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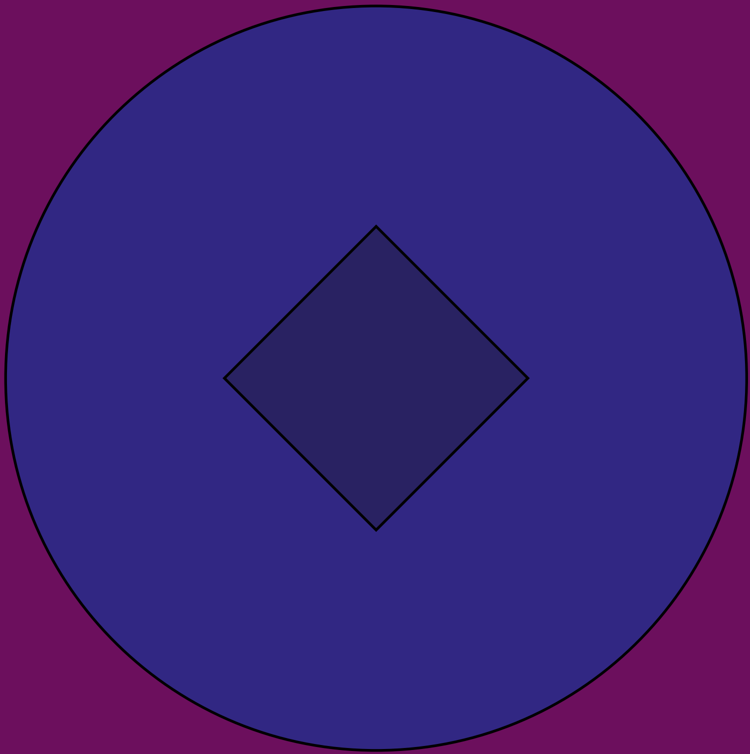

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CYCLIC THEORY OF LIE ALGEBROIDS

ACADEMISCH PROEFSCHRIFT

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Contents

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Introduction

Lie algebroids

The central objects of interest of this thesis are Lie algebroids. They play a prominent role in geometry as generalized infinitesimal symmetries of spaces. In the smooth category, it was Pradines who introduced the concept of a Lie algebroid in [Pr], and he studied its relation to the corresponding global symmetries of spaces, called Lie groupoids. However, the core idea, the study of vector fields preserving some geometry or PDE which are closed under the Lie bracket, has a much longer history and was initiated in the works of Lie and Cartan. Lie algebroids can be defined in different contexts, but they always play the role of generalized symmetries. In this thesis, we consider:

1. The differential geometric version; smooth Lie algebroids.
2. The algebraic notion of Lie–Rinehart algebras
3. The sheaf-theoretic version; sheaves of Lie algebroids over a ringed space.

The notion of Lie–Rinehart algebras is introduced by a number of authors under different names, see [M] for an extensive overview. Rinehart showed in [Rin] that the cohomology theory of Lie–Rinehart algebras can be subsumed in the theory of homological algebra, and he proved the analogue of the Poincaré–Birkhoff–Witt theorem for the associated universal object of a Lie–Rinehart algebra, which will be discussed later in this introduction. The first two sections of [BB] treat Lie algebroids in algebraic geometry, which are examples of sheaves of Lie algebroids over ringed spaces. To stay close to the geometric intuition, we mostly consider smooth Lie algebroids in this introduction:

Definition. *A smooth Lie algebroid is a vector bundle* $\pi : A \to M$*, together with the following data:*

1. *A smooth vector bundle map* $\rho : A \to TM$*, called the anchor map of the Lie algebroid.*
2. *A Lie bracket on the space of smooth sections* $\Gamma(A)$ *of* A *such that for all* $s, t \in A$ *and* $f \in C^\infty(M)$*, the following Leibniz identity holds:*

$$[s, ft]_A = L_{\rho(s)}(f)t + f[s, t]_A.$$

Here L *denotes the Lie derivative of vector fields.*

Perhaps the most well-known examples of smooth Lie algebroids are tangent bundles of smooth manifolds (where the anchor map is just the identity) and finite dimensional, real Lie algebras (where M is a point). Other examples of Lie algebroids are given by regular foliations, which correspond (by the Frobenius theorem) to Lie algebroids with an injective anchor map, actions of Lie algebras on a manifold M, and manifolds with a Poisson structure. Many of the concepts in the theory around Lie algebroids are generalizations of concepts in either the theory around tangent bundles or the theory around Lie algebras. For example, one has the A-de Rham complex

$$\Omega^0_A \xrightarrow{d_A} \Omega^1_A \xrightarrow{d_A} \Omega^2_A \xrightarrow{d_A} \dots$$

associated to a Lie algebroid A (so $d_A^2 = 0$), which both generalizes the Chevalley–Eilenberg complex of Lie algebras and the de Rham complex of smooth manifolds. The cohomology of this complex, denoted by $H^\bullet(A)$, is called Lie algebroid cohomology and reduces to de Rham cohomology and Lie algebra cohomology in the aforementioned examples. Dual to the A-de Rham forms, one has the A-polyvector fields T^A_{poly} with a graded Lie bracket $[\ ,\ ]_{\mathrm{SN}}$ -called the Schouten–Nijenhuis bracket, obtained as the natural extension of the Lie bracket on A. This generalizes the multivector vector fields $T_{\mathrm{poly}}(M)$ on a smooth manifold, and in fact all the structures which comprise the Cartan calculus on the pair $(T_{\mathrm{poly}}(M), \Omega(M))$: the wedge products on $T_{\mathrm{poly}}(M)$ and $\Omega(M)$, the insertion operator ι for polyvector fields in differential forms, the bracket $[\ ,\ ]_{\mathrm{SN}}$ on $T_{\mathrm{poly}}(M)$, the de Rham differential d and the Lie derivative $L = d\iota \pm \iota d$, can be defined in the same way for the pair $(T^A_{\mathrm{poly}}, \Omega_A)$. We refer to §1.1 for the precise definitions and a discussion of these constructions.

The universal enveloping algebra $\mathcal{U}(A)$ of a Lie algebroid A is the algebra generated by $\Gamma(A)$ and $C^\infty(M)$, subject to relations

$$[s,t]_A = st - ts, \qquad \rho(s)(f) = [s,f], \qquad f \cdot s = fs.$$

Consult §1.2 for a precise definition. It generalizes the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ associated to a Lie algebra $\mathfrak{g}$. For $A = TM$, the universal enveloping algebra is given by the differential operators $\mathrm{Diff}(M)$ on M. Like the universal enveloping algebra of a Lie algebra, it comes equipped with a natural ascending filtration. It is clear from this definition that the universal enveloping algebra is noncommutative when the Lie bracket and the anchor map of A are non trivial. When A is a Lie algebroid with trivial bracket and anchor, i.e. a vector bundle, one has $\mathcal{U}(A) = \mathrm{Sym}\,A$; the symmetric algebra bundle of A.

Finite dimensional Lie algebras arise as infinitesimal objects of Lie groups; more precisely as the collection of right (or left) invariant vector fields with respect to the right (or left) action of the group on itself. The notion of Lie groupoids, which generalizes smooth Lie groups, provides for an analogon of this construction, in the sense that they are smooth objects together with several operations to which one can associate infinitesimal, invariant vector fields which form smooth Lie algebroids. One can thus associate a Lie algebroid to each Lie groupoid, The converse statement however, which is Lie's third theorem in the case of Lie groups, does not hold. The precise obstructions to "integrability" are given in [CF1].

Cyclic theory

Given a unitary, associative algebra C, one can define its Hochschild and cyclic homology as well as their dual cohomology theories, commonly referred to as the "cyclic theory of an algebra". They are a fundamental ingredient in noncommutative geometry. Their definition is given in terms of explicit complexes, see §1.2.2, equipped with two differentials; the Hochschild differential b and the cyclic differential B. The latter was already contained in the work of Rinehart, c.f. [Rin], but lay dormant until it was rediscovered in the works of Connes [Co] and Feigin and Tsygan [FT]. The importance of cyclic theory is explained by considering the commutative case:

For a smooth commutative algebra C, the Hochschild homology $H_\bullet(C)$ of C is isomorphic to the algebraic de Rham forms [Rin], courtesy of the Hochschild–Konstant–Rosenberg theorem of [HKR], and the cyclic differential B corresponds to the de Rham differential d. The Hochschild cohomology $H^\bullet(C,C)$ of C is isomorphic to the algebraic polyvector fields on C, and one can define operations which give the pair $(\mathsf{H}^\bullet(\mathsf{C},\mathsf{C}),\mathsf{H}_\bullet(\mathsf{C}))$ the structure of a Cartan calculus, see [DTTs]. These operations correspond to the Cartan calculus structure on the algebraic polyvector fields and differential forms. When $\mathsf{C} = \mathsf{C}^\infty(\mathsf{M})$, one has to take into account the locally convex topology on the algebra $\mathsf{C}^\infty(\mathsf{M})$ in the definition of the cyclic theory, and Connes proved in [Co], that

$$(\mathsf{H}^\bullet(\mathsf{C},\mathsf{C}),\mathsf{H}_\bullet(\mathsf{C})) \cong (\mathsf{T}_{\mathrm{poly}}(\mathsf{M}),\Omega(\mathsf{M})).$$

The virtue of cyclic theory is that it is naturally defined for *noncommutative* algebras C, leading to "de Rham cohomology for noncommutative algebras". This can be applied to certain pathalogical spaces which do not have a smooth structure, but to which one can associate a natural noncommutative algebra. The cyclic theory of this algebra is a replacement for the usual Cartan calculus for smooth spaces. One of the earliest applications of cyclic theory is index theory for foliations, in which the cyclic homology is the recipient of the noncommutative Chern character, c.f. [Co].

In general, the cyclic theory of a noncommutative algebra is quite hard to compute, but in certain cases the result is known. In [Wo], Wodzicki computed the Hochschild and cyclic homology of the algebra of differential operators on a manifold, and Kassel [Ka] did the same for the universal enveloping algebra of a Lie algebra. Both of these examples are almost commutative, i.e. the algebras admit a filtration such that the associated graded algebras are commutative, and the associated graded algebras inherit a Poisson structure measuring the non commutativity of the original algebra. In the first case, this Poisson structure corresponds to the well-known symplectic structure on $\mathsf{T}^*\mathsf{M}$, and in the latter case it is given by the linear Poisson structure on $\mathfrak{g}^*$. In both cases the cyclic theory of the noncommutative algebra is computed in terms of Poisson (co)homology. A great part of this thesis is devoted to the cyclic theory of the universal enveloping algebra of a Lie algebroid, of which the two previous cases are examples. The Poisson structure on $\mathsf{A}^\vee$ plays a similar role in the computation of the cyclic theory of the universal enveloping algebra.

The Poincaré–Birkhoff–Witt theorem

The Poincaré–Birkhoff–Witt theorem for Lie algebras states that $\mathrm{gr}(\mathcal{U}(\mathfrak{g})) \cong \mathrm{Sym}(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$, where $\mathrm{Sym}(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. If $\mathbb{Q} \subset \mathbb{K}$, one can define the symmetrization map $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ inducing the isomorphism on the associated graded spaces. In [Rin], Rinehart proved in a more algebraic setting that the associated graded object to the universal enveloping algebra of a projective Lie–Rinehart algebra L is isomorphic to the symmetric algebra generated by L, which implies that $\mathrm{gr}(\mathcal{U}(\mathsf{A})) \cong \mathrm{Sym}(\mathsf{A})$ for a smooth Lie algebroid A. However, he did not construct a map $\mathrm{Sym}(\mathsf{A}) \longrightarrow \mathcal{U}(\mathsf{A})$ inducing this isomorphism. In [NWX] the concept of a local integrating groupoid was used define such a map in the smooth case.

We generalize the map of [NWX] to locally free sheaves of Lie algebroids $\mathcal{L}$ of constant rank r over a locally ringed space $(\mathsf{X},\mathcal{O}_\mathsf{X})$ over a field $\mathbb{K} \supset \mathbb{Q}$, provided they are *equipped with a global* $\mathcal{L}$*-connection* $\nabla^\mathcal{L}$ *on* $\mathcal{L}$. This condition is certainly satisfied for all smooth Lie algebroids. Roughly speaking, we replace the local integrating groupoid of [NWX] by the sheaf of $\mathcal{L}$-Jets over X

$$\mathcal{J}(\mathcal{L}) := \mathrm{Hom}_{\mathcal{O}_\mathsf{X}}(\mathcal{U}(\mathcal{L}),\mathcal{O}_\mathsf{X}),$$

Using the $\mathcal{L}$-connection, we define an exponential map which we use to prove the *dual* version of the PBW theorem:

Theorem. *Let* $\mathcal{L}$ *be a locally free sheaf of Lie algebroids of constant rank* r *together with an* $\mathcal{L}$*-connection* $\nabla^\mathcal{L}$*, then there exists a map*

$$\mathsf{j}_\nabla : \mathcal{J}(\mathcal{L}) \longrightarrow \widehat{\mathrm{Sym}}_{\mathcal{O}_\mathsf{X}}(\mathcal{L}^\vee),$$

which is an $\mathcal{O}_X$-linear isomorphism of sheaves of algebras with respect to the first $\mathcal{O}_X$-module structure on $\mathcal{J}(\mathcal{L})$.

Dualizing this result gives an isomorphism $\mathrm{Sym}(\mathcal{L}) \to \mathcal{U}(\mathcal{L})$ which respects the coalgebra structures. We give a recursive relation that defines this map, and this relation turns out to be the same as the one in [LSX14]. In the applications of the PBW theorem that we consider; the construction of a trace density map and a formality result, we use the version of the PBW theorem as stated in the theorem, so in our view this is the more fundamental result. Also, the definition of the morphism is natural and easy, which makes the study of the properties of the PBW map more accessible.

Cyclic theory of the universal enveloping algebra

Extending the results of Kassel and Wodzicki, we compute the cyclic theory of the universal enveloping algebra of a smooth Lie algebroid $A \to M$. First we introduce the structures that are needed to phrase the main results.

Representations up to homotopy
Representations of Lie algebroids $A \to M$ are given by flat Lie algebroid connections on vector bundles $E \to M$. In this sense, Lie algebroids do not have many representations. For example, there is no adjoint representation in general. A more flexible notion of a *representation up to homotopy* was defined in [AC]. Informally speaking, the vector bundle E is replaced by a complex of vector bundles $\ldots \stackrel{\partial}{\to} E^i \stackrel{\partial}{\to} E^{i+1} \stackrel{\partial}{\to} \ldots$, and the flat A connection on E is replaced by not necessarily flat connections on each bundle E^i which are compatible with the maps ∂, i.e. $\nabla\partial - \partial\nabla = 0$. The data is completed by a collection of tensors satisfying relations which imply that, together with ∂ and ∇, they form a derivation D of degree 1 which squares to zero on the total complex $\Omega_A(\bigoplus_i E^i)$. Such differentials are also known as *flat superconnections* on a $\mathbb{Z}$-graded bundle and they were introduced in the $\mathbb{Z}/2$-graded case by Quillen in [Qu]. One of the virtues of this definition is that for any smooth Lie algebroid A, the adjoint representation of A, denoted by $\mathrm{ad}\, A$, can be defined. Although it depends on the choice of a connection on A, the associated cohomology is canonical. Moreover, the category of representations up to homotopy is symmetric and monoidal, hence one can define $\mathrm{Sym}(\mathrm{ad}\, A)$, the symmetric algebra of the adjoint representation of A. One of the main results of this thesis computes the Hochschild cohomology and homology of the universal enveloping algebra.

Theorem. *Let A be a smooth Lie algebroid of rank r over a smooth manifold of dimension n with universal enveloping algebra $\mathcal{U}(A)$.*

$i)$ *For the Hochschild cohomology, there is a natural isomorphism*

$$H^\bullet(\mathcal{U}(A), \mathcal{U}(A)) \cong H^\bullet_{\mathrm{Lie}}((\mathrm{Sym}(\mathrm{ad}\, A)).$$

$ii)$ *When both M and A are orientable, there is a natural isomorphism*

$$H_\bullet(\mathcal{U}(A)) \cong H^{n+r-\bullet}_{\mathrm{Lie}}(\mathrm{Sym}(\mathrm{ad}\, A) \otimes Q_A)$$

for the Hochschild homology.

The bundle Q_A is a natural representation of A, sometimes referred to as the bundle of transverse volume forms. In the special cases $A = TM$ and $A = \mathfrak{g}$ we recover exactly the results of Wodzicki and Kassel, c.f. [Ka, Wo]. Let us give an informal discussion of the proof. For simplicity we restrict ourselves to part $i)$ of the theorem.

Discussion of the proof: step 1
The PBW theorem induces a natural Poisson structure on the associated graded algebra of

$\mathcal{U}(A)$, and the differential of the first page of the spectral sequence that can be associated to the filtration on the Hochschild cocomplex of $\mathcal{U}(A)$ turns out to be the Poisson cohomology differential associated with this Poisson structure, c.f. [Br].

Our first step is to identify the Poisson cohomology complex associated to the linear Poisson structure on the dual of A with the complex $(\Omega_A(\mathrm{Sym}(\mathrm{ad}\,A), D)$. In the second step we construct a quasi-isomorphism between the Hochschild cohomology complex for $\mathcal{U}(A)$ and the Poisson cohomology complex of the aforementioned Poisson structure.

Let us first discuss the first step. We start with the observation that sections of $\mathrm{Sym}\,A$ correspond to functions on $A^\vee$ which are polynomial along the fibers. The Lie bracket and the anchor map ρ define a Poisson structure on $\mathrm{Sym}\,A$ in a natural way, and this can be extended to all smooth functions on $A^\vee$. The derivations $L := \mathrm{Der}_\mathbb{R}(\mathrm{Sym}\,A)$, which are a model for the vector fields on $A^\vee$ which are polynomial along the fibers of the projection $A^\vee \to M$, form a sheaf of Lie algebroids over the locally ringed space $(M, \mathrm{Sym}\,A)$, and it fits into the following short exact sequence of sheaves of Lie algebroids over $(M, \mathrm{Sym}\,A)$:

$$0 \longrightarrow \mathrm{Sym}\,A \otimes A^\vee \longrightarrow \mathrm{Der}_\mathbb{R}(\mathrm{Sym}\,A) \longrightarrow \mathrm{Sym}\,A \otimes TM \longrightarrow 0.$$

The choice of a connection on A splits this sequence, and such a splitting defines a Lie bracket on $\mathrm{Sym}\,A \otimes (A^\vee \oplus TM)$. One can do the same for the sheaves of polyvector fields $T^L_{\mathrm{poly}}(A^\vee)$ and differential forms $\Omega_L(A^\vee)$ associated to L. These are models for the polyvector fields and differential forms on $A^\vee$ which are polynomial along the fibers. The linear Poisson structure on $\mathrm{Sym}\,A$ corresponds to a Maurer–Cartan element $\theta \in T^L_{\mathrm{poly}}(A^\vee)$, and we show that under the identification induced by the connection:

- The Poisson cohomology differential $[\theta,\ ]_{\mathrm{SN}}$ corresponds to the differential D of the complex $\Omega^\bullet(\mathrm{Sym}(\mathrm{ad}\,A))$ of the symmetric algebra of the adjoint representation.
- The Poisson homology differential L_θ corresponds to the differential D of the complex $\Omega^\bullet(\mathrm{Sym}(\mathrm{ad}\,A) \otimes Q_A)$

The linear part of the identification between the Poisson cohomology complex and the adjoint representation of A was already contained in [CM].

Step 2: formality

The second step relies on the formality theorem for Lie algebroids. Let us first bring the original formality theorem that Kontsevich proved in [K] to the attention of the reader. Given a smooth manifold, M one can construct the complex of polydifferential operators $D_{\mathrm{poly}}(M)$, which admits a graded Lie algebra bracket $[\ ,\ ]_G$ and a Hochschild differential $[m,\ ] = b^H$. Hence, it is a differential graded Lie algebra (dg Lie algebra). It forms a local model for the Hochschild cohomology of $C^\infty(M)$. By the smooth HKR theorem, the symmetrization map

$$\mathrm{HKR}:\ T_{\mathrm{poly}}(M) \longrightarrow D_{\mathrm{poly}}(M)$$

defines an isomorphism on the cohomology, however it does not respect the Lie algebra structures $[\ ,\ ]_{\mathrm{SN}}$ and $[\ ,\ ]_G$. The formality theorem states that this defect can be cured if one considers the more flexible notion of L_∞-morphisms; i.e. Kontsevich proved the existence of an L_∞-quasi-isomorphism

$$\mathcal{K}:\ T_{\mathrm{poly}}(M) \longrightarrow D_{\mathrm{poly}}(M).$$

One of the main features of L_∞-morphisms is that they relate Maurer–Cartan elements. If one introduces a formal variable $\hbar$ on both sides, Maurer–Cartan elements in the LHS are given by (formal) Poisson structures on M, and on the RHS they define formal deformations of $C^\infty(M)$, i.e. $\star$-products on $C^\infty(M)[[\hbar]]$. Given a Maurer–Cartan element $\pi \in T_{\mathrm{poly}}(M)$, one can twist the L_∞-morphism $\mathcal{K}$ by this element, which gives a morphism of complexes:

$$\mathcal{K}^\pi:\ (T_{\mathrm{poly}}(M)[[\hbar]],\ [\pi,\ ]_{\mathrm{SN}}) \longrightarrow \left(D_{\mathrm{poly}}(M)[[\hbar]],\ b^H + [\tilde{\pi},\ ]_G\right).$$

Moreover, there exists a natural map from the complex on the RHS to the Hochschild cohomology complex of the deformed algebra $(C^\infty(M)[[\hbar]], \star)$.

In the appendix we revisit the formality theorem for a certain type of sheaves of Lie algebroids, which encompasses the sheaf of Lie algebroids $L = \mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\,A)$ over $(M, \mathrm{Sym}\,A)$. The proof that we give stipulates the role of the Poincaré–Birkhoff–Witt theorem for Lie algebroids. The result is an L_∞-quasi-isomorphism of dg Lie algebras:

$$U : T^L_{\mathrm{poly}}(A^\vee) \longrightarrow D^L_{\mathrm{poly}}(A^\vee)$$

with the RHS given by the L-polydifferential operators on Sym A. We twist this L_∞-morphism by the Maurer–Cartan element θ:

$$U^\theta : \left(T^L_{\mathrm{poly}}(A^\vee), [\theta, \ ]_{SN}\right) \longrightarrow \left(D^L_{\mathrm{poly}}(A^\vee), b^H + [\tilde\theta, \ ]_G\right).$$

Then we show that the deformation of Sym A which is associated to the element $\tilde\theta \in D^L_{\mathrm{poly}}(A^\vee)$ is isomorphic to $\mathcal{U}(A_\hbar)$, where $A_\hbar$ is the adiabatic Lie algebroid associated to A c.f. [NWX]. In particular, setting $\hbar = 1$, we obtain a natural morphism:

$$\left(D^L_{\mathrm{poly}}(A^\vee)[[\hbar]], \ b^H + [\theta, \ ]_G\right) \longrightarrow \left(C^\bullet(\mathcal{U}(A), \mathcal{U}(A)), \ b^H\right).$$

Finally, the formality map composed with this morphism gives a map

$$\left(T^L_{\mathrm{poly}}(A^\vee), [\theta, \ ]_{SN}\right) \longrightarrow C^\bullet(\mathcal{U}(A), \mathcal{U}(A), b^H).$$

This map respects the natural filtrations on both complexes and using a spectral sequence argument we show that the morphisms are in fact quasi-isomorphisms, which implies that

$$H^\bullet_\theta(A^\vee) \cong H^\bullet(\mathcal{U}(A), \mathcal{U}(A)),$$

where the LHS is the cohomology of the complex $\left(T^L_{\mathrm{poly}}(A^\vee), [\theta, \ ]_{SN}\right)$, i.e. the Polynomial Poisson cohomology of $A^\vee$. Combining this result with step 1) gives part i) of the theorem. The dual version of the formality theorem which is, in the local case, proved by Shoikhet in [S], leads to a proof of part ii).

The trace density map

The notion of a trace density map comes from deformation quantization and the algebraic index theorem proved by Nest and Tsygan in [NeTs95]. The same authors described the relation between the algebraic index theorem and the analytic index theorem in [NeTs96]. We refer to the introduction of chapter 4 for a longer discussion. From now on, we assume that $\mathcal{L}$ is either the sheaf of sections of a holomorphic Lie algebroid or a smooth Lie algebroid. The main difference is that the former does, in general, not admit global $\mathcal{L}$-connections on $\mathcal{L}$. To phrase the main theorem of the last chapter, we use the language of derived categories:

Theorem. *Let $\mathcal{L}$ be a either a smooth or a holomorphic Lie algebroid and $\mathcal{E}$ a locally free $\mathcal{L}$-module.*

i) *In the derived category $\mathcal{D}(X)$ of $\mathbb{K}$-sheaves on X there exists a canonical morphism*

$$\Phi \in \mathrm{Hom}_{\mathcal{D}(X)}\left(\left(CC_\bullet(\mathcal{U}(\mathcal{L})), b + u^{-1}B\right), \left(\mathrm{Tot}\Omega^{2r-\bullet}_{\mathcal{L}}, \ d_{\mathcal{L}}\right)\right).$$

ii) *Restricted to the structure sheaf $\mathcal{O}_X \subset \mathcal{U}(\mathcal{L}, \mathcal{E})$, the following diagram commutes:*

$$\begin{array}{ccc} \left(CC_\bullet(\mathcal{O}_X), \ b + u^{-1}B\right) & \xrightarrow{\mathrm{HKR}^u_{\mathcal{L}}} & \left(\Omega^\bullet_{\mathcal{L}}, \ u^{-1}d_{\mathcal{L}}\right) \\ \big\downarrow i & & \big\downarrow \cup \mathrm{Td}_{\mathcal{L}}\mathrm{Ch}_{\mathcal{L}}(\mathcal{E}) \\ \left(CC_\bullet(\mathcal{U}(\mathcal{L}, \mathcal{E})), \ b + u^{-1}B\right) & \xrightarrow{\Phi} & \left(\Omega^{2r-\bullet}_{\mathcal{L}}, \ d_{\mathcal{L}}\right). \end{array}$$

Let us clarify this theorem. When $\mathcal{L} = A$ is smooth, part i) implies that there exists a canonical morphism

$$\Phi:\ \mathrm{HP}_\bullet(\mathcal{U}(A)) \longrightarrow \mathrm{H}_{\mathrm{Lie}}(A)$$

from the periodic cyclic homology (part of the "cyclic theory") of $\mathcal{U}(A)$ to the Lie algebroid cohomology of A.

When $\mathcal{L} = \mathcal{A}$ is the sheaf of sections of a holomorphic Lie algebroid, part i) of the theorem implies that there is a canonical morphism between the hyper(co)homology

$$\mathbb{H}_\bullet(\mathrm{CC}_\bullet(\mathcal{U}(\mathcal{A})),\ b + u^{-1}B) \longrightarrow \mathbb{H}^{2r-\bullet}(\Omega_{\mathcal{A}},\ d_{\mathcal{A}}).$$

The RHS is the Lie algebroid cohomology of $\mathcal{A}$, and the LHS is the periodic cyclic homology of the sheaf $\mathcal{U}(\mathcal{A})$. In the holomorphic case part ii) of the theorem measures the failure of the trace density map to be compatible with the $\mathrm{HKR}_{\mathcal{A}}$ map in terms of Lie algebroid characteristic classes. The hyper(co)homology of both sides can be computed by Dolbeault-type resolutions, defined in [LSX12].

Outline of the chapters

Let us give a brief description of what is done in each chapter. We start each chapter with a more extensive description of its contents.

Chapter 1 In the first chapter we define Lie algebroids from three different points of view; smooth Lie algebroids, Lie–Rinehart algebras and sheaves of Lie algebroids. Moreover, we give three equivalent definitions of the structure of a Lie algebroid, after which we briefly discuss Lie algebroid cohomology and give some examples.. We finish with the definitions of the cyclic theory of an algebra.

Chapter 2 The second chapter is devoted to the PBW theorem for Lie algebroids. We start with a discussion of the various structures on the universal enveloping algebra and the jet algebra associated to a Lie algebroid. Then we prove the PBW theorem. Finally, we discuss the time-dependence of the PBW theorem given a linear combination of connections and incorporate the so-called twisting by a locally free $\mathcal{O}_X$-module $\mathcal{E}$.

Chapter 3 In this chapter we compute the cyclic theory of the universal enveloping algebra of a smooth Lie algebroid. It starts with identification of the Poisson (co)homology complexes for the linear Poisson structure on $A^\vee$ with the complex of the symmetric algebra of the adjoint representation of A (twisted by the bundle Q_A). Then we discuss the application of the formality map to prove the main results.

Chapter 4 We discuss the construction of the trace density morphism for Lie algebroids. It starts with the notion of a Fedosov resolution and the application of the PBW theorem. Since global connections are not necessarily given, we have to use Čech resolution of complexes, hence the trace density morphism is defined in the derived category. To prove that it commutes with the differentials we study the time dependence of the PBW theorem in more detail. Finally, we introduce characteristic classes and compute the class of 1 under the trace density map, and discuss the compatibility with the HKR map.

Chapters 2 and 4 are based on the preprint [BP] and chapter 3 is based on a forthcoming paper.

Chapter 1

Preliminaries

1.1 Lie algebroids

The central objects of interest of this thesis are Lie algebroids. In general, Lie algebroids can be defined as sheaves of Lie algebroids over a ringed space, encompassing for example smooth Lie algebroids, holomorphic Lie algebroids and algebraic Lie algebroids. We will also define the notion of Lie–Rinehart algebras, which are the algebraic counterparts of Lie algebroids, and we will discuss the various relations between smooth Lie algebroids, Lie–Rinehart algebras and sheaves of Lie algebroids.

Perhaps the most well-known examples of smooth Lie algebroids are tangent bundles of smooth manifolds and finite dimensional, real Lie algebras. The theory around smooth Lie algebroids, and more generally sheaves of Lie algebroids, is for a great deal analogous to the theory of tangent bundles and the theory of Lie algebras. For example, one has the de Rham complex associated to a Lie algebroid, which both generalizes the Chevalley–Eilenberg complex of Lie algebras and the de Rham complex of smooth manifolds, and thus Lie algebra cohomology and de Rham cohomology are special instances of Lie algebroid cohomology. Moreover, analogous to the dg Lie algebra of multivector vector fields of a smooth manifold, one can define the dg Lie algebra of A-polyvector fields for a given Lie algebroid A. After reviewing these constructions, we will give three equivalent definitions of the structure of a sheaf of Lie algebroids on a given $\mathcal{O}_X$-module $\mathcal{L}$, where $(X, \mathcal{O}_X)$ is a ringed space.

Finite dimensional Lie algebras arise as infinitesimal objects of Lie groups; more precisely as the collection of right (or left) invariant vector fields with respect to the right (or left) action of the group on itself. The notion of Lie groupoids, which generalizes smooth Lie groups, provides for an analogon of this construction, in the sense that they are smooth objects together with several operations to which one can associate infinitesimal, invariant vector fields which form smooth Lie algebroids. Lie's third theorem however, that gives a unique correspondence between real, finite dimensional Lie algebras and simply connected Lie groups, cannot be extended to the realm of Lie groupoids and Lie algebroids, because not all smooth Lie algebroids admit an integration.

1.1.1 Three types of Lie algebroids

In this thesis we discuss three types of Lie algebroids: the differential geometrical variant, which are given by smooth Lie algebroids; the algebraic notion of Lie–Rinehart algebras and the sheaf-theoretic version, provided by sheaves of Lie algebroids over ringed spaces. We start with the smooth case. Let M be a smooth manifold.

Definition 1.1.1. *A smooth Lie algebroid is a vector bundle* $\pi : A \to M$ *together with the following data:*

- *A smooth vector bundle map* $\rho : A \to TM$.
- *A Lie bracket on the space of smooth sections* $\Gamma(A)$ *of* A *such that for all* $\alpha, \beta \in A$ *and* $f \in C^\infty(M)$, *the follolwing Leibniz identity holds:*

$$[\alpha, f\beta]_A = L_{\rho(\alpha)}(f)\beta + f[\alpha, \beta]_A.$$

where L *denotes the usual Lie derivative of vector fields.*

Remark 1.1.2.
1) A computation in [Mar], involving the Jacobi identity for the bracket $[\ ,\ ]_A$, shows that ρ induces a Lie algebra morphism $\rho : \Gamma(A) \to \Gamma(TM)$ on spaces of sections, where the RHS is endowed with the well-known Lie bracket of vector fields.
2) A computation involving the Leibniz identity, for which we again refer to [Mar] shows that the Lie bracket is *local*; this means that the value $\{\alpha, \beta\}(x)$ for two sections $\alpha, \beta \in \Gamma(A)$ at a point $x \in M$ is determined by the first order jets of α and β. In partcicular, this means that the bracket can equivalently defined as a bracket on the sheaf of sections of A.

Example 1. The tangent bundle of M with the Lie bracket given by the commutator of vector fields, and the anchor map equal to the identity is a smooth Lie algebroid.

Example 2. The same holds for a finite dimension real Lie algebra $\mathfrak{g}$ over a point $\{pt\}$ with the Lie bracket of $\mathfrak{g}$ and the trivial anchor map.

The algebraic counterpart of Lie algebroids are given by Lie–Rinehart algebras, [Rin]. Their definition reads as follows:

Definition 1.1.3. *Given a commutative ring* $\mathbb{K}$, *a Lie–Rinehart algebra* (L, R) *is given by a commutative* $\mathbb{K}$*-algebra* R *and a Lie algebra* $(L, [\ ,\ ])$ *over* $\mathbb{K}$, *which is a left* R*-module and which acts on* R *by derivations. Denoting the Lie bracket by* $[\ ,\ ]$, *the module structure by* $r \otimes l \mapsto rl$, *and the action by* $l \otimes r \mapsto l(r)$ *for* $r \in R$ *and* $l \in L$, *the structures are compatible in the following way:*

$$\begin{aligned}(r_1 l)(r_2) =& r_1(l_1(r_2)),\\ [l_1, rl_2] =& l_1(r)l_2 + r[l_1, l_2].\end{aligned}$$

Remark 1.1.4. In this thesis we assume that R is commutative and unital, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ and L is projective over R. Many of the constructions for Lie–Rinehart algebras can be done in greater generality, i.e. when $\mathbb{K}$ is replaced by a commutative, unital ring k. The condition that L is projective implies the PBW theorem of [Rin], and, under the extra condition that L has constant, finite rank, is used in [Hue] and references therein to prove duality results for (co)homology. Lie–Rinehart algebras are also called Lie–Rinehart pairs.

Example 3. A standard example of a Lie–Rinehart algebra is given by a commutative ring R and the Lie algebra of derivations $\mathrm{Der}_k(R)$ of R with the commutator bracket. This is the algebraic counterpart of the tangent bundle example.

Example 4. The global sections $L = \Gamma(A)$ of a smooth Lie algebra $A \to M$ and the algebra of smooth functions $R = C^\infty(M)$ form a Lie–Rinehart algebra $(\Gamma(A), C^\infty(M))$. Conversely, due to the Serre–Swan theorem, any finitely generated, projective module L over $C^\infty(M)$ such that $(L, C^\infty(M))$ is a Lie–Rinehart algebra, is given by the sections of a smooth Lie algebroid.

The most general form of Lie algebroids that we will study consists of sheaves of Lie algebroids over a given (locally) ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_X$ is a sheaf of $\mathbb{K}$-algebras. A detailed introduction into Lie algebroids in algebraic geometry, which are examples of sheaves of Lie algebroids over ringed spaces, can be found in the first two sections of [BB].

Definition 1.1.5. *A Lie algebroid $\mathcal{L}$ over $(X, \mathcal{O}_X)$ is both a sheaf of $\mathcal{O}_X$-modules and a sheaf of $\mathbb{K}$-Lie algebras over X, which acts by derivations on $\mathcal{O}_X$. The compatibilities between these structures are given by*

$$f_1(l_1(f_2)) = f_1 l_1(f_2)$$
$$[l_1, f_1 l_2] = l_1(f_1)l_2 + f_1[l_1, l_2]$$

for all $f_1, f_2 \in \mathcal{O}_X$ and $l_1, l_2 \in \mathcal{L}$, where we used similar notations as in Definition 1.1.3.

Remark 1.1.6. We assume that $\mathcal{L}$ is locally free and of finite, constant rank over $\mathcal{O}_X$. There are interesting examples of sheaves of Lie algebroids that do not have this property, e.g. in [P] Lie algebroids are defined as coherent sheaves over analytic spaces.

Let us now give some examples and relations between smooth Lie algebroids, Lie–Rinehart algebras and sheaves of Lie algebroids.

Examples

1. Any smooth Lie algebroid $A \to M$ gives rise to the Lie–Rinehart algebra $(\Gamma(A), C^\infty(M))$ given by the global sections and to a sheaf of Lie algebroids Γ_A over the locally ringed space (C^∞_M, M), where C^∞_M is the sheaf of smooth functions on M.

2. Given a Lie–Rinehart algebra (L, R), we denote by $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ the associated locally ringed space to R. The R-module L defines an $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\mathcal{L}$, and $\mathcal{L}$ is a sheaf of Lie algebroids over $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$. Conversely, the algebraic version of the Serre–Swan theorem states that any finitely generated, projective module L over R is given by the global sections of a locally free $\mathcal{O}_{\mathrm{Spec}(R)}$-module of finite rank. This implies that locally free Lie algebroids of constant rank over $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ correspond to finitely generated, projective Lie–Rinehart algebras L over R, i.e. the structure maps also correspond.

3. *Holomorphic Lie algebroids.* Given a complex manifold X, a holomorphic vector bundle $\mathcal{A} \to X$ is a holomorphic Lie algebroid if the sheaf of sections $\Gamma_{\mathcal{A}}$ of $\mathcal{A}$ has the structure of a sheaf of complex Lie algebras together with a holomorphic bundle map $\rho : \mathcal{A} \to TX$, such that
$$[X_1, fX_2] = L_{\rho(X_1)}(f)X_2 + f[X_1, X_2]$$
for all $f \in \mathcal{O}_X(U)$, $X_1, X_2 \in \mathcal{A}(U)$ and all $U \subset X$. The pair $(\Gamma(\mathcal{A}), \mathcal{O}(X))$ given by the global sections of $\mathcal{A}$ and the global holomorphic functions on X is a Lie–Rinehart algebra, but it is better to view them as sheaves of Lie algebroids, since holomorphic Lie algebroids cannot be constructed from their global sections in general.

4. *Foliations.* Given a smooth manifold M, an involutive subbundle $F \subset TM$, i.e. a subbundle which is closed under the Lie bracket, forms a Lie algebroid with the anchor map equal to the natural inclusion $i : F \to TM$. By the Frobenius theorem it is integrable, which means that M is equipped with a regular foliation such that F is given by the vector fields that are tangent to the foliation c.f. [MoMr] for an extensive discussion. This example extends to the holomorphic category: Given complex manifold X, an involutive subbundle of the sheaf T_X of holomorphic vector fields is a holomorphic Lie algebroid over X.

5. *Action algebroids.* Let $\mathfrak{g}$ be a finite dimensional real Lie algebra equipped with an action on M, i.e. a Lie algebra morphism $a: \mathfrak{g} \to \Gamma(TM)$. The trivial vector bundle $\mathfrak{g} \times M$ admits the structure of a Lie algebroid as follows. The anchor map on constant sections is given by the action a, and by $C^\infty(M)$-linearity this determines the anchor map on all smooth sections $\Gamma(\mathfrak{g} \times M)$. For constant sections $\tilde{s}, \tilde{t}$, corresponding to elements $s, t \in \mathfrak{g}$ the Lie algebroid bracket is defined as $\widetilde{[s,t]}$, and this determines the bracket on all smooth sections $\Gamma(\mathfrak{g} \times M)$ by the Leibniz identity. This Lie algebroid is called the *action algebroid* of the action of $\mathfrak{g}$ on M and denoted by $\mathfrak{g} \ltimes M$. Again, this example works in the holomorphic category as well: Given a complex Lie algebra $\mathfrak{g}$, and a Lie algebra morphism $a: \mathfrak{g} \to \Gamma(T_X)$ to the space of global holomorphic vector fields, one can equip the trivial bundle $\mathfrak{g} \times X$ in a similar fashion with the structure of a holomorphic Lie algebroid.

6. *Poisson structures.* Recall that a Poisson manifold is a smooth manifold P together with a Poisson bracket $\{\ ,\ \}$ on the smooth functions $C^\infty(P)$. A Poisson bracket is a bilinear operation which is a Lie bracket and satisfies the Leibniz rule:

$$\{fg, h\} = f\{g, h\} + g\{f, h\}, \qquad f, g, h \in C^\infty(P).$$

This bracket is equivalent to a bivector field $\Pi \in \Gamma(\Lambda^2 TP)$ characterized by $\Pi(df, dg) = \{f, g\}$, which satisfies $[\Pi, \Pi]_{SN} = 0$. The Poisson structure defines a Lie algebroid structure on T^*P as follows: The anchor map ρ is contraction by the Poisson tensor, i.e.

$$\rho(\alpha)(\beta) = \Pi(\alpha, \beta), \quad \alpha, \beta \in \Omega_1(P)$$

and the algebroid bracket is given by

$$[\alpha, \beta]_{T^*P} := L_{\rho(\alpha)}(\beta) - L_{\rho(\beta}(\alpha) - d(\Pi(\alpha, \beta)).$$

Locally, i.e. when we can write $\alpha = df$ and $\beta = dg$, the anchor and brackets are given by

$$\rho(df) = X_f \qquad\qquad [df, dg]_{T^*P} = \{f, g\}.$$

where X_f is the Hamiltionian vector field associated to f with respect to the Poisson structure, and in fact these equations determine the bracket globally. Given a complex manifold X, a holomorphic Poisson structure is defined as a Poisson structure on the sheaf of holomorphic functions, i.e. for each open $U \subset X$ the algebra $\mathcal{O}_X(U)$ of holomorphic functions is a Poisson algebra. It is characterized by a global section of the sheaf of holomorphic biderivations of X, which we denote by $\Gamma(\Lambda^2 T_X)$. The formulas that define a Lie algebroid structure on $T_X^\vee$, the sheaf of holomorphic one forms on X, carry over from the smooth case without adaptations.

7. *Atiyah algebroids.* Given a smooth vector bundle $E \to M$, the subspace $\mathrm{Diff}^{\leqslant 1}(\Gamma(E)) \subset \mathrm{End}_{\mathbb{R}}(\Gamma(E))$ of order 1 differential operators is defined by

$$h \in \mathrm{Diff}^{\leqslant 1}(\Gamma(E)) \Leftrightarrow [h, f] \in \mathrm{End}(E) \quad \forall f \in C^\infty(M).$$

The map $f \mapsto [h, f]$ is a derivation and thus it defines an element $\sigma(h) \in TM \otimes \mathrm{End}(E)$. It defines the symbol map

$$\sigma: \mathrm{Diff}^{\leqslant 1}(\Gamma(E)) \longrightarrow TM \otimes \mathrm{End}(E), \qquad h \to \sigma(h),$$

which is a $C^\infty(M)$-linear map of vector bundles with kernel $\mathrm{End}(E)$. A TM-connection (or affine connection) on E, which we will define in the next subsection, defines a splitting of the exact sequence

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{Diff}^{\leqslant 1}(\Gamma(E)) \longrightarrow TM \otimes \mathrm{End}(E) \longrightarrow 0$$

by the formula $X \otimes e \mapsto \nabla_X e$ where $X \otimes e \in TM \otimes \mathrm{End}(E)$, and this gives the middle $C^\infty(M)$-module the structure of a vector bundle. The Atiyah algebroid $\mathrm{At}(E)$ is defined as the subbundle of $\mathrm{Diff}^{\leqslant 1}(\Gamma(E))$ given by differential operators with scalar symbol, i.e. it fits into the exact sequence

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{At}(E) \longrightarrow TM \longrightarrow 0. \tag{1.1.1}$$

The anchor map is provided by the sequence above, and the Lie bracket is given by the commutator of differential operators. It is easy to check that the commutator of two operators with scalar symbol again has a scalar symbol, so this is well-defined. Again, the definition in the holomorphic category is the same; then the Atiyah algebroid is given as the sheaf of order 1 differential operators with scalar symbol.

8. *Lie algebroids associated with hypersurfaces.* Let N be a hypersurface in smooth manifold M. Let $\Gamma(TM(-\log N))$ be the space of global vector fields that are tangent to N. It is a $C^\infty(M)$-module which acts by derivations on $C^\infty(M)$, and one can prove, c.f. [Me] for a proof in the case of manifolds with boundary, that it is projective, i.e. it is given by the sections of a vector bundle. We denote this vector bundle by $TM(-\log N)$. The Lie bracket is given by the commutator of vector fields and the anchor map is given by the inclusion. Let $(x, y_2, \dots y_n)$ be local coördinates on M such that N is defined by $x = 0$. Then it is clear that
$$x\frac{\partial}{\partial x}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_n}$$
forms a local basis for $TM(-\log N)$. Hence, when $x > 0$, the image of $x\partial/\partial x$ under the anchor map evaluates to 1 in the differential form $d(\log(x))$.

   In the holomorphic setting the definition is as follows. Let $D \subset X$ be a smooth holomorphic hypersurface. The $\mathcal{O}_X$-module $T_X(-\log D)$ is defined as the sheaf of sections of T_X which preserve functions that vanish on D, i.e. when f is a function such that $f|_D = 0$, then $V \in T_X(-\log D)$ iff $V(f)|_D = 0$. When D is not smooth, the definition still applies and it gives an involutive subbundle of T_X, however it will not be locally free in general, see [P] for more details. The condition that D has normal crossing singularities, i.e. one can find coördinates $x_1, \dots, x_k$ around singularities of D such that D is defined by $x_1 x_2 \cdots x_k = 0$, implies that $T_X(-\log D)$ is locally free.

9. *Lie algebroids associated to Lie groupoids.* In this example we discuss Lie groupoids, which can be viewed as generalized, global symmetries of spaces. First we give the definition of a smooth Lie groupoid. We refer to [M] for more details and many examples of Lie groupoids. Replacing all the adjective 'smooth' by 'holomorphic' in the following definition defines the notion of a holomorphic Lie Groupoid. Moreover, one can associate a holomorphic Lie algebroid to a holomorphic Lie groupoid, and the examples that we discuss are also defined in the holomorphic category.

   Definition 1.1.7. *A Lie groupoid is given by a pair of manifolds* (G_1, G_0) *and a collection of smooth maps*
$$s, t\colon G_1 \to G_0, \qquad u\colon G_0 \to G_1, \qquad \mathrm{inv}\colon G_1 \to G_1, \qquad m\colon G_1 {}_s\times_t G_1 \to G_1.$$
   called the source map s*, the target map* t*, the unit map* u*, the inverse* inv *and the multiplication* m*. The source and target map are submersions (hence* $G_1 {}_s\times_t G_1$ *is a smooth manifold). Moreover, the pair* (G_1, G_0) *is given by the set of arrows and the set of objects of a category in which all arrows are invertible, and the structure maps of this category coincide with* s, t, u, inv *and* m*.*

We remark that G_1 is not required to be Hausdorff, as opposed to G_0 and s and t fibers. Note that the fibers of s and t are smooth manifolds. It is customary to write $G_1 \rightrightarrows G_0$ to denote a Lie groupoid and to call points of G_1 arrows and points of G_0 objects.

To each Lie groupoid one can associate a Lie algebroid over G_0 which encodes much of the information of the Lie groupoid. The construction is as follows: We denote the subbundle $\ker(Ts) \subset TG$ which fiber consists of vectors in TG which are tangent to the s-fibers by T^sG, and claim that

$$\mathrm{id}^*(T^sG),$$

the pull-back bundle of T^sG_1 under the map id, is a Lie algebroid over G_0. An element $g \in G_1$ defines a smooth diffeomorphism of fibers:

$$R_g : s^{-1}(y) \longrightarrow s^{-1}(x)$$

where $x = s(g)$ and $y = t(g)$. Because it is a diffeomorphism, it induces an isomorphism

$$TR_g : T^s_h(s^{-1}(y)) \longrightarrow^{\cong} T^s_{hg}(s^{-1}(y))$$

where $s(h) = y$. It follows that a section $X \in \Gamma(\mathrm{id}^*(T^sG_1))$ can be uniquely extended to a right invariant vector field X^{inv} in T^sG_1 by the formula

$$X^{\mathrm{inv}}_g := TR_g(X_{\mathrm{id}(y)})$$

The Lie bracket of two invariant vector fields on G_1 is again an invariant vector field, hence one obtains a well-defined bracket on sections of $\mathrm{id}^*(T^sG_1)$. The anchor map is given by $\rho = Tt$. More precisely, given a section $X \in \Gamma(\mathrm{id}^*(T^sG_1))$, the anchor is given by $Tt(X^{\mathrm{inv}})$, which is well-defined since X^{inv} is invariant. The fact that it satisfies the Leibniz identity follows from the Leibniz identity for vector fields on G_1.

10. *Some concrete examples of Lie groupoids.* Let M be a smooth manifold. The pair groupoid is defined as $G_1 \rightrightarrows G_0 = M \times M \rightrightarrows M$. The source and targent maps s and t are the left and right projections, the identity id is given by diagonal inclusion $M \to M \times M$, the multiplication m is defined by $m((x, y), (y, z)) = (x, z)$ and the inverse inv by $\mathrm{inv}(x, y) = (y, x)$. Its associated Lie algebroid is TM with the standard structure maps.

    Let M be a smooth manifold and F an involutive subbundle of TM. The following procedure associates a Lie groupoid, called the *monodromy groupoid*, $\mathrm{Mon}(M, \mathcal{F}) \rightrightarrows M$ to F. Points of $\mathrm{Mon}(M, \mathcal{F})$ (arrows) are given by homotopy classes of paths with fixed end points which map into leaves of the foliation. The multiplication m is given by concatenation of paths, the source and target send a path to its starting respectively end point, the inverse reverses paths and the inclusion sends a point to a constant path in that point. The Lie algebroid which is associated to $\mathrm{Mon}(M, \mathcal{F}) \rightrightarrows M$ is given by F. See [MoMr] for a discussion.

    Let $a : G \times M \to M$ be a smooth action of a Lie group G on a manifold M. Set $G \times M = G_1$ and $G_0 = M$ and define $s(g, x) = x$, $t(g, x) = a(g, x)$, $\mathrm{id}(x) = (e, x)$, $\mathrm{inv}(g, x) = (g^{-1}, m)$ and $m((gx, ghx)(x, gx) = (x, ghx))$ where $e \in G$ is the identity. Then $G_1 \rightrightarrows G_0$ is a Lie groupoid called the action groupoid of the action of G on M. It is denoted by $G \ltimes M \rightrightarrows M$, and its associated Lie algebroid is the action algebroid of the action of $\mathfrak{g}$ on M obtained by differentiating the action of G on M, where $\mathfrak{g}$ is the Lie algebra of G.

11. *A remark about integrability* A Lie groupoid $G_1 \rightrightarrows G_0$ with associated algebroid A is called an integration of A. We have seen that the tangent bundle and Lie algebroids with an injective anchor map always admit an integration, and Lie's third theorem provides

the result in the case of Lie algebras. In general it is not possible to find integrations of Lie algebroids. The obstructions to integrability were determined [CF1], which also provides an extensive overview of the subject.

1.1.2 Alternative definitions of Lie algebroids

It is often possible to generalize the differential geometric constructions that are performed with tangent bundles to the realm of Lie algebroids, and moreover several notions of Lie algebra theory can be extended to Lie algebroids. We will start with constructions that can be performed in the greatest generality, namely that of sheaves of Lie algebroids and discuss particular cases later.

The complex of $\mathcal{L}$-de Rham forms

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{L}$ a sheaf of Lie algebroids over $(X, \mathcal{O}_X)$

Definition 1.1.8. *The de Rham complex of $\mathcal{L}$-forms is a sheaf of differential graded algebras (dg algebras) given by*

$$\Omega^\bullet_{\mathcal{L}} := \Lambda^\bullet_{\mathcal{O}_X}\mathcal{L}^\vee$$

with the product given by the wedge product $\wedge$ and the differential $d_{\mathcal{L}}$ defined in low degrees by

$$\begin{aligned} d_{\mathcal{L}} f(l_1) &:= l_1(f) \\ d_{\mathcal{L}} \alpha(l_1, l_2) &:= l_1(\alpha(l_2)) - l_2(\alpha(l_1)) - \alpha([l_1, l_2]), \end{aligned}$$

where $f \in \mathcal{O}_X$, $l_i \in \mathcal{L}$ and $\alpha \in \mathcal{L}^\vee$, and extended by the derivation property.

Conversely, one can show that a differential of degree 1 which acts by derivations on $\Omega^\bullet_{\mathcal{L}}$ defines a structure of a Lie algebroid on $\mathcal{L}$, see for example [Xu99]. Given three $\mathcal{O}_X$-modules $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ together with an $\mathcal{O}_X$-linear operation

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{G}$$

it follows easily that one can define an operation

$$\Omega^\bullet_{\mathcal{L}}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{\mathcal{L}}(\mathcal{F}) \longrightarrow \Omega^\bullet_{\mathcal{L}}(\mathcal{G})$$

where $\Omega^\bullet_{\mathcal{L}}(\mathcal{E}) := \Omega^\bullet_{\mathcal{L}} \otimes_{\mathcal{O}_X} \mathcal{E}$. One of the simplest examples is the $\mathcal{O}_X$-module structure on $\mathcal{E}$ itself, which equips $\Omega^\bullet_{\mathcal{L}}(\mathcal{E})$ with the structure of an $\Omega^\bullet_{\mathcal{L}}$-module. Other examples are the composition of endomorphisms $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$, the action of endomorphisms $\mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$ on $\mathcal{E}$ and the multiplication m for a sheaf of algebras $\mathcal{A}$.

Definition 1.1.9. *An $\mathcal{L}$-connection on a $\mathcal{O}_X$ module $\mathcal{E}$ is a $\mathbb{K}$-linear map*

$$\nabla : \mathcal{E} \longrightarrow \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{E}$$

satisfying the Leibniz rule

$$\nabla(fs) = d_{\mathcal{L}} f \otimes s + f\nabla(s), \quad f \in \mathcal{O}_X(U),\ s \in \mathcal{E}(U)$$

for all sections and opens $U \subset X$.

As remarked earlier, the $\mathcal{O}_X$-module structure on $\mathcal{E}$ defines an $\Omega^\bullet_{\mathcal{L}}$-module structure on $\Omega_{\mathcal{L}}(\mathcal{E})$, and the formula

$$d_\nabla(\alpha \otimes s) = d_{\mathcal{L}}(\alpha) \otimes s + (-1)^{|\alpha|}\alpha \wedge \nabla(s)$$

defines a map d_∇ of degree 1 on $\Omega^\bullet_{\mathcal{L}}(X, \mathcal{E})$, which is more explicitly given by the Koszul formula

$$\begin{aligned} d_\nabla \omega(l_1, \dots, l_{k+1}) :=& \sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{l_i}(\omega(l_1, \dots, \widehat{l_i}, \dots, l_{k+1})) \\ &+ \sum_{i,j=1}^{k+1} (-1)^{i+j} \omega([l_i, l_j], l_1, \dots, \widehat{l_i}, \dots, \widehat{l_j}, \dots, l_{k+1}). \end{aligned} \tag{1.1.2}$$

One can show that $d_\nabla^2 = R_\nabla \circ$, where $R_\nabla \in \Omega^2_{\mathcal{L}}(\mathrm{End}(\mathcal{E}))$ is the curvature of the connection, defined by

$$R_\nabla(l_1, l_2) := [\nabla_{l_1}, \nabla_{l_2}] - \nabla_{[l_1, l_2]}, \qquad l_i \in \mathcal{L}.$$

If $R_\nabla = 0$, ∇ is *flat* and defines a *representation* of $\mathcal{L}$ on $\mathcal{E}$. The complex $(\Omega^\bullet_{\mathcal{L}}(\mathcal{E}), d_\nabla)$ for a flat connection ∇ is called the complex of $\mathcal{L}$ de Rham forms with values in $\mathcal{E}$.

Example 5. The action of $\mathcal{L}$ on $\mathcal{O}_X$ defines a flat representation and the resulting complex is just the de Rham complex $(\Omega^\bullet_{\mathcal{L}}, d_{\mathcal{L}})$.

The $\mathcal{L}$-polyvectorfields.

We start by recalling the definition of a Gerstenhaber algebra:

Definition 1.1.10. *A Gerstenhaber algebra* A *is a graded vector space equipped with a Lie bracket* $[\ ,\ ]$ *of degree* -1 *and a (super)commutative, associative product* $\cdot$ *of degree* 0 *such that*

$$[\alpha, \beta \cdot \gamma] = [\alpha, \beta] \cdot \gamma + (-1)^{|\alpha|(|\beta|-1)} \beta \cdot [\alpha, \gamma]$$

for all $\alpha, \beta, \gamma \in A$.

Dual to the $\mathcal{L}$-de Rham forms one has the sheaf of $\mathcal{L}$-polyvector fields defined by

$$T^{\mathcal{L}}_{\text{poly}} := \Lambda^\bullet_{\mathcal{O}_X} \mathcal{L}.$$

The $\mathcal{L}$-polyvectorfields are endowed with the graded commutative wedge product $\wedge$, and the Lie bracket on $\mathcal{L}$ can be extended to a bracket on $\mathcal{L}$–polyvectorfields by the following definition:

$$[l_1 \wedge \cdots \wedge l_m, k_1 \wedge \cdots \wedge k_n] := \sum_{i,j} (-1)^{i+j} [l_i, k_j] l_1 \wedge \cdots \widehat{l_i} \cdots \wedge l_m \wedge k_1 \wedge \cdots \widehat{k_j} \cdots \wedge k_n \tag{1.1.3}$$

for all $l_i, k_j \in \mathcal{L}$, and

$$[f, l_1 \wedge \cdots \wedge l_m] = -\iota_{d_{\mathcal{L}} f} l_1 \wedge \cdots \wedge l_m$$

where $f \in \mathcal{O}_X$, and $\iota_{d_{\mathcal{L}} f}$ is defined by

$$\iota_{d_{\mathcal{L}} f}(l_1 \wedge \cdots \wedge l_m)(\alpha_1, \dots, \alpha_{m-1}) := l_1 \wedge \cdots \wedge l_m(d_{\mathcal{L}} f, \alpha_1, \dots, \alpha_{m-1})$$

for $\alpha_i \in \mathcal{L}^\vee$. The bracket is called the Schouten–Nijenhuis bracket and denoted by $[\ ,\ ]_{SN}$ or sometimes simply by $[\ ,\ ]$. In low degrees the bracket is given by

$$\begin{aligned} [l_1, f]_{SN} :=& l_1(f) \\ [l_1, l_2]_{SN} :=& [l_1, l_2] \end{aligned}$$

for $f \in \mathcal{O}_X$ and $l_i \in \mathcal{L}$ and the full bracket can also be defined as the extension of this bracket using the Leibniz rule:

$$[\gamma_1, \gamma_2 \wedge \gamma_3]_{SN} := [\gamma_1, \gamma_2]_{SN} \wedge \gamma_3 + (-1)^{|\gamma_1|(|\gamma_2|+1)} \gamma_2 \wedge [\gamma_1, \gamma_3]_{SN}$$

where $\gamma_i \in T^{\mathcal{L}}_{\text{poly}}$ are homogeneous elements and $|\ |$ denotes the degree of polyvectorfields.

Proposition 1.1.11. *The triple* $(T^{\mathcal{L}}_{\text{poly}}, \wedge, [\ ,\ ]_{SN})$ *is a sheaf of Gerstenhaber algebras.*

Given an $\mathcal{O}_X$-module $\mathcal{L}$, a structure of a sheaf of Gerstenhaber algbras on the sheaf $\Lambda^{\bullet}_{\mathcal{O}_X}\mathcal{L}$ determines a Lie algebroid structure on $\mathcal{L}$ on $\mathcal{O}_X$. It follows that

Proposition 1.1.12. *Given a locally free* $\mathcal{O}_X$*-module* $\mathcal{L}$ *on a ringed space* $(X, \mathcal{O}_X)$*, the following three structures are equivalent:*

1. *The structure of a sheaf of Lie algebroids on* $\mathcal{L}$*.*

2. *A bracket* $[\ ,\ ]_{SN}$ *for which* $(T^{\mathcal{L}}_{\text{poly}}, \wedge, [\ ,\ ]_{SN})$ *is a sheaf of Gerstenhaber algebras.*

3. *A degree 1 differential* $d_{\mathcal{L}}$ *for which* $(\Omega_{\mathcal{L}}, \wedge, d_{\mathcal{L}})$ *is a sheaf of dg algebras.*

A proof for the smooth case can be found in [Xu99]; the argument extends easily to the more general case.

Definition 1.1.13. *Given a Gerstenhaber algebra* $(A, \wedge, [\ ,\ ])$*, a precalculus over* A *is given by a graded vector space* M *and operations* ι *and* L *such that:*

1. *The operation*
$$\iota : A \otimes M \longrightarrow M$$
gives M *the structure of a graded module over* $(A, \wedge)$*.*

2. *The operation*
$$L : A \otimes M \longrightarrow M$$
gives M *the structure of a graded Lie module over* $(A[1], [\ ,\ ])$*.*

3. *The following compatibilities hold for all* $a, b \in A$*:*
$$[L_a, \iota_b] = \iota_{[a,b]} \qquad\qquad \iota_b L_a + (-1)^{|b|} L_a \iota_b = L_{a \wedge b}$$
where we used the graded commutators on the LHS.

A precalculus is a calculus when there exists a derivation d of degree -1 on M such that
$$[d, \iota_a] = (-1)^{|a|-1} L_a.$$

The natural contraction operation between $\mathcal{L}$-differential forms and $\mathcal{L}$-polyvectorfields is given by
$$\iota_{\gamma_1}\omega(\gamma_2) := \omega(\gamma_1 \wedge \gamma_2), \qquad \omega \in \Omega_{\mathcal{L}},\ \gamma_1,\ \gamma_2 \in T^{\mathcal{L}}_{\text{poly}}$$

Proposition 1.1.14. *The de Rham differential* $d_{\mathcal{L}}$ *on* $\mathcal{L}$*-forms turns* $(\Omega_{\mathcal{L}}, \iota, d_{\mathcal{L}}, L)$ with the negative grading *into a sheaf of calculi over the sheaf of Gerstenhaber algebras* $(T^{\mathcal{L}}_{\text{poly}}, \wedge, [\ ,\ ])$

The proof of this proposition in the smooth case can be found in [Xu99] or [ELW], although there the notion of a calculus is not mentioned. Again it extends to the general case without difficulties.

Remark 1.1.15. The latter structure is, for this thesis, merely a convenient way to summarize all the structures on the $\mathcal{L}$-forms and $\mathcal{L}$-polyvectorfields that we will use.

1.1.3 Lie algebroid cohomology

Given a Lie algebroid $\mathcal{L}$ over $(X, \mathcal{O}_X)$, the cohomology of $\mathcal{L}$ is given by the hypercohomology of the complex of sheaves

$$\mathcal{O}_X \xrightarrow{d_{\mathcal{L}}} \Omega^1_{\mathcal{L}} \xrightarrow{d_{\mathcal{L}}} \Omega^2_{\mathcal{L}} \xrightarrow{d_{\mathcal{L}}} \dots$$

which we denote by $H^\bullet(\mathcal{L}) = \mathbb{H}^\bullet(\Omega^\bullet_{\mathcal{L}}, d_{\mathcal{L}})$. As the case of Lie algebras already shows, this complex is not locally acyclic. In the smooth case, this implies that the complex is in general not a resolution of fine sheaves of the sheaf of constant functions.

Example 6.
1) When $\mathcal{L}$ is smooth, the sheaves $\Omega^\bullet_{\mathcal{L}}$ are fine. Hence, the cohomology can be computed as the cohomology of the complex of global sections of the sheaves.
2) Let (L, R) be a Lie–Rinehart algebra such that L is projective over R. One can define the complex of differential forms as $\mathrm{Hom}_R(\Lambda^\bullet L, R)$ and the differential by the standard Koszul formula. Note that $\Lambda^\bullet L^\vee$ is not the right notion if L is not finitely generated. In [Rin], Rinehart showed that the cohomology $H^\bullet(L, R)$ of this complex is equal to $\mathrm{Ext}_{UL}(R, R)$, where UL is the universal enveloping algebra (see §1.2.1), i.e. the cohomology theory forms a part of homological algebra.
3) For a holomorphic Lie algebroid $\mathcal{L}$, the Dolbeault resolution of [LSX12] can be applied. We discuss this below. An alternative is provided by the standard Čech resolution of sheaves which also applies in the more general setting of sheaves of Lie algebroids over ringed spaces.

Given an $\mathcal{L}$-module $\mathcal{E}$, i.e. an $\mathcal{O}_X$-module with a flat $\mathcal{L}$-connection, the hypercohomology $\mathbb{H}^\bullet(\Omega^\bullet_{\mathcal{L}}(\mathcal{E}), d_\nabla)$ of the complex

$$\mathcal{O}_X \xrightarrow{d_\nabla} \Omega^1_{\mathcal{L}}(\mathcal{E}) \xrightarrow{d_\nabla} \Omega^2_{\mathcal{L}}(\mathcal{E}) \xrightarrow{d_\nabla} \dots$$

is defined to be $H^\bullet(\mathcal{L}, \mathcal{E})$, the cohomology of $\mathcal{L}$ with values in $\mathcal{E}$.

Example 7.
1) Let $\mathcal{L} = TM$ for M a smooth manifold. Representations of TM are given by flat vector bundles E over M. The cohomology with values in E is given by the usual cohomology of M with local coefficients given by the flat sections of E. In particular, when E is the trivial line bundle, one recovers the de Rham cohomology of M.
2) When $A = \mathfrak{g}$, a Lie algebra over a point, representations of A are the same as Lie algebra representations of $\mathfrak{g}$, and the algebroid cohomology agrees with the Lie algebra cohomology obtained from the Chevalley–Eilenberg complex. A natural example, besides the trival representation, is the adjoint representation of $\mathfrak{g}$, which leads to a cohomology theory controlling deformations of $\mathfrak{g}$. In chapter 3 we will discuss a broader notion of representations that encompasses the adjoint representation for any smooth Lie algebroid.

Since it will be used in chapter 4, we discuss the example of a holomorphic Lie algebroid in more detail:

Example 8. The Dolbeault resolution for a holomorphic Lie algebroid $\mathcal{A} \to X$ over a complex manifold X was constructed in [LSX08]. Similar to the holomorphic tangent bundle, we consider the Lie algebroid $\mathcal{A}$ as a smooth Lie algebroid $A_{\mathbb{R}}$ together with an almost complex structure, i.e. a bundle map

$$J : A_{\mathbb{R}} \longrightarrow A_{\mathbb{R}}$$

which squares to -1 and is compatible with the anchor map and the bracket. The complexified Lie algebroid $A_{\mathbb{R}} \otimes \mathbb{C}$ splits into $A_{\mathbb{R}} \otimes \mathbb{C} \cong A^{1,0} \oplus A^{0,1}$ according to the $\pm i$ eigenvalues of J, and $(A^{1,0}, A^{0,1})$ forms a *matched pair*, i.e. a pair of Lie algebroids such that $A^{1,0}$ is an $A^{0,1}$-module and $A^{0,1}$ is an $A^{1,0}$-module. The module structures and the anchor map need to satisfy a number of compatibility relations, see [LSX08].

It is a well-known fact that a holomorphic structure on a vector bundle is equivalent to a flat $T^{0,1}X$-connection on the bundle (the holomorphic sections are given by flat sections), hence $A^{1,0}$ is a representation of $T^{0,1}X$. The formula

$$\nabla_s(X) = \mathrm{pr}^{1,0}([\rho(s), X]), \qquad s \in A^{1,0},\ X \in T^{0,1}X$$

gives $T^{0,1}X$ the structure of an $A^{1,0}$-representation, and these two representation give $(T^{0,1}X,\ A^{1,0})$ the structure of a matched pair.

A crucial property of matched pairs is that their direct sum admits the structure of a Lie algebroid, which is, in the case at hand, denoted by $T^{0,1}X \bowtie A^{1,0}$. The cohomology of the latter Lie algebroid is equal to the cohomology of the total complex associated to the following double complex:

$$\begin{array}{ccccc}
\cdots & & \cdots & & \\
{\scriptstyle d^{1,0}_{\mathcal{A}}}\uparrow & & {\scriptstyle d^{1,0}_{\mathcal{A}}}\uparrow & & \\
\mathcal{D}^{1,0}_{\mathcal{A}} & \xrightarrow{\overline{\partial}} & \mathcal{D}^{1,1}_{\mathcal{A}} & \xrightarrow{\overline{\partial}} & \cdots \\
{\scriptstyle d^{1,0}_{\mathcal{A}}}\uparrow & & {\scriptstyle d^{1,0}_{\mathcal{A}}}\uparrow & & \\
\mathcal{D}^{0,0}_{\mathcal{A}} & \xrightarrow{\overline{\partial}} & \mathcal{D}^{0,1}_{\mathcal{A}} & \xrightarrow{\overline{\partial}} & \cdots
\end{array}$$

where $\mathcal{D}^{p,q}_{\mathcal{A}} := \wedge^p (A^{1,0})^\vee \otimes \wedge^q (T^{0,1}X)^\vee$, and the differentials are induced by the module structures. In this particular case the flat $T^{0,1}_X$-connection on $A^{1,0}$ gives rise to the Dolbeault differential $\overline{\partial}$ for $\mathcal{A}$, hence the $\overline{\partial}$-Poincare lemma shows that the double complex above is a resolution of

$$\mathcal{O}_X \xrightarrow{d_{\mathcal{A}}} \Omega^1_{\mathcal{A}} \xrightarrow{d_{\mathcal{A}}} \Omega^2_{\mathcal{A}} \xrightarrow{d_{\mathcal{A}}} \cdots$$

and one has:

Theorem 1.1.16 ([LSX08]). *Let $\mathcal{A}$ be a holomorphic Lie algebroid. Then we have*

1. *The holomorphic de Rham isomorphism: $H^\bullet(\mathcal{A}) = H^\bullet(T^{0,1}_X \bowtie A^{1,0})$, where the LHS is the hypercohomology of the complex of holomorphic de Rham forms, and the RHS is the cohomology of the smooth Lie algebroid $T^{0,1}_X \bowtie A^{1,0}$.*

2. *The Dolbeault isomorphism: $H^q(\Omega^p_{\mathcal{A}}) = H^q(\mathcal{D}^{p,\bullet}_{\mathcal{A}}, \overline{\partial})$, where the LHS is the sheaf cohomology of $\Omega^p_{\mathcal{A}}(X)$.*

1.2 Noncommutative geometry of Lie algebroids

The universal enveloping algebra of a Lie algebra

First we describe the construction of a universal enveloping algebra associated to a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K} \supset \mathbb{Q}$. The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is defined as quotient of the tensor algebra:

$$U(\mathfrak{g}) := T(\mathfrak{g})/I$$

where I is the ideal generated by the set $\{i(x) \otimes i(y) - i(y) \otimes i(x) - i([x,y]),\ x, y \in \mathfrak{g}\}$ and $i: \ \mathfrak{g} \to T(\mathfrak{g})$ is the natural inclusion. The universal property of the universal enveloping algebra is a bijection $\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A)) \simeq \mathrm{Hom}_{\mathrm{Ass}}(U(\mathfrak{g}), A)$, where $\mathrm{Lie}(A)$ is A equipped with the commutator bracket. When the Lie bracket on $\mathfrak{g}$ is trivial, the universal enveloping algebra of $\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}$, denoted $S(\mathfrak{g})$. The symmetric algebra can also be realized as the subalgebra of the tensor algebra consisting of symmetric tensors.

The natural filtration on the tensor algebra descends to the universal enveloping algebra, denoted by $U(\mathfrak{g})^0 \subset U(\mathfrak{g})^1 \subset \ldots$. Let $\mathrm{Gr}(U(\mathfrak{g}))$ be its associated graded algebra: $\mathrm{Gr}(U(\mathfrak{g})) := \bigoplus_{i=0}^{\infty} U(\mathfrak{g})^i / U(\mathfrak{g})^{i-1}$

Theorem 1.2.1 (Poincaré–Birkhoff–Witt)**.** *The composition of the maps*

$$S(\mathfrak{g}) \longrightarrow T(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \longrightarrow \mathrm{Gr}(U(\mathfrak{g}))$$

given by the symmetrization map

$$x_1 \cdots x_k \mapsto \frac{1}{k!} \sum_{\sigma \in S^k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)} \tag{1.2.1}$$

and the natural projections, is an isomorphism of graded algebras.

A proof can be found in [Se]. Because its associated graded algebra is commutative, the universal enveloping algebra is called *almost commutative*. The associated graded algebra inherits a Poisson structure from the commutator bracket on $\mathcal{U}(\mathfrak{g})$ as follows: Let $x \in \mathrm{Gr}^i(\mathcal{U}(\mathfrak{g}))$ and $y \in \mathrm{Gr}^j(\mathcal{U}(\mathfrak{g}))$, and choose representatives $\tilde{x} \in \mathcal{U}(\mathfrak{g})^i$ and $\tilde{y} \in \mathcal{U}(\mathfrak{g})^j$. We have $[\tilde{x}, \tilde{y}] \in \mathcal{U}^{i+j-1}(\mathfrak{g})$ because $\mathcal{U}(\mathfrak{g})$ is almost commutative. Then the Poisson bracket between x and y is defined as the class of $[\tilde{x}, \tilde{y}]$ in $\mathrm{Gr}^{i+j-1}(\mathcal{U}(\mathfrak{g}))$, and it is straightforward to show that this definition does not depend on the choice of lifts.

1.2.1 The universal enveloping algebra of a Lie algebroid

Let $\mathcal{L}$ be a sheaf of Lie algebroids over a ringed space $(X, \mathcal{O}_X)$. The sheaf $\mathcal{O}_X \oplus \mathcal{L}$ is a sheaf of Lie algebras with the bracket defined by

$$[(f_1, l_1), (f_2, l_2)] := (l_1(f_2) - l_2(f_1), [l_1, l_2])$$

for $f_i \in \mathcal{O}_X$ and $l_i \in \mathcal{L}$. The sheaf of universal enveloping algebras of this sheaf of Lie algebras has a subsheaf which is generated by $i(\mathcal{O} \oplus \mathcal{L})$, and the quotient of this sheaf by the relation $f \otimes (g \oplus l) = fg \otimes fl$ for $f \in \mathcal{O}_X$ and $l \in \mathcal{L}$ is defined to be the sheaf of universal enveloping algebras, sometimes abbreviated by universal enveloping algebra, and denoted by $\mathcal{U}(\mathcal{L})$.

The universal enveloping algebra is unital and is equipped with a $\mathbb{K}$-algebra morphism $i : \mathcal{O}_X \to \mathcal{U}(\mathcal{L})$, and a morphism $i : \mathcal{L} \to \mathrm{Lie}(\mathcal{U}(\mathcal{L}))$ of $\mathbb{K}$-Lie algebras, subject to the conditions

$$i(f)i(X) = i(fX), \qquad i(X)i(f) - i(f)i(X) = i(X(f)), \quad f \in \mathcal{O}_X,\ X \in \mathcal{L}.$$

The universal enveloping algebra satisfies the following universal property: for any triple $(\mathcal{A}, \phi_{\mathcal{L}}, \phi_{\mathcal{O}})$ consisting of a sheaf of $\mathbb{K}$-algebras $\mathcal{A}$, a homomorphisms $\phi_{\mathcal{O}_X} : \mathcal{O}_X \to \mathcal{A}$, and a Lie algebra morphism $\phi_{\mathcal{L}} : \mathcal{L} \to \mathrm{Lie}(\mathcal{A})$, such that for all $f \in \mathcal{O}_X,\ l \in \mathcal{L}$

$$\phi_{\mathcal{O}_X}(f)\phi_{\mathcal{L}}(X) = \phi_{\mathcal{L}}(fX), \qquad \phi_{\mathcal{L}}(X)\phi_{\mathcal{O}_X}(f) - \phi_{\mathcal{O}_X}(f)\phi_{\mathcal{L}}(X) = \phi_{\mathcal{O}_X}(X(f)),$$

there is a unique morphism $\Phi : \mathcal{U}(\mathcal{L}) \to \mathcal{A}$ of sheaves of $\mathbb{K}$-algebras such that $\Phi \circ i_{\mathcal{O}} = \phi : \mathcal{O}_X \to \mathcal{U}(\mathcal{L})$ and $\Phi \circ i_{\mathcal{L}} = \phi : \mathcal{L} \to \mathcal{U}(\mathcal{L})$.

Recall that a locally free $\mathcal{O}_X$-module $\mathcal{E}$ is an $(\mathcal{O}_X, \mathcal{L})$-module iff there exists a flat $\mathcal{L}$-connection on $\mathcal{E}$. The map $\mathcal{L} \to \mathrm{End}_{\mathbb{K}}(\mathcal{E})$ given by $X \to \nabla_X$ exactly satisfies the universal property given above, hence the natural functor

$$\mathsf{Mod}(\mathcal{U}(\mathcal{L})) \to \mathsf{Mod}(\mathcal{O}_X, \mathcal{L}) \tag{1.2.2}$$

is an equivalence of categories.

Example 9.
1) Given a smooth manifold M, the universal enveloping algebra of TM is given by the sheaf of differential operators $\mathrm{Diff}(M)$ on M. Likewise, the universal enveloping algebra of the holomorphic tangent bundle TX of a complex manifold X is given by the sheaf of holomorphic differential operators on X.
2) For $G \rightrightarrows M$ a Lie groupoid with associated Lie algebroid $A \to M$, the universal enveloping algebra of A is a model for the algebra of differential operators which are tangent to the s-fibers invariant under the right action on the s-fibers. For differential operators of order 1 this is the correspondence between invariant fiberwise vector fields and the Lie algebroid A.
3) Given a regular foliation of a smooth manifold M with associated Lie algebroid A, the universal enveloping algebroid of A is given by the leafwise differential operators on M, i.e. the subalgebra of differential operators on M which are tangent to the foliation.

1.2.2 Hochschild and Cyclic theory

Basic definitions

We briefly recall the basic notions of Hochschild and cyclic homology as well as their dual cohomology theories of an associative, unital algebra C, to which we will refer by the *cyclic theory* of C. Then we give a few examples. An extensive introduction into the subject is given in [Lo]. We fix a unital algebra C over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. The fact that C is unital allows for considerable simplifications of the definitions, and in this thesis all the algebras that we consider are unital. Let M be an C-bimodule and write $\bar{C} := C/1 \cdot \mathbb{K}$. If C and M are graded, one has to apply the Koszul sign rule in all the definitions that follow. Set

$$C_\bullet(C;M) := M \otimes \bar{C}^{\otimes \bullet}$$
$$C^\bullet(C;M) := \mathrm{Hom}_{\mathbb{K}}(\bar{C}^{\otimes \bullet}, M).$$

Let $a_i \in C$ and $m \in M$. As one can easily check, the formula

$$\begin{aligned} b_H(m \otimes a_1 \otimes \ldots \otimes a_k) &:= m \cdot a_1 \otimes a_2 \otimes \ldots \otimes a_k \\ &+ \sum_{i=1}^{k-1} (-1)^i m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_k + (-1)^k a_k \cdot m \otimes a_1 \otimes \ldots \otimes a_{k-1} \end{aligned}$$

defines a differential b_H of degree -1 on $C_\bullet(C;M)$, and the formula

$$\begin{aligned} b^H \phi(a_1, \ldots, a_k) :=& a_1 \phi(a_2, \ldots, a_k) \\ &+ \sum_{i=1}^{k-1} (-1)^i \phi(a_1, \ldots, a_i a_{i+1}, \ldots, a_k) + (-1)^k \phi(a_1, \ldots, a_{k-1}) a_k \end{aligned}$$

endows $C^\bullet(C;M)$ with a differential b^H of degree 1. The resulting homology of the first complex is called the Hochschild homology of C with values in M, written $H_\bullet(C;M)$, and the second complex defines the Hochschild cohomology $H^\bullet(C;M)$ of C with values in M. When $M = C$, the Hochschild homology complex is denoted by $C_\bullet(C) := C_\bullet(C,C)$ and the Hochschild homology by $HH_\bullet(C) =: H_\bullet(C;C)$. The Hochschild cohomology $H^\bullet(C;C)$ is called the Hochschild cohomology of C and denoted by $HH^\bullet(C)$. The formula

$$B(a_0 \otimes \ldots \otimes a_k) := \sum_{i=0}^{k} (-1)^{ik} 1 \otimes a_i \otimes a \ldots \otimes a_k \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

defines a differential of degree 1 on the Hochschild chain complex $C_\bullet(C;C)$. Besides the relation $B^2 = 0$ it satisfies $[b, B] = 0$. The dual of B, which we also denote by B, defines a differential of degree -1 on $C^\bullet(C, C^*)$, the Hochschild cochain complex with values in the dual of C. To define the chain complexes for the various versions of cyclic homology, we introduce, following [GJ], a formal variable u of degree 2 and a formal variable u^{-1} of degree -2 and set

$$CC_\bullet^W(C) := C_\bullet(C;C)[[u^{-1}]] \otimes_{\mathbb{K}[u^{-1}]} W$$
$$CC^\bullet_W(C) := C^\bullet(C;C^*)[[u]] \otimes_{\mathbb{K}[u]} W.$$

and equip them with the differentials $b_H + u^{-1}B$ and $b^H + uB$. Here, W denotes a $\mathbb{K}[u^{-1}]$-module in the first line and a $\mathbb{K}[u]$-module in the second. Various choices for W lead to the following theories, of which the last three are homology theories.

1. $W = \mathbb{K}$ gives the Hochschild homology of C and the Hochschild cohomology with values in C^*.
2. $W = \mathbb{K}((u^{-1}))/u^{-1}\mathbb{K}[[u^{-1}]] = \mathbb{K}[u]$ leads to (ordinary) cyclic homology, denoted by $HC_\bullet(C)$.
3. $W = \mathbb{K}((u^{-1}))$ leads to periodic cyclic homology, denoted $HP_\bullet(C)$. It has period 2 with periodicity operator induced by multiplication by u.
4. $W = \mathbb{K}[u^{-1}]$ leads to negative cyclic homology, denoted $HC_\bullet^-(C)$.

Short exact sequences of $\mathbb{K}[u]$-modules lead to long exact sequences in homology. For example, the usual SBI-sequence is induced from the short exact sequence

$$0 \longrightarrow \mathbb{K}((u^{-1}))/\mathbb{K}[[u^{-1}]] \longrightarrow \mathbb{K}((u^{-1}))/u^{-1}\mathbb{K}[[u^{-1}]] \longrightarrow \mathbb{K} \longrightarrow 0.$$

On the other hand, the short exact sequence

$$0 \longrightarrow u^{-1}\mathbb{K}[u^{-1}] \longrightarrow \mathbb{K}((u^{-1})) \longrightarrow \mathbb{K}((u^{-1}))/u^{-1}\mathbb{K}[[u^{-1}]] \longrightarrow 0$$

leads to the long exact sequence

$$\ldots \longrightarrow HC^-_{n+2}(C) \longrightarrow HP_n(C) \longrightarrow HC_n(C) \longrightarrow HC^-_{n+1} \longrightarrow \ldots$$

relating negative and periodic cyclic homology.

Additional structures on the Hochschild (co)complex

We define a few operations on the Hochschild and cyclic (co)chain complexes that we will use in chapter 4. They are part of the set of operations which defines the structure of a calculus on the associated (co)homology complexes, c.f. [DTTs]. Let $a \in C$.

1. The insertion operator $\iota_a : C_\bullet(C) \to C_{\bullet+1}(C)$ is defined by

$$\iota_a a_0 \otimes \cdots \otimes a_k := \sum_{i=0}^{k} (-1)^{i+1} a_0 \otimes \cdots \otimes a_i \otimes a \otimes a_{i+1} \otimes \cdots \otimes a_k).$$

2. Let Lie(C) be the Lie algebra given by C equipped with the commutator bracket. The formula

$$L_a = [b, i_a] \tag{1.2.3}$$

equips $C_\bullet(C)$ with a structure of a Lie algebra module over Lie(C). Explicitly, it is given by the following formula:

$$L_a(a_0 \otimes \cdots \otimes a_k) = \sum_{i=0}^{k} (a_0 \otimes \cdots \otimes [a, a_i] \otimes \ldots, a_k).$$

The dual formulas induce actions on $C^\bullet(C, C^*)$.

3. The shuffle product on the Hochschild chain complex is defined by the formula:

$$(a_0 \otimes \cdots \otimes a_p) \times (b_0 \otimes \cdots \otimes b_q) = a_0 b_0 \otimes Sh_{p,q}(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q),$$

where $a_0 \otimes \cdots \otimes a_p \in C_p(C)$ and $b_0 \otimes \cdots \otimes b_q \in C_q(C)$ and $S_{p,q} \subset S_{p+q}$ consists of the p, q shuffle permutations in the group of permutations S_{p+q}.

Remark 1.2.2. Recall that Hochschild (co)homology can be computed using methods of homological algebra. Given an algebra C as above, and an C-bimodule, one has:

$$H_\bullet(C, M) \cong \mathrm{Tor}_\bullet^{C^e}(M, C)$$
$$H^\bullet(C, M) \cong \mathrm{Ext}_{C^e}^\bullet(C, M).$$

Here, $C^e := C \otimes C^{op}$ is the enveloping algebra of C, which has the property that left C^e-modules are equivalent to C-bimodules. A proof can be found in [We]. As we will see in the examples, sometimes it is convenient to find the appropriate projective or injective resolution to compute the Hochschild (co)homology.

Example 10. (Algebraic HKR theorem) In this example we discuss the Hochschild homology of a commutative, unital algebra C over a field $\mathbb{K}$ such that $\mathbb{Q} \subset \mathbb{K}$. We will see that it is given by the algebraic de Rham forms, which indicates that the Hochschild homology can serve as a model for de Rham theory for noncommutative algebras.

The C-module of algebraic de Rham forms of degree 1 of a commutative algebra, denoted by $\Omega^1_{C|\mathbb{K}}$, is defined to be the C-module presented by elements da where $a \in C$, subject to the relations

$$d\lambda = 0 \qquad d(a_1 + a_2) = da_1 + da_2 \qquad d(a_1 a_2) = a_1 da_2 + a_2 da_1$$

for all $\lambda \in \mathbb{K}$ and $a_i \in C$. They are also referred to by Kähler differentials. The algebraic de Rham forms in degree n are defined as

$$\Omega^n_{C|\mathbb{K}} := \Lambda^n_C \Omega^1_{C|\mathbb{K}}$$

and the natural derivation

$$C \to \Omega^1_{C|\mathbb{K}}, \quad a \mapsto da,$$

can be uniquely extended to a derivation d of degree 1 on $\Omega^\bullet_{C|\mathbb{K}}$ given by

$$a_0 da_1 \cdots d_n \mapsto da_0 da_1 \cdots da_n$$

which satisfies $d^2 = 0$. The resulting complex is called the algebraic de Rham complex of C and the cohomology of this complex is the algebraic de Rham cohomology of C, denoted by $H^\bullet_{dR}(C)$. There are two natural maps relating the Hochschild chain complex and the complex of algebraic de Rham forms:

$$\epsilon_n : \ \Omega^n_{C|\mathbb{K}} \longrightarrow C_n(C) \qquad \epsilon_n(a_0 da_1 \cdots da_n) := \sum_{\sigma \in S^n} (-1)^{|\sigma|} a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$
$$\mu_n : \ C_n(C) \longrightarrow \Omega^n_{C|\mathbb{K}} \qquad \mu(a_0 a_1 \otimes \cdots \otimes a_n) := \frac{1}{n!} a_0 da_1 \cdots da_n.$$

One can show that the map μ_n defines a morphism of complexes

$$\mu : \ (CC^W_\bullet(C),\ b + u^{-1}B) \longrightarrow (\Omega^W_{C|\mathbb{K}},\ 0 + u^{-1}d)$$

where $\Omega^W_{C|\mathbb{K}} = \Omega_{C|\mathbb{K}} \otimes_{\mathbb{K}[u^{-1}]} W$. If the algebra C is smooth, see [Lo] for the precise definition, one has the following theorem.

Theorem 1.2.3 ([HKR, LoQ])**.** *If* C *is a smooth algebra over* $\mathbb{K}$*, the map* ϵ *induces an isomorphism*

$$\Omega^{\bullet}_{C|\mathbb{K}} \longrightarrow H_{\bullet}(C).$$

Moreover, if $\mathbb{Q} \subset \mathbb{K}$*, one has canonical isomorphisms*

$$HC_{\bullet}(C) \cong \Omega^{\bullet}_{C|\mathbb{K}}/d\Omega^{\bullet}_{C|\mathbb{K}} \oplus H^{\bullet-2}_{\mathrm{dR}}(C) \oplus H^{\bullet-4}_{\mathrm{dR}}(C) \oplus \dots$$
$$HP_{\bullet}(C) \cong \bigoplus_{i\in\mathbb{Z}} H^{\bullet+2i}_{\mathrm{dR}}(C)$$

where $HC_{\bullet}$ *stands for cyclic homology and* $HP_{\bullet}$ *stands for periodic cyclic homology.*

Example 11. (Connes' theorem.) When the algebra C admits a topology which is compatible with the algebra structures, the Hochschild complex is defined using a different version of the tensor product, which takes the topology into account. This is done because the cyclic theory is rather hard to compute with the algebraic definitions, e.g. in the case of smooth functions on a manifold M. In this case, the Hochschild complex can be defined as

$$C_{\bullet}(C^{\infty}(M)) = C^{\infty}(M^{\times\bullet})$$

but also as the complex of germs or even jets of functions on the diagonals in $M^{\times\bullet}$. Connes proved the following theorem.

Theorem 1.2.4 ([Co])**.** *Let* M *be a smooth compact manifold and let* $\Omega^k(M)$ *be space of smooth* k*-forms on* M*. Then one has isomorphisms*

$$HH_{\bullet}(C^{\infty}(M)) \cong \Omega^{\bullet}(M)$$
$$HP_{\bullet}(C^{\infty}(M)) \cong H^{\mathrm{even}}_{\mathrm{dR}}(M) \oplus H^{\mathrm{odd}}_{\mathrm{dR}}(M).$$

Dualizing both complexes yields the space of de Rham currents, i.e. the continuous linear functionals on the de Rham forms, and the Hochschild cohomology with values in the continuous dual $C^{\infty}(M)^*$ respectively. The dualized morphism of complexes μ^* is a quasi-isomorphism on these complexes. The extension of the HKR theorem to multivector fields will be discussed later, and we also define other, smaller, Hochschild complexes, namely the polydifferential operators and jets. In fact, the HKRC theorem is proved by considering a stable subcomplex of the given Hochschild complex consisting of jets along the diagonal $M \subset M^{\times k}$.

Example 12 (Almost commutative algebras)**.** This computation was done in [Ka] by Kassel. Let C be an associative, unital algebra with a filtration $\dots \subset F_nC \subset F_{n+1}C \subset \dots$ such that

$$\cup F_nC = C \quad \text{and} \quad \cap FC_n = 0.$$

The algebra C is called almost commutative if the associated graded algebra

$$\mathrm{Gr}(C) := \bigoplus_{n+1} F_nC/F_nC$$

is isomorphic to the symmetric algebra $\mathrm{Sym}(V)$ of V, where $V = F_1C/F_0C$. As in the case of the universal enveloping algebra of a Lie algebra, $\mathrm{Gr}(C)$ is a Poisson algebra. The bracket is defined by the same formula:

$$\{f, g\} := \sigma_{m+n-1}([a, b]),$$

where σ_n is the projection $\sigma_n : F_nC \to F_nC/F_{n-1}C$ and a and b are lifts of f and g, i.e.

$$\sigma(a) = f \quad \text{and} \quad \sigma(b) = g.$$

The algebraic de Rham forms of the algebra $\mathrm{Sym}(V)$ are given by

$$\Omega^{\bullet}_{\mathrm{Sym}(V)} \cong \mathrm{Sym}(V)) \otimes \Lambda^{\bullet}V,$$

and in [Br, L] it was observed that the Poisson structure on Sym(V) can be used to define the Poisson homology differential δ of degree -1 on the de Rham complex:

$$\delta(f_0 df_1 \dots df_n) = \sum_{i=1}^{n} (-1)^{i+1}\{f_0, f_i\} df_1 \dots \widehat{f_i} \dots df_n + \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j} f_0 d\{f_i, f_j\} df_1 \dots \widehat{df_j} \dots \widehat{df_j} \dots df_n.$$

This differential anti commutes with the de Rham differential d, hence one can form a mixed complex $\left(\Omega^{\bullet, W}_{\mathrm{Sym}(V)},\ \delta,\ u^{-1} d\right)$. This mixed complex is related to the cyclic theory of C by the following theorem:

Theorem 1.2.5. *For* C *an almost commutative algebra one has the following isomorphisms:*

$$H_\bullet(C) \cong H_\bullet(\Omega^\bullet_{\mathrm{Sym}(V)},\ \delta)$$
$$HC_\bullet(C) \cong HC_\bullet(\Omega^\bullet_{\mathrm{Sym}(V)},\ \delta,\ u^{-1} d)$$
$$HP_\bullet(C) \cong HP_\bullet(\Omega^\bullet_{\mathrm{Sym}(V)},\ \delta,\ u^{-1} d).$$

Remark 1.2.6.

1) The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ and the Weyl algebra, which is defined in appendix §B.1, are particular examples of almost commutative algebras. This gives an easy way to determine the cyclic theory of the Weyl algebra, because the associated Poisson structure is symplectic. The computation of the RHS in the case of the universal enveloping algebra is more intricate.

2) The essential part of the proof that Kassel gave is the observation that the following formula defines a map of complexes

$$\phi : (\Omega^\bullet_{S(\mathfrak{g})}, \delta) \to (C_\bullet(\mathcal{U}(\mathfrak{g})), b_H) \qquad \phi(P dx_1 \dots dx_n) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} \eta(P) x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)},$$

where j is the symmetrization map of (1.2.1) induces a quasi isomorphism of complexes. In chapter 3, we will construct a similar map in the case a smooth Lie algebroid using the formality theorem that Kontsevich proved in [K].

Chapter 2

The Poincaré–Birkhoff–Witt theorem

The core of this chapter is the dual Poincaré–Birkhoff–Witt theorem for sheaves of Lie algebroids. Recall from the previous chapter that the PBW theorem for Lie algebras states that $\mathrm{gr}(\mathcal{U}(\mathfrak{g})) \cong \mathrm{Sym}(\mathfrak{g})$, and that, if $\mathbb{Q} \subset \mathbb{K}$, one can define the symmetrization map $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ inducing an isomorphism on the associated graded spaces. In [Rin], Rinehart proved that $\mathrm{gr}(\mathcal{U}(L, R)) \cong \mathrm{Sym}_R(L)$ for a Lie–Rinehart pair (L, R) under the condition that L is projective over R. However, he did not construct a map $\mathrm{Sym}_R(L) \longrightarrow \mathcal{U}(L, R)$ inducing this isomorphism. In [NWX] the authors used the concept of a local integrating groupoid to define a local exponential map, which induced a lift

$$\mathrm{Sym}(A) \longrightarrow \mathcal{U}(A)$$

of the PBW theorem in the case of a smooth Lie algebroid $A \to M$. In this chapter we generalize this result to locally free sheaves of Lie algebroids $\mathcal{L}$, provided they are *equipped with a global $\mathcal{L}$-connection on $\mathcal{L}$*. Roughly speaking, we replace the local integrating groupoid by $\mathcal{J}(\mathcal{L})$, the sheaf of $\mathcal{L}$-jets associated to $\mathcal{L}$. Then, using the $\mathcal{L}$-connection, we define the exponential map j_∇ of the *dual* version of the PBW theorem:

Theorem. *Let $\mathcal{L}$ be a locally free sheaf of Lie algebroids of constant rank r together with an $\mathcal{L}$-connection $\nabla^{\mathcal{L}}$. There exists an $\mathcal{O}_X$-linear isomorphism of sheaves of algebras*

$$j_\nabla : \ \mathcal{J}(\mathcal{L}) \longrightarrow \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

with respect to the first $\mathcal{O}_X$-module structure on $\mathcal{J}(\mathcal{L})$.

For the definition of $\mathcal{J}(\mathcal{L})$ and its two $\mathcal{O}_X$-module structures we refer to §2.1.2. The algebra structure on $\mathcal{J}(\mathcal{L})$ is induced by the coalgebra structure on $\mathcal{U}(\mathcal{L})$, thus it follows that the dual map

$$j_\nabla^\vee : \ \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}) \longrightarrow \mathcal{U}(\mathcal{L}), \qquad \phi(j_\nabla^\vee(X)) = j_\nabla(\phi)(X)$$

where $X \in \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$ and $\phi \in \mathcal{J}(\mathcal{L})$, is an isomorphism of filtered coalgebras in the category of $\mathcal{O}_X$-modules. Both the induced maps on the associated graded spaces are isomorphisms.

Now we briefly discuss the different subsections. To prove the PBW theorem we need parts of the structures that are defined on $\mathcal{U}(\mathcal{L})$ and $\mathcal{J}(\mathcal{L})$, and we recollect them in §2.1. These structures are described in more detail in [KP] and [CRvdB10]; the unversal enveloping $\mathcal{U}(\mathcal{L})$ is a sheaf of so-called left Hopf algebroids, and $\mathcal{J}(\mathcal{L})$ is a sheaf of Hopf algebroids. Although not formulated in this language, most of the structure maps of the (left) Hopf algebroids $\mathcal{U}(\mathcal{L})$ and $\mathcal{J}(\mathcal{L})$ were already contained in [NeTs01]. In §2.2 we give the definition of the exponential map j_∇ and we prove the dual PBW theorem. We also give recursive relations for the definitions of j_∇ and $j_\nabla^\vee$. It turns out that the recursive relation that we find for $j_\nabla^\vee$ coincides with the recursive relation that was used in [LSX14] to prove the PBW theorem. Then we discuss the straightforward extension of the PBW theorem to an isomorphism of differential operators on $\mathcal{J}(\mathcal{L})$ and $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$, which we use in chapter 4 and the appendix about formality for Lie algebroids. Finally, we discuss how the PBW theorem depends on a linear combination of connections and how it can be twisted by a locally free $\mathcal{O}_X$-module $\mathcal{E}$.

Recently there has been an increased interest in the PBW theorem, see [LiS, LiSX, SX]. The application of the PBW theorem to the formality theorem that the authors consider is very close to our appendix.

2.1 The universal enveloping algebra and the jet algebra

2.1.1 The universal enveloping algebra revisited

Let $\mathcal{L}$ be a locally free sheaf of Lie algebroids of constant, finite rank over a ringed space $(X, \mathcal{O}_X)$. The (sheaf of) universal enveloping algebra(s) $\mathcal{U}(\mathcal{L})$ has both a natural left and right $\mathcal{O}_X$-module structure defined by left and right multiplication respectively. This leads to different notions of tensor products, which we will denote as follows:

$$\begin{aligned} \mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L}) :=&(\mathcal{U}(\mathcal{L}) \otimes_{\mathbb{K}} \mathcal{U}(\mathcal{L}))/\left\langle rD \otimes E - D \otimes rD \right\rangle \\ \mathcal{U}(\mathcal{L}) \otimes^{rl} \mathcal{U}(\mathcal{L}) :=&(\mathcal{U}(\mathcal{L}) \otimes_{\mathbb{K}} \mathcal{U}(\mathcal{L}))/\left\langle Dr \otimes E - D \otimes rD \right\rangle . \end{aligned}$$

This should make the general rule clear. In addition to the algebra structure on $\mathcal{U}(\mathcal{L})$, there also exists a counital, cocommutative and coassociative $\mathcal{O}_X$-coalgebra structure on $\mathcal{U}(\mathcal{L})$. The comultiplication $\Delta : \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L})$ is defined by

$$\begin{aligned} \Delta(r) =& r \otimes 1, & r \in \mathcal{O}_X \\ \Delta(X) =& X \otimes 1 + 1 \otimes X, & X \in \mathcal{L} \end{aligned} \tag{2.1.1}$$

by an application of the universal property of $\mathcal{U}(\mathcal{L})$. The counit $\epsilon : \mathcal{U}(\mathcal{L}) \to \mathcal{O}_X$ is defined by

$$\epsilon(D) = D(1), \qquad D \in \mathcal{U}(\mathcal{L}).$$

Remark 2.1.1. The multiplication on $\mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L})$ by the obvious formula is not well-defined. However, on the subspace of $\mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L})$ which satisfies $D \otimes E(f \otimes 1 - 1 \otimes f) = 0$, the naive product is well-defined. One can easily show that

$$\begin{aligned} \Delta(f)(g \otimes 1 - 1 \otimes g) = 0 \\ \Delta(X)(g \otimes 1 - 1 \otimes g) = 0, \end{aligned}$$

hence the map Δ takes values in the subspace of $\mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L})$ where the multiplication is defined, thus one can apply the universal property to obtain $\Delta : \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L})$. For elements $D, E \in \mathcal{U}(\mathcal{L})$ it is given by $\Delta(DE) = \Delta(D)\Delta(E) = D_{(1)}E_{(1)} \otimes D_{(2)}E_{(2)}$ using Sweedler notation.

The anchor map $\rho : \mathcal{L} \to \mathrm{Der}(\mathcal{O}_X)$ extends, again by the universal property, to a map $\rho : \mathcal{U}(\mathcal{L}) \to \mathrm{End}(\mathcal{O}_X)$. In fact, the image of this map is contained in the differential operators of $\mathcal{O}_X$, i.e. $\rho : \mathcal{U}(\mathcal{L}) \to \mathrm{Diff}(\mathcal{O}_X)$, therefore the algebra $\mathcal{U}(\mathcal{L})$ is sometimes called the algebra of $\mathcal{L}$-differential operators. This map will be used in chapter 3. The universal enveloping algebra admits a natural ascending filtration, denoted by

$$F^0\mathcal{U}(\mathcal{L}) \subset F^1\mathcal{U}(\mathcal{L}) \subset \dots, \tag{2.1.2}$$

where elements of degree zero are given by $i(\mathcal{O}_X)$ and $\mathcal{U}(\mathcal{L})^p$ is defined to be the $\mathcal{O}_X$-submodule of $\mathcal{U}(\mathcal{L})$ which is generated by elements of $i(\mathcal{L})^p$. Because $\mathcal{L}$ is locally free, the associated graded algebra is isomorphic to $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$.

Remark 2.1.2. The universal enveloping algebra is a so-called left Hopf algebroid, c.f. [KK]. The notion of a left Hopf algebroid is weaker than the notion of a Hopf algebroid, as the authors explained in [KP]; the difference being that left Hopf algebroids are not required to have an antipode. Let us summarize the structures of the left Hopf algebroid $\mathcal{U}(\mathcal{L})$.

1. The universal enveloping algebra is endowed with a multiplication m. The unit is given by the natural inclusion of $\mathcal{O}_X$ in $\mathcal{U}(\mathcal{L})$.

2. The comultiplication
$$\Delta : \ \mathcal{U}(\mathcal{L}) \longrightarrow \mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L})$$
which is defined in (2.1.1), is an $\mathcal{O}_X$-bimodule morphism, where both the left and right module structures are given by left multiplication. The image of this map is contained in the subspace on which the entrywise multiplication is defined. The counit for the comultiplication is the composition of $\rho : \mathcal{U}(\mathcal{L}) \to \mathrm{End}(\mathcal{O}_X)$ and the evaluation in 1:
$$\epsilon : \ \mathcal{U}(\mathcal{L}) \longrightarrow \mathcal{O}_X, \qquad D \mapsto D(1).$$
It is an $\mathcal{O}_X$-bimodule morphism, where both the left and right module structures are given by left multiplication.

3. The coproduct is multiplicative, i.e.
$$\Delta(DE) = \Delta(D)\Delta(E).$$

4. The Hopf–Galois map:
$$\beta : \ \mathcal{U}(\mathcal{L}) \otimes^{rl} \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L}) \otimes^{ll} \mathcal{U}(\mathcal{L}), \qquad D \otimes E \mapsto D_{(1)} \otimes D_{(2)} E$$
is an isomorphism.

2.1.2 The jet algebra

The structures

The jet algebra is simply defined as the dual of the universal enveloping algebra:
$$\mathcal{J}(\mathcal{L}) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{U}(\mathcal{L}), \mathcal{O}_X),$$
To be more precise, recall the ascending filtration
$$\ldots \subset F^k\mathcal{U}(\mathcal{L}) \subset F^{k+1}\mathcal{U}(\mathcal{L}) \subset \ldots,$$
on $\mathcal{U}(\mathcal{L})$. Define $F^k\mathcal{J}(\mathcal{L}) := \mathrm{Hom}_{\mathcal{O}_X}(F_k\mathcal{U}(\mathcal{L}), \mathcal{O}_X)$, where the $\mathcal{O}_X$-linearity holds with respect to the left $\mathcal{O}_X$-module structure on $\mathcal{U}(\mathcal{L})$. The jet algebra $\mathcal{J}(\mathcal{L})$ is defined as the projective limit:
$$\mathcal{J}(\mathcal{L}) = \mathrm{proj}\lim_k F^k\mathcal{J}(\mathcal{L}).$$
The jet algebra is a Hopf algebroid, c.f. [CvdB, KP, NeTs01]. Let us discuss the structures of this concept. We can dualize the cocommutative coproduct on $\mathcal{U}(\mathcal{L})$ to obtain a product on $\mathcal{J}(\mathcal{L})$:
$$(\phi_1\phi_2)(D) := \phi_1(D_{(1)})\phi_2(D_{(2)}), \quad \phi_1, \phi_2 \in \mathcal{J}(\mathcal{L}), \ D \in \mathcal{U}(\mathcal{L}). \tag{2.1.3}$$

This defines a commutative algebra structure on $\mathcal{J}(\mathcal{L})$. The dual of the counit ϵ of $\mathcal{U}(\mathcal{L})$ is the unit

$$\epsilon : \mathcal{O}_X \longrightarrow \mathcal{J}(\mathcal{L}), \qquad 1 \mapsto (D \mapsto D(1)).$$

Analogous to the right and left $\mathcal{O}_X$-module structures on $\mathcal{U}(\mathcal{L})$, there exist two $\mathcal{O}_X$-module structures on $\mathcal{J}(\mathcal{L})$, defined by

$$\begin{aligned} \alpha_1(r)(D) &:= \epsilon(rD) = r\epsilon(D) = rD(1), \\ \alpha_2(r)(D) &:= \epsilon(Dr) = D(r), \qquad r \in \mathcal{O}_X,\ D \in \mathcal{U}(\mathcal{L}). \end{aligned}$$

Remark 2.1.3. We will use the different $\mathcal{O}_X$-module structures on $\mathcal{J}(\mathcal{L})$ frequently throughout this thesis. When they occur in tensor products or homomorphisms, we will write this as follows:

$$\mathrm{Hom}_{\mathcal{O}_{X_i}}(\mathcal{J}(\mathcal{L}), \mathcal{E}) \qquad \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_i}} \mathcal{E},$$

where $\mathcal{E}$ is an $\mathcal{O}_X$ module. In the rare occasions that we consider the tensor product of $\mathcal{J}(\mathcal{L})$ with itself, we will either explicitly mention which $\mathcal{O}_X$-module structures we use, or we will write

$$\mathcal{J}(\mathcal{L})_{\alpha_i} \otimes_{\alpha_j} \mathcal{J}(\mathcal{L}).$$

These $\mathcal{O}_X$-module structures can be upgraded to two $\mathcal{U}(\mathcal{L})$-module structures. We formulate this using the notion of a flat connection, since we have seen in (1.2.2) that $\mathcal{U}(\mathcal{L})$-modules corresponds to flat $\mathcal{L}$-connections.

Lemma 2.1.4. *There are two flat $\mathcal{L}$-connections $\nabla^{(1)}$ and $\nabla^{(2)}$ on $\mathcal{J}(\mathcal{L})$ with respect to the two $\mathcal{O}_X$-module structures α_1 and α_2. They are defined by*

$$\nabla^{(1)}_X \phi(D) := \phi(XD) - X(\phi(D)) \tag{2.1.4a}$$

$$\nabla^{(2)}_X \phi(D) := \phi(DX) \tag{2.1.4b}$$

for $X \in \mathcal{L}$, $\phi \in \mathcal{J}(\mathcal{L})$, $D \in \mathcal{U}(\mathcal{L})$, and act by derivations. As $\nabla^{(1)}$ is compatible with α_1 and $\nabla^{(2)}$ with α_2, the flat connections define two commuting $\mathcal{U}(\mathcal{L})$-module structures on $\mathcal{J}(\mathcal{L})$, written $\cdot_1$ and $\cdot_2$, which are explicitly given by

$$\begin{aligned} D \cdot_1 \phi(E) =& D_-(\phi(D_+) \\ D \cdot_2 \phi(E) =& \phi(ED) \end{aligned}$$

Proof. In the literature, the connection $\nabla^{(1)}$ is sometimes denoted by ∇^G and referred to as the *Grothendieck connection*. To prove that $\nabla^{(i)}$ are connections, we have to show that

$$\nabla^{(i)}_{rX} \phi(D) = \alpha_i(r) \nabla^{(i)}_X \phi(D) \tag{2.1.5}$$

$$\nabla^{(i)}_X \alpha_i(r)\phi(D) = \alpha_i(X(r))\phi(D) + \alpha_i(r)\nabla^{(i)}_X \phi(D). \tag{2.1.6}$$

First we give some useful identities:

$$\begin{aligned} \alpha_1(r)\phi(D) =& \alpha_1(r)(D_{(1)})\phi(D_{(2)}) & \alpha_2(r)\phi(D) =& \alpha_2(r)(D_{(1)})\phi(D_{(2)}) \\ =& r\alpha_1(D_{(1)})\phi(D_{(2)}) & =& (Dr)_{(1)}(1)\phi((Dr)_{(2)}) \\ =& r\phi(D) & =& \phi(Dr). \end{aligned}$$

For both identities we used that α_1 is the unit for $\mathcal{J}(\mathcal{L})$, and on the RHS we also used $\Delta(r) = r \otimes 1$. Let us now prove Equation (2.1.5):

$$\begin{aligned} \nabla^{(1)}_{rX} \phi(D) =& r(\nabla^{(1)}_X \phi(D)) & \nabla^{(2)}_{rX} \phi(D) =& \phi(DrX) \\ =& \alpha_1(r)\nabla^{(1)}_X \phi(D) & =& \alpha_2(r)\nabla^{(2)}_X \phi(D). \end{aligned}$$

Before proving that Equation (2.1.6) holds, we show that $\nabla_X^{(i)}$ act by derivations. We use that $\Delta(XD) = XD_{(1)} \otimes D_{(2)} + D_{(1)} \otimes XD_{(2)}$ and $\Delta(DX) = D_{(1)}X \otimes D_{(2)} + D_{(1)} \otimes D_{(2)}X$.

$$\begin{aligned}\nabla_X^{(1)}(\phi_1\phi_2)(D) =& X(\phi_1\phi_2(D)) - \phi_1\phi_2(XD)\\ =& X(\phi_1(D_{(1)})\phi_2(D_{(2)})) - \phi_1((XD)_{(1)})\phi_2((XD)_{(2)})\\ =& X(\phi_1(D_{(1)}))\phi_2(D_{(2)}) + \phi_1(D_{(1)})X(\phi_2(D_{(2)})\\ &- \phi_1(XD_{(1)})\phi_2(D_{(2)}) - \phi_1(D_{(1)})\phi_2(XD_{(2)})\\ =& (\nabla_X^{(1)}\phi_1)\phi_2(D) + \phi_1(\nabla_X^{(1)}\phi_2)(D)\end{aligned}$$

and

$$\begin{aligned}\nabla_X^{(2)}(\phi_1\phi_2)(D) =& \phi_1\phi_2(DX)\\ =& \phi_1(D_{(1)}X)\phi_2(D_{(2)}) + \phi_1(D_{(1)})\phi_2(D_{(2)}X)\\ =& (\nabla_X^{(2)}\phi_1)\phi_2(D) + \phi_1(\nabla_X^{(2)}\phi_2)(D).\end{aligned}$$

These properties, together with the easy fact that $\nabla_X^{(i)}\alpha_i(r) = \alpha_i(X(r))$ for $i = 1, 2$, imply Equation (2.1.6):

$$\begin{aligned}\nabla_X^{(i)}(\alpha_i(r)\phi)(D) =& \nabla_X^{(i)}\alpha_i(r)\phi(D) + \alpha_i(r)\nabla_X^{(i)}\phi(D)\\ =& \alpha_i(X(r))\phi(D) + \alpha_i(r)\nabla_X^{(i)}\phi(D).\end{aligned}$$

The fact that both connections are flat is straightforward and is omitted. The expression for the second $\mathcal{U}(\mathcal{L})$-module structure is obvious, an the expression for the first $\mathcal{U}(\mathcal{L})$-module structure follows from the fact that it holds on degree 0 and 1 terms, and the fact that the action factorizes through $\mathcal{U}(\mathcal{L}) \otimes^{rl} \mathcal{U}(\mathcal{L})$, together with the universal property. This finishes the proof. □

We give a very easy, but important lemma that finds its application in the Fedosov resolutions of chapter 4 and the appendix.

Lemma 2.1.5. *The two $\mathcal{U}(\mathcal{L})$-module structures on $\mathcal{J}(\mathcal{L})$ that were defined in the previous lemma commute. In particular this implies that for all $r \in \mathcal{O}_X$ the following holds:*

$$\nabla^{(1)}(\alpha_2(r)) = 0 \qquad\qquad \nabla^{(2)}(\alpha_1(r)) = 0.$$

Proof. Left to the reader. □

The filtration

The ascending filtration on $\mathcal{U}(\mathcal{L})$ can be dualized to a descending filtration:

$$\mathcal{J}(\mathcal{L}) \supset \mathcal{J}(\mathcal{L})_1 \supset \mathcal{J}(\mathcal{L})_2 \supset \dots,$$

where $\mathcal{J}(\mathcal{L})_k : \{\phi \in \mathcal{J}(\mathcal{L}) : \phi(D) = 0 \text{ for all } D \in F^{k-1}\mathcal{U}(\mathcal{L})\}$. An alternative description of this filtration is given by the following: The unit $\epsilon : \mathcal{J}(\mathcal{L}) \to \mathcal{O}_X$ satisfies $\ker(\epsilon) = \mathcal{J}(\mathcal{L})_1$, and a computation shows that $(\ker(\epsilon))^k = \mathcal{J}(\mathcal{L})_k$. The inclusion to the right is fairly easy if one uses that the coproduct on $\mathcal{U}(\mathcal{L})$ respects the filtration. For the left inclusion a local argument suffices. We say that the descending filtration on $\mathcal{J}(\mathcal{L})$ is equal to the adic filtration with respect to $\ker(\epsilon)$. It is easy to see that the associated graded algebra to the filtration on $\mathcal{J}(\mathcal{L})$ is isomorphic to $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. The adic filtration on $\mathcal{J}(\mathcal{L})$ defines the adic topology. The open sets in this topology are given by $\mathcal{J}(\mathcal{L})_k$. One easily sees that the map

$$H : \mathcal{U}(\mathcal{L}) \longrightarrow \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}), \mathcal{O}_X), \quad D \mapsto (\phi \mapsto \phi(D))$$

is well-defined, and we have

Lemma 2.1.6 ([CRvdB10], lemma 5.1)**.** *The map* H *is an isomorphism of* $\mathcal{O}_X$*-bimodules.*

Let us now summarize the Hopf algebroid structure on $\mathcal{J}(\mathcal{L})$ that is defined in [KP] and, in a slightly different way, [CRvdB10]. We combine the two results:

Theorem 2.1.7 ([KP, CRvdB10])**.** *The commutative, associative algebra* $\mathcal{J}(\mathcal{L})$ *is a Hopf algebroid, i.e. it has the following structure maps*

1. *A counit* $\epsilon : \mathcal{J}(\mathcal{L}) \to \mathcal{O}_X$*, defined by* $\epsilon(\phi) := \phi(1)$*, which is an algebra morphism of* $\mathcal{O}_X$*-bimodules.*
2. *Two unit maps* $\alpha_i : \mathcal{O}_X \to \mathcal{J}(\mathcal{L})$ *which are algebra morphisms. They satisfy* $\epsilon \circ \alpha_i = \mathrm{id}|_{\mathcal{O}_X}$.
3. *A comultiplication* $\Delta : \mathcal{J}(\mathcal{L}) \to \mathcal{J}(\mathcal{L})_{\alpha_2} \widehat{\otimes}_{\alpha_1} \mathcal{J}(\mathcal{L})$ *which is a coassociative* $\mathcal{O}_X$*-bimodule morphism, with the the left* $\mathcal{O}_X$*-module structure defined by* α_1*, and the right* $\mathcal{O}_X$*-module structure by* α_2*, both on* $\mathcal{J}(\mathcal{L})$ *and* $\mathcal{J}(\mathcal{L}) \to \mathcal{J}(\mathcal{L})_{\alpha_2} \widehat{\otimes}_{\alpha_1} \mathcal{J}(\mathcal{L})$.
4. *An antipode* $S : \mathcal{J}(\mathcal{L}) \to \mathcal{J}(\mathcal{L})$ *interchanging the two* $\mathcal{U}(\mathcal{L})$*-module structures and satisfying* $S^2 = \mathrm{id}|_{\mathcal{J}(\mathcal{L})}$.

The structure maps satisfy, for all $\phi \in \mathcal{J}(\mathcal{L})$,

$$\begin{aligned} \alpha_1(\epsilon(\phi_{(1)}))\phi_{(2)} =& \phi = \phi_{(1)}\alpha_2(\epsilon(\phi_{(2)})) \\ \alpha_1 \circ \epsilon(\phi) =& S(\phi_{(1)})\phi_{(2)} \\ \alpha_2 \circ \epsilon(\phi) =& \phi_{(1)}S(\phi_{(2)}). \end{aligned}$$

We omit the proof, but make a few remarks. In [KP] it is proved that the map $\phi \otimes \psi \mapsto (D \otimes E \mapsto \phi(D\psi(E)))$ is an isomorphism

$$\mathcal{J}(\mathcal{L})_{\alpha_2} \widehat{\otimes}_{\alpha_1} \mathcal{J}(\mathcal{L}) \cong \varprojlim_p \mathrm{Hom}(F^p(\mathcal{U}(\mathcal{L}) \otimes^{rl} \mathcal{U}(\mathcal{L})), \mathcal{O}_X).$$

Therefore the coproduct on $\mathcal{J}(\mathcal{L})$ can be defined by dualizing the product:

$$\phi(DE) =: \phi_{(1)}(D\phi_{(2)}(E))$$

where we used Sweedler notation. The $\mathcal{O}_X$-linearity of the multiplication on $\mathcal{U}(\mathcal{L})$ implies that the comultiplication on $\mathcal{J}(\mathcal{L})$ is $\mathcal{O}_X$-bilinear. The antipode S is defined by

$$S\phi(D) := D._1\phi(1)$$

and it is easy to check that it interchanges the two $\mathcal{U}(\mathcal{L})$-module structures on $\mathcal{J}(\mathcal{L})$.

2.2 Incarnations and properties of the PBW theorem

2.2.1 The exponential map

In this section we construct a PBW map for the jet algebra of a Lie algebroid, depending on the choice of a connection. Recall that Rinehart proved in [Rin] that $\mathrm{gr}(\mathcal{U}(L, R)) \cong \mathrm{Sym}_R(L)$ for a Lie–Rinehart pair (L, R) under the condition that L is projective over R. However, he did not construct a map $\mathrm{Sym}_R(L) \longrightarrow \mathcal{U}(L, R)$ inducing this isomorphism. Such a map is the content of our PBW theorem. We fix a locally free sheaf of Lie algebroids $\mathcal{L}$ of constant rank r over a ringed space $(X, \mathcal{O}_X)$. We assume that the sheaf $\mathcal{L}$ admits an $\mathcal{L}$-connection $\nabla^{\mathcal{L}}$, and fix one.

Definition 2.2.1. *Consider the sheaf* $\mathcal{L}$ *equipped with the zero bracket and the trivial action:* $(\mathcal{L}, [\ ,\ ] = 0, \rho = 0)$ *over* $(X, \mathcal{O}_X)$*. The symmetric algebra* $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$ *is defined as the associated universal enveloping algebra of this algebroid. The associated jet algebra is given by*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}), \mathcal{O}_X) \cong \varprojlim_k \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee)/\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee)_{\geqslant k} =: \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee).$$

We denote the descending filtration on this jet algebra by $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}) \supset J \supset J^2 \supset \dots$. The $\mathcal{L}$-connection $\nabla^{\mathcal{L}}$ on $\mathcal{L}$ induces $\mathcal{L}$-connections on tensor products of copies of $\mathcal{L}$ and $\mathcal{L}^\vee$ by the usual formulas, which we all denote by $\nabla^{\mathcal{L}}$. In particular, we obtain connections on the tensor algebra $T_{\mathcal{O}_X}(\mathcal{L})$, the symmetric algebra $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$ and on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$.

The connection acts by derivations on the tensor algebra and the symmetric algebra. Moreover, the product on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ is induced by the coproduct on $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$ and the connection acts by coderivations with respect to this coproduct, as can be easily checked. Hence the connection acts by derivations on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. The connection on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ can be symmetrized;

$$\nabla^{\mathcal{L}}_s \beta(X_1 \cdots X_k) := \frac{1}{k}\sum_{i=1}^{k} (\nabla^{\mathcal{L}}_{X_i}\beta)(X_1 \cdots \widehat{X}_i \cdots X_k), \qquad \beta \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee),\ \ X_i \in \mathcal{L}. \tag{2.2.1}$$

We claim that this is a derivation of $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L})^\vee$ with the property

$$\nabla^{\mathcal{L}} :\ J^k \longrightarrow J^{k+1}. \tag{2.2.2}$$

The latter fact is clear, and the derivation property follows from the fact that the connection $\nabla^{\mathcal{L}}$ acts by derivations. It is explained in more detail in the lemma below. Consider the algebra

$$\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

with entrywise multiplication. We define a connection D on this algebra by the formula:

$$D_X(\phi \otimes \beta) = \nabla^{(2)}_X \phi \otimes \beta + \phi \otimes \nabla^{\mathcal{L}}_X \beta.$$

It is well-defined because

$$\begin{aligned} D_X(\alpha_2(r)\phi \otimes \beta) =& \alpha_2(X(r))\phi \otimes \beta + \alpha_2(r)\nabla^{(2)}_X \phi \otimes \beta + \alpha_2(r)\phi \otimes \nabla^{\mathcal{L}}_X \beta \\ =& \phi \otimes X(r)\beta + \nabla^{(2)}_X \phi \otimes r\beta + \phi \otimes r\nabla^{\mathcal{L}}_X \beta \\ =& D_X(\phi \otimes r\beta) \end{aligned}$$

and acts by derivations since both $\nabla^{(2)}$ and $\nabla^{\mathcal{L}}$ both act by derivations. Let us remark that the connection can be viewed as the natural connection on $\mathrm{Hom}_{\mathcal{O}_{X_2}}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}), \mathcal{J}(\mathcal{L}))$.

Lemma 2.2.2. *The symmetrized version of this connection, denoted by* D_s *and defined by*

$$D_s(\phi \otimes \beta)(X_1 \cdots X_k) := \sum_{i=1}^{k} D_{X_i}(\phi \otimes \beta)(X_1 \cdots \widehat{X}_i \cdots X_k),$$

is a derivation with the property (2.2.2) *for the component in* $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$.

Proof. We will compute the left and right hand side of $D_s((\phi \otimes \beta_1)(\psi \otimes \beta_1)) = (D_s(\phi \otimes \beta_1))(\psi \otimes \beta_2) + (\phi \otimes \beta_1)D_s(\psi \otimes \beta_2)$ evaluated on elements $X_i \in \mathcal{L}$. Using that

$$\beta_1\beta_2(X_1 \cdots X_k) := \sum_{\sigma \in S^k} \beta_1(X_{\sigma(1)} \cdots X_{\sigma(i)})\beta_2(X_{\sigma(i+1)} \cdots X_{\sigma(k)})$$

for $\beta_1, \beta_2 \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ and the fact that D acts by derivations it follows that

$$\begin{aligned} &D_s((\phi \otimes \beta_1)(\psi \otimes \beta_2))(X_1 \cdots X_k) \\ &\quad = \sum_{i=1}^{k} D_{X_i}((\phi \otimes \beta_1)(\psi \otimes \beta_2))(X_1 \cdots \widehat{X}_i \cdots X_k) \\ &\quad = \sum_{\sigma \in S^k} \Big((D_{X_{\sigma(1)}}(\phi \otimes \beta_1))(X_{\sigma(2)} \cdots X_{\sigma(i)})(\psi \otimes \beta_2)(X_{\sigma(i+1)} \cdots X_{\sigma(k)}) \\ &\qquad\qquad + (\phi \otimes \beta_1)(X_{\sigma(1)} \cdots X_{\sigma(i-1)})(D_{X_{\sigma(i)}}(\psi \otimes \beta_2))(X_{\sigma(i+1)} \cdots X_{\sigma(k)}) \Big). \end{aligned}$$

Moreover, we have

$$\begin{aligned}(D_s(\phi\otimes\beta_1))(\psi\otimes\beta_2)(X_1\cdots X_k)\\ &= \sum_{\sigma\in S^k}(D_s(\phi\otimes\beta_1))(X_{\sigma(1)}\cdots X_{\sigma(i)})(\psi\otimes\beta_2)(X_{\sigma(i+1)}\cdots X_{\sigma(k)})\\ &= \sum_{\sigma\in S^k}(D_{X_{\sigma(1)}}(\phi\otimes\beta_1))(X_{\sigma(2)}\cdots X_{\sigma(i)})(\psi\otimes\beta_2)(X_{\sigma(i+1)}\cdots X_{\sigma(k)}),\end{aligned}$$

and a similar expression for $(\phi\otimes\beta)(D\psi\otimes\beta)$. This proves the lemma. □

The PBW map j_∇ is now defined as the composition

$$\mathcal{J}(\mathcal{L}) \xrightarrow{-\otimes 1} \mathcal{J}(\mathcal{L})\otimes_{\mathcal{O}_{X_2}} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \xrightarrow{e^D} \mathcal{J}(\mathcal{L})\otimes_{\mathcal{O}_{X_2}} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \xrightarrow{ev_1} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee), \tag{2.2.3}$$

where ev_1 means evaluation of a jet in $1\in\mathcal{U}(\mathcal{L})$. Note that this map is well-defined since the operator D_s has property (2.2.2) and hence the exponent e^D is well-defined. To be precise, we pair the image of an element $\phi\in\mathcal{J}(\mathcal{L})$ under the PBW morphism with $X_1\cdots X_k\in\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$:

$$j^k_\nabla(\phi)(X_1\cdots X_k) := j_\nabla(\phi)(X_1\cdots X_k) := \frac{1}{k!}D^k_s(\phi\otimes 1)(1,X_1\cdots X_k) =: \frac{1}{k!}D^k_s(\phi)(X_1\cdots X_k).$$

We introduced some notation along the way. We describe a recursive formula for this map in the following proposition:

Proposition 2.2.3. *For all* $k\in\mathbb{N}$ *the following identity holds:*

$$\begin{aligned} j^k_\nabla(\phi)(X_1\cdots X_k) &= \frac{1}{k}\sum_{i=1}^{k-1}\nabla^{\mathcal{L}}_{X_i}(j^{k-1}_\nabla(\phi))(X_1\cdots\hat{X}_i\cdots X_k)\\ &-\frac{1}{k}\sum_{i=1} j^{k-1}_\nabla(\nabla^{(1)}_{X_i}\phi)(X_1\cdots\hat{X}_i\cdots X_k)).\end{aligned}\tag{2.2.4}$$

Proof. This follows by a straightforward computation:

$$\begin{aligned} D^k_s(\phi)(X_1\cdots X_k)(1) &= \frac{1}{k}\sum_{i=1}^{k}\nabla^{(2)}_{X_i}(D^{k-1}_s\phi(X_1\cdots\hat{X}_i\cdots X_k))(1)\\ &-\frac{1}{k}\sum_{i=1}^{k}D^{k-1}_s(\phi)(\nabla^{\mathcal{L}}_{X_i}(X_1\cdots\hat{X}_i\cdots X_k))(1)\\ &=\frac{1}{k}\sum_{i=1}^{k}X_i(D^{k-1}_s\phi(X_1\cdots\hat{X}_i\cdots X_k)(1))\\ &-\frac{1}{k}\sum_{i=1}^{k}D^{k-1}_s\phi(X_1\cdots\hat{X}_i\cdots X_k)(X_i)\\ &-\frac{1}{k}\sum_{i=1}^{k}D^{k-1}_s(\phi)(\nabla^{\mathcal{L}}_{X_i}(X_1\cdots\hat{X}_i\cdots X_k))(1),\end{aligned}$$

where we see that the first and third part of the last expression combine to the first part of Equation (2.2.4), and the middle part of the last expression is equal to the second part of Equation (2.2.4). □

Theorem 2.2.4. *Let* $\mathcal{L}$ *be a locally free sheaf of Lie algebroids of constant rank* r *together with an* $\mathcal{L}$*-connection* $\nabla^{\mathcal{L}}$*. The map*

$$j_\nabla:\ \mathcal{J}(\mathcal{L})\longrightarrow\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee),$$

defined in (2.2.3), *is an* $\mathcal{O}_X$*-linear isomorphism of sheaves of algebras with respect to the first* $\mathcal{O}_X$*-module structure on* $\mathcal{J}(\mathcal{L})$. *The dual map*

$$j_\nabla^\vee : \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{L}) \longrightarrow \mathcal{U}(\mathcal{L}) \qquad \phi(j_\nabla^\vee(X)) = j_\nabla(\phi)(X),$$

where $X \in \operatorname{Sym}(\mathcal{L})$, $\phi \in \mathcal{J}(\mathcal{L})$, *is an isomorphism of filtered coalgebras in the category of* $\mathcal{O}_X$*-modules. Furthermore, the induced map*

$$\operatorname{gr}(j_\nabla^\vee) : \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{L}) \longrightarrow \operatorname{gr}(\mathcal{U}(\mathcal{L}))$$

is the canonical morphism defined considered by [Rin].

Proof. The $\mathcal{O}_X$-linearity of j_∇ follows from the fact that the actions of $\nabla^{(2)}$ and α_1 commute on $\mathcal{J}(\mathcal{L})$. Since the maps $-\otimes 1$, ev_1 and the exponent of a derivation on an algebra are algebra morphisms, j_∇ is an algebra morphism. Recall the adic filtrations on $\mathcal{J}(\mathcal{L})$ and $\widehat{\operatorname{Sym}}_{\mathcal{O}_X}(\mathcal{L})$

$$\mathcal{J}(\mathcal{L}) \supset I \supset I^2 \supset \dots, \qquad \widehat{\operatorname{Sym}}_{\mathcal{O}_X}(\mathcal{L}) \supset J \supset J^2 \supset \dots$$

Note that $\mathcal{J}(\mathcal{L})$ and $\widehat{\operatorname{Sym}}_{\mathcal{O}_X}(\mathcal{L})$ are complete with respect to those filtrations. Moreover, they induce topologies on $\mathcal{J}(\mathcal{L})$ and $\widehat{\operatorname{Sym}}_{\mathcal{O}_X}(\mathcal{L})$ and in [CRvdB10, lemma 5.1] it is shown that

$$\mathcal{U}(\mathcal{L}) \cong \operatorname{Hom}^{\text{cont}}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}), \mathcal{O}_X), \qquad \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{L}) \cong \operatorname{Hom}^{\text{cont}}_{\mathcal{O}_X}(\widehat{\operatorname{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee), \mathcal{O}_X).$$

Explicitly, it means that $\tilde{D} \in \operatorname{Hom}^{\text{cont}}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}), \mathcal{O}_X)$ if and only if there exists an n such that $\tilde{D}(\phi) = 0$ for all $\phi \in I^n$. Now we show that j_∇ respects these filtrations. Let $\phi \in \mathcal{J}(\mathcal{L})$. Using Proposition 2.2.3 it is easy to prove the euqation

$$\begin{aligned} j_\nabla(\phi)(X_1 \cdots X_k) &= \\ &= \frac{1}{k!}\left(\sum_{\sigma \in S^k} \phi(X_{\sigma(1)} \cdots X_{\sigma(k)}) - \sum_{\sigma \in S^k} \phi(\nabla^{\mathcal{L}}_{X_{\sigma(1)}}(X_{\sigma(2)} \cdots X_{\sigma(k)})) + \phi(\tilde{X}) \right), \end{aligned} \tag{2.2.5}$$

where $\tilde{X} \in F^{k-2}\mathcal{U}(\mathcal{L})$ is a complicated term involving the connection. In the remark after this proof we give a more explicit inductive formula from which the previous one can be deduced. Hence, if $\phi \in I^k$, i.e. $\phi(D) = 0$ for all $D \in F_{k-1}\mathcal{U}(\mathcal{L})$,

$$j_\nabla(\phi)(X_1 \cdots X_m) = 0, \qquad \text{forall } m < k,$$

i.e. $j_\nabla(\phi) \in J^l$. Because j_∇ respects the filtrations, the dual map $j_\nabla^\vee$ is well-defined. Moreover, $j_\nabla^\vee$ respects the coalgebra structures because j_∇ respects the product structures and it is $\mathcal{O}_X$-linear because j_∇ is. This leaves us to show that j_∇ and its dual are isomorphisms, which follows once we show that the associated graded morphism $\operatorname{gr}(j_\nabla^\vee)$ is an isomorphism. According to [Rin], the canonical morphism

$$\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{L}) \longrightarrow \operatorname{gr}(\mathcal{U}(\mathcal{L})), \qquad X_1 \cdots X_k \mapsto [i(X_1) \cdots i(X_k)]$$

is an isomorphism, hence if we show that $\operatorname{gr}(j_\nabla^\vee)$ is the canonical morphism, the required statements follow. Equation (2.2.5) shows that the morphism $j_\nabla^\vee$ is given by

$$j_\nabla^\vee(X_1 \cdots X_k) = \frac{1}{k!}\left(\sum_{\sigma \in S^k} \left(X_{\sigma(1)} \cdots X_{\sigma(k)} - \nabla^{\mathcal{L}}_{X_{\sigma(1)}}(X_{\sigma(2)} \cdots X_{\sigma(k)}) \right) + \tilde{X} \right),$$

and this shows that $\operatorname{gr}(j_\nabla^\vee)$ is indeed equal to the canonical morphism. □

Example 13. The adjoint representation ad of a Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-connection on $\mathfrak{g}$. For elements $X_1 \cdots X_k \in \operatorname{Sym}^k(\mathfrak{g})$, the PBW map is inductively defined by

$$j_{\text{ad}}^\vee(X_1 \cdots X_k) := \frac{1}{k} \sum_{i=1}^{k} \left(X_i \cdot j_{\text{ad}}^\vee(X_1 \cdots \hat{X}_i \cdots X_k) - j_{\text{ad}}^\vee(\text{ad}(X_i)(X_1 \cdots \hat{X}_i \cdots X_k)) \right),$$

see Equation (2.2.6). We claim that the second term vanishes for each k. This follows easily from the observation that each term $X_1 \cdots [X_i, X_j] \cdots X_k$ is cancelled by the term $X_1 \cdots [X_j, X_i] \cdots X_k$ because the connection is symmetrized. The PBW map $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ induced by the adjoint representation is therefore equal to the symmetrization map.

2.2.2 Compatibility with the smooth approach

Our approach to the PBW theorem via the dual map defined on jets can be related to that of [LSX14]. There, the authors work in the smooth category and consider pairs of Lie algebroids, which we will define below. First we relate our definition of the dual PBW map to their recursive definition of the PBW map. Consider $X := X_1 \cdots X_k \in \mathrm{Sym}^k(\mathcal{L})$ and $\phi \in J(\mathcal{L})$. Then

$$\begin{aligned}\phi(j_\nabla^\vee(X)) &= \langle j_\nabla^\vee(\phi), X\rangle \\ &= \frac{1}{k!} D^k \phi(X)(1) \\ &= \frac{1}{k!}\sum_{i=1}^k \nabla^{(2)}_{X_i}(D^{k-1}\phi(X_1\cdots\hat{X}_i\cdots X_k))(1) - D^{k-1}\phi(\nabla^{\mathcal{L}}_{X_i}(X_1\cdots\hat{X}_i\cdots X_k)) \\ &= \frac{1}{k}\sum_{i=1}^k \Big(\phi(X_i \cdot j_\nabla^\vee(X_1\cdots\hat{X}_i\cdots X_k)) - \phi(j_\nabla^\vee\nabla^{\mathcal{L}}_{X_i}(X_1\cdots\hat{X}_i\cdots X_k))\Big).\end{aligned}$$

Therefore, we find the recursive formula

$$j_\nabla^\vee(X_1\cdots X_k) = \frac{1}{k}\sum_{i=1}^k \Big(X_i \cdot j_\nabla^\vee(X_1\cdots\hat{X}_i\cdots X_k)) - j_\nabla^\vee(\nabla_{X_i}(X_1\cdots\hat{X}_i\cdots X_k))\Big). \tag{2.2.6}$$

This is precisely the definition of the PBW morphism in [LSX14, Remark 2.3] for the inclusion of the trivial Lie algebroid into $\mathcal{L}$.

For the remaining part of this subsection we work in the smooth category. As mentioned in the introduction of this chapter, in [NWX] the authors use the concept of a local Lie groupoid to define an exponential map in a geometrical way. The fact that each smooth Lie algbroid admits a local integration implies that this construction works for all smooth Lie algebroids. In [LSX14], the authors showed that their algebraic recursive definition for a map

$$\mathrm{Sym}(A) \longrightarrow \mathcal{U}(A)$$

coincides with the exponential map that is provided by the local Lie groupoid of [NWX]. We shortly discuss the geometrical construction from [NWX] and [LSX14]. To be precise, we work in the setting of [LSX14] and we use their notations for this subsection. Their initial data consists of an exact sequence of smooth, integrable Lie algebroids

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0,$$

together with a splitting $j : A/B \longrightarrow A$ and an A-connection on A/B which extends the Bott connection. The Bott connection is the B connection on A/B defined by

$$\nabla^{\mathrm{Bott}}_a q(a) := q([b, a]) \qquad a \in A,\ b \in B.$$

Standard examples of such exact sequences come from holomorphic Lie algebroids and foliations. They first observe that, given a vector bundle $E \to M$, an A-connection on E can be defined as a horizontal lifting $h : L \times_M E \to TE$ using the formula:

$$h(a_m, e_m) := e_*(\rho(a_m)) - \tau_{e_m}((\nabla_a e)_m).$$

In the above formula, $m \in M$, a and e are extensions of $a_m \in A_m$ and $e_m \in E_m$, e_* is the pushforward along e, and τ_{e_m} is the identification of E_m with its tangent space at the point e_m. This interpretation gives rise to the *geodesic vector field* Ξ on A/B, defined by

$$\Xi_x := h(j(x), x), \quad x \in A/B \tag{2.2.7}$$

where $h : A \times_M A/B \to T(A/B)$ is the horizontal lifting of the A-connection on A/B extending the Bott connection. Before we describe how the geodesic vector field is used to define the exponential map, let us relate it to the degree one derivation on $\widehat{\mathrm{Sym}}(A^\vee)$ that we defined to construct the PBW map. The formula for the derivation on $\widehat{\mathrm{Sym}}(A^\vee)$ can be generalized to a formula for a derivation on $\widehat{\mathrm{Sym}}((A/B)^\vee)$:

$$D(\beta)(x_1 \cdots x_n) := \sum_{i=1}^{n} \nabla_{j(x_i)} \beta(x_1 \cdots \hat{x}_i \cdots x_n) \tag{2.2.8}$$

for $x_i \in A/B$ and $\beta \in \mathrm{Sym}((A/B^\vee)$. Note that $\mathrm{Sym}((A/B)^\vee)$ maps injectively into $C^\infty(A/B)$ by the standard formula

$$\iota(\beta)(x_m) := \left\langle \beta, \underbrace{x_m \cdots x_m}_{n \text{ times}} \right\rangle$$

for $\beta \in \mathrm{Sym}^n((A/B)^\vee)$ and $x_m \in (A/B)_m$. For degree 0 elements it is given by

$$\iota = \pi^* : C^\infty(M) \to C^\infty(A/B).$$

The completion $\widehat{\mathrm{Sym}}((A/B)^\vee)$ however, does not have this property, hence we can only relate our derivation to the geodesic vector field on the subalgebra of polynomial functions. The following lemma was suggested by Ping Xu.

Lemma 2.2.5. *On the subalgebra* $\mathrm{Sym}((A/B)^\vee) \subset C^\infty((A/B)$, *the geodesic vector field as defined in* (2.2.7) *is equal to the derivation defined in* (2.2.8).

Proof. Since both the geodesic vector field and the derivation act by derivations, and since $\mathrm{Sym}((A/B)^\vee)$ is generated by degree 0 and 1 elements, we only have to check the claim for these degrees. Let $\pi^*(f) \in C^\infty(A/B)$, where $\pi : A/B \to M$ denotes the projection. We have to prove that $\Xi_{x_m}(\pi^*(f))(x_m) = \iota(D(f))(x_m)$, where $x_m \in A/B$. We compute the action of the geodesic vector field:

$$\begin{aligned}
\Xi_{x_m}(\pi^*(f))(x_m) :=& x_*(\rho(j(x_m))(\pi^*(f))(x_m) - \tau_{x_m}(\nabla_{j(x)}x)(\pi^*(f))(x_m) \\
=& \rho(j(x_m))(\pi^*(f) \circ x)(m) \\
=& \rho(j(x_m))(f)(m) \\
=& \pi^*(\rho(j(x_m))(f))(x_m)
\end{aligned}$$

where we used that the second term on the RHS involves a vector tangent to the fiber $(A/B)_m$, which annihilates $\pi^*(\mathbb{C}^\infty(M))$, and where x stands for a section of A/B extending x_m. On the other hand, Equation 2.2.8 immediately gives that

$$\begin{aligned}
\iota(D(f))(x_m) =& \langle D(f), x_m \rangle (m) \\
=& \rho(j(x_m))(f)(m) \\
=& \pi^*(\rho(j(x_m)(f))(x_m).
\end{aligned}$$

Now we turn to de degree 1 case. Let $\beta \in \mathrm{Sym}^1((A/B)^\vee)$. We have to prove that $\Xi_{x_m}(\iota(\beta))(x_m) = \iota(D(\beta))(x_m)$, which is shown by the following computation:

$$\begin{aligned}
\Xi_{x_m}(\iota(\beta))(x_m) =& x_*(\rho(j(x_m))(\iota(\beta)))(x_m) - \tau_{x_m}(\nabla_{j(x)}x)(\iota(\beta))(x_m) \\
=& \rho(j(x_m))(\iota(\beta) \circ x)(m) - \left\langle \beta, \nabla_{j(x)}x \right\rangle (m) \\
=& \rho(j(x_m))(\langle \beta, x_m \rangle)(m) - \left\langle \beta, \nabla_{j(x)}x \right\rangle (m) \\
=& \langle D(\beta), x_m \otimes x_m \rangle = \iota(D(\beta))(x_m).
\end{aligned}$$

Hence, the lemma is proved. □

Now we briefly discuss the exponential map using integrations. Let $\mathscr{A}$ be a Lie groupoid integrating A and $\mathscr{B}$ be a Lie groupoid integrating B.

Definition 2.2.6. *Let $\mathscr{A}$ be a Lie groupoid integrating a Lie algebroid A.*

1. *An s-path is a smooth curve $\gamma : I \to \mathscr{A}$ emanating from a point in $M \subset \mathscr{A}$ and fully contained in an s-fiber.*

2. *A ρ-path is a smooth curve $\beta : I \to A$ satisfying*

$$\rho(\beta(t)) = \pi_*(\beta'(t)).$$

Every ρ-path β uniquely determines an s-path γ and vice versa, by the following formula:

$$\beta(t) = \frac{d}{d\tau}(\gamma^{-1}(t)\gamma(\tau))_t.$$

Now we describe the exponential map as defined in [NWX, LSX14]. Let x be an element in A/B in the neighborhood of the zero section. Consider the integral curve $\beta_x(t)$ emanating from x defined by the geodesic vector field. The composition $j(\beta_x(t))$ is a ρ-path in B, hence it lifts to an s-path $g_x(t)$ in $\mathscr{A}$. The composition of this path with the canonical projection $p : \mathscr{A} \to \mathscr{A}/\mathscr{B}$, and finally the evaluation in $t = 1$, defines the exponential map, i.e. $\exp^{\nabla,j}(x) := p(g_x(1))$. It is proved in [LSX14] that it induces a local diffeomorphism $A/B \to \mathscr{A}/\mathscr{B}$ around the zero sections.

Given a smooth submersion $p : P \to M$ together with a section ϵ, consider the space $\mathcal{D}(P, M)$, which are maps $C^\infty(P) \to C^\infty(M)$ given by the composition of a differential operator on P tangent to the fiber, and ϵ^*.

To relate the exponential map to the PBW map $\mathrm{Sym}(A/B) \to \mathcal{U}(A)/\mathcal{U}(A)\Gamma(B)$, the following two identifications are used:

- The algebra $\Gamma(\mathrm{Sym}(A/B))$ is isomorphic to the algebra $\mathcal{D}(A/B, M)$.
- The algebra $\mathcal{U}(A)/\mathcal{U}(A)\Gamma(B)$ is isomorphic to the algebra $\mathcal{D}(\mathscr{A}/\mathscr{B}, M)$ with respect to the smooth submersion $s : \mathscr{A}/\mathscr{B} \to M$.

Theorem 2.2.7 ([LSX14])**.** *The local diffeomorphism of fiber bundles*

$$\exp^{\nabla,j} : A/B \to \mathscr{A}/\mathscr{B}$$

induces an isomorphism of coalgebras $\mathcal{D}(A/B, M) \to \mathcal{D}(\mathscr{A}/\mathscr{B}, M)$. This isomorphism coincides with the isomorphism that is defined by the recursion (2.2.6)*.*

2.2.3 The noncommutative PBW theorem

In this subsection we describe a noncommutative version of the PBW theorem for Lie algebroids. It is also contained in the appendix, but we include it here because it is used in chapter 4. Given a locally free sheaf of Lie algebroids of constant rank r, the inclusion $\alpha_2 : \mathcal{O}_X \to \mathcal{J}(\mathcal{L})$ can be extended to inclusions

$$\begin{aligned} \mathcal{L} &\longrightarrow \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{L}, & X &\mapsto \alpha_2(1) \otimes X \\ \mathcal{U}(\mathcal{L}) &\longrightarrow \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}), & D &\mapsto \alpha_2(1) \otimes D. \end{aligned}$$

Lemma 2.2.8 ([CvdB], Lemma 4.3.2). *Given a locally free Lie algebroid over $\mathcal{O}_X$, one has the following isomorphisms:*

$$\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{L} \cong \mathrm{Der}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) \qquad \phi \otimes X \mapsto (\psi \mapsto \phi\nabla^{(2)}\psi)$$
$$\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}) \cong \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) \qquad \phi \otimes D \mapsto (\psi \mapsto \phi D._2\psi)$$

where $D._2$ *indicates the* $\mathcal{U}(\mathcal{L})$*-module structure* (2.1.4b) *on* $\mathcal{J}(\mathcal{L})$*. The first is a* $J(\mathcal{L})$*-module isomorphism of Lie algebras, and the second is a* $\mathcal{J}(\mathcal{L})$*-linear isomorphism of algebras.*

It is proved by studying the associated graded spaces with respect to the filtrations on $\mathcal{U}(\mathcal{L})$ and $\mathcal{J}(\mathcal{L})$. The composition of differential operators in $\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$, which is the algebra structure, corresponds, under the isomorphism above, to the product

$$(\phi \otimes D) \cdot (\psi \otimes E) := \phi D_{(1)}(\psi) \otimes D_{(2)}E \tag{2.2.9}$$

on $\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} \mathcal{U}(\mathcal{L})$. The PBW map, defined in 2.2.4, is an $\mathcal{O}_X$-linear isomorphism

$$\mathcal{J}(\mathcal{L}) \cong \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

with respect to the first $\mathcal{O}_X$-module structure on $\mathcal{J}(\mathcal{L})$, hence the map $j_\nabla(K)(\beta) = j_\nabla(K(j_\nabla^{-1}(\beta)))$ induces an algebra isomorphism

$$j_\nabla : \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) \xrightarrow{\cong} \mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

for $K \in \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$ and $\beta \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. This proves the following theorem.

Theorem 2.2.9. *Let $\mathcal{L}$ be a locally free Lie algebroid of constant, finite rank over a ringed space* $(X, \mathcal{O}_X)$*. Any $\mathcal{L}$-connection ∇ on $\mathcal{L}$ induces an isomorphism of sheaves of algebras*

$$j_\nabla : \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}) \xrightarrow{\cong} \mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))$$

2.2.4 The derivative of the PBW map

In subsection we study the time-dependence of the PBW map on the family of connections

$$\nabla_t = \nabla_1 + t\gamma, \qquad t \in [0,1].$$

where $\gamma \in \Omega^1_{\mathcal{L}}(\mathrm{End}(\mathcal{L}))$. Clearly, this family of connections induces a family

$$j_t := j_{\nabla_t} : \mathcal{J}(\mathcal{L}) \longrightarrow \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

of PBW isomorphisms. We define θ_t via the equation:

$$\frac{d}{dt}(j_t(\phi)) = \theta_t(j_t(\phi)).$$

One has the following isomorphism for the $\mathcal{O}_X$-linear derivations on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$:

$$\mathrm{Der}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \cong \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \mathcal{L}.$$

Derivations $X \in \mathrm{Der}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ decompose as

$$X = \sum_{i=-1}^{\infty} X_i$$

where X_i raises the degree of elements $\beta \in \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ by i.

Proposition 2.2.10. *The map* θ_t *is an* $\mathcal{O}_X$*-linear derivation of* $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$*. If we write* $\theta_t = \sum_{k=-1}^{\infty} \theta_t^i$ *as above, the formulas*

$$\theta_t^{-1} = 0 = \theta_t^0, \quad \text{and} \quad \theta_t^1 = \gamma^s$$

hold for all $t \in [0,1]$.

Proof. The fact that θ_t is an $\mathcal{O}_X$-linear derivation follows from the fact that j_t is an algebra morphism which is $\mathcal{O}_X$-linear. Now recall that $j_t = \mathrm{ev}_1 \circ \exp(D_t^s) \circ - \otimes 1$, where

$$D_t^s := \left(\nabla^{(2)} \otimes 1 + 1 \otimes \nabla_t\right)^s$$

is a derivation of degree 1 on $\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ with respect to the degree in $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. The derivative of $\exp(D_t^s)$ is given by the following general formula, which follows from a straightforward computation:

$$\frac{d}{dt}(\exp D_t^s) = \frac{\exp \mathrm{ad}_{D_t^s} - 1}{\mathrm{ad}_{D_t^s}} \left(\frac{dD_t^s}{dt}\right) \circ \exp D_t^s. \tag{2.2.10}$$

Note that D_t^s and dD_t^s/dt do not commute in general. It is easy to see that the derivative $dD_t^s/dt = \mathrm{id} \otimes \gamma^s$, where γ^s is the $\mathcal{O}_X$-linear derivation of degree 1 on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ that corresponds to $\gamma \in \Omega^1_{\mathcal{L}}(\mathrm{End}(\mathcal{L})) = (\mathcal{L}^\vee)^{\otimes 2} \otimes \mathcal{L}$ via the natural projection $(\mathcal{L}^\vee)^{\otimes 2} \otimes \mathcal{L} \to \widehat{\mathrm{Sym}}^2_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \mathcal{L}$. Hence, since D_t^s has degree 1, γ^s is the term with the lowest degree in θ, and the statement follows since the evaluation map ev_1 intertwines $\mathrm{id} \otimes \gamma^s$ and γ^s. □

2.2.5 Twisting by a locally free sheaf

In the final section on the PBW theorem, we discuss how to incorporate a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules of finite, constant rank into the theory. Following [CvdB], we define the sheaf of $\mathcal{L}$-jets of $\mathcal{E}$ of order k as

$$\mathcal{J}^k(\mathcal{L};\mathcal{E}) := \mathcal{J}^k(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{E}.$$

We denote the full sheaf of infinite jets of $\mathcal{E}$, which is a projective limit, by $\mathcal{J}(\mathcal{L};\mathcal{E})$, and define the universal enveloping algebra with coefficients in $\mathcal{E}$ as

$$\mathcal{U}(\mathcal{L};\mathcal{E}) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L};\mathcal{E}),\mathcal{E}).$$

The following lemma follows directly from the definitions.

Lemma 2.2.11. *The sheaf* $\mathcal{J}(\mathcal{L};\mathcal{E})$ *of* $\mathcal{L}$*-jets of* $\mathcal{E}$ *has the structures of:*

1. *An* $\mathcal{O}_{X_1}$*-linear* $\mathcal{J}(\mathcal{L})$*-module by the assignment* $(\phi, \psi \otimes s) \mapsto \phi\psi \otimes s$.
2. *An* $\mathcal{O}_{X_1}$*-linear* $\mathcal{J}(\mathcal{L})$*-comodule by the assignment* $\phi \otimes s \mapsto \phi_{(1)} \otimes \phi_{(2)} \otimes s$.

Remark 2.2.12. The first oder jets fit into a short exact sequence

$$0 \to \mathcal{L}^\vee \otimes \mathcal{E} \to \mathcal{J}^1(\mathcal{L};\mathcal{E}) \to \mathcal{E} \to 0,$$

and a splitting $\sigma : \mathcal{E} \to \mathcal{J}^1(\mathcal{L};\mathcal{E})$ of this sequence in the category of sheaves of $\mathcal{O}_{X_1}$-modules is equivalent to an $\mathcal{L}$-connection on $\mathcal{E}$.

Lemma 2.2.13. *The universal enveloping algebra with coefficients in* $\mathcal{L}$ *admits a product by the following formula:*

$$D \circ E(\phi \otimes s) = D(\phi_{(1)} \otimes E(\phi_{(2)} \otimes s)), \qquad \phi \otimes s \in \mathcal{J}(\mathcal{L};\mathcal{E}),\ D, E \in \mathcal{U}(\mathcal{L};\mathcal{E})$$

where $\phi_{(1)} \otimes \phi_{(2)} = \Delta(\phi)$ *is the coproduct on* $\mathcal{J}(\mathcal{L})$.

Proof. The fact that $D \circ E$ is $\mathcal{O}_{X_1}$-linear follows from the the fact that the coproduct

$$\Delta\colon\ \mathcal{J}(\mathcal{L}) \longrightarrow \mathcal{J}(\mathcal{L}) \otimes \mathcal{J}(\mathcal{L})$$

is $\mathcal{O}_{X_1}$-linear with respect to the $\mathcal{O}_{X_1}$-module structure on the left copy of $\mathcal{J}(\mathcal{L})$ on the RHS. □

One should think about $\mathcal{U}(\mathcal{L};\mathcal{E})$ as $\mathcal{L}$-valued differential operators acting on $\mathcal{E}$, with the product simply given by composition. In fact, there is an equivalent definition of $\mathcal{U}(\mathcal{L};\mathcal{E})$ which is based on the Atiyah algebroid which is closer to this interpretation. It seems however easier to generalize the PBW theorem to the twisted case via the approach above. Recall that the Atiyah algebroid $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ of $\mathcal{E}$ with values in $\mathcal{E}$ fits into the following diagram

$$0 \longrightarrow \mathrm{End}(\mathcal{E}) \longrightarrow \mathrm{At}_{\mathcal{L}}(\mathcal{E}) \longrightarrow \mathcal{L} \longrightarrow 0$$

with exact rows and $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ defined as in Example 1.1.1. One can also define $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ as the subsheaf $\mathrm{End}_{\mathbb{K}}(\mathcal{E})$ of elements Y such that

$$[Y, f] = \sigma_Y(f) \qquad f \in \mathcal{O}_X,$$

for $\sigma_Y \in \mathcal{L}$. It is easy to check that the commutator and the morphism $Y \mapsto \sigma_Y$ together with the anchor map endow $\mathrm{At}_{\mathcal{L}}(\mathcal{E})$ with the structure of a Lie algebroid.

Lemma 2.2.14. *Given a Lie algebroid $\mathcal{L}$ over $(X, \mathcal{O}_X)$ and a locally free $\mathcal{O}_X$-module $\mathcal{E}$, we have the following canonical isomorphism:*

$$\mathcal{U}(\mathrm{At}_{\mathcal{L}}(\mathcal{E})) \cong \mathcal{U}(\mathcal{L};\mathcal{E}).$$

Proof. The proof is based on the universal property for $\mathcal{U}(\mathrm{At}_{\mathcal{L}}(\mathcal{E}))$. We define maps

$$\begin{aligned} \mathcal{O}_X &\longrightarrow \mathcal{U}(\mathcal{L};\mathcal{E}) \qquad && f \mapsto (\tilde{f}\colon\ \phi \otimes s \mapsto f\phi(1)s) \\ \mathrm{At}_{\mathcal{L}}(\mathcal{E})) &\longrightarrow \mathcal{U}(\mathcal{L};\mathcal{E}) \qquad && Y \mapsto (\tilde{Y}\colon\ \phi \otimes s \mapsto \phi(1)Y(s) + \phi(\sigma_Y)s). \end{aligned}$$

It is easy to see that $\tilde{f} \in \mathcal{U}(\mathcal{L};\mathcal{E})$ and that $\tilde{Y}$ is $\mathcal{O}_{X_1}$-linear. Moreover, we have

$$\begin{aligned} \tilde{Y}(\alpha_2(f)\phi \otimes s) &= f\phi(1)Y(s) + \phi(\sigma_Y f)s \\ &= f\phi(1)Y(s) + \phi(1)\sigma_Y(f)s + f\phi(\sigma_X)s \\ &= \phi(1)Y(fs) + \phi(f\sigma_X)s \\ &= \tilde{Y}(\phi \otimes fs), \end{aligned}$$

thus $\tilde{Y} \in \mathcal{U}(\mathcal{L};\mathcal{E})$. To apply the universal property, we have to show that

$$\widetilde{fY} = \tilde{f} \circ \tilde{Y}, \qquad \widetilde{\sigma_Y(f)} = [\tilde{Y}, \tilde{f}], \qquad \widetilde{[Y, X]} = [\tilde{Y}, \tilde{X}].$$

The first equality is a straightforward check. We compute:

$$\begin{aligned} \tilde{Y} \circ \tilde{f}(\phi \otimes s) &= \tilde{Y}(\phi_{(1)} \otimes \phi_{(2)}(1)fs) \\ &= \phi_{(1)}(1)Y(\phi_{(2)}(1)fs) + \phi_{(1)}(\sigma_Y)\phi_{(2)}(1)fs \\ &= \left(\phi_{(1)}(1)\sigma_Y(\phi_{(2)}(1)) + \phi_{(1)}(\sigma_Y)\phi_{(2)}(1)\right) fs \\ &\quad + \phi_{(1)}(1)\phi_{(2)}(1))fY(s) + \phi_{(1)}(1)\phi_{(2)}(1))\sigma_Y(f)s \\ &= \phi(\sigma_Y)fs + \phi(1)Y(s) + \phi(1)\sigma_Y(f)s \\ &= \tilde{f} \circ \tilde{Y}(\phi \otimes s) + \widetilde{\sigma_Y(f)}(\phi \otimes s), \end{aligned}$$

where we used the definition $\phi_{(1)}(D\phi_{(2)}(E) = \phi(DE)$ for the coproduct where $D,\ E \in \mathcal{U}(\mathcal{L})$ in the fourth equality. This implies the second equation. For the third equation we compute:

$$\begin{aligned} \tilde{Y} \circ \tilde{X}(\phi \otimes s) &= \tilde{Y}(\phi_{(1)} \otimes (\phi_{(2)}(1)X(s) + \phi_{(2)}(\sigma_X)s)) \\ &= \phi_{(1)}(1)\left(\sigma_Y(\phi_{(2)}(1))X(s) + \phi_{(2)}(1)YX(s) + \sigma_Y(\phi_{(2)}(\sigma_X))s + \phi_{(2)}(\sigma_X)Y(s)\right) \\ &\quad + \phi_{(1)}(\sigma_Y)\left(\phi_{(2)}(1)X(s) + \phi_{(2)}(\sigma_X)s\right). \end{aligned}$$

The terms in the commutator $[\tilde{Y}, \tilde{X}]$ which end with s are given by

$$\begin{aligned}&\phi_{(1)}(1)\sigma_Y(\phi_{(2)}(\sigma_X))s + \phi_{(1)}(\sigma_Y)\phi_{(2)}(\sigma_X)s\\ &- \phi_{(1)}(1)\sigma_X(\phi_{(2)}(\sigma_Y))s - \phi_{(1)}(\sigma_X)\phi_{(2)}(\sigma_Y)s\\ &= \phi(\sigma_Y\sigma_X - \sigma_X\sigma_Y)\end{aligned}$$

where we applied the rule

$$\begin{aligned}\phi(\sigma_X\sigma_Y) =&\phi_{(1)}(\sigma_X\phi_{(2)}(\sigma_Y))\\ =&\phi_{(1)}(\sigma_X(\phi_{(2)}(\sigma_Y))) + \phi_{(1)}\sigma_X)\phi_{(2)}(\sigma_Y).\end{aligned}$$

The terms ending with $Y(s)$ and $X(s)$ vanish as can be shown by a similar computation, and the terms ending with $YX(s)$ and $XY(s)$ are

$$\phi(1)YX(s) - \phi(1)XY(s).$$

Since the commutator $[Y, X]$ in the Atiyah algebroid has $\sigma_{[Y,X]} = [\sigma_Y, \sigma_X]$ as associated elements in $\mathcal{L}$, this proves the third equality. The universal property implies that there exists an algebra homomorphism

$$\mathcal{U}(\mathrm{At}_{\mathcal{L}}(\mathcal{E})) \longrightarrow \mathcal{U}(\mathcal{L}; \mathcal{E}).$$

The RHS has a natural ascending filtration defined by the descending filtration on $\mathcal{J}(\mathcal{L})$, and the LHS has a filtration which is defined by defining

$$D \in F^k\mathcal{U}(\mathcal{L}; \mathcal{E}) \quad \text{if } D = X_1 \cdots X_i Y_{i+1} \cdots Y_l, \quad i \leqslant k$$

where $\sigma_{Y_j} = 0$ for all j. It is easy to see that the map on the associated graded spaces is an isomorphism . □

Lemma 2.2.15. *Let $\nabla^{\mathcal{E}}$ be a flat $\mathcal{L}$-connection on $\mathcal{E}$. Then there exists an algebra morphism:*

$$a_{\nabla^{\mathcal{E}}} : \mathcal{U}(\mathcal{L}) \longrightarrow \mathcal{U}(\mathcal{L}, \mathcal{E}).$$

Proof. Since a flat connection is given by a morphism of Lie algebras

$$\nabla^{\mathcal{E}} : \mathcal{L} \longrightarrow \mathrm{At}_{\mathcal{L}}(\mathcal{E})$$

which is compatible with the $\mathcal{O}_X$-module structures and the action on $\mathcal{O}_X$, this follows from the universal property. □

Theorem 2.2.16. *Let $\mathcal{L}$ be a Lie algebroid over a ringed space $(X, \mathcal{O}_X)$, and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of constant finite rank. Any pair $(\nabla^{\mathcal{L}}, \nabla^{\mathcal{E}})$ of $\mathcal{L}$-connections on $\mathcal{L}$ and $\mathcal{E}$ induces an $\mathcal{O}_{X_1}$-linear isomorphism*

$$j_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}} : \mathcal{J}(\mathcal{L}; \mathcal{E}) \to \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}$$

such that

$$j_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}}(\phi\psi \otimes e) = j_{\nabla^{\mathcal{L}}}(\phi) j_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}}(\psi \otimes e) \tag{2.2.11}$$

holds for any $\phi \in \mathcal{J}(\mathcal{L})$, $\psi \otimes e \in \mathcal{J}(\mathcal{L}; \mathcal{E})$.

Proof. The connections $\nabla^{\mathcal{L}}, \nabla^{\mathcal{E}}, \nabla^{(2)}$ define a connection on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}; \mathcal{E})$ and symetrizing this connection leads to a map of degree 1

$$D_s : \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}; \mathcal{E}) \to \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}; \mathcal{E})$$

which has the following Leibniz property:

$$D_s(rs) = D_s(r)s + rD_s(s), \qquad r \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}), \quad s \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}; \mathcal{E}).$$

Here $D_s(r)$ stands for the derivation on $\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ defined in (2.2.4) acting on r, and $D_s(rs)$ and $D_s(s)$ stands for the map that we just defined. Moreover, we used the natural $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L})$-module structure on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L};\mathcal{E})$. The PBW map with coefficients is now defined as the composition

$$\mathcal{J}(\mathcal{L};\mathcal{E}) \xrightarrow{1\otimes -} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L};\mathcal{E}) \xrightarrow{e^D} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L};\mathcal{E}) \xrightarrow{ev_1} \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \mathcal{E}$$

The Leibniz property of D_s gives property (2.2.11). The $\mathcal{O}_{X_1}$-linearity and the fact that it is an isomorphism follows in the same way as in the proof of Theorem 2.2.4. ☐

Corollary 2.2.17. *Using the notation from the previous theorem, any pair $(\nabla^{\mathcal{L}}, \nabla^{\mathcal{E}})$ of $\mathcal{L}$-connections on $\mathcal{L}$ and $\mathcal{E}$ induces an isomorphism*

$$j^*_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}} : \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathrm{End}(\mathcal{E}) \to \mathcal{U}(\mathcal{L};\mathcal{E})$$

Proof. This follows from $\mathrm{Sym}(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathrm{End}(\mathcal{E}) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \mathcal{E}, \mathcal{E})$ and the fact that $j_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}}$ is $\mathcal{O}_{X_1}$-linear. ☐

There is also a noncommutative, twisted PBW theorem, which we describe now.

Theorem 2.2.18. *The data of $\mathcal{L}$-connections $\nabla^{\mathcal{L}}$ and $\nabla^{\mathcal{E}}$ on $\mathcal{L}$ and $\mathcal{E}$ induces an isomorphism of sheaves of algebras:*

$$\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L};\mathcal{E}) \longrightarrow \mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}).$$

Proof. Similar to the untwisted case, there is a canonical morphism

$$\begin{aligned} &\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L};\mathcal{E}) \to \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L};\mathcal{E})) \\ &\quad (\phi \otimes e \otimes D \otimes f) \mapsto (\psi \otimes s \mapsto \phi D \cdot_2 (\alpha_2(f(s))\psi) \otimes e). \end{aligned}$$

Where $e, s \in \mathcal{E}$, $f \in \mathcal{E}^\vee$, $\phi, \psi \in \mathcal{J}(\mathcal{L})$, $D \in \mathcal{U}(\mathcal{L})$ and D acts on $\mathcal{J}(\mathcal{L})$ using $\nabla^{(2)}$. Recall that differential operators of order k on the $\mathcal{J}(\mathcal{L})$-module $\mathcal{J}(\mathcal{L};\mathcal{E})$ are endomorphisms of $\mathcal{J}(\mathcal{L};\mathcal{E})$ satisfying $[\phi_0, [\ldots, [\phi_k, d] \ldots] = 0$ where $\phi_i \in \mathcal{J}(\mathcal{L})$ are considered as endomorphisms of $\mathcal{J}(\mathcal{L};\mathcal{E})$. It is straightforward to check that the morphism above is well-defined and has the range that is indicated. Moreover, it respects the filtrations from $\mathcal{U}(\mathcal{L})$ and $\mathcal{J}(\mathcal{L})$ on the LHS and the degree of differential operators and the $\mathcal{J}(\mathcal{L})$ filtration on the RHS. The fact that it is an isomorphism follows from the same argument as in the untwisted case. Using now that $\mathrm{Diff}(\widehat{\mathrm{Sym}}(\mathcal{L}^\vee) \otimes \mathcal{E}) \cong \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L};\mathcal{E}))$ by the morphism $D \mapsto j_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}} \circ D \circ j^{-1}_{\nabla^{\mathcal{L}},\nabla^{\mathcal{E}}}$, the claim follows. ☐

Chapter 3

Cyclic theory of the universal enveloping algebra

This chapter can be divided into two parts. In the first, we relate the linear Poisson geometry of $A^\vee$, the dual of a smooth Lie algebroid $A \to M$, to the symmetric powers of the adjoint representation (up to homotopy) of the Lie algebroid A. The result is an identification between the polynomial Poisson cohomology complex and the complex of the representation. Moreover, we show that the Poisson homology complex of $A^\vee$ can be related the tensor product of the symmetric powers of the adjoint representation and the natural representation Q_A -the bundle of A-densities.

In the second part we compute the Hochschild and cyclic (co)homology of the universal enveloping algebra $\mathcal{U}(A)$ of A. First we define the Hochschild and cyclic (co)homology taking into account the natural topology on $\mathcal{U}(A)$. The natural filtration on the universal enveloping algebra leads to a spectral sequence that, in principle, allows one to compute the Hochschild and cyclic (co)homology. However, it is not a priori clear that it degenerates. We solve this problem by constructing maps on the level of chains from the polynomial Poisson (co)homology complex to the Hochschild (co)complex of $\mathcal{U}(A)$. Finally, we discuss the relation between our results and the existing literature. More extensive introductions can be found at the beginning of the two sections.

3.1 Poisson (co)homology and representations up to homotopy

In this section we consider the Poisson geometry of $A^\vee$, the dual of a smooth Lie algebroid $A \to M$. Indeed, it is well-known that Lie algebroid structures on a given vector bundle $A \to M$ are in one-to-one correspondence with Poisson structures on $A^\vee$ which are linear along the fibers, see [CM, prop. 7] for a proof of this fact. Here we shall prove that his observation can be extended to a correspondence between (symmetric powers of) the adjoint representation (up to homotopy) of the Lie algebroid A, and the linear (polynomial) Poisson cohomology complex of the Poisson structure on $A^\vee$. Moreover, the Poisson homology complex of the Poisson structure on $A^\vee$ can be related to yet another representation up to homotopy; the tensor product of the symmetric powers of the adjoint representation and the natural representation Q_A -the bundle of A-densities.

3.1.1 Polynomial polyvector fields and differential forms

First we assume $A \to M$ to be a smooth vector bundle. Any section $s \in \Gamma(A)$ induces a fiberwise linear function s on $A^\vee$ by the formula

$$s(\alpha) := \left\langle s_{\pi(\alpha)}, \alpha_{\pi(\alpha)} \right\rangle_{\pi(\alpha)}, \quad \alpha \in A^\vee,$$

where $\langle \ , \ \rangle_x$ denotes the natural pairing between A_x and its dual $A_x^\vee$, for any $x \in M$. Sections of the symmetric algebra bundle $\mathrm{Sym}\, A$ likewise induce fiberwise polynomial functions on $A^\vee$ by the formula $\hat{g}(\alpha) := s_1 \cdots s_k(\alpha) :=< s_1, \alpha > \cdots < s_k, \alpha >$, where $g = s_1 \cdots s_k \in \Gamma(\mathrm{Sym}\, A)$ is the symmetric product of sections $s_i \in \Gamma(A)$. We will sometimes write:

$$C^\infty_{\mathrm{lin}}(A^\vee) = \Gamma(A), \qquad C^\infty_{\mathrm{pol}}(A^\vee) = \Gamma(\mathrm{Sym}\, A).$$

It is clear that the functions on $A^\vee$ which are polynomial along the fibers form a graded algebra, and the degree of elements can be detected as follows. The $\mathbb{R}_{>0}$-action by scaling A along the fibers of the projection to the base M is generated by the Euler vector field $\mathcal{E}$, which, choosing a local frame $\{e_i\}$ for A, can be expressed as

$$\mathcal{E} = \sum \hat{e}_i \frac{\partial}{\partial \hat{e}_i}. \tag{3.1.1}$$

It acts on $C^\infty(A)$ using the Lie derivative and detects the degree of a monomial, i.e., $L_{\mathcal{E}}(\hat{g}) = k\hat{g}$, for $g \in \Gamma(M, \mathrm{Sym}^k A)$.

Although we can identify the sections of A or $\mathrm{Sym}\, A$ with linear, respectively polynomial functions along fibers on $A^\vee$, we prefer to only use this observation as inspiration, and to work with the bundles themselves as well as their sections.

In this way, we can view the pair $(M, \mathrm{Sym}\, A)$ as a locally ringed space. Regarding the notation, we will not distinguish between the vector bundle $\mathrm{Sym}\, A$ and its sheaf of sections. The $\mathbb{R}$-linear derivations of this sheaf

$$\mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A) =: L$$

form a $\mathrm{Sym}\, A$-module. It is the space of sections of a vector bundle $\mathrm{Der}(\mathrm{Sym}\, A)$ over M, as the following lemma shows. It is analogous to [L-BM, prop. 4.1].

Proposition 3.1.1. *There is a short exact sequence of sheaves of Lie algebroids over* $(M, \mathrm{Sym}\, A)$*:*

$$0 \longrightarrow \mathrm{Sym}\, A \otimes A^\vee \longrightarrow \mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A) \longrightarrow \mathrm{Sym}\, A \otimes TM \longrightarrow 0. \tag{3.1.2}$$

A choice of a splitting of the global sections of this exact sequence is equivalent to a choice of a TM*-connection on* A*. In particular,* $\mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A)$ *is given by the sheaf of sections of a vector bundle* $\mathrm{Der}(\mathrm{Sym}\, A)$ *over* M*.*

Proof. First we describe the maps in the sequence above. The inclusion $\mathrm{Sym}\, A \otimes A^\vee \hookrightarrow \mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A)$ is defined by the contraction $\iota_\alpha : \mathrm{Sym}^k A \to \mathrm{Sym}^{k-1} A$ for $\alpha \in A^\vee$. The map $\mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A) \otimes T^*M \to \mathrm{Sym}\, A$ given by $D \otimes df \mapsto D(f)$ defines the projection $\mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A) \longrightarrow \mathrm{Sym}\, A \otimes TM$. These maps are $\mathrm{Sym}\, A$-linear and their composition is equal to 0. The injectivity of the first map is clear, whereas the surjectivity of the second follows from the fact that a choice of a connection ∇ on A defines a $\mathrm{Sym}\, A$-linear splitting by $g \otimes X \mapsto g\nabla_X$ where $g \otimes X \in \mathrm{Sym}\, A \otimes TM$. In particular, this splitting gives an isomorphism of the $C^\infty(M)$-modules $\mathrm{Der}_{\mathbb{R}}(\Gamma(\mathrm{Sym}\, A)) \cong \Gamma(\mathrm{Sym}\, A \otimes (A^\vee \oplus TM)$, and hence $\mathrm{Der}_{\mathbb{R}}(\Gamma(\mathrm{Sym}\, A))$ is given by the sheaf of sections of a vector bundle $\mathrm{Der}(\mathrm{Sym}\, A)$. □

Hereafter we will use the notation L for $\mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A)$. The grading on $\mathrm{Sym}\, A$ induces a grading on L. Let us describe the exact sequence of the lemma above in low degrees. It is clear that $L^{<-1}$ is zero. In degree -1 it only consists of two copies of $A^\vee$, and in degree 0 it is given by the short exact sequence

$$0 \longrightarrow A \otimes A^\vee \longrightarrow L^0 \longrightarrow TM \longrightarrow 0$$

where the middle bundle is the Atiyah algebroid, see equation (1.1.1), which is often used to *define* connections as splittings of this sequence.

Recall from proposition 1.1.12 that the Lie algebroid structure on L defines the Schouten–Nijenhuis bracket $[\cdot,\cdot]_{SN}$, the de Rham differential d_L and the Lie module structure L on the L-polyvectorfields and the L-differential forms:

$$\left(T^L_{\text{poly}}(A^\vee), \wedge, [\cdot,\cdot]_{SN}\right) \overset{L}{\rightsquigarrow} \left(\Omega_L(A^\vee), \wedge, d_L\right). \tag{3.1.3}$$

The notation is slightly inconsistent here; the polyvector fields associated to the sheaf of Lie algebroids L over $(M, \operatorname{Sym} A)$ should be denoted by $T^L_{\text{poly}}(M)$ if we would follow the notation from the last chapter, but we chose $T^L_{\text{poly}}(A^\vee)$ to indicate that the structure sheaf is $\operatorname{Sym} A$, the polynomial functions on $A^\vee$. A similar remark holds for $\Omega_L(A^\vee)$. Note that we view the objects as sheaves over M. The isomorphism $L \cong \operatorname{Sym} A \otimes (A^\vee \oplus TM)$ from proposition 3.1.1 can be extended to the polyvector fields $T^L_{\text{poly}}(A^\vee)$ and the dual of the exact sequence (3.1.2) gives a similar isomorphism for the L-de Rham forms:

$$T^L_{\text{poly}}(A^\vee) \overset{\nabla}{\cong} \bigoplus_k \bigoplus_{i+j=k} \operatorname{Sym} A \otimes \left(\bigwedge^i A^\vee \otimes \bigwedge^j TM\right) \tag{3.1.4}$$

$$\Omega_L(A^\vee) \overset{\nabla}{\cong} \bigoplus_k \bigoplus_{i+j=k} \operatorname{Sym} A \otimes \left(\bigwedge^i A \otimes \bigwedge^j T^*M\right), \tag{3.1.5}$$

both of which are $\operatorname{Sym} A$-linear. Perhaps this a good point to stipulate that there are *two* natural gradings on these algebras; the polynomial one which is given by assigning

$$\deg(A) = 1 \qquad \deg(TM) = 0$$
$$\deg(A^\vee) = -1 \qquad \deg(T^*M) = 0$$

and the "cohomological" degree which is given by $i + j$ in equations (3.1.4) and (3.1.5).

It is easy to see that the graded commutative products of the LHS correspond to the products on the RHS which are given by the natural symmetric product on $\operatorname{Sym} A$ and the wedge product on the graded commuting parts. The insertion operator ι of polyvector fields into differential forms is given by the natural pairings of A and $A^\vee$, TM and T^*M, and the multiplication on $\operatorname{Sym} A$. The following lemma describes the other natural structures on the polyvector fields and differential forms of L on the RHS of equations (3.1.4) and (3.1.5)

Lemma 3.1.2. *Under the identifications* (3.1.4) *and* (3.1.5)*, we have the following:*

1. *The Schouten–Nijenhuis bracket is determined by the formula*

$$[(\alpha, X), (\beta, Y)] = \nabla_X \beta - \nabla_Y \alpha + [X, Y] + R_\nabla(X, Y),$$

where $X, Y \in \wedge^1 TM$ *and* $\alpha, \beta \in \wedge^1 A^\vee$, R_∇ *is the curvature and* $R_\nabla(X,Y) \in \operatorname{Sym}^1 A \otimes \wedge^1 A^\vee$.

2. *The de Rham differential* $d_{A^\vee}$ *on* $\Omega_L(A^\vee)$ *corresponds to* d_∇*, which, on polynomial functions, splits as*

$$\begin{array}{ccc} \operatorname{Sym}^k A & \xrightarrow{\ \delta\ } & \operatorname{Sym}^{k-1} A \otimes \Lambda^1 A \\ & \searrow^{\nabla} & \\ & & \operatorname{Sym}^k A \otimes \Lambda^1 T^*M \end{array}$$

where δ *is the Koszul differential on* $\mathrm{Sym}\, A$. *On differential forms it is detemined by the formulas*

$$d_M : \Lambda^k T^*M \longrightarrow \Lambda^{k+1} T^*M$$

and on $\bigwedge A$ *it splits as*

$$\begin{array}{ccc} \Lambda^k A & \xrightarrow{\nabla} & \Lambda^k A \otimes \Lambda^1 T^*M \\ & \searrow^{R_\nabla} & \\ & & \mathrm{Sym}^1 A \otimes \Lambda^{k-1} A \otimes \Lambda^2 T^*M. \end{array}$$

Together with the compatibility with the wedge product this determines the de Rham differential.

3. *The Lie module structure* L *is determined by the following formulas:*

$$\begin{aligned} L_X(s) &= \nabla_X s - R_\nabla(X, s) & L_\alpha(s) &= \nabla \alpha(s) \\ L_X(\xi) &= L_X^M \xi & L_\alpha(\xi) &= 0, \end{aligned}$$

where $X \in \bigwedge^1 TM$, $\alpha \in \bigwedge^1 A^\vee$, $s \in \bigwedge^1 A$ *and* $\xi \in \bigwedge^1 T^*M$.

Proof. The Schouten–Nijenhuis bracket is determined, using Equation (1.1.3), by the bracket evaluated on elements of cohomological degree 1, hence the given formula indeed suffices to describe it. To arrive at the formula one simply computes the commutator of $\iota_\alpha + \nabla_X$ and $\iota_\beta + \nabla_Y$, viewed as derivations of $\mathrm{Sym}\, A$.

Now we compute the de Rham differential. First, we remark that

$$\delta := \sum \iota_{e^i} \otimes e_i$$

is the standard Koszul differential for the algebra $\mathrm{Sym}\, A$, expressed in local basis $\{e_i\}$ of A. Moreover, the curvature $R_\nabla \in \bigwedge^2 T^*M \otimes A \otimes A^\vee$ of the connection naturally induces a morphism

$$R_\nabla : \bigwedge^k A \longrightarrow \mathrm{Sym}^1 A \otimes \bigwedge^{k-1} A \otimes \bigwedge^2 T^*M.$$

To prove the formulas for the de Rham differential, one uses that it is defined by

$$\begin{aligned} d_{A^\vee}(g)(V_1) &= V_1(g) \\ d_{A^\vee}(\gamma)(V_1, V_2) &= V_1(\gamma(V_2)) - V_2(\gamma(V_1)) + \gamma([V_1, V_2]_{SN}) \end{aligned}$$

where $g \in \mathrm{Sym}\, A$, $V_i \in L$ and $\gamma \in \Omega^1_L(M)$. The fact that the de Rham differential squares to 0 is equivalent to the Bianchi identity. Finally, the Lie algebra module structure on $\Omega_L(M)$ is determined in low degrees because it is compatible with the wedge product on forms and the identity $[L_{\gamma_1}, L_{\gamma_2}] = L_{[\gamma_1,\gamma_2]}$ for $\gamma_i \in T^L_{\mathrm{poly}}(M)$. The given formulas are all direct consequences of the following expression:

$$L_V(\gamma) = (\iota_\gamma d_{A^\vee} + d_{A^\vee} \iota_\gamma) V,$$

where $V \in L$ and $\gamma \in \Omega^1_L(M)$. □

3.1.2 The Poisson structure on $A^\vee$

From now on we assume that $\pi : A \to M$ is a smooth Lie algebroid. It is well-known that the Lie algebroid structure on A defines a canonical Poisson structure on the total space of the dual vector bundle $A^\vee$ by the following formula:

$$\{\pi^* f, \pi^* g\} := 0, \quad \{\hat{s}, \pi^* f\} := \pi^*(\rho(s)(f)), \quad \{\hat{s}, \hat{t}\} := \widehat{[s,t]}, \qquad f \in C^\infty(M),\ s, t \in \Gamma(A). \quad (3.1.6)$$

This defines the Poisson bracket on functions which are constant or linear along the fibers. These functions generate the functions which are polynomial along the fibers, which are in turn a dense subalgebra of the smooth functions on $A^\vee$, so the Poisson bracket can be extended to $C^\infty(A^\vee)$ using the Leibniz rule.

The properties of the Lie bracket and the anchor map imply that Equation (3.1.6) defines a Poisson structure, i.e. that it satisfies the Leibniz and Jacobi identities.

Recall that a Poisson structure on $A^\vee$ is equivalent to an element $\theta \in T^2(A^\vee)$ such that $[\theta, \theta]_{SN} = 0$. The relation to the Poisson structure is given by

$$\{h, k\} = \theta(dh, dk), \quad h, k \in C^\infty(A^\vee). \tag{3.1.7}$$

Remark 3.1.3. We differ from the usual sign convention for the Poisson tensor defined on $A^\vee$, since it seems more natural for most of our constructions.

We view the Poisson tensor as element in $T^{L,1}_{\text{poly}}(A^\vee)$, i.e. as a multiderivation of $\operatorname{Sym} A$. It is important to note that we used the *shifted* degree on the space of polyvectorfields; i.e. its cohomological degree as in (3.1.4) is equal to 2. The shifted degree, which gives the bracket $[\ ,\ ]_{SN}$ degree 0, is conventional in the theory around formality that we will use later.

Given a connection ∇ on A and $s, t \in \Gamma(A)$, the formula $(s, t) \mapsto \nabla_{\rho(s)} t$ defines an A-connection $\nabla_{\rho(\ )}$ on A. The A-torsion T^A_∇ of $\nabla_{\rho(\ )}$ is defined by the following equation:

$$T^A_\nabla : \ s \wedge t \mapsto [s, t] - \nabla_{\rho(s)} t + \nabla_{\rho(t)} s.$$

We view it as an element in $\operatorname{Sym}^1 A \otimes \wedge^2 A^\vee$. Also, we view the anchor ρ as an element of $\wedge^1 A^\vee \otimes \wedge^1 TM$.

Remark 3.1.4. In [NeWa] the authors note that one can shift A-connections on A by half of the torsion to obtain an A-torsion free A-connection. However, the torsion free A-connection associated to an A-connection of the form $\nabla_{\rho(\ )}$ does not in general have the form $\tilde{\nabla}_{\rho(\ )}$ for a TM-connection $\tilde{\nabla}$ on A.

Lemma 3.1.5. *The Poisson tensor defined by the Equations in* (3.1.6) *is, under the isomorphism*

$$T^{L,1}_{\text{poly}}(A^\vee) \overset{\nabla}{\cong} \operatorname{Sym} A \otimes (\wedge^2 A^\vee \oplus (\wedge^1 A^\vee \otimes \wedge^1 TM) \oplus \wedge^2 TM),$$

given by $\theta_\nabla := T^A_\nabla + \rho$.

Proof. We use the identification of (3.1.7). Let $f, g \in C^\infty(M)$ and $s, t \in \operatorname{Sym}^1 A$. It follows easily that

$$\theta_\nabla(d_\nabla f, d_\nabla g) = \theta_\nabla(d_M f, d_M g) = 0.$$

Moreover we, have

$$\begin{aligned}
\theta_\nabla(d_\nabla s, d_\nabla f) =& \theta_\nabla(\delta(s) + \nabla s, d_M f)\\
=& \rho(s)(d_M f)\\
\theta_\nabla(d_\nabla s, d_\nabla t) =& \theta_\nabla(\delta(s), \delta(t)) + \theta_\nabla(\nabla s, \delta(t))\\
&+ \theta_\nabla(\delta(s), \nabla t) + \theta_\nabla(\nabla s, \nabla t)\\
=& [s, t] - \nabla_{\rho(s)} t + \nabla_{\rho(t)} s - \nabla_{\rho(t)} s + \nabla_{\rho(s)} t\\
=& [s, t].
\end{aligned}$$

Comparing this to the Poisson structure defined by the Equations in (3.1.6), the lemma is proved. ∎

The Leibniz and Jacobi identities for the Poisson bracket, which are concisely captured by the Maurer–Cartan equation

$$[\theta, \theta] = 0,$$

imply, together with the Jacobi identity for the Schouten–Nijenhuis bracket on polyvectorfields, that the map

$$[\theta, \ ] : \ T^{L,k}_{\text{poly}}(A^{\vee}) \longrightarrow T^{L,k+1}_{\text{poly}}(A^{\vee}), \qquad \gamma \mapsto [\theta, \gamma]$$

squares to 0, and this complex defines the so-called Poisson cohomology. This is explained in more detail in subsection 3.1.4.

3.1.3 Representations up to homotopy

Let us recall that representations of Lie algebroids $A \to M$ are given by flat Lie algebroid connections on vector bundles $E \to M$. The major drawback of this definition of a representation is its rigidity, so in general, the category of representations of a Lie algebroid is quite small, and in particular may not contain anything that can be called the adjoint representation. This can already be seen in the case of the tangent bundle; any vector bundle which has non zero characteristic classes does not admit flat connections, so cannot be a TM-representation. As a remedy, the more flexible notion of a *representation up to homotopy* was introduced in [AC], which was based on earlier ideas of, for example, [ELW, CM, Qu]. In this section, we shortly describe how a representation up to homotopy is defined, for a more thorough treatment we refer to [AC]. Informally speaking, the vector bundle E is replaced by a complex of vector bundles $\ldots \xrightarrow{\partial} E^i \xrightarrow{\partial} E^{i+1} \xrightarrow{\partial} \ldots$, and the flat A-connection on E is replaced by not necessarily flat connections on each bundle E^i which are compatible with the maps ∂, i.e. $\nabla\partial - \partial\nabla = 0$. The data is completed by a collection of tensors satisfying relations which imply that, together with ∂ and ∇, they form a derivation D of degree 1 which squares to zero on the total complex $\Omega_A(\bigoplus_i E^i)$, which is of course analogous to the fact that flat A-connections on a vector bundle E induce differentials on $\Omega_A(E)$. Since we deal with graded vector bundles, plenty of signs are introduced, but in fact they all boil down to the usual Koszul sign rule, which says that for any transposition of elements x, y of degree $|x|, |y|$, a sign $(-1)^{|x||y|}$ is introduced.

Definition 3.1.6. *Given a smooth Lie algebroid $A \to M$, a representation up to homotopy of A is given by a pair (E, D), where $E = \bigoplus_{i \in \mathbb{Z}} E^i$ is a graded vector bundle, and a map*

$$D : \ \Omega_A(E) \longrightarrow \Omega_A(E)$$

which increases the total degree by 1, squares to 0 and is a graded derivation, i.e.

$$D(\omega \wedge \eta) = d_A(\omega) \wedge \eta + (-1)^k \omega D(\eta), \qquad \textit{for } \omega \in \Omega^k_A, \ \eta \in \Omega_A(E).$$

The cohomology of the resulting complex is denoted by $H^\bullet(A; E)$.

With respect to the degree in E, we can decompose D as a sum

$$D = \partial + \nabla + \sum_{i \geqslant 2} \omega_i,$$

where:

1. The map ∂ is a degree 1 operator on E turning E into a complex.
2. The map ∇ is an A-connection on E such that $[\nabla, \partial] = 0$.
3. The maps ω_i are given by tensors $\omega_i \in \Omega^i_A(\text{End}^{1-i}(E))$, with the usual correspondence between such tensors and Ω_A-module morphisms $\Omega^k_A(E^j) \to \Omega^{k+i}_A(E^{j+1-i})$.

These maps satisfy, in addition to $\partial^2 = 0$ and $[\nabla, \partial] = 0$, the relations $\partial \circ \omega_2 + R_\nabla = 0$ and

$$\partial \circ \omega_i + \nabla \omega_{i-1} + \omega_2 \circ \omega_{i-2} + \ldots + \omega_{i-2} \circ \omega_2 = 0$$

Example 14 (The adjoint representation, [AC]). The graded vector bundle of the adjoint representation up to homotopy is given by $A \oplus TM$, where A has degree 0 and TM has degree 1. Given a TM-connection on the vector bundle $A \to M$ the associated basic A-connection on $A \oplus TM$ is defined by the formulas

$$\begin{aligned} \nabla^{\text{bas}}_s t :=& \nabla_{\rho(t)} s + [s, t] \\ \nabla^{\text{bas}}_s X :=& \rho(\nabla_X s) + [\rho(s), X] \end{aligned}$$

where $X \in \Gamma(TM)$ and $s, t \in \Gamma(A)$. The basic curvature of the connection is defined by

$$R^{\text{bas}}_\nabla(s,t)(X) = \nabla_X([s,t]) - [\nabla_X s, t] - [s, \nabla_X t] - \nabla_{\nabla^{\text{bas}}_t X}(s) + \nabla_{\nabla^{\text{bas}}_s X}(t).$$

A straightforward check shows that this is a tensor $R^{\text{bas}}_\nabla \in \Omega^2_A(\text{Hom}(TM, A))$. In [AC] the following proposition was proved:

Proposition 3.1.7. *The basic curvature R^{bas}_∇ of a TM-connection ∇ on A satisfies the following identities:*

1. *For the curvature $R_{\nabla^{\text{bas}}}$ of the A-connection ∇^{bas} on A one has $R^{\text{bas}}_\nabla \circ \rho = R_{\nabla^{\text{bas}}}$, and for the curvature $R_{\nabla^{\text{bas}}}$ of the A-connection ∇^{bas} on TM one has $R_{\nabla^{\text{bas}}} = -\rho \circ R^{\text{bas}}_\nabla$.*

2. *The basic curvature is closed with respect to the differential $d_{\nabla^{\text{bas}}}$ on $\Omega_A(\text{Hom}(TM, A))$, i.e. $d_{\nabla^{\text{bas}}} R^{\text{bas}}_\nabla = 0$.*

The differential of the adjoint representation is now given by

$$D_\nabla = \rho + \nabla^{\text{bas}} + R^{\text{bas}}_\nabla.$$

The previous proposition together with the property $\rho \circ \nabla^{\text{bas}} = \nabla^{\text{bas}} \circ \rho$ are equivalent to the fact that $D^2_\nabla = 0$. The complex thus obtained will be denoted by $(\Omega^\bullet_A(\text{Ad}(A)), D_\nabla)$.

Example 15 (The double representation [AC]). Let $E \to M$ be a vector bundle. Any A connection on E defines a representation of A on $E \xrightarrow{\text{id}} E$, where the copies of E have degree 0 and 1 by setting

$$D = \text{id} + \nabla + R_\nabla.$$

This representation, called the *double of a vector bundle*, is denoted by $\mathfrak{D}_E$.

As mentioned in the introduction, one can define tensor products of representations. More generally, one can define direct sums, wedge products, duals, symmetric powers and, given two representations up to homotopy E and F, the representation $\text{Hom}(E, F)$. With these operations one can construct a wealth of representations up to homotopy, which is done in [AC], but here we only discuss a few examples which are related to the polynomial polyvector fields and differential forms. A particular operation, which we will need later, is the following.

Example 16 (Conjugation [AC]). Given a representation up to homotopy E with operator D, on can form a representation with the same underlying vector bundle E, and the structure operator given by

$$\overline{D} = -\partial + \nabla - \omega_2 + \omega_3 - \ldots.$$

This representation is isomorphic to the original one under the map Φ given by $(-1)^n\text{Id}$ on E^n.

Example 17 (Symmetric powers)**.** The k-th symmetric power of the adjoint representation of the Lie algebroid $A \to M$ from example 14 gives rise to the following complex of vector bundles:

$$0 \longrightarrow \mathrm{Sym}^k A \longrightarrow \mathrm{Sym}^{k-1} A \otimes TM \longrightarrow \ldots \longrightarrow \mathrm{Sym}^1 A \otimes \wedge^{k-1} TM \longrightarrow \wedge^k TM \longrightarrow 0,$$

where the degree of $\mathrm{Sym}^{k-i} A \otimes \wedge^i TM$ is i. The differential D of the representation acts on $\mathrm{Sym}^i A$ by the extensions of ρ and ∇^{bas} by derivations, on $\wedge^j TM$ by the extensions of ∇^{bas} and $R^{\mathrm{bas}}_{\nabla}$ by derivations, and on $\wedge^l A^\vee$ by d_A. We denote the representation by $\Omega_A(\mathrm{Sym}^k(\mathrm{ad}\, A)), D_\nabla)$. The direct sum of representations $\Omega_A(\bigoplus_k \mathrm{Sym}^k \mathrm{ad}\, A), D_\nabla)$ admits a product which is a combination of the symmetric product on $\mathrm{Sym}\, A$, the wedge product on $\wedge TM$ and the wedge product on $\wedge A^\vee$. It is clear from the form of the differential that $(\Omega_A(\bigoplus_k \mathrm{Sym}^k \mathrm{ad}\, A), \wedge, D_\nabla) =: (\Omega_A(\mathrm{Sym}\,\mathrm{ad}\, A), \wedge, D_\nabla)$ is a differential graded algebra.

Example 18 (Symmetric powers of the double)**.** The k-th symmetric power of the double vector bundle representation $E \xrightarrow{\mathrm{id}} E$ of A from example 15 give rise to the complex

$$0 \longrightarrow \mathrm{Sym}^k E \longrightarrow \mathrm{Sym}^{k-1} E \otimes E \longrightarrow \ldots \longrightarrow \mathrm{Sym}^1 E \otimes \wedge^{k-1} E \longrightarrow \wedge^k E \longrightarrow 0$$

of vector bundles, where the differential of the complex is given by the extension of the identity map on E by derivations to $\mathrm{Sym}^i E$, which turns out to the be standard Koszul differential for the bundle $\mathrm{Sym}\, E \otimes \wedge^\bullet E$. The part of D which preserves the degree of the graded vector bundle is simply given by the extension of the connection ∇ to $\mathrm{Sym}^i E$ and $\wedge^j E$, whereas the part of D in $\Omega^2_A(\mathrm{End}^{-1}(\mathrm{Sym}^k(E \xrightarrow{\mathrm{id}} E)))$ is given by the extension of the curvature $R_\nabla \in \Omega^2_A(\mathrm{End}(E))$ to a map $R_\nabla : \mathrm{Sym}^i E \otimes \wedge^j E \longrightarrow \mathrm{Sym}^{i+1} E \otimes \wedge^{j-1} E \otimes \wedge^2 A^\vee$, which defines an element in $\Omega^2_A(\mathrm{End}^{-1}(\mathrm{Sym}(E \xrightarrow{\mathrm{id}} E)))$. The direct sum of representations $\mathrm{Sym}(E \xrightarrow{\mathrm{id}} E) := \bigoplus_k \mathrm{Sym}^k(E \xrightarrow{\mathrm{id}} E)$ again is a differential graded algebra.

3.1.4 Poisson (co)homology and the modular class

In this section we recall some facts about Poisson manifolds and Lie algebroids, and in particular the definition of the modular class. The material is standard and can for example be found in [Br], [ELW] or [CdSW]. Let P be a Poisson manifold of dimension n, i.e., a manifold together with a Poisson tensor $\pi \in \Lambda^2 TP$ such that $[\pi, \pi]_{SN} = 0$. The Poisson cohomology complex of P is given by the polyvector fields on P with differential $\partial_\pi := [\pi, \ ]$, and the resulting cohomology groups are denoted by $H^\bullet_\pi(P)$. The differential forms $\Omega(P)$ over P form a module over the polyvector fields, and this enables one to define $L_\pi : \Omega^\bullet(P) \to \Omega^{\bullet-1}(P)$, which is the Poisson homology differential. The resulting homology groups are denoted by $H^\pi_\bullet(P)$. For simplicity we assume that P is orientable i.e., there exists a non-vanishing global volume form $\mu \in \Omega^n(P)$ on P. The volume form induces an isomorphism

$$\mu : \Lambda^\bullet TP \xrightarrow{\cong} \Omega^{n-\bullet}(P), \qquad \gamma \mapsto \iota_\gamma \mu \tag{3.1.8}$$

which can be used to define the *modular vector field* of P, by comparing the Poisson cohomology differential to the Poisson homology differential transferred to the polyvector fields, denoted by $\mu(L_\pi) := \mu^{-1} \circ L_\pi \circ \mu$.

Definition 3.1.8. *Given a Poisson manifold P with a non-vanishing volume form $\mu \in \Omega^n(P)$, the modular vector field ν_μ is defined by*

$$(\mu(L_\pi) - \partial_\pi)\gamma = \nu_\mu \wedge \gamma.$$

Proposition 3.1.9. *The modular vector field ν_μ of a Poisson manifold associated to a volume form μ is well-defined. Moreover, it can equivalently be characterized by:*

i) *Given a function* $f \in C^\infty(P)$ *and its Hamiltonian vector field* X^f, *one has*

$$L_{X^f}\mu = \nu_\mu(f)\mu.$$

ii) *The following relation between the Poisson tensor, the volume form, and the modular vector:*

$$L_\pi \mu = \iota_{\nu_\mu}\mu.$$

Given two volume forms μ_1 *and* μ_2 *for which* $\mu_1 = f\mu_2$ *holds for* $f \in C^\infty(P)$, *one has*

$$\nu_{\mu_2} = \nu_{\mu_1} - X^{\log|f|},$$

where $X^{\log|f|}$ *is the Hamiltonian vector field of* $\log|f|$. *The modular vector field satisfies* $\partial_\pi(\nu_\mu) = 0$, *hence it defines a class* $[\nu] \in H^1_\pi(P)$ *in the Poisson cohomology, called the* modular class.

Proof. Although this lemma is often stated in the literature, explicit proofs are not so abundant, therefore we decided to give some details. First we prove that the first characterization defines a vector field. To do this we show that it is a derivation:

$$\begin{aligned} L_{X^{fg}}\mu =& L_{fX^g + gX^f}\mu = (fL_{X^g} + gL_{X^f} + df \wedge L_{X^g} + dg \wedge L_{X^f})\mu \\ =& (fL_{X^g} + gL_{X^f} + L_{X^g}(df) + L_{X^f}(dg))\mu \\ =& (fL_{X^g} + gL_{X^f})\mu \end{aligned}$$

where we used the antisymmetric property of the Poisson tensor in the last line. The following equalities show that the first and second characterization are equivalent:

$$\begin{aligned} L_{X^f}\mu =& - d\iota_{\iota_{df}\pi}\mu = -d\iota_\pi(df \wedge \mu) + d(df \wedge \iota_\pi)\mu = df \wedge L_\pi\mu \\ \nu(f)\mu =& \iota_\nu(df)\mu = df \wedge \iota_\nu\mu. \end{aligned}$$

The next computation shows that assertion ii) from the proposition and the definition of the modular vector field are equivalent:

$$\begin{aligned} \mu(\partial_\pi\gamma + \nu_\mu \wedge \gamma) =& \iota_{[\pi,\gamma]}\mu + \iota_{\nu\wedge\gamma}\nu \\ =& L_\pi\iota_\gamma\mu - \iota_\gamma L_\pi\mu + \iota_\gamma\iota_\nu\mu \\ =& L_\pi\iota_\gamma\mu - \iota_\gamma L_\pi\mu + \iota_\gamma L_\pi\mu \\ =& L_\pi(\mu(\gamma)). \end{aligned}$$

Now let $\mu_1 = f\mu_2$ where $\mu_i \in \Omega^n(P)$ are non-vanishing. Then we have

$$\begin{aligned} L_\pi(f\mu_1) &= d\iota_\pi(f\mu_1) \\ &= df \wedge \iota_\pi\mu_1 + fL_\pi\mu_1 \\ &= \iota_\pi(df \wedge \mu_1) + \iota_{\iota_{df}\pi}\mu_1 + fL_\pi\mu_1 \\ &= -\iota_{X^f}\mu_1 + fL_\pi\mu_1. \end{aligned}$$

Hence it follows that $\nu_{\mu_2} = \nu_{\mu_1} - X^{|\log f|}$. For the last part, we compute

$$\iota_{[\pi,\nu]}\mu = (L_\pi\iota_\nu + \iota_\nu L_\pi)\mu = (L_\pi^2 + \iota_\nu^2)\mu = 0.$$

□

In the same way as we did with the Poisson homology differential, we can transfer the de Rham differential to the polyvectorfields, i.e. $\mu(d) = \mu^{-1} \circ d \circ \mu$. The second characterization from the last proposition is then also equivalent to

$$\mu(d)(\pi) = \nu_\mu.$$

The modular class of a Poisson manifold is in fact part of a more general story for Lie algebroids, which we will describe now.

As mentioned in the discussion about representations up to homotopy, Lie algebroids admit few representations, i.e. flat A-connections on vector bundles. The following construction, in which a canonical representation on a line bundle is constructed, provides an exception to this rule. Representations on line bundles give rise to characteristic cohomology classes in $H^1(A)$. In the special case that we consider, these classes are particular examples of the secondary characteristic classes considered in, for example, [CF2]. Let $A \to M$ be a smooth Lie algebroid of rank r over a manifold of dimension n. We define the line bundle

$$Q_A := \wedge^r A \otimes \wedge^n T^*M.$$

For simplicity, let us assume that it admits a non-vanishing global section $T \otimes \mu$.

Lemma 3.1.10 ([ELW]). *The equation*

$$\nabla_s^{Q_A}(T \otimes \mu) := [s, T] \otimes \mu + T \otimes L_{\rho(s)}\mu$$

for $s \in \Gamma(A)$ *defines a representation of* A *on* Q_A.

Given a (trivial) line bundle L over M together with a flat A-connection D on L, one can define the *characteristic class* $[\mu_L] \in H^1(A)$ of this representation by the equation

$$D_s l = \mu_L^l(s) l$$

where $s \in \Gamma(A)$, $l \in \Gamma(L)$ is non-vanishing and $\mu_L^l \in \Gamma(A^\vee)$. The fact that $\mu_L^l \in \Gamma(A^\vee)$ is straightforward, whereas $d_A \mu_L^l = 0$ follows directly from the fact that D is flat. Given two non-vanishing sections $l_1, l_2 \in \Gamma(L)$ which are related by $fl_1 = l_2$, an easy computation shows that $\mu_L^{l_2}(a) = \mu_L^{l_1}(a) + d(\log|f|)(a)$, so up to exact forms the definition is independent of the choice of a section, hence $[\mu_L] \in H^1(A)$ is well-defined. Choosing $L = Q_A$, this construction gives the *modular class* of the Lie algebroid A. The following proposition gives an interpretation of the characteristic class of a Lie algebroid which is similar to Definition 3.1.8.

Proposition 3.1.11. *Given a representation of* A *on a trivial line bundle* L*, the element* $\mu_L^l \in \Gamma(A^\vee)$ *associated to the non-vanishing section* $l \in \Gamma(L)$ *satisfies*

$$d_A + \mu_L^l \wedge = D^l$$

where D^l *is the differential on* Ω_A *obtained by transferring* d_D *on* $\Omega_A(L)$ *to* Ω_A *by the isomorphism*

$$\Omega_A \xrightarrow{\cong} \Omega_A(L), \qquad \alpha \mapsto \alpha \otimes l.$$

Proof. Writing $d_D = D : \Gamma(L) \longrightarrow \Omega_A^1(L)$, we see that $D(l) =: \mu_L \otimes l \in A^\vee \otimes L$. Then the claim follows directly from the general requirement for the de Rham differentials with coefficients that

$$d_D(\alpha \otimes l) = d_A(\alpha) \otimes l + \alpha \wedge d_D(l).$$

□

Let P be a Poisson manifold and $A = T^*P$ the Lie algebroid associated to P. There are two definitions of the modular class; one for the Poisson manifold P and one for the Lie algebroid T^*P. These are related as follows.

Proposition 3.1.12 ([ELW]). *Given a Poisson manifold* P*, the modular class* θ_P *of the Poisson manifold and the modular class* θ_{T^*P} *of the Lie algebroid* T^*P *satisfy*

$$\theta_P = \frac{1}{2}\theta_{T^*P}.$$

Moreover, the bundle $\wedge^{\mathrm{top}} T^*P$ *admits a representation which characteristic class is equal to the modular class of* P.

Let us remark that $Q_{T^*P} \cong (\wedge^{top} T^*P)^2$. The second part of the proposition is a consequence of the fact that, given a representation on the square of a line bundle L one can define a representation on L itself. Given $\mu \in \Gamma(\wedge^{top} T^*P)$ and $\alpha \in \Gamma(T^*P)$, the representation of T^*P on $\wedge^{top} T^*P$ constructed in this way is given by:

$$D_\alpha \mu = [\alpha, \mu]_{T^*P} - \pi(d\alpha)\mu \tag{3.1.9}$$

$$= \alpha \wedge L_\pi \mu. \tag{3.1.10}$$

The de Rham complex associated with this representation is given by

$$(\Omega^\bullet_{T^*P}(\wedge^n T^*P), d_D) = (\Gamma(\wedge^\bullet TP \otimes \wedge^n T^*P), d_D)$$

where d_D satisfies, given $\gamma \in \Gamma(\wedge^k TP)$,

$$\begin{aligned} D(\gamma \otimes \mu) &= d_{T^*P}\gamma \otimes \mu + (-1)^k \gamma \wedge D\mu) \\ &= [\pi, \gamma]_{SN} \otimes \mu + (-1)^k \gamma \wedge D\mu) \end{aligned}$$

The contraction $\gamma \otimes \mu \mapsto \iota_\gamma(\mu)$ defines an isomorphism

$$\tau : \Omega^\bullet_{T^*P}(\wedge^n T^*P) \xrightarrow{\cong} \Omega^{n-\bullet}(P)$$

and this isomorphism leads to the following theorem:

Theorem 3.1.13 ([ELW], Thm. 4.5). *For $\gamma \otimes \mu \in \Omega^\bullet_{T^*P}(\wedge^n T^*P)$ one has*

$$\tau(d_D(\gamma \otimes \mu)) = (-1)^{\bullet+1} L_\pi(\tau(\gamma \otimes \mu)),$$

hence $H^\bullet_{\mathrm{Lie}}(T^*P, \wedge^n T^*P) \cong H^\pi_{n-\bullet}(P)$.

3.1.5 Poisson (co)homology vs. representations up to homotopy

In this subsection we finally relate the polynomial Poisson complexes of the dual of a Lie algebroid $A \to M$ to representations up to homotopy. Recall the Lie algebroid $L = \mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\, A)$ over $(M, \mathrm{Sym}\, A)$, its polyvector fields and differential forms, and the Poisson (co)homology differentials.

Theorem 3.1.14. *Given a Lie algebroid $A \to M$ and a connection ∇ on A, we have the following isomorphism:*

$$H^\theta_\bullet(T^L_{\mathrm{poly}}(A^\vee)) \cong H^\bullet(A, \mathrm{Sym}(\mathrm{ad}\, A)).$$

Theorem 3.1.15. *Let $A \to M$ be an orientable smooth Lie algebroid of rank r over an orientable manifold M of dimension n, then we have the following isomorphism:*

$$H^{n+r-\bullet}_\theta\left(\Omega^\bullet_L(A^\vee)\right) \cong H^\bullet\left(A, \mathrm{Sym}(\mathrm{ad}\, A) \otimes Q_A\right).$$

Before giving the proofs of these theorems, we prove a simple lemma that, given a vector bundle $E \to M$, relates the polynomial differential forms on E^* with the de Rham differential to the representation up to homotopy from example 18 in the case $A = TM$.

Lemma 3.1.16. *Given a vector bundle $E \to M$ and an affine connection ∇ on E, the de Rham differential $d_{\mathrm{Sym}\, E}$ of the polynomial differential forms $\Omega_{\mathrm{Sym}\, E}$ is equal to D_∇ of the representation $\mathrm{Sym}(E \xrightarrow{\mathrm{id}} E)$ of example 18 under the isomorphism induced by the connection ∇ of (3.1.5).*

Proof. Recall from lemma 3.1.2 that the de Rham differential d_∇ on $\mathrm{Sym}\, A \otimes \wedge A \otimes \wedge T^*M$, which has polynomial degree 0 and cohomological degree 1, is given by:

i) The sum of the Koszul differential and the connection on the $\mathrm{Sym}(E)$ part.

ii) The de Rham differential on the $\wedge T^*M$ part.

iii) The sum of the connection and the curvature R_∇, considered as a map $R_\nabla : \wedge^k A \to \mathrm{Sym}^1 A \otimes \wedge^{k-1} A \wedge^2 T^*M$.

This exactly agrees with the discussion of the differential D_∇ of the symmetric power of the double vector bundle representation $E \xrightarrow{\mathrm{id}} E$ in the last example. □

The following lemma not only proves theorem 3.1.14, but is in fact a stronger statement since it gives an isomorphism between the two complexes on the level of cochains. This isomorphism depends on the choice of a connection on A.

Lemma 3.1.17. *Given a Lie algebroid $A \to M$ and a connection ∇ on A, the Poisson cohomology complex $(T^L_{\mathrm{poly}}(A^\vee), [\theta, \ ])$ is isomorphic to the symmetric power of the adjoint representation $(\Omega_A(\mathrm{Sym}(\mathrm{ad}\, A)), D_\nabla)$.*

Proof. We will prove that, under the identification

$$T^L_{\mathrm{poly}}(A^\vee) \overset{\nabla}{\cong} \mathrm{Sym}\, A \otimes \wedge A^\vee \otimes \wedge TM$$

the Poisson differential $[\theta, \ ]$ translates to $\overline{D}_\nabla$, the differential of the conjugate representation of $\mathrm{Sym}(\mathrm{ad}\, A)$. Since representations are isomorphic to their conjugates this proves the lemma. Note that this lemma is a natural extension of lemma 3.1.2. The differential $[\theta, \ ]_{SN}$ on $T^L_{\mathrm{poly}}(A^\vee)$ has polynomial degree -1 and cohomological degree 1, hence, under the isomorphism induced by ∇ it gives a map

$$[\theta_\nabla, \ ]_{SN} : \bigoplus_{\substack{s+t=j \\ r-s=k}} \mathrm{Sym}^r A \otimes \wedge^s A^\vee \otimes \wedge^t TM \longrightarrow \bigoplus_{\substack{\tilde{s}+\tilde{t}=j+1 \\ \tilde{r}-\tilde{s}=k-1}} \mathrm{Sym}^{\tilde{r}} A \otimes \wedge^{\tilde{s}} A^\vee \otimes \wedge^{\tilde{t}} TM$$

where j and $j+1$ indicate the cohomological degree and k and $k-1$ indicate the polynomial degree. Recall from lemma 3.1.5 that the Poisson tensor $\theta \in T^{L,2}(A^\vee)$ corresponds to $T^A_\nabla + \rho$ with $T^A_\nabla \in \mathrm{Sym}^1 A \otimes \wedge^2 A^\vee$ and $\rho \in \wedge^1 A^\vee \otimes \wedge^1 TM$ under the isomorphism induced by ∇.

We have to prove that $\overline{D}_\nabla = [\theta_\nabla, \ ]_{SN}$. Let us give two simple but useful formulas from the Poisson cohomology complex. Let $g \in \mathrm{Sym}\, A$, $\gamma \in T^{L,1}(A^\vee)$ and $\omega_i \in \Omega^1_L(A^\vee)$, then

$$\begin{aligned} [\theta, g](\omega_1) &= -\theta(dg, \omega_1) \\ [\theta, \gamma](\omega_1, \omega_2) &= -L_\gamma \theta(\omega_1, \omega_2) \\ &= -\gamma(\theta(\omega_1, \omega_2)) + \theta(L_\gamma \omega_1, \omega_2) + \theta(\omega_1, L_\gamma \omega_2) \end{aligned} \tag{3.1.11}$$

To compute $[\theta_\nabla, \ ]_{\mathrm{SN}}$, we evaluate it on one forms using the formula above and the expressions for the de Rham differential and the Lie bracket from 3.1.2. Let $s, t \in \mathrm{Sym}^1 A$, $\alpha \in \wedge^1 A^\vee$ and $X, Y \in \wedge^1 TM$.

- We start with computing the action of $[\theta_\nabla, \ ]$ on $\mathrm{Sym}\, A$. This is clearly determined by the action on $\mathrm{Sym}^1 A$, thus we have to compute $[\rho, s] \in \mathrm{Sym}^1 A \otimes \wedge^1 A^\vee \oplus \wedge^1 TM$ and $[T^A_\nabla, s] \in \mathrm{Sym}^1 A \otimes \wedge^1 A^\vee$:

$$\begin{aligned} [\rho, s](\xi) &= -\rho(d_\nabla s, \xi) = -\rho(\delta s + \nabla s, \xi) = -\xi(\rho(s)) \\ [\rho, s](t) &= -\rho(d_\nabla s, t) = -\rho(\delta s + \nabla s, t) = \nabla_{\rho(t)} s \\ [T^A_\nabla, s](t) &= -T^A_\nabla(d_\nabla s, t) = -T^A_\nabla(\delta s, t) = -[s, t]_A + \nabla_{\rho(s)} t - \nabla_{\rho(t)} s. \end{aligned}$$

These equalities show that $[\theta_\nabla, s] = -\rho(s) + \nabla^{\mathrm{bas}} s$.

- Now we compute the action of $[\theta_\nabla, \ ]$ on polynomial vector fields. Let $\alpha \in \wedge^1 A^\vee$. From the expression for the Schouten–Nijenhuis bracket it follows easily that $[\theta_\nabla, \alpha] \in \mathrm{Sym}^1 A \otimes \wedge^2 A^\vee$. The equalities

$$\begin{aligned}
[\rho, \alpha](s,t) &= -\iota_\alpha(\rho(s,t)) + \rho(L_\alpha(s), t) + \rho(s, L_\alpha(t)) \\
&= \rho(\nabla\alpha(s), t) + \rho(s, \nabla\alpha(t)) \\
&= -\rho(t)(\alpha(s)) + \alpha(\nabla_{\rho(t)}s) + \rho(s)(\alpha(t)) - \alpha(\nabla_{\rho(s)}t) \qquad (3.1.12)\\
[T^A_\nabla, \alpha](s,t) &= -\iota_\alpha([s,t]_\nabla) + [L_\alpha s, t]_\nabla + [s, L_\alpha t]_\nabla \\
&= -\alpha([s,t]) + \alpha(\nabla_{\rho(s)}t) - \alpha(\nabla_{\rho(t)}(s))
\end{aligned}$$

imply that $[\theta_\nabla, \alpha] = d_A\alpha$, where d_A is the de Rham differential of Ω_A.

- Finally, let us compute $[\theta_\nabla, X]$. This time, the expression for the Schouten–Nijenhuis bracket implies that $[\theta_\nabla, X] \in (\wedge^1 A^\vee \otimes \wedge^1 TM) \oplus (\mathrm{Sym}^1 A \otimes \wedge^2 A^\vee)$. We assume for simplicity that $\xi_i = d_\nabla f_i = d_M f_i$ for $f_i \in C^\infty(M)$.

$$\begin{aligned}
[\rho, X](s, d_\nabla f) &= - X(\rho(s, d_\nabla f) + \rho(L_X s, d_\nabla f) + \rho(s, L_X(d_\nabla f)) \\
&= - X(\rho(s)(f)) + \rho(\nabla_X s + R(X)(s), d_\nabla f) + \rho(s, d_\nabla(X(f)) \\
&= [\rho(s), X](f) + \rho(\nabla_X s)(f) \\
[\rho, X](s,t) &= \rho(-R_\nabla(X)(s), t) + \rho(s, -R_\nabla(X)(t))) \\
&= R_\nabla(X, \rho(t))(s) - R_\nabla(X, \rho(s))(t) \\
&= \left([\nabla_X, \nabla_{\rho(t)}] - \nabla_{[X,\rho(t)]}\right) s - \left([\nabla_X, \nabla_{\rho(s)}] - \nabla_{[X,\rho(s)]}\right) t \\
[T^A_\nabla, X](s,t) &= - \nabla_X(T^A_\nabla(s,t)) + T^A_\nabla(L_X s, t) + T^A_\nabla(s, L_X t) \\
&= - \nabla_X([s,t]) + \nabla_X\nabla_{\rho(s)}t - \nabla_X\nabla_{\rho(t)}s) \\
&+ \ [\nabla_X s, t] - \nabla_{\rho(\nabla_X s)}t + \nabla_{\rho(t)}\nabla_X s \\
&+ \ [s, \nabla_X t] - \nabla_{\rho(s)}\nabla_X t + \nabla_{\rho(\nabla_X t)}s \\
&= - \nabla_X([s,t]) + [\nabla_X s, t] + [s, \nabla_X t] \\
&+ \ [\nabla_X, \nabla_{\rho(s)}]t - [\nabla_X, \nabla_{\rho(t)}]s \\
&+ \ \nabla_{\rho(\nabla_X t)}s - \nabla_{\rho(\nabla_X s)}t.
\end{aligned}$$

The first equality implies that $[\theta_\nabla, X]|_{\wedge^1 A^\vee \otimes \wedge^1 TM} = \nabla^{\mathrm{bas}} X$. Moreover, a careful analysis of the signs in the second and third equalities reveal that

$$[\theta_\nabla, X](s,t) = -R^{\mathrm{bas}}_\nabla(s,t)(X).$$

This finalizes, using the expression for D_∇ from example 17, the proof that $\overline{D}_\nabla$ coincides with $[\theta_\nabla, \ ]$. □

The last part of this section is devoted to a proof of theorem 3.1.15. Again, this follows from a stronger statement on the level of chains:

Lemma 3.1.18. *Given a smooth, orientable Lie algebroid A of rank r over an orientable manifold M of dimension n, we have the following isomorphism:*

$$\left(\Omega_L^{n+r-\bullet}(A^\vee), L_\theta\right) \cong \left(\Omega^\bullet_A(\mathrm{Sym}\,\mathrm{ad}\,A \otimes Q_A), \tilde{D}_\nabla\right)$$

where $\tilde{D}_\nabla$ is the differential of the tensor product of the representations $\mathrm{Sym}\,\mathrm{ad}\,A$ and Q_A.

Loosely speaking, the way to prove this lemma is to:

- use the identification

$$\Omega_L(A^\vee) \overset{\nabla}{\cong} \mathrm{Sym}\,A \otimes \wedge A \otimes \wedge T^*M,$$

- and relate the representation of A on Q_A and the representation of $\Omega^1_L(A^\vee)$ on $\Omega^{\mathrm{top}}_L(A^\vee)$ to each other, which also implies that the modular class of the Lie algebroid A lifts in a natural way to the modular class of the Poisson manifold $A^\vee$.

The first part is already extensively described in lemma 3.1.2 and 3.1.17, so we focus on the second part. Recall that the connection on A induces

$$L^* = \Omega^1_L(A^\vee) \overset{\nabla}{\cong} \operatorname{Sym} A \otimes (A \oplus T^*M). \tag{3.1.13}$$

Consider the following maps from A into this bundle:

$$\begin{aligned} d_\nabla :\ \Gamma(A) &\longrightarrow \operatorname{Sym} A \otimes (\wedge^1 A \oplus \wedge^1 T^*M) \qquad && s \mapsto d_\nabla(s) = \delta s + \nabla s \\ \delta :\ \Gamma(A) &\longrightarrow \operatorname{Sym} A \otimes (\wedge^1 A \oplus \wedge^1 T^*M) \qquad && s \mapsto \delta s. \end{aligned}$$

The first map is given by $s \mapsto d_{A^\vee}(s)$ under the identification from (3.1.13). Note that δs is the Koszul differential of $\operatorname{Sym} A$ applied to $s \in \operatorname{Sym}^1 A$, and one has $\delta s = s \in \wedge^1 A$. The following lemma will be used throughout this section.

Lemma 3.1.19. *The natural inclusion* $(\Omega^\bullet_A, d_A) \longrightarrow \left(T^L_{\mathrm{poly}}(A^\vee), [\theta,\ ]\right)$ *defined by* $\alpha \mapsto \iota_\alpha$ *for* $\alpha \in \Gamma(A^\vee)$ *is a morphism of differential graded algebras.*

Proof. This is a direct consequence of formula 3.1.12. □

As we will see in the following two lemmas, the maps d_∇ and ι both respect only part of the structures, i.e. they are not morphisms of Lie algebroids. In the first lemma we will view $\Gamma(A)$ as a Lie algebra together with a Lie algebra action on $C^\infty(M)$, and similarly we will view $L^* = \Omega^1_L(A^\vee)$ as a Lie algebra (with the Lie bracket induced from the Poisson structure $\theta \in T^{L,2}_{\mathrm{poly}}(A^\vee)$) with a Lie algebra action on $\operatorname{Sym} A$. We will write $d_{A^\vee} = d$ for aesthetic reasons.

Lemma 3.1.20. *The following diagrams define morphisms of Lie algebras and their modules:*

$$\begin{array}{ccc} \Gamma(A) & \overset{\rho_A}{\rightsquigarrow} & C^\infty(M) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \iota} \\ L^* & \overset{\rho_{L^*}}{\rightsquigarrow} & \operatorname{Sym} A \end{array} \qquad \begin{array}{ccc} \Gamma(A) & \overset{\mathrm{ad}}{\rightsquigarrow} & \Gamma(A) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \iota} \\ L^* & \overset{\rho_{L^*}}{\rightsquigarrow} & \operatorname{Sym} A \end{array} \tag{3.1.14}$$

Proof. We will not include ι in the notation since it is clear from the context where it should be written. The fact that d is a morphism follows from the definition of the Poisson structure on $A^\vee$:

$$[ds, dt]_{L^*} = d\{s,t\} = d([s,t]).$$

The fact that the first diagram is commutative also follows from the definition:

$$ds(f) = \theta(ds, df) = \{s, f\} = \rho(s)(f),$$

and the equation

$$\rho_{L^*}(ds)(t) = \theta(ds, dt) = [s,t]$$

proves the fact that the second diagram is commutative.

Lemma 3.1.21. *The inclusion* δ *is* $C^\infty(M)$*-linear and satisfies*

$$\begin{aligned} [\delta s, \delta t]_{L^*} &= \delta([s,t]_A) + R^{\mathrm{bas}}_\nabla(s,t) \\ \rho_{L^*}(\delta s)(f) &= \rho(s)(f). \end{aligned}$$

Recall that, given two one forms $\alpha_1,\ \alpha_2 \in \Omega^1_L(A^\vee)$, the Lie algebroid structure on $\Omega^1_L(A^\vee)$ is defined by

$$\begin{aligned}[\alpha_1, \alpha_2]_{L^*} &= -d\theta(\alpha_1, \alpha_2) + L_{\theta(\alpha_1,\ )}\alpha_2 - L_{\theta(\alpha_2,\ )}\alpha_1 \\ &= \ \ d\theta(\alpha_1, \alpha_2) + \iota_{\theta(\alpha_1,\ )}d\alpha_2 - \iota_{\theta(\alpha_2,\ )}d\alpha_1.\end{aligned}$$

Given $s,\ t \in \Gamma(A)$, we will compute $[\delta(s), \delta(t)]_{L^*} - \delta([s,t]_A)$ using the above formula. We know from the formulas for θ, the Lie derivatives and d_∇ that

$$[\delta(s), \delta(t)]_{L^*} - \delta([s,t]_A) \in \wedge^1 A \oplus (\mathrm{Sym}^1 A \otimes \wedge^1 T^*M),$$

so we can evaluate in $\beta \in \Gamma(A^\vee)$ or $X \in \Gamma(TM)$. First we compute

$$\begin{aligned} d\theta(\delta(s), \delta(t))(\beta) &= \beta([s,t] - \nabla_{\rho(s)}t + \nabla_{\rho(t)}s) \\ \iota_{\theta(s,\ )}dt(\beta) &= dt(\rho(s), \beta) = \beta(\nabla_{\rho(s)}t) \\ -\iota_{\theta(t,\ )}ds(\beta) &= -ds(\rho(t), \beta) = -\beta(\nabla_{\rho(t)}s)\end{aligned}$$

from which it follows that $([\delta(s), \delta(t)]_{L^*} - \delta([s,t]_A))|_{\wedge^1 A} = 0$. Then we evaluate in X:

$$\begin{aligned} d\theta(\delta(s), \delta(t))(X) &= \nabla_X([s,t] - \nabla_{\rho(s)}t + \nabla_{\rho(t)}s) \\ \iota_{\theta(s,\ )}dt(X) &= \iota_{\rho(s)}dt(X) + \iota_{[s,\ ]_\nabla}dt(X) \\ &= R(X, \rho(s))(t) - ([s, \nabla_X t] + \nabla_{\rho(s)}\nabla_X t + \nabla_{\rho(\nabla_X t)}s) \\ -\iota_{\theta(t,\ )}ds(X) &= -R(X, \rho(t))(s) + ([t, \nabla_X s] + \nabla_{\rho(t)}\nabla_X s + \nabla_{\rho(\nabla_X s)}t)\end{aligned}$$

from which it follows that $([\delta(s), \delta(t)]_{L^*} - \delta([s,t]_A))|_{\mathrm{Sym}^1 A \otimes \wedge^1 T^*M} = R^{\mathrm{bas}}_\nabla(s,t)$. □

Remark 3.1.22. The previous lemma is related to [AC][Section 3.2]. The authors consider the short exact sequence for the jet bundle of A

$$0 \longrightarrow \mathrm{Hom}(TM, A) \longrightarrow J^1(A) \longrightarrow A \longrightarrow 0.$$

This is the degree 1 part of the short exact sequence for the polynomial differential forms that is induced by the connection. The splitting j on the level of sections which is defined by $s \mapsto j(s)$ is not $C^\infty(M)$-linear, and corresponds in our story to the map induced by the de Rham differential. In [AC], the Lie algebra structure on $J^1(A)$ is defined by requiring j to be a Lie algebra morphism, which is similar to our definition of the Lie algebroid structure on $\Omega^1_L(A^\vee)$. The splitting $j_\nabla = j + \nabla$ of the short exact sequence for the jet bundle is $C^\infty(M)$-linear and corresponds to our map δ. This map does not respect the Lie algebra structures on $\Gamma(A)$ and $\Gamma(J(A))$, but satisfies the same equation involving the basic connection.

The maps d and δ can both be extended to maps

$$\begin{aligned} d:\ Q_A &\longrightarrow \Omega^{\mathrm{top}}_L(A^\vee) \\ \delta:\ Q_A &\longrightarrow \Omega^{\mathrm{top}}_L(A^\vee)\end{aligned}$$

defined by

$$\begin{aligned} t_1 \wedge \cdots \wedge t_r \otimes \mu_1 \wedge \cdots \wedge \mu_n &\longmapsto d(t_1) \wedge \cdots \wedge d(t_r) \otimes \mu_1 \wedge \cdots \wedge \mu_n \\ t_1 \wedge \cdots \wedge t_r \otimes \mu_1 \wedge \cdots \wedge \mu_n &\longmapsto \delta(t_1) \wedge \cdots \wedge \delta(t_r) \otimes \mu_1 \wedge \cdots \wedge \mu_n\end{aligned}$$

where $t_i \in \Gamma(A)$, $\mu_i \in T^*M$. These two maps are equal because the terms involving $\nabla(t_i) \in \mathrm{Sym}^1 A \otimes \wedge^1 T^*M$ cancel for degree reasons. We will denote this map by d. Recall that Q_A is an A-representation with the flat connection denoted by $\nabla^{Q_A} = D$, and $\Omega^{n+r}_L(A^\vee)$ is a representation of the Lie algebroid $\Omega^1_L(A^\vee)$ with connection D.

Lemma 3.1.23. *The following diagram defines a morphism of Lie algebras and their modules:*

$$\begin{array}{ccc} \Gamma(A) & \xrightarrow{\quad d \quad} & \Omega^1_L(A^\vee) \\ {\scriptstyle D}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ Q_A & \xrightarrow{\quad d \quad} & \Omega^{n+r}_L(A^\vee) \end{array}$$

Proof. Recall that the action of A on Q_A is given by

$$D_s(T \otimes \mu) = [s, T]_{SN} \otimes \mu + T \otimes L_{\rho(s)}\mu$$

where $s \in \Gamma(A),\ T \in \wedge^r A,\ \mu \in \wedge^n T^*M$. To compute the action of $\Omega^1_L(A^\vee)$ on $\Omega^{n+r}_L(A^\vee)$ we work locally, and write $T = t_1 \wedge \cdots \wedge t_r$ and $\mu = f_0 df_1 \wedge \cdots \wedge df_n$ for $f_i \in C^\infty(M)$. Given $g \in \operatorname{Sym} A$, the action of $\Omega^1_L(A^\vee)$ on a section $\Omega \in \Omega^{n+r}_L(A^\vee)$ has the form

$$D_{dg}\Omega = [dg, \Omega]_{L^*} = L_{\theta(dg,\ )}\Omega$$

which follows from the fact that the Schouten–Nijenhuis bracket on $\Omega_L(A^\vee)$ has the property that $[dg, dh]_{L^*} = d(\theta(dg, dh)) = L_{\theta(dg,\ )}dh$ where $g, h \in \operatorname{Sym} A$. Let $s, t \in \Gamma(A)$, then

$$L_{\theta(ds,\ )}dt = d(\theta(ds, dt)) = d([s, t])$$

from which we can conclude that $L_{\theta(ds,\ )}(dt_1 \wedge \cdots \wedge dt_r) = d\left([s, t_1 \wedge \cdots \wedge t_r]_A\right)$. Moreover, we have

$$L_{\theta(ds,\ )}df = d(\theta(ds, df)) = d(\rho(s)(f)) = L_{\rho(s)}df$$

from which we can conclude that $L_{\theta(ds,\ )}(f_0 df_1 \wedge \cdots \wedge df_n) = L_{\rho(s)}(f_0 df_1 \wedge \cdots \wedge df_n)$. Finally we have

$$L_{\theta(ds,\ )}(d(T \otimes \mu)) = d\left([s, T]_{SN} \otimes \mu + t \otimes L_{\rho(s)}\mu\right) = d(D_s(t \otimes \mu)),$$

which proves the lemma. □

Corollary 3.1.24. *Given a non vanishing section $\Omega \in Q_A$, the modular form $\nu_\Omega \in \Omega^1_A$ associated to the Lie algebroid A satisfies*

$$d\nu_\Omega = \nu_{d\Omega},$$

where $\nu_{d\Omega}$ is the modular vector field of the Poisson structure on $A^\vee$ associated to the volume form $d\Omega$.

Proof. First we remark that, given $\alpha \in \Gamma(A^\vee)$ and $s \in \Gamma(A)$, one has $\langle ds, d\alpha \rangle = \alpha(s) \in \operatorname{Sym}^0 A$ where $\langle \cdot, \cdot \rangle$ is the natural pairing between L^* and L.

Using the last lemma and the definitions for the modular vector fields of Poisson structures and Lie algebroids, we compute:

$$\langle ds, d\nu_\Omega \rangle\, d\Omega = d(\nu_\Omega(s)\Omega) = d(D_s(\Omega)) = D_{ds}(d\Omega) = L_{\theta(ds,\ )}d\Omega = \langle ds, \nu_{d\Omega} \rangle\, d\Omega$$

which proves the lemma. □

Proof of lemma 3.1.18. The differential of the representation $\operatorname{Sym}(\operatorname{ad} A) \otimes Q_A$ is defined by $D_\nabla \otimes 1 + 1 \otimes D$, where D_∇ is the differential of $\operatorname{Sym}(\operatorname{ad} A)$ and D defines the representation of A on Q_A. From lemma 3.1.17 we know that under the isomorphism

$$\Omega^\bullet_A(\operatorname{Sym}(\operatorname{ad} A)) \overset{\nabla}{\cong} T^L_{\mathrm{poly}}(A^\vee)$$

induced by the connection the differential D_∇ corresponds to $[\theta,\ ]$ up to signs, and from lemma 3.1.23 we know that the action of A on Q_A and the action of L^* on $\Omega_L^{n+r}(A^\vee)$ are intertwined under the maps $d: A \longrightarrow L^*$ and $d: Q_A \longrightarrow \Omega_L^{n+r}(A^\vee)$. It follows that

$$\left(\Omega_A^\bullet(\mathrm{Sym}(\mathrm{ad}\,A)\otimes Q_A),\ \tilde{D}_\nabla\right) \overset{\nabla}{\cong} \left(T_{\mathrm{pol}}^{L,\bullet}(A^\vee)\otimes\Omega_{\mathrm{pol}}^{n+r}(A^\vee),\ [\theta,\ ]\otimes 1 + 1\otimes D_\theta\right)$$

is an isomorphism of differential graded algebras, where $\tilde{D}_\nabla$ is the differential for the representation $\mathrm{Sym}(\mathrm{ad}\,A)\otimes Q_A$. Finally, the composition of this isomorphism with the isomorphism from 3.1.13 leads to

$$\left(\Omega_A^\bullet(\mathrm{Sym}(\mathrm{ad}\,A)\otimes Q_A),\ \tilde{D}_\nabla\right) \overset{\nabla}{\cong} \left(\Omega_{\mathrm{pol}}^{n+r-\bullet}(A^\vee),\ L_\theta\right)$$

which, up to signs, intertwines the differentials, and hence induces the desired isomorphism of complexes. □

3.2 Application of formality

In this section we compute the Hochschild and cyclic (co)homology of the universal enveloping algebra $\mathcal{U}(A)$ of a smooth Lie algebroid A over a manifold M. We define the Hochschild and cyclic (co)homology taking into account the natural topology on $\mathcal{U}(A)$. The natural filtration on the universal enveloping algebra leads to a spectral sequence that, in principle, allows one to compute the Hochschild and cyclic (co)homology. However, it is not a priori clear that it degenerates. As discussed in 1.2.6 Kassel circumvented this problem in [Ka] by constructing a morphism of complexes

$$j:\ (\Omega^\bullet_{S(\mathfrak{g})}, L_\theta) \longrightarrow (C_\bullet(A), b) \qquad j(P dx_1\cdots dx_n) = \sum_{\sigma\in S_n}(-1)^{|\sigma|}\eta(P)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)}$$

where η is the symmetrization map $\mathrm{Sym}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})$, which induces an isomorphism in homology. It requires a more sophisticated approach to construct a direct morphism in the general case of smooth Lie algebroids, which relies on the formality map for Lie algebroids, c.f. [D, C]. Let us briefly explain this. Given a smooth Lie algebroid $A\to M$, we showed in the previous subsection that $(\mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\,A), \mathrm{Sym}\,A) =: (L, R)$ forms a sheaf of Lie algebroids over M. In fact, L is triangular, i.e., there exists a Maurer–Cartan element $\theta\in\Lambda^2_R L = T^{L,1}_{\mathrm{poly}}(A^\vee)$, which is given by the L-Poisson tensor. In the appendix, to which we refer if notations are unclear, we use the PBW theorem for Lie algebroids to revisit the proof for the formality theorem for Lie algebroids, which says that there exists an L_∞-quasi-isomorphism $U = (U_1, U_2, \ldots)$

$$U:\ T^L_{\mathrm{poly}}(A^\vee)\longrightarrow D^L_{\mathrm{poly}}(A^\vee)$$

with the first component equal to the HKR map. The RHS is given by the L-polydifferential operators on R, which are the polydifferential operators on $A^\vee$ that are polynomial along the fibers, viewed as a sheaf over M. This morphism equips $C_L(A^\vee)$, the complex of L-Hochschild chains, with the structure of an L_∞-module over $T^L_{\mathrm{poly}}(A^\vee)$, and there exists a quasi-isomorphism $V = (V_1, V_2, \ldots)$ of L_∞-modules

$$V:\ C_L(A^\vee)\longrightarrow\Omega_L(A^\vee)$$

over $T^L_{\mathrm{poly}}(A^\vee)$, with V_1 equal to the HKR map for chains. It is well-known that Maurer–Cartan elements in $T^L_{\mathrm{poly}}(A^\vee)$ can be used to define Maurer–Cartan elements in $D^L_{\mathrm{poly}}(A^\vee)$, which in turn define an L-deformations of R. We prove that the formality map and θ define a deformation $R_\hbar$ of $R = \mathrm{Sym}\,A$ which is isomorphic to $\mathcal{U}(A_\hbar)$, where $A_\hbar$ is (a formal version of) the adiabatic Lie algebroid discussed in [NWX], and that the product is well-defined for $\hbar = 1$.

This is an extension of the discussion for Lie algebras in [K, §8.3.1]. Then we show that the formality maps twisted by θ and composed with the natural morphisms induced by the anchor ρ of A gives morphisms

$$(T^{L,\bullet}_{\text{poly}}(A^\vee), [\theta, \ ]) \longrightarrow C^\bullet(\mathcal{U}(A), \mathcal{U}(A), b)$$
$$(C_\bullet(\mathcal{U}(A), b) \longrightarrow (\Omega^\bullet_L(A^\vee), L_\theta).$$

Using a spectral sequence argument it is then showed that these morphism induce isomorphisms in (co)homology. Finally, since the formality map for chains intertwines the de Rham differential d for $\Omega_L(A^\vee)$ and the cyclic differential B for cyclic homology, we can show that

$$(C^W_\bullet(\mathcal{U}(A)), b + u^{-1}B) \longrightarrow (\Omega^\bullet_L(A^\vee)^W, L_\theta + u^{-1}d)$$

is a morphism of complexes, where W is a $\mathbb{K}[u^{-1}]$-module. This proves the result for the (periodic) cyclic homology.

3.2.1 Deformation quantization

Recall that a formal deformation quantization of a commutative ring R is given by a $\mathbb{K}[[\hbar]]$-linear, associative product $\star$ on $R[[\hbar]]$ satisfying

$$1 \star r = r = r \star 1, \quad \text{for all } r \in R.$$

Writing out such a $\star$-product in powers of $\hbar$,

$$r_1 \star r_2 = r_1 r_2 + \sum_{k \geqslant 1} \frac{\hbar^k}{k!} \tilde{B}_k(r_1, r_2) \tag{3.2.1}$$

we see that it is given by a formal series $\tilde{B} := \sum_{k\geqslant 1} \hbar^k \tilde{B}_k/k! \in \hbar C^2(R, R)[[\hbar]]$. It is well known that associativity of the product is equivalent to B satisfying the Maurer–Cartan equation in the (shifted) Hochschild cochain complex:

$$\delta\tilde{B} + \frac{1}{2}[\tilde{B}, \tilde{B}]_G = 0.$$

We encounter formal deformations controlled by the Lie–Rinehart algebra L: this means that we can find $B_k \in \mathcal{U}(L) \otimes_R \mathcal{U}(L)$ such that $\rho(B_k) = \tilde{B}_k$ where ρ denotes the canonical extension of the anchor to a morphism of cochain complexes

$$\begin{aligned} \rho : D^{L,\bullet}_{\text{poly}}(R)[[\hbar]] &\longrightarrow C^\bullet(R, R)[[\hbar]] \\ \rho(D_1 \otimes \cdots \otimes D_{n+1})(r_1 \otimes \cdots \otimes r_{n+1}) &:= D_1(r_1) \cdots D_{n+1}(r_{n+1}), \end{aligned} \tag{3.2.2}$$

where $D^{L,n}_{\text{poly}}(R)$ is the degree n component of the L-polydifferential operators as defined in the appendix. Following [NeTs01], we shall call such deformations L-*deformations*.

In the remaining part of this section we set $R = \operatorname{Sym} A$ and $L = \operatorname{Der}_{\mathbb{R}}(\operatorname{Sym} A)$ for a smooth Lie algebroid A. Recall the so-called adiabatic Lie algebroid $A_\hbar$ [NWX]. As a Lie–Rinehart algebra, it is given by $(\Gamma(A)[[\hbar]], C^\infty(M)[[\hbar]]$ with Lie bracket $\hbar[\ ,\ ]$ and anchor $\hbar\rho$, but it can also be considered as a sheaf of Lie algebroids over M.

Proposition 3.2.1. *Let $A \to M$ be a smooth Lie algebroid with associated triangular Lie algebroid (L, R, θ). The formality L_∞-morphism for L gives rise to a quantization $R_\hbar$ of R which is isomorphic to $\mathcal{U}(A_\hbar)$. Moreover, this isomorphism restricts to the subalgebras which are polynomial in $\hbar$.*

Proof. Let $U : T^L_{\text{poly}}(A^\vee) \rightsquigarrow D^L_{\text{poly}}(A^\vee)$ be the L_∞-morphism defined in the appendix. The Maurer–Cartan element θ defines a Maurer–Cartan element in $\mathcal{U}(L)\otimes_R\mathcal{U}(L)[[\hbar]]$ by the formula:

$$\sum_{k\geqslant 1} \frac{\hbar^k}{k!} U_k(\theta,\ldots,\theta). \tag{3.2.3}$$

This follows from the general theory for L_∞-algebras as discussed in the appendix. Moreover, $\sum_{k\geqslant 1} \hbar^k/k! B_k$, where $B_k = U_k(\theta,\ldots,\theta)$, defines a deformation quantization on $R[[h]]$ by (3.2.1). We will denote this deformed algebra by $R_\hbar$.

The Maurer–Cartan element θ has polynomial degree -1 and we claim that the bidifferential operators B_k have polynomial degree $-k$. This fact follows from the following observations. Recall that the polynomial degree is detected by the formula

$$L_{\mathcal{E}}(f) = nf, \qquad f \in \text{Sym}^n A, \quad \mathcal{E} = \text{Euler vector field on } A^\vee.$$

The action of the Euler vector field can naturally extended to all the sheaves that are involved in the proof of the formality morphism in the appendix. To show that the formality map preserves the polynomial degree, it suffices to note that all the morphism that are used to define the formality map respect the action of $L_{\mathcal{E}}$. For the natural resolutions using the jet bundle this is clear and for the PBW map it follows because we can choose the L-connection on L, c.f. (A.1.1) to be *homogeneous*:

$$L_{\mathcal{E}}(\nabla_X Y) = \nabla_{L_{\mathcal{E}}X} Y + \nabla_X(L_{\mathcal{E}} Y), \qquad \text{for all } X, Y \in L.$$

Finally, for the fact that the twisted map preservers the degree we refer to the argument given in [D] (there the property is described in terms of an action of a smooth Lie group, which is in this case just the $\mathbb{R}^*$ action on the fibers of $A^\vee$).

Concretely, this means that B_k is bidifferential operator mapping polynomials of degree p and q to a polynomial of degree $p + q - k$, which in turn implies that we can restrict the deformed product to the subalgebra of elements which are polynomial in $\hbar$.

Now recall that the formal version of the adiabatic Lie algebroid is defined as the sheaf of Lie algbroids $A[[\hbar]]$ over the ringed space $C^\infty(M)[[h]])$ with bracket and anchor defined by

$$[s,t]_{A[[\hbar]]} := \hbar[s,t], \qquad \rho_{A[[\hbar]]}(s)(f) := \hbar\rho(s)(f).$$

We denote it by $(A_\hbar, C^\infty(M)_\hbar)$. Its universal enveloping algebra $\mathcal{U}(A_\hbar)$ is, as an $\mathbb{R}[[h]$-module, isomorphic to Sym$(A)[[h]])$, and one can easily see that it is a deformation of the Poisson algebra (Sym$(A), \{\ ,\ \}$). Because $\mathcal{U}(A_\hbar)$ and $R_\hbar$ are deformations of (Sym$(A),\{\ ,\ \}$), i.e. of the same Poisson bracket, they are, by an argument of Kontsevich in [K], isomorphic als algebras.

We can restrict the product in both deformations to the polynomial subalgebra, hence the claim is proved. □

Remark 3.2.2. The core of the proof above, the homogeneity of the bidifferential operators B_k, shows that the quantization of the Poisson manifold $A^\vee$ defined by applying the formality map for the Lie–Rinehart algebra (L, R) to the Lie–Poisson structure, is *homogeneous* as considered in [NeWa]. For our purposes, it also shows that we can put $\hbar = 1$ and consider R itself with the deformed product defined by B. This is then isomorphic to the universal enveloping algebra $\mathcal{U}(A)$ itself.

3.2.2 The basic complexes

Here we briefly recall the definitions of the (co)chain complexes computing the Hochschild and cyclic theories of the universal enveloping algebras $\mathcal{U}(A)$ of a smooth Lie algebroid A. The algebra $\mathcal{U}(A)$ has a natural locally convex topology defined as follows: we write $\mathcal{U}(A)$ as the direct limit of the increasing family of filtered subspaces $F_k \subset \mathcal{U}(A)$, c.f. (2.1.2). By the PBW theorem 2.2.4, F_k is the space of smooth sections of the vector bundle $\bigoplus_{i=0}^k \mathrm{Sym}^i(A)$. It therefore has a natural locally convex topology given by seminorms of the form

$$||f||_{p,K} := \sup_{\substack{x\in K \\ X_1,\ldots,X_p}} ||\nabla_{X_1}\cdots\nabla_{X_p} f||_{\mathrm{Sym}^k(A)_x}, \qquad K\subset M \text{ compact},$$

where we have chosen a connection ∇ and a metric on A. With respect to this topology, the inclusions $F_k \subset F_{k+1}$ are continuous and we give $\mathcal{U}(A)$ the direct limit topology. As such, $\mathcal{U}(A)$ is a nuclear LF-space, as each of the pieces F_k is nuclear Fréchèt. c.f. [Tr] for definitions. By Proposition 3.2.1, the product $f\star g$ of $f, g \in F_k$ involves a finite number -at most k- of derivatives of f and g. For each $p \in \mathbb{N}$ and $K \subset M$ compact, we can find $p' = p + k$ such that

$$||f\star g||_{p,K} \leqslant C||f||_{p',K}||g||_{p',K}, \quad \text{for some } C > 0.$$

The product $\star : \mathcal{U}(A)\times\mathcal{U}(A) \to \mathcal{U}(A)$ is therefore continuous for the topology and extends to a map $\mathcal{U}(A)\hat{\otimes}\mathcal{U}(A) \to \mathcal{U}(A)$, where $\hat{\otimes}$ denotes the completed projective tensor product, so $\mathcal{U}(A)$ is a locally convex algebra as in [Co][Appendix B].

The (topological) Hochschild cohomology is defined as the cohomology of the complex defined on the graded vector space

$$C^\bullet(\mathcal{U}(A),\mathcal{U}(A)) := \mathrm{Hom}_{\mathrm{cts}}\left(\mathcal{U}(A)^{\hat{\otimes}\bullet},\mathcal{U}(A)\right),$$

with differential $\delta := [\ ,\star]$. The dual Hochschild chain complex is given by

$$C_\bullet(\mathcal{U}(A)) := \mathcal{U}(A)^{\hat{\otimes}(\bullet+1)},$$

with differential written as b. As $\mathcal{U}(A)$ is a unital algebra, the B-operator defining the cyclic homology of $\mathcal{U}(A)$ is given by the usual formula.

3.2.3 Computation of the cyclic theory

In this section we use the previous results to compute the topological Hochschild cohomology of $\mathcal{U}(A)$. The final result is:

Theorem 3.2.3. *Let $A \to M$ be a smooth Lie algebroid over a manifold M. There is a canonical isomorphism*

$$H^\bullet(\mathcal{U}(A)) \cong H^\bullet(A, \mathrm{Sym}(\mathrm{ad}\ A)).$$

Proof. The tangent map of the formality morphism, c.f. A.2 associated to the Maurer–Cartan element $\theta \in T^{L,1}_{\mathrm{poly}}(A^\vee)$ is given by

$$T_\theta U : \left(T^L_{\mathrm{poly}}(A^\vee), [\theta,\ ]\right) \longrightarrow \left(D^L_{\mathrm{poly}}(A^\vee)[\hbar], \delta + [\tilde{\theta},\ ]\right),$$

where we could restrict to polynomials in $\hbar$ because of proposition 3.2.1. In fact, this morphism is an L_∞-moprhism, but we don't need the Lie algebra structures for this proof. The same definition as the natural morphism $D^{L,\bullet}_{\mathrm{poly}}(A^\vee)[\hbar] \longrightarrow C^\bullet(R,R)[\hbar]$ from (3.2.2) gives a morphism of complexes

$$\left(D^L_{\mathrm{poly}}(A^\vee)[\hbar], \delta + [\tilde{\theta},\ ]\right) \longrightarrow \left(C(R^{\mathrm{pol}}_\hbar, R^{\mathrm{pol}}_\hbar), \delta_\hbar\right). \tag{3.2.4}$$

This is not entirely obvious, but it follows from a rather straightforward computation relating the differentials $\delta + [\tilde{\theta},\ ]$ and $\delta_\hbar$ -the Hochschild differential for the deformed algebra $R^{\mathrm{pol}}_\hbar$. We set $\hbar$ equal to 1, and use proposition 3.2.1 to obtain a morphism of complexes

$$\left(T^L_{\mathrm{poly}}(A^\vee), [\theta,\ ]\right) \longrightarrow \left(C^\bullet(\mathcal{U}(A),\mathcal{U}(A)), \delta\right).$$

Let us now consider the filtration on the Hochschild complex induced by the filtration on $\mathcal{U}(A)$: first define

$$F_p(\mathcal{U}(A)^{\hat{\otimes}k}) = \sum_{i_1+\ldots i_k=p} F_{i_1}(\mathcal{U}(A)) \otimes \ldots \otimes F_{i_k}(\mathcal{U}(A)).$$

And with this we put

$$F_p(C^k(\mathcal{U}(A),\mathcal{U}(A)) = \sum_l \mathrm{Hom}_{\mathrm{cts}}\left(F_l(\mathcal{U}(A)^{\hat{\otimes}k}), F_{l+p}(\mathcal{U}(A))\right).$$

This induces a spectral sequence with

$$E_0^{p,q} = F_p C^{p+q}/F_{p-1}C^{p+q} \cong \mathrm{Hom}^p_{\mathrm{cts}}\left(R^{\otimes(p+q)}, R\right).$$

The differential $d_0:\ E_0^{p,q} \to E_0^{p,q+1}$ is just the standard Hochschild differential of the commutative topological algebra R. By the topological version of the Hochschild–Kostant–Rosenberg theorem we therefore find

$$E_1^{p,q} \cong T^L_{\mathrm{poly}}(A^\vee).$$

The differential on this page of the spectral sequence is defined using the first term B_1 and is, by the same reasoning as in [Br], equal to the Poisson cohomology differential $[\theta,\ ]$. If we now view $(T^L_{\mathrm{poly}}(A^\vee),\ [\theta,\ ])$ as a filtered complex on its own, with the filtration by polynomial degree, we can see that $T_\theta U$ is a morphism of filtered complexes as follows: The formality morphism preserves the polynomial degree, and the tangent map

$$T_\theta U = U_1 + \text{ terms containing } \theta$$

where U_1 is the HKR map which is of degree 0, and the terms containing θ strictly decrease the degree because θ has degree -1. Because of this, we see that the induced map on the spectral sequence is an isomorphism from the first page on. By the comparison theorem for spectral sequences, we conclude that $T_\theta U$ is a quasi-isomorphism itself.

Finally, the theorem follows from the identification of the polynomial Poisson cohomology with the Lie algebroid cohomology of symmetric powers of the adjoint, c.f. Theorem 3.1.14. □

Theorem 3.2.4. *Let A be a smooth Lie algebroid over a manifold M. There exists an isomorphism*

$$H_\bullet((\mathcal{U}(A),\mathcal{U}(A)) \cong H^\bullet_\theta\left(\Omega^\bullet_L(A^\vee)\right)$$

where $H^\bullet_\theta(\Omega^\bullet_L(A^\vee))$ is the Poisson homology of the complex $(\Omega^\bullet_L(A^\vee), L_\theta)$.

Proof. This follows by an argument which is very similar to the proof of the last theorem, but here one uses the globalized and twisted version of Shoikhets formality map:

$$\left(C_L(A^\vee), b_H + L_{\tilde\theta}\right) \xrightarrow{\mathcal{V}^\theta} \left(\Omega^\bullet_L(A^\vee),\ L_\theta\right). \tag{3.2.5}$$

Moreover, we use the inclusion $(C_\bullet(\mathcal{U}(A),\mathcal{U}(A)), b_H) \longrightarrow \left(C_L(A^\vee), b_H + L_{\tilde\theta}\right)$ which is the composition of the natural injective map

$$\alpha_1:\ C_\bullet(R,R) \longrightarrow C_\bullet(\mathcal{J}(L),\mathcal{J}(L))$$

and the projection (A.4.2). The description of the first page of the spectral sequence associated to the filtrated Hochschild complex of $\mathcal{U}(A)$ can be found in [Br]. □

Theorem 3.2.5. *Let A be a smooth Lie algebroid over a manifold M. There exists a quasi-isomorphism of complexes*

$$\left(C_\bullet(\mathcal{U}(A)), b + u^{-1}B\right) \longrightarrow \left(\Omega^\bullet_L(A^\vee), L_\theta + u^{-1}d\right).$$

Proof. This is done in a similar fashion, but now the differentials B and d are involved. We refer to lemma A.6.1 for the fact that the twisted formality map of modules in (3.2.5) intertwines the differentials B and d. Moreover, in [BrGe, Wo], the authors proved that the differential on the second page of the spectral sequence of the cyclic complex of $\mathcal{U}(A)$ coincides with the de Rham differential d. There are no further difficulties. □

For the periodic theory, the homology simplifies:

Corollary 3.2.6. *For a smooth Lie algebroid $A \to M$ we have*

$$HP_\bullet(\mathcal{U}(A)) \cong H^\bullet(M)((u^{-1})).$$

Proof. Recall that the periodic cyclic homology is computed from the complex $C_\bullet(\mathcal{U}(A))((u^{-1}))$ with differential $b+u^{-1}B$. The quasi-isomorphism of the previous theorem extends to a quasi-isomorphism to the complex $\Omega_L^\bullet(A^\vee)((u^{-1}))$ equipped with the differential $L_\theta + u^{-1}d$. The invertible map

$$e^{u\iota_\theta} : \Omega_L^\bullet(A^\vee)((u^{-1})) \to \Omega_L^\bullet(A^\vee)((u^{-1}))$$

simplifies the differential to just d. But the cohomology of the complex $\Omega_L^\bullet(A^\vee)((u))$ is easy to compute: Any homogeneous element $\alpha \in \Omega_L^\bullet(A^\vee)((u^{-1}))$ of degree p is an eigenvector of the Euler vector field: $L_\mathcal{E}\alpha = p\alpha$. But this implies that when α is d-closed, it is also exact: $\alpha = p^{-1}d\iota_\mathcal{E}\alpha$. This argument shows that the inclusion $\pi^* : \Omega^\bullet(M)((u^{-1})) \hookrightarrow \Omega_L^\bullet(A^\vee)((u^{-1}))$ -the pull-back along the projection map π, is a quasi-isomorphism. This proves the statement. □

3.2.4 Some examples

Trivial Lie algebroids

For the trivial Lie algebroid $A = \{0\}\times M$, the universal enveloping algebra reduces to the algebra of smooth functions $C^\infty(M)$. In this case the adjoint representation has only one nonzero component in degree 1 where it is just TM, so $\mathrm{Sym}(\mathrm{ad}\,A) = \Lambda^\bullet TM$ with zero differential. The Lie algebroid $\mathrm{Der}_\mathbb{R}(\mathrm{Sym}\,A)$ is equal to TM and the Poisson structure $\theta \in \Lambda^2 L$ is given by the trivial one, hence $[\theta,\ ] = L_\theta = 0$. Clearly, we have

$$(\Omega_L^\bullet, L_\theta + d_L) = (\Omega(M), d),$$

so in this case our computations reduce to the well-known isomorphisms of Connes [Co]:

$$\begin{aligned} HH^\bullet(C^\infty(M)) &\cong \Lambda^\bullet TM \\ HH_\bullet(C^\infty(M)) &\cong \Omega^\bullet(M) \\ HC_\bullet(C^\infty(M)) &\cong H^\bullet(\Omega(M)[u^{-1}], u^{-1}d) \\ HP_\bullet(C^\infty(M)) &\cong \bigoplus_{k\geqslant 0} H_{dR}^{\bullet+2k}(M). \end{aligned}$$

Lie algebras

A Lie algebroid over a point is the same as an ordinary Lie algebra $\mathfrak{g}$. In this case the adjoint representation exists on $\mathfrak{g}$ as an honest representation. The isomorphism between the Poisson cohomology complex of the Poisson algebra $\mathrm{Sym}\,\mathfrak{g}$ and the Lie algebra cohomology complex with values in the symmetric algebra of the adjoint representation is given by:

$$(T_{\mathrm{poly}}^{L,\bullet}(A^\vee), [\theta,\ ]) \cong (C^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g}), d_{\mathrm{Lie}}).$$

Here $L = \mathrm{Sym}\,\mathfrak{g} \otimes \mathfrak{g}^*$. It has a dual counterpart, namely an isomorphism between the Poisson homology complex of the Poisson algebra $\mathrm{Sym}\,\mathfrak{g}$ and the Lie algebra homology complex with values in the symmetric algebra of the adjoint representation:

$$(\Omega_L^\bullet, L_\theta) \cong (C_\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g}), d_{\mathrm{Lie}}).$$

Our computations of the (co)homology of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ coincide with the results from Kassel [Ka]

$$\begin{aligned} H^\bullet(\mathcal{U}(\mathfrak{g})) &\cong H^\bullet_{\mathrm{Lie}}(\mathfrak{g}; \mathrm{Sym}(\mathfrak{g})) \\ H_\bullet(\mathcal{U}(\mathfrak{g})) &\cong H^{\mathrm{Lie}}_\bullet(\mathfrak{g}; \mathrm{Sym}(\mathfrak{g})) \\ HP_\bullet(\mathcal{U}(\mathfrak{g})) &\cong \mathbb{C}((u^{-1})) \\ HC_\bullet(\mathcal{U}(\mathfrak{g})) &\cong H_\bullet\left(\Omega_L^\bullet(\mathfrak{g}^*)[u^{-1}], L_\theta + u^{-1} d\right). \end{aligned}$$

Tangent spaces

Consider the Lie algebroid $A = TM$. The universal enveloping algebra of TM is given by the differential operators $\mathrm{Diff}(M)$ on M. The Lie algebroid $L = \mathrm{Der}_{\mathbb{R}}(\mathrm{Sym}\,TM)$ has a Poisson structure which is non degenerate, and this simplifies the discussion. We have the following lemma:

Lemma 3.2.7. *The maps* i *and* π

$$(\Omega(M), d) \xrightarrow{\ i\ } (\Omega_L, d_L) \qquad\qquad (\Omega_L, d_L) \xrightarrow{\ \pi\ } (\Omega(M), d)$$

given by the natural injection and projection are homotopy inverses with respect to the given differentials. Hence

$$H^\bullet_{\mathrm{dR}}(M) \cong H^\bullet_{\mathrm{Lie}}(L)$$

Proof. This a copy of the proof of proposition 4.5 from [L-BM]. □

Brylinski proved in [Br] that the 'hodge star' map for a symplectic manifold gives an isomorphism of the de Rham complex with the Poisson homology complex:

$$(\Omega_L^\bullet, d_L) \longrightarrow (\Omega_L^{2n-\bullet}, L_\theta).$$

Since the modular class of a symplectic manifold vanishes, the Poisson homology complex and the Poisson cohomology complex are isomorphic as well:

$$(\Omega_L^{2n-\bullet},\ L_\theta) \cong (T^{L,\bullet}_{\mathrm{poly}},\ [\theta, \cdot]),$$

so we conclude that $H^\bullet_{\mathrm{dR}}(M) \cong H^\bullet(A, \mathrm{Sym}(\mathrm{ad}\,A))$ and $H^\bullet_{\mathrm{dR}}(M) \cong H^\theta_{2n-\bullet}(\mathrm{Sym}\,A)$, where the last notation means the Poisson homology of the sheaf of Poisson algebras $\mathrm{Sym}\,A$ over M. Hence, in this case our computations reduce to the results of Wodzicki [Wo]:

$$\begin{aligned} HH^\bullet(\mathrm{Diff}(M)) &\cong H^\bullet_{\mathrm{dR}}(M) \\ HH_\bullet(\mathrm{Diff}(M)) &\cong H^{2n-\bullet}_{\mathrm{dR}}(M) \\ HC_\bullet(\mathrm{Diff}(M)) &\cong H^{2n-q}_{\mathrm{dR}}(M) \oplus H^{2n-q+2i}_{\mathrm{dR}}(M) \oplus \dots \\ HP_\bullet(\mathrm{Diff}(M)) &\cong H^{\mathrm{ev}/\mathrm{odd}}_{\mathrm{dR}}(M). \end{aligned}$$

Chapter 4

The trace density map

In this chapter we construct a trace density map for the universal enveloping algebra of a Lie algebroid. We assume that $\mathcal{L}$ is either a smooth Lie algebroid or a holomorphic Lie algebroid, but we think that most of the constructions can be done in greater generality. In the previous chapter we abstractly identified the cyclic theory of the universal enveloping algebroid $\mathcal{U}(A)$ of a smooth Lie algebroid; the result was that it is given by the cohomology of the symmetric algebra of the adjoint representation (twisted by the bundle Q_A). The trace density map, which is extensively studied in index theory, gives a more explicit construction.

The trace density in deformation quantization made its first appearance in the fundamental papers [NeTs95, NeTs96, BNT] on the algebraic index theorem. There, the authors used Čech methods and an abstract description of the cyclic cohomology of the Weyl algebra to prove an index theorem. In [FFS] the authors constructed an explicit cocycle in the Hochschild cohomology of the Weyl algebra with invariance properties and used it to define a map from the Hochschild homology of a deformation of the smooth functions on a symplectic manifold to de Rham forms of top degree of that manifold. The *canonical trace* associated to the deformation was then defined as the integral of the image of smooth functions with compact support over M. Using the local Riemann–Roch theorem, c.f. [FT, FFS], which identifies the image of the Hochschild cocycle under a natural map to the Lie algebra cohomology as certain characteristic classes, the authors proved that the trace applied to the canonical class of 1 in the Hochschild homology is equal to the integral of certain characteristic classes, i.e. topological data. In [EnFe], the Hochschild cocycle was used to define a morphism from the Hochschild homology of holomorphic differential operators on a complex vector bundle $E \to X$ to the sheaf of smooth de Rham forms on X, and they used it to prove that the supertrace of a holomorphic differential operator can be expressed as an integral over topological data. In [PoPfTa10] and [W15] the fundamental Hochschild cocycle was extended to a cyclic cocycle of the Weyl algebra, and the authors of [PoPfTa10] used it to give a proof of the algebraic index theorem of [NeTs95]. Roughly speaking, given a deformation of the smooth functions on a symplectic manifold, the algebraic index theorem computes the pairing of the Chern character of an element in the K_0 theory of the deformation with a smooth de Rham form as an integral over topological data. These ideas were used in [PoPfTa10] to prove the higher index theorem of [CoMo]. Generalizing to Lie groupoids, in [PoPfTa15] the authors applied similar techniques to construct a pairing for elliptic elements of the universal enveloping algebroid of a smooth, integrable Lie algebroid. This pairing was expressed as an integral over A^*. The image of our trace density map is given by the $\mathcal{L}$-valued forms, and it would be interesting to investigate whether similar pairings can be constructed for our trace density map.

The trace density map that we construct is a morphism from the sheaf complex of cyclic chains of the universal enveloping algebra to a Čech resolution of the de Rham complex of $\mathcal{L}$-

forms. When $\mathcal{L}$ is smooth, the construction can be performed globally, providing a morphism from the cyclic complex of (global sections of) the universal enveloping algebra to the complex of global section of $\mathcal{L}$-forms.

Now we outline our construction. Our trace density map depends on the PBW isomorphism, and hence on the choice of a connection, which explains that it can be globally defined when $\mathcal{L}$ is smooth. We show that the associated map in (co)homology does not depend on the choice of a connection in the smooth case. When $\mathcal{L}$ is holomorphic, global connections do not always exist. However, we can choose an open cover $\mathscr{U}$ such that $\mathcal{L}$ is locally trivial over each $U_i \in \mathscr{U}$, which implies that we can choose local connections and we obtain a morphism of sheaves (restricted to U_i)

$$\Phi_{U_i} : \mathrm{CC}_\bullet(\mathcal{U}(\mathcal{L}))|_{U_i} \longrightarrow \mathrm{Tot}^{2r-\bullet}\Omega_{\mathcal{L}}|_{U_i}.$$

Here $\mathrm{Tot}^\bullet\Omega_{\mathcal{L}} := \bigoplus_{i\geqslant 0}\Omega_{\mathcal{L}}^{\bullet-2i}$. Over intersections $U_{i_0,\ldots i_k}$, we consider an affine connection ∇^{aff} depending on a variable t of the k-simplex, and the associated *family* of PBWisomorphisms. We use the derivatives of this family, c.f. 2.2.10, to define maps

$$\Phi_{U_{i_0,\ldots,i_k}} : \mathrm{CC}_\bullet(\mathcal{U}(\mathcal{L}))|_{U_{i_0,\ldots,i_k}} \longrightarrow \mathrm{Tot}^{2r-\bullet}\Omega_{\mathcal{L}}|_{U_{i_0,\ldots,i_k}}$$

This implies that, given a section D of the sheaf $\mathrm{CC}_\bullet(\mathcal{U}(\mathcal{L}))$, we can send it to the Čech resolution of $\Omega^\bullet_{\mathcal{L}}$ with respect to $\mathscr{U}$, and we obtain a morphism

$$\Phi_{\mathscr{U}} : \mathrm{CC}_\bullet(\mathcal{U}(\mathcal{L})) \longrightarrow \check{C}^{2r-\bullet}(\mathscr{U}, \mathrm{Tot}\Omega_{\mathcal{L}}).$$

We will show that this morphism intertwines the differentials $b + u^{-1}B$ on the LHS and $d + (-1)^k\delta$ on the RHS, so in fact it is a morphism of complexes of sheaves. Moreover, by the same reasoning as in the smooth case, this morphism does not depend on the choice of connections on U_i. The hypercohomology of $\Omega^\bullet_{\mathcal{L}}$ coincides with the hypercohomology of $\check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{L}})$ for any cover of X and we show that the trace density map on the level of hypercohomology

$$\Phi : \mathbb{H}_\bullet(\mathrm{CC}_\bullet(\mathcal{U}(\mathcal{L})),\ b + u^{-1}B) \longrightarrow \mathbb{H}^\bullet(\mathrm{Tot}\Omega_{\mathcal{L},W}, d_{\mathcal{L}})$$

does not depend on the choice of a cover and local connections.

In section 4.2 we discuss characteristic classes in the setting of Čech cohomology, in order to compare them with the image of our trace density map. This construction is a particular example of the characteristic classes for simplicial manifolds, c.f. [Du]. Then we identify the image of the canonical class of 1 in the cyclic homology under the trace density map in the twisted case, i.e., we consider the universal enveloping algebra with coefficients in a vector bundle $\mathcal{E}$. It is given by the product of the Todd class of $\mathcal{L}$ and the $\mathcal{L}$-valued Chern character of the vector bundle $\mathcal{E}$. Note that we complexify the bundle in the case where $\mathcal{L}$ and $\mathcal{E}$ are smooth to define the Todd class. Finally, we show that the trace density map is compatible with the HKR map.

4.1 Definition and properties of the trace density map

4.1.1 The set up

We assume that $(X, \mathcal{O}_X)$ is either a smooth manifold with the sheaf of smooth functions, or a holomorphic manifold with the sheaf of holomorphic functions. Moreover, $\mathcal{L}$ is either a smooth or a holomorphic Lie algebroid of rank r over $(X, \mathcal{O}_X)$. We define the sheaf complex of the cyclic homology of $\mathcal{O}_X$ as the sheaf associated to the following presheaf:

$$U \mapsto (\mathrm{CC}^W_\bullet(\mathcal{O}_X(U)),\ b + u^{-1}B), \qquad U \subset X \text{ open}.$$

Recall that u^{-1} is a formal variable of degree -2 (so that $b+u^{-1}B$ has degree -1) and that we write $CC^W_\bullet(\mathcal{O}_X(U)) = CC_\bullet(\mathcal{O}_X(U))[[u^{-1}]] \otimes_{\mathbb{K}[u^{-1}]} W$ where W is one of the $\mathbb{K}[u^{-1}]$-modules discussed in the section about cyclic theory. The usual Hochschild–Kostant–Rosenberg map

$$\mathrm{HKR}: f_0 \otimes \ldots \otimes f_k \mapsto f_0 df_1 \wedge \ldots \wedge df_k, \quad f_i \in \mathcal{O}_X(U),\ i = 0, \ldots, k, \tag{4.1.1}$$

sheafifies to define a morphism $(CC^W_\bullet(\mathcal{O}_X), b, u^{-1}B) \mapsto (\Omega^\bullet_{\mathcal{O}_X,W}, 0, u^{-1}d)$ of mixed complexes of sheaves, where $\Omega^\bullet_{\mathcal{O}_X,W} = \Omega^\bullet_{\mathcal{O}_X} \otimes_{\mathbb{K}[u^{-1}]} W$.

Remark 4.1.1. When X is a manifold, it is customary to complete the sheaf of cyclic chains by taking the germs or jets of functions around the diagonal in $X^{\times(k+1)}$. We do not consider such completions here, neither are we concerned with the question whether the HKR map is a quasi-isomorphism. However, it is not difficult to prove that in this case our index theorem extends to the completed chain complexes.

We define the cyclic chain complex of sheaves associated to the sheaf of universal enveloping algebras for a given Lie algebroid $\mathcal{L}$ in a similar fashion as the sheaf associated to the following presheaf of complexes:

$$U \mapsto (CC^W_\bullet(\mathcal{U}(\mathcal{L}(U))),\ b + u^{-1}B).$$

The associated sheaf of chain complexes is simply denoted by $(CC_\bullet(\mathcal{U}(\mathcal{L})),\ b + u^{-1}B)$.

The aim of this section is to construct a map between this complex of sheaves and the de Rham complex of sheaves $(\Omega^\bullet_{\mathcal{L}}, d_{\mathcal{L}})$. It is the analogue of the so-called "trace density" [BNT] or "character map" [PoPfTa10, Theorem 3.9] in deformation quantization. In our setting, the proper formulation is in terms of the derived category $\mathcal{D}(X)$ of $\mathbb{K}$-sheaves on X. To state the precise theorem, we define, $\Omega^\bullet_{\mathcal{L},W} := \Omega^\bullet_{\mathcal{L}}[[u]^{-1}] \otimes_{\mathbb{K}[u^{-1}]} W$, for any $\mathbb{K}[u^{-1}]$-module W. In the language of derived categories, the main theorem of this section then reads as follows:

Theorem 4.1.2. *Let $(\mathcal{L}, \mathcal{O}_X)$ be a Lie algebroid. There exists a canonical morphism in the derived category of $\mathbb{K}$-sheaves:*

$$\Phi \in \mathrm{Hom}_{\mathcal{D}(X)}\left(\left(CC^W_\bullet(\mathcal{U}(\mathcal{L})),\ b + u^{-1}B\right), \left(\mathrm{Tot}^{2r-\bullet}\Omega_{\mathcal{L},W},\ d_{\mathcal{L}}\right)\right).$$

Recall that the derived category of $\mathbb{K}$-sheaves is constructed out of the category of chain complexes of sheaves by 1) identifying homotopic chain maps (i.e., going over to the homotopy category and 2) localization with respect to quasi-isomorphisms. For any cover $\mathscr{U}$ of X, the natural map from the Lie algebroid chain complex $(\Omega^\bullet_{\mathcal{L},W}, d_{\mathcal{L}})$ to its Čech resolution $\check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{L},W})$ is a quasi-isomorphism, hence a morphism

$$\Phi_{\mathscr{U}}:\ \left(CC^W_\bullet(\mathcal{U}(\mathcal{L})),\ b + u^{-1}B\right) \longrightarrow \check{C}^{\bullet-2r}(\mathscr{U}, \mathrm{Tot}\Omega_{\mathcal{L},W},\ d_{\mathcal{L}} + (-1)^{2r-\bullet}\delta)$$

determines a morphism in the derived category:

$$[\Phi_{\mathscr{U}}] \in \mathrm{Hom}_{\mathcal{D}(X)}\left(\left(CC^W_\bullet(\mathcal{U}(\mathcal{L})),\ b + u^{-1}B\right), \left(\mathrm{Tot}^{2r-\bullet}\Omega_{\mathcal{L},W},\ d_{\mathcal{L}}\right)\right).$$

The proof of the theorem that we give consists of three steps: 1) The construction of a morphism $\Phi_{\mathscr{U}}$ for any open cover $\mathcal{U}$ such that $\mathcal{L}$ is trivial over all $U_i \in \mathscr{U}$). 2) We show that $\Phi_{\mathcal{U}}$ intertwines the differential of the complexes of sheaves. 3) We show that the morphism in the derived category does not depend on the choices that were made.

When $\mathcal{L}$ is smooth, it implies that there exists a canonical morphism on the cohomology and when $\mathcal{L}$ is holomorphic it implies that there exists a canonical map in hypercohomology.

4.1.2 The Fedosov resolution

In this subsection we assume that $\mathcal{L}$ admits a global $\mathcal{L}$-connection ∇. We construct the analogue of the Fedosov resolution [Fed] for $\mathcal{U}(\mathcal{L})$. In the smooth category, such a resolution was constructed in [NeWa] from the point of view of formal deformation quantization of the dual of a Lie algebroid equipped with the Lie–Poisson bracket. Here we use the PBW theorem of chapter 2. Recall from this chapter that

$$\alpha_2 : \mathcal{U}(\mathcal{L}) \longrightarrow \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}) \cong \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})),$$

where the first map is given by $D \mapsto \alpha_2(1) \otimes D$ and the second map is defined using the second $\mathcal{U}(\mathcal{L})$-module structure on $\mathcal{J}(\mathcal{L})$. There is a natural product on the RHS given by composition of differential operators. The flat connection $\nabla^{(1)}$ on $\mathcal{J}(\mathcal{L})$ induces in a natural way a flat connection on $\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$, which is given by $\nabla^{(1)} \otimes 1$ on $\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L})$. We denote the associated Chevalley–Eilenberg complex by

$$\Omega^\bullet_{\mathcal{L}}(\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))) := \left(\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L})\right) \otimes_{\mathcal{O}_{X_1}} \Lambda^\bullet_{\mathcal{O}_X} \mathcal{L}^\vee$$

and the differential by $d_{\nabla^{(1)}}$.

Lemma 4.1.3. *The Chevalley–Eilenberg complex* $\left(\Omega^\bullet_{\mathcal{L}}\left(\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))\right), d_{\nabla^{(1)}}\right)$ *is a sheaf of differential graded algebras.*

Proof. By construction, $\Omega^\bullet_{\mathcal{L}}\left(\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))\right)$ is both a graded algebra (combining the multiplication with the wedge product) and a cochain complex. Therefore, the only thing to check is the compatibility between both structures. The action of $\nabla^{(1)}$ on $\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$ is defined by

$$\nabla^{(1)}_X(D)(\phi) := [\nabla^{(1)}_X, D](\phi), \qquad X \in \mathcal{L},\ D \in \mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) \tag{4.1.2}$$

and hence it follows easily that $\nabla^{(1)}$ acts by derivations on $\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$, which proves the lemma. □

Proposition 4.1.4. *The map* $\mathcal{U}(\mathcal{L}) \to \Omega^\bullet_{\mathcal{L}}(\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})))$ *is a quasi-isomorphism of complexes of sheaves.*

Proof. The proof is very similar to the proof of proposition 4.2.4 in [CvdB]. The adic filtration on $\mathcal{J}(\mathcal{L})$, c.f chapter 2, induces a natural filtration on the sequence:

$$\mathcal{U}(\mathcal{L}) \xrightarrow{\alpha_2} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}) \xrightarrow{d_{\nabla^{(1)}}} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} \Lambda^1_{\mathcal{O}_X} \mathcal{L}^\vee \xrightarrow{d_{\nabla^{(1)}}} \dots$$

Note that $\mathcal{L}^\vee$ has degree 1. The associated graded exact sequence is given by

$$\mathcal{U}(\mathcal{L}) \xrightarrow{i} \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \mathcal{U}(\mathcal{L}) \xrightarrow{-d_K} \mathrm{Sym}_{\mathcal{O}_X} \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{O}_X} \Lambda^1_{\mathcal{O}_X} \mathcal{L}^\vee \xrightarrow{-d_K} \dots$$

where $-d_K$ is (minus) the standard Koszul differential on $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \Lambda_{\mathcal{O}_X} \mathcal{L}^\vee$ given by $\sum(\iota_{l_i} \otimes l_i^*)$ in local coördinates l_i on $\mathcal{L}$, and the identity on $\mathcal{U}(\mathcal{L})$. Since $-d_K$ is exact, the original sequence is exact. □

Remark 4.1.5. In the smooth category, it is a well-known fact that the universal enveloping algebra of a Lie algebroid is isomorphic to the algebra of invariant differential operators on an integrating Lie groupoid. The preceeding Proposition should be viewed as an algebraic counterpart of that statement, with the integrating Lie groupoid being replaced by the Hopf algebroid of jets, and instead of taking invariants, we consider the cohomology of $\Omega^\bullet_{\mathcal{L}_1}(\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})))$.

The PBW theorem, which is a $\mathcal{O}_{X_1}$-linear algebra isomorphism, induces the following natural isomorphisms:

$$(\Omega_{\mathcal{L}}(\mathcal{J}(\mathcal{L})),\ d_{\nabla^{(1)}}) \longrightarrow \left(\Omega_{\mathcal{L}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)),\ \tilde{d}_{\nabla^{(1)}}\right)$$
$$\left(\Omega_{\mathcal{L}}(\mathrm{Diff}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))),\ d_{\nabla^{(1)}}\right) \longrightarrow \left(\Omega_{\mathcal{L}}(\mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))),\ \tilde{d}_{\nabla^{(1)}}\right)$$

where $\tilde{d}_{\nabla^{(1)}} = j_\nabla \circ d_{\nabla^{(1)}} \circ j_\nabla^{-1}$. This equips the right hand sides with a dg algebra structure. The $\mathcal{L}$-connection ∇ induces natural derivations of degree 1 on the algebras $\Omega_{\mathcal{L}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))$ and $\Omega_{\mathcal{L}}(\mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)))$. These derivations, which we denote by ∇ as well, do not square to 0 unless ∇ happened to be flat. Let us explicitly compare the operators ∇ and $\tilde{d}_{\nabla^{(1)}}$ on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$:

$$A(\nabla) := j_\nabla \circ d_{nabla^{(1)}} \circ j_\nabla^{-1} - \nabla. \tag{4.1.3}$$

Since $\tilde{d}_{\nabla^{(1)}}$ and ∇ act by derivations, it follows that $A(\nabla) \in \Omega^1_{\mathcal{L}}(\mathrm{Der}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)))$. The difference of the two operators ∇ and $\tilde{d}_{\nabla^{(1)}}$ on the graded algebra $\mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)))$ is given by

$$\mathrm{ad}(A(\nabla)) = [A(\nabla),\ ] = j_\nabla \circ d_{\nabla^{(1)}} \circ j_\nabla^{-1} - \nabla.$$

This follows from the fact that ∇ and $d_{\nabla^{(1)}}$ are defined as in (4.1.2). The crucial property of $A(\nabla)$ is that it satisfies the Maurer–Cartan equation, which is the content of the next lemma.

Lemma 4.1.6. *Let* $A(\nabla)$ *be as defined in* (4.1.3). *The following formula holds:*

$$R(\nabla) + \nabla A(\nabla) + \frac{1}{2}[A(\nabla), A(\nabla)] = 0. \tag{4.1.4}$$

Proof. This follows from the fact that $\nabla + A(\nabla)$ is a differential. Note that $R(\nabla)$ is the curvature of ∇. □

4.1.3 A proof of Theorem 4.1.2

As explained after theorem 4.1.2, we will define the trace density map as a morphism from the sheaf complex of the cyclic complex of $\mathcal{U}(\mathcal{L})$ to a Čech resolution of the comples of $\mathcal{L}$-forms. Then we show that the induced morphism in the derived category does not depend on the choices that are made. Since we apply the PBW theorem, we choose a cover $\mathscr{U} = \{U_i\}_{i\in I}$ of X by opens U_i such that $\mathcal{L}|_{U_i}$ is trivial for each $U_i \in \mathscr{U}$. This implies that there exist $\mathcal{L}|_{U_i}$-connections on $\mathcal{L}|_{U_i}$. A concise way to formulate the Čech complex is provided by the notion of the Čech groupoid:

$$X_{\mathscr{U}} := \coprod_{i,j\in I} U_i \cap U_j \rightrightarrows \coprod_{i\in I} U_i.$$

For each intersection in the k-th nerve $X_{\mathscr{U},k} := \coprod_{i_0,\dots,i_k \in I} U_{i_0} \cap \ldots \cap U_{i_k}$, we define the family of connections

$$\nabla^{\mathrm{aff}}_{U_{i_0,\dots,i_k}} := \sum_{j=0}^{k} t_{i_j} \nabla_{i_j}, \qquad \text{where } (t_{i_0} \ldots t_{i_k}) \in \Delta^k = \{(t_{i_0} \ldots t_{i_k}) \in \mathbb{R}^{k+1} : \sum_{j=0}^{k} t_{i_j} = 1\}$$

In the following, we simply write $\bar{t} = (t_{i_0}, \ldots, t_{i_k}) \in \Delta^k$. To this "simplicial connection" $\nabla^{\mathrm{aff}}_{U_{i_0,\dots,i_k}}$ we can associate the following elements:

1. The family of connection forms $A^{\mathrm{aff}}_{U_{i_0,\dots i_k}}(\bar{t})$ as in formula (4.1.3).

2. The derivations $\theta_j^{U_{i_0,\dots,i_k}}(\bar{t})$, $j = 1, \ldots, k$ by applying propostion 2.2.10 to the family of PBW isomorphisms induced by the affine connection. To be precise, $\theta_j^{U_{i_0,\dots,i_k}}(\bar{t})$ is the $\bar{t}$-dependent derivation of $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ that satisfies

$$\theta_j^{U_{i_0,\dots,i_k}}(\bar{t}) \circ j_{\nabla^{\mathrm{aff}}} = (\partial_{t_{i_j}} - \partial_{t_{i_{j-1}}}) j_{\nabla^{\mathrm{aff}}}(\bar{t}). \tag{4.1.5}$$

These elements, and the cyclic cocycle τ_{2r} in the cohomology complex of the Weyl algebra, c.f. B, appear in the definition of the trace density map. Let $D = D_0 \otimes \ldots \otimes D_l \in CC^W_l(\mathcal{U}(\mathcal{L}))$.

Definition 4.1.7. *Given a cover $\mathscr{U}$ such that $\mathcal{L}$ is locally trivial over each $U_i \in \mathscr{U}$, the trace density map*

$$\Phi_{\mathscr{U}} : CC_\bullet(\mathcal{U}(\mathcal{L})) \to \check{C}^{2r-\bullet}(\mathscr{U}, \mathrm{Tot}\Omega_{\mathcal{L}})$$

is defined by the following equation:

$$\Phi_{U_{i_0 \dots i_k}}(D) := (-1)^\alpha \int_{\Delta^k} \iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2r}(1 \otimes \exp(\wedge A^{\mathrm{aff}}(\bar{t})) \times D) d\bar{t}. \tag{4.1.6}$$

where $U_{i_0, \dots i_k} \subset X_{\mathscr{U},k}$, the k-th Čech nerve of the cover $\mathscr{U}$.

Let us clarify this definition.

1. We suppressed a number of super and subscripts that we used before to keep the notation readable.
2. The symbol $d\bar{t}$ indicates the natural volume form on the simplex induced by the standard volume form on $\mathbb{R}^n$.
3. The symbol D stands for $j_{\nabla^{\mathrm{aff}}}(D) \in \mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))$.
4. The cyclic cocycle τ_{2r} is a cocycle for the Weyl algebra $\mathcal{W}(\mathcal{L})$, thus, to apply it to elements in $\mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))$, we use that $\mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \cong \mathcal{W}(\mathcal{L})$.
5. The product is given by the graded shuffle product of elements in $\Omega_{\mathcal{L}}(\mathcal{W}(\mathcal{L}))$. The grading is given by the degree of differential forms.
6. The element $\exp(\wedge A)$ is an $\mathcal{L}$-form with values in the Weyl algebra such that
$$\exp(\wedge A)(X_1, \dots, X_n) = A(X_1) \wedge \dots \wedge A(X_n)$$
7. Finally, the sign is given by $\alpha = \sum_{i=1}^k i + \sum_{j=1}^l j$.

This was step 1 of the proof of theorem 4.1.2; the following proposition deals with step 2.

Proposition 4.1.8. *The map $\Phi_{\mathscr{U}}$ defined in equation* (4.1.6) *is a morphism of complexes:*

$$\Phi_{\mathscr{U}} : \left(CC_\bullet^W(\mathcal{U}(\mathcal{L})),\ b + u^{-1}B\right) \longrightarrow \left(\check{C}^{2r-\bullet}(\mathscr{U}, \mathrm{Tot}\Omega_{\mathcal{L},W}, d + (-1)^k\delta\right).$$

Let us simply write j^{aff} for the family of PBW isomorphisms $j_{\nabla^{\mathrm{aff}}}$ induced by the connection ∇_k^{aff} parametrized by $\bar{t} \in \Delta^k$. We parametrize the k-simplex Δ^k by $(t_1, \dots, t_k) \in \mathbb{R}^n : t_j \geqslant 0, \sum t_j \leqslant 1$ which allows to write ∂_{t_j} instead of $\partial_{t_j} - \partial_{t_{j-1}}$, c.f. (4.1.5). The following Lemma is an easy consequence of proposition 2.2.10.

Lemma 4.1.9. *The following equations hold true:*

$$\begin{aligned} \partial_{t_i} j^{\mathrm{aff}} &= \mathrm{ad}_{\theta_i}(j^{\mathrm{aff}}) \\ \partial_{t_i} A^{\mathrm{aff}} &= \mathrm{ad}_{\theta_i}(A^{\mathrm{aff}}) - \nabla^{\mathrm{aff}}\theta_i - \partial_{t_i}\nabla^{\mathrm{aff}} \\ \partial_{t_i}\theta_j - \partial_{t_j}\theta_i &= [\theta_i, \theta_j]. \end{aligned}$$

Proof. This follows from straightforward computations. Let $K \in \mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. We compute:

$$\begin{aligned} \partial_{t_i}(j^{\mathrm{aff}}(K)) =& \partial_{t_i}(j^{\mathrm{aff}} \circ K \circ (j^{\mathrm{aff}})^{-1}) \\ =& \mathrm{ad}_{\theta_i} \circ j^{\mathrm{aff}}(K), \end{aligned}$$

which is the first equality. The second equality follows from the identities

$$\begin{aligned}\partial_{t_i}(A^{\mathrm{aff}}+\nabla^{\mathrm{aff}}) =&\partial_{t_i}(j^{\mathrm{aff}}\circ d_{\nabla(1)}\circ (j^{\mathrm{aff}})^{-1})\\ =&\mathrm{ad}_{\theta_i}(j^{\mathrm{aff}}\circ d_{\nabla(1)}\circ (j^{\mathrm{aff}})^{-1})\\ =&\mathrm{ad}_{\theta_i}(A^{\mathrm{aff}})-\nabla^{\mathrm{aff}}\theta_i.\end{aligned}$$

Finally, the chain rule implies that

$$\partial_{t_i}\partial_{t_j}j^{\mathrm{aff}}_k=(\partial_{t_i}\theta_j+\theta_j\theta_i)j^{\mathrm{aff}}$$

which gives, together with the same equality where i and j are interchanged, the last expression. □

Next, we need some equalities involving the Maurer–Cartan element $A(\nabla)$ in the sheaf of unital, graded algebras $\Omega^\bullet(\mathcal{W}(\mathcal{L}))$. We set $(A)^k=1\otimes A\otimes\ldots\otimes A$. Consult section 1.2.2 for the definitions of cyclic homology and the graded shuffle product.

Lemma 4.1.10. *For all* $a=a_0\otimes\cdots\otimes a_p\in C_p(\Omega^\bullet(\mathcal{W}(\mathcal{L})))$ *the* b *operator satisfies the following equalities:*

$$\begin{aligned}b((A)^k\times a)=&b((A)^k)\times a+(-1)^k(A)^k\times b(a)\\ &+(-1)^k\sum_i(A)^{k-1}\times(a_0\otimes\ldots\otimes[A,a_i]\otimes\ldots a_l),\\ b((A)^k)=&\nabla((A)^{k-1}).\end{aligned}$$

The cyclic differential B *satisfies the following relation:*

$$B((A)^k\times a)=(-1)^k(A)^k\times B(a).$$

This lemma holds for any dg algebra with a Maurer–Cartan element.

Proof. Entirely analogous to [EnFe, Lemma 2.5 & 2.6] and [PoPfTa10, Lemma 3.3]. □

In the following remark we provide some details on the identification of the Weyl algebra with the differential operators, which we will use in the proof.

Remark 4.1.11. Recall that the sheaf of Weyl algebras $\mathcal{W}(\mathcal{L})$ is constructed as follows. The direct sum $\mathcal{L}\oplus\mathcal{L}^\vee$ has a natural symplectic structure, and $\mathcal{W}(\mathcal{L})$ is the sheaf of Weyl algebras with fibers $\mathcal{W}(\mathcal{L})|_x$ given by the Weyl algebra associated to the symplectic vector space $(\mathcal{L}\oplus\mathcal{L}^\vee)|_x$. There is a natural fiberwise identification

$$\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)\otimes\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})\cong\mathcal{W}(\mathcal{L})\cong\mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)).$$

We introduce local coördinates l_i on $\mathcal{L}$ and dual coördinates l_i^* on $\mathcal{L}^\vee$, moreover we denote the coördinates on the symplectic vector spaces $(\mathcal{L}\oplus\mathcal{L}^\vee)|_x$ by q_i and p_i. Under the identification above we have

$$(l_i^*\otimes l_i)_x\mapsto p_iq_i+\frac{1}{2},\qquad l_i^*\otimes l_i\in\widehat{\mathrm{Sym}}^1_{\mathcal{O}_X}(\mathcal{L}^\vee)\otimes\mathrm{Sym}^1_{\mathcal{O}_X}(\mathcal{L}).$$

Proof of Proposition 4.1.8. It suffices to check the proposition on a fixed intersection $U_{i_0,\ldots,i_k}$. Unraveling the definitions of the trace density map, we need to show that

$$\Phi_{U_{i_0,\ldots,i_k}}(b+u^{-1}B)(D)=d_{\mathcal{L}}(\Phi_{U_{i_0,\ldots,i_k}}(D))+(-1)^k\delta(\Phi_{U_{i_0,\ldots,\hat{i}_j,\ldots i_k}}(D)).\qquad(4.1.7)$$

] We denote the coordinates on the simplex Δ^k by $t_1, \ldots\ldots t_k$, where we used the embedding $\Delta^k \subset \mathbb{R}^k$. The coördinates on the simplices corresponding to the intersections $U_{i_0,\ldots\hat{i}_j,\ldots i_k}$ are denoted by $t_1, \ldots, \hat{t}_j, \ldots, t_k$ for $j \neq 0$ and by $t_2, \ldots t_k$ for $j = 0$. Stokes' theorem implies:

$$\begin{aligned}
\delta(\Phi_{U_{i_0,\ldots,\hat{i}_j,\ldots i_k}}(D)) &= \sum_{j=0}^{k} (-1)^j \Phi_{U_{i_0,\ldots,\hat{i}_j,\ldots i_k}}(D) \\
&= \sum_{j=0}^{k} (-1)^j \int_{\partial_j \Delta^k} \left(\sum_{i=1}^{k} \iota_{\theta_1} \ldots \hat{\iota}_{\theta_i} \ldots \iota_{\theta_k} \tau_{2s}(A \times D) dt_1 \ldots \widehat{dt_i} \ldots dt_k \right) \\
&= \int_{\Delta^k} d_{\Delta^k} \left(\sum_{j=1}^{k} \iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D) dt_1 \ldots \widehat{dt_i} \ldots dt_k \right) \\
&= \int_{\Delta^k} \sum_{j=1}^{k} (-1)^{j+1} \partial_{t_j} \left(\iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D) \right) dt_1 \ldots dt_k.
\end{aligned}$$

The symbol τ_{2s} indicates the $2s$ component of the cyclic cocycle τ_{2r} and we wrote A for $(A)^j$ The number j can be deduced from the formula, but sometimes we will write it for clarity. Hereafter we will be concerned with rewriting the integrand in the last expression. We omit the term $dt_1 \ldots dt_k$. First we split the integrand into two terms:

$$\begin{aligned}
&\sum_{j=1}^{k} (-1)^{j+1} \iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \partial_{t_j} (\tau_{2s}(A \times D)) && (*) \\
&+ \sum_{j \neq i} (-1)^{j+1} \iota_{\theta_1} \ldots \iota_{\partial_{t_j} \theta_i} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \partial_{t_j} \tau_{2s}(A \times D) && (**)
\end{aligned}$$

The relations for the derivatives of j^{aff} and A^{aff}, c.f. lemma 4.1.9, imply that

$$\begin{aligned}
(*) &= \sum_j (-1)^{j+1} \iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(L_{\theta_j}(A \times D)) \\
&+ \sum_j (-1)^{j+1} \sum_i \iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} (\tau_{2s}((1 \otimes A \otimes \ldots \otimes \underbrace{-\nabla\theta_j}_{i} \otimes \ldots \otimes A) \times D) \\
&= \sum_j (-1)^{j+1} L_{\theta_j} \iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D)) \\
&+ \sum_j (-1)^{j+1} \iota_{\nabla\theta_j} \iota_{\theta_1} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D)),
\end{aligned}$$

where we omitted the terms involving $\partial_{t_i} \nabla^{\text{aff}} \in \Omega^1_{\mathcal{L}}(\widehat{\text{Sym}}^1_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \text{Sym}^1_{\mathcal{O}_X}(\mathcal{L})$. The reason is the following: the identification of remark 4.1.11 shows that $\partial_{t_i} \nabla^{\text{aff}}$ corresponds to a constant term and quadratic terms in the Weyl algebra. The definition of the cocycle shows that the constant terms vanish, and the quadratic terms are zero because τ_{2r} is basic, c.f. equation (B.2.3). The relation $\partial_{t_i} \theta_j - \partial_{t_j} \theta_i = [\theta_i, \theta_j]$ can be used to show that

$$\begin{aligned}
(**) &= \sum_{j \neq i} (-1)^{j+1} \iota_{\theta_1} \ldots \iota_{\partial_{t_j} \theta_i} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D) \\
&= \sum_{i<j} (-1)^j \iota_{\theta_1} \ldots \iota_{[\theta_i, \theta_j]} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D).
\end{aligned}$$

Moreover, we have an equality derived from $[L_a, \iota_b] = \iota_{[b,a]}$:

$$\begin{aligned}
\sum_j (-1)^{j+1} \iota_{\theta_1} \ldots L_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D) &= \sum_j (-1)^{j+1} L_{\theta_j} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D) \\
&+ \sum_{i<j} (-1)^j \iota_{\theta_1} \ldots \iota_{[\theta_i, \theta_j]} \ldots \hat{\iota}_{\theta_j} \ldots \iota_{\theta_k} \tau_{2s}(A \times D).
\end{aligned}$$

Collecting the previous formulas amounts to

$$(*) + (**) = \sum_j (-1)^{j+1} \iota_{\nabla\theta_j} \iota_{\theta_1} \dots \hat{\iota}_{\theta_j} \dots \iota_{\theta_k} \tau_{2s}((A)^{2s-k-l} \times D))$$
$$+ \sum_j (-1)^{j+1} \iota_{\theta_1} \dots L_{\theta_j} \dots \iota_{\theta_k} \tau_{2s}((A)^{2s-k-l+1} \times D).$$

Applying the relation $b\iota_a + \iota_a b = L_a$ several times gives

$$\sum_j (-1)^{j+1} \iota_{\theta_1} \dots L_{\theta_j} \dots \iota_{\theta_k} = b\iota_{\theta_1} \dots \iota_{\theta_k} + (-1)^{k+1} \iota_{\theta_1} \dots \iota_{\theta_k} b.$$

Moreover, we have the following two equalities, obtained from lemma 4.1.10 and a simple relation between the shuffle operation to the insertion operator:

$$b((A)^{2s-k-l+1} \times D) = \nabla(A)^{2s-k-l} \times D - (-1)^{k+l}(A)^{2s-k-l+1} \times b(D)$$
$$-(-1)^{k+l} \sum_i (A)^{2s-k-l} \times D_0 \otimes \dots \otimes [A, D_i] \otimes \dots \otimes D_l$$

$$\iota_{\theta_1} \dots \iota_{\theta_k} b\tau_{2s}(A \times D) = (-1)^k \tau_{2s+2}(B((\theta_k) \times \dots \times (\theta_1) \times A \times D))$$
$$= (-1)^{k+l+1} \tau_{2s+2}((\theta_k) \times \dots \times (\theta_1) \times A \times B(D))$$
$$= (1)^{l+1} \iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2s+2}(A \times B(D)).$$

We know that D is a flat section for the connection $\tilde{\nabla}^{(1)}$. This implies that $\nabla D_i + [A, D_i] = 0$ and hence the following identies hold:

$$(*) + (**) = \sum_j \iota_{\theta_1} \dots \iota_{\nabla\theta_j} \dots \iota_{\theta_k} \tau_{2s}((A)^{2s-k-l} \times D))$$
$$+ \iota_{\theta_1} \dots \iota_{\theta_k} \left(\tau_{2s}(\nabla(A)^{2s-k-l} \times D) + (-1)^{k+l} \tau_{2s}((A)^{2s-k-l} \times \nabla(D))\right)$$
$$+ (-1)^{k+l+1} \iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2s}((A)^{2s-k-l+1} \times b(D))$$
$$+ (-1)^{k+l+1} \iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2s+2}((A)^{2s-k-l+1} \times B(D))$$
$$= d_{\mathcal{L}} (\iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2s}(A \times D)) + (-1)^{k+l+1} \iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2s}(A \times b(D))$$
$$+ (-1)^{k+l} \iota_{\theta_1} \dots \iota_{\theta_k} \tau_{2s+2}(A \times B(D)),$$

which is equivalent to Equation (4.1.7). Of course, the sign α in Definition 4.1.6 has to be taken into account. □

We are ready to give the proof of the main theorem:

Proof of Theorem 4.1.2. Recall that the derived category of sheaves is constructed out of the category of chain complexes of sheaves by 1) identifying homotopic chain maps (i.e., going over to the homotopy category and 2) localization with respect to quasi-isomorphisms. For any cover $\mathscr{U}$, the natural map from the Lie algebroid chain complex $(\Omega^{\bullet}_{\mathcal{L},W}, d_{\mathcal{L}})$ to the Čech resolution $\check{C}^{\bullet}(\mathscr{U}, \Omega_{\mathcal{L},W}, d_{\mathcal{L}} + (-1)^k \delta)$ with respect to the cover is a quasi-isomorphism, so by Proposition 4.1.8 the character map $\Phi_{\mathscr{U}}$ defines a morphism in the derived category as stated in Theorem 4.1.2. It therefore remains to be shown that this map is independent of the cover and the compatible connections chosen.

Suppose we are given two covers $(\mathscr{U}, \nabla_{\mathscr{U}})$ and $(\mathscr{V}, \nabla_{\mathscr{V}})$ and compatible families of connections. We can merge the two covers $\mathscr{U}$ and $\mathscr{V}$ into one cover $\mathscr{U} \coprod \mathscr{V}$ which consists of the intersections $U_i \cap V_j$ for all $U_i \in \mathscr{U}$ and $V_j \in \mathscr{V}$. We denote the cover $\mathscr{U} \coprod \mathscr{V}$ with all the connections induced by the connections on $\mathscr{U}$ by $(\mathscr{U}\coprod\mathscr{V},\nabla_{\mathscr{U}})$, and we write $(\mathscr{U}\coprod\mathscr{V},\nabla_{\mathscr{V}})$ for the cover $\mathscr{U}\coprod\mathscr{V}$ with the connections induced by the connections on $\mathscr{V}$. It is obvious that the following diagram is commutative:

$$\begin{array}{ccc} CC_\bullet(\mathcal{U}(\mathcal{L})) & \xrightarrow{\Phi_{\mathscr{U}}} & \left(\check{C}^{2r-\bullet}(\mathscr{U},\mathrm{Tot}\Omega_{\mathcal{L}},d_{\mathcal{L}}+(-1)^k\delta\right) \\ & \searrow^{\Phi_{\mathscr{U}\coprod\mathscr{V},\nabla^{\mathscr{U}}}} & \downarrow \\ & & \left(\check{C}^{2r-\bullet}(\mathscr{U}\coprod\mathscr{V},\mathrm{Tot}\Omega_{\mathcal{L}},d_{\mathcal{L}}+(-1)^k\delta\right). \end{array}$$

Since the right arrow is a quasi-isomorphism, the morphisms $\Phi_{\mathscr{U}}$ and $\Phi_{\mathscr{U}\coprod\mathscr{V},\nabla^{\mathscr{U}}}$ induce the same morphism in the derived category. A similar diagram shows the same for $\Phi_{\mathscr{V}}$ and $\Phi_{\mathscr{U}\coprod\mathscr{V},\nabla^{\mathscr{V}}}$, hence it suffices to show that $\Phi_{\mathscr{U}\coprod\mathscr{V},\nabla^{\mathscr{U}}}$ and $\Phi_{\mathscr{U}\coprod\mathscr{V},\nabla^{\mathscr{V}}}$ define the same morphism in the derived category. Since the covers of these morphisms are the same we can assume that the covers are trivial, i.e. we have prove that Φ_{∇^1} and Φ_{∇^2} are homotopic for ∇^1 and ∇^2 global $\mathcal{L}$-connections on a space X. This follows if we can write the difference $\Phi_{\nabla^1}-\Phi_{\nabla^2}$ as $d_{\mathcal{L}}(\phi)-\phi(b+u^{-1}B)$, but such a ϕ is exactly given by the formula

$$(-1)^\alpha\int_{\Delta^k}\iota_{\theta_1}\dots\iota_{\theta_k}\tau_{2r}(1\otimes\exp(\wedge A^{\mathrm{aff}}(\bar{t}))\times D)d\bar{t}$$

in the case $k=1$, as the proof of Proposition 4.1.8 showed. □

Example 19. As an example of this construction, let us consider the Weyl algebra W_r introduced in Appendix B.1. Indeed, W_r is just the universal enveloping algebra of the Lie–Rinehart algebra $(L_r,\mathcal{O}_r)$, where $L_r=\mathrm{Der}(\mathcal{O}_r)=\mathcal{O}_r\left\langle\partial/\partial x^1,\dots,\partial/\partial x^r\right\rangle$. We denote the dual basis of $L_r^\vee$ by $\{dx^i\}$, $i\in\{1,\dots r\}$. This Lie–Rinehart algebra admits a canonical flat L_r-connection defined by

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right)=0,\qquad i,j=1,\dots,r.$$

It is straightforward to deduce that A is given by $\sum_i 1\otimes\frac{\partial}{\partial x^i}\otimes dx^i$. The Hochschild and cyclic (co)homology of W_r are known, see B, and the image of the only nontrivial generator c_{2r} defined in B is, using equation B.2.2, equal to 1, the only generator of the de Rham cohomology. We conclude that in this case the character map Φ induces an isomorphism for Hochschild and cyclic homology.

4.2 The index theorem

4.2.1 Characteristic classes

Recall that $\mathcal{L}$ denotes a smooth (resp. holomorphic) Lie algebroid of rank r over a smooth (resp. holomorphic) manifold X with structure sheaf $\mathcal{O}_X$ and that $\mathcal{E}$ denotes a smooth (resp. holomorphic) vector bundle of rank n over X. The usual Chern–Weil construction of characteristic classes extends to Lie algebroids in a straightforward manner if $\mathcal{E}$ admits a global $\mathcal{L}$-connection. When this is not the case, which is possible when $\mathcal{L}$ is holomorphic, the construction of characteristic classes for simplicial manifolds can be applied. In our case this approach is favourable because it is exactly the absence of a global connection that is the reason why the trace density map maps to the Čech resolution of the Lie algebroid differential forms. Therefore, the characteristic classes that we construct precisely fit to appear as the image of the trace density map applied to the trivial element 1 as in the index theorem. We remark that smooth bundles will be complexified in this section.

First we assume that there exists a global $\mathcal{L}$-connection on $\mathcal{E}$ and we denote it by $\nabla^{\mathcal{E}}$. We write $R_{\nabla^{\mathcal{E}}}\in\Omega_{\mathcal{L}}(\mathrm{End}(\mathcal{E}))$ for its curvature. For any $GL(\mathbb{K},n)$-invariant polynomial P_k :

$M_n(\mathbb{K}) \to \mathbb{K}$ of degree k, the form

$$\chi(P_k)(R(\nabla^{\mathcal{E}})) \in \Omega^{2k}_{\mathcal{L}}$$

is well-defined (because of the invariance of the polynomial) and closed (because of the Bianchi identity). The usual arguments of Chern–Weil theory apply verbatim to show that the induced cohomology class $[P_k(R(\nabla^{\mathcal{E}}))] \in H^{2k}(\mathcal{L})$ is independent of the chosen connection (in fact this follows from the simplest application of the simplicial methods). In the smooth setting this version of Chern–Weil theory has been considered in [Cr, Fer].

The simplicial construction

Now we assume that there does not exist a global $\mathcal{L}$-connection on $\mathcal{E}$. Choose a cover $\mathscr{U}$ of X such that $\mathcal{E}$ is trivializes over each $U_i \in \mathscr{U}$, which implies that there exist local $\mathcal{L}$-connections ∇_i on $\mathcal{E}$. Consider the pullback of the restricted Lie algebroid $\mathcal{L}|_{U_{i_0,\dots i_k}}$

$$\begin{array}{ccc} \pi^!\mathcal{L}|_{U_{i_0,\dots,i_k}} & \longrightarrow & \mathcal{L}|_{U_{i_0,\dots,i_k}} \\ \downarrow & & \downarrow \\ U_{i_0,\dots,i_k} \times \Delta^k & \xrightarrow{\pi} & U_{i_0,\dots,i_k} \end{array}$$

as in [MaHi]. The formula

$$\nabla^{\mathrm{tot}}_{U_{i_0,\dots,i_k}} := \sum_{i=0}^{k} t_i \nabla^i, \qquad (t_0,\dots,t_k) \in \Delta^k$$

defines a connection on the pullback $\pi^*\mathcal{E}|_{U_{i_0,\dots,i_k}}$ of the restricted vector bundle $\mathcal{E}|_{U_{i_0,\dots,i_k}}$. The Chern–Weil construction for Lie algebroids is summarized in the following proposition.

Proposition 4.2.1. *Let I^{inv} be the ring of $GL(\mathbb{K}, n)$-invariant polynomials $P :\ M_n(\mathbb{K}) \to \mathbb{K}$.*

i) *For $P \in I^{\mathrm{inv}}$ homogeneous of degree m, the forms*

$$P(\mathscr{U})_{U_{i_0,\dots,i_k}} := \int_{\Delta^k} P(R(\nabla^{\mathrm{tot}}_{U_{i_0,\dots,i_k}})) \in \Omega^{2m-k}_{\mathcal{L}}|_{U_{i_0,\dots,i_k}}$$

define a Čech cocycle in the Čech resolution $\check{C}^\bullet(\mathscr{U}, \Omega^\bullet_{\mathcal{L}}, d_{\mathcal{L}} + (-1)^k\delta)$.

ii) *The cohomology class in the hypercohomology does not depend on the choice of a cover and the choices of connections.*

iii) *The resulting map*

$$I^{\mathrm{inv}} \to H^{2\bullet}_{\mathrm{Lie}}(\mathcal{L})$$

is a ring homomorphism.

Proof. The first claim follows from an application of Stokes' theorem and the fact that $P(R(\nabla^{\mathrm{tot}}_{U_{i_0,\dots,i_k}}))$ is closed w.r.t. the Lie algebroid differential $d_{\pi^!\mathcal{L}}$:

$$\begin{aligned} d_{\mathcal{L}}\left(\int_{\Delta^k} P(\nabla^{\mathrm{tot}}_{U_{i_0,\dots,i_k}})\right) &= (-1)^k \int_{\Delta^k} d_{\pi^!\mathcal{L}}(P(\nabla^{\mathrm{tot}}_{U_{i_0,\dots,i_k}}) + \int_{\Delta^k} d_{\Delta^k}(P(\nabla^{\mathrm{tot}}_{U_{i_0,\dots,i_k}})) \\ &= \sum_{i=0}^{k} (-1)^j \int_{\partial_i\Delta^k} P(\nabla^{\mathrm{tot}}_{U_{i_0,\dots,\hat{i}_j,\dots,i_k}}). \end{aligned}$$

To prove the second claim, we argue as in the proof of Theorem 4.1.2. Namely, given two covers $\mathscr{U}$ and $\mathscr{V}$, we consider the intersection of this cover; $\mathscr{U} \coprod \mathscr{V}$. A diagram such as in the proof of Theorem 4.1.2 shows that we can reduce to the case of a trivial cover with two connections. Then the usual homotopy argument, c.f. [CF2], applies. For the last claim, we first remark that the product on the Čech resolution of $\Omega^\bullet_{\mathcal{L}}$ is given by the Alexander-Whitney product, which is graded commutative on cohomology. The Chern–Weil map factorizes through the complex $\check{C}(\mathscr{U}, \Omega_{\pi^!\mathcal{L}}, d_{\pi^!\mathcal{L}})$, c.f. [Du], and since the Chern–Weil map to this complex is a homomorphism, the result follows. $\square$

In the following, we only need the following invariant polynomials:

$$\mathrm{Ch}(X) := \mathrm{Tr}\left(e^X\right), \quad \mathrm{Td}(X) := \det\left(\frac{x}{1-e^x}\right), \quad \hat{A}(X) = \det\left(\frac{x/2}{\sinh(x/2)}\right). \tag{4.2.1}$$

As usual, it is meant that one takes the power series expansion around zero of the analytic functions used in these definitions, and applies the Chern–Weil construction to each homogeneous component. The resulting characteristic classes are called the Chern class $\mathrm{Ch}_{\mathcal{L}}(\mathcal{E})$ and the Todd class $\mathrm{Td}_{\mathcal{L}}(\mathcal{E})$.

4.2.2 The index theorem

Some notes on the twisted case

Before we state and prove the index theorem, we make some remarks about the twisted case. Recall that the twisted PBW map, which depends on the $\mathcal{L}$-connections $\nabla^{\mathcal{L}}$ on $\mathcal{L}$ and $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, is given by an isomorphism

$$j^{\mathcal{E}}_{\nabla} : \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{E} \longrightarrow \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \mathcal{E}.$$

It has the property that $j^{\mathcal{E}}_{\nabla}(\phi\psi \otimes s) = j_{\nabla}(\phi) j^{\mathcal{E}}_{\nabla}(\psi \otimes s)$ for $\phi \in \mathcal{J}(\mathcal{L})$ and $\psi \otimes s \in \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{E}$. The flat connection $\nabla^{(1)}$ on $\mathcal{J}(\mathcal{L})$ extends to $\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{E}$ in a natural way. We denote the corresponding connection on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \mathcal{E}$ by D and define the difference

$$A^{\mathcal{E}} := D - \nabla^{\mathcal{L}} \otimes 1 - 1 \otimes \nabla^{\mathcal{E}}.$$

Using the property above, one can show that $A^{\mathcal{E}}$ is a one form with values in

$$\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathrm{Der}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \subset \mathrm{Diff}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}). \tag{4.2.2}$$

This one form can be used in exactly the same way as in the previous section to define the twisted character map, which we will denote by $\Phi^{\mathcal{E}}$. The proof that it is well-defined and does not depend on the choices that are made applies verbatim.

Back to the theorem.

We are now in a position to state and prove the index theorem for Lie algebroids. Recall that the character map $\Phi^{\mathcal{E}}$ of Theorem 4.1.2 maps -in the derived category- the sheaves of Hochschild and cyclic complexes to the Lie algebroid complexes. Applied to the cycle $1 \in C_0(\mathcal{U}(\mathcal{L};\mathcal{E}))$, we obtain a class

$$[\Phi^{\mathcal{E}}(1)] \in H^{\mathrm{ev}}_{\mathrm{Lie}}(\mathcal{L})$$

in Lie algebroid cohomology which is canonically defined, i.e., an invariant of the Lie algebroid $\mathcal{L}$ and the vector bundle $\mathcal{E}$. The main theorem of this section identifies this class in terms of the characteristic classes of the previous section:

Theorem 4.2.2. $\Phi^{\mathcal{E}}(1) = \mathrm{Td}_{\mathcal{L}}(\mathcal{L})\,\mathrm{Ch}_{\mathcal{L}}(\mathcal{E})$.

Proof. Again we fix a cover $\mathscr{U}$ such that $\mathcal{L}$ and $\mathcal{E}$ are trivial when restricted to any $U_i \in \mathscr{U}$. Let $U_{i_0,\ldots,i_k}$ be an intersection. We have to compare the following two forms in $\Omega^{2n-k}_{\mathcal{L}}|_{U_{i_0,\ldots,i_k}}$:

$$\int_{\Delta^k} P(R(\nabla^{\mathrm{tot}}_{U_{i_0,\ldots,i_k}})) \quad \text{and} \quad (-1)^{\alpha} \int_{\Delta^k} \iota_{\theta_1} \ldots \iota_{\theta_k} \tau_{2r}(1 \otimes \exp(\wedge A^{\mathrm{aff}}(\bar{t}))d\bar{t}.$$

Consider the integrand of the RHS. One can check that the evaluation morphism to Lie algebra cohomology of Appendix B.3 commutes with the insertion operators ι_{θ_i} on the Hochschild complex and the relative Lie algebra complex. This fact, and the local Riemann–Roch Theorem B.5.1, allows us to rewrite this integrand as

$$\begin{aligned} \iota_{\theta_{i_1}} \dots \iota_{\theta_{i_k}} \tau_{2r}(1, A, \dots A) &= \mathrm{ev}_1(\iota_{\theta_{i_1}} \dots \iota_{\theta_{i_k}} \tau_{2r})(A, \dots, A) \\ &= \iota_{\theta_{i_1}} \dots \iota_{\theta_{i_k}} \mathrm{ev}_1(\tau_{2r})(A, \dots, A)) \\ &= \iota_{\theta_{i_1}} \dots \iota_{\theta_{i_k}} \chi(P)(A, \dots, A), \end{aligned} \tag{4.2.3}$$

where P is the invariant polynomial corresponding to the product of the $\widehat{A}$-genus and the Chern class. Note that we used the stronger version of the local Riemann–Roch theorem described in Proposition B.5.2. This is possible because of the specific form of $A^{\mathcal{E}}$ that we gave in (4.2.2). Next, in the evaluation of the Chern–Weil homomorphism, three types of curvature terms appear:

$$C(A, A), \quad C(\theta_i, A), \quad C(\theta_i, \theta_j).$$

As explained in B, one has $C(a, b) = [\pi(a), \pi(b)] - \pi([a, b])$ where $\pi : \mathcal{W}_r^n \to \mathfrak{sp}_{2r} \oplus \mathfrak{gl}_n$ is the projection. Before we give a lemma, applicable in the non-twisted case, that allows us to compute these terms, we fix some notation for the one form $A \in \Omega^1_{\mathcal{L}}(\mathrm{Der}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))$ that was defined in Equation (4.1.3). We have seen that $\mathrm{Der}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \cong \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes_{\mathcal{O}_X} \mathcal{L}$ and thus we can write

$$A = \sum_{k \geqslant -1} A_k \qquad A_k \in \Omega^1_{\mathcal{L}}(\widehat{\mathrm{Sym}}^{k+1}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \mathcal{L}).$$

Lemma 4.2.3. *When the $\mathcal{L}$-connection is torsion free, the lowest degree components of A are given by*

$$A_{-1} = -1 \otimes \mathrm{id}_{\mathcal{L}} \qquad A_0 = 0,$$

where we view $\mathrm{id}_{\mathcal{L}}$ as an $\mathcal{L}$-one-form with values in $\mathcal{L}$.

Proof. We will prove this by evaluating A in an element $X \in \mathcal{L}$, which gives a derivation of $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. We apply this derivation to an element $\alpha \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. Since derivations are determined by their actions in low degrees, we can assume that $\alpha \in \mathrm{Sym}^1_{\mathcal{O}_X}(\mathcal{L}^\vee)$. Let us recall the PBW isomorphism in low degrees:

$$j_\nabla(\phi)(1) = \phi(1), \quad j_\nabla(\phi)(X) = \phi(X), \quad j_\nabla(\phi)(XY) = \frac{1}{2}\phi(XY + YX - \nabla_X Y - \nabla_Y X).$$

This implies that the inverse of the PBW map satisfies

$$j_\nabla^{-1}(\alpha)(X) = \alpha(X), \quad j_\nabla^{-1}(\alpha)(XY + YX - \nabla_X Y - \nabla_Y X) = \alpha(XY) = 0. \tag{4.2.4}$$

We use these identities to compute:

$$\begin{aligned} A_{-1}(X)(\alpha)(1) =& j_\nabla \circ d_{\nabla^{(1)}} \circ j_\nabla^{-1}(\alpha)(1, X) \\ =& X(j_\nabla^{-1}(\alpha)(1)) - j_\nabla^{-1}(\alpha)(X) \\ =& -\alpha(X). \\ A_0(X)(\alpha)(Y) =& j_\nabla \circ d_{\nabla^{(1)}} \circ j_\nabla^{-1}(\alpha)(Y, X) - \nabla_X \alpha(Y) \\ =& X(j_\nabla^{-1}(\alpha)(Y)) - j_\nabla^{-1}(\alpha)(XY) - X(\alpha(Y)) + \alpha(\nabla_X Y) \\ =& X(\alpha(Y)) - j_\nabla^{-1}(\alpha)(XY) - X(\alpha(Y)) + \alpha(\nabla_X Y). \end{aligned}$$

When the torsion $\nabla_X Y - \nabla_Y X - [X, Y]$ vanishes Equation (4.2.4) implies that $j_\nabla^{-1}(\alpha)(XY) = \alpha(\nabla_X Y)$, and hence the result follows. ∎

In the non-twisted case, the previous lemma shows that $\pi(A) = 0$. For the twisted case, one can show that the zeroth degree of the first component of $A^{\mathcal{E}}$ in (4.2.2) can be set to zero by changing the connection $\nabla^{\mathcal{E}}$. This implies $\pi(A^{\mathcal{E}}) = 0$.

We proceed with the proof of Theorem 4.2.2. We assume that the $\mathcal{L}$-connection is torsion free, hence $\pi(A^{\mathcal{E}}) = 0$. In Lemma 2.2.10 we obtained an explicit expression for the low degrees of θ_i:

$$\theta_i^{-1} = 0, \quad \theta_i^0 = 0, \quad \theta_i^1 = \gamma_i.$$

Note that θ is not a one-form, thus γ is viewed as derivation on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. The other components of θ are derivations of higher degrees, hence $\pi(\theta_i) = 0$. This implies that

$$\begin{aligned} C(A, A) &= -\pi([A, A]) \\ C(A, \theta_i) &= -\pi([A, \theta_i]) \\ C(\theta_i, \theta_j) &= \ \pi([\theta_i, \theta_j]). \end{aligned}$$

Since the commutator of derivations of degree p and q has degree $p + q$, it follows that $\pi([\theta_i, \theta_j]) = 0$. Using the expressions from Lemma 4.1.9 we obtain:

$$\begin{aligned} C(A, A) &= -\pi([A, A] = \pi(\nabla A + R(\nabla^{\mathrm{aff}})) = \pi(R(\nabla^{\mathrm{aff}}_{\mathcal{L}}) \otimes \mathrm{id}_{\mathcal{E}} + R(\nabla^{\mathrm{aff}}_{\mathcal{E}})) \\ C(\theta_i, A) &= -\pi([\theta_i, A]) = -\pi(\partial_{t_i} A + \nabla^{\mathrm{aff}} \theta_i + \partial_{t_i} \nabla^{\mathrm{aff}}) = -\gamma_i^{\mathcal{L}} - \gamma_i^{\mathcal{E}}. \end{aligned} \tag{4.2.5}$$

We used here that both the operators ∇^{aff} and ∂_{t_j} preserve the degrees of derivations. Let us now discuss the terms in the Chern–Weil cocycle. We consider the (local) $\mathcal{L}$-connections $\nabla = \nabla^{\mathcal{L}} + \nabla^{\mathcal{E}}$ on the direct sum $\mathcal{L} \oplus \mathcal{E}$. We can write the total connection in the following form:

$$\nabla^{\mathrm{tot}} = d_{\pi^! \mathcal{L}} + \sum_{i=0}^{k} \gamma_i.$$

An easy computation shows that

$$R(\nabla^{\mathrm{aff}}) + \sum_{i=0}^{k} dt_i \gamma_i.$$

Recall that the Chern–Weil cocycle restricted to an intersection is given by

$$P(\mathscr{U})_{U_{i_0,\ldots,i_k}} := \int_{\Delta^k} P(R(\nabla^{\mathrm{tot}}_{U_{i_0,\ldots,i_k}})).$$

Observe that the only terms in the integrand that have a contribution are the ones with exactly k terms containing a one form dt_i. These terms all contribute a term γ_i. On the other hand, Equation (4.2.3) shows that the class $\Phi^{\mathcal{E}}(1)$ is given by

$$\Phi^{\mathcal{E}}_{U_{i_0\ldots i_k}}(1) = \int_{\Delta^k} \chi(P)(\theta_1, \ldots, \theta_k, A, \ldots, A) d\bar{t}.$$

The Equations in (4.2.5) show that the two integrals coincide. Finally, using the same argument as in the proof of [W15, Prop. 20], relating the $\hat{A}$-genus and the Todd class, the theorem is proved. □

4.2.3 Compatibility with the HKR map

In this section we shall refine the index Theorem 4.2.2 of the previous section by showing that the Lie algebra cohomology class it determines measures the obstruction for our character map to be compatible with the Hochschild–Kostant–Rosenberg map (4.1.1). Define $\mathrm{HKR}_{\mathcal{L}} := \rho^* \circ \mathrm{HKR}$, the pre-composition of the HKR map with the pull-back along the anchor, i.e.,

$$\mathrm{HKR}_{\mathcal{L}}(f_0 \otimes \ldots \otimes f_k) := f_0 d_{\mathcal{L}} f_0 \wedge \ldots \wedge d_{\mathcal{L}} f_k \in \Omega^k_{\mathcal{L}}, \quad f_i \in \mathcal{O}_X, \ i = 0, \ldots k.$$

Using the obvious inclusion $\mathcal{O}_X \hookrightarrow \mathcal{U}(\mathcal{L})$, we can compare this map with the character map of Theorem 4.1.2:

Theorem 4.2.4. *Let $\mathcal{L}$ be a smooth resp. holomoprhic Lie algebroid over a smooth resp. holomorphic manifold X with structure sheaf $\mathcal{O}_X$. The diagram*

$$\begin{array}{ccc} (CC_\bullet(\mathcal{O}_X), b+u^{-1}B) & \xrightarrow{HKR_{\mathcal{L}}} & (\Omega^\bullet_{\mathcal{L}}, u^{-1}d_{\mathcal{L}}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \wedge \mathrm{Td}_{\mathcal{L}}(\mathcal{L})\mathrm{Ch}_{\mathcal{L}}(\mathcal{E})} \\ (CC_\bullet(\mathcal{U}(\mathcal{L};\mathcal{E})), b+u^{-1}B) & \xrightarrow{\Phi^{\mathcal{E}}} & (\Omega^{2r-\bullet}_{\mathcal{L}}, d_{\mathcal{L}}) \end{array}$$

commutes in the derived category.

Proof. We fix the usual cover $\mathscr{U}$ of X. We will show that the trace density map $\Phi^{\mathcal{E}}_{\mathscr{U}} \circ i^* = \mathrm{Td}_{\mathcal{L}}(\mathcal{L})\mathrm{Ch}_{\mathcal{L}}(\mathcal{E})|_{\mathscr{U}} \wedge j_{\mathscr{U}} \circ \mathrm{HKR}_{\mathcal{L}}$ where $j_{\mathscr{U}}$ is the natural inclusion $\Omega^\bullet_{\mathcal{L}} \to \check{C}(\mathscr{U}, \Omega^\bullet_{\mathcal{L}})$. We work over an intersection $U_{i_0,\dots i_k}$. First remark that in the definition of the character map $f \in \mathcal{O}_X$ is embedded in the sheaf of Weyl algebras $\mathcal{W}$ as

$$j_\nabla(f) = j^0_\nabla(f) + j^1_\nabla(f) + \ldots \in \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

where j_∇ is the PBW isomorphism. Analogous to the proof of Theorem 4.2.4, we now have to consider the forms

$$\iota_{\theta_1}\ldots\iota_{\theta_p}\tau_{2r}(A\otimes\cdots\otimes A\times j_\nabla(f_0)\otimes\ldots\otimes j_\nabla(f_k)). \qquad (\star)$$

We will evaluate these forms using the expansion of $A(\nabla)$ from Lemma 4.2.3 and the description of θ_i given in Lemma 2.2.10. Write the cocycle (B.2.1) as $\tau_{2r} = \mu_{2r} \circ S_{2r} \circ \pi_{2r}$. Fix a local basis $\{e_i,\ i=1,\dots,r\}$ of $\mathcal{L}$, and denote the dual basis of $\mathcal{L}^\vee$ by $\{de_i\}$. Let $\deg_{\mathcal{L}^\vee}(a)$ and $\deg_{\mathcal{L}}(a)$ denote the degree in the first respectively the second component for $a \in \mathcal{W}(\mathcal{L}) \cong \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)\otimes\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L})$. For elements $a_0\otimes\ldots\otimes a_k$ we have $\deg_{\mathcal{L}}(a_0\otimes\ldots\otimes a_k) = \sum_i \deg_{\mathcal{L}}(a_i)$ and $\deg_{\mathcal{L}^\vee}(a_0\otimes\ldots\otimes a_k) = \sum_i \deg_{\mathcal{L}^\vee}(a_i)$. Note that μ_{2r} is the projection on the $\deg_{\mathcal{L}} = \deg_{\mathcal{L}^\vee} = 0$ part of $a_0\otimes\ldots\otimes a_{2r}$ followed by multiplication. We claim that we can make, in the expression of $\star$, the following replacements;

$$\begin{aligned} A &\rightsquigarrow A_{-1}+A_1 \\ j_\nabla(f_0) &\rightsquigarrow j^0_\nabla(f_0) \\ j^1_\nabla(f_i) &\rightsquigarrow j^1_\nabla(f_i), \quad i \geqslant 1 \\ \theta_l &\rightsquigarrow \theta_l^2 \end{aligned}$$

without changing the expression. By θ_l^2 we mean the $\deg_{\mathcal{L}^\vee} = 2$ part of θ_l, which is equal to γ_l. Consider A_k with $k \geqslant 2$ in the i'th slot. We have $\deg_{\mathcal{L}}(A_k) = 1$ and $\deg_{\mathcal{L}^\vee}(A_k) = k+1$, hence an application of π_{2r} will give an element of either $(\deg_{\mathcal{L}^\vee}, \deg_{\mathcal{L}}) = (k+1, 0)$ or $(k,1)$. An application of $e^{s\alpha_{ij}}$ will only give a nonzero $(0,0)$ term if the term in the j'th slot has degree $(0,k)$ or $(1,k-1)$, and these terms are not present. For θ_l an analogous reasoning applies. Now consider $j_\nabla(f_i)$ in the i'th slot. The term $j^0_\nabla(f_i)$ does not survive π_{2r}, and π_{2r} applied to terms j^k_∇ with $k \geqslant 2$ gives terms of degree $(k-1,0)$, to which an application of $e^{s\alpha_{ij}}$ only gives a nonzero $(0,0)$ term if the term in the j'th slot has degree $(0,k-1)$. Again, these terms are not present; here it is used that $A_0 = 0$. In conclusion:

$$(\star) = \iota_{\theta_1^2}\ldots\iota_{\theta_p^2}\tau_{2r}((A_{-1}+A_1)\otimes\cdots\otimes(A_{-1}+A_1)\times j^0_\nabla(f_0)\otimes j^1_\nabla(f_1)\otimes\ldots\otimes j^1_\nabla(f_k)).$$

We can still eliminate terms from this expression. We have for general elements $a_0\otimes\ldots\otimes a_{2r} \in \mathcal{W}(\mathcal{L})\otimes\ldots\otimes\mathcal{W}(\mathcal{L})$:

$$\begin{aligned} \deg_{\mathcal{L}^\vee}(\alpha_{ij}(a_0\otimes\ldots\otimes a_{2r})) &= \deg_{\mathcal{L}^\vee}(a_0\otimes\ldots\otimes a_{2r}) - 1 \\ \deg_{\mathcal{L}^\vee}(\pi_{2r}(a_0\otimes\ldots\otimes a_{2r})) &= \deg_{\mathcal{L}^\vee}(a_0\otimes\ldots\otimes a_{2r}) - r \end{aligned}$$

and similar formulas for the $\mathcal{L}$-degree. This implies that the only terms in $a_0 \otimes \ldots \otimes a_{2r}$ that have a nonzero contribution after an application of τ_{2r} have equal $\mathcal{L}$ and $\mathcal{L}^\vee$ degrees. It follows that

$$(\star) \;= \tau_{2r}((\theta_p^2) \times \ldots \times (\theta_1^2) \times (A_1)_{r-k-p} \times (A_{-1})_r \times j^0_\nabla(f_0) \otimes j^1_\nabla(f_1) \otimes \ldots \otimes j^1_\nabla(f_k)).$$

The first order jets $j^1_\nabla(f_i) = \sum_j \rho(e_j)(f_i)de_j$, $i = 1, \ldots, k$ of the f_i combine with $A_{-1} = \sum_i 1 \otimes e_i \otimes de_i$, after applying π_{2r}, to the forms $\rho^*(df_i)$, so that we have:

$$\begin{aligned}(\star) \;=& \frac{1}{k!} f_0 \rho^* df_1 \wedge \ldots \wedge \rho^* df_k \pi_{2r}((\theta_p) \times \ldots \times (\theta_1) \times \underbrace{\pi \times \ldots \times \pi}_{\#=k} \times (A_{-1})_{r-k} \times (A_1)_{r-k-p}) \\ =& f_0 \rho^* df_1 \wedge \ldots \wedge \rho^* df_k \iota_{\theta_1} \ldots \iota_{\theta_p} \frac{\iota_\pi^k}{k!} \tau_{2r}((A_{-1})_{r-k} \times (A_1)_{r-k-p}) \\ =& f_0 \rho^* df_1 \wedge \ldots \wedge \rho^* df_k u^{-n} [\mathrm{Td}_{\mathcal{L}}(\mathcal{L}) \, \mathrm{Ch}(\mathcal{E})].\end{aligned}$$

We used the definition of the cyclic cocycle using the insertion ι_π, and Theorem 4.2.2. □

4.2.4 Holomorphic Lie algebroids

In this subsection we give (a sketch of) an alternative derivation of our main index theorem using the Dolbeault-type resolution to compute the Lie algebroid cohomology constructed in [LSX12], c.f. Example 6. We use the notations of that example, e.g. we denote the holomorphic Lie algebroid by $\mathcal{A}$. Recall the Dolbeault complex

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow {\scriptstyle d^{1,0}_{\mathcal{A}}} & & \uparrow {\scriptstyle d^{1,0}_{\mathcal{A}}} & & \\ \mathcal{D}^{1,0}_{\mathcal{A}} & \xrightarrow{\bar\partial} & \mathcal{D}^{1,1}_{\mathcal{A}} & \xrightarrow{\bar\partial} & \ldots \\ \uparrow {\scriptstyle d^{1,0}_{\mathcal{A}}} & & \uparrow {\scriptstyle d^{1,0}_{\mathcal{A}}} & & \\ \mathcal{D}^{0,0}_{\mathcal{A}} & \xrightarrow{\bar\partial} & \mathcal{D}^{0,1}_{\mathcal{A}} & \xrightarrow{\bar\partial} & \ldots \end{array}$$

where $\mathcal{D}^{p,q}_{\mathcal{A}} := \wedge^p (A^{1,0})^\vee \otimes \wedge^q (T^{0,1}X)^\vee$. The kernel of the differential $\bar\partial : \mathcal{D}^{\bullet,0}_{\mathcal{A}} \to \mathcal{D}^{\bullet,1}_{\mathcal{A}}$ is precisely the holomorphic Lie algebroid complex $(\Omega^\bullet_{\mathcal{A}}, d_{\mathcal{A}})$ and the total complex computes the Lie algebroid cohomology of $\mathcal{A}$.

To apply this resolution for our construction of the character map, we tensor with the smooth jet bundle and universal enveloping algebra to obtain the complex

$$\left(\mathcal{J}(A^{1,0}) \otimes_{C^\infty_{X_2}} \mathcal{U}(A^{1,0}) \otimes_{C^\infty_{X_1}} \mathcal{D}^{p,q}_L, \nabla^{(1)} + \bar\partial_A \right). \tag{4.2.6}$$

This time the kernel of the differential $\bar\partial_A$ in degree 0 is precisely the sheaf of dg algebras of Lemma 4.1.3 for the Lie algebroid $\mathcal{A}$.

Next we choose a torsion free $A^{1,0}$-connection ∇ on $A^{1,0}$ Remark that in this C^∞-setting, such a connection always exists. By the PBW theorem, this induces an isomorphism

$$j_\nabla : \mathcal{J}(A^{1,0}) \xrightarrow{\cong} \widehat{\mathrm{Sym}}((A^{1,0})^\vee).$$

The non commutative version of the PBW theorem induces an isomorphism of the total space of the dg algebra of (4.2.6) with $\mathcal{W}(A^{1,0}) \otimes_{C^\infty_X} \mathcal{D}^{\bullet,\bullet}_{\mathcal{A}}$, also denoted j_∇, and allows us to introduce

$$A(\nabla) := j_\nabla \circ \nabla^{(1)} \circ j_\nabla^{-1} - (\nabla + \bar\partial_A).$$

Remark that the flat connection $\nabla^{(1)} + \bar\partial_A$ on $\mathcal{J}^\infty_{A^{1,0}} \otimes_{C^\infty_{X_2}} \mathcal{U}(A^{1,0})$, when transferred to $\mathcal{W}_{A^{1,0}}$, does not necessarily have the form $j_\nabla \circ \nabla^{(1)} \circ j_\nabla^{-1} + \bar\partial_A$ and hence A has both components in the $T^{0,1}$ and $A^{1,0}$ direction.

Theorem 4.2.5. *Let $\mathcal{A} \to X$ be a holomorphic Lie algebroid over a complex manifold, and $\mathcal{E} \to X$ a holomorphic vector bundle.*

i) *A pair $(\nabla^{\mathcal{A}}, \nabla^{\mathcal{E}})$ of $\mathcal{A}$-connections on $\mathcal{A}$ and $\mathcal{E}$ compatible with the holomorphic structure, with $\nabla^{\mathcal{A}}$ torsion free, induces a character map*

$$\Phi_{\nabla} : \left(CC_{\bullet}(\mathcal{U}(\mathcal{A};\mathcal{E})), b + u^{-1}B\right) \to \left(\bigoplus_{p+q=2r-\bullet} \mathcal{D}^{p,q}(E)_{\mathcal{A}}[u], d^{1,0}_{\mathcal{A}} + \bar{\partial}_L\right).$$

ii) *For different choices of connections $(\nabla^{\mathcal{A}}, \nabla^{\mathcal{E}})$ and $(\tilde{\nabla}^{\mathcal{A}}, \tilde{\nabla}^{\mathcal{E}})$, the maps Φ_{∇} and $\Phi_{\tilde{\nabla}}$ are chain homotopic.*

iii) (Index Theorem) *The following equality of differential forms holds true:*

$$\Phi_{\nabla}(1) = \mathrm{Td}_A \mathrm{Ch}_A(E).$$

Appendix A

Formality

A.1 Introduction

In this section we revisit the formality theorem for Lie algebroids as developed in a series of papers [C, CDH, CvdB, CRvdB12, D]. The formality theorems in [CvdB, CRvdB12] are stated and proven in great generality, namely for locally free sheaves of Lie–Rinehart algebras $\mathcal{L}$ of finite constant rank over a ringed site X. The version that we will prove reads as follows.

Theorem A.1.1. *Let $\mathcal{L}$ be a locally free sheaf of O_X-modules, where O_X is a fine structure sheaf of a ringed space. There exist an L_∞-quasi-isomorphism $U = (U_1, U_2, \dots)$*

$$U: \ T^{\mathcal{L}}_{\text{poly}}(X) \longrightarrow D^{\mathcal{L}}_{\text{poly}}(X)$$

with the first component equal to the HKR map. This morphism equips $C_{\mathcal{L}}(X)$ with the structure of an L_∞-module over $T^{\mathcal{L}}_{\text{poly}}(X)$, and there exists a quasi-isomorphism $V = (V_1, V_2, \dots)$ of L_∞-modules

$$V: \ C_{\mathcal{L}}(X) \longrightarrow \Omega_{\mathcal{L}}(X)$$

over $T^{\mathcal{L}}_{\text{poly}}(X)$, with V_1 equal to the HKR map for chains.

Here, U is an L_∞-morphism from the dg Lie algebra of $\mathcal{L}$-polyvectorfields $T^{\mathcal{L}}_{\text{poly}}(X)$ to the dg Lie algebra of $\mathcal{L}$-polydifferential operators $D^{\mathcal{L}}_{\text{poly}}(X)$ over X, and V is an L_∞-morphism from the dg module $C_{\mathcal{L}}(X)$ of $\mathcal{L}$-Hochschild chains to the dg module of $\mathcal{L}$-differential forms. The first part of this section will be a brief summary of results about (curved) L_∞ algebras, their morphisms and twists. We will prove this theorem in a more restrictive case than in [CvdB, CRvdB12], since we don't need it in its full generality and since our proof requires less technical machinery. Our only application of the theorem is in the following case.

The locally ringed space $(X, \mathcal{O}_X)$ is given by $(M, \operatorname{Sym} A)$, where M is a smooth manifold and $A \to M$ a smooth vector bundle. The sheaf of Lie algebroids $\mathcal{L} = \operatorname{Der}_{\mathbb{R}}(\operatorname{Sym}(A))$. Local trivializations of A and the tangent bundle TM, together with the isomorphism

$$\operatorname{Der}_{\mathbb{R}}(\operatorname{Sym}(A)) \cong \Gamma(\operatorname{Sym}(A) \otimes (A^\vee \oplus TM))$$

induced by a connection on A, c.f. §3.1.2, show that $\mathcal{L}$ is locally free of finite, constant rank, and, given a TM-connection ∇^A on A and a TM-connection ∇^{TM} on TM, the following formula defines an $\mathcal{L}$-connection on $\mathcal{L}$:

$$\nabla_{(\alpha,X)}(\beta, Y) = (\nabla^A_X \beta, \nabla^{TM}_X Y), \qquad (\alpha, X),\ (\beta, Y) \in A^\vee \oplus TM. \tag{A.1.1}$$

Finally, the sheaves $\operatorname{Sym} A$ and $\operatorname{Der}_{\mathbb{R}}(\operatorname{Sym} A)$ are *fine* sheaves, because partitions of unity exist. Hence, this example satisfies the conditions in the theorem.

Considering the second point, the proof that we provide depends crucially on the PBW theorem for Lie algebroids, which in turn depends on an $\mathcal{L}$-connection on $\mathcal{L}$. Moreover, to globalize locally defined PBW morphisms, we use that $\mathcal{O}_X$ and $\mathcal{L}$ are fine sheaves.

Remark A.1.2. Another important example is of course given by $A \to M$ a smooth Lie algebroid, where $\mathcal{O}_X = C^\infty(M)$, but this case is exactly the result from [C].

Remark A.1.3. Let us finally make some remarks about the terminology. We will be a bit sloppy when talking about sheaves. Many of the constructions need the process of 'sheafification', e.g. the tensor product of sheaves is not a sheaf but a presheaf, and one defines the tensor product of two sheaves to be the associated sheaf to this presheaf. We will usually not mention this. Moreover, operations such as the product, contraction etc. of two given sheaves are defined locally, i.e. over an open $U \subset X$, but again we usually won't mention this.

A.2 Curved L_∞-algebras

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a $\mathbb{Z}$-graded vector space. In the following we write $|X|$ for the degree of a homogeneous element $X \in \mathfrak{g}$. We consider the symmetric bialgebra of $\mathfrak{g}$, shifted by one; $S(\mathfrak{g}[1])$, with the coproduct Δ defined on generators by

$$\Delta: \ X \mapsto X \otimes 1 + 1 \otimes X \qquad X \in \mathfrak{g}[1].$$

Since $\mathfrak{g}[1]$ is graded, there are signs involved in the shuffle products

$$\Delta: \ X_1 \cdots X_n \mapsto \sum_{i=0}^{n} \sum_{\sigma \in Sh(i,n-i)} \pm X_{\sigma(1)} \cdots X_{\sigma(i)} \otimes X_{\sigma(i+1)} \cdots X_{\sigma(n)}.$$

To be more precise, these signs can be reconstructed from the Koszul sign rule, which says that for each transposition of two elements X, Y, a sign $(-1)^{|X||Y|}$ is introduced.

Definition A.2.1. *A* curved L_∞-structure *on* $\mathfrak{g}$ *is given by a differential* Q *of degree* 1 *on* $S(\mathfrak{g}[1])$ *which is a coderivation with respect to* Δ.

Remark A.2.2. The décalage isomorphism

$$\mathrm{dec}_n: \ S^n(\mathfrak{g}[1]) \cong \wedge^n \mathfrak{g}[n], \qquad X_1 \cdots X_n \mapsto (-1)^{\sum_{i=1}^n (n-i)(|X_i|-1)} X_1 \wedge \cdots \wedge X_n$$

allows to define L_∞-algebras in terms of the algebra $\Lambda\mathfrak{g}$, but with that definition the signs are less natural. However, when we will discuss the Taylor components, which will be introduced hereafter, of the coderivation, morphisms of L_∞-algebras, L_∞-modules etc., we will use this identification.

Since the exterior algebra $S(\mathfrak{g}[1])$ is cofreely generated by $\mathfrak{g}$, The differential Q is completely determined by its projections

$$Q_n: \ \Lambda^n \mathfrak{g} \to \mathfrak{g}[2-n], \qquad n \geqslant 0,$$

called *multibrackets* or Taylor components of Q. These multibrackets have to satisfy identities of the form

$$\sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} \pm Q_{q+1}\left(Q_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)}) \wedge X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)}\right) = 0,$$

for $k \geqslant 0$, resulting from the condition $Q^2 = 0$. The element $Q_0 \in \mathfrak{g}[2]$ is called the *curvature* of the L_∞-algebra, and when $Q_0 = 0$, we simply speak of a (non curved) L_∞-*algebra*.

All the curved L_∞-algebras in this work satisfy $Q_k = 0,\ k \geqslant 3$, and are called curved differential graded Lie algebras (dg Lie algebras). It is customary to write the data as $Q_0 := \omega \in \mathfrak{g}_2$, a degree 1 map $Q_1 := d : \mathfrak{g} \to \mathfrak{g}$ and a bracket $Q_2 := [\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of degree 0. The L_∞-relations above now amount to the following conditions:

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]] \quad \text{(Jacobi identity)} \tag{A.2.1a}$$
$$d([x,y]) = [dx,y] + (-1)^{|x|}[x,dy] \quad \text{(derivation property)} \tag{A.2.1b}$$
$$d^2 = [\omega,-] \qquad \text{(Curvature equation)} \tag{A.2.1c}$$
$$d\omega = 0 \qquad \text{(Bianchi identity).} \tag{A.2.1d}$$

A natural example of such a curved dg Lie algebra is given by $\Omega^\bullet(M;\mathrm{End}(E))$, for a smooth vector bundle E over a manifold M. The bracket is given by the wedge product of differential forms combined with the Lie bracket on $\mathrm{End}(E)$. The two remaining structure maps are induced by the choice of a connection ∇ on E: this induces a derivation on the associated bundle $\mathrm{End}(E)$ and the curvature of ∇ gives the element $R(\nabla) \in \Omega^2(M;\mathrm{End}(E))$ satisfying the Bianchi identity.

Morphisms of L_∞-algebras.

A *morphism of curved* L_∞*-algebras* $U : \mathfrak{g}_1 \to \mathfrak{g}_2$ is simply a morphism of coalgebras $S(\mathfrak{g}_1[1]) \to S(\mathfrak{g}_2[1])$ compatible with differentials. Again, using cofreeness, such a morphism is determined by its projections

$$U_n : \Lambda^n \mathfrak{g}_1 \to \mathfrak{g}_2[1-n].$$

As before, compatibility with the differentials imposes conditions on the maps U_n. When $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simply curved dg Lie algebras, these take the form $U_1(\omega_1) = \omega_2$ and

$$\begin{aligned} U_{n+1}(\omega_1 \wedge X_1 \wedge \cdots \wedge X_n) \quad &= \quad \pm dU_n(X_1 \wedge \cdots \wedge X_n) \\ &+ \sum_{i=1}^{n} \pm U_n(X_1 \wedge \cdots \wedge dX_i \wedge \cdots \wedge X_n) \\ &\quad + \frac{1}{2} \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} \pm \left[U_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)}), U_q(X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)})\right] \\ &+ \sum_{i<j} \pm U_{n-1}([X_i,X_j] \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n). \end{aligned} \tag{A.2.2}$$

If both L_∞-algebras are not curved, these relations have a simple meaning in low degrees: for $n = 1$ we see that U_1 is a morphism of complexes. For $n = 2$ we find that the failure of U_1 to be compatible with Lie brackets is exact, so it induces a morphism of Lie algebras on the level of cohomology.

L_∞-modules

Let $(\mathfrak{g}, Q)$ be a curved L_∞-algebra and M a $\mathbb{Z}$-graded vector space. An L_∞-module structure on M is given by a degree 1 differential φ which is a coderivation on the cofree $\Lambda\mathfrak{g}$-module $\Lambda\mathfrak{g} \otimes M$. Again we can write φ out in components

$$\varphi_k : \Lambda^k \mathfrak{g} \otimes M \to M[1-k], \quad k \geqslant 0,$$

and the coderivation property is given by the relations

$$\sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} \pm X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)} \wedge \varphi_q(X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m)$$
$$+ \sum_{\substack{p+q=n \\ \sigma \in Sh(p,q)}} \pm(Q_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)}) \wedge X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m)) = 0.$$

In degree $k = 0$ this means that $\varphi(m) = \pm\varphi_0(m) \pm Q_0 \otimes m$. The fact that φ is a differential leads to the following quadratic relations:

$$\sum_{\substack{p+q=n\\ \sigma\in Sh(p,q)}} \pm\, \varphi_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)} \wedge \varphi_q(X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m)) + \sum_{\substack{p+q=n\\ \sigma\in Sh(p,q)}} \pm\varphi_{q+1}(Q_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)}) \wedge X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m)) = 0.$$

For $n = 0$ this means that $\varphi_0^2 = \varphi_1(Q_0 \otimes -)$, so M is a complex when the curvature of $\mathfrak{g}$ vanishes.

Given two modules M and N over $\mathfrak{g}$, a morphism f from M to N of modules is a morphism of comodules commuting with the coderivations. In components

$$f_k : \Lambda^k\mathfrak{g} \otimes M \to N[-k]$$

this means that

$$\begin{aligned}&\sum_{\substack{p+q=n\\ \sigma\in Sh(p,q)}} \pm f_{q+1}(Q_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)}) \wedge X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m)\\ &+ \sum_{\substack{p+q=n\\ \sigma\in Sh(p,q)}} \pm f_p(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)} \wedge \varphi_q(X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m))\\ &= \sum_{\substack{p+q=n\\ \sigma\in Sh(p,q)}} \pm\varphi_q(X_{\sigma(1)} \wedge \cdots \wedge X_{\sigma(p)} \wedge f_q(X_{\sigma(p+1)} \wedge \cdots \wedge X_{\sigma(n)} \otimes m)).\end{aligned}$$

Twisting by Maurer–Cartan elements

Given a curved dg Lie algebra $\mathfrak{g}$ with structure maps $Q_0 = \omega$, $Q_1 = d$, $Q_2 = [\ ,\ ]$, and an element $X \in \mathfrak{g}_1$, the structure maps can be twisted in the following way:

$$\begin{aligned}\omega_X &= \omega - dX + \frac{1}{2}[X, X]\\ d_X &= d + [X,\ ]\\ [\ ,\ ]_X &= [\ ,\ ].\end{aligned} \tag{A.2.3}$$

These new structure maps define another curved dg Lie algebra, as can be checked by a computation. A classical example is given by a uncurved dg Lie algebra $\mathfrak{g}$ together with a Maurer–Cartan element $X \in \mathfrak{g}_1$ which satisfies

$$dX + \frac{1}{2}[X, X] = 0,$$

which means that the twisted dg Lie algebra also has zero curvature. In a curved dg Lie algebra $(\mathfrak{g}, \omega, d, [\ ,\ ])$ a Maurer–Cartan element is an element $X \in \mathfrak{g}_1$ satisfying

$$dX + \frac{1}{2}[X, X] = \omega.$$

In this case the twisted differerential $d_X := d + [X,\ ]$ has vanishing curvature: $d_X^2 = 0$, therefore the twisted dg Lie algebra $(\mathfrak{g}, d_X, [\ ,\ ])$ is uncurved. Given an L_∞-morphism $U : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ of two curved dg Lie algebras one can check that the formal sum

$$\tilde{X} := \sum_{k\geqslant 1} \frac{U_k(X, \ldots, X)}{k!}$$

is an element in $\mathfrak{h}^1$ which satisfies the Maurer–Cartan equation if X does, provided the sum converges. In case it doesn't converge one can throw in a formal parameter $\hbar$ and work in $\mathfrak{h}[[\hbar]]$ instead. Since the twisted codifferential Q^X is related to the originial codifferential Q by the equation

$$Q^X = \exp(-X\wedge) \circ Q \circ \exp(X\wedge),$$

we can twist the morphism U to an L_∞-morphism $U^X : (\mathfrak{g}, \omega^{\mathfrak{g}}_X, d_X, [\ ,\ ]) \rightsquigarrow (\mathfrak{h}, \omega^{\mathfrak{h}}_{\tilde{X}}, d_{\tilde{X}}, [\ ,\ ])$ defined by

$$U^X := \exp(-\tilde{X}\wedge) \circ U \circ \exp(X\wedge).$$

Similarly, an L_∞-module M over $(\mathfrak{g}, \omega^{\mathfrak{g}}, d, [\ ,\ ])$ is twisted to a module over $(\mathfrak{g}, \omega^{\mathfrak{g}}_X, d_X, [\ ,\ ])$ by the codifferential

$$\varphi^X := \exp(-X\wedge) \circ \varphi \circ \exp(X\wedge),$$

and therefore a morphism $F : M \rightsquigarrow N$ of L_∞-modules over $(\mathfrak{g}, d, [\ ,\ ])$ is twisted by the formula

$$F^X := \exp(-X\wedge) \circ F \circ \exp(X\wedge)$$

which is a morphism of $(\mathfrak{g}, \omega^{\mathfrak{g}}_X, d_X, [\ ,\ ])$-modules.

The tangent map

Given a Maurer–Cartan element X in $\mathfrak{g}$, the underlying cochain complex $T_X\mathfrak{g} := \mathfrak{g}[1]$ of the twisted dg Lie algebra with differential $d_X := d + [X,\ ]$ is called the *tangent complex.* For an L_∞-morphism $U : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ the first component of the twisted morphism U^X defines a *tangent map* $T_X U : T_X\mathfrak{g} \to T_{\tilde{X}}\mathfrak{h}$ of cochain complexes given explicitly by

$$T_X U(Y) = \sum_{k\geqslant 1} \frac{U_k(Y, X, \ldots, X)}{(k-1)!}.$$

A.3 Polyvector fields and polydifferential operators

We assume to be given a sheaf of Lie algebroids $\mathcal{L}$ over $(X, \mathcal{O}_X)$ as described in the introduction of this appendix, and an $\mathcal{L}$-connection on $\mathcal{L}$. Recall from the chapter 1 that the $\mathcal{L}$-valued polyvector fields, which form a sheaf of graded Lie algebras, are defined as

$$T^{\mathcal{L}}_{\text{poly}}(X) := \bigoplus_{n\geqslant -1} \bigwedge^{n+1}_{\mathcal{O}_X} \mathcal{L},$$

with the zero differential and the Lie bracket given by the natural extension of the Lie bracket on $\mathcal{L}$, called the Schouten–Nijenhuis bracket. The Lie bracket has degree 0 due to the shift in the degree. We denote this dg Lie algebra by $\left(T^{\mathcal{L}}_{\text{poly}}(X), 0, [\cdot,\cdot]_S\right)$.

Another natural sheaf of dg Lie algebras associated to a Lie algebroid $\mathcal{L}$ is the algebra of $\mathcal{L}$-polydifferential operators, defined by

$$D^{\mathcal{L}}_{\text{poly}}(X) := \bigoplus_{-1\leqslant n} \mathcal{U}(\mathcal{L})^{\otimes n+1} = \bigoplus_{-1\leqslant n} \underbrace{\mathcal{U}(\mathcal{L})\otimes \ldots \otimes \mathcal{U}(\mathcal{L})}_{n+1 \text{ times}},$$

where we use the left $\mathcal{O}_X$-module structure on $\mathcal{U}(\mathcal{L})$ in the definition of the tensor products. On homogeneous elements $D := D_1 \otimes \ldots \otimes D_{p+1}$ and $E := E_1 \otimes \ldots \otimes E_{q+1}$, one defines $D \circ_i E \in D^{\mathcal{L}}_{\text{poly}}(X)$ by the formula

$$D \circ_i E := D_1 \otimes \ldots \otimes D_i \otimes D^{(1)}_{i+1} E_1 \otimes \ldots \otimes D^{(p+1)}_{i+1} E_{q+1} \otimes D_{i+2} \otimes D_{p+1},$$

where we used Sweedler notation. With this, the *Gerstenhaber bracket* (of degree 0) is defined as

$$[D, E]_G := D \circ E - (-1)^{pq} E \circ D,$$

where

$$D \circ E := \sum_{i=0}^{p} (-1)^{iq} D \circ_i E.$$

As for the ordinary shifted Hochschild complex, the Gerstenhaber bracket defines a graded Lie algebra structure on $D^{\mathcal{L}}_{\text{poly}}(X)$. The element $B_0 = 1 \otimes 1 \in D^{\mathcal{L},0}_{\text{poly}}(X)$ induces the product on $\mathcal{O}_X$ and satisfies $[B_0, B_0]_G = 0$. It is easy to check that $[B_0, \cdot]_G$ defines a differential, which is explicitly given by

$$\begin{aligned}\delta(D_1 \otimes \ldots \otimes D_n) := {} & 1 \otimes D_1 \otimes \ldots \otimes D_n \\ & + \sum_{i=1}^{n} (-1)^i D_1 \otimes \ldots \otimes \Delta D_i \otimes \ldots \otimes D_n + (-1)^{n+1} D_1 \otimes \ldots \otimes D_n \otimes 1.\end{aligned} \tag{A.3.1}$$

The triple $\left(D^{\mathcal{L}}_{\text{poly}}(X), \delta, [\cdot,\cdot]_G\right)$ is a dg Lie algebra. This dg Lie algebra also admits a cup product, which gives $D^{\mathcal{L}}_{\text{poly}}(X)$ the structure of a Gerstenhaber algebra up to homotopy, c.f [CRvdB12].

A.4 Differential forms and Hochschild chains

In this subsection we describe two natural dg Lie algebra-modules over the $\mathcal{L}$-polyvectorfields and $\mathcal{L}$-polydifferential operators. Recall the sheaf of $\mathcal{L}$-differential forms (with an unusual grading)

$$\Omega_{\mathcal{L}}(X) := \bigoplus_{n \leqslant 0} \Lambda^{-n}_{\mathcal{O}_X} \mathcal{L}^*,$$

which, equipped with the *trivial* differential and the Lie derivative of polyvectorfields, defined by the Cartan formula:

$$L_\gamma = d_L \circ \iota_\gamma + (-1)^k \iota_\gamma \circ d_L, \quad \gamma \in T^{\mathcal{L},k}_{\text{poly}}(X),$$

forms a dg Lie algebra module over $T^{\mathcal{L}}_{\text{poly}}(X)$. The complex of Hochschild $\mathcal{L}$-chains is defined as:

$$C_{\mathcal{L}}(X) := \bigoplus_{n \leqslant 0} \mathcal{J}(\mathcal{L})^{\hat{\otimes}_{\mathcal{O}_{X_1}} n}.$$

To give this complex the structure of a dg Lie algebra module over $D^{\mathcal{L}}_{\text{poly}}(X)$, we proceed as in [CDH, CRvdB12]. There it is explained that the pair $(D^{\mathcal{L}}_{\text{poly}}(X), C_{\mathcal{L}}(X))$ can be realized as the flat sections of a pair of Hochschild cochains and chains of the sheaf of algebras $\mathcal{J}(\mathcal{L})$. The differentials and the module structure of the Hochschild (co)chains commute with the flat connection, and thus these structures descend to flat sections. First, we consider the Hochschild $\mathcal{L}$-cochains. The second $\mathcal{U}(\mathcal{L})$-module structure on $\mathcal{J}(\mathcal{L})$ defines an injective map

$$r: \; D^{\mathcal{L}}_{\text{poly}}(X) \longrightarrow \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} D^{\mathcal{L}}_{\text{poly}}(X) \overset{\nabla^{(2)}}{\cong} \mathcal{D}_{\text{poly}}(\mathcal{J}(\mathcal{L})), \tag{A.4.1}$$

where the RHS is given by the polydifferential operators of $\mathcal{J}(\mathcal{L})$ which are linear with respect to the first $\mathcal{O}_X$-module structure. The isomorphism which is decorated by $\nabla^{(2)}$ is given by

$$\phi \otimes D \mapsto (\psi \mapsto \phi D._2(\psi)), \qquad \phi \otimes D \in \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{U}(\mathcal{L}), \; \psi \in \mathcal{J}(\mathcal{L})$$

where $D._2$ is the action by the second $\mathcal{U}(\mathcal{L})$-module structure on $\mathcal{J}(\mathcal{L})$. The $\mathcal{L}$-connection $\nabla^{(1)}$ extends to $\mathcal{D}_{\text{poly}}(\mathcal{J}(\mathcal{L}))$, and the map given above induces an isomorphism of $D^{\mathcal{L}}_{\text{poly}}(X)$ and the flat sections of $D_{\text{poly},\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$. For the Hochschild $\mathcal{L}$-chains we proceed as follows. The complex of Hochschild chains for the algebra $\mathcal{J}(\mathcal{L})$ is given by

$$\mathcal{C}^{\mathcal{L}}_\bullet(X) := \mathcal{J}(\mathcal{L})^{\hat{\otimes}_{\mathcal{O}_{X_1}} -(\bullet+1)}.$$

It is related to the $\mathcal{L}$-chains by the following map:

$$\begin{aligned}\mathcal{C}(\mathcal{J}(\mathcal{L})) &\xrightarrow{\epsilon} C_{\mathcal{L}}(X) \\ \phi_0 \hat{\otimes} \cdots \hat{\otimes} \phi_k &\longmapsto \epsilon(\phi_0)\phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_k.\end{aligned} \tag{A.4.2}$$

Extending $\nabla^{(1)}$ to $\mathcal{C}(\mathcal{J}(\mathcal{L}))$, this map restricts to an isomorphism on the module of flat sections on the LHS. The dg Lie algebra structure on $D_{\text{poly},\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$ is in the same way defined as for $D^{\mathcal{L}}_{\text{poly}}(X)$ and it is clear that the inclusion r from (A.4.1) is compatible with the structures on the LHS and the RHS. The standard Hochschild differential for chains is defined by

$$b(\phi_0 \otimes \cdots \otimes \phi_k) := \phi_0\phi_1 \otimes \cdots \otimes \phi_k + \sum_{i=1}^{k-1}(-1)^i \phi_0 \otimes \cdots \otimes \phi_i\phi_{i+1} \otimes \cdots \otimes \phi_k + (-1)^k \phi_k\phi_0 \otimes \cdots \otimes \phi_k,$$

whereas the $D_{\text{poly},\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$-module structure on the chain complex is defined by

$$L_D(\phi_0 \otimes \cdots \otimes \phi_n) := \sum_{i=n-k+1}^{n} (-1)^{n(i+1)} D(\phi_{i+1}, \cdots, \phi_0, \cdots) \otimes \phi_{k+i-n} \otimes \cdots \otimes \phi_i + \sum_{i=0}^{n-k}(-1)^{(k-1)(i+1)}\phi_0 \otimes \cdots \otimes \phi_i \otimes D(\phi_{i+1}, \cdots, \phi_{i+k}) \otimes \phi_{i+k+1} \otimes \cdots \otimes \phi_n.$$

Since $\nabla^{(1)}$ acts by derivations on $\mathcal{J}(\mathcal{L})$ the differential b commutes with $\nabla^{(1)}$, and one can easily check that the $D_{\text{poly},\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L}))$-module structure L on $\mathcal{C}(\mathcal{J}(\mathcal{L}))$ defines a $D^{\mathcal{L}}_{\text{poly}}(X)$ module structure on $C_{\mathcal{L}}(X)$. These considerations imply that the subpair $(D^{\mathcal{L}}_{\text{poly}}(X), C_{\mathcal{L}}(X))$ of $(\mathcal{D}_{\text{poly}}(\mathcal{J}(\mathcal{L})), \mathcal{C}^{\mathcal{L}}_{\bullet}(X))$ forms a dg Lie algebra together with a dg Lie algebra module, i.e we have a commuting diagram

$$\begin{array}{ccc} D^{\mathcal{L}}_{\text{poly}}(X) & \xrightarrow{r} & \mathcal{D}_{\text{poly}}(\mathcal{J}(\mathcal{L})) \\ \downarrow L & & \downarrow L \\ C_{\mathcal{L}}(X) & \xrightarrow{s} & \mathcal{C}^{\mathcal{L}}_{\bullet}(X) \end{array} \tag{A.4.3}$$

in which the left vertical arrow is the restriction of the right one and in which s is the map to the flat sections with respect to $\nabla^{(1)}$ such that $\epsilon \circ s = \text{id}$, c.f. [KP]. The wavy arrows indicate dg Lie algebra module structures.

To summarize, we described the dg Lie algebra of $\mathcal{L}$-polyvector fields $T^{\mathcal{L}}_{\text{poly}}(X)$ with the dg Lie algebra module of $\mathcal{L}$-differential forms $\Omega_{\mathcal{L}}(X)$ over $T^{\mathcal{L}}_{\text{poly}}(X)$, and the dg Lie algebra of $\mathcal{L}$-polydifferential operators $D^{\mathcal{L}}_{\text{poly}}(X)$ with the dg Lie algebra module $C^{\mathcal{L}}_{\text{poly}}(X)$ of Hochschild $\mathcal{L}$-chains. In a diagram this looks as follows:

$$\begin{array}{ccc} T^{\mathcal{L}}_{\text{poly}}(X) & & D^{\mathcal{L}}_{\text{poly}}(X) \\ \downarrow L & & \downarrow L \\ \Omega_{\mathcal{L}}(X) & & C_{\mathcal{L}}(X) \end{array} \tag{A.4.4}$$

The next paragraph will discuss the formality theorem, which relates these two dg Lie algebras and their modules.

A.5 Formality for Hochschild $\mathcal{L}$-(co)chains

In this section we discuss the relation between the $\mathcal{L}$-polyvector fields and the $\mathcal{L}$-polydifferential operators. There is an HKR type map $U_1 : T^{\mathcal{L}}_{\text{poly}}(X) \to D^{\mathcal{L}}_{\text{poly}}(X)$ given by antisymmetrization

$$U_1(X_0 \wedge \ldots \wedge X_k) := \frac{1}{(k+1)!} \sum_{\sigma \in S_{k+1}} (-1)^{\sigma} X_{\sigma(0)} \otimes \ldots \otimes X_{\sigma(k)}. \tag{A.5.1}$$

This is a quasi-isomorphism of sheaves of complexes, but it is not compatible with the Lie brackets. There is also a HKR type chain map $V_1 : C^{\mathcal{L}}(X) \to \Omega_{\mathcal{L}}(X)$ given by

$$V_1(\phi_1 \otimes \ldots \otimes \phi_k) := \phi_0 d\phi_1 \wedge \ldots \wedge d\phi_k, \tag{A.5.2}$$

which is a quasi-isomorphism of sheaves of complexes. However, it is not compatible with the dg Lie algebra-module structures.

In his seminal paper [K], Kontsevich proved that, in the case of a smooth manifold M and $\mathcal{L} = TM$, this morphism can be "corrected" to a quasi isomorphism of L_∞-algebras with Taylor components $U_k : \Lambda^k(T^{\mathcal{L}}_{\text{poly}}(X)) \to D^{\mathcal{L}}_{\text{poly}}(X)$. Amongst many other consequences, this allows to relate the Maurer–Cartan elements in the two dg Lie algebras, which gives a correspondence between Poisson structures and deformations of the product on $C^\infty(M)$. We formulate the most fundamental version of the so-called formality theorem, because we need it for our proof.

Local formality results

Consider the trivial Lie algebroid $\mathbb{R}^d \to 0$ with coordinates $y^1, \dots, y^d$. Its algebra of jets is given by $\mathcal{J}(\mathbb{R}^d) \cong \mathbb{R}[[y^1, \dots, y^d]]$ and the formal polyvectorfields $T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$ and the formal polydifferential operators $D_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$ are defined algebraically, e.g. a formal polyvector field of degree k can be written as

$$X = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} \frac{\partial}{\partial y_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial y_{i_k}}, \qquad a_{i_1 \dots i_k} \in \mathcal{J}(\mathbb{R}^d).$$

The formality theorem [K, §7] reads as follows.

Theorem A.5.1. *There exists an L_∞-quasi-isomorphism*

$$\mathcal{K} : T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d)) \longrightarrow D_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$$

with the following properties:

1. *The first structure map $\mathcal{K}_1$ is given by the HKR map* (4.1.1) .
2. *$\mathcal{K}$ is $GL_d(\mathbb{R})$-equivariant*
3. *Let $n > 1$. For any vector fields $v_1, \dots, v_n \in T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$ one has*
$$\mathcal{K}_n(v_1, \dots, v_n) = 0.$$
4. *If $n > 1$, then for any vector field $v \in T^0_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$ linear in the coordinates y^i and any polyvector fields $X_2, \dots, X_n \in T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$ one has*
$$\mathcal{K}_n(v, X_2, \dots, X_n) = 0.$$

The formula for $\mathcal{K}_n$ is given by a sum over certain graphs, with weights that are given by integrals over configuration spaces. In [K], Kontsevich already globalized the formality map to a smooth manifold M, and discussed the compatibility of the morphism with other structures such as the wedge product on polyvector fields.

We denote the differential forms and Hochschild chains related to the algebra $\mathcal{J}(\mathbb{R}^d)$ by $\Omega_{\mathcal{J}(\mathbb{R}^d)}$ and $C(\mathcal{J}(\mathbb{R}^d))$. Since $C(\mathcal{J}(\mathbb{R}^d))$ is a dg Lie module over $D_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$, the formality morphism $\mathcal{K}$ of Kontsevich equips $C(\mathcal{J}(\mathbb{R}^d))$ with the structure of an L_∞-module over $T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d)$. The formality conjecture for chains, formulated by Tsygan in [Ts] and proved by Shoikhet in [S], states that the two L_∞-modules $\Omega_{\mathcal{J}(\mathbb{R}^d)}$ and $C(\mathcal{J}(\mathbb{R}^d))$ over $T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d)$ are related as follows:

Theorem A.5.2 ([S]). *There exists a quasi-isomorphism $\mathcal{S}$ of L_∞-modules over $T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$*

$$\mathcal{S} : C(\mathcal{J}(\mathbb{R}^d)) \longrightarrow \Omega_{\mathcal{J}(\mathbb{R}^d)}$$

with the following properties:

1. *The first structure map* $\mathcal{S}_1$ *is given by the dual of the HKR map* (4.1.1) *.*
2. $\mathcal{S}$ *is* $\mathsf{GL}_d(\mathbb{R})$*-equivariant*
3. *If* $n > 1$ *then for any vector field* $v \in T^0_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$ *linear in the coordinates* y^i *and any polyvector fields* $X_2, \ldots, X_n \in T_{\text{poly}}(\mathcal{J}(\mathbb{R}^d))$*, and any chain* $j \in C(\mathcal{J}(\mathbb{R}^d))$
$$\mathcal{S}_n(v, X_2, \ldots, X_n; j) = 0.$$

Again, the maps $\mathcal{S}_n$ are defined using sums over graphs with weights given by integrals over configuration spaces. The theorem was globalized to the case of smooth manifolds, and the compatibility between the de Rham differential and the cyclic differential B was studied by Willwacher in [W15].

Theorem A.5.3 ([CDH, CRvdB12, C, CvdB, D])**.** *Let* $(\mathcal{L}, \mathcal{O}_X)$ *be a locally free, fine sheaf of Lie algebroids of constant rank* r *over* $\mathbb{R}$*. There exist an* L_∞*-quasi-isomorphism* $U = (U_1, U_2, \ldots)$
$$U: \ T^{\mathcal{L}}_{\text{poly}}(X) \longrightarrow D^{\mathcal{L}}_{\text{poly}}(X)$$
with the first component equal to the HKR map U_1 *above. This morphism equips* $C_{\mathcal{L}}(X)$ *with the structure of an* L_∞*-module over* $T^{\mathcal{L}}_{\text{poly}}(X)$*, and there exists a quasi-isomorphism* $V = (V_1, V_2, \ldots)$ *of* L_∞*-modules*
$$V: \ C_{\mathcal{L}}(X) \longrightarrow \Omega_{\mathcal{L}}(X)$$
over $T^{\mathcal{L}}_{\text{poly}}(X)$*, with* V_1 *equal to the HKR map* V_1 *for chains.*

Proof. The remainder of this subsection is devoted to a proof of Theorem A.5.3. It follows from the following claim: There exist two sheaves of dg Lie algebras L_1 and L_2 over X, two dg Lie modules M_1, M_2 over L_1 and L_2 respectively, and a commutative diagram
$$\begin{array}{ccccccc} T^{\mathcal{L}}_{\text{poly}}(X) & \hookrightarrow & L_1 & \longrightarrow & L_2 & \hookleftarrow & D^{\mathcal{L}}_{\text{poly}}(X) \\ \wr\downarrow L & & \wr\downarrow L & & \wr\downarrow L & & \wr\downarrow L \\ \Omega_{\mathcal{L}}(X) & \hookrightarrow & M_1 & \longleftarrow & M_2 & \hookleftarrow & C_{\mathcal{L}}(X). \end{array} \tag{A.5.3}$$

The hooked arrows in the upper rows are quasi-isomorphisms of dg Lie algebras and the middle arrow in the upper row is an L_∞-quasi-isomorphism. The wavy arrows indicate the dg Lie module structures. The hooked arrows in the lower row are dg Lie module quasi-isomorphisms, and the middle arrow in the lower row is a quasi-isomorphism of L_∞-modules.

To be precise, this diagram does not provide a complete proof of the formality theorem as stated above, since one also needs to invert the upper right and lower left arrows to obtain the L_∞-morphims. This matter is properly dealt with in [D] and [CFW], so we refer to these articles. The proof for the existence of the diagram is divided into three steps, and we give it because it differs slightly from existing proofs. Firstly, we stress the role played by the dual PBW theorem for jets, and secondly, we work with *curved* L_∞-algebras.

Step 1: Fedosov resolutions

Detailed proofs for all the statements in this step can be found in [CvdB, section 4]. Recall the sheaf $\mathcal{J}(\mathcal{L})$ of $\mathcal{L}$-jets, with its two $\mathcal{L}$-module structures from chapter 2. Consider the sequence
$$\mathcal{O} \xrightarrow{\alpha_2} \mathcal{J}(\mathcal{L}) \xrightarrow{d_{\nabla^{(1)}}} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{L}^\vee \xrightarrow{d_{\nabla^{(1)}}} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \Lambda^2\mathcal{L}^\vee \xrightarrow{d_{\nabla^{(1)}}} \ldots \tag{A.5.4}$$
where $d_{\nabla^{(1)}}$ is the de Rham $\mathcal{L}$-differential associated to the flat connection $\nabla^{(1)}$. The filtration on $\mathcal{J}(\mathcal{L})$ induces a filtration on the sequence, and the differential induced on the associated

graded spaces $\mathrm{Sym}_{\mathcal{O}}(\mathcal{L}^\vee) \otimes_{\mathcal{O}} \Lambda^k\mathcal{L}$ is the Koszul differential for $\mathrm{Sym}_{\mathcal{O}}(\mathcal{L}^\vee)$. Since the Koszul differential is exact, the sequence (A.5.4) is exact. Replacing the sheaf $\mathcal{O}_X$ in this sequence by $T^{\mathcal{L}}_{\mathrm{poly}}(X)$, $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ or $\Omega_{\mathcal{L}}(X)$, one obtains exact sequences

$$\begin{gathered}
T^{\mathcal{L}}_{\mathrm{poly}}(X) \xhookrightarrow{\alpha_2} T^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) \xrightarrow{d_{\nabla^{(1)}}} T^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} \mathcal{L}^\vee \xrightarrow{d_{\nabla^{(1)}}} \dots \\
D^{\mathcal{L}}_{\mathrm{poly}}(X) \xhookrightarrow{\alpha_2} D^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) \xrightarrow{d_{\nabla^{(1)}}} D^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} \mathcal{L}^\vee \xrightarrow{d_{\nabla^{(1)}}} \dots \\
\Omega_{\mathcal{L}}(X) \xhookrightarrow{\alpha_2} \Omega_{\mathcal{L}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) \xrightarrow{d_{\nabla^{(1)}}} \Omega_{\mathcal{L}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} \mathcal{L}^\vee \xrightarrow{d_{\nabla^{(1)}}} \dots
\end{gathered} \tag{A.5.5}$$

The map $\nu\colon \phi \otimes X \mapsto \left(\psi \mapsto \phi\nabla^{(2)}_X\psi\right)$ is an isomorphism of $\mathcal{J}(\mathcal{L})$-modules

$$\nu\colon \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \mathcal{L} \xrightarrow{\cong} \mathrm{Der}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})). \tag{A.5.6}$$

Again, this is proved by studying the map on the associated graded spaces. Note that the inclusion $X \mapsto \nabla^{(2)}_X$ is a morphism of Lie algebras. This isomorphism can be extended to isomorphisms of dg Lie algebras:

$$\begin{aligned}
T^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) &\cong T_{\mathrm{poly},\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) =: \mathcal{T}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L})) \\
D^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes_{\mathcal{O}_{X_2}} \mathcal{J}(\mathcal{L}) &\cong D_{\mathrm{poly},\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) =: \mathcal{D}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L})).
\end{aligned} \tag{A.5.7}$$

Dual to the map (A.5.6), we have the isomorphism

$$u\colon \Omega^1_{\mathcal{J}(\mathcal{L})} := \Omega^1_{\mathcal{J}(\mathcal{L})/\mathcal{O}_{X_1}} \xrightarrow{\cong} \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \Lambda^1\mathcal{L}^\vee$$

which can be extended to an isomorphism of dg algebras

$$u\colon \left(\mathit{\Omega}_{\mathcal{J}(\mathcal{L})} := \mathit{\Omega}_{\mathcal{J}(\mathcal{L})}, d_{\mathrm{dR}}\right) \xrightarrow{\cong} \left(\mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_2}} \Lambda\mathcal{L}^\vee, \nabla^{(2)}\right). \tag{A.5.8}$$

Recall that the module structure L of $\Omega_{\mathcal{L}}(X)$ over $T^{\mathcal{L}}_{\mathrm{poly}}(X)$ is defined as the graded commutator of the contraction operation and the de Rham differential. The same holds for the module structure of $\mathit{\Omega}_{\mathcal{J}(\mathcal{L})}$ over $\mathcal{T}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L}))$. The inverse of the map (A.5.8) preserves the dg algebra structure, i.e. it commutes with the de Rham differential, and we recall the following commuting diagram [CvdB, (4.18)]

$$\begin{array}{ccc}
\mathcal{L}^\vee \otimes \mathcal{L} & \longrightarrow & \mathcal{O} \\
\big\downarrow & & \big\downarrow {\scriptstyle \alpha_2} \\
\mathit{\Omega}_{\mathcal{J}(\mathcal{L})} \hat{\otimes} \mathrm{Der}_{\mathcal{O}_{X_1}}(\mathcal{J}(\mathcal{L})) & \longrightarrow & \mathcal{J}(\mathcal{L})
\end{array}$$

where the horizontal arrows are the natural contractions, and the left vertical arrow is given by ν and u^{-1}. The resolutions of the polyvector fields and differential forms in (A.5.5), the isomorphisms ν and u^{-1} and the compatibility of the dg Lie algebra module structures with these morphisms are summarized in the following diagram:

$$\begin{array}{ccc}
\left(T^{\mathcal{L}}_{\mathrm{poly}}(X), 0, [\cdot,\cdot]\right) & \xrightarrow{\nu\circ\alpha_2} & \left(\mathcal{T}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L})) \otimes_{\mathcal{O}_{X_1}} \Lambda\mathcal{L}^\vee, d_{\nabla^{(1)}}, [\cdot,\cdot]\right) \\
\big\downarrow {\scriptstyle L} & & \big\downarrow {\scriptstyle L} \\
(\Omega_{\mathcal{L}}(X), 0) & \xrightarrow{u^{-1}\circ\alpha_2} & \left(\mathit{\Omega}_{\mathcal{J}(\mathcal{L})} \otimes_{\mathcal{O}_{X_1}} \Lambda\mathcal{L}^\vee, d_{\nabla^{(1)}}\right),
\end{array} \tag{A.5.9}$$

which is a commuting diagram of dg Lie algebras and their modules, such that the horizontal arrows are quasi-isomorphisms.

Remark A.5.4. The brackets on the RHS are $\wedge\mathcal{L}^\vee$-linearly extended to the tensor product. To be more precise, one has

$$T_{\mathrm{poly},\wedge\mathcal{L}^\vee}\left(\mathcal{J}(\mathcal{L})\otimes_{\mathcal{O}_{X_1}}\wedge\mathcal{L}^\vee\right)\cong\mathcal{T}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L}))\otimes\wedge\mathcal{L}^\vee$$

where the LHS is given by the polyvector fields of the algebra $\mathcal{J}(\mathcal{L}))\otimes_{\mathcal{O}_{X_1}}\wedge\mathcal{L}^\vee$ which are $\wedge\mathcal{L}^\vee$-linear, and the corresponding Schouten–Nijenhuis bracket on the RHS is given by the $\wedge\mathcal{L}^\vee$-linear extension of the bracket on $\mathcal{T}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L}))$ to the tensor product.

To define the required structures on the Hochschild $\mathcal{L}$-(co)chains we already realized them as $\nabla^{(1)}$-flat sections of another dg Lie algebra with a module over it. Let us recall diagram (A.4.3) and include the relevant operations:

$$\begin{array}{ccc}\left(D^{\mathcal{L}}_{\mathrm{poly}}(X),\delta,[\cdot,\cdot]\right) & \xrightarrow{\ r\ } & \left(\mathcal{D}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L}))\otimes\wedge\mathcal{L}^\vee,\delta+d_{\nabla^{(1)}},[\cdot,\cdot]\right)\\ \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle L}\\ (C_{\mathcal{L}}(X),b) & \xrightarrow{\ \ s\ \ } & \left(\mathcal{C}^{\mathcal{L}}_\bullet(X)\otimes\wedge\mathcal{L}^\vee,b+d_{\nabla^{(1)}}\right).\end{array}\tag{A.5.10}$$

Step 2: The PBW isomorphism

Recall that the PBW isomorphism is an algebra isomorphism

$$j_\nabla:\ \mathcal{J}(\mathcal{L})\longrightarrow\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$$

which is $\mathcal{O}_X$-linear with respect to the first $\mathcal{O}_X$-module structure on $\mathcal{J}(\mathcal{L})$ and depends on the choice of an $\mathcal{L}$-connection on $\mathcal{L}$. It induces the following commuting diagram of dg Lie algebras and their modules:

$$\begin{array}{ccc}\mathcal{T}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L})) & \xrightarrow{\ j_\nabla\ } & \mathcal{T}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))\\ \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle L}\\ \Omega_{\mathcal{J}(\mathcal{L})} & \xrightarrow{\ j_\nabla\ } & \Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)}\end{array}\qquad\begin{array}{ccc}\mathcal{D}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L})) & \xrightarrow{\ j_\nabla\ } & \mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))\\ \big\downarrow{\scriptstyle L} & & \big\downarrow{\scriptstyle L}\\ \mathcal{C}(\mathcal{J}(\mathcal{L})) & \xrightarrow{\ j_\nabla\ } & \mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))\end{array}$$

where the horizontal arrows are isomorphisms. The flat connection $\nabla^{(1)}$ induces a flat connection $\tilde{\nabla}^{(1)}=j_\nabla^{-1}\circ\nabla^{(1)}\circ j_\nabla$ on $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$, and taking the tensor product of each of the objects in the above diagram with $\wedge\mathcal{L}^\vee$, the differentials $d_{\nabla^{(1)}}$ and $d_{\tilde{\nabla}^{(1)}}$ can be included in the dg Lie algebra (module) structures. E.g., we obtain

$$\left(\mathcal{D}_{\mathrm{poly}}(\mathcal{J}(\mathcal{L}))\otimes\wedge\mathcal{L}^\vee,\delta+d_{\nabla^{(1)}},[\cdot,\cdot]_G\right)\overset{j_\nabla}{\cong}\left(\mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))\otimes\wedge\mathcal{L}^\vee,\delta+d_{\tilde{\nabla}^{(1)}},[\cdot,\cdot]_G\right).$$

Now set

$$\begin{aligned}L_1&:=\mathcal{T}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))\otimes_{\mathcal{O}}\wedge\mathcal{L}^\vee &\qquad M_1&:=\Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)}\\ L_2&:=\mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))\otimes_{\mathcal{O}}\wedge\mathcal{L}^\vee &\qquad M_2&=\mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)).\end{aligned}$$

The quasi-isomorphisms from (A.5.9) and (A.5.10), together with the maps induced by the PBW map that we just discussed, combine to the hooked arrows in (A.5.3). In step 3) we will prove the existence of the middle arrows in (A.5.3).

Step 3: Twisting the local L_∞-maps

The final step of the proof is quite similar to [CDH, section 4.1], however, we include the notion of curved L_∞-algebras, which is different from their approach.

Using the fact that $\mathcal{L}$ is locally trivial, we can choose a trivializing basis $\{e_i\}_{i=1}^r$ of $\mathcal{L}$ over an open $U \subset X$. Hence we obtain $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)|_U \cong \mathcal{O}|_U \otimes \mathbb{R}^d_{\mathrm{formal}}$, and similar local isomorphisms for the polyvector fields, the polydifferential operators, the differential forms and the Hochschild chains.

Kontsevich' formality morphism from Theorem A.5.1 and Shoikhets formality morphisms from Theorem A.5.2 can be used to define an $\mathcal{O}|_U$-linear extension of the formality morphisms for $\mathcal{J}(\mathbb{R}^d)$. By property (2) of Theorem A.5.1 and property (2) of Theorem A.5.2, these local definitions do not depend on the choice of basis, hence they globalize to $\mathcal{O}_X$-linear L_∞-morphisms:

$$\begin{array}{ccc} \left(\mathcal{T}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, 0, [\cdot,\cdot]\right) & \xrightarrow{\mathcal{K}} & \left(\mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \delta, [\cdot,\cdot]\right) \\ \downarrow L & & \downarrow L \\ \left(\Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)} \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee\right) & \xleftarrow{\mathcal{S}} & \left(\mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, b\right). \end{array}$$

We will twist these L_∞-morphisms locally and argue that the twisted morphisms do not depend on the local choices and thus define global L_∞-morphisms.

We denote the dual basis $\{e_i\}$ of $\mathcal{L}^\vee$ either by $\{f^i\}$ or by $\{de^i\}$. The first notation is used when we describe local sections of $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$, the second is used when we denote local sections of $\Lambda\mathcal{L}^\vee$. The de Rham differential $d_\mathcal{L}$ for $\mathcal{L}$-forms with values in bundles such as $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$ is well-defined for $\mathcal{L}|_U$, although dependent on a choice of basis of $\mathcal{L}$. Because $\mathcal{K}$ and $\mathcal{S}$ are $\mathcal{O}_X$-linear, they commute with the de Rham differential, thus we have local L_∞-morphisms

$$\begin{array}{ccc} \left(\mathcal{T}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, d_\mathcal{L}, [\cdot,\cdot]\right) & \xrightarrow{\mathcal{K}} & \left(\mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, d_\mathcal{L}+\delta, [\cdot,\cdot]\right) \\ \downarrow L & & \downarrow L \\ \left(\Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)} \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, d_\mathcal{L}\right) & \xleftarrow{\mathcal{S}} & \left(\mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, d_\mathcal{L}+b\right). \end{array}$$

The horizontal arrows are quasi-isomorphisms, as can be shown by a spectral sequence argument based on the filtration defined by the exterior degree, c.f. [CDH, section 4]. Locally we can write the $\mathcal{L}$ connection on $\mathcal{L}$ as the sum of the de Rham differential and an endomorphism valued one form $\Gamma \in \Omega^1_\mathcal{L}(\mathrm{End}(\mathcal{L}))$:

$$\nabla = d_\mathcal{L} + \Gamma^i_{jk} f^j \frac{\partial}{\partial f^i} de^k.$$

The element Γ can be used to twist the L_∞ structures and the L_∞ morphisms, as described in (A.2.3). This gives a (local) morphism of curved L_∞-algebras:

$$\begin{array}{ccc} \left(\mathcal{T}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \nabla^\mathcal{L}, [\cdot,\cdot]\right) & \xrightarrow{\mathcal{K}} & \left(\mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \delta+\nabla^\mathcal{L}, [\cdot,\cdot]\right) \\ \downarrow L & & \downarrow L \\ \left(\Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)} \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \nabla^\mathcal{L}\right) & \xleftarrow{\mathcal{S}} & \left(\mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, b+\nabla^\mathcal{L}\right), \end{array}$$

where the curvature term is, on both sides, given by R_∇.

Lemma A.5.5. *The morphisms $\mathcal{K}$ and $\mathcal{S}$ in the diagram are* global L_∞*-quasi-isomorphims of curved dg Lie algebras and their modules.*

Proof. Property (4) from Theorem A.5.1 and property (3) from Theorem A.5.2 imply that $\mathcal{K} = \mathcal{K}^\Gamma$ and $\mathcal{S} = \mathcal{S}^\Gamma$, as Γ is linear in the f^i-coordinates. Hence the moprhisms $\mathcal{K}$ and $\mathcal{S}$ are globally defined morphisms of curved L_∞-algebras and their modules. □

For the final step we need another twist of the L_∞-morphism by a Maurer–Cartan element in the curved dg Lie algebra on the LHS. In Lemma 4.2.3 it is shown that, for a torsion free $\mathcal{L}$-connection, one has

$$d_{\tilde{\nabla}^{(1)}} = \nabla - \delta_K + \sum_{k\geqslant 2} A_k$$

where $A_k \in \mathrm{Der}^{k-1}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee))$ and δ_K is the Kozsul differential for $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. Recall that a given connection can be modified to a torsion free connection. The term $B := -\delta_K + \sum_{k\geqslant 2} A_k$ is globally defined and is clearly a Maurer–Cartan element for the curved dg Lie algebra structure given by $(\nabla,\ [\cdot,\cdot])$ because $\tilde{d}_{\nabla^{(1)}}$ is a differential. Hence we can twist the L_∞-morphisms $\mathcal{K}$ and $\mathcal{S}$ by B. This gives global L_∞-morphisms (of uncurved) dg Lie algebras and their modules:

$$\begin{array}{ccc}
\left(\mathcal{T}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \tilde{d}_{\nabla^{(1)}}, [\cdot,\cdot]\right) & \xrightarrow{\mathcal{K}^B} & \left(\mathcal{D}_{\mathrm{poly}}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \delta + \tilde{d}_{\nabla^{(1)}}, [\cdot,\cdot]\right) \\
\big\downarrow L & & \big\downarrow L \\
\left(\Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)} \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, \tilde{d}_{\nabla^{(1)}}\right) & \xleftarrow{\mathcal{S}^B} & \left(\mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathcal{O}_X} \Lambda\mathcal{L}^\vee, b + \tilde{d}_{\nabla^{(1)}}\right).
\end{array}$$

Remark that the L_∞-algebra structure on the RHS is twisted by the MC-element

$$\tilde{B} := \sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{K}_k(B,\ldots,B)$$

where B is the Maurer–Cartan element of the LHS. In this particular case property (3) of Theorem A.5.1 implies that $\tilde{B} = B$. From the general theory, c.f. [D], it follows that, since $\mathcal{K}$ and $\mathcal{S}$ are quasi-isomorphisms, $\mathcal{K}^B$ and $\mathcal{S}^B$ are quasi isomorphims as well. Thus, $\mathcal{K}^B$ and $\mathcal{S}^B$ are the middle horizontal arrows of diagram (A.5.3), which existence we showed now. Hence, apart from inverting certain quasi-isomorphisms, Theorem A.5.3 is proved.

Step 4: Inverting the L_∞-quasi isomorphisms.

Since L_∞-quasi isomorphisms are invertible up to homotopy, the upper right arrow and the lower left arrow in diagram (A.5.3) can be inverted to obtain the L_∞-morphisms $\mathcal{U}$ and $\mathcal{V}$ from the theorem. However, this is not completely straightforward, and for a discussion of the subtleties we refer to [CFW, D]. In [D] an explicit inverse is constructed via a recursive procedure. This explicit choice of an inverse allows Dolgushev to prove that the formality map thus obtained is invariant under an action of a smooth Lie group, provided the connection is invariant under this action. We will use this property in our main application to prove that the formality map preserves the polynomial degrees , c.f . §3.2. □

Remark A.5.6. We would like to make a short comparison with the techniques which are based on formal geometry that Fedosov, and later for example [C] and [D] used. In that approach, one starts with the observation that

$$T^{\mathcal{L}}_{\mathrm{poly}}(X) \longrightarrow \left(T^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \otimes \Lambda\mathcal{L}^\vee, -\delta_K\right) \tag{A.5.11}$$

is a resolution, where $-\delta_K$ is the fiberwise Koszul differential of $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{L}^\vee)$. The isomorphism $T^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee) \cong T_{\mathrm{poly},\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$, equips the RHS with a fiberwise Lie algebra structure, but the inclusion in (A.5.11) is *not* a morphism of Lie algebras. Given a torsion free $\mathcal{L}$-connection on $\mathcal{L}$, the differential $-\delta_{\mathrm{fiber}} + \nabla$ can be, using an iterative procedure, adjusted to a differential

$$D \;=\; -\delta_{\mathrm{fiber}} + \nabla + A$$

where $A \in \Omega^1_{\mathcal{L}}(\mathrm{Der}_{\mathcal{O}_X}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L})))$. Moreover, the inclusion in (A.5.11) can be modified to a resolution

$$T^{\mathcal{L}}_{\mathrm{poly}}(X) \longrightarrow \left(T^{\mathcal{L}}_{\mathrm{poly}}(X) \otimes \widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^*) \otimes \Lambda\mathcal{L}^*, D\right)$$

which respects the Lie algebra structures. The PBW isomorphism allows for more direct construction of D, which in our case is just given by $\tilde{d}_{\nabla^{(1)}}$. The constructions are carried out in a similar fashion for $D^{\mathcal{L}}_{\mathrm{poly}}(X)$ etc., and Fedosov applied it to the Weyl algebra bundle associated to a symplectic manifold.

A.6 Cyclic $\mathcal{L}$-chains

Finally, let us discuss the extension to cyclic chains. It is a straightforward extension, mostly based on [W11]. It starts with the observation that the complex $\mathcal{C}(\mathcal{J}(\mathcal{L}))$ carries an additional cyclic differential B turning it into a *mixed complex*: we just take the standard B-operator defined by

$$B(\phi_0 \otimes \cdots \otimes \phi_k) := \sum_{j=0}^{n} (-1)^{nj} 1 \otimes \phi_j \otimes \cdots \otimes \phi_n \otimes \phi_0 \otimes \cdots \otimes \phi_{j-1}$$

with the remark that $\phi_{-1} := \phi_n$. Note that we use the augemented complex given by $\mathcal{C}(\mathcal{J}(\mathcal{L}))^{-k}_{\mathrm{aug}} := \mathcal{J}(\mathcal{L}) \otimes_{\mathcal{O}_{X_1}} (\mathcal{J}(\mathcal{L})/\mathcal{O}_{X_1})^k$, which gives the same homology as the complex $\mathcal{C}(\mathcal{J}(\mathcal{L}))$, but has a simpler expression for the cyclic differential B. This operator commutes with the Grothendieck connection, i.e.,

$$[\nabla^{(1)}_X, B] = 0, \quad \text{for all } X \in \mathcal{L}$$

since $\nabla^{(1)}_X(1) = 0$. Thus it defines an operator, which we also denote by B, on $C_{\mathcal{L}}(X)$. Recall from 1.2.2 that cyclic complexes are defined as $C_{\mathcal{L}}(X) \otimes_{\mathbb{K}[u^{-1}]} W$, equipped with the differential $b + u^{-1}B$ where W is a $\mathbb{K}[u^{-1}]$-module.

Proposition A.6.1. *The u^{-1}-linear extension of V induces a morphism*

$$V : \left(C^{\mathcal{L}}(\mathcal{L})[[u^{-1}]] \otimes_{\mathbb{C}[u^{-1}]} W,\ b + u^{-1}B\right) \to \left(\Omega^\bullet_{\mathcal{L}}[[u^{-1}]] \otimes_{\mathbb{C}[u^{-1}]} W,\ u^{-1}d\right)$$

of L_∞-modules over $T^{\mathcal{L}}_{\mathrm{poly}}(X)$.

Proof. The proof is very similar to the proof of [W11, cor. 4]; we give a sketch. We need to show that the lower arrows of diagram (A.5.3);

$$\Omega_{\mathcal{L}}(X)^W \longleftrightarrow M_1^W \xleftarrow{\ \mathcal{S}^B\ } M_2^W \longleftrightarrow C_{\mathcal{L}}(X)^W \tag{A.6.1}$$

intertwine $u^{-1}d_{\mathcal{L}}$ on $\Omega^W_{\mathcal{L}} = \Omega_{\mathcal{L}}[[u^{-1}]] \otimes_{\mathbb{K}[u^{-1}]} W$, the $\mathcal{O}_X$-linear de Rham differential $u^{-1}d_{dR}$ on $M_1^W = \Omega_{\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)} \otimes_{\mathbb{K}[u^{-1}]} W$, the $\mathcal{O}_X$-linear cyclic differential $u^{-1}B$ on $M_2^W = \mathcal{C}(\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)) \otimes_{\mathbb{K}[u^{-1}]} W$, and $u^{-1}B$ on $C_{\mathcal{L}}(X) \otimes_{\mathbb{K}[u^{-1}]} W$ respectively.

- The morphism $\Omega_{\mathcal{L}}(X)^W \hookrightarrow M_1^W$ is the composition of the inclusion $u^{-1} \circ \alpha_2 : \Omega_{\mathcal{L}}(X) \to \Omega_{\mathcal{J}(\mathcal{L})}$ and the PBW map j_∇. It is easy to check that $u^{-1} \circ \alpha_2$ -note that u^{-1} indicates a morphism, and not a formal variable, intertwines the differentials $d_{\mathcal{L}}$ and d_{dR}, and since j_∇ is an algebra isomorphism it intertwines d_{dR} for $\mathcal{J}(\mathcal{L})$ and d_{dR} for $\widehat{\mathrm{Sym}}_{\mathcal{O}_X}(\mathcal{L}^\vee)$.
- The main theorem in [W11] states that the middle arrow commutes with the respective differentials.
- Finally, the morphism $C_{\mathcal{L}}(X)^W \to M_2^W$ is given by the composition $j_\nabla \circ s$. The map s intertwines the differentials by definition, and j_∇ commutes with the differentials because it is an algebra isomorphism.

□

Appendix B

The local Riemann–Roch theorem

In this appendix we describe, following [PoPfTa10, W15] the fundamental cyclic cocycle on the Weyl algebra and recall the Riemann–Roch theorem for this cocycle.

B.1 The Weyl algebra

Let $\mathcal{O}_r = \mathbb{K}[x^1, \ldots, x^r]$ be the polynomial algebra on r generators. The Weyl algebra $\mathcal{W}_r$ is, by definition, the algebra of differential operators on $\mathcal{O}_r$. Over $\mathcal{O}_r$, it is generated by the fundamental derivations $\partial/\partial x^i$, $i = 1, \ldots, r$ subject to the well-known commutation relations

$$[\frac{\partial}{\partial x^i}, x^j] = \delta_{ij}.$$

It therefore admits another presentation as the space of polynomials $\mathbb{K}[q^1, \ldots, q^r, p_1, \ldots, p_r]$ equipped with the Moyal–Weyl product

$$f \star g := m \circ e^{\pi}(f \otimes g), \quad f, g \in \mathbb{K}[q^1, \ldots, q^r, p_1, \ldots, p_r], \tag{B.1.1}$$

where m is the commutative product of polynomials and

$$\pi := \sum_{i=1}^{r} \left(\frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i} - \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} \right).$$

The isomorphism with $\mathcal{W}_r$ is simply given by sending $x^i \mapsto q^i$, $p_i \mapsto \partial/\partial x^i$.

B.2 The Hochschild cocycle

Using the spectral sequence associated to the filtration given by the degree of a polynomial, it was proved in [FT] that

$$HH_\bullet(\mathcal{W}_r) = \begin{cases} \mathbb{K} & \bullet = 2r \\ 0 & \bullet \neq 2r. \end{cases}$$

A generator for the nontrivial class in degree $2n$ is given by

$$c_{2r} := \sum_{\sigma \in S_{2r}} (-1)^{\sigma} 1 \otimes y_{\sigma(1)} \otimes \ldots y_{\sigma(2r)}, \quad y_{2i} = q_i, \; y_{2i-1} = p_i, \; i = 1, \ldots, r.$$

In [FFS], a dual Hochschild cocycle generating the only nontrivial class was defined by the formula

$$\tau_{2r}^{\text{Hoch}}(a) := \mu_{2r} \int_{\Delta^{2r}} \prod_{0 \leqslant i < j \leqslant 2r} e^{(t_j - t_i - \frac{1}{2})\pi_{ji}} (1 \otimes \pi^{\wedge r})(a) dt_1 \cdots dt_{2r}, \tag{B.2.1}$$

where $a = a_0 \otimes \ldots a_{2r} \in W_r^{\otimes(2r+1)}$ and $\mu_{2r}(a_0 \otimes \ldots a_{2r}) = a_0(0) \cdots a_{2r}(0)$ is the evaluation at zero followed by the commutative multiplication. That this cocycle exists follows simply by duality, and indeed we have the property that

$$\tau_{2r}(c_{2r}) = 1. \tag{B.2.2}$$

However, the explicit expression above allows to prove the surprising fact the cocycle τ_{2r}^{Hoch} is $\mathfrak{gl}(r, \mathbb{K})$-invariant and basic:

$$L_X \tau_{2r}^{\text{Hoch}} = 0, \quad \tau_{2r}^{\text{Hoch}}(\ldots, X, \ldots) = 0, \quad X \in \mathfrak{gl}(r, \mathbb{K}). \tag{B.2.3}$$

The center of the Weyl algebra is clearly equal to $\mathbb{K}$, and, since every derivation is inner, as an explicit calculation shows, we have the exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{K} \longrightarrow W_r \xrightarrow{\text{ad}} \text{Der}(W_r) \longrightarrow 0. \tag{B.2.4}$$

B.3 The map to Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The Chevalley–Eilenberg cochain complex, written $C^\bullet_{\text{Lie}}(\mathfrak{g})$, is a special case of the complex of $\mathcal{L}$-differential forms (1.1.2) in the case $\mathcal{L} = \mathfrak{g}$ and $(X, \mathcal{O}_X) = (\{pt\}, \mathbb{K})$. Let C be a unital, associative algebra over $\mathbb{K}$. Consider the cyclic cochain complex $CC^\bullet_W(C)$ for a $\mathbb{K}[u]$–module W. The evaluation map

$$\text{ev}_1(\phi)(a_1, \ldots, a_k) := \sum_{\sigma \in S_k} (-1)^\sigma \phi(1 \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(k)}), \quad a_1, \ldots a_k \in C,$$

defines a morphism of cochain complexes

$$\text{ev}_1 \colon (CC^\bullet_W(C), b + uB) \longrightarrow \left(C^\bullet_{\text{Lie}}(\mathfrak{gl}(C))[[u]] \otimes_{\mathbb{K}[u]} W, d_{\text{Lie}}\right), \tag{B.3.1}$$

where $\mathfrak{gl}(C)$ denotes the Lie algebra associated to C, i.e., C equipped with the commutator.

B.4 The fundamental cyclic cocycle

In [PoPfTa10] and [W15] the Hochschild cocycle τ_{2r}^{Hoch} was extended to a full cyclic cocycle in the (b, B)-complex. Here we follow the presentation of [W15]. Define the insertion operator

$$\iota_\pi = \sum_{i=1}^r \iota_{p_i} \iota_{q^i} : C^k(W_r, W_r^*) \to C^{k-2}(W_r, W_r^*).$$

With this notation, for each choice of $w \in W \backslash \{0\}$, the cochain

$$\tau_{2r}^w := e^{-u\iota_\pi} \tau_{2r}^{\text{Hoch}} \otimes w \in CC^{2r}_W(W_r)$$

is closed: $(b + uB)\tau_{2r}^w = 0$. For $w = 1$ in ordinary cyclic cohomology one can expand

$$\tau_{2r}^1 = \tau_{2r}^{\text{Hoch}} + u\tau_{2r-2} + \ldots + u^r \tau_0 \in C^{2r}(W_r)[u]$$

with $\tau_{2r-2k} = \iota_\pi^k \tau_{2r}^{\text{Hoch}}/k!$. It is possible to perform the integration to obtain an explicit formula for for the lower degrees τ_{2k} as an integral over the simplex Δ^{2k} analogous to (B.2.1), c.f. [PoPfTa10].

For the twisted case, we need a slight generalization of this cocyle. The algebra $\mathcal{W}_r^n := M_n(\mathcal{W}_r)$ is Morita equivalent to $\mathcal{W}_r$, thus has the same Hochschild and cyclic homology. The cyclic cocyle can be generalized by the formula

$$\tau^{\mathcal{W},n}((a_0 \otimes M_0) \otimes \ldots \otimes (a_k \otimes M_k)) := \tau_{2r}^{\mathcal{W}}(a_0 \otimes \ldots \otimes a_k)\mathrm{tr}(M_0 \ldots M_k).$$

The algebra $\mathcal{W}_r^n$ contains the Lie subalgebra $\mathfrak{sp}_{2r} \oplus \mathfrak{gl}_n$, and the cocycle is basic and invariant w.r.t. this subalgebra, c.f. [W15].

B.5 The local Riemann–Roch theorem

The most fundamental property of the cyclic cocycle $\tau_{2r}^{\mathcal{W}}$ is the local Riemann–Roch theorem which links it in Lie algebra cohomology to the characteristic classes appearing in the index theorem. This theorem has a long history dating back to [FT, Thm 5.1.1.], where it first appeared in abstract form for a Hochschild class of $\tau_{2r}^{\mathrm{Hoch}}$. Other appearances of the theorem are in [BNT, FFS, PoPfTa10, W15]. We shall state the theorem below as an equality on the level of *chains* (the last proposition), the proof follows as in [W15] by evaluating the integrals appearing in the formula for $\tau_{2r}^{\mathcal{W}}$.

Let us first recall the Chern–Weil homomorphism in Lie algebra cohomology: Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras, and suppose that there exists an $\mathfrak{h}$-equivariant projection $\pi : \mathfrak{g} \to \mathfrak{h}$. Then π determines, in the language of Lie algebroids, a $\mathfrak{g}$-connection on $\mathfrak{h}$ by the formula

$$\nabla_X Z = \pi([X, Z]), \quad \text{for } X \in \mathfrak{g},\ Z \in \mathfrak{h},$$

with curvature $R \in \mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathfrak{h})$ given by

$$C(X, Y) = [\pi(X), \pi(Y)] - \pi([X, Y]), \quad X, Y \in \mathfrak{g}.$$

The Chern–Weil construction then leads to a homomorphism

$$\chi : \left(\mathrm{Sym}^\bullet \mathfrak{h}^\vee\right)^{\mathfrak{h}} \to C_{\mathrm{Lie}}^{2\bullet}(\mathfrak{g}, \mathfrak{h}),$$

where the RHS is the relative Chevalley–Eilenberg complex. Explicitly, it is given by the formula

$$\chi(P)(X_1, \ldots, X_{2k}) := \frac{1}{k!} \sum_{\substack{\sigma \in S_{2k} \\ \sigma(2i-1) < \sigma(2i)}} (-1)^\sigma P\left(C(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, C(X_{\sigma(2k-1)}, X_{\sigma(2k)})\right),$$

where $X_1, \ldots, X_{2k} \in \mathfrak{g}$ and $P \in \mathrm{Sym}^k \mathfrak{h}^\vee$. For the Riemann–Roch theorem we need this construction for the inclusion

$$\mathfrak{gl}_r \oplus \mathfrak{gl}_n \subset \mathcal{W}_r^n$$

where $\mathfrak{gl}_r$ is embedded as diagonal matrices (by the inclusion $\mathfrak{gl}_r \to \mathfrak{sp}_{2r}$) and $\mathfrak{gl}_n$ as constant matrices. Recall the invariant polynomials defining the Chern and Todd classes:

$$\mathrm{Ch}(X) := \mathrm{Tr}\left(e^X\right), \quad \mathrm{Td}(X) := \det\left(\frac{x}{1 - e^X}\right), \quad \hat{A}(X) = \frac{x/2}{\sinh x/2} \tag{B.5.1}$$

As usual, it is meant that one takes the power series expansion around zero of the analytic functions used in these definitions, and applies the Chern–Weil construction to each homogeneous component.

Theorem B.5.1 (Local Riemann–Roch)**.** *Under the* u*-linear extension of the morphism* (B.3.1)*, we have the following equality of Lie algebra classes:*

$$[ev_1(\tau^n)]_{2k} = (-1)^k[\chi(\hat{A}_r\,\mathrm{Ch}_n)]_{2k} \in H_{\mathrm{Lie}}^{2k}\left(\mathfrak{gl}(\mathcal{W}_r^n) \oplus \mathfrak{gl}_n(\mathcal{O}_r); \mathfrak{gl}_r \oplus \mathfrak{gl}_n\right).$$

This theorem is proved by the following proposition, which is in fact stronger than the theorem itself. Let $C_{n,r}$ be the subalgebra of W_r^n consisting of polynomials of the form $\sum_{j=1}^r f_j(q) \otimes 1 + \sum_{j\geqslant 1} g_j(q) \otimes M_j$ for f_j and g_j polynomials in q and $M_j \in \mathfrak{gl}_n$.

Proposition B.5.2 ([FFS, W15])**.** *On the subalgebra* $C_{n,r}$ *the cocycles* $ev_1(\tau^n)_{2k}$ *and* $\chi((\hat{A}_r \operatorname{Ch}_n)_{2k})$ *agree.*

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Samenvatting

Cyklische theorie van Lie algebroïden

Dit proefschrift richt zich op de studie van de cyklische theorie van de universeel omhullende algebras van Lie algebroïden. Lie algebroïden zijn meetkundige objecten die infinitesimale symmetrieën beschrijven, en het concept omvat een groot aantal klassieke begrippen uit de meetkunde, zoals Poisson variëteiten, foliaties en acties van Lie algebras op variëteiten. Het bestuderen van meetkundige objecten is in veel gevallen equivalent aan het bestuderen van de algebras van functies op deze objecten, en deze observatie heeft geleid tot de niet-commutatieve meetkunde, waarbij men niet-commutatieve algebras, die niet noodzakelijk zijn gerelateerd aan een meetkundig object, bestudeert met technieken die voortkomen uit de meetkunde. Voor elke Lie algebroïde kan men een niet-commutatieve algebra -de universeel omhullende algebra- definiëren, welke onder andere de algebra van differentiaaloperatoren op een variëteit en de universeel omhullende algebra van een Lie algebra veralgemeniseert. In dit proefschrift laten we zien dat de cyklische theorie van deze algebra gelijk is aan de Poisson (co)homologie van de duale van de Lie algebroïde, welke op haar beurt gelijk is aan de Lie algebroïde cohomologie met waarden in de symmetrische algebra van de geadjungeerde representatie tot op homotopy (gedraaid door een lijnbundel). Verder definiëren we een spoor-dichtheid afbeelding van de cyklische theorie van de universeel omhullende algebra naar het de Rham complex van de Lie algebroïde, wat een veralgemenisering is van bekende resultaten voor de raakbundel van een variëteit. We gebruiken een Čech resolutie van het de Rham complex, wat de constructie ook toepasbaar maakt op holomorfe Lie algebroïden. Zowel de berekening van de cyklische theorie van de universeel onhullende algebra als de constructie van de spoor-dichtheid afbeelding is gebaseerd op de Poincaré–Birkhoff–Witt stelling voor Lie algebroïden, die we dan ook eerst bewijzen.

Summary

Cyclic theory of Lie algebroids

In this thesis we study the cyclic theory of universal enveloping algebras of Lie algebroids. Lie algebroids are geometrical objects that encode infinitesimal symmetries, and the concept encompasses many classical objects from geometry, such as Poisson manifolds, foliations and actions of Lie algebras on manifolds. The study of geometrical objects is in many cases equivalent to the study of the algebras of functions on these objects, and this observation led to the field of noncommutative geometry, where one studies noncommutative algebras, that are not necessarily related to geometrical objects, with techniques from geometry. For each Lie algebroid on can define a noncommutative algebra, called the universal enveloping algebra, which generalizes the algebra of differential operators on a manifold and the universal enveloping algebra of a Lie algebra. In this thesis we show that the cyclic theory of this algebra is equal to the Poisson (co)homology of the dual of the Lie algebroid, which in turn is equal to the Lie algebroid cohomology with values in the symmetric algebra of the adjoint representation up to homotopy (twisted by a line bundle). Moreover, we define a trace-density map from the cyclic theory of the universal enveloping algebra to the de Rham complex of the Lie algebroid, which generalizes known results for the tangent bundle of a manifold. We use a Čech resolution of the de Rham complex, which makes the construction suitable for holomorphic Lie algebroids as well. Both the calculation of the cyclic theory of the universal enveloping algebra as well as the construction of the trace-density map is based on the Poincaré–Birkhoff–Witt theorem for Lie algebroids, which we therefore prove first.