-1}) - \Lambda(K+1).$$

Proof. Given the k^{th} jump arrival $u_k, k = 1, \dots, K$, the intensity of the point process at $t \in [u_{k-1}, u_k]$ follows

$$d\lambda_t = \alpha(\lambda_\infty - \lambda_t) dt, \quad u_{k-1} \leq t < u_k,$$

with

$$\lambda_{u_k} = \lambda_{u_k^-} + \beta.$$

The differential equation admits the solution

$$\lambda_t = \begin{cases} (\lambda_{u_{k-1}} - \lambda_\infty)e^{-\alpha(t-u_{k-1})} + \lambda_\infty, & \text{if } u_{k-1} \leq t < u_k, \\ (\lambda_{u_{k-1}} - \lambda_\infty)e^{-\alpha(t-u_{k-1})} + \lambda_\infty + \beta, & \text{if } t = u_k. \end{cases}$$

Conditional on the jump arrival u_{k-1} , until the next jump arrival, the point process is a time-inhomogeneous Poisson jump process with exponentially decaying intensities. Denote the

probability of a jump occurrence at time u_k and no jump occurrences between time u_{k-1} and u_k^- by $\mathbb{P}(u_k|\mathcal{F}_{u_{k-1}})$. It holds that

$$\begin{aligned}\mathbb{P}(u_k|\mathcal{F}_{u_{k-1}}) &= \mathbb{P}(\text{waiting time} = (u_k - u_{k-1})|\mathcal{F}_{u_{k-1}}) \\ &= \lambda_{u_k^-} \exp\left(-\int_{u_{k-1}}^{u_k^-} \lambda_s ds\right).\end{aligned}$$

Define

$$\Lambda(k) := \int_{u_{k-1}}^{u_k^-} \lambda_s ds.$$

It holds that

$$\begin{aligned}\Lambda(k) &= \int_{u_{k-1}}^{u_k} \left((\lambda_{u_{k-1}} - \lambda_\infty)e^{-\alpha(t-u_{k-1})} + \lambda_\infty\right) dt \\ &= -\frac{1}{\alpha}(\lambda_{u_{k-1}} - \lambda_\infty)(e^{-\alpha(u_k-u_{k-1})} - 1) + \lambda_\infty(u_k - u_{k-1}).\end{aligned}$$

When $k = K + 1$, the probability of no jump occurrences until T can be computed as

$$\begin{aligned}\mathbb{P}(u_{K+1}|\mathcal{F}_{u_K}) &:= \mathbb{P}(N_{u_{K+1}} - N_{u_K} = 0|\mathcal{F}_{u_K}) \\ &= \mathbb{P}(\text{waiting time} > (T - u_K)|\mathcal{F}_{u_K}) \\ &= \exp\left(-\int_{u_K}^T \lambda_s ds\right) \\ &= \exp(-\Lambda(K+1)).\end{aligned}$$

□

The algorithm can be easily generalized to a multivariate setting.

Algorithm 3.2 (Multivariate). *The algorithm of computing the transition densities for a D -dimensional multivariate mutually exciting jump process, given the K joint jump times $\{u_1, \dots, u_K\}$ within a time span $[0, T]$, conditional on $\lambda_0 = \lambda_\infty$, $\mathbf{N}_0 = \mathbf{0}$:*

1. *Set the initial conditions: $u_0 = 0$, $\lambda_t = \lambda_\infty$, $t \in [0, u_1^-]$, and $k \in \{1, \dots, K\}$. Define $u_{K+1} := T$.*
2. *Decide u_k belongs to which jump component. Denote the jump component by d . The log transition probability of u_k is*

$$f(u_k|\mathcal{F}_{u_{k-1}}) = \log \lambda_{d, u_k^-} - \sum_{j=1}^D \Lambda(k, j), \quad (3.48)$$

where

$$\Lambda(k, j) := -\frac{1}{\alpha}(\lambda_{j, u_{k-1}} - \lambda_{j, \infty})(e^{-\alpha(u_k - u_{k-1})} - 1) + \lambda_{j, \infty}(u_k - u_{k-1}). \quad (3.49)$$

3. *Record the jump intensity at u_k to be*

$$\lambda_{u_k} = \lambda_{u_k^-} + \beta_d,$$

where β_d is the d^{th} column of the excitation matrix β .

4. Compute the intensities just before the next jump arrival u_{k+1} : for $j = 1, \dots, D$,

$$\lambda_{j,u_{k+1}^-} = (\lambda_{j,u_k} - \lambda_{j,\infty})e^{-\alpha(u_{k+1}-u_k)} + \lambda_{j,\infty}.$$

5. Repeat step 2-4 until $k = K$.

6. The total log likelihood L is

$$L = \sum_1^K f(u_k | \mathcal{F}_{k-1}) - \sum_{j=1}^D \Lambda(K+1, j). \quad (3.50)$$

3.E Small sample behavior

To examine the small sample behavior of the maximum likelihood estimators, we run 5,000 Monte Carlo simulation experiments. We set the data generating parameters (DGP) to be the MEJD parameter estimates given in Table 3.4. We generate 43 years of jump arrivals using the exact simulation algorithm proposed by Dassios and Zhao [43]. For each simulated sample, we estimate α, β using the maximum likelihood, using the same starting values as in the empirical estimation. Table 3.9 reports the mean, standard error and the quartiles of the estimates.

MEJD estimation										
	α	$\beta_{1,1}$	$\beta_{2,1}$	β_{31}	$\beta_{1,2}$	$\beta_{2,2}$	β_{32}	β_{13}	β_{23}	β_{33}
DGP	29.3	12.6	9.3	24.1	0.0	7.4	2.7	5.8	2.3	8.1
Mean	29.6	11.8	9.4	24.5	0.3	7.0	2.7	5.7	2.5	7.6
Std	3.9	3.5	3.2	5.2	0.6	2.4	1.7	2.2	1.8	2.9
25% quantile	27.1	9.5	7.2	21.0	0.0	5.4	1.6	4.2	1.2	5.6
Median	29.5	12.0	9.3	24.2	0.0	7.0	2.6	5.6	2.3	7.5
75% quantile	32.1	14.2	11.5	27.9	0.4	8.6	3.8	7.1	3.6	9.5

Table 3.9: Mean and quartiles of parameter estimates from 5,000 Monte Carlo experiments.

3.F Robustness checks

Risk premium calibration

All parameters which are used to produce Table 3.8 are estimated from the historical data except for the risk premium parameters η, κ . In this section, we vary the risk premium parameters, while keeping the sum of the variance and jump premium equal to the historical equity premium. We will see in Table 3.10 and 3.11 that varying the risk premiums does not have a qualitative impact on the US bias.

Table 3.10 reports the model predicted jump exposure for different combinations of the variance premium and the jump premium. The variance premium is restricted such that the predicted Brownian exposure θ^{W*} coincides with the Brownian risk exposure of the market portfolio.

Optimal portfolio exposure to jump risks (imposing θ^{W*} to be equal to the market portfolio exposure to Brownian risks)						
Scenario	VP	JP	Diffusion	PJD	SEJD	MEJD
I	0.74	5.10	0.38	0.37	0.38	0.60
	0.25	5.00	0.31	0.32	0.16	0.12
	0.45	5.24	0.31	0.31	0.46	0.28
II	1.30	4.54	0.37	0.36	0.35	0.59
	0.44	4.81	0.32	0.32	0.17	0.13
	0.78	4.90	0.31	0.32	0.48	0.28
III	1.86	3.99	0.35	0.35	0.32	0.58
	0.63	4.62	0.34	0.34	0.19	0.14
	1.12	4.57	0.32	0.32	0.49	0.28
IV	2.79	3.06	0.31	0.31	0.27	0.55
	0.94	4.31	0.37	0.36	0.24	0.17
	1.68	4.01	0.32	0.32	0.49	0.29
V	3.72	2.13	0.26	0.27	0.22	0.49
	1.25	4.00	0.41	0.40	0.29	0.21
	2.23	3.45	0.33	0.33	0.49	0.29

Table 3.10: Model predicted jump exposure for different combinations of variance premiums (VP) and the jump premiums (JP). The models under consideration are pure diffusion (“Diffusion”), Poisson jump diffusion (“PJD”), self exciting jump diffusion (“SEJD”) and mutually exciting jump diffusion (“MEJD”). Within each scenario, every column is in the order of “US, Japan, Europe”. The “Variance premium” and “Jump premium” are reported in percentage per annum. The sum of the variance premium and the jump premium is equal to the equity premium, which is held fixed at historical levels. Variance premium is calibrated such that the model predicted Brownian exposure coincides with that of the market portfolio. Parameter values are contained in Table 3.4 with $\gamma = 5$, $T = 10$. The figures are normalized so that the exposure to the three regions adds up to 1.

Table 3.11 reports both the model predicted Brownian and jump exposure without imposing the restriction that the Brownian exposure θ^{W*} is equal to the Brownian risk exposure of the market portfolio. Observe that given the variance premium, all four models predict the same optimal Brownian risk exposure θ^{W*} . We report the optimal risk exposure under equal jump risk premiums across all regions (the first scenario), under equal variance premium across all regions (the second scenario), and when the total equity premium in each region is divided equally into the variance premium and jump premium (the third scenario).

Optimal portfolio exposure to risk factors (without imposing θ^{W*} to be equal to the market portfolio exposure to Brownian risks)							
Scenario	VP	JP	θ^{W*}	Diffusion	PJD	SEJD	MEJD
Equal	2.85	3.00	0.36	0.38	0.37	0.38	0.55
jump	2.25	3.00	0.26	0.32	0.32	0.23	0.16
premium	2.69	3.00	0.39	0.30	0.30	0.39	0.28
Equal	3.00	2.85	0.31	0.42	0.41	0.41	0.56
variance	3.00	2.25	0.32	0.27	0.28	0.20	0.15
premium	3.00	2.69	0.37	0.31	0.31	0.38	0.29
50-50	2.92	2.92	0.33	0.40	0.39	0.39	0.55
premium	2.62	2.62	0.29	0.30	0.30	0.22	0.16
division	2.84	2.84	0.37	0.31	0.31	0.39	0.29

Table 3.11: Model predicted portfolio exposure to Brownian and jump risk factors for different combinations of variance premiums (VP) and jump premiums (JP). The models under consideration are pure diffusion (“Diffusion”), Poisson jump diffusion (“PJD”), self exciting jump diffusion (“SEJD”) and mutually exciting jump diffusion (“MEJD”). Within each scenario, every column is in the order of “US, Japan, Europe”. The “Variance premium” and “Jump premium” are reported in percentage per annum. The sum of the variance premium and the jump premium is equal to the equity premium, which is held fixed at historical levels. The first block reports the optimal risk exposure under equal jump risk premiums across all regions; the second block reports the optimal risk exposure under equal variance premium across all regions; the last block reports the optimal risk exposure when the total equity premium in each region is divided equally into the variance premium and jump premium. Parameter values are contained in Table 3.4 with $\gamma = 5$, $T = 10$. The figures are normalized so that the exposure to the three regions adds up to 1.

Time zone differences

The econometric estimation is conducted using daily data on international equity returns. To account for differences in market opening times, we re-estimate the excitation parameter estimates by lagging the US returns by one day. Table 3.12 reports the resulting excitation parameters of the SEJD and MEJD models. Observe that the SEJD parameters are not affected, because the SEJD model does not take into account the interdependence structure among jumps. With respect to the parameter estimates of the MEJD model, compared to Table 3.4, the self excitor of the US is smaller and not significant at the 95% level. Nevertheless, the cross section exciters from the US to Japan and EU are statistically significant and large in magnitude compared to the reverse directions.

Excitation parameter estimates: US returns lagged by one day									
	SEJD						MEJD		
α	17.7***						30.9***		
β	$\begin{pmatrix} 11.8*** & 0 & 0 \\ 0 & 8.8*** & 0 \\ 0 & 0 & 13.6*** \end{pmatrix}$						$\begin{pmatrix} 4.4 & 0.00 & 13.2*** \\ 9.4*** & 7.8*** & 2.6 \\ 14.2*** & 3.5^* & 13.7*** \end{pmatrix}$		

Table 3.12: Excitation parameter estimates when the US returns are lagged by one day. “SEJD” represents the self exciting jump diffusion model and “MEJD” represents the mutually exciting jump diffusion model. *, **, *** indicate significance at 95%, 97.5%, and 99.5% confidence levels, respectively.

Sub-sample estimation

As a third robustness check, we estimate the excitation parameters over a subsample of the full sample, excluding data from the first one-third of the sample. Table 3.13 reports the subsample parameter estimates. The excitation structure is not qualitatively different from the full-sample estimation results.

Excitation parameter estimates: latest two-thirds of the sample									
	SEJD						MEJD		
α	18.4***						32.8***		
β	$\begin{pmatrix} 12.1*** & 0 & 0 \\ 0 & 8.9*** & 0 \\ 0 & 0 & 13.4*** \end{pmatrix}$						$\begin{pmatrix} 14.9*** & 0.00 & 6.6** \\ 9.0** & 6.5** & 2.8 \\ 24.2*** & 3.0 & 7.0*** \end{pmatrix}$		

Table 3.13: Excitation parameter estimates over a subsample of the full sample, excluding the starting one third of the sample. “SEJD” represents the self exciting jump diffusion model and “MEJD” represents the mutually exciting jump diffusion model. *, **, *** indicate significance at 95%, 97.5%, and 99.5% confidence levels, respectively.

Chapter 4

Equilibrium Currency Hedging under Equity-Currency Contagion¹

4.1 Introduction

The last two decades witnessed several episodes of financial and currency crises, most notably the 1994 Mexico peso crisis, the 1997 Asian crisis, the 1998 Russian crisis, the more recent 2008 global financial crisis, and the subsequent 2009 European debt crisis. A common feature in these currency crises is that, among other things, the currency devaluation in a crisis is usually accompanied by dramatic capital market drops. During the Asian crisis, for instance, initiated by sharp currency devaluations in Southeast Asia, the Dow Jones Industrial Average plummets 554 points for its biggest point loss by then. Shortly after, the Korean won hits a record low in December, followed by Indonesian rupiah's free fall in January 1998 and the collapse of Russia's financial system in mid-1998.

The equity-currency contagion is a well-documented phenomenon in the literature. For example, Caramazza et al. [29] conclude that financial linkages are significant causes of currency crises after controlling for the role of domestic and external fundamentals, trade spillovers, and financial weaknesses in the affected countries. A strong financial linkage to the crisis country of origin not only raises the probability of contagion substantially but also helps explain the observed regional concentration of currency crises. Pesenti and Tille [102] study the Asian currency crisis and find that while weak or unsustainable economic policies provide a partial explanation of the currency crisis, they cannot account for the severity of the crises. One also need to take into account the volatile capital markets.² Fratzscher [64] finds that the Latin American crisis in 1994-95 and the Asian crisis of 1997 spread across emerging markets are not primarily due to the weakness of those countries' fundamentals but rather to a high degree of financial interdependence among affected economies. Brunnermeier et al. [25] link the crash risk of carry trade strategies to funding constraints of speculators, with funding constraints measured by the implied volatility of the S&P 500 stock index. Ferreira Filipe and Suominen [59] investigate how the financial market conditions in a major carry trade funding country, Japan, affect the global currency markets and currency trading and find that funding risks in Japan (measured by the stock options implied volatility and crash risk in the stock market in Japan) affect the global currency market. Consistent with

¹Professor Roger J. A. Laeven made helpful comments and suggestions.

²Practitioners also share a similar view. For example, Bluford Putnam, the managing director and chief economist at CME Group claims that the emerging market currency contagion in 2013-2014 was driven by asset allocation shifts from emerging markets to US equities and other mature industrial markets.

these findings, De Bock and de Carvalho Filho [44] find that during the risk-off episodes, currency markets exhibit recurrent patterns, as the Japanese yen, Swiss franc, and US dollar appreciate against other G-10 and emerging market currencies. Lettau et al. [89] study the cross section of currency returns using the downside risk capital asset pricing model. They find that high-yield currencies earn higher excess returns than low-yield currencies because their co-movement with aggregate market returns is stronger conditional on bad market returns than it is conditional on good market returns. Francis et al. [63] and Chernov et al. [35] find that spillover from equity to currency market exists not only in mean but also in volatility.

The interdependence between equities and currencies leads to the concept of “safe haven” currencies and “investment currencies”. Rinaldo and Söderlind [103] define an asset to be “safe haven” if it offers hedging benefits on average or in times of stress. They find that safe haven currencies tend to have low yield but immune to market downturns. “Investment currencies”, on the other hand, are like the mirror image of safe haven currencies – high yield and high exposure to systemic risks. When the global market is in stress, investors tend to move into safe haven currencies [31]. They identify safe haven currencies by regressing currency returns on current or lagged risk factors such as stock returns and bond returns. They conclude that the Swiss franc, the Japanese yen, and the British pound display safe currency characteristics. Nevertheless, the covariance between currency returns and equity returns can be time varying, and can even change signs over time. Cenedese [31] finds that during periods of bear, volatile world equity markets, currencies provide different hedging benefits than in bull markets. The 2008 financial crisis emerged as an important case study where safe haven effects went against typical patterns partially in contrast with the results of Rinaldo and Söderlind [103]. During the crisis, a large number of currencies that were not at the center of the turmoil depreciated, even those which were regarded as safe haven currencies preceding the crisis [85]. Habib and Stracca [74] study what makes a safe haven currency in a systematic way and find that only a few factors are robust associated to a safe haven status.

The interplay between the equity market and the currency market poses challenges on optimal currency hedging. So far, there is no consensus on how much currency risks to hedge and even whether to hedge currency risks at all. The complicated dependence structure between the equity and currency returns calls for more realistic models that captures equity and currency returns jointly. Indeed, as pointed out in Backus et al. [11], the gross return of a foreign currency is the ratio of the return of the foreign stochastic discount factor and that of the domestic one. As long as risk factors are compensated differently in the two economies, priced risk factors that drive the equity returns should in principle drive exchange rates. Therefore the equity market and the foreign exchange rate are interconnected in theory. Modeling equity and exchange rate jointly is not only empirically interesting but also of theoretical relevance.

We contribute to the equity-currency literature by bridging this gap. We revisit the Black’s equilibrium currency hedging problem under the context of equity-currency contagion. We propose a realistic model that generates equity-currency contagion, which enables a theoretical characterization of the “safe haven” properties of a risky currency. We derive the equilibrium currency hedging strategies under this context.

In particular, we propose a mutually exciting jump diffusion model to describe equity and exchange rate processes jointly. In this model, an equity price jump today increases the probability of experiencing further price jumps in the equity market in the future as well as the probability of experiencing price jumps in the exchange rates, and similar for the

exchange rate jumps. The model, therefore, produces a rich dependence structure between equities and currencies – the normal dependence is captured by instantaneous covariance and the dependence during market downturns is generated by jump excitation. Jump excitation is a better candidate than time-varying covariance for two reasons in this case. First, although there appears to be excess dependence between the equity market and the exchange rates, the comovement is neither simultaneous nor certain. By mutually exciting jumps, a crash in the equity market only increase the *probability* of *future* currency jumps. Second, investment currencies, which are more prone to capital market turmoil, not necessarily have an equally strong impact on equities as equities on them, especially during recessions. Dependence generated by covariance is symmetric in nature, in the sense that if a currency were of the “investment” type measured by covariance, then its movement should also impact the equity market equally well. By capturing tail dependence using jump excitation, we allow for asymmetric excitation structure, in which case a currency that barely influences the equity market may respond to equity market downturns sharply.

While deviating from the log normal stochastic discount factors in the carry trade literature, the model complies with the foreign exchange literature findings that both global and country-specific risk factors are essential ingredients to generate the observed carry trade return patterns. In particular, our model is consistent with Brusa et al. [26], in which three types factors are driving the stochastic discount factors – an equity factor that only drives the equity, a currency factor which only appears in the currency returns, and an equity-currency factor, which drives both the equity market and the foreign exchange rate. Our model can be regarded as an extension and variation of Farhi et al. [57], where we allow the equity market and the exchange rate to be mutually exciting, while maintaining the factor structure that prevails the exchange rate literature.

We first solve the portfolio optimization problem with country-specific stocks and currencies in closed form in a partial equilibrium framework, taking equity and exchange rate dynamics as given. To see the implication on the equilibrium currency hedging strategies under the equity-currency contagion context, we impose security market clearing conditions. Our equilibrium currency hedging strategy differs from that of Black [15] in the following aspects. Investors with different home currencies will have different hedging ratios towards a risky currency in general. Moreover, all else equal, investors have a larger hedging ratio for investment currencies, those that are prone to equity market turmoil than that for the safe haven currencies, those that are less susceptible to equity market downturns.

This chapter is organized as follows: Chapter 4.2 proposes the equity and exchange rate dynamics. We show that our model generates equity-currency contagion and complies with the extant literature. Chapter 4.3 solves the optimal asset allocation problem in a partial equilibrium framework. Chapter 4.4 studies the property of the optimal net currency weights. Chapter 4.5 imposes the security market clearing conditions and derives the equilibrium currency hedging strategies. Chapter 4.6 illustrates the safe haven bias, that is, all else equal, investors will have a larger hedging ratio towards investment currencies than the safe haven currencies. The preference towards safe haven currencies cannot be directly replicated using linear correlation in classic models. Chapter 4.7 concludes.

4.2 A parsimonious model that allows for equity-currency contagion

In this section, we propose a model of equity and exchange rate dynamics that generates equity-currency contagion. This is achieved by including both cross sectionally and serially dependent jump components, namely, mutually exciting jumps, in equity and currency returns. We specify country-specific stocks, pricing kernels and exchange rates in Chapter 4.2.1. For market completeness, we also introduce country-specific stock derivatives. In Chapter 4.2.2, we give the pricing formula for call options in this context. Chapter 4.2.3 discusses how our model relates to the factor models in the extant literature.

4.2.1 Set up

The equity market

In this section, we propose an equity-currency model that generates tail risk contagion between equity risk and currency risk. Let there be $n + 1$ countries. Each country has its own currency. Let there be a risk free money market account, a country-specific stock index, and a derivative written on the stock index in each country, all denominated in the domestic currency. We use superscript to denote in which currency the quantities are measured and subscript to denote the referred object. Variables without a superscript are prices denominated in the domestic currency. For instance, we denote the money market account of country i by $B_i(t)$ when denominated in currency i and $B_i^j(t)$ when denominated in currency j . We adopt the convention of denoting vectors and matrices using boldface characters to distinguish them from scalars. It holds that

$$dB_i(t) = B_i(t)r_i(t) dt, \quad (4.1)$$

where $r_i(t)$ is the continuously compounded risk free rate of country i at time t .

Suppose the country stocks are exposed to country-specific Brownian risks $W_i, i = 0, \dots, n$, and a global equity crash risk, modeled by a counting process N_m with intensity $\lambda_m(t)$ at time t . Denote the domestic stock of country i by S_i . S_i follows the dynamics

$$\frac{dS_i(t)}{S_i(t^-)} = r_i(t) dt + \mu_{s_i} \lambda_m(t) dt + \sigma_{s_i} \sqrt{\lambda_m(t)} dW_i(t) + j_{s_i} (dN_m(t) - \lambda_m(t) dt), \quad (4.2)$$

where μ_{s_i} is the expected excess return, and j_{s_i} is the jump amplitude of country i , assumed to be a negative constant. The Brownian risk $W_i, i = 0, \dots, n$, are independent of the jump risk N_m . The Brownian motions that drive the stocks of different countries are allowed to be correlated. Denote the correlation between W_i and W_j by ρ_{ij} .

In addition to the country stock, we introduce a stock option $O_i(t)$ of the European type in each country. If the market is free of arbitrage opportunities, there exists a risk neutral measure Q_i , under which

$$O_i(t) = \mathbb{E}_t^{Q_i}[g(S_i(\tau), \lambda_m(\tau))],$$

for any $t \leq \tau$, where τ is the time to expiration.

The stock option provides exposure to the same risk factors as the stock return does. As we will see later, the introduction of stock options completes the equity market in the sense that (1) the risk premiums of the equity Brownian motions and the equity jump component can be uniquely pinned down; (2) a portfolio that belongs to the \mathcal{H}^2 space with any exposure

to the equity Brownian motions and jump component can be replicated using the stocks and the stock options. To illustrate the latter point, as Liu and Pan [91] explain, one can start with the stock and add out-of-the-money put options to the portfolio which provide more exposure to jump risk, in order to separate exposure to jump risk from that to diffusive risk.

Let $E_i^j(t)$ be the exchange rate between currency j and currency i , understood as the currency j price per unit of currency i . We choose currency $i = 0$ as the base currency. A superscript of 0 indicates denomination in the base currency.

If the exchange rate is stochastic, the money market account of country i is a risky investment for investor from country j , $j \neq i$. The money market account of country i denominated in currency j has price $B_i^j(t) = B_i(t)E_i^j(t)$. It holds that

$$\frac{dB_i^j(t)}{B_i^j(t)} = r_j(t) + \frac{dE_i^j(t)}{E_i^j(t)}.$$

Similarly, equity i denominated in currency j has price $S_i^j(t) = S_i(t)E_i^j(t)$ at time t . Define the currency-hedged stock $\hat{S}_i^j(t)$ as

$$\frac{d\hat{S}_i^j(t)}{\hat{S}_i^j(t)} = \frac{dS_i^j(t)}{S_i^j(t)} - \left(\frac{dB_i^j(t)}{B_i^j(t)} - \frac{dB_j(t)}{B_j(t)} \right). \quad (4.3)$$

The currency-hedged stock i for investor j can be constructed by a continuously-rebalanced portfolio that invests 100% in the unhedged stock i , borrowing 100% from country i and lending domestically (to country j). Effectively, borrowing abroad and lending domestically mimics a currency forward contract (see Campbell et al. [28]).

Define the currency-hedged global equity index as the weighted average of country s -stocks. Denoted in the base currency, it holds that

$$\hat{M}^0 = \sum_{i=0}^n h_i \hat{S}_i^0, \quad (4.4)$$

where h_i is country i 's market capital as a proportion of the global capital, with $\sum_{i=0}^n h_i = 1$. Note that h_i is a currency-independent variable.

The return on the global equity index in the base currency is given by

$$\begin{aligned} \frac{d\hat{M}^0(t)}{\hat{M}^0(t^-)} &= \sum_{i=0}^n \left(h_i \hat{\mu}_{s_i}^0 \lambda_m dt + \sigma_{s_i} \sqrt{\lambda_m} dW_i + \hat{j}_{s_i}^0 (dN_m - \lambda_m dt) \right) \\ &=: \mu_m^0 \lambda_m dt + \sigma_m \sqrt{\lambda_m} dW_m + j_m^0 (dN_m - \lambda_m dt), \end{aligned} \quad (4.5)$$

where

$$\sigma_m \sqrt{\lambda_m} dW_m(t) = \sum_{i=0}^n h_i \sigma_{s_i} \sqrt{\lambda_m} dW_i(t), \quad j_m^0 = \sum_{i=1}^n h_i \hat{j}_{s_i}^0.$$

Define σ_s as a diagonal matrix containing σ_{s_i} , $i = 0, \dots, n$, on the diagonal, \mathbf{h} a vector containing h_i , $i = 0, \dots, n$, and $\mathbf{W}(t)$ a vector containing $W_i(t)$, $i = 0, \dots, n$. We can see that $W_m(t) = \frac{1}{\sigma_m} \mathbf{h}' \sigma_s \mathbf{W}(t)$, $\sigma_m = \sqrt{\mathbf{h}' \Sigma \mathbf{h}}$. Here, Σ is the covariance matrix of the stock returns of different countries. Define \mathbf{L} as the Choleski decomposition of the correlation

matrix, meaning that LL' is a matrix with ones on the diagonal and correlation coefficients off-diagonal,

$$LL' = \begin{pmatrix} 1 & \rho_{01} & \dots & \rho_{0n} \\ \rho_{01} & 1 & \dots & \rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0n} & \rho_{1n} & \dots & 1 \end{pmatrix}.$$

It holds that $\Sigma = \sigma_s LL' \sigma_s'$.

Pricing kernel and exchange rate processes

The exchange rate return is effectively the ratio of the change in pricing kernel processes of the two countries [11]. We first specify the pricing kernel process of each country and then derive the consistent exchange rate process thereafter.

Following Pan [101], we specify the pricing kernel process of country i , $i = 1, \dots, n$, to be of the following parametric form

$$\begin{aligned} \frac{d\pi_i(t)}{\pi_i(t^-)} = & - (r_i(t) dt + \eta_i \sqrt{\lambda_m(t)} dW_m(t) + v_i \sqrt{\lambda_i(t)} dZ_i(t)) \\ & + \kappa_i (dN_m(t) - \lambda_m(t) dt) + (y_i dN_i(t) - \mathbb{E}[y_i] \lambda_i(t) dt). \end{aligned} \quad (4.6)$$

Here, the pricing kernel also prices risks that do not drive equity returns. We introduce two new priced risk factors: one is a country-specific Brownian motion Z_i , independent of other risk factors as well as Z_j , $j \neq i$; and the other is a jump component N_i with intensity $\lambda_i(t)$ at time t . In Equation (4.6), η_i , κ_i are equity Brownian and jump risk premium in country i ; v_i is a constant that represents the risk premium of Brownian motion Z_i ; y_i is allowed to be a random variable.

The literature has shown that there are risk dimensions that influence currency returns in international economies but are absent in a single-economy equity market.³ Our pricing kernel specification (4.6) is consistent with that of Bakshi et al. [12] and Brusa et al. [26], in that apart from domestic equity risk factors, the pricing kernel process of a country is also driven by foreign equity risk factors and risk factors not spanned in the international equity market.

Notice that the pricing kernel is driven by the global equity Brownian risk W_m and jump risk N_m , rather than the country-specific ones (although the exposure to country-specific equity risk factors can be further inferred). This is intuitive, the pricing kernel process π_i determines the risk premium of risky investment for investors in country i . Consistent with the international CAPM (see Solnik [112]), only systematic equity risks are compensated. In our case, the systematic equity risk factors are the Brownian risk W_m and jump risk N_m , as those that drive the global equity index.

The equity jump component has deterministic a jump size and is compensated only with the jump timing risk. Under the risk neutral measure, the jump component N_m has intensity $(1 + \kappa_i) \lambda_m$ under measure Q_i . With respect to the currency jump component, if we restrict that the jump risks are compensated for jump size risk but not for jump timing risk, as in Pan [101], then only the jump size distribution changes under Q_i (determined by the distribution of y_i) and the jump intensity remains the same after the measure change.

³See, for example, Brandt et al. [20].

As a normalization, we assume that the base currency is free of currency-specific risks. The pricing kernel process of the base country is given by

$$\frac{d\pi_0(t)}{\pi_0(t^-)} = -\eta_0\sqrt{\lambda_m(t)}dW_m + \kappa_0(dN_m(t) - \lambda_m(t)dt), \quad (4.7)$$

where η_0, κ_0 represent equity Brownian and jump risk premium in the base country. Exposure to currency risks does not come with any risk premium in the base country since only systematic equity risk factors are priced. In this sense, the base currency can be regarded as a reserve currency. We refer to Appendix 4.B for more detail on the normalization in the base country.

According to Backus et al. [11], if the international markets are integrated, exchange rates reflect differences in pricing kernels in the associated markets,

$$E_i^0(t) = \pi_i(t)/\pi_0(t), \quad (4.8)$$

or, in SDE representation,

$$\begin{aligned} \frac{dE_i^0(t)}{E_i^0(t^-)} &= \left(r_0(t) - r_i(t) + \eta_0(\eta_0 - \eta_i)\lambda_m(t) \right) dt + (\eta_0 - \eta_i)\sqrt{\lambda_m(t)}dW_m(t) \\ &\quad - v_i\sqrt{\lambda_i(t)}dZ_i(t) + \frac{\kappa_i - \kappa_0}{1 + \kappa_0}(dN_m(t) - \lambda_m(t)dt) \\ &\quad + (y_i dN_i(t) - \mathbb{E}[y_i]\lambda_i(t)dt) \\ &=: \left(r_0(t) - r_i(t) + \mu_{e_i}^0\lambda_m(t) \right) dt + \sigma_{e_i}\sqrt{\lambda_m(t)}dW_m(t) \\ &\quad - v_i\sqrt{\lambda_i(t)}dZ_i(t) + j_{e_i}^0(dN_m(t) - \lambda_m(t)dt) \\ &\quad + (y_i dN_i(t) - \mathbb{E}[y_i]\lambda_i(t)dt), \end{aligned} \quad (4.9)$$

where

$$\mu_{e_i}^0 = \eta_0(\eta_0 - \eta_i) - \kappa_i j_i, \quad \sigma_{e_i} = \eta_0 - \eta_i, \quad j_{e_i}^0 = \frac{\kappa_i - \kappa_0}{1 + \kappa_0}. \quad (4.10)$$

Notice that we may kill the equity jump component in the exchange rate process by setting $\kappa_i = \kappa_0, \forall i$. In other words, currencies are free of equity jump risks only if the equity jump risk is compensated the same way in every market.

By the same token, the exchange rate between currency i and j , for all i, j , can be calculated using

$$E_i^j(t) = \pi_i(t)/\pi_j(t).$$

One can easily verify the triangular equality that $E_i^j(t) = E_i^l E_l^j, \forall l$. In other words, converting from currency i to currency j is the same as first converting to any currency l and then to currency j .

It is clear from Equation (4.6) and (4.7) that $\pi_i, i = 0, \dots, n$, are local martingales under the real world measure. If π_i are actually martingales, one can verify according to the Lenglart-Girsanov Theorem that the pricing kernels can serve as the Radon-Nikodym derivatives that change the physical measure P to risk neutral measures Q_i , under which the global

equity index and country stocks denominated in currency j follow

$$\begin{aligned}\frac{d\hat{M}^j(t)}{\hat{M}^j(t^-)} &= (\hat{\mu}_m^j - \eta_j \sigma_m + \hat{j}_m^j \kappa_j) \lambda_m(t) dt + \sigma_m \sqrt{\lambda_m(t)} dW_m^{Q_j}(t) \\ &\quad + \hat{j}_m^j (dN_{m,t}^{Q_j} - (1 + \kappa_j) \lambda_m(t) dt), \\ \frac{d\hat{S}_i^j(t)}{\hat{S}_i^j(t^-)} &= (\hat{\mu}_{s_i}^j - \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l} + \hat{j}_{s_i}^j \kappa_j) \lambda_m(t) dt + \sigma_i \sqrt{\lambda_m(t)} dW_i^{Q_j}(t) \\ &\quad + \hat{j}_{s_i}^j (dN_{m,t}^{Q_j} - (1 + \kappa_j) \lambda_m(t) dt),\end{aligned}$$

where $W_i^{Q_j}$ is a standard Brownian motion under the risk neutral measure of country j , with $W_i^{Q_j}(t) = W_i(t) - \eta_j \int_0^t \sqrt{\lambda_m(s)} ds$. The jump process $N_{m,t}^{Q_j}(t)$ has intensity $(1 + \kappa_j) \lambda_m(t)$ under the martingale measure Q_j . In order that $\hat{M}^j(t)$, $\hat{S}_i^j(t)$ are local martingales under the risk neutral measure of country j , it should hold that

$$\begin{cases} \hat{\mu}_m^j = \sigma_m \eta_j - \kappa_j \hat{j}_m^j, \\ \hat{\mu}_{s_i}^j = \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l} - \kappa_j \hat{j}_{s_i}^j. \end{cases} \quad (4.11)$$

Note that the expected excess return of a country's stock consists of the risk premium of (I) the country-specific Brownian risk, and (II) the global equity crash risk. In particular, $\sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l}$ is the instantaneous covariance between the returns of the market equity and stock i . Then the premium for country-specific Brownian risk is the premium for the market equity times the covariance between the market equity and stock i divided by the instantaneous variance of the market equity. Similarly, the equity jump premium in stock i is the ratio of the jump amplitude of stock i and that of the market equity. That is,

$$\begin{aligned}\hat{\mu}_{s_i}^j &= \hat{\mu}_{s_i}^j(I) + \hat{\mu}_{s_i}^j(II), \\ \hat{\mu}_{s_i}^j(I) &:= \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{s_i} \sigma_{s_l} = \frac{\text{Cov}(\hat{R}_i^{j,c}, \hat{R}_m^{j,c})}{\text{Var}(\hat{R}_m^{j,c})} \hat{\mu}_m^j(I), \\ \hat{\mu}_{s_i}^j(II) &:= -\kappa_j \hat{j}_{s_i}^j = \frac{\hat{j}_{s_i}^j}{\hat{j}_m^j} \hat{\mu}_m^j(II).\end{aligned}$$

Here, $\hat{R}_i^{j,c}$, $\hat{R}_m^{j,c}$ denote the continuous part of the return of stock i and the market equity, respectively. $\hat{\mu}_{s_i}^j(I)$ is the country-specific volatility risk premium that exhibits a CAPM structure, and $\hat{\mu}_{s_i}^j(II)$ is the jump premium in stock i for investor j .

In addition, free of arbitrage opportunities implies that similar structure applies to the expected excess return of countries' derivatives,

$$\hat{\mu}_{o_i}^j = \frac{\eta_j}{\sigma_m} \sum_{l=0}^n h_l \rho_{il} \sigma_{o_i} \sigma_{o_l} - \kappa_j \hat{j}_{o_i}^j, \quad \forall i, j.$$

Equity-currency contagion

We allow for jump propagation between equity and currencies by letting N_m, N_i to be mutually exciting with intensities $\lambda_m(t), \lambda_i(t)$ that follow

$$\begin{aligned} d\lambda_m(t) &= \alpha_m(\lambda_{m,\infty} - \lambda_m(t)) dt + \beta_{m,m} dN_m(t) + \sum_{l=1}^n \beta_{l,m} dN_l(t), \\ d\lambda_i(t) &= \alpha_i(\lambda_{i,\infty} - \lambda_i(t)) dt + \beta_{m,i} dN_m(t) + \sum_{l=1}^n \beta_{l,i} dN_l(t), \end{aligned}$$

where $\alpha_m, \alpha_i, \lambda_{m,\infty}, \lambda_{i,\infty}, \beta_{i,m}, \beta_{m,i}, \beta_{i,j}, \beta_{m,m} \geq 0, \forall i, j$.

The occurrence of a jump in the equity market at time t , i.e., $dN_m(t) = 1$, not only raises the intensity of the equity jump component, $\lambda_m(t)$, by a non-negative amount $\beta_{m,m}$, but also increases the intensity of the currency jump component, $\lambda_i(t)$, by a non-negative amount $\beta_{m,i}$. After being excited, both equity jump intensity $\lambda_m(t)$ and currency jump intensity $\lambda_i(t)$ mean revert to their respective steady state, $\lambda_{m,\infty}, \lambda_{i,\infty}$, at exponential decaying rates α_m, α_i , until they get excited by a next jump occurrence.

In the remainder, we call β , defined as

$$\beta := (\beta_m, \beta_1, \dots, \beta_n) = \begin{pmatrix} \beta_{m,m} & \beta_{1,m} & \dots & \beta_{n,m} \\ \beta_{m,1} & \beta_{1,1} & \dots & \beta_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m,n} & \beta_{1,n} & \dots & \beta_{n,n} \end{pmatrix},$$

the *excitation matrix* between equity and currency i ; $\beta_{m,m}$ is called the *equity self excitor*; $\beta_{i,i}$ is called the *currency self excitor* of currency i ; $\beta_{m,i}$ is called the *equity-currency excitor* of currency i , which measures the excitation from the equity jump component to the jump component of currency i ; $\beta_{i,m}$ is called the *currency-equity excitor* of currency i , which measures the excitation from the jump component of currency i to the equity jump component.

Let $\lambda(t) = (\lambda_m(t), \lambda_1(t), \dots, \lambda_n(t))'$. The unconditional expectation of the jump intensity is given by

$$\mathbb{E}[\lambda(t)] = (\mathbf{I}_n - \beta / (\alpha \iota'))^{-1} \lambda_\infty,$$

where \mathbf{I}_n is an n by n identity matrix; α, λ_∞ are vectors of $\alpha_i, \lambda_{i,\infty}, i = m, 1, \dots, n$, respectively; ι is a column vector of all ones. The intensity processes can be made stationary by imposing

$$(\mathbf{I}_n - \beta / (\alpha \iota'))^{-1} > 0.$$

This is a general yet parsimonious model which generates contagion between the equity market and the foreign exchange market. The equity and currency model given in (4.2) and (4.9) is a natural extension of the classic geometric Brownian motion models (see Solnik [112], Black [15], Campbell et al. [28]). The model also generates stochastic volatility driven by jump intensity processes.

The mutually exciting jump components in Equation (4.2) and (4.9) are able to produce important stylized facts of equity-currency behavior. For example, the stock returns exhibit jump clustering as a result of the time series excitation, and equity-currency contagion as a consequence of the cross section excitation between these two asset classes. The market equity and currencies have an instantaneous covariance of $\sigma_m \sigma_{e_i} \lambda_m(t)$, which is stochastic, and increases when the equity market is in turmoil. This is consistent with the empirical findings of stochastic covariance between the equity market and the foreign exchange market.

In addition, during market downturns, equity market turbulence can lead to currency turmoil and vice versa, thus creating a non-linear excess dependence between the equity market and foreign exchange market.

The dependence generated by mutually exciting jumps has two distinctive features. The first is that the dependence between equity and currency is not simultaneous. Under contagious equity-currency risks, the exchange rate is likely to experience a jump in succession of an equity market plunge. Different from the dependence generated by common risk factors, the dependence of the extreme movements in these two markets is neither simultaneous nor certain. The second property is that the model allows for asymmetric excitation: the impact of an equity plunge on a currency value can be different from that of the currency depreciation on the equity market. It allows for separate analysis on the two-way equity-currency contagion. A currency whose value remains relatively stable during the equity market turbulence (i.e., low equity-currency excitor) has the property of a safe haven currency.

4.2.2 Option pricing

Suppose the market is free of arbitrage. The price of an option $O_j(t)$ written on stock $S_j(t)$ with payoff function $f(S_j(\tau))$ is given by

$$O_j(t) = e^{-r_j(\tau-t)} \mathbb{E}_t^{Q_j}[f(S_j(\tau))]. \quad (4.12)$$

In this section, we consider the price of a standard call option on the domestic equity of each country. Appendix 4.D also gives the pricing formula for put options and straddles. The payoff function for the call option is given by

$$f(S_j(\tau)) = (S_j(\tau) - K)^+ =: C_j(\tau),$$

where K is the strike price.

The following proposition gives the call option price $C_j(t)$, $t \leq \tau$, as a function of the stock price and the equity jump intensity at time t .

Proposition 4.1. *The call option price $C_j(t)$, $t \leq \tau$, is given by*

$$C_j(t) = G_{1,-1}(-\log K) - KG_{0,-1}(-\log K), \quad (4.13)$$

where

$$G_{a,b}(w) = \frac{1}{2}\psi_t(a) - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[e^{-i u w} \psi_t(a + i u b)]}{u} du, \quad (4.14)$$

with

$$\psi_t(u) = S_j(t)^u \exp(\mathcal{P} + \mathcal{Q}\lambda_m(t)), \quad (4.15)$$

Here, $\mathcal{P} = \mathcal{P}(t)$, $\mathcal{Q} = \mathcal{Q}(t)$,

$$\begin{aligned} \frac{d}{dt}\mathcal{Q}(t) &= \left(\frac{1}{2}\sigma_{e_j}^2 + j_{e_j}(1 + \kappa_j)\right)u + \alpha_m\mathcal{Q}(t) - \frac{1}{2}u^2\sigma_{e_j}^2 - (1 + \kappa_j)\left((1 + j_{e_j})^u e^{\beta_{m,m}\mathcal{Q}(t)} - 1\right), \\ \frac{d}{dt}\mathcal{P}(t) &= -\alpha_m\lambda_{m,\infty}\mathcal{Q}(t). \end{aligned}$$

with $\mathcal{P}(\tau) = \mathcal{Q}(\tau) = 0$.

Therefore the dynamics of the option prices are given by

$$\begin{aligned} \frac{dO_j(t)}{O_j(t^-)} = & r_j(t) dt + \mu_o(S_j(t), \lambda_m(t)) \lambda_m(t) dt + \sigma_o(S_j(t), \lambda_m(t)) \sqrt{\lambda_m(t)} dW_m(t) \\ & + j_o(S_j(t), \lambda_m(t)) (dN_m(t) - \lambda_m(t) dt), \end{aligned}$$

where

$$\begin{aligned} \sigma_{o_j}(S_j(t), \lambda_m(t)) &= \frac{\sigma_m S_j(t)}{O(t^-)} \frac{\partial f(S_j(t), \lambda_m(t))}{\partial S_j(t)} \bigg|_{(S_j(t), \lambda_m(t))}, \\ j_{o_j}(S_j(t), \lambda_m(t)) &= \frac{1}{O_j(t^-)} \left(f((1 + j_{s_j}) S_j(t), \lambda_m(t) + \beta_{m,m}) - f(S_j(t), \lambda_m(t)) \right). \end{aligned}$$

We will drop the arguments of $\mu_{o_j}(S_j(t), \lambda_m(t))$, $\sigma_{o_j}(S_j(t), \lambda_m(t))$, $j_{o_j}(S_j(t), \lambda_m(t))$ and simply denote them by $\mu_{o_j}(t)$, $\sigma_{o_j}(t)$ for notation simplicity, with the time t argument indicating state dependence.

4.2.3 Relation to factor models

Should the jump factors $N_m, N_i, i = 1, \dots, n$, be Poissonian, the pricing kernel of country i would admit an orthogonal decomposition into an equity component π_i^m and a currency component π_i^s as in Bakshi et al. [12],

$$\begin{cases} \frac{d\pi_i^{(m)}(t)}{\pi_i^{(m)}(t^-)} = -\eta_i \sqrt{\lambda_m} dW_m + \kappa_i (dN_m(t) - \lambda_m dt), \\ \frac{d\pi_i^{(s)}(t)}{\pi_i^{(s)}(t^-)} = -v_i \sqrt{\lambda_i} dZ_i + (y_i dN_i(t) - \mathbb{E}[y_i] \lambda_i(t) dt), \\ \pi_{i,t} = \exp\left(-\int_0^t r_i(s) ds\right) \pi_i^{(m)}(t) \pi_i^{(s)}(t). \end{cases} \quad (4.16)$$

Not all risk factors are priced in the pricing kernel of country j . The consequence of this is that exchange rates of different currencies are exposed to different risk factors. Similar assumptions regarding the pricing kernel (that the pricing kernels are driven by both global factors and country-specific factors) can be found in Lustig et al. [93], Bakshi et al. [12], and Farhi et al. [57].

The pricing kernel processes given in Equation (4.6) exhibit a factor structure. Consistent with Bates [13] and Carr and Wu [30], the model has Gaussian and non-Gaussian factors. We allow the pricing kernels to price risk factors other than equity risk factors, as in Bakshi et al. [12].

Our model is also consistent with Brusa et al. [26], in which three global factors drive the stochastic discount factors. The first is a global equity factor N_m which can be priced the same way in every country by setting $\kappa_i = \kappa_j, \forall i, j$. In Brusa et al. [26], this factor does not appear in the exchange rate process but drives the world equity return. The second is a country-specific currency factor N_i . This factor only drives the exchange rate but not the equity returns, capturing the crash risk in the carry trade. The third is a common factor W_m that drives both equity and currency returns, mimicking the “dollar factor” in Brusa et al. [26].

4.3 Optimal asset allocation

Let there be a representative investor from each country. In this section we define and solve the optimal asset allocation problem for every investor. Instead of raw assets, which are

foreign assets quoted in the domestic currency, we will work with currency-hedged asset prices, which has a one-to-one correspondence to the raw prices. Chapter 4.3.1 derives the dynamics of currency-hedged asset returns. Chapter 4.3.2 solves the asset allocation problem in the universe of stocks, stock options and bonds from all countries. Chapter 4.3.3 presents the Separation Theorem which states that the asset universe can be collapsed into a global equity, a global derivative portfolio and countries' bonds without incurring utility cost for investors.

4.3.1 Returns on the currency-hedged assets

When investing in a foreign stock, the investor is faced with not only the equity risk but also the currency risk. We will formulate the optimal asset allocation problem in terms of currency-hedged assets instead of the original assets. There is a one-to-one correspondence between the allocation strategy on the currency-hedged assets and that on the original assets. In the extreme case, an unhedged position in foreign stock j corresponds to a long position in currency j equal to the holding of stock j , whereas a fully hedged stock position corresponds to a net zero position in that foreign currency.

Recall Equation (4.3), the return of currency-hedged stock j for investor i follows

$$\begin{aligned} \frac{d\hat{S}_j^i(t)}{\hat{S}_j^i(t^-)} &= \frac{d(S_j(t)E_j^i(t))}{S_j^i(t^-)E_j^i(t^-)} - \left(\frac{dB_j^i(t)}{B_j^i(t^-)} - \frac{dB_i(t)}{B_i(t)} \right) \\ &=: \hat{\mu}_{s_j}^i \lambda_m(t) dt + \sigma_{s_j} \sqrt{\lambda_m(t)} dW_j(t) + \hat{j}_{s_j}^i (dN_m(t) - \lambda_m(t) dt). \end{aligned}$$

And similar for the currency-hedged option returns,

$$\begin{aligned} \frac{d\hat{O}_j^i(t)}{\hat{O}_j^i(t^-)} &= \frac{d(O_j(t)E_j^i(t))}{O_j(t^-)E_j^i(t^-)} - \left(\frac{dB_j^i(t)}{B_j^i(t^-)} - \frac{dB_i(t)}{B_i(t)} \right) \\ &=: \hat{\mu}_{o_j}^i \lambda_m(t) dt + \sigma_{o_j} \sqrt{\lambda_m(t)} dW_j(t) + \hat{j}_{o_j}^i (dN_m(t) - \lambda_m(t) dt). \end{aligned}$$

One can easily work out that $\hat{\mu}_{s_j}^i = \mu_{s_j} + \frac{(\sigma_{e_i} - \sigma_{e_j})\sigma_{s_j}}{\sigma_m} \sum_{l=1}^n \rho_{jl} h_l \sigma_{s_l}$, $\hat{j}_{s_j}^i = j_{s_j}(1 + j_{e_i})$, $\hat{\mu}_{o_j}^i = \mu_{o_j} + \frac{(\sigma_{e_i} - \sigma_{e_j})\sigma_{o_j}}{\sigma_m} \sum_{l=1}^n \rho_{jl} h_l \sigma_{e_l}$, $\hat{j}_{o_j}^i = j_{o_j}(1 + j_{e_i})$.

4.3.2 Solving the optimal asset allocation problem

Define the portfolio weights vector

$$\hat{\mathbf{w}}^j(t) = (\hat{w}_{s_0}^j(t), \dots, \hat{w}_{s_n}^j(t), \hat{w}_{o_0}^j(t), \dots, \hat{w}_{o_n}^j(t), \hat{w}_{e_0}^j(t), \dots, \hat{w}_{e_n}^j(t))',$$

to be a $3(n+1) \times 1$ vectored process, which are adapted, càglàd, and bounded in \mathcal{L}^2 . Problem 4.1 defines the asset allocation problem.

Problem 4.1. *Let there be a representative investor from each country j with initial wealth x_j , who has expected power utility with risk aversion $u(x_j) = \frac{1}{1-\gamma_j} x_j^{1-\gamma_j}$, $\gamma_j > 0, \forall j$. Each investor is allowed to invest in foreign as well as domestic risk-free and (currency-hedged) risky assets. Investors neither consume nor receive any intermediate income. Assume that investors can rebalance their portfolios in continuous time without incurring any transaction*

costs. The objective is to maximize the expected utility over terminal wealth $X_j(T)$ through optimal continuous time trading.

$$\sup_{\hat{w}^j} \mathbb{E}_0 \left[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j} \right], \quad (4.17)$$

subject to the budget constraint,

$$\frac{dX_j(t)}{X_j(t^-)} = \sum_{i=0}^n \hat{w}_{s_i}^j \frac{d\hat{S}_i^j(t)}{\hat{S}_i^j(t^-)} + \sum_{i=0}^n \hat{w}_{o_i}^j \frac{d\hat{O}_i^j(t)}{\hat{O}_i^j(t^-)} + \sum_{i=0}^n \hat{w}_{e_i}^j \frac{dB_i^j(t)}{B_i^j(t^-)}. \quad (4.18)$$

Let J be the indirect utility function for investor j at time $t = 0$. Here, we drop the investor's identity j in the indirect utility function to avoid notation clustering.

$$J(t, x, \boldsymbol{\lambda}) = \sup_{\hat{w}^j} \mathbb{E} \left[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j} \right],$$

where $x = X_j(t)$, $\boldsymbol{\lambda} = (\lambda_m(t), \lambda_1(t), \dots, \lambda_n(t))'$ are current values of the wealth and jump intensities. Define $\theta_{w_i}^j, \theta_n^j$ as the exposure to equity risk factors W_i, N_m in investor j 's portfolio,

$$\begin{cases} \theta_{w_i}^j = \hat{w}_{s_i}^j \sigma_{s_i} + \hat{w}_{o_i}^j \sigma_{o_i} + \sum_{l=1}^n \hat{w}_{e_l}^j \sigma_{e_l} h_i \sigma_{s_i} / \sigma_m, \\ \theta_n^j = \sum_{i=0}^n (\hat{w}_{s_i}^j \hat{j}_{s_i}^j + \hat{w}_{o_i}^j \hat{j}_{o_i}^j) + \sum_{i=1}^n \hat{w}_{e_i}^j \hat{j}_{e_i}^j. \end{cases} \quad (4.19)$$

Note that $\sigma_{e_l} dW_m = \sigma_{e_l} \sum_{i=0}^n h_i \sigma_{s_i} / \sigma_m dW_i$. Therefore $\sigma_{e_l} h_i \sigma_{s_i} / \sigma_m$ is the exposure to the Brownian motion W_i through investing in currency l . Write in matrix notation $\boldsymbol{\theta}_w^j = (\theta_{w_0}^j, \dots, \theta_{w_n}^j)'$.

We are going to solve for the optimal $\boldsymbol{\theta}_w^j, \theta_n^j, \hat{w}_{e_i}^j, i = 1, \dots, n$, by first conjecturing (which we later verify) that the indirect utility function is of the form

$$J(t, x, \boldsymbol{\lambda}) = \frac{(x_j)^{1-\gamma_j}}{1-\gamma_j} \exp(P_j(t) + \boldsymbol{Q}_j(t)' \boldsymbol{\lambda}), \quad (4.20)$$

where $P_j(t)$ and $\boldsymbol{Q}_j(t)$ are functions of time but not of the state variables x and $\boldsymbol{\lambda}$.

The following proposition provides an analytical solution to the optimal portfolio strategy.

Proposition 4.2. *There exists a solution*

$$\hat{\boldsymbol{w}}^j(t) = (\hat{w}_{s_0}^j(t), \dots, \hat{w}_{s_n}^j(t), \hat{w}_{o_0}^j(t), \dots, \hat{w}_{o_n}^j(t), \hat{w}_{e_0}^j(t), \dots, \hat{w}_{e_n}^j(t))'$$

to Problem 4.1. The optimal portfolio weight is given by solving the following equations for the elements of $\hat{\boldsymbol{w}}^j$,

$$\begin{cases} -\mathbb{E}[y_i] - \gamma_j \hat{w}_{e_i}^j v_i^2 + e^{\boldsymbol{Q}_j^t \boldsymbol{\beta}_i} \mathbb{E}[(1 + \hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i] = 0, & i = 1, \dots, n, \\ \begin{pmatrix} \hat{w}_{s_0}^j \\ \vdots \\ \hat{w}_{s_n}^j \\ \hat{w}_{o_0}^j \\ \vdots \\ \hat{w}_{o_n}^j \end{pmatrix} = \begin{pmatrix} \sigma_{s_0} & 0 & \dots & 0 & \sigma_{o_0} & 0 & \dots & 0 \\ 0 & \sigma_{s_1} & \dots & 0 & 0 & \sigma_{o_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{s_n} & 0 & 0 & \dots & \sigma_{o_n} \\ \hat{j}_{s_0}^j & \hat{j}_{s_1}^j & \dots & \hat{j}_{s_n}^j & \hat{j}_{o_0}^j & \hat{j}_{o_1}^j & \dots & \hat{j}_{o_n}^j \end{pmatrix}^{-1} \begin{pmatrix} \theta_{w_0}^j - \sum_{l=1}^n \hat{w}_{e_l}^j \sigma_{e_l} h_0 \sigma_{s_0} / \sigma_m \\ \vdots \\ \theta_{w_n}^j - \sum_{l=1}^n \hat{w}_{e_l}^j \sigma_{e_l} h_n \sigma_{s_n} / \sigma_m \\ \theta_n^j - \sum_{l=1}^n \hat{w}_{e_l}^j \hat{j}_{e_l}^j \end{pmatrix}, \\ \hat{w}_{e_0}^j = 1 - \sum_{l=0}^n (\hat{w}_{s_l}^j + \hat{w}_{o_l}^j) - \sum_{l=1}^n \hat{w}_{e_l}^j \hat{j}_{e_l}^j, \end{cases} \quad (4.21)$$

where

$$\begin{cases} \theta_w^j = \frac{\eta_j}{\gamma_j \sigma_m} \sigma'_s \mathbf{h}, \\ \theta_n^j = (1 + \kappa_j)^{-\frac{1}{\gamma_j}} \exp(\frac{1}{\gamma_j} \mathbf{Q}_j' \beta_m) - 1. \end{cases} \quad (4.22)$$

Here, $\hat{w}_{e_i}^j = \hat{w}_{e_i}^j(t = 0)$, $\mathbf{Q}_j = \mathbf{Q}_j(t = 0)$, in which $\mathbf{Q}_j(t) = (Q_{j,m}, Q_{j,1}, \dots, Q_{j,n})$ is a deterministic process defined by the ordinary differential equations, for $i = 1, \dots, n$,

$$\begin{aligned} \dot{Q}_{j,m}(t) &= \alpha_m Q_{j,m}(t) + \frac{\gamma_j - 1}{2\gamma_j} \eta_j^2 + (\gamma_j - 1)\kappa_j - \gamma_j(1 + \kappa_j)^{\frac{\gamma_j - 1}{\gamma_j}} \exp\left(\frac{1}{\gamma_j} \mathbf{Q}_j(t)' \beta_m\right) + \gamma_j \\ \dot{Q}_{j,i}(t) &= \alpha_i Q_{j,i}(t) + (1 - \gamma_j) \hat{w}_{e_i}^j(t) \mathbb{E}[y_i] + \frac{1}{2} \gamma_j (1 - \gamma_j) \hat{w}_{e_i}^j(t)^2 v_i^2 \\ &\quad - \mathbb{E}[(1 + \hat{w}_{e_i}^j(t) y_i)^{1 - \gamma_j}] \exp(\mathbf{Q}_j(t)' \beta_i) + 1, \\ \dot{P}_j(t) &= - (1 - \gamma_j) r_j - \alpha_m \lambda_{m,\infty} Q_{j,m}(t) - \sum_{i=1}^n \alpha_i \lambda_{i,\infty} Q_{j,i}(t). \end{aligned} \quad (4.23)$$

with $P_j(T) = 0$, $\mathbf{Q}_j(T) = \mathbf{0}$.

From Proposition 4.2, the optimal portfolio weights for any investor can be calculated by solving simultaneously a set of equations – Equation (4.21) and (4.23). Specifically, for a given investor j , one starts with the terminal condition $\mathbf{Q}_j(T) = \mathbf{0}$ to derive the optimal weights at the terminal time T , $\hat{w}_{e_i}^j(T)$, $\forall i$. Then go back a small time interval Δ , and calculate $\mathbf{Q}_j(T - \Delta)$ using $\hat{w}_{e_i}^j(T)$. Continue with the recursive algorithm until time zero. A step size as small as a quarter of a day is enough to generate the desired accuracy. The computation burden in Proposition 4.2 is almost negligible compared to numerically solving the multi-dimensional HJB equation.

Here, $\hat{w}_{e_i}^j$ only measures the *direct* position on currency i in investor j 's portfolio. Even if $\hat{w}_{e_i}^j = 0$, investor j still indirectly invests in currency j through investing in the currency-hedged stock j .

The pricing kernel of a country j only prices equity and currency j risks. Why would investor j invest in a foreign currency whose risk is not compensated domestically? Especially for the base investor, who only earns risk premium through equity risks, it is tempting to think that stocks and stock derivatives are sufficient assets to optimize his/her portfolio. Why would the base investor directly invest in foreign currencies at all, apart from those for currency hedging purposes? The answer lies in the fact that equity risk and currency risk are contagious, making currency jump intensities stochastic state variables of the economy. Investors would demand currency exposure to hedge state variable risks.

As one may expect, if the jump components were Poissonian, the base investor would not invest in any foreign currency (provided that he/she has access to stocks and stock options). In other words, the base investor would hedge 100% of the currency risk in his/her portfolio.⁴

One important distinction between the optimal currency hedging strategy predicted by our model and that of Campbell et al. [28] is that the optimal currency demand in our model is home currency dependent. In Campbell et al. [28], for example, the residents of both the United States and Germany will have the same optimal demand for Australian dollar. In our model, however, since the currency demand is generated through nonlinear dependence between currencies and the equity market, investors from different home currencies would have different demand for a foreign currency in general.

⁴Same would apply to investors from other countries if the market were complete. Since stocks and stock derivatives are not sufficient to complete the market in a non-base country, we cannot draw the same conclusion for other investors.

4.3.3 The Separation Theorem

Solnik [112] proves a three-fund separation theorem in the context of a geometric Brownian motion model. In particular, he shows that investors are indifferent between the country stocks and a market equity index. Observe that in Equation (4.18), while there are $n + 2$ equity risk factors – ($n + 1$) country-specific Brownian risk factors and one global equity jump factor) – there are $2(n + 1)$ equity assets (one stock and one stock option from each country). One can also see from Equation (4.21) that the matrix to be inverted has full row rank but not full column rank, indicating redundant equity assets.

The next theorem presents the $n + 1 + 2$ fund separation result of our model.

Theorem 4.1. *Every investor is indifferent between choosing portfolios from the original $3(n + 1)$ assets or from $(n + 1) + 2$ funds. From the perspective of investor j , a possible choice for those funds is*

- the market equity index (hedged against currency risk) \hat{M}^j , as defined in Equation (4.4).
- a portfolio of stock derivatives (hedged against currency risk) \hat{D}^j , defined as

$$\hat{D}^j = \sum_{i=0}^n k_i \hat{O}_i^j,$$

with

$$(k_0, \dots, k_n)' = \frac{\sigma_o^{-1} \sigma_e' h}{\iota' (\sigma_o^{-1} \sigma_e' h)}, \quad (4.24)$$

- the $n + 1$ bonds of each country.

In light of Theorem 4.1, the investable asset universe for every investor is the $n + 1$ bonds of each country (the domestic bond is regarded as the risk-free asset), a currency-hedged global equity index

$$\frac{d\hat{M}^j(t)}{\hat{M}^j(t)} = \mu_m^j \lambda_m(t) dt + \sigma_m \sqrt{\lambda_m(t)} dW_m(t) + j_m^j (dN_m(t) - \lambda_m(t) dt),$$

and a currency-hedged portfolio of stock derivatives

$$\frac{d\hat{D}^j(t)}{\hat{D}^j(t)} = \mu_d^j \lambda_m(t) dt + \sigma_d \sqrt{\lambda_m(t)} dW_m(t) + j_d^j (dN_m(t) - \lambda_m(t) dt),$$

with

$$\sigma_d = \frac{\sigma_m}{\iota' (\sigma_o^{-1} \sigma_e' h)}, \quad j_d^j = \frac{h' \sigma_e \sigma_o^{-1}}{\iota' (\sigma_o^{-1} \sigma_e' h)} j_o^j.$$

We may redefine the optimal asset allocation problem in terms of the market equity, market derivative portfolio and the bonds.

Problem 4.2. *Let there be a representative investor from each country j , who has expected power utility with risk aversion γ_j and aims to maximize his/her expected utility at time $t = 0$ through optimally investing:*

$$\sup_{\hat{w}^j} \mathbb{E}_0 \left[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j} \right], \quad (4.25)$$

subject to the budget constraint:

$$\frac{dX_j(t)}{X_j(t^-)} = \hat{w}_m^j \frac{d\hat{M}^j(t)}{\hat{M}^j(t^-)} + \hat{w}_d^j \frac{d\hat{D}^j(t)}{\hat{D}^j(t^-)} + \sum_{i=0}^n \hat{w}_{e_i}^j \frac{dB_i^j(t)}{B_i^j(t^-)}. \quad (4.26)$$

The following proposition solves the above portfolio choice problem.

Proposition 4.3. *The asset allocation problem in Problem 4.2 has a solution*

$$\hat{\mathbf{w}}^j = (\hat{w}_m^j, \hat{w}_d^j, \hat{w}_{e_0}^j, \dots, \hat{w}_{e_n}^j).$$

The optimal portfolio weight is given by solving the following nonlinear equation for $\hat{\mathbf{w}}^j$,

$$\begin{cases} -\mathbb{E}[y_i] - \gamma_j \hat{w}_{e_i}^j v_i^2 + e^{Q'_j \beta_i} \mathbb{E}[(1 + \hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i] = 0, & i = 1, \dots, n, \\ \begin{pmatrix} \hat{w}_m^j \\ \hat{w}_d^j \end{pmatrix} = \begin{pmatrix} \sigma_m & \sigma_o \\ \hat{j}_m^j & \hat{j}_d^j \end{pmatrix}^{-1} \begin{pmatrix} \theta_m^j - \sum_{l=1}^n \hat{w}_{e_l}^j \sigma_{e_l} h_0 \sigma_{s_0} / \sigma_m \\ (1 + \hat{j}_{e_j}^j) \theta_n^j - \sum_{l=1}^n \hat{w}_{e_l}^j j_{e_l}^j \end{pmatrix}, \end{cases} \quad (4.27)$$

where

$$\begin{cases} \theta_m^j = \frac{1}{\gamma_j} \eta_j, \\ \theta_n^j = (1 + \kappa_j)^{-\frac{1}{\gamma_j}} \exp(\frac{1}{\gamma_j} \mathbf{Q}'_j \beta_m) - 1. \end{cases} \quad (4.28)$$

Here, $\hat{w}_{e_i}^j = \hat{w}_{e_i}^j(t = 0)$, $\mathbf{Q}_j = \mathbf{Q}_j(t = 0)$, in which $\mathbf{Q}_j(t)$ is a deterministic vectored process given by Equation (4.23).

4.4 Properties

In Chapter 4.3.3 we show that the asset allocation problem boils down to optimally investing in the global equity index, the global derivative portfolio and currencies. In this section, we focus on the optimal weights on this simplified universe of assets (instead of the country-specific stocks and derivatives), especially the optimal weights on currencies. In Chapter 4.4.1, we decompose the optimal net currency weight into four components, among which the intertemporal hedging component is of particular interests. In Chapter 4.4.2, we conduct comparative statics analysis of the intertemporal hedging demand with respect to jump risk parameters.

4.4.1 Decompose the currency weight

Now that we have solved the asset allocation problem for investors from each country, we study the property of the optimal net currency weights in their portfolios in this section. Note that the solutions given by Proposition 4.3 are general results where exchange rates are exposed to both the equity jump component and the currency-specific jump component. In this section, we make the simplified assumption that $j_i = 0, \forall i = 1, \dots, n$, so that the equity jump component N_m does not drive currency returns.

We can write the HJB equation in terms of portfolio weights on the global equity index, the derivative portfolio and the risky currencies. We suppress the time dependence if no

confusion is caused.

$$\begin{aligned}
0 = \sup_{\hat{w}^j} \Big\{ & J_t + \left(r_j + \hat{w}_m^j (\hat{\mu}_m^j - \hat{j}_m^j) \lambda_m + \hat{w}_d^j (\hat{\mu}_d^j - \hat{j}_d^j) \lambda_m + \sum_{i=1}^n \hat{w}_{e_i}^j (\hat{\mu}_{e_i}^j \lambda_m - \mathbb{E}[y_i] \lambda_i) \right) J_x x \\
& + \alpha_m (\lambda_{m,\infty} - \lambda_m) J_{\lambda_m} + \sum_{i=1}^n \alpha_i (\lambda_{i,\infty} - \lambda_i) J_{\lambda_i} + \frac{1}{2} \left((\hat{w}_m^j \sigma_m)^2 \lambda_m + (\hat{w}_d^j \sigma_d)^2 \lambda_m \right. \\
& + \sum_{i=1}^n (\hat{w}_{e_i}^j)^2 (\sigma_{e_i}^2 \lambda_m + v_i^2 \lambda_i) + 2 \hat{w}_m^j \hat{w}_d^j \sigma_m \sigma_d \lambda_m + 2 \sum_{i=1}^n \hat{w}_m^j \hat{w}_{e_i}^j \sigma_m \sigma_{e_i} \lambda_m \\
& + \sum_{i=1}^n \sum_{l=1}^{n \setminus i} \hat{w}_{e_i}^j \hat{w}_{e_l}^j \sigma_{e_i} \sigma_{e_l} \lambda_m + 2 \sum_{i=1}^n \hat{w}_d^j \hat{w}_{e_i}^j \sigma_o \sigma_{e_i} \lambda_m \Big) J_{xx} x^2 \\
& + \lambda_m \left(J(x(1 + \hat{w}_m^j \hat{j}_m^j + \hat{w}_d^j \hat{j}_d^j), \boldsymbol{\lambda} + \boldsymbol{\beta}_m) - J \right) \\
& + \sum_{i=1}^n \lambda_i \mathbb{E} \left[\left(J(x(1 + \hat{w}_{e_i}^j y_i), \boldsymbol{\lambda} + \boldsymbol{\beta}_i) - J \right) \right] \Big\}.
\end{aligned}$$

If $\hat{w}_m^j, \hat{w}_d^j, \hat{w}_{e_i}^j$ given by Proposition 4.3 are optimal, then by substituting J for its functional form (4.20), $\hat{w}_{e_i}^j$ must satisfy the following first order conditions

$$\begin{aligned}
0 = & \hat{\mu}_{e_i}^j \lambda_m - \mathbb{E}[y_i] \lambda_i - \gamma_j \left(\hat{w}_d^j \sigma_d \sigma_{e_i} \lambda_m + \hat{w}_m^j \sigma_m \sigma_{e_i} \lambda_m + \sum_{l=1}^n \sigma_{e_l} \sigma_{e_i} \lambda_m \right. \\
& + \left. \hat{w}_{e_i}^j (\sigma_{e_i}^2 \lambda_m + v_i^2 \lambda_i) \right) + \lambda_i e^{\mathbf{Q}_j^T \boldsymbol{\beta}_i} \mathbb{E}[(1 + \hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i], \\
= & :a_i^j - \gamma_j \left(\hat{w}_d^j \sigma_{id} + \hat{w}_m^j \sigma_{im} + \sum_{l=1}^n \hat{w}_{e_l}^j \sigma_{jl} \right) + \lambda_i (Y_i - \mathbb{E}[y_i]) + \lambda_i (e^{\mathbf{Q}_j^T \boldsymbol{\beta}_i} - 1) Y_i,
\end{aligned} \tag{4.29}$$

where $a_i^j := \hat{\mu}_{e_i}^j \lambda_m$ is the expected excess return of currency i for investor j ; $\sigma_{id} := \sigma_d \sigma_{e_i} \lambda_m$ is the covariance between the derivative portfolio and currency i ; $\sigma_{im} := \sigma_m \sigma_{e_i} \lambda_m$ is the covariance between the market equity portfolio and currency i ; $\sigma_{il} := \sigma_{e_i} \sigma_{e_l} \lambda_m$ is the covariance between currency j and currency l ; $b_i := \sigma_{e_i}^2 \lambda_m + v_i^2 \lambda_i$ is the instantaneous variance of currency i ; and $Y_i := \mathbb{E}[(1 + \hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i]$ is the marginal utility increase induced by jump component N_i from investing in one unit of foreign currency i .

Rearrange Equation (4.29) and get

$$\hat{w}_{e_i}^j = \frac{1}{\gamma_j b_i} \left\{ \underbrace{a_i^j}_{I} - \underbrace{\gamma_j (\hat{w}_d^j \sigma_{id} + \hat{w}_m^j \sigma_{im} + \sum_{l=1}^{n \setminus i} \hat{w}_{e_l}^j \sigma_{e_l})}_{II} + \underbrace{\lambda_i Y_i}_{III} + \underbrace{\lambda_i (e^{\mathbf{Q}_j^T \boldsymbol{\beta}_i} - 1) Y_i}_{IV} \right\}. \tag{4.30}$$

The optimal portfolio weights consists of a risk premium demand (I), a risk management demand (II), a myopic buy-and-hold demand (III), and an intertemporal hedging demand (IV).

The risk premium demand (I) is determined by the expected excess return on investing in foreign currency i . It is a return-driven demand. A larger expected excess return indicates larger appreciation of the currency in expectation with respect to the domestic currency. The risk management demand (II) exploits the diversification benefit of investing in the risky currency. The diversification potential is measured by its covariance with the market equity,

equity derivative portfolio and other risky currencies. This is the demand that has been extensively studied in the international empirical finance literature. For example, Glen and Jorion [71], Campbell et al. [28], and De Roan et al. [45] all base their currency hedging strategies solely on the risk management demand (covariance with the market equity portfolio).

The myopic buy-and-hold demand (III) arises due to currency jumps. As explained by Liu et al. [92], unlike continuous fluctuations, jumps may occur before the investor has the opportunity to adjust the portfolio. Jump risks, therefore, are similar to “illiquidity risk”: the investor has to hold the asset until the jump has occurred. Observe that

$$Y_i^j \propto \nabla_{\hat{w}_{e_i}^j} \mathbb{E}[u(X_j(t)) - u(X_j(t^-)) | N_i(t) - N_i(t^-) = 1].$$

$\mathbb{E}[u(X_j(t)) - u(X_j(t^-)) | N_i(t) - N_i(t^-) = 1]$ is the expected utility gain at time t conditional on an occurrence in jump component i at time t . Therefore (III) is the expected marginal utility increase induced by jump component i from investing in one unit of currency i at time t . The buy-and-hold demand is “myopic” in the sense that it does not take into account the uncertainties of future jump intensities. Note that in case of $\gamma_j = 0$, meaning that the investor is risk neutral, this term is zero.

The last term (IV) is tailored to account for the fact that the jumps are mutually exciting. Since currency returns and state variables $\lambda(t)$ are driven by the jump component $N_i(t)$, currency i can be used to hedge future realizations of the state variables.

As one may expect, currency weights predicted by special cases of our model are combinations of the decomposed terms. In particular, if the currency returns are independent of equity returns, as assumed by Solnik [112], the risk management demand (II) of the currency is zero. If the economy is free of jumps as in Sercu [109], Adler and Dumas [2], and Black [15], both the myopic buy-and-hold demand (III) and the intertemporal hedging demand (IV) are zero. If the jumps are Poissonian with constant jump intensities as in Torres [115], the intertemporal hedging demand (IV) is zero.

4.4.2 Comparative statics

The risk premium demand, risk management demand and the buy-and-hold demand components of the currency weights can be interpreted in a straightforward way by observing Equation (4.30). One can immediately tell that the risk premium demand (I) increases when the investor can earn a higher expected excess return from investing in the risky currency; the risk management demand (II) is negative when there is positive correlation between other assets and the currency and positive otherwise; the buy-and-hold demand (III) is negative when the currency jumps downward and positive otherwise.

The intertemporal hedging demand (IV), however, depends on the excitation structure between the equity jump component and the currency jump component. To see how (IV) is determined by the jump excitation parameters, we conduct comparative statics analysis in Figure 4.1 and 4.2.

We consider three countries: a base country with the base currency, Country I, and Country II. We make the simplified assumption that these three countries represent the global financial market. We denote the currencies from these countries by the base currency, Currency I, and Currency II, respectively. Similarly, we call the representative investors from these countries the base investor, Investor I and Investor II, respectively.

Here, we only study the comparative statics of the optimal net weight on Currency I from the perspective of the base investor. The behaviour of the optimal net weight on Currency

I for investor II has a qualitatively similar pattern. To keep the analysis clean, we adopt a deterministic currency jump size $y_1, y_2 < 0$ in this section.

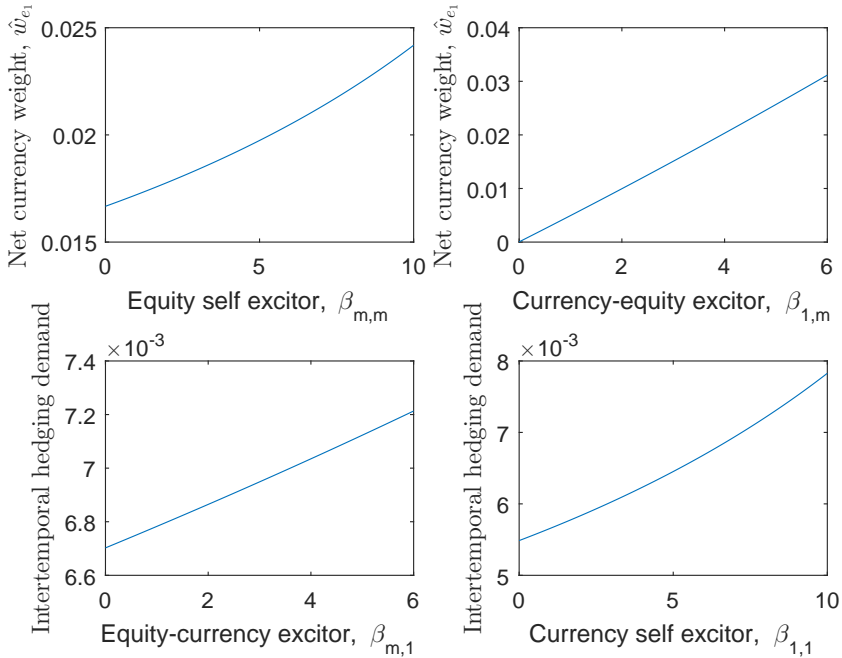


Figure 4.1: The intertemporal hedging demand (IV) of Currency I for the base investor as functions of elements in the excitation matrix β . The base case parameters are $\eta_0 = 0.1$, $\sigma_m = 0.2$, $\sigma_d = 0.1$, $\sigma_{e1} = \sigma_{e2} = -0.1$, $v_1 = v_2 = 0.05$, $j_m = -0.03$, $j_d = 0.1$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $\beta = (15, 6, 6; 6, 8, 0; 6, 0, 8)$, $T = 1$, $\kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$, $\lambda_m = \lambda_1 = 2$.

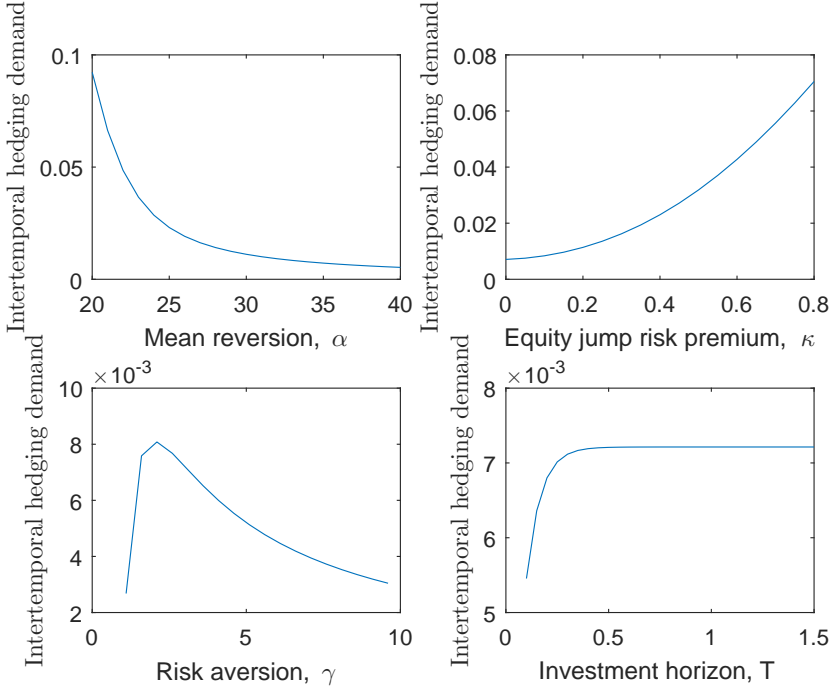


Figure 4.2: The intertemporal hedging demand (IV) of Currency I for the base investor as functions of the mean reversion rate α_m (top left), equity jump risk premium κ_0 (top right), risk aversion γ_0 (bottom left) and investment horizon T (bottom right). The base case parameters are $\eta_0 = 0.1$, $\sigma_m = 0.2$, $\sigma_d = 0.1$, $\sigma_{e1} = \sigma_{e2} = -0.1$, $v_1 = v_2 = 0.05$, $j_m = -0.03$, $j_d = 0.1$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $\beta = (15, 6, 6; 6, 8, 0; 6, 0, 8)$, $T = 1$, $\kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$, $\lambda_m = \lambda_1 = 2$.

We plot the intertemporal hedging demand (IV) of Currency I for the base investor as functions of elements in the excitation matrix β in Figure 4.1. The figure shows that increasing any element of the excitation matrix β increases the hedging demand (IV) of Currency I in the base investor's portfolio, whether it be the self excitor of the market equity, $\beta_{m,m}$ (top left), the equity-currency excitor, $\beta_{m,1}$ (bottom left), the currency-equity excitor, $\beta_{1,m}$ (top right), or the self excitor of Currency I, $\beta_{1,1}$ (bottom right).

Figure 4.2 plots the intertemporal hedging demand (IV) of Currency I for the base investor as functions of the mean reversion rate α_m (top left), equity jump risk premium κ_0 (top right),⁵ domestic risk aversion γ_0 (bottom left) and the investment horizon T (bottom right). Larger jump risk premium and longer investment horizon result in increasing hedging demand for Currency I. On the contrary, faster mean reversion rate decreases the hedging demand for Currency I for the base investor. Interestingly, increasing the risk aversion first increases then decreases the base investor's hedging demand.

When $y_1 < 0$, from the perspective of the base investor, the foreign currency jumps downward, opposite to the jumps in equity and currency jump intensities. Currency I, therefore, can be used as a static hedge against the state variables. As a result, the base investor

⁵For every κ_0 , we maintain that $\kappa_0 = \kappa_1 = \kappa_2$.

has a positive hedging demand for Currency I. The larger the hedging potential the risky currency has against the state variables, the larger the hedging demand.

The currency jump intensity process $\lambda_1(t)$ is more volatile under larger excitation and slower mean reversion. As one may expect, the more uncertain the state variables are, the larger hedging incentive investors have. Loosely speaking, as the equity jump risk premium increases, more weight is assigned to the market equity, which leads to more jump risks to be hedged. Similarly, longer investment horizon leads to increased sensitivity of indirect utility to state variables. In short, hedging demand rises when there are increasing uncertainties in the indirect utility.

The effect of increasing the risk aversion, however, is not clear. On the one hand, increasing the risk aversion decreases the demand for Currency I in general, implying a smaller amount to be hedged. On the other hand, a more risk averse investor is more inclined to hedge the changes in the state variable, and may have a larger hedging demand consequently. The final result depends on which effect is larger. Figure 4.2 shows that the effect of increasing risk aversion is not monotone: it first increases and then reduces the jump risk demand.

An interesting phenomenon is that while the currency weight $\hat{w}_{e_t}^j$ does not display market timing, its components do. Figure 4.3, 4.4 plot the volatility-scaled four components of the weight on Currency I in the base investor's portfolio as functions of the equity jump intensity λ_m and the currency jump intensity λ_1 , respectively.

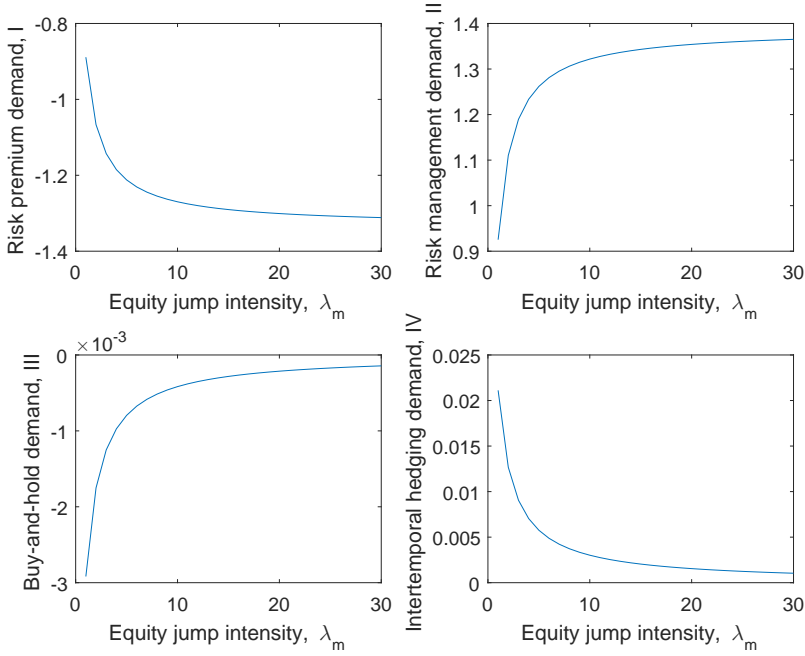


Figure 4.3: Comparative statics of the components (I, II, III, IV) of the optimal net weight on Currency I for the base investor as functions of the current equity jump intensity λ_m . The upper left panel plots the risk premium demand I, the upper right panel plots the risk management demand II, the bottom left panel plots the buy-and-hold demand III, and the bottom right panel plots the intertemporal hedging demand IV. The base case parameters are $\eta_0 = 0.2$, $\sigma_m = 0.2$, $\sigma_d = 0.1$, $\sigma_{e1} = \sigma_{e2} = -0.1$, $v_1 = v_2 = 0.05$, $j_m = -0.03$, $j_d = 0.02$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $\beta = (15, 6, 6; 6, 8, 0; 6, 0, 8)$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma = 3$, $\lambda_1 = \lambda_2 = 2$.

Figure 4.3 plots the components of the optimal weight on Currency I for an investor from the base country as functions of the current equity jump intensity λ_m . The figure shows that the risk premium demand I (upper left panel) increases with the equity jump intensity. Since the expected excess return is proportional to the jump intensity, larger jump intensity increases the compensation for the base investor. It is not surprising that given the negative covariance with other assets, the risk management component II (upper right panel) decreases with the equity jump intensity. Larger equity intensity increases the covariance, resulting in more negative risk management demand. Both the myopic buy-and-hold demand III (bottom left) and the intertemporal hedging demand IV (bottom right) approach zero as the equity jump intensity increases. Larger equity jump intensity increases the currency volatility b_i but leaves the myopic demand and intertemporal hedging demand unchanged. Therefore after volatility scaling, both components approach zero as volatility increases.

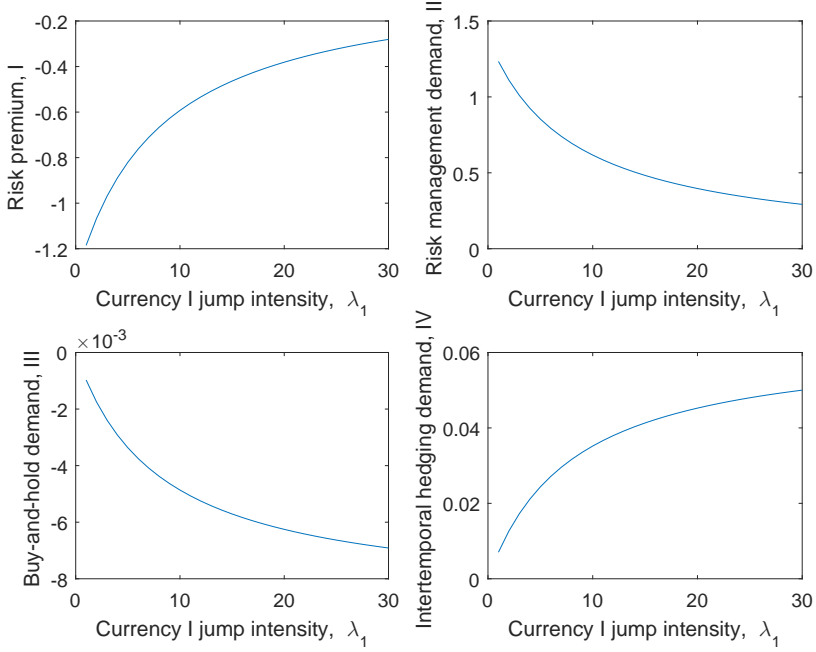


Figure 4.4: Comparative statics of the components (I, II, III, IV) of the optimal net weight on Currency I for the base investor as functions of the current Currency I jump intensity λ_1 . The upper left panel plots the speculation demand I, the upper right panel plots the risk management demand II, the bottom left panel plots buy-and-hold demand III, and the bottom right panel plots the hedging demand IV. The base case parameters are $\eta_0 = 0.2$, $\sigma_m = 0.2$, $\sigma_d = 0.1$, $\sigma_{e_1} = \sigma_{e_2} = -0.1$, $v_1 = v_2 = 0.05$, $j_m = -0.03$, $j_d = 0.02$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $\beta = (15, 6, 6; 6, 8, 0; 6, 0, 8)$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma = 3$, $\lambda_m = \lambda_2 = 2$.

Figure 4.4 plots the components of the optimal weight on Currency I for the base investor as functions of the current currency jump intensity λ_1 . We see opposite patterns to Figure 4.3. Both the risk premium demand I (upper left) and the risk management demand II (upper right) converge to zero as the currency jump intensity increases. This is because larger currency jump intensity raises the currency volatility but does not affect the expected excess return or covariance matrix. Both buy-and-hold demand III (bottom left) and intertemporal hedging demand IV (bottom right) increase in absolute value as currency jump intensity rises. This is because increasing the currency jump intensity magnifies the currency jump effect, thereby increasing the demands related to the currency jump component.

4.5 Market equilibrium

This chapter derives the equilibrium currency hedging strategies. Previous to this chapter, we work with the net currency weight $\hat{w}_{e_i}^j$. The raw currency weights are given by the net weights plus the implicit currency investment in the currency-hedged assets. In order to calculate the equilibrium currency hedging strategy, one needs to get to the raw currency

weights. In Chapter 4.5.1, we introduce market clearing conditions. We give equilibrium currency hedging formula in Chapter 4.5.2.

4.5.1 Equilibrium condition

Denote country j 's wealth as a proportion to the world wealth as $f_j = X_j/M^j$. Following Wang [118] and Bongaerts et al. [17], Definition 4.1 defines the market equilibrium as the condition that the security markets clear.

Definition 4.1 (Market Equilibrium). *Market equilibrium consists of the asset price processes $(M^j(t), D^j(t))$ and the trading strategies (w^j) for $j = 0, \dots, n$, such that investor j 's expected utilities is maximized*

$$w^j = \arg \sup \mathbb{E}_0 \left[\frac{X_j(T)^{1-\gamma_j}}{1-\gamma_j} \right],$$

subject to their respective wealth dynamics:

$$\frac{dX_j(t)}{X_j(t^-)} = w_m^j \frac{dM^j(t)}{M^j(t^-)} + w_d^j \frac{dD^j(t)}{D^j(t^-)} + \sum_{l=0}^n w_{e_l}^j \frac{dB_l^j(t)}{B_l^j(t^-)}, \quad j = 0, \dots, n. \quad (4.31)$$

and the security markets clear

$$\begin{cases} \sum_{i=0}^n f_i = 1, \\ \sum_{i=0}^n h_i = 1, \\ \sum_{i=0}^n f_i w_d^i = 0, \\ \sum_{i=0}^n f_i w_{e_l}^i = 0, \quad l = 0, \dots, n. \end{cases} \quad (4.32)$$

The first equation implies that the sum of the market capitalization of each country equals the total market capitalization. The second condition says that the total capital in the market comes from the wealth of nations. The third and fourth equations impose that the net supply of the equity derivatives and bonds should be zero, meaning that the gross lending in the stock derivatives and bonds should be equal to the gross borrowing.

The security market clearing conditions imply that the wealth distribution f , country's market capitalization h and each country's share in the derivative portfolio need to be consistent with the return dynamics of equity and exchange rate return dynamics.

In Chapter 4.3, we have derived the optimal asset allocation on currency-hedged assets. The following lemma shows how to compute the weights on the raw assets from the weights on the hedged assets.

Lemma 4.1. *The portfolio weights on the currency-unhedged assets are given by*

$$w_m^j = \hat{w}_m^j, \quad \hat{w}_o^j = \hat{w}_o^j, \quad j = 0, \dots, n \quad (4.33)$$

$$w_{e_i}^j = \hat{w}_{e_i}^j - h_i \hat{w}_m^j - k_i \hat{w}_d^j, \quad i, j = 1, \dots, n, \quad i \neq j, \quad (4.34)$$

$$w_{e_j}^j = 1 + \hat{w}_{e_j}^j - h_j \hat{w}_m^j - k_j \hat{w}_d^j, \quad i, j = 1, \dots, n, \quad (4.35)$$

$$w_{e_0}^j = - \left(\sum_{i=1}^n \hat{w}_{e_i}^j + h_0 \hat{w}_m^j + k_0 \hat{w}_d^j \right), \quad j = 1, \dots, n. \quad (4.36)$$

Therefore we can equivalently construct the market clearing conditions using the weights on the currency-hedged assets.

Theorem 4.2. For $f_i, h_i \in [0, 1]$, Equation (4.32) is equivalent to

$$\begin{cases} \sum_{i=0}^n f_i = 1, \\ \sum_{i=0}^n h_i = 1, \\ \sum_{i=0}^n f_i \hat{w}_m^i = 1, \\ \sum_{i=0}^n f_i \hat{w}_d^i = 0, \\ \sum_{i=0}^n \hat{w}_{e_j}^i f_i - h_j + f_j = 0, \quad \forall j = 1, \dots, n. \end{cases} \quad (4.37)$$

4.5.2 Equilibrium currency hedging

In Black [15], the equilibrium hedging strategy of currency i for investor j is defined as the negative of the investment on currency i per unit of the global equity index invested,

$$H_i^j := -\frac{w_{e_i}^j}{w_m^j}. \quad (4.38)$$

In terms of weights on the currency-hedged assets, Equation (4.38) can be expressed as

$$H_i^j = -\frac{\hat{w}_{e_i}^j - (h_i \hat{w}_m^j + k_i \hat{w}_d^j)}{\hat{w}_m^j}. \quad (4.39)$$

Proposition 4.4 (Black's formula). *If all prices follow geometric Brownian motion processes and all investors have the same risk aversion coefficient γ , then the equilibrium hedging strategy of currency i for any investor $j, j \neq i$ is given by*

$$H_i^j\text{-Black} = f_i(1 - 1/\gamma), \quad \forall j \neq i. \quad (4.40)$$

Equation (4.40) is the well-known universal hedging formula derived by Black [15]. The two key implications are: (1) In equilibrium, every investor hedges the same amount of any risky currency i regardless of his/her home currency j ; (2) The universal currency hedging ratio of currency i only depends on two variables: the coefficient of relative risk aversion and the total wealth held by investors in country i . This means that the currency's expected excess return, volatility or correlation with the equity market do not have a direct impact on the hedging ratio of the currency, as long as the wealth holdings and risk attitude are fixed.

4.6 Safe haven vs. investment currencies

In Ranaldo and Söderlind [103], a safe haven currency is a currency that offers hedging benefits on average. For instance, Campbell et al. [28] show that Swiss franc and Euro are negatively related to the equity market. However, correlations between currencies and the equity market turn out to be unstable and may switch between positive and negative values periodically. Even worse, during the 2007-2009 financial crisis, as Kohler [85] notes, "a large number of currencies that were not at the center of the turmoil depreciated, even those which were regarded as safe haven currencies preceding the crisis". Ranaldo and Söderlind [103] also confirm that safe haven effects went against typical patterns during the 2008 financial crisis.

Therefore we focus on the alternative definition of safe haven currencies in Ranaldo and Söderlind [103]: a currency is considered safe haven if it gives hedging benefits in times of stress.

We can intuitively distinguish a safe haven currency and an investment currency in our framework. A safe haven currency provides a safe haven to investors during a recession. Therefore a safe haven currency should be relatively immune to capital market turmoil. In our model, the excitor $\beta_{m,i}$ measures how large a jump occurrence in the equity market N_m raises the intensity of the currency jump component λ_i . A safe haven currency, therefore, should have a relatively small $\beta_{m,i}$. An investment currency, on the contrary, is like the mirror image of safe haven currencies and is characterized by a relatively large $\beta_{m,i}$. As a consequence, a safe haven currency is not as prone to the equity market downturns as an investment currency.

Whether a currency is of the “safe haven” type or “investment” type has important implication in determining the optimal currency exposure. Observe that $\beta_{m,i}$ plays a different role from $\beta_{i,m}$ in determining the currency demand. Recall that the intertemporal hedging demand (IV) is a function of β_i , in which $\beta_{i,m}$ and $\beta_{m,i}$ are not weighted symmetrically. Therefore imagine a safe haven currency and an investment currency with identical risk profile (including expected return, covariance, jump size, jump intensity, etc.) except that the safe haven currency has smaller equity-currency excitor $\beta_{m,i}$, the demand for these two risky currencies would be in general different.

4.6.1 Equilibrium net currency weight

In this section, we are going to illustrate investors’ preferences towards the safe haven currency numerically. Similar to the numerical studies in Chapter 4.4.2, we consider a three-currency scenario including a base currency.

Figure 4.5 and Figure 4.6 plot the equilibrium net weight on Currency I for the base investor, $\hat{w}_{e_1}^0$, when the equity-currency contagion structure changes, using the first Equation of (4.27). We fix the first and third row of the excitation matrix. The equity-currency contagion structure for Currency II does not vary. In Figure 4.5, we let the equity-currency excitor $\beta_{m,1}$ increase while varying the currency self excitor $\beta_{1,1}$ so that the expected jump intensity $\mathbb{E}[\lambda_1]$ is kept the same. Conversely in Figure 4.6, we let the currency-equity excitor $\beta_{1,m}$ increase while varying the currency self excitor $\beta_{m,m}$ so that the expected equity jump intensity $\mathbb{E}[\lambda_m]$ does not vary. The equilibrium net weights on Currency I are plotted in the solid curves, and those on Currency II are depicted in dotted curves.

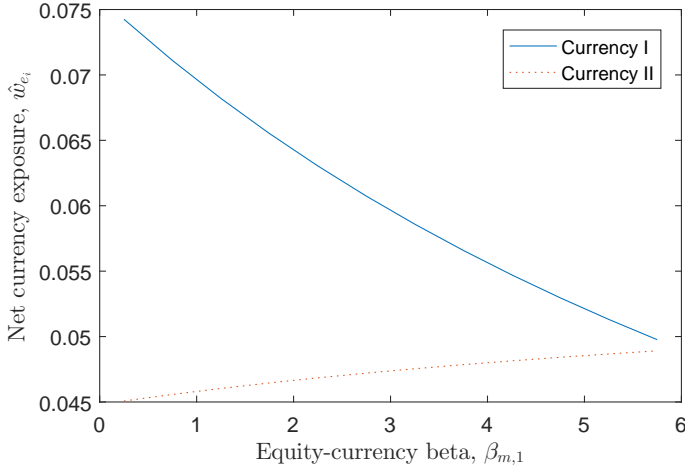


Figure 4.5: Equilibrium net weight on Currency I (solid line) and II (dotted line) of the base investor as a function of the equity-currency excitor $\beta_{m,1}$. The equilibrium net currency weight for the base investor is computed using Equation (4.27). The excitation matrix is $\beta = (15, 6, 6; \beta_{m,1}, \beta_{1,1}, 0; 6, 0, 8)$. We let $\beta_{m,1}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. All the other parameters are kept constant at $\eta_0 = 0.3$, $\sigma_{e_1} = \sigma_{e_2} = -0.1$, $v_1 = v_2 = 0.05$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$.

Jump excitation structure determines the intertemporal hedging demand for currencies. The x -axis in Figure 4.5 starts with 0, indicating that an occurrence in the equity jump component does not increase the probability of a depreciation of Currency I. In comparison, at the end point of the x -axis, an occurrence in the equity jump component raises the jump intensity $\lambda_1(t)$ by 6. As $\beta_{m,1}$ increases, the impact of a price plunge in the equity market on Currency I increases, making Currency I less safe haven. When Currency I moves away from a safe haven currency and towards an investment currency, the base investor decreases the net weight on Currency I and slightly increases that on the other risky currency, Currency II.

Figure 4.6 plots the equilibrium net weight on Currency I (solid line) and II (dotted line) of the base investor as a function of the currency-equity excitor $\beta_{1,m}$. Similar to Figure 4.5, the expected jump intensities of the equity and currencies are kept constant.

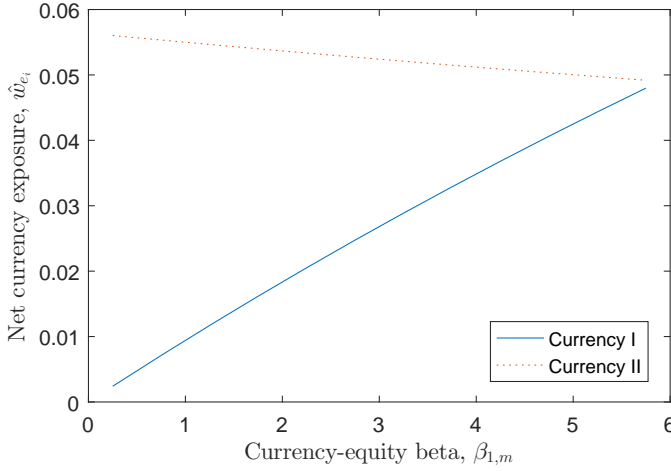


Figure 4.6: Equilibrium net weight on Currency I (solid line) and II (dotted line) of the base investor as a function of the currency-equity excitor $\beta_{1,m}$. The equilibrium net currency weight for the base investor is computed using Equation (4.27). The excitation matrix is $\beta = (\beta_{m,m}, \beta_{1,m}, 6; 6, 8, 0; 6, 0, 8)$. We let $\beta_{1,m}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. All the other parameters are kept constant at $\eta_0 = 0.3$, $\sigma_{e1} = \sigma_{e2} = -0.1$, $v_1 = v_2 = 0.05$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$.

Note that in both Figure 4.5 and 4.6, the two non-base currencies have the same risk profile (volatility, covariance with the equity, jump amplitude, expected jump intensity) except the excitation structure with the equity market. Figure 4.5 shows what happens when Currency I moves from a safe haven currency to an investment currency. When $\beta_{m,1}$ is small, the currency has the safe haven characteristic thus is stable during market downturns. As $\beta_{m,1}$ increases, the currency becomes more liable to depreciate during the capital market turmoil. We observe from the figure that the base investor demands more currency exposure when the currency is of the safe haven type. In Figure 4.6, even though the dependence between the equity and Currency I increases all the same (just like Figure 4.5), the optimal net currency weight displays an opposite pattern to Figure 4.5. Comparing Figure 4.5 and 4.6, we conclude that when it comes to portfolio choices, the direction of excitation matters. In particular, a currency is only safe haven when the *equity-currency* excitor is small.

4.6.2 Equilibrium currency hedging strategy

In this section, we study what happens to the equilibrium currency hedging strategy given in Equation (4.39) as the equity-currency excitor increases.

For a cleaner illustration of the distinction between the equilibrium currency hedging prediction of our model and that of Black [15], we fix the risk aversion parameter γ and the wealth distribution vector \mathbf{f} , such that the prediction of Black [15] is not affected by the variation in the equity-currency contagion. The equilibrium needs to be restored at every new excitor value. To do this, we allow the global equity index and derivative portfolio to be endogenous. The detailed algorithm of finding and restoring the market equilibrium can be found in Appendix 4.C.

Figure 4.7 compares the equilibrium currency hedging ratio of our model to the universal hedging ratio given in Equation (4.40) of Black [15]. The left panel plots the model prediction of the hedging ratio of currency I in equilibrium when Currency I moves from a safe haven currency to an investment currency, using Equation (4.39). The hedging ratio of Currency I for the base investor is plotted in the solid curve, while that for Investor II is plotted in the dotted curve. The figure is produced in the same way as Figure 4.5 but with the dependent variable being the hedging ratio of Currency I. We see that as Currency I becomes more prone to equity market downturns, both investors from the base country and Country II hedge a larger proportion of the risk of Currency I.

The right panel of Figure 4.7 plots the currency hedging prediction calculated using the Black hedging formula (4.40). We see that the equilibrium currency hedging in the Black model does not change when Currency I is no longer safe haven. Notice that in the right panel, one curve is visible because the base investor and Investor II have the same hedging ratio of Currency I, namely, the universal hedging ratio.

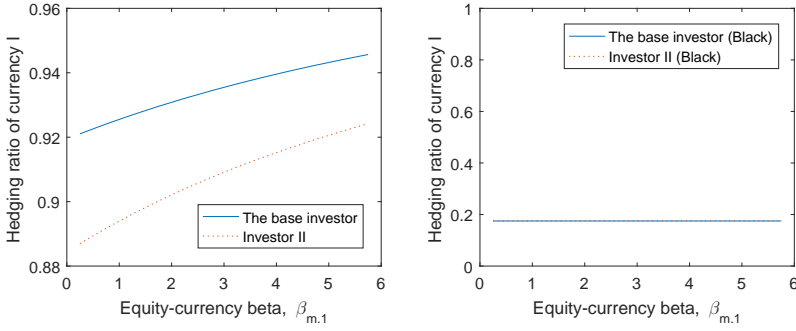


Figure 4.7: Equilibrium hedging ratio of Currency I as a function of the equity-currency excitor $\beta_{m,1}$. The left panel plots the hedging ratio calculated by Equation (4.39). The hedging ratio for the base investor is plotted in the solid curve, while that for Investor II is plotted in the dotted curve. The right panel plots Black's universal hedging ratio for Currency I (see Equation (4.40)). Here, one curve is visible because the base investor and Investor II have the same hedging ratio. The excitation matrix is $\beta = (15, 6, 6; \beta_{m,1}, \beta_{1,1}, 0; 6, 0, 8)$. We let $\beta_{m,1}$ increase and find the corresponding $\beta_{m,m}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. The following parameters are kept constant at $\eta_0 = 0.3$, $\sigma_{e_1} = \sigma_{e_2} = -0.1$, $v_1 = v_2 = 0.05$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$.

Figure 4.8 plots the equilibrium hedging ratio of Currency I as a function of the currency-equity excitor $\beta_{1,m}$. Similar to Figure 4.7, we keep the risk aversion parameter γ , the wealth distribution vector \mathbf{f} , and the expected jump intensities constant. We see that increasing the currency-equity excitor leads to opposite patterns as increasing the equity-currency excitor, although both leads to larger equity-currency dependence. Similar to the right panel of Figure 4.7, increasing the currency-equity beta does not have an impact on the Black's hedging strategy.

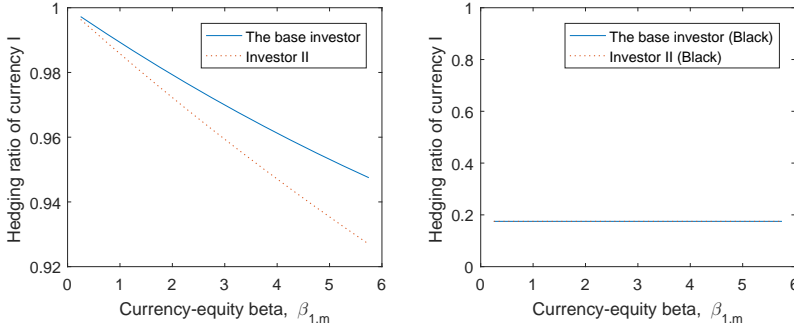


Figure 4.8: Equilibrium hedging ratio of Currency I as a function of the currency-equity excitor $\beta_{1,m}$. The left panel plots the hedging ratio calculated by Equation (4.39). The hedging ratio for the base investor is plotted in the solid curve, while that for Investor II is plotted in the dotted curve. The right panel plots Black's universal hedging ratio for Currency I (see Equation (4.40)). Here, one curve is visible because the base investor and Investor II have the same hedging ratio. The excitation matrix is $\beta = (15, 6, 6; \beta_{1,m}, \beta_{1,1}, 0; 6, 0, 8)$. We let $\beta_{1,m}$ increase and find the corresponding $\beta_{1,1}$ such that the expected equity and currency jump intensities do not vary with the excitation matrix. The following parameters are kept constant at $\eta_0 = 0.3$, $\sigma_{e_1} = \sigma_{e_2} = -0.1$, $v_1 = v_2 = 0.05$, $\alpha_m = \alpha_1 = \alpha_2 = 35$, $T = 1$, $\kappa_0 = \kappa_1 = \kappa_2 = 0.02$, $y_1 = y_2 = -2\%$, $\gamma_0 = \gamma_1 = \gamma_2 = 3$.

In Figure 4.5, 4.6, 4.7, and 4.8, we use call options as the derivative assets. In Appendix 4.D, we show that if we use put options and straddles instead of call options, the patterns are qualitatively similar.

All else equal, investors prefer safe haven currencies to investment currencies regardless of their home currencies. The latter is more likely to go through a substantial depreciation once the equity market experiences a price plunge. Investors with exposure to investment currencies have to take the risk that the currency investment will go down during financial crises. Exposure to safe haven currencies, however, can act as a shield to the equity investment: when the equity market is in turmoil, the value of the safe haven currencies typically remains stable.

4.7 Conclusion

Inspired by the empirical findings that there exists risk spillover from the equity market to the currency market, we revisit the classic equilibrium currency hedging problem established by Solnik [112] and Black [15] under the context of equity-currency contagion. We postulate a mutually exciting jump diffusion model to jointly model equity returns and currency returns. Our model is consistent with the extant literature in that (1) the currency returns are subjected to country-specific risk factors as well as global risk factors, and (2) currency returns are subjected, but not limited, to equity risks. On top of these features, we further allow for cross excitation among the equity jump component and the currency jump components.

We assume that the global market is integrated and free of arbitrage opportunities, in which case the return of a currency is equal to the difference in the returns of the pricing kernels of the two countries. We first solve analytically the asset allocation problem for every representative investor in terms of the currency-hedged assets. We show that the optimal net currency weights can be decomposed into four components: (1) a risk premium demand that

earns the expected excess returns by taking currency risks, (2) a risk management demand that exploits the diversification benefits embedded in the instantaneous covariance matrix with other assets in the portfolio, (3) a myopic buy-and-hold demand which is induced by discontinuities (jumps) in the returns, and (4) an intertemporal hedging demand that hedges the state variable risks. The intertemporal hedging demand is a result of the mutually exciting nature of the jump components. Loosely speaking, the intertemporal hedging demand for currencies increases when there is more uncertainty in the state variables and when there is more jump risk to hedge.

Next, we impose security market clearing conditions to derive the equilibrium currency hedging strategy, defined as the negative of the investment on a risky currency per unit of global equity index invested. Compared with the classic equilibrium currency hedging ratio of Black [15], our prediction has two distinctive features. First, the universal hedging ratio no longer holds: investors with different domestic currencies will have different currency hedging ratios in general. Second, the dependence structure between the equity market and the currency market does matter: everything else equal, investors hedge more investment currency risk than the safe haven currency risk, whereas investors can be indifferent in Black [15].

Appendices

4.A Proofs

Proof for Proposition 4.1. Define

$$X_j(t) = \log S_j(t),$$

$$\psi(u) = e^{-r_j(\tau-t)} \mathbb{E}^{Q_j}[\exp(uX_j(\tau)) | \mathcal{F}_t].$$

Under the risk neutral measure Q_j of country j , the dynamics of $X_j(t)$ follows

$$dX_j(t) = \left(r_j - \frac{1}{2} \sigma_{s_j}^2 \lambda_m(t) \right) dt + \sigma_{s_j} dW_j^{Q_j}(t) \\ + \log(1 + j_{s_j}) (dN_m^{Q_j}(t) - (1 + \kappa_j) \lambda_m(t) dt).$$

The jump process $N_m^{Q_j}(t)$ has intensity $(1 + \kappa_j) \lambda_m(t)$ under the risk neutral measure of country j . Duffie et al. [50] show that the price of a call option $C_j(t)$ is given by

$$C_j(t) = G_{1,-1}(-\log K_j) - K_j G_{0,-1}(-\log K_j),$$

where $G_{a,b}(y)$ denotes the price of a security that pays $e^{aX_j(T)}$ at time T in case of $bX_j(t) \leq y$. The Fourier transform $\mathcal{G}_{a,b}(\cdot)$ is defined as

$$\mathcal{G}_{a,b}(u) := \int_{-\infty}^{+\infty} e^{izy} dG_{a,b}(y) \\ = \mathbb{E}_t^{Q_j}[\exp((a + iub)X_j(T))] \\ = \psi_t(a + iub).$$

Employ the Duffie et al. [50] transform analysis, define

$$\mathbf{K}_0 = \begin{pmatrix} 0 \\ \alpha_m \lambda_\infty \end{pmatrix}, \quad \mathbf{K}_1 = \begin{pmatrix} 0 & -\frac{1}{2} \sigma_{s_j}^2 - j_{s_j}(1 + \kappa_j) \\ 0 & -\alpha_m \end{pmatrix}, \\ (\mathbf{H}_1)_{11} = (0, \sigma_{s_j}^2)', \quad \mathbf{H}_0 = 0, \\ l_0 = 0, \quad \mathbf{l}_1 = (0, 1 + \kappa_j)', \quad \theta(c) = \exp(j_{s_j} c_1 + c_2 \beta_{m,m}).$$

It holds that

$$\psi_t(u) = S_j(t)^u \exp(\mathcal{P} + \mathcal{Q} \lambda_m(t)),$$

where $\mathcal{P} = \mathcal{P}(t)$, $\mathcal{Q} = \mathcal{Q}(t)$,

$$\frac{d}{dt} \mathcal{Q}(t) = \left(\frac{1}{2} \sigma_{s_j}^2 + j_{s_j}(1 + \kappa_j) \right) u + \alpha_m \mathcal{Q}(t) - \frac{1}{2} u^2 \sigma_{s_j}^2 \\ - (1 + \kappa_j) \left((1 + j_{s_j})^u e^{\beta_{m,m} \mathcal{Q}(t)} - 1 \right), \\ \frac{d}{dt} \mathcal{P}(t) = -\alpha_m \lambda_{m,\infty} \mathcal{Q}(t).$$

$G_{a,b}(y)$ can be recovered by applying the inverse Fourier transform formula

$$G_{a,b}(y) = \frac{1}{2} \psi_t(a) - \frac{1}{\pi} \int_0^\infty \frac{\text{Im}[e^{-iuy} \psi_t(a + iub)]}{u} du.$$

□

Proof of Proposition 4.2. Since the market is incomplete, we employ the stochastic control method to solve the portfolio optimization problem. We can rewrite the budget constraint (4.18), replacing the portfolio weight on stocks and stock options by the portfolio exposure to equity risk factors, while keeping the portfolio weight on the foreign currency:

$$\begin{aligned} \frac{dX_j(t)}{X_j(t^-)} = & r_j(t) dt + \theta_w^{j'} \left(\sqrt{\lambda_m} d\mathbf{W} + \frac{1}{\sigma_m} \eta_j \mathbf{h}' \sigma_s(\mathbf{L}\mathbf{L}') \theta_w^j \lambda_m dt \right) \\ & + \theta_n^j (dN_m - (1 + \kappa_j) \lambda_m dt) + \sum_{i=1}^n \hat{w}_{e_i}^j (\lambda_i dt - v_i \sqrt{\lambda_i} dZ_i + y_i dN_i - \mathbb{E}[y_i] \lambda_i dt). \end{aligned}$$

Bellman's optimality principle implies that

$$0 = \sup_{\hat{w}^j} \mathcal{A}J,$$

where \mathcal{A} denotes the infinitesimal generator operator. The Hamilton-Jacobi-Bellman (HJB) equation reads

$$\begin{aligned} 0 = \sup_{\theta_w^j, \theta_n^j, \hat{w}_{e_i}^j} \Big\{ & J_t + \left(r_j + \eta_j \mathbf{h}' \sigma_s(\mathbf{L}\mathbf{L}') \theta_w^j \lambda_m / \sigma_m - \theta_n^j (1 + \kappa_j) \lambda_m - \sum_{i=1}^n \hat{w}_{e_i}^j \mathbb{E}[y_i] \lambda_i \right) J_x x \\ & + \alpha_m (\lambda_{m,\infty} - \lambda_m) J_{\lambda_m} + \sum_{i=1}^n \alpha_i (\lambda_{i,\infty} - \lambda_i) J_{\lambda_i} \\ & + \frac{1}{2} \left(\theta_w^{j'} \mathbf{L}\mathbf{L}' \theta_w^j \lambda_m + \sum_{i=1}^n (\hat{w}_{e_i}^j v_i)^2 \lambda_i \right) J_{xx} x^2 \\ & + \lambda_m \left(J(x(1 + \theta_n^j), \boldsymbol{\lambda} + \boldsymbol{\beta}_m) - J \right) \\ & + \sum_{i=1}^n \lambda_i \mathbb{E} \left[J(x_j(1 + \hat{w}_{e_i}^j y_i), \boldsymbol{\lambda} + \boldsymbol{\beta}_i) - J \right] \Big\}. \end{aligned}$$

We use $J_t, J_x, J_{\lambda_m}, J_{\lambda_i}$ to denote the partial derivatives of J with respect to $t, x, \lambda_m, \lambda_i$, and similarly for the higher order derivatives.

We take derivatives of $J(t, x, \lambda)$ with respect to its arguments, substitute into the HJB equation, and differentiate with respect to the portfolio risk exposure θ_w^j, θ_n^j , and the currency weights $\hat{w}_{e_i}^j, i = 1, \dots, n$, to obtain the following first-order conditions:

$$0 = \frac{1}{\sigma_m} \eta_j \mathbf{h}' \sigma_s(\mathbf{L}\mathbf{L}') \lambda_m - \gamma_j (\mathbf{L}\mathbf{L}') \theta_w^j \lambda_m, \quad (4.41)$$

$$0 = -(1 + \kappa_j) \lambda_m + \lambda_m e^{\mathbf{Q}_j' \boldsymbol{\beta}_m} (1 + \theta_n^j)^{-\gamma_j}, \quad (4.42)$$

$$0 = -\mathbb{E}[y_i] \lambda_i - \gamma_j \hat{w}_{e_i}^j v_i^2 \lambda_i + \lambda_i e^{\mathbf{Q}_j' \boldsymbol{\beta}_i} \mathbb{E}[(1 + \hat{w}_{e_i}^j y_i)^{-\gamma_j} y_i]. \quad (4.43)$$

which results in Equation (4.21).

It should be noted that $\theta_w^j, \theta_n^j, \hat{w}_{e_i}^j$ are independent of $X_t, \boldsymbol{\lambda}(t)$ and are functions of \mathbf{Q}_j . We now proceed to derive the ordinary differential equations for the time-varying coefficients $P_j(t)$ and $\mathbf{Q}_j(t)$, under which the conjectured form (4.20) for the indirect utility function J indeed satisfies the HJB equation. For this, we substitute (4.20), (4.41) and (4.42) into the

HJB equation and obtain,

$$\begin{aligned}
 0 = & \dot{P}_j + \dot{Q}_j' \lambda + (1 - \gamma_j) \left(r_j(t) + \eta_j \mathbf{h}' \sigma_s (\mathbf{L} \mathbf{L}') \boldsymbol{\theta}_w^j \lambda_m / \sigma_m - (1 + \kappa_j) \theta_n^j \lambda_m \right. \\
 & - \sum_{i=1}^n \hat{w}_{e_i}^j \mathbb{E}[y_i] \lambda_i \Big) + \alpha_m (\lambda_{m,\infty} - \lambda_m) Q_{j,m} + \sum_{i=1}^n \alpha_i (\lambda_{i,\infty} - \lambda_i) Q_{j,i} \\
 & - \frac{1}{2} \gamma_j (1 - \gamma_j) (\boldsymbol{\theta}_w^{j'} \mathbf{L} \mathbf{L}' \boldsymbol{\theta}_w^j \lambda_m + \sum_{i=1}^n (\hat{w}_{e_i}^j v_i)^2 \lambda_i) \\
 & + \lambda_m \left((1 + \theta_n)^{1-\gamma} \exp(\mathbf{Q}_j' \boldsymbol{\beta}_m) - 1 \right) \\
 & + \sum_{i=1}^n \lambda_i \left(\mathbb{E}[(1 + \hat{w}_{e_i} y_i)^{1-\gamma_j}] \exp(\mathbf{Q}_j' \boldsymbol{\beta}_i) - 1 \right),
 \end{aligned}$$

where \dot{P}_j, \dot{Q}_j denote the derivatives of $P_j(t), Q_j(t)$ with respect to time t . The RHS of this expression is an affine function in λ_m, λ_i . For this expression to hold for all λ_m, λ_i , the constant term and the linear coefficients of λ_m, λ_i on the RHS must be set equal to zero separately, which leads to the ordinary differential equation for $\mathbf{Q}_j(t)$ given in (4.23). \square

Proof of Theorem 4.1. By replacing the country-specific equities with a global market equity, Equation (4.21) can be written as

$$\begin{pmatrix} \hat{w}_m^j \\ \hat{w}_{o_0}^j \\ \vdots \\ \hat{w}_{o_n}^j \end{pmatrix} = \begin{pmatrix} h_0 \sigma_{s_0} & \sigma_{o_0} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_n \sigma_{s_n} & 0 & \cdots & \sigma_{o_n} \\ j_m & j_{o_0} & \cdots & j_{o_n} \end{pmatrix}^{-1} \begin{pmatrix} \theta_{w_0}^j - \sum_{i=1}^n \hat{w}_{e_i}^j \sigma_{e_i} \\ \vdots \\ \theta_{w_n}^j - \sum_{i=1}^n \hat{w}_{e_i}^j \sigma_{e_i} \\ \theta_n^j - \sum_{i=1}^n \hat{w}_{e_i}^j j_{e_i}^j \end{pmatrix}. \quad (4.44)$$

Notice that the matrix to be inverted is of full column rank. Therefore the equity weights vector exists and is unique.

Next we show that all investors, regardless of their home currencies, will invest in the same global derivative portfolio. Denote investor j 's position on the currency-hedged global equity index by \hat{w}_m^j . Multiplying each country equity's weight in the market equity index, we get $h_i \hat{w}_m^j$ which gives the weight on the country equities in investor j 's portfolio, i.e.,

$$h_i \hat{w}_m^j = \hat{w}_{s_i}^j.$$

Define $\boldsymbol{\sigma}_o$ as an $(n+1) \times (n+1)$ diagonal matrix with σ_{o_i} on the diagonal. Further define $\boldsymbol{\sigma}_e$ as an $n \times (n+1)$ matrix with the $[i, j]^{\text{th}}$ element containing currency i 's exposure to country j 's equity Brownian motion. The first equation of (4.19) implies that

$$\boldsymbol{\sigma}_s' \mathbf{h} \hat{w}_m^j + \boldsymbol{\sigma}_o \hat{w}_o^j = \boldsymbol{\theta}_w^j - \boldsymbol{\sigma}_e' \hat{w}_e^j. \quad (4.45)$$

Note that $\boldsymbol{\sigma}_e$ can be written as

$$\boldsymbol{\sigma}_e' = \begin{pmatrix} \frac{\eta_0 - \eta_1}{\sigma_m} \boldsymbol{\sigma}_s' \mathbf{h} & \cdots & \frac{\eta_0 - \eta_n}{\sigma_m} \boldsymbol{\sigma}_e' \mathbf{h} \end{pmatrix}.$$

In addition, Proposition 4.2 shows that

$$\boldsymbol{\theta}_w^j = \frac{\eta_j}{\sigma_m \gamma_j} \boldsymbol{\sigma}_s' \mathbf{h}.$$

Therefore we have

$$\begin{aligned}\sigma'_e \hat{w}_e^j &= \sigma'_s \mathbf{h} \left(\frac{\eta_0 - \eta_1}{\sigma_m} \quad \dots \quad \frac{\eta_0 - \eta_n}{\sigma_m} \right) \hat{w}_e^j \\ &=: (\mathbf{c} \hat{w}_e^j) \sigma'_s \mathbf{h},\end{aligned}$$

where \mathbf{c} denote the $1 \times n$ vector $\left(\frac{\eta_0 - \eta_1}{\sigma_m} \quad \dots \quad \frac{\eta_0 - \eta_n}{\sigma_m} \right)$. Equation (4.45) becomes

$$\left(\hat{w}_m^j - \frac{\eta_i}{\sigma_m \gamma_j} + \mathbf{c} \hat{w}_e^j \right) \sigma'_s \mathbf{h} + \sigma_o \hat{w}_o^j = 0,$$

from which we get

$$\hat{w}_o^j = \left(\frac{\eta_i}{\sigma_m \gamma_j} - \hat{w}_m^j - \mathbf{c} \hat{w}_e^j \right) \sigma_o^{-1} \sigma'_s \mathbf{h}.$$

Notice that $\left(\frac{\eta_i}{\sigma_m \gamma_j} - \hat{w}_m^j - \mathbf{c} \hat{w}_e^j \right)$ is a single number and $\sigma_o^{-1} \sigma'_s \mathbf{h}$ is an $(n+1) \times 1$ vector. Both terms are independent of the investor identity j . Therefore all investors will invest in the same global derivative portfolio. \square

Proof for Lemma 4.1. The weights on the unhedged equity and equity derivatives should be equal to the weights on the hedged ones,

$$w_m^j = \hat{w}_m^j, \quad w_d^j = \hat{w}_d^j.$$

The raw weight on currency i , $\hat{w}_{e_i}^j$ should be the net currency weight $\hat{w}_{e_j}^j$ plus the currency position embedded in the hedged assets,

$$w_{e_i}^j = \hat{w}_{e_i}^j - h_i \hat{w}_m^j - k_i \hat{w}_d^j, \quad j = 1, \dots, n.$$

The budget constraint (4.26) can be written as

$$\begin{aligned}\frac{dX_j(t)}{X_j(t^-)} &= \hat{w}_m^j \frac{d\hat{M}^j(t)}{\hat{M}^j(t^-)} + \hat{w}_d^j \frac{d\hat{D}^j(t)}{\hat{D}^j(t^-)} + \sum_{i=1}^n \hat{w}_{e_i}^j \frac{dB_i^j(t)}{B_i^j(t^-)} + \left(1 - \hat{w}_m^j - \hat{w}_d^j - \sum_{i=1}^n \hat{w}_{e_i}^j \right) \frac{dB_0^j(t)}{B_0^j(t^-)} \\ &= \hat{w}_m^j \frac{dM^j(t)}{M^j(t^-)} + \hat{w}_d^j \frac{dD^j(t)}{D^j(t^-)} + \sum_{i=1}^{n \setminus j} \left(\hat{w}_{e_i}^j - h_i \hat{w}_m^j - k_i \hat{w}_d^j \right) \frac{dB_i^j(t)}{B_i^j(t^-)} \\ &\quad - \left(h_0 \hat{w}_m^j + k_0 \hat{w}_d^j + \sum_{i=1}^n \hat{w}_{e_i}^j \right) \frac{dB^j(t)}{B^j(t^-)} + \left(1 + \hat{w}_{e_j}^j - h_j \hat{w}_m^j - k_j \hat{w}_d^j \right) \frac{dB_j(t)}{B_j(t)}.\end{aligned}$$

Collecting the coefficients of the raw asset returns, we get the desired equations in Lemma 4.1. \square

Proof for Theorem 4.2. The third equation is obtained by multiplying the third equation in (4.32) by h_j on both sides. And since $w_m^i h_j = w_{s_j}^i$, we get $\sum_{i=0}^n f_i \hat{w}_m^i = 1$. The other equations can be easily verified by replacing the weights on the unhedged assets by the hedged counterparts using Lemma 4.1. \square

Proof for Proposition 4.4. See Black Black [15]. \square

4.B Normalization in the base country

The pricing kernel specification of the base country can also be regarded as a normalization. Suppose the jump components N_m, N_i were Poisson jumps with intensity λ_m and λ_i . Then the assumption on the base currency can be regarded as a normalization without loss of generality. To see this, observe the exchange rate of currency i against currency j ,

$$\begin{aligned} \frac{dE_{i,t}^j}{E_{i,t}^j} &= \left(r_j(t) - r_i(t) + (\mu_{e_i} - \mu_{e_j} - \sigma_{e_i}\sigma_{e_j} + \sigma_{e_j}^2)\lambda_m + (v_j^2 - \mathbb{E}\left[\frac{y_j^2}{1+y_j}\right])\lambda_j \right) dt \\ &\quad + (\sigma_i - \sigma_j)\sqrt{\lambda_m} dW_m(t) - v_i\sqrt{\lambda_i} dZ_i(t) + v_j\sqrt{\lambda_j} dZ_j(t) \\ &\quad + \frac{j_{e_i} - j_{e_j}}{1 + j_{e_j}} (dN_m(t) - \lambda_m(t) dt) + (y_i dN_i(t) - \mathbb{E}[y_i]\lambda_i(t) dt) \\ &\quad - \left(\frac{y_j}{1 + y_j} dN_j(t) - \mathbb{E}\left[\frac{y_j}{1 + y_j}\right]\lambda_j(t) dt \right) \\ &= \left(r_j(t) - r_i(t) + \bar{\mu}_{e_i}\bar{\lambda}_m \right) dt + \bar{\sigma}_{e_i}\sqrt{\bar{\lambda}_m} d\bar{W}_m(t) + \bar{v}_i\sqrt{\bar{\lambda}_i} d\bar{Z}_i(t) \\ &\quad + (\bar{y}_i d\bar{N}_i(t) - \mathbb{E}[\bar{y}_i]\bar{\lambda}_i(t) dt). \end{aligned}$$

The global equity denominated in currency j is given by

$$\begin{aligned} \frac{d(\hat{M}^j/E_j^0)(t)}{(\hat{M}^j/E_j^0)(t^-)} &= r_j(t) dt + \left(\mu_m - \sigma_m\sigma_j - \mu_j + \sigma_j^2 \right) \lambda_m(t) dt + (v_j^2 - \mathbb{E}\left[\frac{y_j^2}{1+y_j}\right])\lambda_j dt \\ &\quad + (\sigma_m - \sigma_j)\sqrt{\lambda_m(t)} dW_m(t) - v_j\sqrt{\lambda_j} dZ_j + \frac{j_m - j_j}{1 + j_j} (dN_m(t) - \lambda_m(t) dt) \\ &\quad - \left(\frac{y_j}{1 + y_j} dN_j - \mathbb{E}\left[\frac{y_j}{1 + y_j}\right]\lambda_j dt \right) \\ &=: r_j(t) dt + \bar{\mu}_m\bar{\lambda}_m dt + \bar{\sigma}_m\sqrt{\bar{\lambda}_m} d\bar{W}_m + \bar{j}_m(d\bar{N}_m - \lambda_m dt). \end{aligned}$$

Here, $\bar{W}_m(t), \bar{Z}_i(t)$ are independent and standard Brownian motions. \bar{N}_m, \bar{N}_i are Poissonian jumps with intensities $\bar{\lambda}_m, \bar{\lambda}_i$ given by

$$\bar{\lambda}_m = \lambda_m + \lambda_j, \quad \bar{\lambda}_i = \lambda_i + \lambda_j. \quad (4.46)$$

In addition,

$$\begin{aligned} \bar{\mu}_m &= \frac{(\mu_m - \sigma_m\sigma_j + \sigma_j^2)\lambda_m + \left(v_j^2 - \mathbb{E}\left[\frac{y_j^2}{1+y_j}\right] \right)\lambda_j}{\lambda_m + \lambda_j}, \\ \bar{\sigma}_i^2 &= \frac{(\sigma_i - \sigma_j)^2\lambda_m + v_j^2\lambda_j}{\lambda_i + \lambda_j}, \\ \bar{v}_i &= v_i^2 \frac{\lambda_i}{\lambda_i + \lambda_j}, \\ \bar{\sigma}_m^2 &= \frac{(\sigma_m - \sigma_j)^2\lambda_m + v_j^2\lambda_j}{\lambda_m + \lambda_j}. \end{aligned}$$

From which we see that currency j can be regarded as the base currency and the model can be rewritten as the parametric form of Equation (4.7) by redefining the parameters.

The model proposed in Chapter 4.2.1, therefore, can be regarded as a natural extension of the standard Poissonian jump diffusion model. We let the Poissonian jumps to be mutually exciting to generate equity-currency contagion.

4.C Numerical equilibrium calculation

In this section, we explain the numerical algorithms we use for the equilibrium calculations. We will introduce the algorithm to find an initial equilibrium and the algorithm to restore the equilibrium. To produce Figure 4.7 and Figure 4.8, we employ the former to find an initial equilibrium. In particular, we take the return dynamics and investors' preferences as given and use Algorithm 4.1 to find the wealth distributor \mathbf{f} and market capitalization ratio \mathbf{h} that clear the security market.

At every new excitor value, we use Algorithm 4.2 to restore the equilibrium. There, we take the wealth distributor \mathbf{f} and currency dynamics as given and find the equilibrium market equity process.

Algorithm 4.1 (Algorithm to find an initial equilibrium). *Let the equity returns and exchange rate dynamics be given. The purpose is to find a wealth distributor \mathbf{f} and a market capital distributor \mathbf{h} that clear the security markets. We use the following algorithm to find an initial equilibrium, where Figure 4.7 and Figure 4.8 start with.*

1. Solve for the optimal net currency holding $\hat{w}_{e_i}^j$ for each investor $j = 0, \dots, n$, and for each currency $i = 1, \dots, n$, using Proposition 4.2.
2. According to the security market clearing conditions given by (4.37), the clearing of the bonds market implies that

$$\mathbf{h}^- = \hat{\mathbf{w}}_e^0 \mathbf{f} + \mathbf{f}^-, \quad (4.47)$$

where \mathbf{h}^- is a vector containing h_1, \dots, h_n ; \mathbf{f}^- is a vector containing f_1, \dots, f_n ; $\hat{\mathbf{w}}_e$ is an $n \times (n+1)$ matrix defined as

$$\hat{\mathbf{w}}_e = \begin{pmatrix} \hat{w}_{e_1} & \dots & \hat{w}_{e_1}^n \\ \vdots & \vdots & \vdots \\ \hat{w}_{e_n} & \dots & \hat{w}_{e_n}^n \end{pmatrix}.$$

Here, since $\mathbf{h} = (1 - \mathbf{t}'\mathbf{h}^-, \mathbf{h}^-)'$, $\mathbf{f} = (1 - \mathbf{t}'\mathbf{f}^-, \mathbf{f}^-)'$, we see that the global equity index composition vector \mathbf{h} can be expressed as a function of the wealth distribution vector \mathbf{f} and the net currency holdings.

3. According to the third and forth equations of (4.37), the equity and derivative market clearing condition imply that

$$\begin{cases} \sigma_m = \sum_{j=0}^n f_j \theta_m^j + \sum_{i=1}^n \sigma_{e_i} (f_i - h_i), \\ j_m = \sum_{j=0}^n (1 + j_{e_j}) f_j \theta_n^j + \sum_{i=1}^n j_{e_i} (f_i - h_i). \end{cases} \quad (4.48)$$

4. But the global equity index needs to be a weighted average of countries' equities with the weights h , therefore

$$\sigma_m^2 = \mathbf{h}' \Sigma \mathbf{h}, \quad j_m^j = \sum_{i=0}^n h_i j_{s_i} (1 + j_{e_j}). \quad (4.49)$$

5. Substitute σ_m and j_m on the LHS of Equation (4.49) by Equation (4.48) and get

$$\left(\sum_{j=0}^n f_j \theta_m^j + \sum_{i=1}^n \sigma_{e_i} (f_i - h_i) \right)^2 = \mathbf{h}' \Sigma \mathbf{h}, \quad (4.50)$$

$$\sum_{j=0}^n (1 + j_{e_j}) f_j \theta_m^j + \sum_{i=1}^n j_{e_i} (f_i - h_i) = \sum_{i=0}^n h_i j_{s_i} (1 + j_{e_j}), \quad (4.51)$$

with \mathbf{h} a function of \mathbf{f} given by Equation (4.47). We hence arrive at two equations of the vector \mathbf{f} (which has $n - 1$ unknown elements). With carefully specified exogenous parameters, one can easily find an n -dimensional simplex $\mathbf{f} \in R^{n+1}$ such that the above equations hold. Note that in case of $n \geq 2$, the solution is not necessarily unique.

6. Once a solution \mathbf{f} to Equation (4.50) and (4.51) is found, one can calculate the corresponding \mathbf{h} using Equation (4.47).

Algorithm 4.2 (Algorithm to restore equilibrium). *Let the international market be in equilibrium. Now we independently change the equity-currency excitor $\beta_{m,i}$. The new equilibrium is found as follows Variables that vary with $\beta_{m,i}$ are denoted by a bar to be distinguished from the constant variables.*

1. Solve for the new optimal net currency holdings $\bar{w}_{e_i}^j$ for each investor $j = 0, \dots, n$, and for each currency $i = 1, \dots, n$, using Proposition 4.3.
2. For fixed \mathbf{f} , the bond market clearing condition implies that

$$\bar{\mathbf{h}}^- = \bar{\mathbf{w}}_e \mathbf{f} + \mathbf{f}^-. \quad (4.52)$$

3. From the perspective of the base investor, the equity and derivative market clearing condition imply that

$$\begin{cases} \bar{\sigma}_m = \sum_{j=0}^n f_j \theta_m^j + \sum_{i=1}^n \sigma_{e_i} (f_i - \bar{h}_i), \\ \bar{j}_m^j = \sum_{j=0}^n (1 + j_{e_j}) f_j \theta_m^j + \sum_{i=1}^n j_{e_i} (f_i - \bar{h}_i). \end{cases} \quad (4.53)$$

4. But the global equity index needs to be a weighted average of countries' equities with the weights h ,

$$\bar{\sigma}_m^2 = \bar{\mathbf{h}}' \bar{\Sigma} \bar{\mathbf{h}}, \quad \bar{j}_m^j = \sum_{i=0}^n \bar{h}_i \bar{j}_{s_i} (1 + j_{e_j}), \quad (4.54)$$

which leads to renewed countries' stock volatility $\bar{\Sigma}$ and jump amplitude \bar{j}_s , that are compatible with Equation (4.53).

5. For the same type of the derivative contract, updated stock dynamics imply updated option prices, which eventually leads to a new global derivative portfolio. Denote the updated option parameters by $\bar{\sigma}_o, \bar{j}_o$. We have

$$\bar{\sigma}_d = \bar{\mathbf{k}}' \bar{\Sigma}_o \bar{\mathbf{k}}, \quad \bar{j}_d = \sum_{i=0}^n \bar{h}_i \bar{j}_{o_i} (1 + j_{e_i}),$$

with

$$\bar{\mathbf{k}} = \frac{\bar{\sigma}_o^{-1} \bar{\sigma}_e' \bar{\mathbf{h}}}{\nu' (\bar{\sigma}_o^{-1} \bar{\sigma}_e' \bar{\mathbf{h}})}.$$

6. Calculate the new portfolio weights on the global equity index and global derivative portfolio using

$$\begin{pmatrix} \bar{w}_m^j \\ \bar{w}_d^j \end{pmatrix} = \begin{pmatrix} \bar{\sigma}_m & \bar{\sigma}_d \\ \bar{j}_m & \bar{j}_d \end{pmatrix}^{-1} \begin{pmatrix} \theta_m^j - \sum_{i=1}^n \bar{w}_{e_i}^j \sigma_{e_i} \\ (1 + j_{e_j}) \theta_n^j - \sum_{i=1}^n \bar{w}_{e_i}^j j_{e_i} \end{pmatrix}.$$

7. Calculate the new hedging strategy

$$\bar{H}_i^j := -\frac{\bar{w}_i^j}{\bar{w}_m^j} = -\frac{\bar{w}_i^j - \bar{h}_i \bar{w}_m^j - \bar{k}_i \bar{w}_d^j}{\bar{w}_m^j}.$$

4.D Robustness check

In this section, we show that the safe haven preferences in the equilibrium currency hedging strategies is free of particular derivative contract chosen. The previous sections use call options as derivative contracts and presents the safe-haven preference. In this section, we use different derivative contracts and show that investors' preferences for safe-haven currencies in equilibrium are not affected qualitatively.

Similar to the call option price, the put option price $P_j(t)$ with maturity τ and strike price K is given by

$$P_j(t) = KG_{0,1}(\log K) - G_{1,1}(\log K), \quad (4.55)$$

where $G_{a,b}(\cdot)$ can be calculated according to Equation (4.14) in Proposition 4.1.

Having priced the call and put options, we also consider a straddle. Inspired by Liu et al. [92], we consider the following “delta-neutral” straddle:

$$\text{Straddle}_j(t) = C_j(S_j(t), \lambda_m(t); K, \tau) + P_j(S_j(t), \lambda_m(t); K, \tau),$$

where C and P are pricing formulae for call and put options with the same strike price K and time to expiration τ . As Liu et al. [92] comment, the “delta-neutral” straddle is made of call and put options that are typically very close to the money, which can be used to intentionally avoid deep out-of-the-money options due to liquidity issues.

Table 4.1 reports the equilibrium hedging ratio of Currency I and Currency II for the base investor when different derivative contracts are used for a given equity-currency excitation structure. The rows correspond to different derivative contracts. The first row reports the hedging ratios when call options are used. The second row corresponds to put options and the third row straddles. The three major columns are hedging ratios in different equity-currency excitation scenarios. In case of “Large excitation”, Currency I and Currency II have the same risk profile, including the excitation structure with the equity market. “Medium excitation” refers to the case where the equity-currency excitor of Currency I is smaller than that in the “Large excitation” scenario, while the excitation structure involving Currency II remains unchanged from the “Large excitation” scenario. The equity-currency excitor of Currency I is smallest in the “Small excitation” case. Across all three scenarios, the excitation structure between the equity and Currency II does not vary. Also invariant are the expected jump intensities of the equity jump component, Currency I jump component and Currency II jump component.

The hedging ratio of Currency I is always the largest in case of “Large excitation” and smallest in case of “Small excitation”, regardless of which derivative contracts are used. The investor has a preference for the safe haven currency, in the sense that the more immune the currency is to the equity turmoil, the less currency risk the investor hedges away in equilibrium. This conclusion is robust regarding the derivative contracts chosen.

	Large excitation		Medium excitation		Small excitation	
	Currency I	Currency II	Currency I	Currency II	Currency I	Currency II
call	0.942	0.942	0.934	0.943	0.926	0.945
put	0.941	0.941	0.933	0.942	0.924	0.943
straddle	0.936	0.936	0.927	0.937	0.918	0.939

Table 4.1: Equilibrium hedging ratio of Currency I and Currency II for the base investor when different derivative contracts are used. The rows correspond to different derivative contracts: call options in the first row, put options in the second, and straddles in the third. “Large”, “medium” and “small” excitation refer to the different scenarios of equity-currency excitor of Currency I. The equity-currency excitor of Currency I is the largest in “Large excitation”, and smallest in “Small excitation”. The equity-currency excitation structure of Currency II is the same as that of Currency I in the “Large excitation” scenario and remains the same in the other two scenarios. The expected jump intensities of the equity jump component, Currency I jump component, and Currency II jump component are kept constant in all scenarios. The excitation matrix is set to be $\beta = (15, 6, 6; 6, 8, 0; 6, 0, 8)$ in case of “Large excitation”, $\beta = (15, 6, 6; 4, 11.8, 0; 6, 0, 8)$ in case of “Medium excitation”, and $\beta = (15, 6, 6; 2, 15, 0; 6, 0, 8)$ in case of “Small excitation”. The expected jump intensities are intentionally kept constant with $\mathbb{E}[\lambda_m] = 1.28$, $\mathbb{E}[\lambda_1] = \mathbb{E}[\lambda_2] = 0.67$ in all three scenarios.

Chapter 5

The Term Structure of the Currency Basis

5.1 Introduction

Currency basis, defined as the difference between the currency forward premium (the logarithm of the currency forward rate minus the logarithm of the spot exchange rate) and the interest rate differential, is a term that has emerged only in recent years. Currency basis should in theory be zero as ensured by the Covered Interest rate Parity (CIP) condition. However, during the 2007-2009 financial crisis, this was not the case. The global financial crisis has seen the currency basis between many foreign currencies and the US dollar amounted to hundreds of basis points, beyond the explanation of transaction costs and data imperfection. A statistically significant non-zero currency basis challenges the no arbitrage assumption.

To understand the link between the exclusion of the arbitrage opportunities and the CIP condition, consider a USD/EUR forward contract. One can construct a currency basis trade by entering a forward contract agreeing to buy US dollar using a pre-determined Euro price, converting US dollar into Euro in the spot market, saving in the Euro money market, and changing the Euro back to US dollar at maturity using the forward rate. Traditionally, it is regarded as a risk-free trade, since there is no uncertainty about the trade – the interest rates of the US and Europe and the forward exchange rate are known by the time the transaction is made. In other words, the excess return of the currency basis trade should be zero. The transaction in the cash market perfectly offsets the payoff in the derivatives market, implying that the currency forward rate should be equal to the interest rate differentials of the two countries. If the investor makes money as a result of such transactions, the profit will be “risk free”.

Early studies tend to agree that the markets are efficient in the sense that after taking into account data imperfections, brokerage fees and other transaction costs, the covered interest rate parity holds.¹ In a landmark study, Taylor [114] documents small but potentially exploitable profitable arbitrage opportunities during periods of turbulence. The recent financial crisis starting in late 2007 has again witnessed a break down of the CIP condition, leaving “arbitrage opportunities” unexploited. Many papers have documented persistent and significant deviations from CIP in various currency pairs using different interest rate instruments.² So

¹See Frenkel and Levich [66], Deardorff [46], Rhee and Chang [105], and the survey paper by Officer and Willett [99].

²The documentation of the deviation from CIP in the 2007-2009 crisis can be found in Baba et al. [10], Coffey et al. [37], Fong et al. [60], Genberg et al. [69], Sarkar [107], Hui et al. [79], and Mancini-Griffoli and

far the extant literature has been focusing on the time series behavior of the currency basis. Like interest rates, there exist different maturities for the currency basis. The term structure of the currency basis is relatively less explored.

The first purpose of the paper is to empirically study the term structure of the currency basis. We study the term structure through a panel of currency basis trade profits, constructed as described before. Our data consist of daily zero-coupon bond returns of the US, Canada, Australia and Europe, and FX spot and forward rates of US dollar *vis-à-vis* Australian dollar (USD/AUD), Canadian dollar (USD/CAD), and Euro (USD/EUR), from August 2005 to June 2016. We construct an equally weighted portfolio of currency basis trades in the aforementioned currencies. Specifically, we record the daily profit for a representative US investor who borrows domestically, lends abroad, and hedges the currency risk using currency forward contracts. This way, we obtain a data panel of daily currency basis of different maturities, averaged over three currency pairs. To offer a first impression of the term structure, we plot the annualized mean currency basis trade profits during the market turbulent period in Figure 5.1. This figure, which is studied in more detail later in the paper, shows that the US investor on average could make a larger profit by borrowing domestically and lending abroad in the short term than in the long term. The profit from the currency basis trade decreases as a convex function of time to maturity. Puzzling as it is that there exists a statistically significant currency basis, more surprising is that we find a downward sloping and convex shape of the term structure of currency basis during the financial crisis.

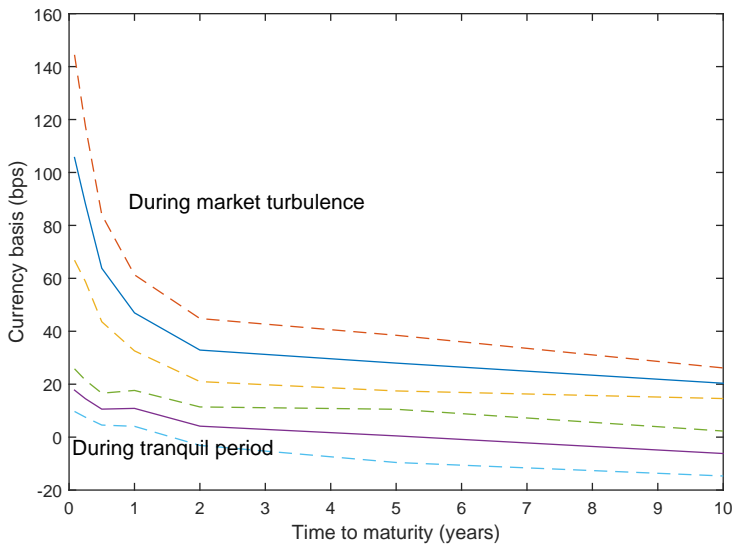


Figure 5.1: The figure plots the sample mean of the returns (annualized) of an equally weighted portfolio of currency basis trades of three currency pairs – USD/AUD, USD/CAD, and USD/EUR, from May 2007 to April 2009. The currency basis trade return for a particular currency pair is calculated as the difference between the forward premium and the interest rate differentials of the US dollar and the foreign country. The US dollar is regarded as the domestic currency.

The next question is why two contracts – the currency forward contract and the replica-
Ranaldo [94].

tion portfolio in the cash account – with essentially the same payoff would have different prices? Or put differently, why would a currency basis trade, which seems to take no financial risk, earn a nontrivial excess return? Note that the currency forward contract is traded over-the-counter in the derivatives market. If the investor forms a replication portfolio, the transaction would be carried out in the cash market. The replication portfolio requires an up-front investment of capital. By contrast, a forward contract in principle requires no payment by the time the contract is made.³ In other words, the derivatives market allows the investor to leverage. With a relatively small initial outlay, the investor is able to take a large speculative position [80]. While it may not lead to problems when the market is liquid, the capital-intensive nature of the cash market may cause price deviations from the derivatives market even with the same payoff when liquidity risk is high.

The capital constraint implicitly imposed by the cash market becomes binding when we consider position unwinding risk, that is, the possibility that an investor needs to exit a position before the contract matures. During tranquil periods, investors are seldom forced to exit a position to free capital. When funding liquidity is tight, e.g., during a financial crisis, however, arbitrageurs are more likely to exit the positions [110]. Such position unwinding can be due to various reasons. For example, a mutual fund may sell securities because it faces capital withdrawals from its shareholders [38]. A bank can increase its holdings of cash sharply at the expense of other investment opportunities (such as lending) due to liquidity preferences. Indeed, He et al. [78] show that from the last quarter of 2007 to the first quarter of 2009, the cash and reserves of commercial banks grew almost tenfold, while the total financial assets only increased by 20%. Shleifer and Vishny [110] also note that fear of future fire sales encourages financial institutions to hoard cash rather than finance investment, making cash-intensive investments less favorable.

In the derivatives market, position unwinding is relatively easy: the investor simply needs to enter another forward contract with a third party, taking the opposite side to the original contract. Suppose the investor replicates a forward contract in the cash market but needs to free the capital before the maturity date for liquidity and unwind his position. He/She faces the risk that the pre-mature bonds/deposits can only be sold at a lower price. For instance, the old bonds usually have a lower price than the newly issued ones with the same time to maturity, since they are less liquid.⁴

We propose an asset pricing model with position unwinding risk and show how position unwinding risk and the on-the-run/off-the-run spread can contribute to a nontrivial currency basis. In particular, we show that the possibility of a position unwinding event, the liquidity differences of the fixed income markets of the two countries, and the interdependence between the position unwinding event and market liquidity are key determinants of the magnitude of the currency basis.

Indeed, the empirical literature finds liquidity risks to be the major driver of the currency basis observed during market turmoil. Baba et al. [10] conclude that Dollar funding shortages of non-US financial institutions were largely responsible for the large deviations from CIP in 2007. As they explain,

...frequently reported were efforts by European financial institutions to secure Dollar funds to support US conduits for which they had committed backup liq-

³Collateral may be required.

⁴In the deposit market, even worse, the investors are not allowed to withdraw pre-matured deposits, in which case the liquidity loss is 100% of the asset price. However, considering that investors can borrow cash pledging the deposits as collateral but with a haircut, we only consider the liquidity loss in the form of a new-bond/old-bond spread without loss of generality.

uidity facilities. At the same time, the usual suppliers of Dollar funds to the inter-bank market were looking to conserve their liquidity, due to their own growing needs and increased concerns over counterparty credit risk.

In spite of the existing empirical studies, the questions (1) why there is a term structure of the currency basis, (2) through which mechanism currency forward contracts are exposed to liquidity risks, and (3) why these risk factors seem to be negligible during tranquil periods, remain unexplored. The second purpose of the paper is to go one step further to the asset pricing theory to see how, in a standard no arbitrage pricing model, market liquidity can contribute to a nontrivial currency basis.

We therefore also contribute to the theoretical literature of the currency basis by answering these questions in an arbitrage-free asset pricing framework. Inspired by the empirical literature, we consider liquidity risk in the form of position unwinding events and the on-the-run/off-the-run spread in our model.

Consider a forward contract deal with intermediate position unwinding. Suppose today an investor enters a currency forward contract agreeing to buy Dollars with a certain amount of Euros. To be able to deliver Euros at the maturity date, the investor starts with an initial Dollar capital, converts it into Euro against the spot exchange rate, and buys a Euro-denominated bond with the same time to maturity as the forward contract. At a certain time before the contract matures, the investor may need to withdraw the capital in the Euro cash market for additional liquidity. The investor therefore unwinds the forward contract by entering a new forward contract to fulfil the original forward contract. By equating the initial capital with the expected payoff under the risk neutral measure, we show that the currency basis today is equal to the expectation of the future liquidity costs weighted by the position unwinding probability, where the liquidity cost is measured by the price differences between a new bond and an old bond with the same time to maturity.

To further derive the currency basis in explicit form, we impose structure on the position unwinding intensity and the dynamics of the on-the-run/off-the-run spread. Consistent with Goldreich et al. [72], we model the on-the-run/off-the-run spread as the risk neutral expectation of future instantaneous liquidity risk. Inspired by the finding that market liquidity and funding liquidity are linked,⁵ we model the position unwinding intensity as a factor process, driven by liquidity state variables.

The convex and downward-sloping term structure of currency basis observed empirically is informative on the nature of the instantaneous illiquidity process: they tend to impose a larger impact on currency basis in the short term than in the long term. To be consistent with this empirical observation, the instantaneous liquidity is modeled using mean reverting processes.

We derive an explicit currency forward rate formula as a function of parameters that characterize the instantaneous liquidity risk and the position unwinding intensity process. In our model, violation of the CIP condition does not indicate risk-free profits. A non-zero currency basis reflects potential liquidity costs the investor is bearing in the currency basis trade. In case that either the probability of incurring a position unwinding event or the liquidity premium attached to the new bonds is zero, we arrive at the CIP condition as a special case.

We show that our model not only produces the time series variations of the currency basis (high basis during crises and low basis in tranquil periods) but also delivers the convex and downward sloping shape of the term structure. To our knowledge, this paper is the first

⁵See Brunnermeier and Pedersen [24], and Shleifer and Vishny [110].

attempt to directly price the currency basis in a no arbitrage framework.

The rest of this chapter is organized as follows. Chapter 5.2 introduces the Covered Interest rate Parity condition and how it implies no arbitrage. Chapter 5.3 documents the term structure of the currency basis. Chapter 5.4 introduces the asset pricing model and shows how market liquidity costs are reflected in the currency basis. Chapter 5.5 imposes structure on the instantaneous liquidity process and the position unwinding intensity process and derives an explicit formula for the currency basis. Chapter 5.6 calibrates the parameters and shows numerically how our model is capable of matching the time series and cross section patterns of the observed currency basis. Chapter 5.7 concludes.

5.2 The Covered Interest rate Parity

The currency forward contract is an agreement between two parties to exchange a certain amount of currencies at a certain rate (forward rate) at a certain time (maturity date). A currency forward contract is essentially a hedging tool that does not involve any upfront cash payment. Currency forward contracts are not traded on a centralized exchange. They are over-the-counter instruments and therefore can be tailored to a particular amount and delivery period, unlike currency futures.

A well-known interest rate parity with the use of a currency forward contract is the Covered Interest rate Parity (CIP). Consider a currency forward contract at time t which specifies an exchange of $F(t, T)$ US dollar for 1 unit of foreign currency at maturity date T . The Covered Interest rate Parity refers to the following equation

$$f(t, T) - s(t) = (T - t)(r(t, T) - r^*(t, T)), \quad (5.1)$$

where $f(t, T)$ is the logarithm of the forward rate $F(t, T)$; $s(t)$ is the logarithm of the spot exchange rate $S(t)$; $r(t, T), r^*(t, T)$ are the annualized continuously compounded risk free return at time t with time to maturity T of the US and the foreign country, respectively.

To see why Equation (5.1) holds, consider a forward contract between Euro and US dollar. Suppose at time t , Party B agrees to exchange Euro with Party A at a forward rate $F(t, T)$ at time T (meaning that one Euro is worth $F(t, T)$ US dollar at time T). From the perspective of Party A , he can perfectly finance the contract by engaging in the following trade. At time t , he converts 1 Dollar to $1/S(t)$ Euro and puts it in the Euro money market account by, say, buying a Euro zero coupon bond that pays 1 Euro at time T with price $P^*(t, T)$ at time t . At the maturity date T , he has in total $\frac{1}{S(t)P^*(t, T)}$ Euros. He converts them back to Dollars with the pre-specified forward rate and gets $\frac{F(t, T)}{S(t)P^*(t, T)}$ Dollars. Since the payoff of this transaction is certain – the spot exchange rate, the forward rate, the price of the Dollar and Euro zero coupon bonds are all known at time t – the investor is taking no risks. Therefore he should earn the Dollar risk free return. Using the Dollar-denominated zero coupon bond as numeraire, it holds that

$$\frac{1}{P(t, T)} = \frac{1}{P^*(t, T)S(t)}F(t, T), \quad (5.2)$$

where $P(t, T)$ is the time t price of the US dollar-denominated zero coupon bond that pays 1 Dollar at time T . On the LHS of Equation (5.2), the investor starts with 1 Dollar as the initial capital. On the RHS, the investor ends up with $\frac{1}{P^*(t, T)S(t)}F(t, T)$ Dollar at time T . Replacing $P(t, T)$ by $\exp(r(t, T)(T - t))$, and $P^*(t, T)$ by $\exp(r^*(t, T)(T - t))$, Equation (5.2) reduces to Equation (5.1).

5.3 Empirics: the cross-section of the currency basis

In this section, we are going to explore the currency basis empirically and study its stylized facts both in the time series dimension and in the cross section dimension. We first introduce some key concepts related to the currency basis in Chapter 5.3.1. Chapter 5.3.2 describes the data we employ for the empirical analysis. We report the empirical statistics in Chapter 5.3.3.

5.3.1 Definitions

Currency forward premium Denote the $(T - t)$ -year log forward exchange rate at time t by $f(t, T)$. The currency forward premium is the difference between the logarithm of the forward exchange rate and the spot exchange rate,

$$\text{Forward Premium}(t, T) = f(t, T) - s(t).$$

(Annualized) Currency basis Denote the time t currency basis of maturity date T by $y(t, T)$. It is defined as the (annualized) difference between the currency forward premium and the interest rate differential,

$$y(t, T) := \frac{1}{(T - t)} \left(f(t, T) - s(t) \right) - (r(t, T) - r^*(t, T)). \quad (5.3)$$

The Covered Interest rate Parity is equivalent to a zero currency basis. A non-zero basis is traditionally regarded as a sign that one can make a sure profit by borrowing one currency and saving in another, while hedging the currency risk with the forward contract. Here, we assume zero transaction costs.

Term spread on the currency basis Fixing the current date t , the term spread of the currency basis of maturity date T_1 and that of maturity T_2 , where $T_1 > T_2$, is the difference between the annualized currency basis of the two maturities,

$$\text{Term Spread}(t; T_1, T_2) = y(t, T_1) - y(t, T_2).$$

Currency basis trade A currency basis trade is a currency forward contract and an offsetting cash portfolio. The forward contract allows the investor to buy currency A with currency B at a future time T with a fixed rate $F(t, T)$. The investor converts currency A into currency B using the spot exchange rate $S(t)$ at time t and saves in the money market of currency B . At time T , the investor withdraws the savings in currency B and converts back to currency A using the forward contract. In short, a currency basis trade is to lend a currency to another country today and hedge the currency risk using the currency forward contract in the future. The excess return of the transaction is therefore equal to the currency basis. The currency basis trade intends to profit from a positive currency basis between currency A and B .

In the remainder, we use the term “currency basis”, “currency basis trade returns”, and “deviations from CIP” interchangeably.

5.3.2 Data

The forward and spot exchange rates can be easily collected from WM/Reuters. We obtain forward exchange rates of the same maturity as the interest rates. All forward and spot exchange rates are sampled at 4 p.m. UK time. The 10-year forward exchange rate is typically available since 2005. Therefore our sample starts in August 2005, by which time the 10 year forward rates become available for all currency pairs we are interested in, even though interest rates data do go back to very early years. We also include the post-crisis periods to act as a relatively “tranquil” period for better comparison. The sample starts from August 2005 and ends in June 2016.

To calculate the currency basis, we obtain interest rate data of different maturities of different countries. We study currency basis of Australian dollar vs. US dollar (USD/AUD), Canadian dollar vs. US dollar (USD/CAD), and Euro vs. US dollar (USD/EUR). Interest rates of longer maturities (2, 5, 10-year) are collected from the Thomson Reuters government bond benchmark indices. We use government bond indices to get rid of the credit risk premium in the long term interest rates. The benchmark indices of countries are based on the selection rules and formulation recommended by EFFAS (European Federation of Financial Analysis Societies). The benchmark index is formed using single bonds, usually the latest issue. Consideration is also given to liquidity, yields, issue size and coupon rates, such that the benchmark bonds are the most representative for a given maturity at each point in time. The constituents of the index are reviewed daily by DataStream.

Interest rates of shorter (1, 3, 6, 12-month) terms of different countries are obtained from different sources. Short term interest rates of Australia are bank accepted bill rates provided by the Reserve Bank of Australia, reported in levels, percentage per annum. We collect short term Canadian interest rates from the Bank of Canada, calculated using pricing data of Canadian treasury bills. Euro interest rates are provided by the European Central Bank. The yields are calculated based on government bonds issued in the Euro area (with changing composition) with triple A ratings. US interest rates are based on treasuries with constant maturity, obtained from the Federal Reserve Bank. For all countries, the interest rates include 1-month, 3-month, 6-month, 1-year, 2-year, 5-year, and 10-year maturities. We end up with a balanced panel of interest rates containing daily observations of 7 maturities of 4 countries. Sample mean and standard deviation of interest rates in these countries of different maturities are reported in Table 5.1. During the sample period, all countries have an upward sloping yield curve, except for Australia, whose yield curve has a slight smile shape.

Mean and standard deviation of interest rates								
Maturity		1m	3m	6m	1y	2y	5y	10y
US	Mean	1.11	1.16	1.26	1.34	1.54	2.26	3.05
	Std	1.78	1.80	1.83	1.77	1.62	1.30	1.01
AUS	Mean	4.29	4.09	4.06	4.02	3.98	4.21	4.53
	Std	1.67	1.67	1.72	1.70	1.57	1.44	1.18
CAN	Mean	1.50	1.56	1.65	1.76	1.88	2.34	2.89
	Std	1.36	1.38	1.40	1.38	1.27	1.15	0.99
EU	Mean	1.13	1.16	1.19	1.27	1.44	2.01	2.77
	Std	1.48	1.53	1.56	1.58	1.56	1.44	1.28

Table 5.1: The table reports the sample mean and standard deviation (Std) of interest rates of the US, Australia (AUS), Canada (CAN) and Europe (EU) of one-month (1m), three-month (3m), six-month (6m), one-year (1y), two-year (2y), five-year (5y) and 10-year (10y) maturities.

We construct currency basis trade returns, i.e., converting US dollar capital into a foreign currency, investing in the government bonds of that foreign country and hedging the currency risk with the forward contract of the same term. We calculate the currency basis of Australian dollar, Canadian dollar and Euro against the US dollar for all maturities using Equation (5.3). Figure 5.2 plots the currency basis of the three currency pairs of three-month maturity (at which term the currency forward contract is the most liquid). Should CIP hold, the currency basis between any currency pair would not exceed the range of trading costs. The transaction cost associated with the CIP arbitrage trade has been estimated to be below 25 bps.⁶ This is clearly contrary to what is observed in the market. From mid 2007 to mid 2009, the currency basis of all three currency pairs stayed beyond 50 bps most of the time, and even went up to as high as 450 bps in late 2008. The currency basis returned to the accepted level after 2010.

⁶The estimated transaction costs of engaging in currency basis trade is around 18 bps by Branson [22], and 15 bps by Frenkel and Levich [66].

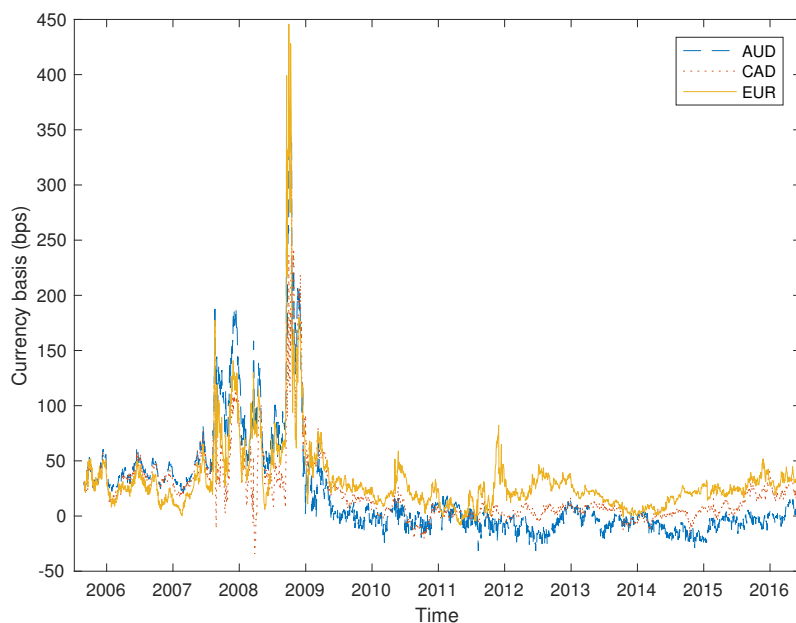


Figure 5.2: The figure plots the 3-month currency basis of Australian dollar against US dollar (dashed line), Canadian dollar against US dollar (dotted line) and Euro against US dollar (solid line) from August 2005 to June 2016.

To see how the currency basis covaries with the spot exchange rates, we plot the 3-month currency basis (solid lines) with the logarithm of the spot FX rates (dashed lines) averaged over USD/AUD, USD/CAD and USD/EUR in Figure 5.3. We normalize both series to have mean zero and standard deviation 1 to facilitate comparison. We see that the spot exchange rates have little comovement with the currency basis.

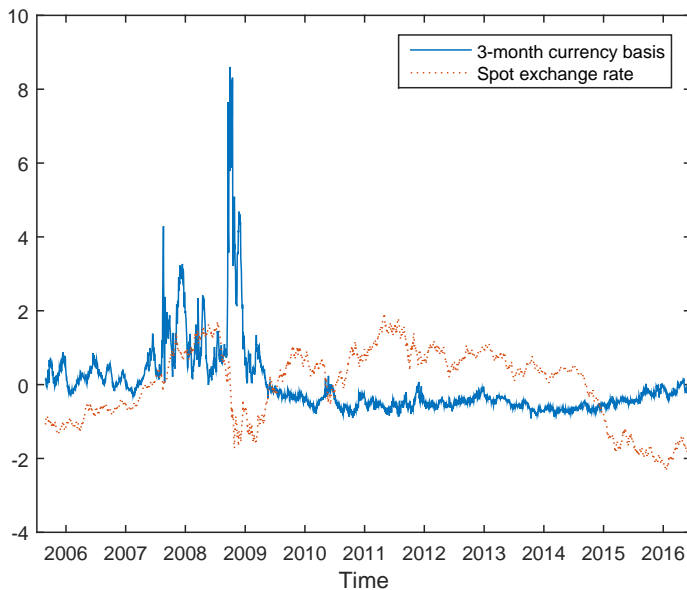


Figure 5.3: The figure plots the average of the logarithm of the FX spot rates (dashed line) and the 3-month average currency basis of USD/AUD, USD/CAD, and USD/EUR from August 2005 to June 2016. Both series are normalized to have mean zero and standard deviation 1.

We admit that the data we employ for the empirical analysis on the currency basis have limitations. A true deviation from CIP suggests making arbitrage profits without any frictions. Ideally, the interest rate data and the exchange rate data used to calculate the CIP deviation should be recorded at the same instant in time at which a trader could have dealt. While the foreign exchange contracts are traded over-the-counter so that we can record spot exchange rates and the forward rates at 4 p.m. London time, the interest rates of the fixed income markets are end-of-the-day closing prices in each country, which suffer from time zone differences. In addition, one needs to account for transaction costs incurred in forming currency basis trades, including bid-ask spread, brokerage fees and other associated costs. A rigorous study on whether investors do make real-life profits through deviation from CIP is beyond the scope of this paper. However, given the limitation of our data, a currency basis beyond 50 bps and persistent for months can hardly be attributed to data imperfections.

5.3.3 The Term structure of the currency basis

We divide the sample into a period when the market is particularly turbulent (June 2007 – December 2008) and a period when the market is relatively tranquil (August 2005 – May 2007 & January 2009 - June 2016). The summary statistics of the currency basis for the two periods are reported in Table 5.2.

5.3. EMPIRICS: THE CROSS-SECTION OF THE CURRENCY BASIS

Panel A: Turbulent period: June, 2007 - December 2008								
Maturity		1m	3m	6m	1y	2y	5y	10y
AUD	Mean	153.64	110.45	76.29	41.58	6.06	-6.43	-3.71
	Std	96.97	65.31	41.40	27.87	23.32	22.18	16.97
	Skewness	2.07	1.81	2.10	1.91	0.75	0.81	0.61
	Excess kurtosis	5.27	3.84	5.97	8.76	1.28	1.04	2.23
CAD	Mean	60.23	66.94	58.00	52.84	37.63	27.82	9.12
	Std	65.81	56.08	38.24	29.05	26.00	24.99	16.93
	Skewness	1.18	1.40	1.73	2.03	1.85	2.35	1.68
	Excess kurtosis	1.34	1.84	2.93	4.58	3.51	5.99	7.45
EUR	Mean	103.08	87.08	57.15	46.46	55.00	62.52	55.66
	Std	94.33	71.91	51.62	37.08	29.46	25.68	20.85
	Skewness	2.54	2.55	2.40	1.84	1.61	1.27	12.24
	Excess kurtosis	8.77	7.51	6.35	2.69	1.83	1.14	198.74
Portfolio	Mean	105.65	88.16	63.81	46.96	32.90	27.97	20.36
	Std	80.11	60.64	41.74	29.66	24.58	21.71	11.94
	Skewness	2.16	1.89	2.05	1.78	1.62	1.54	2.90
	Excess kurtosis	5.55	3.74	4.48	3.37	1.98	1.84	26.08
Panel B: Tranquil period: August, 2005 - May, 2007, January, 2009 - June, 2016								
Maturity		1m	3m	6m	1y	2y	5y	10y
AUD	Mean	18.23	6.05	0.51	-4.33	-21.07	-36.31	-45.42
	Std	24.74	19.82	18.10	18.28	17.80	26.16	24.94
	Skewness	0.51	1.08	0.81	0.57	-0.40	0.02	0.64
	Excess kurtosis	0.12	0.75	0.25	-0.45	0.51	-0.49	1.12
CAD	Mean	9.56	13.91	12.41	14.85	6.62	-2.37	-23.54
	Std	17.51	16.98	15.49	16.98	18.78	26.51	23.51
	Skewness	1.11	1.08	1.96	1.52	1.26	0.81	1.27
	Excess kurtosis	1.87	1.19	6.27	3.35	2.17	0.54	1.30
EUR	Mean	25.48	23.58	18.81	22.07	26.89	40.10	49.34
	Std	19.26	13.80	12.25	15.00	19.34	25.75	22.31
	Skewness	0.78	1.02	1.36	1.16	1.36	1.49	0.73
	Excess kurtosis	1.19	3.22	5.49	3.91	2.83	2.27	0.66
Portfolio	Mean	17.76	14.51	10.58	10.86	4.15	0.47	-6.19
	Std	16.63	14.45	12.42	14.00	14.99	20.76	17.53
	Skewness	1.10	1.31	1.61	1.33	1.40	1.14	1.42
	Excess kurtosis	1.46	2.05	4.66	2.38	2.74	1.50	2.30

Table 5.2: The table reports sample moments of the currency basis of Australian dollar against US dollar (AUD), Canadian dollar against US dollar (CAD) and Euro against US dollar (EUR) of one-month (1m), three-month (3m), six-month (6m), one-year (1y), two-year (2y), five-year (5y) and 10-year (10y) maturities. Panel A reports the statistics during the market turbulent period, from June 2007 to December 2008. Panel B reports the statistics during the relatively tranquil periods, August 2005 – May 2007 and January 2009 – June 2016.

Clearly, the currency basis (at least for short maturities) during the turbulent period is too substantial to be explained by transaction costs or data imperfections. One might wonder why is a currency basis of 100 bps a large amount – 100 bps is no more than 1%, far below the average returns of many asset classes. It is worth noting that in the absence of transaction

costs, a positive currency basis implies a *sure* profit. In other words, a non-zero currency basis, however small it is, suggests a Sharpe ratio of infinity.

More interestingly, the currency basis, like interest rates, displays term structure: at time t , the returns from the currency basis trade of different maturities are in general different, just like interest rates. All returns from the currency basis trades formed during the market turbulent period have a downward sloping curve. By contrast, the term structure of the currency basis trade returns during the tranquil period does not have a monotone shape.

To see how the currency basis behaves in the time series when maturities differ, Figure 5.4 plots the return of the equally weighted portfolio of currency basis trade with 1-month (dashed line), 1-year (dotted line), and 5-year (solid line) maturity. There are three interesting patterns we would like to point out. First, during the market turbulent period, the returns of different maturities exhibit large comovement in the cross section. From the starting of the sample up to mid-2009, returns of the shortest maturity (1-month) have the largest volatility. The volatility of returns decreases when maturity gets longer. Despite volatility differences, the returns of 1-month, 1-year, and 5-year maturity move together closely. Currency basis trade returns reach their highest levels for all maturities after the Lehman Brother default, which took place in September 2008.

Second, the time series of the currency basis displays volatility clustering. During the market turbulent period, when the currency basis is the largest, its volatility is also the largest. When the currency basis is small, the volatility stays low.

Third, after 2010, returns of the currency basis trade portfolio become much more persistent with independent movements of relatively low volatility. The differences between returns of different maturities shrink. The short-term currency basis trade returns, which display high volatility during the turbulent period, are almost indistinguishable from the longer maturity returns. Currency basis trade no longer makes significant profits during the tranquil period.

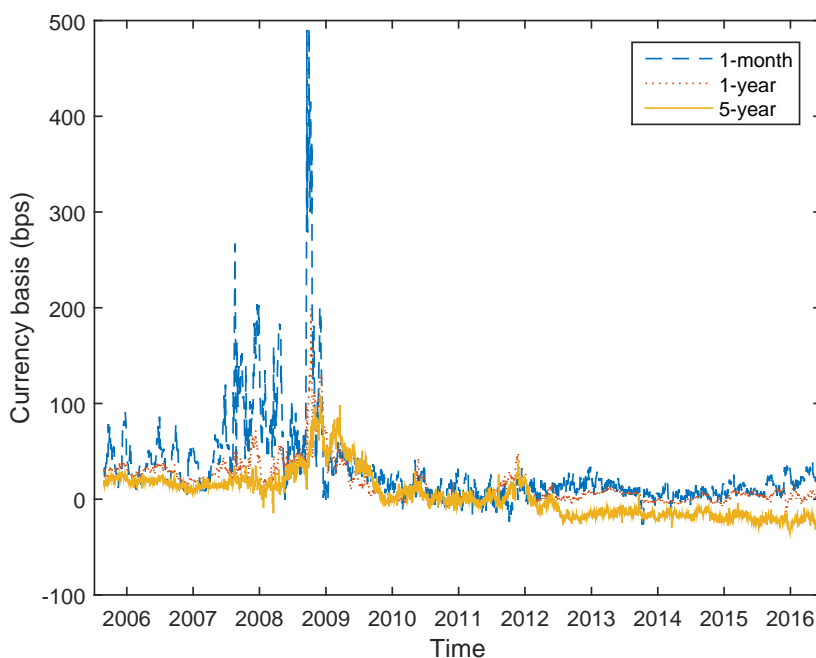


Figure 5.4: The figure plots the returns of the equally weighted portfolio of currency basis trade with 1-month (dashed line), 1-year (dotted line), and 5-year (solid line) maturity from August 2005 to June 2016.

To gain better insights of the term structure of currency basis trade returns, Figure 5.5 plots the 10-year term spread of the three currency pairs over the sample period. From the figure we observe that the term spread during the market turbulent period is negative most of time, as one expects from a downward-sloping curve. The term spread fluctuates mildly around zero since 2010. In short, the term spread is small (in absolute value) when currency basis is small, and gets large (in absolute value) when the currency basis is large.

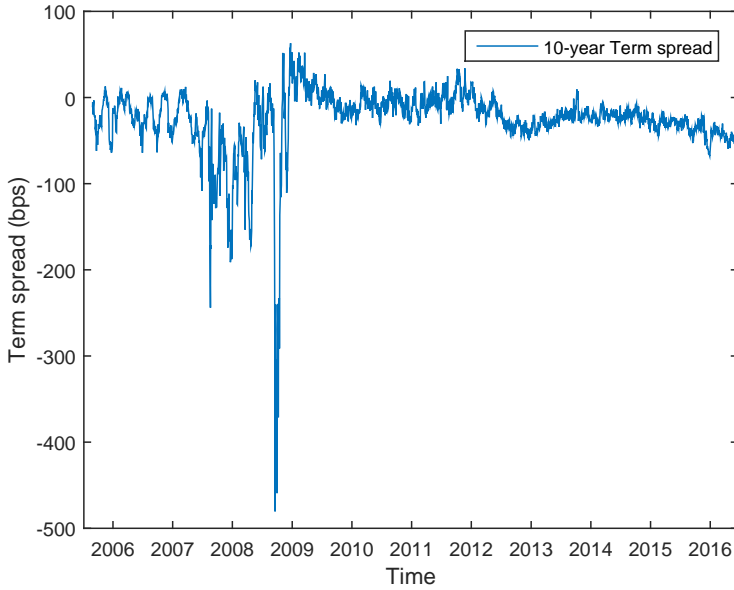


Figure 5.5: The figure plots the 10-year term spread of the returns from the equally weighted portfolio of currency basis trades from August 2005 to June 2016.

The patterns of the term spread, the time series and cross section features of the currency basis have important implications on theoretical modeling of the currency basis. We summarize the empirical stylized facts as follows:

1. The currency basis is large during the crisis, and small during the tranquil periods. And so is the term spread of the currency basis.
2. The currency basis displays volatility clustering: large currency basis tends to experience larger volatility, and vice versa.
3. The term structure of the currency basis during the financial crisis has a convex and downward sloping shape.

5.4 Revisit the Covered Interest rate Parity

In this section, we are going to build a structural model to theoretically derive the currency basis by taking into account additional sources of risks. Inspired by the extant literature, which shows that liquidity risk can explain a large part of the abnormal currency basis observed in the recent financial crisis, we are going to explore the channel through which liquidity risk (or other potential risk factors) can contribute to a nontrivial currency basis.

Classic CIP condition states that since a currency forward contract is replicable, the currency forward rate should be given by

$$\tilde{F}(t, T) = \frac{P^*(t, T)}{P(t, T)} S(t) = S(t) \exp \left((T - t)(r(t, T) - r^*(t, T)) \right). \quad (5.4)$$

We refer to $\tilde{F}(t, T)$ as the CIP implied forward rate. If the market is free of arbitrage opportunities, there exists a risk neutral measure Q , such that

$$0 = \mathbb{E}_t[S(T) - \tilde{F}(t, T)], \quad (5.5)$$

where \mathbb{E}_t denotes the expectation under Q , conditional on the information available at time t . Suppose an investor buys one unit of forward contract. The payoff he/she would get at maturity is the difference between the forward rate and the spot exchange rate at time T . Since there is no initial payment at time t when the contract is made, Equation (5.5) says that risk neutral expectation of the payoff at maturity should be zero. One can show that the CIP implied rate is a Q -martingale, implying

$$\tilde{F}(t, T) = \mathbb{E}_t[\tilde{F}(u, T)], \quad u \in (t, T). \quad (5.6)$$

Equation (5.5) shows that we can only replace the CIP implied rate by the market forward rate if the contract is successfully carried until maturity. As Shleifer and Vishny [110] point out, however, during a financial crisis, investors are more likely to exit their positions, either unwillingly or intentionally.

Now consider the possibility of unwinding the forward contract at some intermediate time $u \in (t, T]$. Suppose at time t , Party B enters a currency forward contract with Party A , agreeing to exchange $F(t, T)$ Dollar for 1 Euro at time T . To be able to deliver 1 Euro at time T , Party A starts with an initial capital of $S(t)P^*(t, T)$ Dollar and converts into Euro using the spot exchange rate and saves in the Euro deposit account. At a certain time $u, u \in (t, T]$, Party A may need to withdraw the Euro deposits for additional liquidity. Party A could unwind the forward contract by entering a new forward contract with Party C , agreeing to exchange $F(u, T)$ Dollar for 1 Euro at time T . This way, Party A can fulfil the contract with Party B using the proceeds from party C at time T . At maturity time T , the difference between $F(t, T)$ and $F(u, T)$ is a net gain to Party A . Figure 5.6 illustrates this process.

To allow for price differences between the recently issued and old bonds, we introduce the bond price notation $P(u; t, T)$, $u \in [t, T]$, which is the price of the domestic zero-coupon bond at time u that is issued at time t and matures at time T .

Free of arbitrage opportunities imply that under the risk neutral expectation, the input capital (the lightly shaded term) should be equal to the total payoff (dark shaded terms) after proper discount.

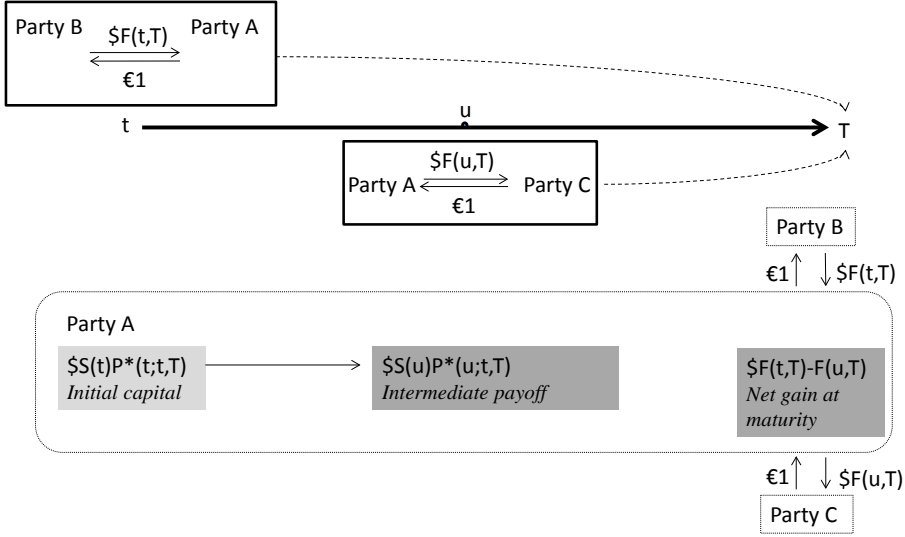


Figure 5.6: Illustration of a forward contract transaction with intermediate position unwinding. Party A and Party B sign a currency forward contract at time t with forward rate $F(t, T)$. Party A has initial capital $S(t)P^*(t, T)$ (lightly shaded on the very left) in the Euro deposit account to finance this forward contract. At a certain time u , $u \in (t, T]$, Party A is faced with a liquidity shock and need to withdraw the Euro deposit for additional liquidity. In order to unwind the contract, Party A enters a new forward contract with Party C with forward rate $F(u, T)$ that matures at time T . At time T , after fulfilling both contracts, Party A is left with $F(t, T) - F(u, T)$ as the net Dollar gain.

We call the time u at which the investor has to withdraw deposits for additional liquidity the position unwinding time. We suppose that the position unwinding time u has a risk-neutral hazard rate process λ , which means that the process Λ , defined as 0 before the position unwinding time and 1 afterwards, takes the following form

$$d\Lambda_t = (1 - \Lambda_t)\lambda_t dt + d\mathcal{M}(t). \quad (5.7)$$

Here, $\mathcal{M}(t)$ is a martingale process. λ_t measures the rate of occurring a position unwinding event at time t , given that no such events have happened up to time t ,

$$\mathbb{E}[d\Lambda_t | \Lambda_t = 0] = \lambda_t dt.$$

Note that Λ_t can also be represented using an indicator function $\Lambda_t = 1_{\{u \leq t\}}$. Using the domestic bond as the numeraire, it holds that

$$\frac{S(t)P^*(t, T)}{P(t, T)} = \mathbb{E}_t \left[F(t, T)(1 - \Lambda_T) + \int_t^T \left(\frac{S(u)P^*(u; T)}{P(u; T)} + F(t, T) - F(u, T) \right) d\Lambda_s \right]. \quad (5.8)$$

The left-hand-side of Equation (5.8) is the initial capital Party A starts with. The first term on the right-hand-side is the payoff at maturity T if no intermediate position unwinding has taken place. The second term on the right-hand-side of Equation (5.8) is the pay off at time u if position unwinding happens at time u .

Rearrange Equation (5.8) to get

$$F(t, T) = \tilde{F}(t, T) + \mathbb{E}_t \left[\int_t^T \left(F(u, T) - \frac{S(u)P^*(u; t, T)}{P(u; t, T)} \right) d\Lambda_u \right]. \quad (5.9)$$

Equation (5.9) says that the currency forward rate is equal to the parity implied rate, plus any expected loss incurred at the time of position unwinding. The loss at the time of position unwinding is the difference between the current currency forward rate and the market value of the cash account.

Now consider the case where the bonds at time u that are issued at time t and mature at time T have the same price as the ones that are just issued at time u with the same maturity,

$$P(u; t, T) = P(u; u, T), \quad P^*(u; t, T) = P^*(u; u, T),$$

in which case Equation (5.9) can be written as

$$F(t, T) = \tilde{F}(t, T) + \mathbb{E}_t \left[\int_t^T \left(F(u, T) - \tilde{F}(u, T) \right) d\Lambda_u \right].$$

Here, the currency forward rate is the implied rate plus any deviation from the parity in the future before maturity. In this case, we can easily see that $F(t, T) = \tilde{F}(t, T)$ is a solution to Equation (5.9). Therefore the position unwinding risk alone does not lead to deviations from CIP.

Now we consider the more general case where the old bond does not necessarily have the same price as the latest ones with the same time to maturity. Studies have shown that old bonds are very often sold at a lower price than the most recent bonds. As Krishnamurthy [86] notes, “One of the striking characteristics of the (long) bond sector is the high premium attached to the new bond”.

The price difference between the old and the new bonds is known as the “on-the-run/off-the-run spread”. The “on the run” bonds, which are the most recently issued, attract most of the liquidity. When the new bonds are issued, the old bonds go “off the run” and become much less liquid. The investor who holds the old bond may have to sell the bond at a lower price (than the on-the-run ones) to compensate the buyer for the loss of liquidity.

Towards maturity, however, the prices of the on-the-run bonds and the off-the-run bonds are expected to converge, as the remaining illiquidity in the off-the-run bonds decreases. Indeed, the on-the-run and off-the-run spread can be significant enough that some hedge funds trade such spread for profits. One example is the Long Term Capital Management. One of the fund’s main strategy was to exploit any difference between the price of a newly issued treasury bond and a similar one issued previously.

Yet a positive on-the-run/off-the-run spread itself does not lead to derivations from CIP. Consider the case where the probability of incurring a position unwinding event before maturity is zero,

$$\mathbb{P}_t(\Lambda_u = 0) = 1, u \in [t, T],$$

with \mathbb{P} representing the probability under the risk neutral measure. In this case, the second term on the RHS of Equation (5.9) is zero. Equation (5.9) therefore reduces to

$$F(t, T) = \tilde{F}(t, T).$$

If there are no intermediate liquidity shocks that force the investor into a position unwinding, however large the on-the-run/off-the-run spread is, the currency forward rate would be equal to the CIP implied rate. Therefore both the position unwinding risk and a nontrivial on-the-run/off-the-run spread in the bond market are essential risk factors that cause the currency forward rate to deviate from the parity condition. We postpone a formal discussion of the CIP as a special case of our model to Chapter 5.5.3.

5.4.1 The currency basis formula

Equation (5.9) gives a recursive formula of the currency forward rate. Define $Y(t, T)$ as the ratio between the forward rate and the CIP implied rate,

$$Y(t, T) := \frac{F(t, T)}{\tilde{F}(t, T)}. \quad (5.10)$$

In case CIP holds, $Y(t, T)$ should be equal to unity for all t, T .

We further define a liquidity discount factor $L(u; t, T)$ to capture the price difference between the old bonds and the new ones, such that

$$P(u; t, T) = P(u; u, T)L(u; t, T), \quad P^*(u; t, T) = P^*(u; u, T)L^*(u; t, T),$$

with $L(u; t, T), L^*(u; t, T) \leq 1$, $u \in [t, T]$, $L(T; t, T) = L^*(T; t, T) = 1$. Here $L(u; t, T), L^*(u; t, T) \leq 1$ implies that the off-the-run bonds are traded at a cheaper price than the on-the-run ones before the bond matures. At maturity, $L(T; t, T) = L^*(T; t, T) = 1$, indicating that the price of the old bonds converge to that of the on-the-run ones towards maturity.

We can rewrite Equation (5.9) as

$$Y(t, T) = 1 + \frac{1}{\tilde{F}(t, T)} \mathbb{E}_t \left[\int_t^T \tilde{F}(u, T) \left(Y(u, T) - \frac{L^*(u; t, T)}{L(u; t, T)} \right) d\Lambda_u \right].$$

If the implied forward rate $\tilde{F}(t, T)$ is independent of the liquidity risk L, L^* , and the position unwinding risk Λ , then it holds that

$$\begin{aligned} Y(t, T) &= 1 + \frac{1}{\tilde{F}(t, T)} \int_t^T \mathbb{E}_t[\tilde{F}(u, T)] \mathbb{E}_t \left[\left(Y(u, T) - \frac{L^*(u; t, T)}{L(u; t, T)} \right) d\Lambda_u \right] \\ &= 1 + \mathbb{E}_t \left[\int_t^T \left(Y(u, T) - \frac{L^*(u; t, T)}{L(u; t, T)} \right) d\Lambda_u \right]. \end{aligned} \quad (5.11)$$

The last equality holds because of Equation (5.6).

The currency basis $Y(t, T)$ is expected to be a function of time t , the liquidity discount factor $L(u; t, T)$, and the position unwinding intensity $\lambda(t)$. For a given maturity T , Equation (5.11) shows that the currency basis at time t is determined by two components. The first is the expected difference in liquidity risks between the bank accounts of the two currencies at the position unwinding time. The second is the expected future currency basis at the position unwinding time.

We further assume that the liquidity discount factor $L(u; t, T)$ is only a function of the current time u and maturity date T , but not a function of t .⁷

Note that Equation (5.11) is of a recursive form. Once the investor enters another forward contract after the position unwinding, he can in principle repeat the same replication

⁷Later we show that this assumption is consistent with the literature.

procedure all over again, which would expose him to the same position unwinding risk in the remaining of the time to maturity. The recursive form of Equation (5.11) saves us from modeling what happens after the first position unwinding, since any future forward rate $F(u, T)$, $u > t$, must reflect all the risk the investor will be faced with during the remaining life of the contract. Rewrite Equation (5.11) as

$$\begin{aligned} Y(t, T) &= 1 + \mathbb{E}_t \left[\int_t^{t+dt} \left(Y(u, T) - \frac{L^*(u)}{L(u)} \right) d\Lambda_u + \int_{t+dt}^T \left(Y(u, T) - \frac{L^*(u)}{L(u)} \right) d\Lambda_u \right] \\ &= \left(Y(t, T) - \frac{L^*(t)}{L(t)} \right) \lambda_t dt + \mathbb{E}_t \left[\int_{t+dt}^T \left(Y(u, T) - \frac{L^*(u)}{L(u)} \right) d\Lambda_u \right] + 1 \\ &= \left(Y(t, T) - \frac{L^*(t)}{L(t)} \right) \lambda_t dt + \mathbb{E}_t[Y(t + dt, T)]. \end{aligned} \quad (5.12)$$

Equation (5.12) implies that

$$\mathcal{A}Y(t, T) + \lambda_t Y - \lambda_t \frac{L^*(t, T)}{L(t, T)} = 0, \quad Y(T, T) = 1, \quad (5.13)$$

where \mathcal{A} is the infinitesimal generator acting on $Y(t, T)$.

Equation (5.13) characterizes the exponential of the currency basis by a stochastic partial differential equation. The following proposition solves the stochastic partial differential equation using standard techniques.

Proposition 5.1. *Under technical conditions, using the Feynman-Kac formula, $Y(t, T; X_t)$ can be represented using the conditional expectation*

$$Y(t, T) = -\mathbb{E}_t \left[\int_t^T \exp \left(\int_t^u \lambda(v) dv \right) \frac{L^*(u)}{L(u)} \lambda(u) du \right] + \mathbb{E}_t \left[\exp \left(\int_t^T \lambda(v) dv \right) Y(T, T) \right]. \quad (5.14)$$

Recall that unless both the position unwinding risk and liquidity risk are present, the currency basis would be zero. As a matter of fact, if the liquidity premium attached to the new bonds are identical in the two countries, the currency basis is zero even if the on-the-run/off-the-run spreads in both countries are nontrivial. The following lemma gives the conditions under which CIP holds.

Lemma 5.1. *If any of the following three situations applies, the currency forward rate satisfies the Covered Interest rate Parity.*

1. *No intermediate position unwinding risk, i.e.,*

$$\mathbb{P}[u < T] = 0.$$

2. *The off-the-run bonds are always sold at the same price as the on-the-run bonds, i.e.,*

$$L(u, T) = L^*(u, T) = 1, \quad \forall u \in [t, T].$$

3. *The on-the-run/off-the-run spreads of the two countries are always equal, that is*

$$L(u, T) = L^*(u, T), \quad \forall u \in [t, T].$$

Lemma 5.1 states that, in addition to the presence of both the position unwinding risk and on-the-run/off-the-run spread, the differences in the spreads of the relevant countries are also essential. This is intuitive, the currency forward rate, similar to the spot exchange rate, is by definition a relative amount. Loosely speaking, it is the *difference* in liquidity that drives the currency basis, not the market liquidity itself. This point is better illustrated in Corollary 5.1 later in Chapter 5.5.3.

5.5 The explicit formula for the currency basis

In the previous section, we have shown that the currency basis can be nontrivial once there are uncertainties (brought by market liquidity risk) in the currency basis trade. The uncertainties of such a transaction come from two aspects: (1) Whether the investor could carry the forward contract to maturity; (2) If a position unwinding event strikes, how much is the cash account worth? To work out the currency basis formula in explicit form, we need to impose structure on the position unwinding intensity process and the liquidity premium attached to the new bonds. In this section, we propose parametric models to capture time series dynamics of the liquidity process and the position unwinding intensity. The resulting currency basis will be a function of the parameters that characterize these stochastic processes.

5.5.1 The on-the-run/off-the run spread

Following Goldreich et al. [72], at time u , the value of the illiquid bond is equal to the value of the liquid bond discounted by the expected future liquidity risk,

$$L(u, T) = \mathbb{E}_u[\exp\left(-\int_u^T q(s) ds\right)], \quad (5.15)$$

where $q(t)$ can be understood as the instantaneous illiquidity at time t . Just like an interest rate of r reduces a cash flow's value at a rate r per period, an instantaneous illiquidity q reduces an old bond's value (relative to the new ones) at a rate q per period.

Similar to Acharya and Pedersen [1], we assume the liquidity process (q_t) follows Cox et al. [41] (CIR) process,

$$dq(t) = a(b - q(t)) dt + \sigma\sqrt{q(t)} dW(t). \quad (5.16)$$

Here, $a > 0$ is the mean reversion rate; $b > 0$ is the long term average of the liquidity level; $\sigma\sqrt{q(t)}$, $\sigma > 0$ is the volatility; and $W(t)$ is a standard Brownian motion under the risk measure Q . The level of the liquidity $q(t)$ drives the volatility of itself. When $q(t)$ is large, the market is relatively illiquid, reflected by a large on-the-run/off-the-run spread and large currency basis ultimately. Large illiquidity leads to large volatility, generating volatility clustering. When $q(t)$ is very small, the volatility will also converge to zero to prevent negative values of $q(t)$.

Similarly, in the foreign country,

$$P^*(u; t, T) = P^*(u; u, T)L^*(u, T) = P^*(u; u, T)\mathbb{E}^Q[\exp\left(-\int_u^T q^*(s) ds\right)],$$

with

$$dq^*(t) = a^*(b^* - q^*(t)) dt + \sigma^*\sqrt{q^*(t)} dW_t^*. \quad (5.17)$$

Similarly, a^* , b^* , σ^* are the mean reversion rate, long term steady state and volatility of the instantaneous illiquidity of the foreign country, respectively.

Besides producing volatility clustering and preventing negative values, CIR process also admits a tractable form of the liquidity discount factor $L(u, T)$.

Proposition 5.2. *Under the Assumptions (5.15), (5.16), (5.17), the liquidity risk discount factor L is given by*

$$L(u, T) = \exp(\alpha + \beta q(u)), \quad (5.18)$$

with

$$\begin{cases} \alpha = \frac{2ab}{\sigma^2} \left(\log(2de^{(a+d)(T-u)/2}) - \log((a+d)(\exp(d(T-u)) - 1) + 2d) \right), \\ \beta = \frac{2(1 - \exp(d(T-u)))}{(a+d)(\exp(d(T-u)) - 1) + 2d}, \end{cases} \quad (5.19)$$

where $d = \sqrt{a^2 + 2\sigma^2}$. $L^*(u, T)$ can be calculated similarly using the corresponding parameters of the foreign country.

5.5.2 The position unwinding intensity

What usually compels an investor to unwind a position is the risk of hitting the funding constraint. More often than not, the funding liquidity risk is likely to worsen when market liquidity is at stake. Brunnermeier and Pedersen [24] show that market liquidity and investors' funding liquidity are linked: when investors' capital is abundant that there is no risk of hitting the funding constraint hence no position unwinding, market liquidity is naturally at its highest level. However, "when speculators hit their capital constraints or risk hitting their capital constraints over the life of a trade then they reduce their positions and market liquidity declines."

Shleifer and Vishny [110] comment that "it can happen that arbitrageurs become more likely to exit their positions, rather than doubling up, when prices are most wrong". When market liquidity is low, the on-the-run/off-the-run spread widens, which result in larger deviations from the parity. Shleifer and Vishny [110] state that it is highly likely that investors reach their funding constraint at precisely the same time as prices move away from fundamental values and arbitrage opportunities improve. In case of currency basis trade, it implies that position unwinding events are more likely to strike when the market liquidity is low, reflected by a widened on-the-run/off-the-run spread.

To allow for dependence between the position unwinding events and market liquidity, we further adopt a factor structure (see Duffie and Singleton [49]) to model the position unwinding intensity λ_t ,

$$\lambda(t) = \lambda_0 + \gamma_1 q(t) + \gamma_2 q^*(t). \quad (5.20)$$

Here, γ_1 and γ_2 are loadings on the instantaneous illiquidity levels of the domestic and foreign countries. A positive $\gamma_1(\gamma_2)$ implies that a rise in the domestic (foreign) illiquidity increases the position unwinding likelihood. If $\gamma_1 = \gamma_2 = 0$, the arrival of position unwinding events is independent of the domestic or foreign market liquidity. In this case, the position unwinding intensity is a constant λ_0 . In principle, the position unwinding intensity can be driven by other sources of uncertainty. We do not consider other stochastic drivers in this model for simplicity.

5.5.3 The currency basis formula

The stochastic process of the position unwinding intensity and the on-the-run/off-the-run spread enable us to derive the currency basis explicitly.

Notice that the conditional expectation in Equation (5.14) falls into the “extended” transform class of the affine processes introduced by Duffie et al. [50], which allows us to evaluate the conditional expectation analytically. The following proposition derives the explicit currency basis up to integration.

Proposition 5.3 (The currency basis formula). *Let the liquidity discount factor be given by Equation (5.18) and the position unwinding intensity by (5.20). Suppose the realizations of the domestic and foreign instantaneous illiquidity at time 0 are q, q^* , respectively. The currency basis at time 0 with maturity date T is given by*

$$\begin{aligned} y(0, T) &:= \frac{1}{T} \log Y(0, T) \\ &= \frac{1}{T} \log \left(\exp(R_0 + R_1 q + R_2 q^*) \right. \\ &\quad \left. - \int_0^T (\lambda_0 + C + D_1 q + D_2 q^*) \exp(\alpha^* - \alpha + A + B_1 q + B_2 q^*) du \right), \end{aligned} \quad (5.21)$$

with $A = A(0), B_1 = B_1(0), B_2 = B_2(0), C = C(0), D_1 = D_1(0), D_2 = D_2(0), R_0 = R_0(0), R_1 = R_1(0), R_2 = R_2(0)$, where

$$\begin{cases} \dot{B}_1(t) = -\gamma_1 + aB_1(t) - \frac{1}{2}B_1^2(t)\sigma^2, & B_1(u) = -\beta, \\ \dot{B}_2(t) = -\gamma_2 + a^*B_2(t) - \frac{1}{2}B_2^2(t)\sigma^{*2}, & B_2(u) = \beta^*, \\ \dot{A}(t) = -\lambda_0 - abB_1(t) - a^*b^*B_2(t), & A(u) = 0, \\ \dot{D}_1(t) = aD_1(t) - B_1(t)D_1(t)\sigma^2, & D_1(u) = \gamma_1, \\ \dot{D}_2(t) = a^*D_2(t) - B_2(t)D_2(t)\sigma^2, & D_2(u) = \gamma_2, \\ \dot{C}(t) = -abD_1(t) - a^*b^*D_2(t), & C(u) = 0, \\ \dot{R}_1(t) = -\gamma_1 + aR_1(t) - \frac{1}{2}R_1^2(t)\sigma^2, & R_1(T) = 0, \\ \dot{R}_2(t) = -\gamma_2 + a^*R_2(t) - \frac{1}{2}R_2^2(t)\sigma^{*2}, & R_2(T) = 0, \\ \dot{R}_0(t) = -\lambda_0 - abR_1(t) - a^*b^*R_2(t), & R_0(T) = 0. \end{cases} \quad (5.22)$$

Here, $\alpha, \alpha^*, \beta, \beta^*$ are given by Proposition 5.2.

To compute the currency basis using Proposition 5.3, one needs to first compute the auxiliary deterministic processes $A(t), B_1(t), B_2(t), C(t), D_1(t), D_2(t), R_0(t), R_1(t)$, and $R_2(t)$ according to Equation (5.22). Starting with the terminal condition and working backwards, one can easily solve the ordinary differential equations numerically.

Before we numerically calculate the currency basis, we first give an impression on what the currency basis would be like in the simplest scenario with constant position unwinding risk and instantaneous market illiquidity.

Corollary 5.1. *In the special case where the market illiquidity of both countries are constant at all times, i.e.,*

$$q = b, \quad q^* = b^*, \quad \sigma = \sigma^* = 0,$$

and that the position unwinding intensity is a constant, i.e., $\gamma_1 = \gamma_2 = 0, \lambda(t) = \lambda_0, \forall t$, it holds that

$$y(t, T) = \frac{1}{T-t} \log \left(\exp(\lambda_0(T-t)) - \frac{\lambda_0}{\lambda_0 + q - q^*} (\exp((\lambda_0 + q - q^*)(T-t)) - 1) \right). \quad (5.23)$$

Interestingly, only the difference between the market illiquidity $q - q^*$ matters in this case. Trivially, when $q - q^* = 0$, Equation (5.23) is equal to 0. Write $q - q^*$ as Δq , Equation (5.23) implies that

$$Y(t, T) = \exp(\lambda_0(T - t)) - \frac{\lambda_0}{\lambda_0 + \Delta q} \left(\exp((\lambda_0 + \Delta q)(T - t)) - 1 \right).$$

The derivative of $Y(t, T)$ with respect to Δq is given by

$$\frac{\partial Y(t, T)}{\partial \Delta q} = \frac{\lambda_0}{(\lambda_0 + \Delta q)^2} \left(\exp((\lambda_0 + \Delta q)(T - t)) (1 - (\lambda_0 + \Delta q)(T - t)) - 1 \right). \quad (5.24)$$

One can easily show that the function defined as $f(x) = \exp(x)(1 - x)$ has a supremum of 1. Therefore Equation (5.24) is negative for all possible values of Δq , meaning that $Y(t, T)$ is a decreasing function of Δq , with $Y(t, T) = 1$ when $\Delta q = 0$. This means that the currency basis is positive when Δq is negative and vice versa. The difference in market illiquidity determines the direction of the currency basis of two countries: when the foreign market is less liquid than the domestic market, the forward currency rate, quoted as the domestic price of the foreign currency, contains a premium. Conversely, when the foreign market is more liquid than the domestic market, the forward currency rate, quoted as the future foreign price of the domestic currency, would contain a premium.

After witnessing the substantial currency basis in the market, there has been a heated discussion among institutional investors on whether to hedge US dollar risk, since it seems that the forward exchange rate is higher than its fair value (implied rate) and therefore imposes a cost on Dollar risk hedging. In this simplest model, the “perceived” extra hedging cost is the liquidity cost a US investor asks for lending abroad.

5.6 Numerical illustrations

Given the highly non-linear form of the currency basis formula (5.21) and the fact that the instantaneous liquidity process $q(t), q^*(t)$ are latent state variables, one may expect the parameter identification to be a challenging task. The parametrization of this model is rich and econometrically challenging. So far, all parameters are specified under the pricing measure Q . A full estimation of the model entails parameter specification under the physical measure, which would result in even higher dimensions of the parameter space. We therefore only calibrate the model parameters to match the cross-section patterns of the currency basis observed empirically. A formal econometric treatment is beyond the scope of this study.

5.6.1 Model calibration

In this section, we will calibrate the parameters for the USD/EUR currency basis. In principle, one can calibrate the other currency pairs in the same fashion. The mean reversion parameter a, a^* are determined to match the convex shape of the term structure. The long term steady states b, b^* are calibrated to match the currency basis at long maturities. The volatility σ, σ^* are determined to deliver the variations observed in the time series of the data. The position unwinding intensity loadings γ_1, γ_2 are set to match the magnitude of the basis. The calibrated parameter values are reported in Table 5.3. The detailed calibration procedure can be found in Appendix 5.B.

Parameter calibration for USD/EUR currency basis					
Panel A: Market illiquidity					
$dq(t) = a(b - q(t))dt + \sigma\sqrt{q(t)}dW(t)$					
a	5.1	b	0.003	σ	0.1
a^*	8.0	b^*	0.005	σ^*	0.2
Panel B: Position unwinding risk					
$\lambda_t = \lambda_0 + \gamma_1 q(t) + \gamma_2 q^*(t)$					
λ_0	0.1	γ_1	18.4	γ_2	0.0

Table 5.3: The table reports the calibrated parameter values of the USD/EUR currency basis. The mean reversion parameter a, a^* are determined to match the convex shape of the term structure. The long term steady states b, b^* are calibrated to match the currency basis at long maturities. The volatility σ, σ^* are determined to deliver the variations observed in the time series. The position unwinding intensity loadings γ_1, γ_2 are set to match the magnitude of the basis. Panel A reports the parameters of the instantaneous illiquidity process. The position unwinding intensity parameters are reported in Panel B.

We see that the mean reversion rates are large to ensure a downward sloping term structure. The steady states are relatively low to be negligible during tranquil periods. Both shocks in illiquidity in the US market and foreign market raise the probability of a position unwinding event.⁸

We can filter the instantaneous illiquidity process by

$$\hat{q}(t) = \frac{1}{N} \sum_{m_i} (\log L(t, t + m_i) - \hat{\alpha}(m_i)) / \hat{\beta}(m_i), \quad (5.25)$$

where the summation is over all available maturities $m_i, i = 1, \dots, m_N$. Here, $\hat{\alpha}, \hat{\beta}$ are calculated using Equation (5.19) with the calibrated parameter values $\hat{a}, \hat{b}, \hat{\sigma}$. Figure 5.7 plots the standardized (with mean zero and a standard deviation of 1) 2-year on-the-run/off-the-run spread of the US treasury and the filtered instantaneous illiquidity process of the US. We see that the two processes co-move closely during the 2007-2010 financial crisis. The instantaneous illiquidity, in particular, is high in market turbulence periods, e.g., the burst of the internet bubble in early 2000, the 2002 Asian financial crisis and the 2008 financial crisis.

⁸Since we do not match the time series moments of the currency basis and that the instantaneous illiquidity processes are unobserved, one should not expect the parameters that deliver the cross-section patterns to be unique. The parameter values reported in Table 5.3 are an example of a set of reasonable parameter values. It is highly likely that another set of parameter values would deliver a similar term structure pattern.

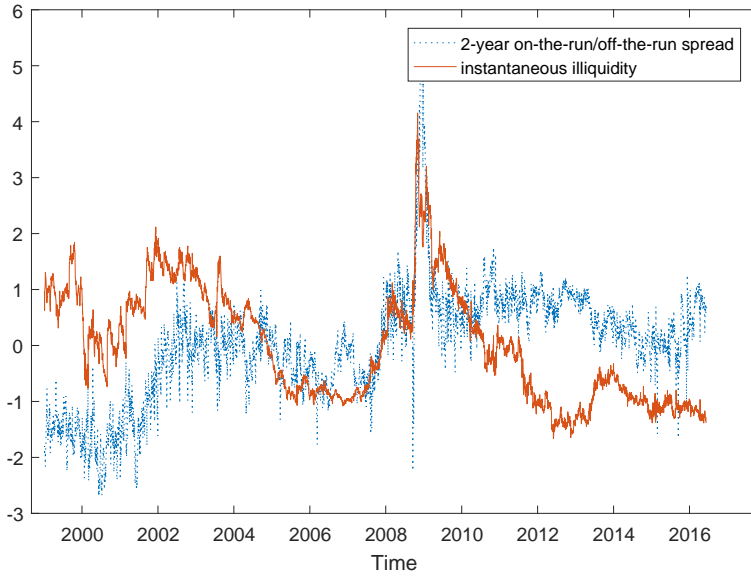


Figure 5.7: The figure plots the calibrated instantaneous illiquidity process against the 2-year on-the-run/off-the-run spread of the US treasury. Both processes are normalized to have zero mean and a standard deviation of 1. The instantaneous illiquidity process is filtered using Equation (5.25).

5.6.2 Comparative statics

In the remainder of this section, we numerically show how the currency basis depends on the stochastic characteristics of the liquidity risk in the off-the-run bonds and the position unwinding risk of investors. We compute the term structure of the currency basis between a foreign currency and US dollar using Proposition 5.3 at different parameter values.

The shape of the term structure is jointly determined by the liquidity risk process as well as the position unwinding risk process. Intuitively, in case of a high instantaneous illiquidity, due to the mean reversion feature of the liquidity risk process, a liquidity shock impacts the shorter maturities more than the longer ones. However, chances are that as time to maturity gets longer, the position unwinding events take place and result in a substantial loss. When maturity is short, a liquidity shock has not yet died out and could potentially lead to large losses. When maturity is long, although the expected future liquidity risk is lower, there could be a bigger chance of experiencing a position unwinding event. The eventual term structure is a result of the competition of these two forces.

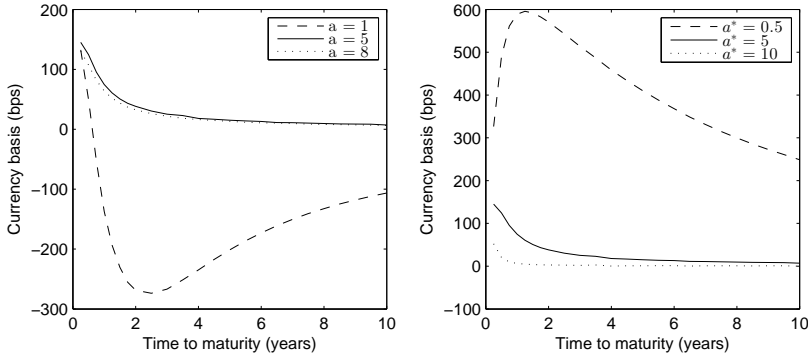


Figure 5.8: The figure plots the term structure of the currency basis for different values of the mean reversion rate of the instantaneous liquidity risk of the US (left panel) and of the foreign country (right panel). The cases with slow mean reversion are plotted in the dashed lines, medium mean reversion are plotted in the solid curves, and fast mean reversion are plotted in the dotted curves. The other parameters are from Table 5.3, with the state variables $q = 0.1$, $q^* = 0.4$.

Figure 5.8 plots the currency basis term structure for different values of the mean reversion rate of the instantaneous liquidity risk in the US, a (left panel), and that in the foreign country a^* (right panel). On the left panel, the case of slow mean reversion, $a = 1$, is plotted in the dashed line, the case of medium mean reversion, $a = 5$, is shown by the solid curve, while the case of fast mean reversion, $a = 10$, is depicted using the dotted curves. In case of slow mean reversion, the term structure of the currency basis is hump-shaped, in which case the currency basis increases with time to maturity at the short end and reaches its peak at round 1-year to maturity and then starts to decline. When mean reversion is fast enough, the term structure is convex and monotonically decreasing. The faster the mean reversion is, the more convex the term structure becomes.

When liquidity risk has a fast mean reversion rate, it is quickly pulled to the long term level once it deviates. The fast mean reversion rate guarantees that the liquidity risk will die out in the future even if there is uncertainty in the liquidity risk. Therefore, the short end of the term structure is affected the most by a large liquidity shock. If the mean reversion rate is not faster than the rate at which position unwinding risk increases with maturity, the currency basis is an increasing function of time to maturity.

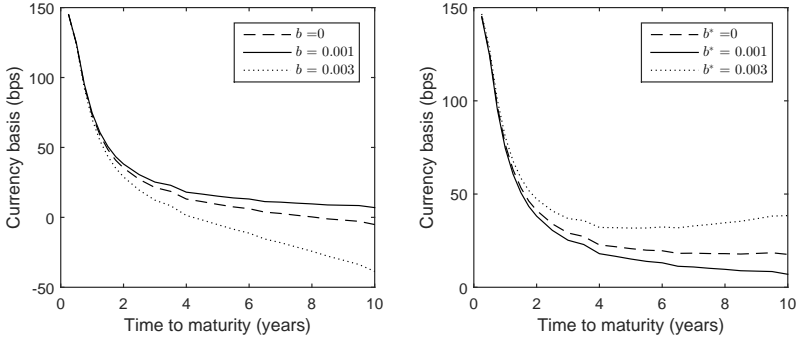


Figure 5.9: The figure plots the term structure of the currency basis for different steady states of the instantaneous liquidity risk of the US (left panel) and of the foreign country (right panel). The case with low steady states $b(b^*) = 0$ is plotted in the dashed line, medium steady states $b(b^*) = 0.001$ is plotted in the solid curve, and high steady states $b(b^*) = 0.003$ is plotted in the dotted curve. The other parameters are from Table 5.3, with the state variables $q = 0.1$, $q^* = 0.4$.

While the mean reversion rate determines the shorter end of the term structure, the steady state of liquidity b determines the longer end. Figure 5.9 plots the term structure for different parameter values for the long term steady state of the instantaneous liquidity risk in the US, b (left panel), and that in the foreign country, b^* (right panel). In the long term, the instantaneous liquidity risk is expected to stay at the level of its long term steady state. Higher level of expected US liquidity risk drives the currency basis down, whereas higher level of expected foreign liquidity risk drives the currency basis up.

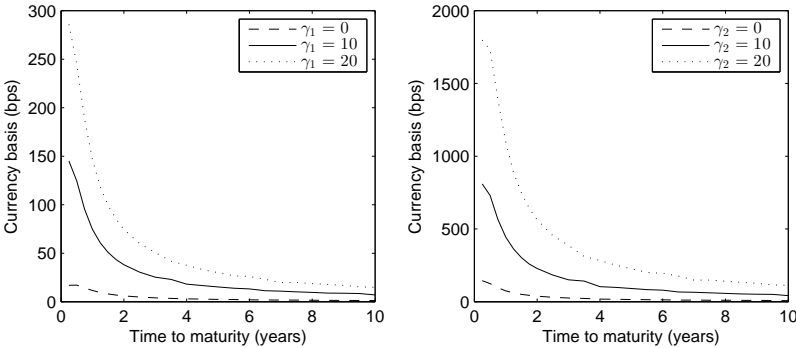


Figure 5.10: The figure plots the term structure of the currency basis for different coefficients of the position unwinding intensity on the instantaneous liquidity risk of the US (left panel) and of the foreign country (right panel). The case with low loadings $\gamma_1(\gamma_2) = 0$ is plotted in the dashed line, medium loadings $\gamma_1(\gamma_2) = 10$ is plotted in the solid curve, and large loadings $\gamma_1(\gamma_2) = 20$ is plotted in the dotted line. The other parameters are from Table 5.3, with the state variables $q = 0.1$, $q^* = 0.4$.

Figure 5.10 plots the currency basis term structure for different coefficients of the position unwinding intensity on the US instantaneous liquidity risk γ_1 (left panel) and on the foreign instantaneous liquidity risk γ_2 (right panel). In case that the loading on the instantaneous liquidity risk is not zero, investors face time-varying position unwinding risks. When either

γ_1 or γ_2 is non-zero, the position unwinding intensity is also mean reverting. When a liquidity shock arrives, not only liquidity risk shoots up and raises the short-term currency basis, the probability that the investor would exit his/her position also increases sharply in the short run. Increasing the coefficient of the position unwinding intensity on the liquidity risk of the foreign country (right panel) has similar effects as increasing the coefficient on the liquidity risk of the US (left panel). In both cases, the currency basis of short maturities has the largest increase, resulting in a more convex and steeper curve of the term structure.

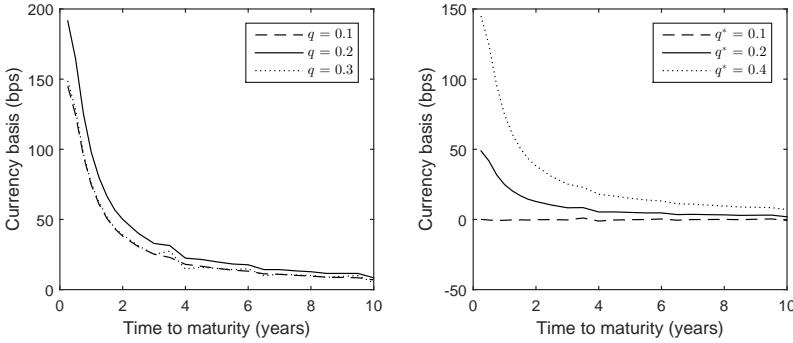


Figure 5.11: The figure plots the term structure of the currency basis for different values of the current instantaneous illiquidity level of the US (left panel) and of the foreign country (right panel). Low current illiquidity cases are plotted in the dashed lines, medium illiquidity cases are plotted in the solid curves, and large illiquidity cases are plotted in the dotted curves. The other parameters are from Table 5.3, with the state variables $q = 0.1$ (right panel), $q^* = 0.4$ (left panel).

Figure 5.11 plots the term structure of the currency basis for different values of the current instantaneous illiquidity level of the US (left panel) and the foreign country (right panel). Since liquidity risks will mean revert to their long term steady states, they affect currency basis in the short term more than they do in the long term.

Note that in our choice of parametrization, the instantaneous illiquidity level in the foreign country only affects the potential losses in case of position unwinding. It does not, however, increase the position unwinding probability itself (since $\gamma_2 = 0$). The instantaneous illiquidity level in the US, by contrast, has two roles. First, it increases the on-the-run/off-the-run spread in the US, narrowing the liquidity risk differences between the two countries, thereby reducing the currency basis for short maturities. Second, higher instantaneous illiquidity level of the US leads to a higher position unwinding intensity, which tends to raise the currency basis. Whether the currency basis goes up as a result of a current US instantaneous illiquidity shock depends on which effect is larger. This explains why we see the term structure with $q = 0.3$ is not so different from that with $q = 0.2$ on the left panel of the curve. The increase in the current US instantaneous illiquidity reduces the liquidity risk differences between the two countries but increases the position unwinding risk. The former tends to increase the currency basis while the latter tends to reduce it. The two effects are comparable, leaving the term structure relatively unchanged as compared to the case where $q = 0.2$.

5.7 Conclusion

This paper studies the behavior of the currency basis, i.e., deviations from the covered interest rate parity condition. We conduct empirical analysis using Australian dollar vs. US dollar, Canadian dollar vs. US dollar and Euro vs. US dollar. We first document the stylized facts of the behavior of the currency basis during the market turbulent period as well as during tranquil period. We establish that the term structure of the currency basis during the financial crisis is downward sloping: the violation of the CIP is much more substantial for short maturity forward contracts than for long term ones. The term structure during the post-crisis period, however, does not have a monotone shape.

Inspired by the extant empirical literature, which finds liquidity risk to be a statistically significant explanatory variable of the observed currency basis, we develop an asset pricing model with no arbitrage opportunities to price the currency basis theoretically. To see how investors are exposed to liquidity risk by entering a forward contract, consider an arbitrageur who engages in a currency basis trade. At a certain time before the contract matures, the investor may be forced to exit the position in order to free the capital in the cash market. Since premature deposits and off-the-run bonds can only be sold at a lower price, the investor incurs a liquidity cost when the market liquidity is low when he/she unwinds the position. We show that when we consider the position unwinding risk, the forward currency rate given by the asset pricing model exceeds the CIP implied forward rate by the risk neutral expectation of future liquidity cost weighted by the position unwinding probability. Here, the future liquidity cost is measured by the on-the-run/off-the-run spread.

In the special cases where either the position unwinding risk is absent, or the on-the-run/off-the-run spread of the two countries are the same at all times (this includes the case where the on-the-run/off-the-run spreads are zero in both countries), our model recovers the CIP condition as a nested case.

We show that our model can not only produce the observed magnitude of the currency basis but also generate the term structure of currency basis that resembles the reality. We therefore contribute to the foreign exchange literature by (1) establishing that the term structure of the currency basis curve has a convex and downward-sloping shape and (2) putting forward a no-arbitrage asset pricing model with liquidity risk. The currency basis predicted by the model matches the observed patterns.

Appendices

5.A Proofs

Proof of Proposition 5.1. When $ab > \sigma$, the liquidity process never touches 0. If we further cap the liquidity process by a large number, for example, define $\bar{q}(t) = q(t) \wedge 10\%$, and replace the unbounded process $q(t)$ with the bounded process $\bar{q}(t)$, then, one can easily verify that we can readily apply the Feynman-Kac formula (see, for example, Duffie [48]) and conclude that Equation (5.14) is the unique solution to Equation (5.12). \square

Proof of Lemma 5.1. 1. $\mathbb{P}[u < T] = \int_0^T \lambda(t) dt = 0$, then $\lambda(t) = 0, t \in [0, T)$. Equation (5.14) therefore becomes

$$Y(t, T) = \mathbb{E}_t[Y(T, T)] = 1,$$

implying that

$$F(t, T) = \tilde{F}(t, T).$$

2. When the on-the-run/off-the-run spread is zero in the domestic country as well as the foreign country, it holds that

$$L(u, T) = L^*(u, T) = 1.$$

Equation (5.14) reduces to

$$Y(t, T) = -\mathbb{E}_t \left[\int_t^T \exp \left(\int_t^u \lambda(v) dv \right) \lambda(u) du \right] + \mathbb{E}_t \left[\exp \left(\int_t^T \lambda(v) dv \right) Y(T, T) \right]. \quad (5.26)$$

Note that

$$d \exp \left(\int_t^u \lambda(v) dv \right) = \exp \left(\int_t^u \lambda(v) dv \right) \lambda(u) du.$$

Applying integration by part on the first term on the RHS of Equation (5.26), we have

$$\begin{aligned} \int_t^T \exp \left(\int_t^u \lambda(v) dv \right) \lambda(u) du &= \int_t^T 1 d \exp \left(\int_t^u \lambda(v) dv \right) \\ &= \exp \left(\int_t^u \lambda(v) dv \right) \Big|_t^T - \int_t^T \exp \left(\int_t^u \lambda(v) dv \right) d1 \\ &= \exp \left(\int_t^T \lambda(v) dv \right) - \exp(0) \\ &= \exp \left(\int_t^T \lambda(v) dv \right) - 1. \end{aligned}$$

Therefore Equation (5.26) can be written as

$$Y(t, T) = -\mathbb{E}_t \left[\exp \left(\int_t^T \lambda(v) dv \right) - 1 \right] + \mathbb{E}_t \left[\exp \left(\int_t^T \lambda(v) dv \right) \right] = 1.$$

If $L^*(u) = L(u), \forall u$, then Equation (5.14) reduces to Equation (5.26). The rest follows the proof for the second condition when there is no on-the-run/off-the-run spread. \square

Proof of Proposition 5.2. See Cox et al. [41]. □

Proof for Proposition 5.3. It holds that

$$\begin{aligned} & \mathbb{E}_0 \left[\exp \left(\int_0^u \lambda(v) dv \right) \frac{L^*(u, T)}{L(u, T)} \lambda(u) \right] \\ &= \mathbb{E}_0 \left[\exp \left(\int_0^u \lambda(v) dv \right) \lambda(u) \exp \left(\alpha^* + \beta^* q^*(u) - \alpha - \beta - q(u) \right) \right] \\ &= \exp \left(\alpha^* - \alpha \right) \mathbb{E}_0 \left[\exp \left(\int_0^u \lambda(v) dv \right) \exp \left(\beta^* q^*(u) - \beta q(u) \right) \lambda(u) \right]. \end{aligned}$$

Write

$$\begin{aligned} & \mathbb{E}_0 \left[\exp \left(\int_0^u \lambda(v) dv \right) \exp \left(\beta^* q^*(u) - \beta q(u) \right) \lambda(u) \right] \\ &= \mathbb{E}_0 \left[\exp \left(\int_0^u \lambda(v) dv \right) \exp \left(\beta^* q^*(u) - \beta q(u) \right) \lambda_0 \right] \\ & \quad + \mathbb{E}_0 \left[\exp \left(\int_0^u \lambda(v) dv \right) \exp \left(\beta^* q^*(u) - \beta q(u) \right) (\gamma_1 q(u) + \gamma_2 q^*(u)) \right] \\ &:= \lambda_0 \psi + \phi. \end{aligned}$$

According to Duffie et al. [50], define

$$\begin{aligned} \mathbf{K}_0 &= \begin{pmatrix} ab \\ a^* b^* \end{pmatrix}, \mathbf{K}_1 = \begin{pmatrix} -a & 0 \\ 0 & -a^* \end{pmatrix}, \\ \rho_0 &= -\lambda_0, \quad \boldsymbol{\rho}_1 = -(\gamma_1, \gamma_2), \\ \mathbf{H}_0 &= 0, \quad (\mathbf{H}_1)_{11} = (\sigma^2, 0)', \quad (\mathbf{H}_1)_{22} = (\sigma^{*2}, 0)'. \end{aligned}$$

Then it holds that

$$\psi = \exp(A + \mathbf{B}' \mathbf{x}(0)), \quad \phi = (C + \mathbf{D}' \mathbf{x}(0)) \psi,$$

with $A = A(0)$, $\mathbf{B} = \mathbf{B}(0)$, $C = C(0)$, $\mathbf{D} = \mathbf{D}(0)$, where

$$\begin{cases} \dot{B}_1(t) = -\gamma_1 + a B_1(t) - \frac{1}{2} B_1^2(t) \sigma^2, & B_1(u) = -\beta \\ \dot{B}_2(t) = -\gamma_2 + a^* B_2(t) - \frac{1}{2} B_2^2(t) \sigma^{*2}, & B_2(u) = \beta^* \\ \dot{A}(t) = -\lambda_0 - ab B_1(t) - a^* b^* B_2(t), & A(u) = 0 \\ \dot{D}_1(t) = a D_1(t) - B_1(t) D_1(t) \sigma^2, & D_1(u) = \gamma_1 \\ \dot{D}_2(t) = a^* D_2(t) - B_2(t) D_2(t) \sigma^{*2}, & D_2(u) = \gamma_2 \\ \dot{C}(t) = -ab D_1(t) - a^* b^* D_2(t), & C(u) = 0. \end{cases}$$

Using the same formula, one can easily show that

$$\mathbb{E}_0 \left[\exp \left(\int_0^T \lambda(u) du \right) \right] = \exp(R_0 + R_1 q(0) + R_2 q^*(0)),$$

with

$$\begin{cases} \dot{R}_1(t) = -\gamma_1 + a R_1(t) - \frac{1}{2} R_1^2(t) \sigma^2, & R_1(T) = 0 \\ \dot{R}_2(t) = -\gamma_2 + a^* R_2(t) - \frac{1}{2} R_2^2(t) \sigma^{*2}, & R_2(T) = 0 \\ \dot{R}_0(t) = -\lambda_0 - ab R_1(t) - a^* b^* R_2(t), & R_0(T) = 0. \end{cases}$$

The currency basis formula is therefore

$$Y(0, T) = \exp(R_0 + R_1 q + R_2 q^*) - \int_0^T (\lambda_0 + C + D_1 q + D_2 q^*) \exp(\alpha^* - \alpha + A + B_1 q + B_2 q^*) du.$$

□

Proof of Corollary 5.1. In case of constant market illiquidity and position unwinding, we may remove the expectation operator, since there is no uncertainty at time t . The liquidity discount factor becomes

$$L(u, T) = \exp(-q(T - u)), \quad L^*(u, T) = \exp(-q^*(T - u)).$$

Then Equation (5.14) becomes

$$\begin{aligned} Y(t, T) &= - \int_t^T \exp(\lambda_0(u - t)) \frac{\exp(-q^*(u - t))}{\exp(-q(u - t))} du + \exp(\lambda_0(T - t)) \\ &= -\lambda_0 \int_t^T \exp((\lambda_0 + q - q^*)(u - t)) du + \exp(\lambda_0(T - t)) \\ &= -\frac{\lambda_0}{\lambda_0 + q - q^*} \exp((\lambda_0 + q - q^*)(u - t)) \Big|_t^T + \exp(\lambda_0(T - t)) \\ &= \exp(\lambda_0(T - t)) - \frac{\lambda_0}{\lambda_0 + q - q^*} \left(\exp((\lambda_0 + q - q^*)(T - t)) - 1 \right). \end{aligned}$$

□

5.B Model Calibration

For parameter calibration, we also take advantage of the off-the-run yields of the US government bonds. The off-the-run yields of the US treasuries are estimated by Gürkaynak et al. [73]. The authors estimate a “synthetic” off-the-run US Treasury yield curve at a daily frequency from 1961 to the present. The data is posted on the Federal Reserve website and is updated regularly.⁹

We first calibrate the liquidity parameters of the US using bond prices. Define the parameter space $\Theta_1 = \{a, b, \sigma\}$. For notation convenience, we define time to maturity $m_i = T_i - t$ and write $L(t; m_i)$ instead of $L(t, T)$. We first compute on-the-run/off-the-run spread on each day for each maturity, using

$$l(t; m_i) = \log L(t; m_i),$$

where $L(t; m)$ is calculated as the ratio between the off-the-run and on-the-run bond prices. Theoretically, according to Proposition 5.2, it holds that

$$l(t; m_i) = \alpha(m_i) + \beta(m_i)q(t).$$

⁹The data is available at <http://www.federalreserve.gov/econresdata/feds/2006>. In the paper, the authors demonstrate how the on-the-run/off-the-run spread can be inferred using this dataset.

Since the dependence between the observed spread and the unobserved latent illiquidity process is linear, for any parameter candidates $\hat{\Theta}_1 = \{\hat{a}, \hat{b}, \hat{\sigma}\}$, we can find the latent illiquidity process as the average of the implied latent states over all available maturities m in the data.

$$\hat{q}(t; \hat{\Theta}_1) = \frac{1}{N} \sum_{m_i} (l(t; m_i) - \hat{\alpha}(m_i, \hat{\Theta}_1)) / \hat{\beta}(m_i, \hat{\Theta}_1), \quad (5.27)$$

where $\hat{\alpha}(m_i, \hat{\Theta}_1), \hat{\beta}(m_i, \hat{\Theta}_1)$ can be calculated using Equation (5.19) with the calibrated parameters $\hat{\Theta}_1$. We are ready to define a calibrated on-the-run/off-the-run spread $\hat{l}(t; m_i, \hat{\Theta}_1)$ as

$$\hat{l}(t; m_i, \hat{\Theta}_1) = \hat{\alpha}(m_i, \hat{\Theta}_1) + \hat{\beta}(m_i, \hat{\Theta}_1) \hat{q}(t; \hat{\Theta}_1).$$

The calibrated parameters can be obtained by minimizing the predicted spread and the observed spread,

$$\hat{\Theta}_1 = \arg \min_{m_i, t} \sum \left(\hat{l}(t; m_i, \hat{\Theta}_1) - l(t; m_i) \right)^2. \quad (5.28)$$

Having obtained the parameter estimates $\hat{\Theta}_1$, one can easily filter the latent states $\hat{q}(t; \hat{\Theta}_1)$ process according to Equation (5.27).

Unfortunately, the on-the-run/off-the-run spreads of other countries are not immediately available. The liquidity process parameters a^*, b^*, σ^* can only be identified using the currency basis. To estimate the remaining parameters, we treat the latent illiquidity process in the foreign country (q_t^*) as parameters and define the parameter space

$$\Theta_2 = \{a^*, b^*, \sigma^*, \gamma_1, \gamma_2, \lambda_0, (q_t^*)\}.$$

Using the calibrated parameters in the first step, we are ready to calculate the predicted currency basis for a given $\hat{\Theta}_2$, denoted by $\hat{y}(t; m_i, \hat{\Theta}_1, \hat{q}(t; \hat{\Theta}_1), \hat{\Theta}_2)$.

Similar to the first-step estimation, we minimize the squared pricing error,

$$\hat{\Theta}_2 = \arg \min_{m_i, t} \sum \left(\hat{y}(t; m_i, \hat{\Theta}_1, \hat{q}(t; \hat{\Theta}_1), \hat{\Theta}_2) - y(t; m_i) \right)^2, \quad (5.29)$$

over all available time t and maturity m_i .

To enhance parameter calibration, the following practical issues are taken into account. In particular, we impose the following restrictions: $a(a^*) > 0, \sigma(\sigma^*) > 0, b(b^*) \geq 0, \lambda_0 > 0$. The first two conditions ensure that both the mean reversion rate and the volatility of instantaneous liquidity risk in all countries are positive, so that the model is well-defined. By requiring a non-negative $b(b^*)$, we make sure that the liquidity risk is always positive, meaning that the off-the-run bond is always sold at a liquidity premium. The last restriction means that the probability of a position unwinding event is non-negative.

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Summary

Title: Essays on International Portfolio Choice and Asset Pricing under Financial Contagion

The global financial crisis that took place in 2007-2009 has posed a challenge to the financial modeling practice where classical models which have been widely applied no longer work. In this doctoral dissertation, we revisit the most relevant financial problems, in particular, portfolio choice, hedging, and pricing, under the context of financial crises.

Chapter 3 and Chapter 4 investigate the equity portfolio choice and currency hedging problem under financial contagion. We employ mutually exciting jumps to generate excess cross-market linkages. We first analyze the optimal equity portfolio choice problem under equity risk contagion across geographical markets, assuming exchange rate risk is fully hedged away. Once allowing for excitation asymmetry, we find that, instead of an optimally diversified equity portfolio, as suggested in the classic asset allocation literature, the optimal portfolio for an expected CRRA utility investor is “under-diversified” in the sense that investors over-invest in the US, whose equity market is more capable of transmitting its jump risks worldwide, a phenomenon that we term “the US bias”. Next, we relax the assumption of 100% currency risk hedging and allow investors to choose their optimal currency exposure. We revisit the equilibrium currency hedging problem under equity-currency contagion. We show that in equilibrium, investors would hedge less “safe haven currency”, the currency that is less susceptible to equity market turmoil.

Chapter 5 studies the market efficiency aspect of the financial crisis – the breakdown of the covered interest rate parity in particular. We find that the currency basis during the crisis has a convex and downward sloping term structure. We propose to explicitly account for liquidity risk when pricing the currency forward contract. We show that the risk that investors may need to exit the position to free the capital for liquidity preference during the crisis can lead to a nontrivial currency basis.

In these cases, predictions made by models that account for financial crises are qualitatively different from those made by classic models. Ignoring the crisis phenomena can lead to substantial losses for institutional and individual investors when a financial crisis strikes.

Samenvatting

Titel: Essays over Internationale Portefeuillekeuze en Activawaardering tijdens Financiële Crises.

De wereldwijde financiële crisis in 2007-2009 heeft voor grote uitdagingen gezorgd in het modelleren van financiële vraagstukken, aangezien de klassieke modellen, die op grote schaal werden toegepast, niet meer werkten. In dit proefschrift kijken wij opnieuw naar de meest relevante financiële problemen, in het bijzonder portefeuillekeuze, het afdekken van financiële risico's, en activawaardering ten tijden van financiële crises.

Hoofdstukken 3 en 4 onderzoeken de vraagstukken omtrent optimale aandelenportefeuillekeuze en de afdekking van valutarisico's in de context van financieel besmettingsgevaar. We gebruiken elkaar wederzijds versterkende sprongprocessen om sterke samenhang tussen verschillende markten te genereren. We analyseren eerst het optimale aandelenportefeuillekeuze probleem, onder de aanname dat valutarisico voor beleggers volledig afgedekt is, in de situatie dat er risicobesmettingsgevaar is tussen geografisch verschillende markten. Zodra we asymmetrie toelaten in de besmettingskanalen, vinden we dat een belegger die zijn verwachte nut maximaliseert, onder de aanname van een CRRA nutsfunctie, een "onder-gediversifieerde" portefeuille aanhoudt ten opzichte van de portefeuille die tot stand zou komen in de klassieke activa allocatieliteratuur. In het bijzonder vinden we dat de belegger over-investeert in de VS, aangezien het aandelenmarktrisico in de VS zich makkelijker verspreidt naar de rest van de wereld. Wij refereren naar deze observatie als de "VS bias". Vervolgens verzachten we de aanname dat er geen valutarisico's zijn en staan we toe dat beleggers hun blootstelling naar verschillende valutarisico's zelf kunnen kiezen. We beschouwen opnieuw het valutarisico afdekkingsprobleem in de context van internationale portefeuillekeuze met besmettingsgevaar tussen aandelen-en valutarisico's. We laten zien dat in de evenwichtssituatie beleggers hun risico met betrekking tot "veilige haven valuta", de valuta die relatief immuun is voor onrust in de aandelenmarkt, minder afdekken.

Hoofdstuk 5 bestudeert de efficiëntie van markten gedurende financiële crises en in het bijzonder de schending van de afgedekte rentegelijkheid. We vinden dat de valutabasis een convexe en neerwaarts lopende termijnstructuur heeft tijdens de crisis. We stellen voor om expliciet rekening te houden met liquiditeitsrisico bij het waarderen van valuta forward contracten. We laten zien hoe het afwikkelen van posities en de on-the-run/off-the-run spreiding bij kan dragen aan een niet-triviale valutabasis die overeenkomt met empirisch waargenomen waarden.

In alle gevallen geldt dat de voorspellingen van modellen die rekening houden met financiële crises kwalitatief anders zijn dan de voorspellingen van klassieke modellen. Het niet in ogenschouw nemen van verschillende crisisaspecten kan leiden tot substantiële verliezen voor zowel institutionele als individuele beleggers als een crisis zich daadwerkelijk manifesteert.

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