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Interfaces with Other Disciplines

Optimal investment and consumption when allowing terminal debt

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ABSTRACT

We analyze a dynamic optimization problem which involves the consumption and investment of an investor with constant relative risk aversion for consumption but with a risk aversion for final wealth which does not necessarily imply that terminal wealth must always be positive. We require risk aversion for terminal wealth to be positive but not monotone: there is a point of maximal risk aversion at zero wealth and the investor may continue to consume when wealth is negative. Using dual optimization methods we can derive explicit solutions and we find that the optimal solution differs in a fundamental way from the case where risk aversion is monotone. It turns out that the optimal consumption function is convex and concave at different wealth levels and that the optimal investment strategy may no longer be monotone as a function of the remaining time to invest and consume.

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1. Introduction

The present paper contributes to the existing literature on optimal portfolio theory by finding explicit solutions for an investor who is risk-averse but who is willing to end up with debts, i.e. a negative wealth at the end of the consumption and investment period. Closed-form optimal policies have, to the best of our knowledge, only been found for the specific class of exponential utilities, where risk aversion is independent of wealth. Instead, we formulate a model to study the consumption patterns for people who try to avoid debt, but do not avoid it at all costs. In this model, there is therefore an explicit tradeoff between a higher probability of debts at the end of the time period under consideration and the utility of immediate consumption.

A specific example which can motivate our approach would be a fund manager who invests to generate cashflows for a client during a finite time period. She would try to avoid negative wealth at the end of the investment period since her performance measurement over the investment period will take both consumption during and wealth after the period into account, but she may not exclude negative wealth on beforehand to allow the possibility to 'gamble for resurrection' in disadvantageous scenarios.

Such behavior could be interpreted as a 'sunk costs effect' (also known as 'escalation of commitment'), which in this case would

mean that the investor, having been unsuccessful enough on the stock market to arrive at negative wealth levels, would feel an incentive to continue investing in stocks to make up for the lost earlier investments.²

In that sense our model is meant to be descriptive rather than normative and it assumes that the investor still has the possibility to borrow money once wealth is negative. This may be the case, for example, when the wealth position is not disclosed during but only after the investment period.³ Our setup also allows us to deal with cases where an investor starts with negative wealth at the initial point in time but still shows (asymptotically) constant relative risk aversion for high positive wealth levels.

The standard investment and consumption model which assumes preferences with constant relative risk aversion (CRRA) cannot be applied for such investors since it assumes that wealth must stay positive and preferences with constant absolute risk aversion (CARA) allow only risk aversion which does not depend on wealth. We thus believe it may be useful to offer an alternative which takes account of such possibilities. In particular, we would like to

² See Arkes and Blumer (1985) for the psychology of sunk costs, Thaler and Johnson (1990) for the break-even effect and the paper by Baghestanian and Massenet (2015) for recent empirical evidence on the incentive to gamble for resurrection.

³ If wealth levels would be disclosed during the investment period, it would be more natural that the interest rate on the cash account would change once wealth becomes negative. We will not make that assumption here; see He and Pagés (1993) and Dybvig and Liu (2010) for results when borrowing is restricted and Korn (1995) for the case where there is a higher interest rate for borrowing than for lending. It seems that in the latter case, our optimization problem can only be solved numerically.

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allow investors to have different preferences for consumption and for terminal wealth, but we still want explicit formulas for the optimal strategies that can be directly compared to strategies in which terminal debts must be avoided at all costs or where investors become risk-seeking when wealth becomes negative.

1.1. SAHARA utility

We therefore assume that an investor has CRRA preferences for consumption and a utility for terminal wealth which exhibits Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA). This class of utility functions allows for positive and negative values of wealth. For very large values the risk preferences correspond to those of the CRRA class but SAHARA investors have a level of wealth at which their risk aversion is maximal.⁴ We take this reference point, without loss of generality,⁵ to be the point of zero wealth and we may interpret the fact that the investor becomes more risk averse when the point of zero wealth is approached from above as a manifestation of zero-risk-bias. However, once an investor has crossed this point and wealth has become negative, she becomes less risk averse if wealth becomes even more negative. This property formalizes the intuition that people may become less concerned about debt once they are already in debt. If that would not be the case, there would, for example, be no people who switch to a new credit card once their first card has a negative balance which is not compensated by other assets they own.

The SAHARA utility functions define a less risk averse attitude for large values of wealth but also for large values of debt, i.e. for very negative wealth. However, we will still assume that there is positive risk aversion once the investor is heavily in debt. We do not include risk seeking behavior such as proposed in models with gain-loss preferences (see [Tversky & Kahneman \(1992\)](#)). We thus retain concavity of utility as a function of terminal wealth on the entire domain, which allows us to find explicit solutions for the optimized behavior in this dynamic setting. We will compare our optimized strategies to those based on gain-loss preferences in the last section before the conclusion.

The optimal consumption and asset allocation problem in a continuous-time setting dates back to [Merton \(1969, 1971\)](#). The solutions to the associated stochastic optimal control problems strongly depend on risk preferences and the assumptions on the asset price dynamics. The most frequently used utility functions exhibit hyperbolic absolute risk aversion (HARA). In particular, one often assumes that either wealth has to be above a certain threshold (for example, by assuming CRRA utility functions) or that individuals' risk aversion does not depend on their wealth (for CARA utility functions). Together with the assumption of Brownian or Geometric Brownian asset dynamics, Merton shows that closed-form solutions can be obtained for these risk preferences.⁶ In

⁴ SAHARA utility assumes a unique reference point at which the absolute risk aversion for terminal wealth reaches its maximum. In this sense, SAHARA utility does not capture the preferences of investors who exhibit multiple reference points where risk aversion levels obtain a local maximum. Some recent experimental studies have shown that investors may indeed display multiple reference points in terminal wealth, see [Knoller \(2016\)](#), and [Koop and Johnson \(2012\)](#). [Wang and Johnson \(2012\)](#) develop a theory based on three reference points: the minimum requirement value, the status quo, and the goal. We believe that our study of preferences with a single point of maximal risk aversion provides insight into the more complicated case where there are more reference points.

⁵ Choosing the reference point d differently for the utility of terminal wealth (and for the levels of consumption) would amount to a translation of the whole optimization problem and the corresponding optimal strategies over the wealth axis, since our asset returns do not depend on wealth. Note, however, that this is no longer true if there would be multiple local maxima for risk aversion.

⁶ Merton's optimal consumption and investment problem has been further investigated and extended in different contexts. Some relevant references are [Dybvig and Huang \(1988\)](#), [Davis and Norman \(1990\)](#), [Sørensen \(1999\)](#), [Brennan and Xia \(2000\)](#),

[Chen, Pelsser, and Vellekoop \(2011\)](#) this closed-form solution for Merton's problem is extended to include SAHARA risk preferences. This class of utility functions also arises naturally in the framework of forward performance criteria: see the earlier formulations in [Zariphopoulou and Musiela \(2009, 2010\)](#).

In contrast to the analysis in [Chen et al. \(2011\)](#) we consider both the utility of intermediate consumption and the utility of terminal wealth in our optimization problem. Since it is possible to invest in riskless and risky assets during this period, we thus allow the investor to postpone the payments of debts to a later time while continuing to consume, in the hope that favorable returns on financial assets may still help to achieve a better level of wealth at the end.

1.2. Dual methods

Since we will not make the Markovian assumption for the asset price dynamics, we rely on the dual approach for optimization in stochastic dynamic systems to derive explicit solutions in the general case. Such methods have been studied for complete markets in [Cox and Huang \(1989\)](#), [Karatzas, Lehoczky, and Shreve \(1987\)](#) and [Pliska \(1986\)](#) and in an incomplete market setting e.g. by [Davis \(1997\)](#), [He and Pearson \(1991a,b\)](#), [Karatzas, Lehoczky, Shreve, and Xu \(1991\)](#), [Schachermayer \(2001\)](#) and [Rogers \(2003\)](#). We apply the dual method in a multi-asset economy which takes survival probabilities of the investor into account. If Markovian dynamics are not assumed we cannot necessarily express the optimal consumption strategy as an explicit function of the level of wealth. But if we assume that the parameters which specify the market dynamics and preferences are deterministic functions of time, closed-form strategies can be derived which provide direct insight in consumption patterns and the optimal portfolio holdings.

We find for example that if asymptotic risk aversion for large values of terminal wealth is larger (or smaller) than the risk aversion for consumption, the wealth level at which investment in risky assets is minimal occurs for a negative (or positive) value. If these two risk aversion coefficients are equal, we can show how much the level of consumption and the allocation to risky assets increase in comparison to an investor who has CRRA preferences for both consumption and terminal wealth. This difference is time-varying and involves several key parameters, such as the strength of the propensity to become less risk averse when going further into debt (measured by a parameter β), the tradeoff between utility from consumption and terminal wealth (measured by a parameter K) and a parameter ν_γ which is a combination of the risk free rate r , asymptotic risk aversion γ , the vector of market prices of risk θ , the discount factor δ and force of mortality ζ (see [Corollary 3.6](#)).

We also find that over long time horizons the investment strategy does not differ that much from the one generated by a loss averse preference, which is one of the characteristic elements in cumulative prospect theory (see [Berkelaar, Kouwenberg, and Post \(2004\)](#) and [Jin and Zhou \(2008\)](#)). One can interpret our choice as in between the classical framework of CRRA preferences, where negative wealth is avoided at all cost, and gain-loss preferences, which are risk seeking for negative wealth. In our case, the investor remains risk averse for negative wealth but less so if wealth becomes more negative, so in that case the investor gets closer to the point where risk aversion becomes risk seeking behavior, without ever reaching it.

The remainder of the paper is structured as follows. [Section 2](#) describes the underlying financial market and formulates

[Munk and Sørensen \(2004, 2010\)](#), [Korn and Steffensen \(2007\)](#), [Dybvig and Liu \(2010\)](#), and [Kraft and Munk \(2011\)](#). For the case of investors which are not assumed to be time-consistent see [Marín-Solano and Navas \(2010\)](#).

the consumption and investment problem. Solutions to optimal consumption, terminal wealth and investment strategies are found in Section 3. Furthermore, some special cases are discussed in which investment strategies can be expressed as an explicit function of wealth. Section 4 compares our results to those found under different preference structures and Section 5 provides some concluding remarks and suggestions for further research. All proofs are collected in the appendix.

2. The optimal consumption and investment problem

In this section, we define the underlying economy and formulate the investor’s performance criteria.

2.1. Financial market

We assume a financial market in continuous time without transaction costs that contains $d \geq 1$ traded risky assets and one risk free asset: the bank account. Let the asset price dynamics for the risky assets⁷ $S_t = (S_t^1, S_t^2, \dots, S_t^d)'$ and bank account B be given by

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sum_{j=1}^d \Sigma_t^{ij} dW_t^j \tag{1}$$

$$\frac{dB_t}{B_t} = r_t dt, \tag{2}$$

where $W_t := (W_t^1, W_t^2, \dots, W_t^d)'$ are d independent Brownian motions on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We fix a time horizon $T > 0$ and define for $t \in [0, T]$ the processes $r_t \geq 0$, $\mu_t = (\mu_t^1, \mu_t^2, \dots, \mu_t^d)'$ and $(\Sigma_t)^{ij}$ which are adapted⁸ with respect to the filtration generated by the Brownian motions, which is denoted by $\{\mathcal{F}_t\}_{t \in [0, T]}$. Furthermore we assume that $\int_0^T \sum_{i=1}^d \sum_{j=1}^d (\Sigma_t^{ij})^2 dt < \infty$ a.s., that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$ with $\|x\| = 1$ we have $\sigma_{min} \leq \|\Sigma_t x\| \leq \sigma_{max}$ for a certain $\sigma_{max} \geq \sigma_{min} > 0$ and that the processes μ and r are bounded by a deterministic constant. This implies that the stochastic differential equations for S and B have a unique solution on $[0, T]$. We assume $B_0 = 1$ and $S_0^i > 0$, all i .

We are in a complete market setting and the unique measure \mathbb{Q} under which all discounted asset prices are martingales can be characterized in terms of its Radon–Nikodym density process $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t}$ which satisfies $Z_0 = 1$ and

$$dZ_t = -Z_t \theta_t' dW_t, \quad \text{with} \quad \theta_t := \Sigma_t^{-1} (\mu_t - r_t \mathbf{1})$$

and $\mathbf{1} = (1, 1, \dots, 1)'$ a d -dimensional vector of ones. Note that our conditions on Σ and the boundedness of μ and r lead to the uniform boundedness of $\|\theta_t\|$. The discounted process $H_t := Z_t/B_t$ is the state price density or deflator process, which satisfies $H_0 = 1$ and

$$dH_t = -H_t (r_t dt + \theta_t' dW_t). \tag{3}$$

2.2. The preferences for consumption

We consider an investor who wants to optimize investment and consumption on the time period $[0, T]$ given that she will be left with a terminal wealth X_T if she is still alive at the time. The investor is initially endowed with a wealth $x_0 > 0$. She can trade in the $d + 1$ assets and consume part of the wealth in the period

$[0, T]$ while alive. We assume that the time of death $\tau \geq 0$ of the investor can be modeled using a stochastic force of mortality process ζ , which is adapted to a filtration $(\mathcal{G}_t)_{t \in [0, T]}$ that is independent of the Brownian filtration $(\mathcal{F}_t)_{t \in [0, T]}$ so

$$\mathbb{P}(\tau \geq t | \mathcal{G}_t) := \zeta_t = \exp \left(- \int_0^t \zeta_u du \right), \tag{4}$$

with $\zeta_t \geq 0$ and $\int_0^t \zeta_s ds < \infty$ for all $t \in [0, T]$ \mathbb{P} -almost surely.

The investor will choose a consumption process $(c_t)_{t \in [0, T]}$ which is adapted with respect to $\{\mathcal{F}_t\}_{t \in [0, T]}$, and such that \mathbb{P} -almost surely

$$c_t \geq 0, \quad \mathbb{E} \left(\int_0^T c_t dt \right)^2 < \infty.$$

He also chooses an investment strategy that we describe in terms of the amounts of wealth invested in all the risky assets at time t , which we denote by $(A_t)_{t \in [0, T]} = (A_t^1, A_t^2, \dots, A_t^d)'$. We assume that this process is adapted to the same filtration and by investing the remaining wealth $A_t^0 = X_t - \sum_{i=1}^d A_t^i$ in the risk free asset we guarantee that our strategy is self-financing.

The wealth process related to the strategy (c_t, A_t) when starting with an initial wealth x_0 is then easily seen to satisfy

$$dX_t^{A,c} = (r_t X_t^{A,c} - c_t) dt + A_t' ((\mu_t - r_t \mathbf{1}) dt + \Sigma_t dW_t), \tag{5}$$

$$X_0^{A,c} = x_0.$$

The process $W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s ds$ is a martingale under \mathbb{Q} since $\|\theta_t\|$ is bounded, so the following process is a local martingale under \mathbb{Q} :

$$Y_t^{A,c} := X_t^{A,c} / B_t + \int_0^t (c_u / B_u) du - x_0 = \int_0^t (1/B_u) A_u' \Sigma_u dW_u^{\mathbb{Q}}. \tag{6}$$

To avoid arbitrage opportunities which are generated by pathological investment strategies, we only allow processes A_t which make this process in fact a \mathbb{Q} -martingale on $[0, T]$. We will call investment strategies which fulfill this condition *martingale-generating*.⁹ We can give an explicit condition on A to achieve this, by proving an extension to the no-consumption case which has been treated in Chen et al. (2011).

Lemma 2.1. *Take a consumption process c which satisfies the conditions mentioned above. If an adapted investment strategy A generates a wealth process $X^{A,c}$ such that for a certain stochastic variable $\bar{K} < \infty$ we have*

$$\|A_t\|^2 \leq \bar{K} (1 + (X_t^{A,c})^2), \tag{7}$$

for all $t \in [0, T]$, then the process A is martingale-generating.

We will often use the shorthand notation $X_t = X_t^{A,c}$ to avoid the more cumbersome version. We also introduce the following notation for admissible wealth processes.

Definition 2.2. We indicate by $\mathcal{X}(x_0)$ all possible wealth processes that can be generated when starting from an initial capital x_0 by using strategies (c_t, A_t) which satisfy the conditions outlined above, including the martingale-generating property.

Preferences of the investor for consumption are modeled using a power utility function which is defined on the positive real line and exhibits a constant relative risk aversion (CRRA):

$$U_1(c) = K \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

⁷ Throughout the paper the prime ' will be used to denote transposition of vectors.

⁸ Note that this is an extension of the results reported in Chen et al. (2011), where these and other parameters are assumed to be deterministic.

⁹ In line with, for example, Karatzas and Shreve (1998).

for a constant $K > 0$ and a risk aversion parameter $\gamma > 0, \gamma \neq 1$. For $\gamma \rightarrow 1$ we find $U_1(c) \rightarrow K \ln c$. The risk aversion function $A_1(c) = -U''(c)/U'(c)$ thus equals γ/c for all positive values of γ . We will frequently use the convex dual function

$$\tilde{U}_i(y) = \sup_{x \in \mathbb{R}} (U_i(x) - xy)$$

and the function which defines the value x for which the supremum is attained: $x = (U'_i)^{-1}(y) := I_i(y)$. For the CRRA utility these functions are

$$\tilde{U}_1(y) = \frac{K}{1-\gamma} (\gamma (y/K)^{1-1/\gamma} - 1), \quad I_1(y) = (y/K)^{-1/\gamma},$$

with domain \mathbb{R}^+ .

2.3. The preferences for terminal wealth

For terminal wealth, we would like to assume that an investor may decide to end up with debts so we would like to allow negative wealth. The investor will avoid the point of zero wealth in the sense that risk aversion increases when this level is approached from above. But there is a possibility that negative wealth levels are indeed reached. If this happens, we will assume the investor will show behavior which is less risk averse but never risk-seeking. We therefore use the SAHARA class of utility functions (Chen et al. (2011) and Zariphopoulou and Musiela (2009, 2010)) which is defined for the entire real line. This class, which incorporates CRRA and CARA utility functions as limiting cases, is in general defined as follows:

Definition 2.3. A utility function U with domain \mathbb{R} is of the SAHARA class if the absolute risk aversion function $A(x) = -U''(x)/U'(x)$ is well-defined on its entire domain and satisfies

$$A(x) = \frac{\alpha}{\sqrt{\beta^2 + (x-d)^2}} > 0 \tag{8}$$

for a given $\beta > 0$ (the scale parameter), $\alpha > 0$ (the risk-aversion parameter) and $d \in \mathbb{R}$ (the threshold wealth).

As state before, we will assume in the remainder of the paper, without loss of generality, that the point of maximal risk aversion d is at zero wealth.

Below the threshold wealth risk aversion becomes smaller for more negative wealth levels since the investor is relatively indifferent between being slightly or severely in debt. Our approach therefore differs from classical expected utility formulations, where positive and monotone risk aversion is assumed, but also from preferences with loss aversion, which exhibit risk aversion that is negative and decreasing for wealth values below a certain threshold value and positive and decreasing above it. This is for example the case in the example given in Tversky and Kahneman (1992), where $\tilde{U}(x) = x^{a_+}$ for $x \geq 0$ and $\tilde{U}(x) = -b(-x)^{a_-}$ for $x < 0$ where b, a_+ and a_- are strictly positive constants.¹⁰ When a_+ and a_- are taken equal in this definition (in the paper both were estimated to be 0.88) we see that, both in their model and in ours, risk aversion converges to zero for large positive and negative wealth levels. This suggests a tendency to try to 'gamble oneself out of trouble'. But in our model there is no risk seeking behavior and risk aversion is finite and smooth around the threshold value of maximal risk avoidance. The risk aversion associated with \tilde{U} is discontinuous across that point, since maximal risk aversion switches to maximal risk seeking behavior the moment the investor crosses this point from above.

¹⁰ In their framework the investor's objective function also involves a possible distortion of the probabilities. We do not consider this in our paper.

If U is a SAHARA utility function with scale parameter $\beta > 0$ and risk aversion parameter $\alpha > 0$ we can easily calculate the convex dual and inverse of marginal utility which are defined at the end of the previous section. Since $A(x)$ is the derivative of $-\ln U'(x)$ one may show by a direct integration that

$$U'(x) = \bar{c}(x + \sqrt{x^2 + \beta^2})^{-\alpha}$$

for a certain constant $\bar{c} > 0$ and that there exists a constant \bar{c}_1 such that $U(x) = \bar{c}_1 + \bar{c}\hat{U}(x)$ with

$$\hat{U}(x) = \begin{cases} -\frac{1}{\alpha^2-1} \left(x + \sqrt{\beta^2 + x^2}\right)^{-\alpha} \left(x + \alpha \sqrt{\beta^2 + x^2}\right) & \alpha \neq 1 \\ \frac{1}{2} \ln(x + \sqrt{\beta^2 + x^2}) + \frac{1}{2} \beta^{-2} x (\sqrt{\beta^2 + x^2} - x) & \alpha = 1 \end{cases} \tag{9}$$

where the domain is \mathbb{R} in both cases. We take $\bar{c}_1 = 0$ and $\bar{c} = 1$ from now on and use $U_{\alpha, \beta}(x)$ to denote $\hat{U}(x)$ in that case.

Both the convex dual $\tilde{U}_{\alpha, \beta}(y) = \sup_{x \in \mathbb{R}} (U_{\alpha, \beta}(x) - xy)$ and the inverse marginal utility function of a SAHARA utility function are easily seen to be a combination of two powers¹¹ of y . The latter one can be written in the explicit form

$$I_{\alpha, \beta}(y) = (U'_{\alpha, \beta})^{-1}(y) = \beta \sinh\left(-\frac{1}{\alpha} \ln y - \ln \beta\right), \tag{10}$$

with domain \mathbb{R}^+ .

3. Solving the optimization problem

The optimization problem of the investor to choose an optimal portfolio and consumption rule can now be stated as¹²

$$V(x_0) := \sup_{(c,A): X^{A,c} \in \mathcal{X}(x_0)} \mathbb{E} \left[\int_0^{T \wedge \tau} \delta_s U_1(c_s) ds + \delta_T U_2(X_T) \mathbf{1}_{\{\tau > T\}} \right], \tag{11}$$

$$U_1(c) = K \frac{c^{1-\gamma}}{1-\gamma}, \quad U_2(x) = U_{\alpha, \beta}(x),$$

for $K, \gamma, \alpha, \beta > 0$. Here δ_t is a subjective discounting function for the investor. It is strictly positive, decreasing and not necessarily Markovian but adapted to the Brownian filtration $(\mathcal{F}_t)_{t \in [0, T]}$, with $\delta_0 = 1$. The constant $K > 0$ allows us to make consumption less or more important than terminal wealth in the total welfare of the investor.

Our definition of strategies which lead to admissible wealth processes allows us to derive an auxiliary result which gives us a so-called *budget constraint*, in analogy to the case where only positive wealth levels can occur.

Lemma 3.1. For every (c, A) such that $X^{A,c} \in \mathcal{X}(x_0)$ we have that

$$\mathbb{E} \left[\int_0^T H_s c_s ds + H_T X_T^{A,c} \right] = x_0. \tag{12}$$

Moreover, for every consumption process c which satisfies the conditions given above and any \mathcal{F}_T -measurable stochastic variable Y which satisfies

$$\mathbb{E} \left[\int_0^T H_s c_s ds + H_T Y \right] = x_0,$$

there exists an investment strategy A such that $X_T^{A,c} = Y$ and $X^{A,c} \in \mathcal{X}(x_0)$.

¹¹ The case $\alpha = 1$ also involves a logarithmic term in the convex dual.

¹² Here and in the sequel, we use the notation $\mathbf{1}_A$ for an indicator function which equals one if A is true and zero if it is not.

3.1. Main result

In a complete market setting, every contingent claim is attainable by using a self-financing trading portfolio when the initial wealth is sufficiently high. The previous lemma shows that this still holds when an appropriate predefined consumption process is included. This means that the dual approach to optimization for our portfolio problem can be extended to the case where risk aversion is no longer assumed to be monotone. It allows us to characterize the optimal strategies in the following theorem.

Theorem 3.2. Assume that $\mathbb{E}[\delta_T^{-2/\alpha}] < \infty$, $\mathbb{E}[\delta_T L(H_T/\delta_T)] < \infty$ for $L(x) := x^{1-1/\alpha} + x^{1+1/\alpha} + \mathbf{1}_{\alpha=1} \ln(x)$ and $\mathbb{E}[\delta_t \tilde{L}(H_t/\delta_t)] < \infty$ for all $t \in [0, T]$ when $\tilde{L}(x) := x^{1-1/\gamma}$. Then there exists a unique constant $c_0 > 0$ such that the optimal consumption and terminal wealth are given by

$$c_t^* = c_0 \left(\frac{\delta_t \mathbb{E}[\zeta_t]}{H_t} \right)^{1/\gamma},$$

$$X_T^* = \beta \sinh \left(\frac{\gamma}{\alpha} \ln c_T^* - \frac{1}{\alpha} \ln K - \ln \beta \right).$$

We see that the SAHARA preference structure allows us to get an explicit solution in this case where the financial market is complete. Note that we have allowed the force of mortality to be stochastic, but the independence of the filtration for the stochastic mortality $(\mathcal{G}_t)_{t \in [0, T]}$ and the filtration for asset prices $(\mathcal{F}_t)_{t \in [0, T]}$ means, in the absence of assets to hedge our mortality risk, that the optimal strategy does not involve the process ζ_t itself but only its expected value $\mathbb{E}[\zeta_t]$.

3.2. The optimal strategies

If we are willing to make some additional assumptions on the structure of our financial market, we can derive an explicit formula for the optimal allocation process A^* . More specifically, we now restrict ourselves to the case where the processes which characterize our financial market and the time preferences of the investor are deterministic, which makes the asset price processes Markovian.

Proposition 3.3. If the processes δ_t , r_t and both $\Sigma_t^{-1} \theta_t$ and θ_t are deterministic then the optimal allocation process has the form

$$A_t^* = \left(\frac{1}{\gamma} f(t) c_t^* + \frac{1}{\alpha} \sqrt{(X_t^* - f(t) c_t^*)^2 + \beta^2 g(t)^2} \right) \Sigma_t^{-1} \theta_t \quad (13)$$

for certain positive deterministic functions f and g with $f(T) = 0$ and $g(T) = 1$. Moreover, if we denote the risk tolerance for our optimized value function V as defined in (11) by $T_0(x_0) := -V'(x_0)/V''(x_0)$, we have that

$$T_0(x_0) = \mathbb{E} \left[\int_0^T H_s T_1(c_s^*) ds + H_T T_2(X_T^*) \right] \quad (14)$$

with $T_1(c) = -U_1'(c)/U_1''(c)$ and $T_2(x) = -U_2'(x)/U_2''(x)$ the risk tolerance functions for consumption and terminal wealth.

The optimal portfolio dictates that one should keep a time-varying part of wealth invested in one fixed fund, and the rest in the bank account, i.e. we find a special version of the usual mutual fund theorem. The percentage invested changes stochastically over time. The form of the optimal portfolio is also more complicated than in the pure power utility case, where we invest a constant proportion of our wealth in risky assets. This special case corresponds to $\beta = 0$ and $\alpha = \gamma$ in the proposition.

In Figs. 1 and 2 optimal consumption and investment in risky assets are shown as a function of wealth at three points in time: $t = 0$, $t = T/2$ and $t = T$. The parameter values used are given in the caption below the figures. We show the cases where terminal

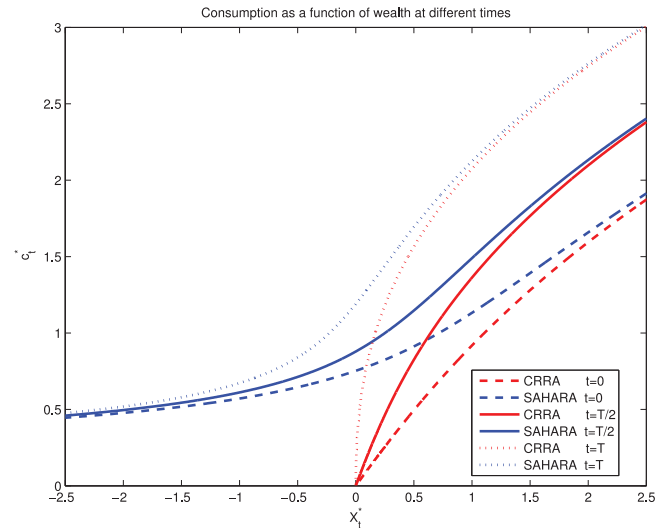


Fig. 1. Consumption as function of wealth for CRRA and SAHARA terminal utility. We assume there are two tradable risky assets and one risk-free asset in the market. The asset price processes considered are $dS_t^1 = S_t^1(\mu_1 dt + \sigma_1 dW_t^1)$, $dS_t^2 = S_t^2(\mu_2 dt + \sigma_2(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2))$. Parameters were chosen to be $X_0 = 1$, $r = 0.05$, $\mu_1 = 0.08$, $\mu_2 = 0.10$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = 0.3$, $T = 1$, $\alpha = 2$, $\beta = \frac{1}{2} X_0$, $K = 10$, $\gamma = 5$, $\delta_t = \exp(-0.03t)$, $\zeta_t = \exp(-t/15)$.

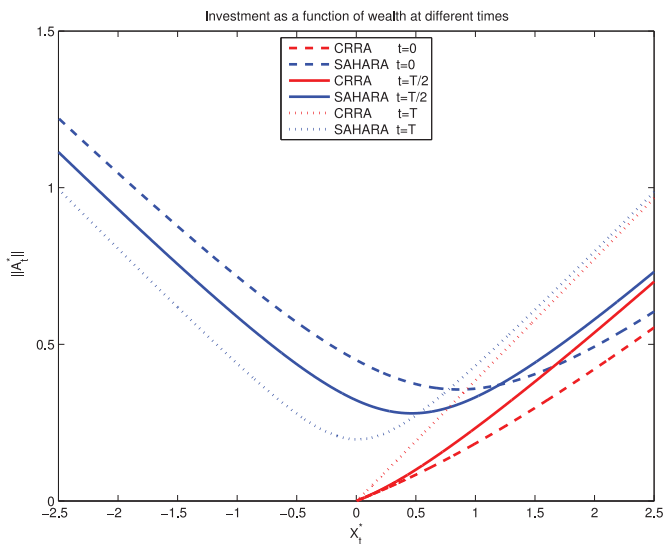


Fig. 2. Investment as a function of wealth for CRRA and SAHARA terminal utility. The asset price processes and parameters were chosen as indicated before.

wealth is characterized by both SAHARA and CRRA utilities and we observe the following.

First, the more wealthy the investor becomes, the more she consumes and the SAHARA-investor always consumes more than the power-investor. The SAHARA-investor is more interested in sustaining a certain consumption level and might consume beyond her means, even when her total wealth is negative. Under favorable economic scenarios, which give rise to a large positive wealth, the consumption pattern of a SAHARA- and a CRRA-investor become similar.

Second, we observe in Fig. 2 that the SAHARA-investor invests more in risky assets than the CRRA-investor (for the same time and wealth levels) in order to achieve the desired higher consumption with a higher probability. There is a unique wealth level where we find a minimal amount of investment in the risky assets and for the parameter values chosen here, this occurs for a positive wealth.

In general the sign of this point depends on the ratio γ/α , as the following result shows.

Corollary 3.4. Assume $\beta > 0$. The minimal amount of investment in risky assets occurs for a unique wealth level \hat{X}_t which is negative if $\alpha > \gamma$, positive if $\alpha < \gamma$ and zero when $\alpha = \gamma$.

Another interesting feature of the CRRA/SAHARA model is the shape of consumption as a function of wealth. This function is usually assumed to be concave¹³. Whether our optimal consumption function can be convex for low wealth levels depends again on the ratio γ/α .

Corollary 3.5. Assume $\beta > 0$. The function for optimal consumption in terms of wealth is convex if $\alpha \geq \gamma$. For $\alpha < \gamma$ there is a point $\hat{X}_t \geq 0$ for every $t \in [0, T]$ such that consumption is convex for wealth levels below that point and concave for wealth levels above that point.

Notice that for $\alpha < \gamma$ we have $\hat{X}_t > 0$ whenever $\beta > 0$ so consumption as a function of wealth always switches between convex and concave at some point. In the CRRA case, where $\beta = 0$, consumption can be convex too as long as risk aversion for consumption is lower than risk aversion for terminal wealth. The occurrence of convex consumption patterns is therefore not a specific feature of the SAHARA utility form alone.¹⁴ If we interpret terminal wealth as a bequest then there is evidence to assume that $\alpha < \gamma$ since relative risk aversion for bequests has been found to be lower than relative risk aversion for consumption.¹⁵ Note, however, that in our model the terminal time is not the time of death. If T is the time of, for example, retirement, then $\alpha < \gamma$ would imply lower risk aversion among the elderly. This has been reported in Bellante and Green (2004).

Figs. 3 and 4 show how the investor should consume and invest for different relationships between the asymptotic risk aversion for consumption (γ) and terminal wealth (α). Moving from $\alpha = 2\gamma$ to $\alpha = \gamma/2$, the investor becomes less risk-averse with respect to the terminal wealth so for a given γ , the total risk aversion decreases, too. As a consequence of a declining α , both CRRA- and SAHARA-investors take more risky assets in their portfolios. Note that Theorem 3.2 shows that consumption at the end of the period c_T^* as a function of terminal wealth X_T^* does not depend on the individual risk aversion parameters α and γ , but only on their ratio. This could perhaps be used to design empirical experiments which test whether this ratio is close to one, as is usual assumed, or that there is a marked difference between the two risk aversion coefficients.

3.3. Development of strategies over time

Fig. 5 shows for a given level of wealth (which is the initial value $X_0 = 1$) how investment in the risky asset varies over time. CRRA preferences with $\alpha = \gamma$ (the red solid line) lead to a constant

¹³ In the formulation of Keynes (1936) in *The General Theory of Employment, Interest and Money* pp. 31: ‘Not only is the marginal propensity to consume weaker in a wealthy community, but, owing to its accumulation of capital being already larger, the opportunities for further investment are less attractive unless the rate of interest falls at a sufficiently rapid rate’.

¹⁴ For a discussion about the curvature of the consumption function in the absence of terminal wealth see Carroll and Kimball (1996).

¹⁵ In Carroll (2000, 2002), the author points out that the bequest function is irrelevant for most of population, but when it is included it should be treated as a luxury good in the sense that the elasticity of intertemporal substitution with respect to bequest is higher than the one for consumption. Assuming power utility functions to characterize the preferences for the bequest and for consumption, this implies that the relative risk aversion for terminal wealth in the form of a bequest is lower than the one for consumption. In Palligkinis (2015), using a dataset of US stockholders drawn from the Health and Retirement Study, the author verifies the results of Carroll (2000, 2002) and finds a lower relative risk aversion for terminal wealth.

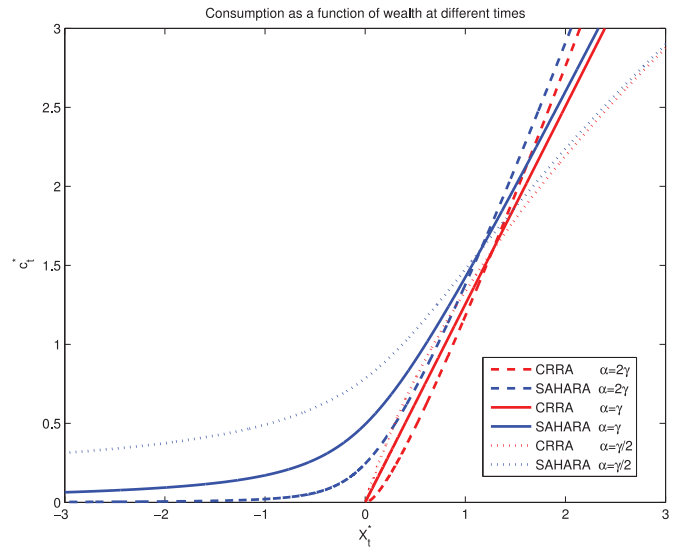


Fig. 3. Consumption as function of wealth for CRRA and SAHARA terminal utility. The asset price processes and parameters were chosen as indicated before, the time of consumption is $t = \frac{1}{2}T$.

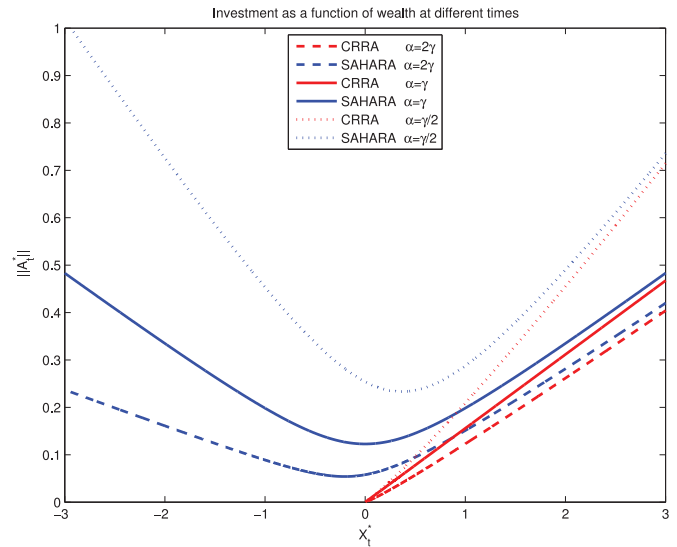


Fig. 4. Investment as a function of wealth for CRRA and SAHARA terminal utility. The asset price processes and parameters were chosen as indicated before, the time of investment is $t = \frac{1}{2}T$.

investment strategy if wealth does not change. This is not consistent with the decreasing investment in stocks over time which is found in the empirical life-cycle literature. Such behavior can be explained using SAHARA preferences. This has been pointed out in work by Bernard and Kwak (2016) for the case without consumption, and they use this property as a motivation to consider utilities with non-monotone risk aversion in the first place. Our current results imply that a decrease in risky investments over the life cycle will still hold when there is a utility for consumption with the same risk aversion parameter ($\alpha = \gamma$), as shown by the solid blue line in Fig. 5.

We also see that if we take risk aversion for terminal wealth smaller than for consumption ($\alpha < \gamma$), equity exposure may increase over time for the CRRA case. For SAHARA utility, we see that the exposure can even be non-monotone over time. Optimal consumption c_t^* will rise over time for a fixed level of wealth, but optimal investment now depends on a term which we can call

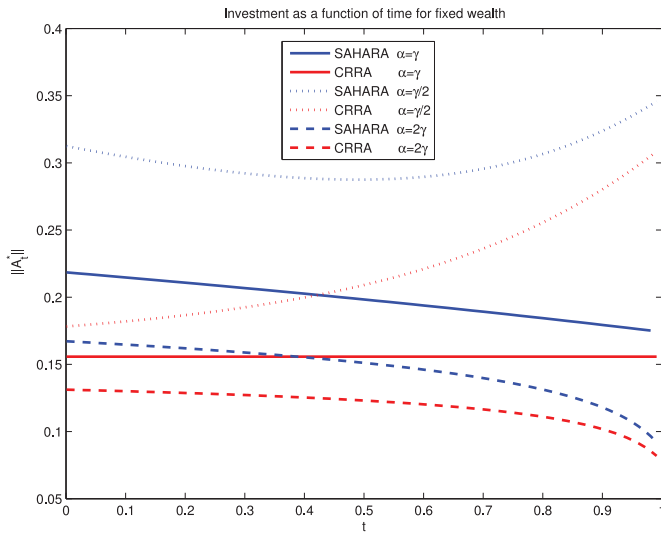


Fig. 5. Investment as a function of time for a fixed wealth level. The asset price processes and parameters were chosen as indicated before.

investment for consumption ($f(t)c_t^*/\gamma$) and a term representing investment for terminal wealth ($\sqrt{(X_t^* - f(t)c_t^*)^2 + \beta^2 g(t)^2}/\alpha$). The function f is decreasing so the combined effect can lead to an allocation of risky investments for consumption which decreases over time and an allocation for terminal wealth which increases over time. The weighting of these two effects varies over time so it can happen that the trend downwards dominates far from the terminal time while the trend upwards dominates towards the end.

Fig. 6 shows this effect by plotting the fraction of wealth that is invested in the risky assets, $\|A_t^*(t, X_t)\|/X_t$, for different values of the risk aversion ratio α/γ (in the graph on the left, where $X_t = 1$) and for different values of wealth X_t (in the graph on the right, where $\alpha/\gamma = \frac{1}{2}$). We see from the second graph that for very low values of wealth, the percentage invested in risky assets decreases over time while for large values of wealth it increases. For wealth around one, it first decreases and then increases. We also note the large equity exposure for low wealth at the beginning of the investment period, which can also be found in the figure of (Bernard & Kwak, 2016) for the optimal investment strategy.

3.4. Development of risk tolerance over time

Eq. (14) for the absolute risk tolerance in Proposition 3.3 allow us to compare risk aversion at earlier and later times. Gollier and Zeckhauser (2002) showed in a finite period consumption model that investors with a longer time horizon are less risk averse if the absolute risk tolerance function for consumption is convex and subhomogeneous.¹⁶ In our setup we have consumption with a risk tolerance function $T_1(x) = x/\gamma$ but also terminal wealth, for which the risk tolerance equals $T_2(x) = \sqrt{x^2 + \beta^2}/\alpha$. Both are convex and subhomogeneous and we find the analogue of Gollier and Zeckhauser’s result in continuous time for the case which includes terminal wealth, since combining (14) with the budget constraint (3.1) we find:

$$T_0(x_0) = \mathbb{E} \left[\int_0^T H_s T_1(c_s^*) ds + H_T T_2(X_T^*) \right] = \mathbb{E} \left[\int_0^T H_s c_s^* ds / \gamma + H_T \sqrt{(X_T^*)^2 + \beta^2} / \alpha \right]$$

¹⁶ A function h is subhomogeneous if for every x in its domain and for every $m \geq 1$ we have that $h(mx) \leq mh(x)$.

$$\geq \min\{1/\gamma, 1/\alpha\} \mathbb{E} \left[\int_0^T H_s c_s^* ds + H_T X_T^* \right] = \min\{1/\gamma, 1/\alpha\} x_0$$

so risk tolerance at time zero is indeed larger than the risk aversion for consumption at later times, and we get an extended result which also involves asymptotic risk tolerance of terminal wealth.

3.5. Parameter sensitivities

When α and γ are equal and interest rates and market prices of risk remain constant, we can derive the optimal consumption and investment strategies in more explicit form.

Corollary 3.6. If $\alpha = \gamma$ and the processes $\Sigma_t^{-1}\theta_t$ and r_t and θ_t are constant and

$$\delta_t = \exp(-\eta t), \quad \mathbb{E}[\zeta_t] = \exp(-\varsigma t),$$

then the optimal consumption and allocation process have the form

$$c_t^* = c_0^* \left(\frac{1}{2} \frac{X_t^*}{x(t)} + \frac{1}{2} \sqrt{\left(\frac{X_t^*}{x(t)} \right)^2 + j(t)} \right)$$

$$A_t^* = \frac{1}{\gamma} \sqrt{(X_t^*)^2 + j(t)x(t)^2} \Sigma_t^{-1}\theta_t$$

with

$$x(t) = c_0^* v_\gamma^{-1} (1 - e^{-v_\gamma(T-t)}) + \frac{1}{2} c_0^* K^{-1/\gamma} e^{-v_\gamma(T-t)}$$

$$v_\gamma := \left(1 - \frac{1}{\gamma} \right) \left(r + \frac{\|\theta\|^2}{2\gamma} \right) + \frac{1}{\gamma} (\eta + \varsigma)$$

$$j(t) := 2\beta^2 K^{1/\gamma} (c_0^*)^{-1} e^{-v_\gamma(T-t)} / x(t)$$

Notice that for $\beta = 0$ we find the classical power utility results since $j(t) \equiv 0$ in that case.

In Fig. 7 we show some parameter sensitivities for this case. The parameter K governs the relative importance of consumption (as opposed to terminal wealth) so making K smaller decreases consumption. Increasing the time preference parameter η (which has the same effect as increasing the mortality rate ς in the optimal solution) increases current consumption and investment, as expected. Consumption becomes smaller when we make the risk aversion parameter for terminal wealth α smaller, since less consumption means that more wealth is saved for the terminal time. If the risk aversion parameter for consumption γ is decreased, consumption hardly changes but investment in risky assets rises for high levels of wealth, i.e. for cases when there is a large probability that we can consume and still have a decent amount of terminal wealth. Making the parameter β smaller brings the SAHARA preference closer to CRRA; we see that for positive wealth values consumption and investment for the two types of preferences become very close to each other, but even for small values of β SAHARA investors may still end up with negative wealth.

4. Comparison of strategies for different preferences

Two investors who have CRRA and SAHARA preferences for terminal wealth respectively but share the same preferences for consumption, will invest differently when they have the same initial capital x_0 at their disposal. We can characterize the difference in their strategies in terms of their terminal total wealth values $X_T^* - X_T^{*,CRRA}$, which in this complete market represents a replicable contingent claim. The claim must have initial value zero since the initial wealth of the two investors is the same. We can characterize it explicitly:

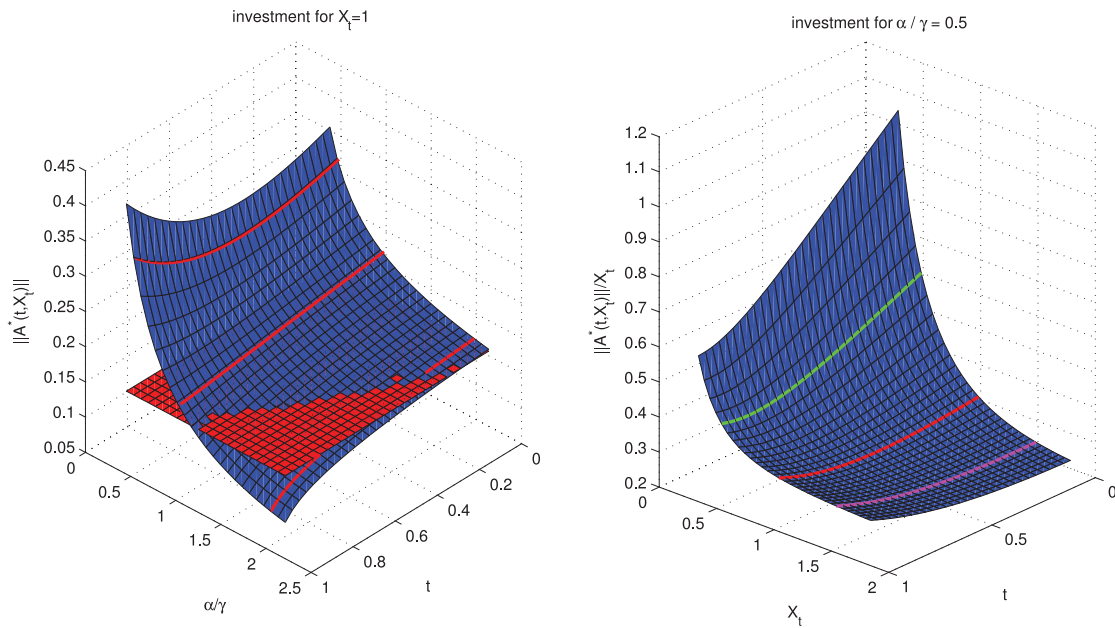


Fig. 6. SAHARA Investment as a function of time and wealth / risk aversion ratio. The asset price processes and parameters were chosen as indicated before. On the left, values for risk aversion ratios 1/2, 1, and 2 have been indicated by red lines and the constant value for CRRA utility is shown as a red surface. On the right we indicate the values at wealth levels 1/2, 1, and 3/2 in green, red and purple. As a function of time they are always decreasing, first decreasing and then increasing, and always increasing, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

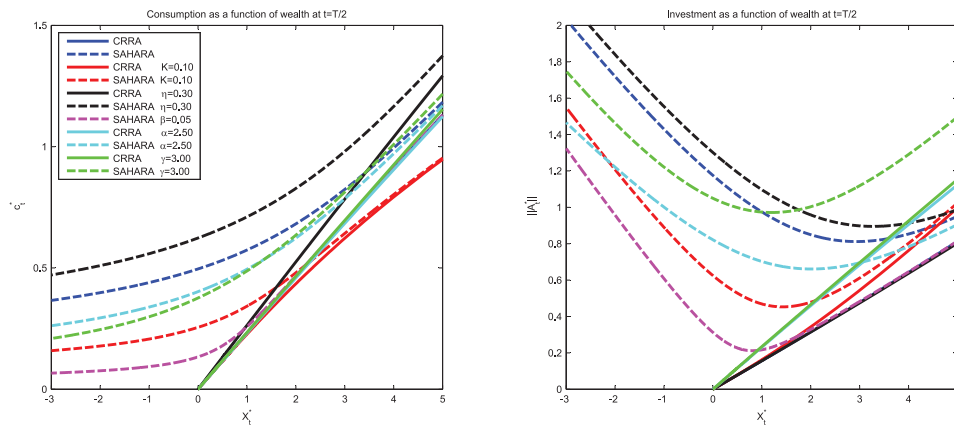


Fig. 7. Parameter sensitivities. Default parameters (corresponding to dark blue lines) are $X_0 = 1$, $r = 0.05$, $\mu_1 = 0.08$, $\mu_2 = 0.10$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = 0.3$, $T = 10$, $\alpha = 2$, $\beta = \frac{1}{2}X_0$, $K = 10$, $\gamma = 5$, $\delta_t = \exp(-\eta t)$, $\zeta_t = \exp(-\varsigma t)$, $\eta = 0.03$, $\varsigma = 1/15$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

Proposition 4.1. Assume that the processes δ_t , $\Sigma_t^{-1}\theta_t$, r_t and θ_t are deterministic. If we define X_T^* as above for a certain $\beta \geq 0$ and $X_T^{*,CRRA}$ as the corresponding solution for $\beta = 0$ then we have

$$X_T^* - X_T^{*,CRRA} = v \cdot \left(\frac{H_T^{-1/\alpha}}{\mathbb{E}H_T^{1-1/\alpha}} - \frac{H_T^{1/\alpha}}{\mathbb{E}H_T^{1+1/\alpha}} \right).$$

for a constant $v \geq 0$ which is given by

$$v = \sqrt{x_0^2 + 4\beta^2 \mathbb{E}H_T^{1-1/\alpha} \mathbb{E}H_T^{1+1/\alpha} - |x_0|}.$$

4.1. Gain-loss preferences

In our approach, risk aversion for terminal wealth is increasing when we approach the point of zero wealth from above, but risk aversion decreases to zero when wealth continues to become more negative. Our investors do therefore not exhibit risk-seeking behavior for negative wealth. It is therefore interesting to compare the behavior of such investors to ours. In Berkelaar et al.

(2004) the problem of maximizing the expected utility of terminal wealth from investment in a complete market is solved for a number of utility functions. These include an example that was proposed by Tversky and Kahneman (1992), from the class of so called gain-loss utility functions which form an important element of cumulative prospect theory¹⁷:

$$U_{TK}(X_T) = u_+(X_T)\mathbf{1}_{\{X_T > 0\}} - u_-(-X_T)\mathbf{1}_{\{X_T \leq 0\}},$$

with u_+ and u_- utility functions which are concave as usual and with $u'_+(0) < u'_-(0)$ to make preferences loss-averse in the sense of (Köberling & Wakker, 2005). In the paper of Tversky and Kahneman, both u_+ and u_- are power functions.

¹⁷ Cumulative prospect theory by Tversky and Kahneman (1992) has been widely used to interpret consumption and investment behaviors. See for example Bowman, Minehart, and Rabin (1999), Dichtl and Drobetz (2011). Also see Eckhoudt, Fiori, Gianin, and R. (2015) for recent results on the influence of loss-averse preferences on portfolio choice.

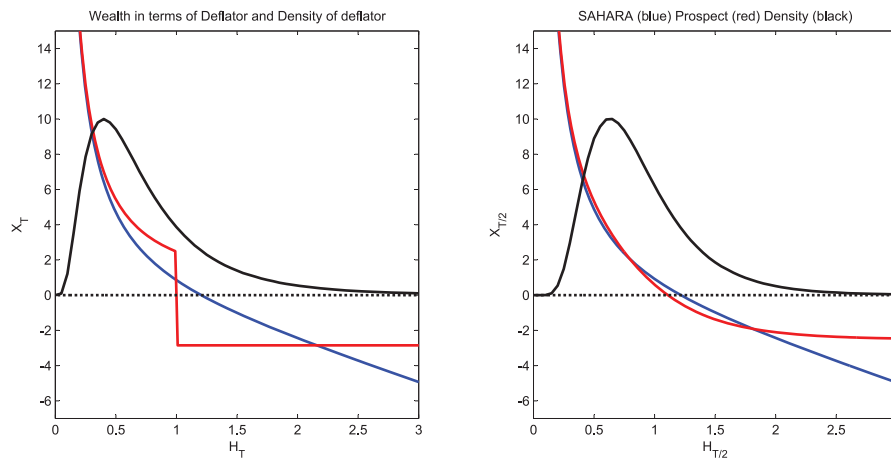


Fig. 8. Wealth as a function of the deflator for SAHARA and gain-loss preferences. Parameters were mostly chosen as before but here $T = 5$, $\beta = 2$ and $\alpha = 0.88$. On the left we show the terminal time $t = T$ while on the right we are halfway through the investment period, so $t = T/2$.

The resulting problem is no longer a concave optimization problem and some restrictions on possible terminal wealth distributions must be imposed to obtain a well-posed problem. In the paper by Berkelaar et al. terminal wealth is restricted to be positive.¹⁸ Another possibility is to use a distortion of the probability measure to model the fact that investors tend to assign more weight to the lefthand (i.e. disadvantageous) tails of probability distributions. Results for that more general case are given in Jin and Zhou (2008) and He and Zhou (2011).

As these papers show, the solutions for such problems of investment without consumption take the general form

$$X_T^* = I_+(yH_T/f(H_T))\mathbf{1}_{\{H_T \leq h\}} - a\mathbf{1}_{\{H_T > h\}}, \tag{15}$$

where I_+ denotes the inverse of $(u_+)'$ and $h > 0$ and $y > 0$ are free parameters that must be chosen optimally. The positive constant a follows from the budget constraint by determining the price of the contingent claim on both sides of the equation at time zero:

$$a = \frac{\mathbb{E}[H_T I_+(yH_T/f(H_T))\mathbf{1}_{\{H_T \leq h\}}] - x_0}{\mathbb{E}[H_T \mathbf{1}_{\{H_T > h\}}]}.$$

The function f depends on the probability distortion and it equals $f \equiv 1$ if there is no such distortion. If wealth is constrained to be positive then a must equal zero.

The gain-loss preference implies risk seeking behavior for negative wealth levels. For the investor it is optimal to invest in a claim which gives a positive payoff in good states of the deflator process (when $H_T \leq h$). The money invested in this claim, with payoff $I_+(yH_T/f(H_T))\mathbf{1}_{\{H_T \leq h\}}$ at time T , exceeds the initial wealth x_0 . This is made possible using the proceeds from selling another claim, which generates a negative payoff in bad states of the world. This claim with payoff $-a\mathbf{1}_{\{H_T > h\}}$ at time T can only attain the values 0 and $-a$ and adding this payoff can thus be interpreted as raising money by selling an insurance product on certain unfavorable outcomes of the deflator process. The extra money is used to create a leveraged position in stocks and bonds.

For SAHARA preferences, a similar interpretation can be given. We can rewrite our solution for that case (without distortion and without consumption, i.e. $K \downarrow 0$) in a similar form¹⁹ to compare the two expressions:

$$X_T^* = I_{\alpha,\beta}(yH_T)\mathbf{1}_{\{H_T \leq (y/\beta)^\alpha\}} - |I_{\alpha,\beta}(yH_T)|\mathbf{1}_{\{H_T > (y/\beta)^\alpha\}}.$$

¹⁸ One could also use the weaker condition that wealth at the final time T must be larger than a certain lower bound with a specified probability; this assumption is made in the paper by Basak and Shapiro (2001) for classical utility functions.

¹⁹ See (25) in the Appendix.

with $I_{\alpha,\beta}$ the inverse marginal utility for a SAHARA function with parameters α and β . For $\beta \downarrow 0$ we converge to the CRRA solution which only includes a positive part but for the general case with $\beta > 0$ we see some differences with the solution for the gain-loss preference structure. Allowing the investor to end up with debts gives her the opportunity to sell a contingent claim that makes her a debtor at time T in bad states, i.e. when $H_T > (y/\beta)^\alpha$. The money earned by this sale increases the amount of money that can be invested to generate payoffs in the good states (where $H_T \leq (y/\beta)^\alpha$).

When we compare the terminal wealth X_T as a function of the final state of the economy H_T for the SAHARA investor and for the loss averse investor, i.e. an investor with an S-shaped utility function, we see a marked difference. The loss averse investor will create a leveraged portfolio with a terminal payoff that is discontinuous around a certain terminal state H_T , meaning that when the economy ends up around that state when time approaches the horizon T , there is a lot of uncertainty about the eventual wealth. The investor will either have a positive wealth or a substantially negative wealth. For the SAHARA case, the debt she owes in bad states and the amount she owns in good states are linked in a continuous fashion. We now have different payoffs in the different bad states $H_T > (y/\beta)^\alpha$, instead of just the fixed values $-a$ and zero. This is a result of our milder behavior for negative wealth: the S-shaped utility implies risk-seeking behavior for negative wealth which implies that the large debt $-a$ is preferred over the smoother profile of the SAHARA case, where investors prefer to be mildly in debt instead of having the possibility to end up heavily indebted.

The optimal terminal wealth as a function of the deflator is thus continuous for SAHARA preferences while it is discontinuous for gain-loss utility functions in the point $H_T = h$. But since the wealth dynamics follow a diffusion process, optimal wealth at earlier times, X_t^* , will be a continuous function of the deflator H_t at that time. As was shown earlier in the work of Berkelaar et al. (2004), the optimal strategies may be quite similar when the terminal time is still far away, even when the terminal wealth profile is different. As an illustration we show in Fig. 8 the optimal wealth as a function of the deflator H_t for $t = T$ and $t = \frac{1}{2}T$ for both preferences. We also plot the density of the deflator at these times to show the probabilities of reaching certain outcomes.

Since the two preferences have the same asymptotic risk aversion at high wealth levels we see that the strategies are very close in the most favorable scenario's (the good states represented by low values of the deflator) but we also see that they differ in very

bad states. While the losses under SAHARA preferences can be unbounded, the investor with gain-loss utility makes sure she never goes below the fixed level $-a$. From the probability density of the deflator H (represented by the black line in Fig. 8) we see that when we are halfway through the optimization period at $t = \frac{1}{2}T$, the strategies are quite close for the most relevant bad outcomes, i.e. for states where the distribution of the deflator has a lot of probability mass.

5. Conclusion

The present paper solves an optimal consumption and investment problem in a multi-asset financial market that explicitly incorporates negative terminal wealth. We assume that the optimizing investor has CRRA preferences for consumption and SAHARA utility for terminal wealth. Relying on the dual approach to stochastic optimization, we are able to derive explicit solutions for the optimal consumption, wealth and investment strategies. It turns out that the qualitative behavior of these strategies critically depends on the ratio of the (asymptotic) risk aversion coefficients for consumption and terminal wealth. Interestingly, we find that for certain values of these parameters the consumption as a function of wealth may be convex and concave at different wealth levels. Moreover, optimal investment may not be monotone as a function of the remaining time to invest and consume.

Comparing our strategies with those generated by gain-loss preferences, we conclude that in the absence of consumption we find very similar investment strategies as long as the terminal time is still far away. Closer to the terminal time, the changes in the latter preferences from risk aversion to risk seeking behavior at zero wealth makes the strategies different in the worst scenarios of the economy since SAHARA investors do not constrain themselves to keep wealth above a certain fixed value.

It would be interesting to compare our results to the case where consumption is included in the optimal investment problem under gain-loss preferences. Results for utility functions of consumption that are kinked or where both u_+ and u_- are power functions have been derived in van Bilsen (2013) but no terminal wealth has been included there. All wealth is therefore consumed over the time horizon and there is no trade-off between consumption during, and a possible debt after, the time period considered. It would be interesting to see if similar results as in our case would be obtained if both consumption and terminal wealth may generate utility, since we have seen in this paper that it is the relative strength of the asymptotic risk aversion in these two components that determines the shape of the optimal strategies for investment and consumption.

Finally, we remark that for SAHARA functions the unique point of maximal risk aversion directly translates into a unique wealth level which corresponds to minimal investment, as shown in Corollary 3.4. It will be interesting to study whether such results can be extended to preferences which exhibit a more complex shape, such as the ones used in the tri-reference point theory of Wang and Johnson (2012).

Acknowledgment

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Appendix

Proof. Lemma 2.1

Denote $Y_t = Y_t^{A,c}$. Due to our assumptions on Σ there exists a constant σ_{max} such that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[Y_T^2] &= \mathbb{E}^{\mathbb{Q}} \left[\int_0^t \|A'_u \Sigma_u / B_u\|^2 du \right] \\ &\leq \sigma_{max}^2 \mathbb{E}^{\mathbb{Q}} \left[\int_0^t \bar{K} (1 + (X_u^{A,c})^2) B_u^{-2} du \right] \\ &\leq \bar{K} \sigma_{max}^2 \mathbb{E}^{\mathbb{Q}} \left[\int_0^t \left[B_u^{-2} + Y_u^2 + \left(x_0 - \int_0^t \frac{c_s}{B_s} ds \right)^2 \right] du \right] \\ &= \hat{K} + \tilde{K} \sigma_{max}^2 \int_0^t \mathbb{E}^{\mathbb{Q}}[Y_u^2] du \end{aligned} \tag{16}$$

for a.s. finite stochastic variables \hat{K} and \tilde{K} , where we have used the definition of $Y_t^{A,c}$ in (6) and the fact that the process r is bounded. This gives, by Gronwall's lemma, that $\mathbb{E}^{\mathbb{Q}}[Y_T^2]$ is finite. For every (\mathcal{F}_t) -stopping time τ we have that $\mathbb{E}^{\mathbb{Q}}[Y_{t \wedge \tau}^2] \leq \mathbb{E}^{\mathbb{Q}}[Y_T^2]$ so the collection of stochastic variables $(Y_{t \wedge \tau})$, for all $t \in [0, T]$ and over all stopping times τ , is uniformly integrable. But this means Y is a \mathbb{Q} -martingale and A is martingale-generating. \square

Proof. Lemma 3.1

The first statement is immediate: let $X^{A,c} \in \mathcal{X}(x_0)$ and define Y_t as in (6) then we have by the Bayes Rule that

$$\begin{aligned} \mathbb{E} \left[\int_0^T H_s c_s ds + H_T X_T^{A,c} \right] &= B_0 \mathbb{E}^{\mathbb{Q}} \left[\int_0^T (c_s / B_s) ds + (X_T^{A,c} / B_T) \right] \\ &= x_0 + B_0 \mathbb{E}^{\mathbb{Q}}[Y_T] = x_0. \end{aligned}$$

For the converse, assume c and Y given as indicated and define the process

$$\bar{X}_t = \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (c_s B_t / B_s) ds + Y B_t / B_T \mid \mathcal{F}_t \right].$$

We then take

$$M_t := (\bar{X}_t / B_t) + \int_0^t (c_s / B_s) ds = \mathbb{E}^{\mathbb{Q}} \left[\int_0^T (c_s / B_s) ds + Y / B_T \mid \mathcal{F}_t \right],$$

which is a \mathbb{Q} -martingale with respect to the Brownian filtration starting at $M_0 = x_0$. By martingale representation there therefore exists an adapted process \tilde{A} such that $M_t = x_0 + \int_0^t \tilde{A}_u^t dW_u^{\mathbb{Q}}$. But taking $A_t = B_t \Sigma_t^{-1} \tilde{A}_t$ then gives, using (6),

$$\begin{aligned} X_t^{A,c} &= B_t (x_0 + \int_0^t (A'_u \Sigma_u / B_u) dW_u^{\mathbb{Q}} + \int_0^t (c_u / B_u) du) \\ &= B_t (M_t + \int_0^t (c_u / B_u) du) = \bar{X}_t. \end{aligned}$$

So $X_T^{A,c} = \bar{X}_T = Y$ as required and since $Y_t^{A,c} = M_t - x_0$ we see that A is indeed martingale-generating. \square

Proof. Theorem 3.2

Using Fubini's theorem we write

$$\begin{aligned} &\mathbb{E} \left[\int_0^{T \wedge \tau} \delta_s U_1(c_s) ds + \delta_T U_2(X_T) \mathbf{1}_{\{\tau > T\}} \right] \\ &= \mathbb{E} \left[\int_0^T \delta_s U_1(c_s) \mathbf{1}_{\{\tau > s\}} ds + \delta_T U_2(X_T) \mathbf{1}_{\{\tau > T\}} \right] \\ &= \mathbb{E} \left[\int_0^T \mathbb{E}[\delta_s U_1(c_s) \mathbf{1}_{\{\tau > s\}} \mid \mathcal{F}_T] ds + \mathbb{E}[\delta_T U_2(X_T) \mathbf{1}_{\{\tau > T\}} \mid \mathcal{F}_T] \right] \\ &= \mathbb{E} \left[\int_0^T \delta_s (\mathbb{E}[\zeta_s]) U_1(c_s) ds + \delta_T (\mathbb{E}[\zeta_T]) U_2(X_T) \right], \end{aligned} \tag{17}$$

where we have used the independence of the process $\mathbf{1}_{\{\tau > t\}}$ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and the fact that both δ and c are adapted to that filtration. Together with the previous lemma this proves that the optimal (c, A) in the optimization problem (11) coincides with the optimal strategy (c, A) of

$$V(x_0) = \max_{(c_u)_{u \in [0, T]}, X_T \in m\mathcal{F}_T} \left(\mathbb{E} \left[\int_0^T \delta_s(\mathbb{E}[\zeta_s]) U_1(c_s) ds \right] + (\mathbb{E}[\zeta_T]) \mathbb{E}[\delta_T U_2(X_T)] \right) \tag{18}$$

s.t. $\mathbb{E} \left[\int_0^T H_t c_t dt + H_T X_T \right] = x_0,$

where $m\mathcal{F}_T$ is the class of all \mathcal{F}_T -measurable stochastic variables and the consumption process c must satisfy the same conditions as before.

By the definition of the convex dual \tilde{U}_i , we have

$$\mathbb{E} \left[\int_0^T \delta_s(\mathbb{E}[\zeta_s]) U_1(c_s) ds \right] + (\mathbb{E}[\zeta_T]) \mathbb{E}[\delta_T U_2(X_T)] + \lambda \left(x_0 - \mathbb{E} \left[\int_0^T H_s c_s ds + H_T X_T \right] \right) \leq \lambda x_0 + \mathbb{E} \left[\int_0^T \delta_s \tilde{U}_1(\lambda H_s / (\delta_s \mathbb{E}[\zeta_s])) ds + \delta_T \tilde{U}_2(\lambda H_T / (\delta_T \mathbb{E}[\zeta_T])) \right], \tag{19}$$

for a Lagrange multiplier λ , with equality if and only if

$$c_t = I_1(\lambda H_t / (\delta_t \mathbb{E}[\zeta_t])) = (\lambda H_t / (K \delta_t \mathbb{E}[\zeta_t]))^{-1/\gamma} \tag{19}$$

$$X_T = I_2(\lambda H_T / (\delta_T \mathbb{E}[\zeta_T])) = I_2((I_1)^{-1}(c_T)) = I_2(U_1^{-1}(c_T)) = I_2(K(c_T)^{-\gamma}). \tag{20}$$

Substitution of these expressions in the utility functions U_1 and U_2 shows that there exist constants K_X and K_c such that $|U_2(X_T)| \leq K_X L(H_T/\delta_T)$ and $|U_1(c_t)| \leq K_c \tilde{L}(H_t/\delta_t)$ so $\mathbb{E}[\delta_T U_2(X_T)]$ and $\mathbb{E}[\delta_t U_1(c_t)]$ are finite for all $t \in [0, T]$ by the assumptions we made in the statement of the Theorem. From (10) we see that it only remains to show that there is a value $\lambda > 0$ such that the budget constraint is satisfied since the previous lemma shows that one can then find a martingale-generating investment strategy to generate this terminal wealth X_T . Notice that $c_0 = (\lambda/K)^{-1/\gamma}$ so the constraint can be rewritten as

$$x_0 = \mathbb{E} \left[\int_0^T H_s c_s ds + H_T X_T \right] = \mathbb{E} \left[\int_0^T H_s c_s ds + H_T I_2(K(c_T)^{-\gamma}) \right] = \mathbb{E} \left[c_0 \int_0^T H_s^{1-1/\gamma} (\delta_s \mathbb{E}[\zeta_s])^{1/\gamma} ds + \frac{1}{2} H_T^{1-1/\alpha} (\delta_T \mathbb{E}[\zeta_T]/K)^{1/\alpha} (c_0)^{\gamma/\alpha} - \frac{1}{2} \beta^2 H_T^{1+1/\alpha} (\delta_T \mathbb{E}[\zeta_T]/K)^{-1/\alpha} (c_0)^{-\gamma/\alpha} \right].$$

Since r and θ are uniformly bounded, $\mathbb{E}[H_T^m]$ exists for all $m \in \mathbb{R}$. The process δ_t decreases while $\delta_0 = 1$ so the condition $\mathbb{E}[(\delta_T)^{-2/\alpha}] < \infty$ guarantees that the righthand-side is a finite expression for all $c_0 > 0$ by the Cauchy-Schwarz inequality. The righthand side is clearly an increasing function of c_0 , which goes to $-\infty$ for $c_0 \downarrow 0$ and to ∞ for $c_0 \rightarrow \infty$ so there must exist a unique positive value of c_0 such that the constraint is satisfied. \square

Proof. Proposition 3.3

Since $d(H^n)/(H^n) = n(dH/H) + \frac{1}{2}n(n-1)d\langle H, H \rangle / (H^2)$ we can define for all $s \geq t$ the functions

$$h_n(t, s) := \mathbb{E}[(H_s/H_t)^n | \mathcal{F}_t] = \exp \left(n \int_t^s (-r_u + \frac{1}{2}(n-1)\|\theta_u\|^2) du \right)$$

and these are deterministic. From Lemma 3.1 we know that

$$\begin{aligned} X_t^* &= \frac{1}{H_t} \mathbb{E} \left[\int_t^T H_s c_s^* ds + H_T X_T^* \middle| \mathcal{F}_t \right] \\ &= \frac{1}{H_t} \left[c_0^* \int_t^T \mathbb{E}[H_s^{1-1/\gamma} | \mathcal{F}_t] (\delta_s \mathbb{E}[\zeta_s])^{1/\gamma} ds + \frac{1}{2} \mathbb{E}[H_T^{1-1/\alpha} | \mathcal{F}_t] (\delta_T \mathbb{E}[\zeta_T]/K)^{1/\alpha} (c_0^*)^{\gamma/\alpha} \right. \\ &\quad \left. - \frac{1}{2} \beta^2 \mathbb{E}[H_T^{1+1/\alpha} | \mathcal{F}_t] (\delta_T \mathbb{E}[\zeta_T]/K)^{-1/\alpha} (c_0^*)^{-\gamma/\alpha} \right] \\ &= c_0^* H_t^{-1/\gamma} \int_t^T h_{1-1/\gamma}(t, s) (\delta_s \mathbb{E}[\zeta_s])^{1/\gamma} ds + \frac{1}{2} H_t^{-1/\alpha} h_{1-1/\alpha}(t, T) (\delta_T \mathbb{E}[\zeta_T]/K)^{1/\alpha} (c_0^*)^{\gamma/\alpha} \\ &\quad - \frac{1}{2} \beta^2 H_t^{1/\alpha} h_{1+1/\alpha}(t, T) (\delta_T \mathbb{E}[\zeta_T]/K)^{-1/\alpha} (c_0^*)^{-\gamma/\alpha} \\ &:= A_1(t, T) H_t^{-1/\gamma} + A_2(t, T) H_t^{-1/\alpha} - A_3(t, T) H_t^{1/\alpha}, \end{aligned} \tag{21}$$

for deterministic functions A_i which are defined in the last equality and which are positive for all $t \leq T$. Applying Itô's Lemma we see that there exists an adapted process L_t such that

$$dX_t^* = L_t dt + \left(\frac{-1}{\gamma} A_1(t, T) H_t^{-1/\gamma} + \frac{-1}{\alpha} A_2(t, T) H_t^{-1/\alpha} + \frac{-1}{\alpha} A_3(t, T) H_t^{1/\alpha} \right) (-\theta_t' dW_t).$$

On the other hand, dX_t^* satisfies (5) so $dX_t^* = L_t dt + (A_t^*)' \Sigma_t dW_t$. This gives, by uniqueness in the martingale representation, that

$$A_t^* = \left(\frac{1}{\gamma} A_1(t, T) H_t^{-1/\gamma} + \frac{1}{\alpha} [A_2(t, T) H_t^{-1/\alpha} + A_3(t, T) H_t^{1/\alpha}] \right) \Sigma_t^{-1} \theta_t. \tag{22}$$

But since $H_t^{-1/\gamma} = (c_t^*/c_0^*) (\delta_t \mathbb{E}[\zeta_t])^{-1/\gamma}$ we can write $(1/\gamma) A_1(t, T) H_t^{-1/\gamma} = (1/\gamma) f(t) c_t^*$ for the positive deterministic function f given by

$$f(t) = \int_t^T h_{1-1/\gamma}(t, s) \left(\frac{\delta_s \mathbb{E}[\zeta_s]}{\delta_t \mathbb{E}[\zeta_t]} \right)^{1/\gamma} ds = \int_t^T e^{-(1-\frac{1}{\gamma}) \int_t^s (r_u + \frac{1}{2} \|\theta_u\|^2) du} \left(\frac{\delta_s \mathbb{E}[\zeta_s]}{\delta_t \mathbb{E}[\zeta_t]} \right)^{1/\gamma} ds$$

and by (21)

$$\begin{aligned} &\frac{1}{\alpha} (A_2(t, T) H_t^{-1/\alpha} + A_3(t, T) H_t^{1/\alpha})^2 \\ &= \frac{1}{\alpha} (A_2(t, T) H_t^{-1/\alpha} - A_3(t, T) H_t^{1/\alpha})^2 + \frac{4}{\alpha} A_2(t, T) A_3(t, T) \\ &= \frac{1}{\alpha} (X_t^* - f(t) c_t^*)^2 + \frac{4}{\alpha} A_2(t, T) A_3(t, T). \end{aligned}$$

The choice

$$g(t) = 2\beta^{-1} \sqrt{A_2(t, T) A_3(t, T)} = \exp \left(- \int_t^T (r_u - \frac{1}{2} \|\theta_u\|^2 / \alpha^2) du \right)$$

then proves the first result.

For the second result we first differentiate (12) with respect to the initial wealth x_0 to find that

$$1 = \mathbb{E} \left[\int_0^T H_s \frac{\partial c_s^*}{\partial x_0} ds + H_T \frac{\partial X_T^*}{\partial x_0} \right] \stackrel{(19)-(20)}{=} \mathbb{E} \left[\int_0^T H_s \frac{\partial \lambda}{\partial x_0} \frac{H_s}{\delta_s \mathbb{E}[\zeta_s]} I_1' \left(\frac{\lambda H_s}{\delta_s \mathbb{E}[\zeta_s]} \right) ds \right]$$

$$\begin{aligned}
 & + H_T \frac{\partial \lambda}{\partial x_0} \frac{H_T}{\delta_T \mathbb{E}[\zeta_T]} \left[\frac{\lambda H_T}{\delta_T \mathbb{E}[\zeta_T]} \right] \\
 = & -\frac{1}{\lambda} \frac{\partial \lambda}{\partial x_0} \mathbb{E} \left[\int_0^T H_s T_1(c_s^*) ds + H_T T_2(X_T^*) \right] \tag{23}
 \end{aligned}$$

where we have used that for all utility functions U with derivatives U' and their inverses $l = (U')^{-1}$ we may show by simple differentiation that $xl'(x) = -T(I(x))$. This shows that we are done when we prove that $\lambda = V'(x_0)$. This follows from differentiating (18) with respect to x_0 since this implies that

$$\begin{aligned}
 V'(x_0) & := \mathbb{E} \left[\int_0^T \delta_s \mathbb{E}[\zeta_s] U'_1(c_s^*) \frac{\partial c_s^*}{\partial x_0} ds + \delta_T \mathbb{E}[\zeta_T] U'_2(X_T^*) \frac{\partial X_T^*}{\partial x_0} \right] \\
 & \stackrel{(19)-(20)}{=} \mathbb{E} \left[\int_0^T \lambda H_s \frac{\partial c_s^*}{\partial x_0} ds + \lambda H_T \frac{\partial X_T^*}{\partial x_0} \right] = \lambda
 \end{aligned}$$

where we have used (23) again in the last step. \square

Proof. Corollary 3.4

Finding the minimum means calculating

$$\begin{aligned}
 0 & = \frac{\partial}{\partial H_t} (\gamma^{-1} A_1(t, T) H_t^{-1/\gamma} + \alpha^{-1} A_2(t, T) H_t^{-1/\alpha} + \alpha^{-1} A_3(t, T) H_t^{1/\alpha}) \\
 & = -H_t^{-1} (\gamma^{-2} A_1(t, T) H_t^{-1/\gamma} + \alpha^{-2} A_2(t, T) H_t^{-1/\alpha} - \alpha^{-2} A_3(t, T) H_t^{1/\alpha}) \\
 & = -H_t^{-1} \alpha^{-2} (X_t^* + (\alpha^2 \gamma^{-2} - 1) H_t^{-1/\gamma} A_1(t, T))
 \end{aligned}$$

which gives the result since H_t and $A_1(t, T)$ are both positive. \square

Proof. Corollary 3.5

We can write $H_t^{-1/\gamma} = c_t^* m(t)$ for the strictly positive deterministic function $m(t) = (c_0^*)^{-1} (\delta_t \mathbb{E}[\zeta_t])^{-1/\gamma}$ and $X_t^* = A_1(t, T) c_t^* m(t) + A_2(t, T) [c_t^* m(t)]^{\gamma/\alpha} - A_3(t, T) [c_t^* m(t)]^{-\gamma/\alpha}$. Twice differentiating the righthand side with respect to c_t^* and making the result strictly positive shows us where the consumption function is strictly concave (since consumption increases in wealth). This gives

$$\begin{aligned}
 0 & < \frac{\gamma}{\alpha} \left(\frac{\gamma}{\alpha} - 1 \right) A_2(t, T) (c_t^*)^{\gamma/\alpha - 2} m(t)^{\gamma/\alpha} \\
 & - \frac{\gamma}{\alpha} \left(\frac{\gamma}{\alpha} + 1 \right) A_3(t, T) (c_t^*)^{-\gamma/\alpha - 2} m(t)^{-\gamma/\alpha} \\
 & = \frac{\gamma}{\alpha^2} A_2(t, T) (c_t^*)^{-\gamma/\alpha - 2} m(t)^{-\gamma/\alpha} ((\gamma - \alpha) [c_t^* m(t)]^{2\gamma/\alpha} \\
 & - (\gamma + \alpha) A_3(t, T) / A_2(t, T)).
 \end{aligned}$$

This proves that for $\alpha \geq \gamma$ consumption is never concave while for $\alpha < \gamma$ the function changes from convex to concave when crossing from below to above consumption level

$$\begin{aligned}
 \hat{c}_t & = m(t)^{-1} \left[\frac{(\gamma + \alpha) A_3(t, T)}{(\gamma - \alpha) A_2(t, T)} \right]^{\frac{\alpha}{2\gamma}} = \left[\beta^2 \frac{(\gamma + \alpha)}{(\gamma - \alpha)} \frac{h_{1+1/\alpha}(t, T)}{h_{1-1/\alpha}(t, T)} \right]^{\frac{\alpha}{2\gamma}} \\
 & = \beta^{\frac{\alpha}{\gamma}} \left(\frac{\gamma + \alpha}{\gamma - \alpha} \right)^{\frac{\alpha}{2\gamma}} e^{-\frac{1}{\gamma} \int_t^T (r_u - \frac{1}{2} \|\theta_u\|^2) du},
 \end{aligned}$$

which corresponds to a unique value $\hat{X}_t > 0$ when $\beta > 0$ and gives $\hat{X}_t = 0$ for $\beta = 0$. \square

Proof. Corollary 3.6

From (21) we know that when $\gamma = \alpha$ we have

$$X_t^* = [A_1(t, T) + A_2(t, T)] H_t^{-1/\gamma} - A_3(t, T) H_t^{1/\gamma}$$

and this equation can be solved to give a solution for $H_t^{-1/\gamma}$:

$$\begin{aligned}
 H_t^{-1/\gamma} & = \frac{1}{2} \frac{X_t^*}{A_1(t, T) + A_2(t, T)} \\
 & + \frac{1}{2} \sqrt{\left(\frac{X_t^*}{A_1(t, T) + A_2(t, T)} \right)^2 + \frac{4A_3(t, T)}{A_1(t, T) + A_2(t, T)}}
 \end{aligned}$$

but $c_t^* = c_0^* (\delta_t \mathbb{E}[\zeta_t])^{1/\gamma} H_t^{-1/\gamma}$ so we have

$$\begin{aligned}
 c_t^* / c_0^* & = \frac{1}{2} \frac{X_t^*}{(A_1(t, T) + A_2(t, T)) (\delta_t \mathbb{E}[\zeta_t])^{-1/\gamma}} \\
 & + \frac{1}{2} \sqrt{\frac{\left(\frac{X_t^*}{(A_1(t, T) + A_2(t, T)) (\delta_t \mathbb{E}[\zeta_t])^{-1/\gamma}} \right)^2}{4A_3(t, T) (\delta_t \mathbb{E}[\zeta_t])^{1/\gamma}} + \frac{1}{(A_1(t, T) + A_2(t, T)) (\delta_t \mathbb{E}[\zeta_t])^{-1/\gamma}}} } \\
 & = \frac{1}{2} \frac{X_t^*}{x(t)} + \frac{1}{2} \sqrt{\left(\frac{X_t^*}{x(t)} \right)^2} + j(t) \tag{24}
 \end{aligned}$$

if we define

$$\begin{aligned}
 x(t) & = (A_1(t, T) + A_2(t, T)) (\delta_t \mathbb{E}[\zeta_t])^{-1/\gamma} \\
 & = c_0^* \nu_\gamma^{-1} (1 - e^{-\nu_\gamma (T-t)}) + \frac{1}{2} c_0^* K^{-1/\gamma} e^{-\nu_\gamma (T-t)} \\
 j(t) & = 4A_3(t, T) (\delta_t \mathbb{E}[\zeta_t])^{1/\gamma} / x(t) = 2\beta^2 K^{1/\gamma} (c_0^*)^{-1} e^{-\nu_\gamma (T-t)} / x(t)
 \end{aligned}$$

and the first result follows. For the second result we see from (21), (22) and (24) that when $\alpha = \gamma$ that

$$\begin{aligned}
 A_t^* & = \gamma^{-1} (X_t^* + 2A_3(t, T) / H_t^{-1/\gamma}) \Sigma_t^{-1} \theta_t \\
 & = \gamma^{-1} \left(X_t^* + \frac{1}{2} j(t) x(t) \frac{c_0^*}{c_t^*} \right) \Sigma_t^{-1} \theta_t \\
 & = \gamma^{-1} \left(X_t^* + \frac{j(t) x(t)^2}{X_t^* + \sqrt{(X_t^*)^2 + j(t) x(t)^2}} \right) \Sigma_t^{-1} \theta_t \\
 & = \frac{1}{\gamma} \sqrt{(X_t^*)^2 + j(t) x(t)^2} \Sigma_t^{-1} \theta_t
 \end{aligned}$$

where we have used once more that $c_t^* = c_0^* (\delta_t \mathbb{E}[\zeta_t])^{1/\gamma} H_t^{-1/\gamma}$. \square

Proof. Proposition 4.1

Substitution of $t = T$ in expression (21) for X_t^* with $\beta \geq 0$ shows that X_T^* takes the form

$$X_T^* = y H_T^{-1/\alpha} - \beta^2 y^{-1} H_T^{1/\alpha} \tag{25}$$

for a certain deterministic value $y \geq 0$ which should thus satisfy the equation

$$x_0 = y \mathbb{E} H_T^{1-1/\alpha} - \beta^2 y^{-1} \mathbb{E} H_T^{1+1/\alpha},$$

which has the solution

$$\begin{aligned}
 y & = \frac{1}{2} (\mathbb{E} H_T^{1-1/\alpha})^{-1} (x_0 + \sqrt{x_0^2 + 4\beta^2 \mathbb{E} H_T^{1-1/\alpha} \mathbb{E} H_T^{1+1/\alpha}}) \\
 y^{-1} & = \frac{1}{2} (\mathbb{E} H_T^{1+1/\alpha})^{-1} \beta^{-2} (-x_0 + \sqrt{x_0^2 + 4\beta^2 \mathbb{E} H_T^{1-1/\alpha} \mathbb{E} H_T^{1+1/\alpha}}).
 \end{aligned}$$

But then

$$\begin{aligned}
 X_T^* & = \left(\sqrt{x_0^2 + 4\beta^2 \mathbb{E} H_T^{1-1/\alpha} \mathbb{E} H_T^{1+1/\alpha}} + x_0 \right) \frac{H_T^{-1/\alpha}}{\mathbb{E} H_T^{1-1/\alpha}} \\
 & + \left(-\sqrt{x_0^2 + 4\beta^2 \mathbb{E} H_T^{1-1/\alpha} \mathbb{E} H_T^{1+1/\alpha}} + x_0 \right) \frac{H_T^{1/\alpha}}{\mathbb{E} H_T^{1+1/\alpha}}.
 \end{aligned}$$

Subtracting the same expression for the special case where $\beta = 0$ then gives the result. \square

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