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Coherent cascade conjecture for collapsing solutions in global AdSBen Freivogel^{1,*} and I-Sheng Yang^{2,†}¹*GRAPPA and ITFA, Institute of Physics, Universiteit van Amsterdam,
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We analyze the gravitational dynamics of a classical scalar field coupled to gravity in asymptotically AdS spacetime, which leads to black hole formation on the shortest nonlinear time scale for some initial conditions. We show that the observed collapse cannot be described by the well-known process of a random-phase cascade in the theory of weak turbulence. This implies that the dynamics on this time scale is highly sensitive to the phases of modes. We explore the alternative possibility of a coherent phase cascade and analytically find stationary solutions with completely coherent phases and power-law energy spectra. We show that these power-law spectra lead to diverging geometric backreaction, which is the likely precursor to black hole formation. In $4 + 1$ dimensions, our stationary solution has the same power-law energy spectrum as the final state right before collapse observed in numerical simulations. We conjecture that our stationary solutions describe the system shortly before collapse in other dimensions, and predict the energy spectrum.

DOI: [10.1103/PhysRevD.93.103007](https://doi.org/10.1103/PhysRevD.93.103007)**I. INTRODUCTION AND SUMMARY**

The nonlinear stability of global anti-de Sitter (AdS) spacetime has received increasing attention in the past few years. The other two maximally symmetric spacetimes, Minkowski and de Sitter, have been shown to be stable [1,2]. Intuitively speaking, the main difference is that in those cases, energy can escape to infinity. Global AdS spacetime, however, comes naturally with reflecting boundary conditions. Any initial perturbation, no matter how small, is confined to gravitationally interact with itself, effectively in a finite region, forever. Therefore it is more likely for the energy distribution to become highly uneven and backreact strongly on the metric. The long-term outcome of such nonlinear dynamics is difficult to predict, which resulted in the richness of the current AdS (in) stability problem [3–34].

Four years ago, Bizon and Rostworowski presented interesting numerical results that spurred recent developments [3]. They showed that initial perturbations with a small amplitude ϵ collapse to a black hole on the nonlinear time scale, $T \sim \epsilon^{-2}$. This result is very interesting. Perturbation theory guarantees that the instability cannot develop at any time scale shorter than $T \sim \epsilon^{-2}$, so the BR result suggests that global AdS may be unstable at the shortest possible time scale allowed by the dynamics. Furthermore, an instability can lead to many different deviations from empty AdS, and it has no particular reason

to directly become a black hole, but the BR result suggests that black hole formation may be the generic outcome.

Black hole formation is a natural end point of the dynamics, since in one way of taking the classical limit, the black hole is the equilibrium configuration in the microcanonical ensemble [21,35]. However, there is certainly no guarantee that the system will equilibrate on this short time scale. In addition, a generic expectation is that as the interaction strength decreases, the ergodic region of phase space decreases in size. This result is proven by the KAM theorem for a wide class of systems [36], but the assumptions of the theorem are not met here. So it is interesting to ask how generic black hole formation is, particularly on the nonlinear time scale. This question is dual to the question of how efficient thermalization is in a large N conformal field theory on a spatial sphere of radius R , at energies in the range

$$R^{-1} \ll E \ll \frac{N^2}{R}. \quad (1.1)$$

Further work showed that although some initial conditions do lead to collapse, black hole formation is not the only possible behavior at the $T \sim \epsilon^{-2}$ time scale. In particular, there exist open sets of initial conditions that avoid collapse on this time scale. Such a strong conclusion is based on two cornerstones: First, the discovery of “islands of stability” [7,8,10], and study of the phase-coherent dynamics of these noncollapsing solutions in the two-time formalism [31,34]; and second, the conditional reliability of the two-time formalism and rescaling

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symmetry that guarantees that these solutions survive in the small amplitude limit, at the ϵ^{-2} time scale [30].

In this paper, we take a step towards a similar understanding for collapsing solutions. Some evidence showed that open sets of stable solutions are anchored special solutions with stationary exponential energy spectra [31,34]. We will present possibly analogous special solutions for collapsing solutions: stationary solutions with power-law spectra which causes the right amount of geometric backreaction for black hole formation.

First of all, we have to point out a confusion caused by how the term “weak turbulence” is used in the AdS (in) stability problem. As described in many pioneering works [3,4,9,24,37], weak gravitational interactions between the AdS eigenstates of linearized perturbations, expanded to the first nonlinear order, lead to quartic couplings between the modes. This type of system has been extensively studied in the theory of weak turbulence. However, it is misguided to claim that the Kolmogorov-Zakharov power law in weak turbulence [38,39] explains the power-law spectrum seen in the collapsing AdS solutions [6].

In the weak turbulence context, the equations of motion are solved under the random phase ansatz. The phases of eigenstates are assumed to be randomly distributed, thus any phase-dependent effect averages to zero. The phase information is therefore totally discarded, and the amplitudes of eigenstates are described by a phase-independent dynamics. A basic result under the random phase ansatz is that for a system with quartic interactions, there is *no energy transfer* between the modes on the ϵ^{-2} time scale; the nontrivial dynamics occurs on the longer ϵ^{-4} time scale [38,39].¹ If one really takes the full analogy to weak turbulence to solve the dynamics of AdS eigenstates, the Kolmogorov-Zakharov power law will be attained at the ϵ^{-4} time scale, which cannot explain the observed black hole formation in the shorter, ϵ^{-2} time scale. Therefore, in order to understand the results of simulations, we must pursue an analytical strategy that keeps track of the phases and their relevance in the dynamics.

Keeping track of the phases makes the problem considerably harder. Fortunately, experience tells us that even the simplest possible solution can provide a lot of insight. Recall that the sets of stable solutions, sometimes called stability islands, are anchored on special “quasiperiodic solutions,” namely exactly stationary solutions, with coherent phases and exponential spectrum, in the two-time approximation [13,24,31,34]. We will show that the two-time approximation contains another type of exactly stationary and coherent solution which likely plays the same

role for collapsing solutions. Although stationary solutions do not really evolve, their possible forms are highly constrained, providing important information of the dynamics. Finding them is often the first step toward understanding other solutions with similar properties [31,34]. The solution we find has the following properties:

- (i) Instead of an exponential spectrum, these solutions have a power-law spectrum.²
- (ii) Within the two-time approximation, these solutions are protected by the rescaling symmetry, thus also persist in the $\epsilon \rightarrow 0$ limit.
- (iii) These solutions come with specific power laws, $E \sim \omega^{2-d}$ as a function of the frequency ω , which agree with the extensive numerical observations in $d = 4$.
- (iv) The backreaction from these coherent power-law solutions is strong enough to give finite deviation from empty AdS even in the $\epsilon \rightarrow 0$ limit. In particular, the deviation is suggestive of black hole formation.

The backreaction calculation is not difficult, but is has been neglected in the recent literature. Although the possibility of black hole formation largely motivated the current developments, much recent work has focused on the scalar field spectrum without establishing an explicit link to the actual geometric backreaction. Such a link is necessary to establish that AdS space is indeed unstable, and to understand the outcome of instability.

In Sec. II, we calculate the relation between the scalar field power spectrum and the geometric backreaction it causes, with particular emphasis on diagnosing whether the backreaction is singular, and when it is, whether it suggests a black hole, or some other type of singularity. We find that the phase coherence between modes strongly affects the backreaction. We compare the fully coherent and the fully incoherent cases and find that phase coherence leads to stronger backreaction from the same power law. In particular, we observe that the power laws found in numerical simulations are probably insufficient to imply black hole formation if the phases are incoherent. On the other hand, the same power laws with coherent phases strongly suggest black hole formation.

In Sec. III, we derive the stationary coherent power-law solutions from the recently reported scaling behavior of the coupling coefficients in [33]. Phase-coherent dynamics explicitly predicts these power laws in AdS _{$d+1$} with arbitrary $d > 3$. The energy per mode as a function of frequency is given by

$$E \sim \omega^{2-d}, \quad (1.2)$$

¹Since [38,39] are quite technical and contain a lot of other information, it may not be straightforward to understand this point directly by reading it. We sketch a simple derivation in Appendix A to show the readers how random-phase ansatz kills any dynamics in the ϵ^{-2} time scale.

²The power-law solutions evaded earlier solution searches [31,34] either due to some starting assumption that excluded power laws from the very beginning, or due to removing solutions by hand if they run into a UV cutoff, which power-law solutions usually do.

where d is the spatial dimension of the bulk theory. The $d = 4$ result exactly agrees with many numerical observations, while higher d can be checked in the future. The analysis in $d = 3$ is more subtle due to anomalies in the scaling and a possible dependence on the UV cutoff. Our result there is in some tension with the scaling reported in simulations, and we specifically point out possible causes to study in the future.

We do not have a precise mathematical argument relating our stationary solutions to the dynamical collapse. However, we expect that our time-independent power law is a good description of the dynamics for a range of wavelengths that are well separated from the long-wavelength scale, where the energy is initially injected, and the (time-dependent) UV scale where the modes have not yet been populated.

Finally, we explore possible relations between the stationary power-law solution we found and the actual dynamical evolution from initial data into high modes. It is likely too naive to imagine all collapsing solutions as “approaching” these stationary coherent power-law solutions. Recall that typical stable solutions do not approach quasiperiodic solutions either. They only evolve around orbits which seem to center on the quasiperiodic solutions [31]. The corresponding behavior of a typical unstable solution is likely more complicated. We suggest a possible first step in this direction by noticing that two-mode initial data seems to be particularly prone to collapse in existing numerical results. In Sec. IV we analytically derive that indeed two-mode initial data directly leads to an initially phase-coherent energy cascade. Although this is only valid for time less than ϵ^{-2} , it might be an interesting starting point. For example, three-mode initial data does not have the same simple phase-coherent structure at early times. One can then numerically study the fate of three-mode initial data to see whether there are significant differences.

II. GRAVITATIONAL BACKREACTION OF COHERENT AND INCOHERENT POWER LAWS

In this section, we analyze the gravitational backreaction when a number of modes are turned on. We find qualitatively different behaviors when the phases of the modes are taken to be coherent than when they are incoherent.³ We particularly focus on a power-law spectrum of amplitudes, since this is the case that is seen in numerical simulations. Our convention is to parametrize the amplitudes as a power law of order $-\alpha$, namely $A_n \sim n^{-\alpha}$. The allowed frequencies for a massless field in global AdS are discrete, with $\omega_n = 2n + d$ in units of the AdS radius, so a power law in

frequency is equivalent to a power law in mode number n . In the following, we work with the mode number n . Some other papers in this field use the corresponding energy spectrum, which will be $E_n \sim n^{2-2\alpha}$. First we present our results in this chart.

	Regular	Naked curvature singularity	Naked redshift singularity	Black hole
$d = 3$				
Incoherent phases	$\alpha > \frac{5}{2}$	$\frac{5}{2} \geq \alpha > \frac{3}{2}$	$\alpha \leq \frac{3}{2}$	Never
Coherent phases	$\alpha > 3$	$3 \geq \alpha > 2$	$\alpha = 2$	$\alpha < 2$
$d > 3$				
Incoherent phases	$\alpha > \frac{d+2}{2}$	$\frac{d+2}{2} \geq \alpha > \frac{d}{2}$	$\alpha = \frac{d}{2}$	$\alpha < \frac{d}{2}$
Coherent phases	$\alpha > \frac{d+3}{2}$	$\frac{d+3}{2} \geq \alpha > \frac{d+1}{2}$	$\alpha = \frac{d+1}{2}$	$\alpha < \frac{d+1}{2}$

We can see that in $(3 + 1)$ dimensions, independent of what power law we have, incoherent phases can never correspond to black hole. Thus an observation of black hole formation together with any power law implies phase coherence. In $(4 + 1)$ dimensions, the numerical collapses reported values of α very close to 2. Since such a value is right at the edge for incoherent phases, one cannot be as conclusive. However, coherent phases still leave less doubt about the connection between this power-law and black hole formation.

Since we are calculating the backreaction using the leading order expansion, and the last three columns in this chart actually correspond to diverging backreaction that invalidates the expansion, we should explain their physical meanings more carefully.

These power-law solutions are always well defined as a dynamically evolving set of harmonic oscillators described by the two-time formalism [13,23,25],⁴ which approximates the actual gravity evolution before backreaction reaches order one. So what we actually calculate is a “fictitious backreaction” which approximates the actual backreaction before it reaches order one. If we start with some initial conditions with small backreactions, evolve them with the two-time formalism and they reach any of the diverging power laws, then before that time, the actual backreaction does become order one. Reaching order-one backreaction is already sufficient to show an instability. Evolving toward a diverging fictitious backreaction in the two-time formalism then guarantees an order-one backreaction for the gravity evolution with an arbitrarily small initial amplitude. In particular, the form of the diverging fictitious backreaction represent the form of the actual backreaction when it reaches order one. So one can ask whether such form is similar to a Schwarzschild metric or not, which can be a good sign of what type of large backreaction it will approach afterward.

³A more precise definition of coherence will become clear later.

⁴In a forthcoming publication, we will explain more explicitly how the oscillating singularity shown in [25] disappears in the boundary time gauge.

A. Geometric deviation

With spherical symmetry, we can demand to always put the metric into a standard form that is easy to compare with empty AdS,⁵

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2. \quad (2.1)$$

We can fix the gauge for the perturbations by enforcing that r is always the area radius of the $(d-1)$ sphere, g_{tt} approaches the empty AdS value at $r \rightarrow \infty$, and the off-diagonal term $g_{tr} = 0$, leaving only the physical quantities δg_{tt} and δg_{rr} . In particular, in defining which solutions have large backreaction, we care about their maximum values in all space and in all time within one AdS period.⁶

In a more general treatment without spherical symmetry, defining which solutions have large backreaction is nontrivial due to the freedom of gauge choice. It could be defined through a max-min scheme: scanning through the entire spacetime for the largest $\delta g_{\mu\nu}$, then all possible gauge choices to minimize it. We will leave such an endeavor to future work.

We are interested in the case that the total energy is small and approaching zero. Thus the only possibility to have a large backreaction is to focus the energy into a small region, much smaller than the AdS radius, which was set to one. Such a region can be effectively described locally by Minkowski space, for which the perturbative expansion of small backreaction is well known,

$$ds^2 = -\left(1 + \frac{2M}{r^{d-2}} + 4V\right)dt^2 + \left(1 + \frac{2M}{r^{d-2}}\right)dr^2 + r^2 d\Omega_{d-1}^2. \quad (2.2)$$

Here M and V are the usual definition of the enclosed mass and gravitational potential,

$$M(r) = \int_0^r \frac{\dot{\phi}^2 + \phi'^2}{2} dr', \quad (2.3)$$

$$V(r) = - \int_r^\infty (d-2) \frac{M(r')}{r'^{d-1}} dr'. \quad (2.4)$$

⁵The AdS radius is set to 1 for convenience throughout this paper.

⁶The reason for scanning through one AdS time was explained in [30]. At time t , a large and dilute shell may be tuned to converge at $r = 0$ and form a black hole within one AdS time, which is not the long time scale instability we are studying. Thus such a finely tuned dilute shell, although it only modifies the metric mildly at that moment, must already mean a large geometric backreaction for our purpose.

Using this approximation, we have

$$\delta g_{tt} = \frac{2M(r)}{r^{d-2}} + 4V(r), \quad \delta g_{rr} = \frac{2M(r)}{r^{d-2}}. \quad (2.5)$$

This not only allows us to estimate whether backreaction is large, but we can further describe the physics of the backreaction, for example whether it is approaching a black hole, which requires $\delta g_{tt} \approx -\delta g_{rr}$. Some may worried that we are only keeping the leading order in the metric deviation, while higher order terms always become important in an actual black hole formation. We remind the reader that the formal mathematical definition of an instability is whether *an infinitesimal initial perturbation leads to a finite perturbation*, in which a *finite perturbation* can be still small and well approximated by the leading order term. It is true that strictly speaking, developing some small but finite $\delta g_{tt} \approx -\delta g_{rr}$ does not guarantee black hole formation. However, it is clearly different and more suggestive of such a possibility, compared to situations in which δg_{tt} is very different from $-\delta g_{rr}$.

It would be very interesting to extend our treatment beyond linearized backreaction. However, this would require a number of nontrivial steps, such as defining the modes of the scalar field in the presence of nonlinear metric perturbations.

B. Single mode

Following the previous section, we can calculate the geometric backreaction of any field configuration. As a warm-up exercise, we first consider the situation where all energy is in one eigenstate, $\phi(t, x) = A_n e_n(x) \cos w_n t$ and $E = w_n^2 A_n^2$. The energy density ρ is given by

$$\rho_n(r) \approx w_n^2 A_n^2 e_n^2 \sim E n^{d-1} \quad \text{for } r \leq n^{-1}, \quad (2.6)$$

$$\sim \frac{E}{r^{d-1}} \quad \text{for } n^{-1} < r < 1. \quad (2.7)$$

This directly follows from the large n , small r behavior of the eigenfunctions $e_n(r)$ in Eq. (3.2). Actually, instead of going through the hypergeometric function for the actual e_n , one can easily derive this energy distribution with the following physical intuition. Spherically symmetric eigenstates are basically standing waves from the superpositions of incoming and outgoing waves. Thus they have roughly uniform energy for every shell of unit thickness. That leads to Eq. (2.7). Note that such an energy density would be divergent at $r = 0$, but at scales shorter than the wavelength n^{-1} , it is smeared out and becomes uniform. That means the central ball of radius n^{-1} has total energy equal to a large shell of thickness n^{-1} , which leads to Eq. (2.6).⁷ In this physical picture, one can imagine a cutoff at $r = 1$ and treat AdS as a finite box.

The enclosed mass in this single mode data can be estimated as

$$M(r) = \int_0^r \rho_n(\bar{r}) \bar{r}^{d-1} d\bar{r} \sim E n^{d-1} r^d \quad \text{for } r \leq n^{-1},$$

$$\sim E r \quad \text{for } n^{-1} < r < 1. \quad (2.8)$$

Assuming the energy is dominated by the highest possible mode, $n \rightarrow \infty$, we can calculate the backreaction to the metric from:

$$\delta g_{rr} \sim \frac{2M(r)}{r^{d-2}} \sim 2E r^{3-d}. \quad (2.9)$$

Note that in $d = 3$, this never diverges no matter how large n is. On the other hand,

$$\delta g_{tt} - \frac{2M}{r^{d-2}} = 4V(r) = -4 \int_r^1 (d-2) \frac{M(\bar{r})}{\bar{r}^{d-1}} d\bar{r} \sim 4E \ln r \quad \text{for } d = 3,$$

$$\text{or } \sim -4E r^{3-d} \quad \text{for } d > 3. \quad (2.10)$$

This diverges even for $d = 3$.

Just from this simple example, we can see how “energy cascade” is a too naive statement to conclude black hole formation. Even if all energy goes to one, infinitely high mode, it only means a black hole in $d > 3$, since $\delta g_{rr} \sim -\delta g_{tt}$ indeed diverges together. In $d = 3$, only δg_{tt} diverges but δg_{rr} does not. It is a large geometric backreaction, but not a black hole.

C. Power-law spectrum

Next we will consider a power-law spectrum with

$$A_n = A_0(n+1)^{-\alpha}. \quad (2.11)$$

Even before any calculation, clearly, there should be some values of α large enough that the contribution from short wavelength modes is insufficient to make any feature in short distance scales. Likewise, there should be values of α small enough that the short distance behavior is singular. Our goal will be to find those thresholds.

First of all, we have to make a technical distinction between a finite power law, $\alpha > 3/2$, and an infinite power law, $\alpha \leq 3/2$. In the finite case, the spectrum has a finite IR amplitude for a finite total energy:

$$E_{\text{tot}} = \sum_{n=0}^{\infty} w_n^2 A_n^2 \sim \frac{A_0^2}{2\alpha - 3}. \quad (2.12)$$

In the infinite case, what we mean by an infinite sum is implicitly a limiting case when the UV cutoff N on the sum goes to infinity, while the IR amplitude, A_0 , drops to zero accordingly, maintaining a finite total energy:

⁷Throughout this paper, we use \approx for an actual approximation, such as dropping subleading terms in large n . We use \sim when we also drop all n -independent factors and are only interested in the scaling with n .

$$E_{\text{tot}} = \sum_{n=0}^N w_n^2 A_n^2 \sim A_0^2 \ln N, \quad \text{for } \alpha = 3/2, \quad (2.13)$$

$$\sim A_0^2 N^{3-2\alpha}, \quad \text{for } \alpha < 3/2. \quad (2.14)$$

In addition, there is a very important physical distinction when multiple eigenstates are involved—whether their phases, B_n in Eq. (3.1), are coherent or not. This directly plays a role in the calculation of mass enclosed:

$$M(r) \sim \int_0^r r_1^{d-1} dr_1 \left(\sum_{n=0}^{\infty} w_n A_n e_n(r_1) \cos(w_n t + B_n) \right)^2,$$

$$\sim \int_0^r r_1^{d-1} dr_1 \left[\sum_{n=r^{-1}}^{\infty} w_n^2 A_n^2 e_n(r_1)^2 \right. \\ \left. + \left(\sum_{n=0}^{r^{-1}} w_n A_n e_n(0) \cos(w_n t + B_n) \right)^2 \right]. \quad (2.15)$$

For all modes with $n > r^{-1}$, they oscillate rapidly within the integration range, thus the cross terms automatically vanish from the integral. However, modes with $n < r^{-1}$ are basically constant within the integration range. If all of their phases are coherent, for example $t = \theta_n = 0$ for all n , then the cross terms contribute significantly to the mass.

1. Incoherent phases

Here we will derive the result when the phases are incoherent, thus all cross terms from Eq. (2.15) can be dropped:

$$\begin{aligned}
 M(r) &\sim A_0^2 \left(r \sum_{n=r^{-1}}^{\infty} n^{2-2\alpha} + r^d \sum_{n=0}^{r^{-1}} n^{d+1-2\alpha} \right) \\
 &\sim E_{\text{tot}} r^d \quad \text{for } \alpha > \frac{d+2}{2}, \\
 \text{or } &\sim -E_{\text{tot}} r^d \ln r \quad \text{for } \alpha = \frac{d+2}{2}, \\
 \text{or } &\sim E_{\text{tot}} r^{2\alpha-2} \quad \text{for } \frac{3}{2} < \alpha < \frac{d+2}{2}, \\
 \text{or } &\sim E_{\text{tot}} r \quad \text{for } \alpha \leq \frac{3}{2}. \tag{2.16}
 \end{aligned}$$

Note that we are only keeping the small r behavior. Not surprisingly, for large α , the behavior of lower modes dominates and the mass scales like volume, which is independent of α . When the value of α drops below $(d+2)/2$, the energy density starts to develop a singularity at $r=0$, which grows more singular as α decreases further. Finally, infinite power laws lead to the same result as a single mode of arbitrarily high n , as seen in Eq. (2.8).

A singular energy density implies a singular curvature tensor, but not always a large perturbation in the metric, which we will calculate here. For a more concise presentation, we only provide the explicit expression in the cases which the deviation can be singular. Any finite deviation will go to zero with E_{tot} , so for our purpose their exact r dependence is not that important:

$$\begin{aligned}
 \delta g_{rr} &\approx \frac{2M(r)}{r^{d-2}} \sim \text{regular}, \quad \text{for } \alpha \geq \frac{d}{2}, \\
 \text{or } &\sim 2E_{\text{tot}} r^{2\alpha-d}, \quad \text{for } \frac{3}{2} < \alpha < \frac{d}{2}, \quad d > 3, \\
 \text{or } &\sim 2E_{\text{tot}} r^{3-d}, \quad \text{for } \alpha \leq \frac{3}{2}. \tag{2.17}
 \end{aligned}$$

$$\begin{aligned}
 \delta g_{tt} - \frac{2M}{r^{d-2}} &= 4V(r) \\
 &= -4(d-2) \int_r^1 \frac{M(r_1)}{r_1^{d-1}} dr_1 \\
 &\sim \text{regular}, \quad \text{for } \alpha > \frac{d}{2}, \\
 \text{or } &\sim 4E_{\text{tot}} \ln r, \quad \text{for } \alpha = \frac{d}{2}, \quad \text{or } \alpha \leq \frac{3}{2}, \quad d = 3, \\
 \text{or } &\sim -4E_{\text{tot}} r^{2\alpha-d}, \quad \text{for } \frac{3}{2} < \alpha < \frac{d}{2}, \quad d > 3, \\
 \text{or } &\sim -4E_{\text{tot}} r^{3-d}, \quad \text{for } \alpha \leq \frac{3}{2}, \quad d > 3. \tag{2.18}
 \end{aligned}$$

As a quick summary, if the phases are incoherent, then approaching an α -power law means

- (i) Geometric deviation goes to zero with E_{tot} when $\alpha > d/2$, but curvature can already get large when $\alpha \leq (d+2)/2$.

- (ii) Geometric deviation gets large but **does not** approach a black hole when $\alpha = d/2$ in any d and $\alpha \leq 3/2$ in $d = 3$.
- (iii) Geometric deviation gets large as if approaching a black hole when $\alpha < d/2$ in $d > 3$, but it never does in $d = 3$.

Just like in the single-mode case, $d = 3$ is special. With incoherent phases, one never gets a black hole-like geometric deviation. Large geometric deviation still occurs when $\alpha \leq 3/2$, namely for infinite power laws. That leads to δg_{tt} blowing up like a log while δg_{rr} stays small. That is also the situation in other dimensions with exactly $\alpha = d/2$. We can plug the behavior of δg_{rr} and δg_{tt} into Eq. (2.2) to get a better physical intuition for what is happening in these cases:

$$ds^2 = (1 - E_{\text{tot}})(-r^{-E_{\text{tot}}} dt^2 + dr^2) + r^2 d\Omega_{d-1}^2. \tag{2.19}$$

There are order-one factors in front of both appearances of E_{tot} in the above equation that we did not keep track of, but those are not very relevant for our analysis. The point $r = 0$ is singular for any positive E_{tot} , but it takes only finite time for light rays to reach $r = 0$ and come back to infinity, so it is not developing a horizon. It is a clear distinction between this deviation and those approaching an AdS-Schwarzschild metric.

2. Coherent phases

In Sec. III we will discuss more thoroughly what phase coherence means in this context. Here let us just assume $t = \theta_n = 0$ in Eq. (2.15). That leads to

$$\begin{aligned}
 M(r) &\sim A_0^2 \left[r \sum_{n=r^{-1}}^{\infty} n^{2-2\alpha} + r^d \left(\sum_{n=0}^{r^{-1}} n^{\frac{d+1}{2}-\alpha} \right)^2 \right], \\
 &\sim E_{\text{tot}} r^d, \quad \text{for } \alpha > \frac{d+3}{2}, \\
 \text{or } &\sim E_{\text{tot}} r^d (\ln r)^2, \quad \text{for } \alpha = \frac{d+3}{2}, \\
 \text{or } &\sim E_{\text{tot}} r^{2\alpha-3}, \quad \text{for } \frac{3}{2} < \alpha < \frac{d+3}{2}. \tag{2.20}
 \end{aligned}$$

We are omitting the technical results for infinite power laws here. Those cases have an ambiguity regarding the order of limits: the mode sum cutoff $N \rightarrow \infty$ and $r \rightarrow 0$; they are also not too relevant for us since geometric deviation is already singular for finite power laws with small enough α . Reducing α further to an infinite power law can only make the result more singular.

Note that when $\alpha \leq (d+3)/2$, the energy density is already singular at $r = 0$. As expected, this happens earlier (for a larger α) compared to the case of incoherent phases in the previous session. It is straightforward to repeat the calculation of metric deviation and find that they also diverge earlier:

$$\begin{aligned} \delta g_{rr} &\approx \frac{2M(r)}{2r^{d-2}} \sim \text{regular}, \quad \text{for } \alpha \geq \frac{d+1}{2}, \\ \text{or } &\sim 2E_{\text{tot}} r^{2\alpha-d-1}, \quad \text{for } \frac{3}{2} < \alpha < \frac{d+1}{2}. \end{aligned} \quad (2.21)$$

$$\begin{aligned} \delta g_{tt} - \frac{2M}{r^{d-2}} &= 4V(r) = -4(d-2) \int_r^1 \frac{M(r_1)}{r_1^{d-1}} dr_1 \\ &\sim \text{regular}, \quad \text{for } \alpha > \frac{d+1}{2}, \\ \text{or } &\sim 4E_{\text{tot}} \ln r, \quad \text{for } \alpha = \frac{d+1}{2}, \\ \text{or } &\sim -4E_{\text{tot}} r^{2\alpha-d-1}, \quad \text{for } \frac{3}{2} < \alpha < \frac{d+1}{2}. \end{aligned} \quad (2.22)$$

We have collected all of these results in the summary table at the beginning of this section.

III. STATIONARY COHERENT POWER-LAW SOLUTIONS

In this section we examine the evolution equations, using the two-time formalism, to establish the existence of phase-coherent power laws as exactly stationary solutions.

A. Two-time analysis

We first review the two-time formalism that is employed to describe the AdS-gravity dynamics at the ϵ^{-2} time scale [13,23,25]. A spherically symmetric free scalar field in a fixed AdS background can be decomposed into eigenstates [40,41],

$$\begin{aligned} \phi(t, r) &= \sum_{n=0}^{\infty} \phi_n(t) e_n(r) \\ &\equiv \sum_{n=0}^{\infty} \bar{A}_n e_n(r) \cos(w_n t + B_n), \end{aligned} \quad (3.1)$$

$$\begin{aligned} e_n(r) &= \sqrt{\frac{(2n+d)n!\Gamma(n+d)}{2^d \Gamma(n+d/2) \Gamma(n+d/2+1)}} \\ &\times (1+r^2)^{-d/2} P_n^{(d/2-1, d/2)} \left(\frac{1-r^2}{1+r^2} \right). \end{aligned} \quad (3.2)$$

The eigenfrequencies are all integers given by $w_n = 2n + d$.

Without gravity, \bar{A}_n and B_n will stay constant forever. Including gravity, the presence of energy from this field modifies the metric, which in turn modifies the evolution of

the field. When such an effect is small, it can be approximated by⁸

$$\ddot{\phi}_n + w_n^2 \phi_n = \sum_{k+l=m+n}^{k+l=m+n} C_{klmn} \phi_k \phi_l \phi_m + \mathcal{O}(\phi^5). \quad (3.3)$$

The stability at the $T \sim \epsilon^{-2}$ time scale, taking the $\epsilon \rightarrow 0$ limit, can always be addressed within the regime that the higher order terms can be safely dropped [30]. This is effectively a collection of quartically coupled harmonic oscillators.

One can rewrite this second order differential equation into two first order equations for \bar{A}_n and B_n :

$$\begin{aligned} 2w_n \frac{d\bar{A}_n}{dt} &= \sum_{klm}^{k+l=m+n} C_{klmn} \bar{A}_k \bar{A}_l \bar{A}_m \\ &\times \sin(B_n + B_m - B_k - B_l), \end{aligned} \quad (3.4)$$

$$\begin{aligned} 2w_n \frac{dB_n}{dt} &= \bar{A}_n^{-1} \sum_{klm}^{k+l=m+n} C_{klmn} \bar{A}_k \bar{A}_l \bar{A}_m \\ &\times \cos(B_n + B_m - B_k - B_l). \end{aligned} \quad (3.5)$$

Note that we can rescale time, $t = \tau \epsilon^{-2}$, and also rescale the amplitudes, $\bar{A}_n = A_n \epsilon$, we can then rewrite the dynamical equations in the ‘‘long time’’ τ .

$$\begin{aligned} 2w_n \frac{dA_n}{d\tau} &= \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \\ &\times \sin(B_n + B_m - B_k - B_l), \end{aligned} \quad (3.6)$$

$$\begin{aligned} 2w_n \frac{dB_n}{d\tau} &= A_n^{-1} \sum_{klm}^{k+l=m+n} C_{klmn} A_k A_l A_m \\ &\times \cos(B_n + B_m - B_k - B_l). \end{aligned} \quad (3.7)$$

This set of equations then represents the evolution of the scale-independent, relative amplitudes of all modes, together with their phases.

Note that in the previous section, we have chosen a gauge that the time at the asymptotic boundary stays the same, thus our equations here are also in such boundary gauge. As discussed in [22], such gauge is intuitively convenient since there exists a Lagrangian (and Hamiltonian) that reproduces the equations of motion. Furthermore, the point $r = 0$ is quite special in the spherically symmetric setup, and using its proper time can be

⁸The constraint $k+l=m+n$ is the combination of two effects. (1) The resonant condition $w_n = \pm w_k \pm w_l \pm w_m$, and (2) the actual evaluation of C_{klmn} which is related to hidden symmetries of AdS and extra conserved quantities in the dynamics [29,32,42].

misleading. For example, the oscillating divergence observed in [25] means that the point $r = 0$ has an infinite redshift with respect to any other point. Whether this means a black hole is unclear, as we already explained in Sec. II. In order to avoid similar confusions, we will stay in this boundary gauge in the rest of this paper, unless otherwise specified.

Physically, the phases are coherent if there is some time during one AdS period where all of the modes are in phase. The phase θ_n of the mode n is related to the “slow phase” B_n by

$$\theta_n(\tau, t) = B_n(\tau) + \omega_n t = B_n(\tau) + (2n + d)t. \quad (3.8)$$

Note that the slow phase B_n depends on the slow time τ , while the full phase θ_n depends on the fast time as well.

For the phases to align at some time during the short time period $\delta t = 2\pi$ requires

$$\theta_n(\tau, t) - \theta_m(\tau, t) = 2\pi N_{nm}, \quad (3.9)$$

where N_{nm} are integers that can depend on the modes involved. Coherence requires that we can solve this equation for the short time t over one cycle $0 < t < 2\pi$, at the same t for all modes. Plugging in the formula for the phases θ_n , we have

$$B_n(\tau) - B_m(\tau) = 2\pi N_{nm} + 2(n - m)t. \quad (3.10)$$

Since the B_n are only defined mod 2π , we can drop the first term on the right side. Define $2t \equiv \theta(\tau)$, the equation becomes simply

$$B_n(\tau) - B_m(\tau) = (n - m)\theta(\tau). \quad (3.11)$$

Solving this equation for all choices of m and n requires

$$B_n(\tau) = n\gamma(\tau) + \delta(\tau), \quad (3.12)$$

where γ, δ are free functions of the slow time that must be independent of the mode number n , and the equation is valid mod 2π .

We are interested in describing the behavior at large mode numbers, so we should allow corrections to this formula. Our final condition for phase coherence is therefore

$$B_n(\tau) = n\gamma(\tau) + \delta(\tau) + \dots \quad (3.13)$$

Here “ \dots ” are just anything that goes to zero in the large n limit. It may be interesting to consider a weaker notion of phase coherence, which would still allow for constructive interference in the gravitational backreaction, but in this paper we will only use the above definition.

B. Asymptotic phase-coherent power laws

We now want to self-consistently solve the slow-time evolution equations, Eqs. (3.6) and (3.7), under the coherent phase condition Eq. (3.13). In order to analyze the equations, we need to know the scaling of the interaction coefficients C_{ijkl} . In [33], it was reported that in the boundary gauge, the coefficients obey the simple scaling law,

$$C_{(\lambda k)(\lambda l)(\lambda m)(\lambda n)} \sim \lambda^d C_{klmn}, \quad (3.14)$$

for greater than three spatial dimensions, $d > 3$.

In a forthcoming publication [43], we find that in fact this scaling is modified for the diagonal terms C_{ijij} and C_{iiii} in $d = 4$ by additional logarithmic factors; however these factors do not appear to affect the final results, so here we use the simple scaling in Eq. (3.14) and defer a more detailed description to [43]. In higher dimensions, $d > 4$, the scaling (3.14) is exact for large mode numbers.

First of all, the phase-locked condition, Eq. (3.13), is already a natural solution to one of the equations of motion, Eq. (3.6). Since the resonant condition is $m + n = k + l$, the phase-locked condition makes $(B_n + B_m - B_k - B_l) = 0$. This makes all the sine terms in Eq. (3.6) zero, thus $dA_n/d\tau = 0$. In other words, this choice of phases guarantees that there is no energy transfer among the modes. This is exactly the same as in the quasiperiodic, noncollapsing solutions [31]. The remaining question is whether the coherent phase assumption is maintained under time evolution.

Examining the equation for the phase evolution (3.7), all the cosine factors there are 1 due to the coherent phase ansatz, so this equation takes a very simple form:

$$2w_n \frac{dB_n}{d\tau} = A_n^{-1} \sum_{k,l,m}^{k+l=m+n} C_{klmn} A_k A_l A_m. \quad (3.15)$$

We need $B_n = n\gamma(\tau) + \delta(\tau) + \dots$ to maintain the phase coherence, and $\omega_n \sim n$, so the left side of the equation must have the n -scaling $n^2\gamma(\tau) + n\delta(\tau)$. As long as $\gamma(\tau) \neq 0$, this means that the right-hand side of the above equation must scale like n^2 . Plugging in the power-law spectrum, $A_n = A_0 n^{-\alpha}$, into the right side of Eq. (3.15), we get

$$2w_n \frac{dB_n}{d\tau} = A_0^2 n^\alpha \sum_{k,l,n}^{k+l=m+n} C_{klmn} [k(m+n-k)m]^{-\alpha}. \quad (3.16)$$

Then we use integrals to approximate the sums:

$$\approx A_0^2 n^\alpha \int dk dm C_{k(m+n-k)mn} [k(m+n-k)m]^{-\alpha}. \quad (3.17)$$

Whether the integral approximation to the sums is a good one depends on the detailed dependence of the coefficients

C_{ijkl} on each one of the indices, not only on the overall scaling. For now, we assume the integral approximation holds, and leave a more careful analysis for future work.

Within the integral approximation, we can utilize the scaling behavior in Eq. (3.14):

$$\begin{aligned} A_0^2 n^\alpha \int dk dm C_{k(m+n-k)mn} [k(m+n-k)m]^{-\alpha} & \quad (3.18) \\ = A_0^2 n^\alpha \int n^2 dx dz n^d C_{x(z+1-x)z1} n^{-3\alpha} [x(z+1-x)z]^{-\alpha} \\ \propto A_0^2 n^{d+2-2\alpha}. & \quad (3.19) \end{aligned}$$

Thus $\alpha = d/2$ is the unique value to provide n^2 scaling, maintaining the phase-locked condition. If we had considered the special case $\gamma = 0$, then the self-consistent solution would be a different power law, $\alpha = d/2 + 1/2$. In the doubly special case $\gamma = \delta = 0$, the value is $\alpha = d/2 + 1$. By examining the early time dynamics in Sec. IV, we believe that the generic case $\gamma \neq 0$ is dynamically selected.

Note that $\alpha = d/2$ we find here, strictly speaking, is necessary but not sufficient for the phases to remain coherent dynamically. It forbids higher order n scaling in B_n , but it is not clear whether there are subleading fractional powers of n or order 1 fluctuating contributions. Those can potentially ruin the phase coherence, but could only be checked given subleading behavior of the coupling coefficients C_{ijkl} . These are difficult to obtain.

Leaving these various caveats aside, we can go ahead and ask whether the power law predicted by our analysis agrees with that observed in the full numerical evolution. Maliborski and Rostworowski [9] suggested a ‘‘preliminary guess’’ for the energy spectrum,

$$E_n \sim n^{-\frac{6}{5} - \frac{4}{5}(d-3)}. \quad (3.20)$$

The energy per mode is related to the amplitude by $E_n \sim \omega_n^2 A_n^2 \sim n^{2-2\alpha}$. Plugging in our values of α , our analysis predicts an energy spectrum

$$E_n \sim n^{2-d}. \quad (3.21)$$

Recall that we have assumed the scaling (3.14), which is valid in $d > 4$, and almost valid (up to logarithmic corrections) in $d = 4$. In $d = 4$, our formula agrees with the Maliborski-Rostworowski (M-R) guess. In $d = 5$, we get $E_n \sim n^{-3}$, while the M-R formula gives $n^{-2.8}$. It is not a big difference, and there has not been a lot of data to accurately determine the actual power yet [44]. Further numerical results in $d \geq 5$ would provide an important check for our predictions.

I. AdS₃₊₁

The situation in AdS₃₊₁ is trickier. As we will explain in [43], the diagonal terms in the boundary gauge are still likely to have peculiar behaviors, thus such a gauge choice does not simplify the matter here. This subsection will be an exception to the rest of the paper, and we will use the central gauge, in which the scaling property of C_{klmn} was analytically derived in [31,34]:

$$C_{jnjn} \sim n^2 j^2 \ln j, \quad C_{(\lambda k)(\lambda l)(\lambda m)(\lambda n)} \sim \lambda^3 C_{klmn}. \quad (3.22)$$

Fortunately, the diagonal terms trivially satisfy the phase-locked requirement due to its n^2 dependence.⁹ The remaining question is the off-diagonal terms, which are the same in either gauge.

These terms can then be analyzed in exactly the same way as above. Naively extending our result to $d = 3$ gives $A_n \sim n^{-3/2}$, or equivalently $E_n \sim n^{-1}$. Note that an energy spectrum n^{-1} is not normalizable at large n , so we need to do a more refined analysis as explained in Sec. II C: Including a UV cutoff N such that the amplitudes go to zero as N goes to infinity to conserve total energy. Keeping that in mind, we can begin with a similar process:

$$\begin{aligned} 2\omega_n \frac{dB_n}{d\tau} &= A_n^{-1} \sum_{k,l,m}^{k+l=m+n} C_{klmn} A_k A_l A_m \\ &\approx A_0^2 n^\alpha \int_0^N dm \int_0^{m+n} dk C_{k(m+n-k)mn} \\ &\quad \times [k(m+n-k)m]^{-\alpha} \\ &\approx A_0^2 n^{5-2\alpha} \int_0^{\frac{N}{n}} dy \int_0^{y+1} dx C_{x(y+1-x)y1} \\ &\quad \times [x(y+1-x)y]^{-\alpha}. \end{aligned} \quad (3.23)$$

Here N is the UV cutoff which will later go to infinity as A_0 goes to zero according to Eq. (2.13). After scaling out n , in the rescaled integral, only at most 3 out of 4 indices will get large, and those particular coefficients scale quite differently. Such scaling behavior was derived in [34]:

$$C_{j(\lambda m+\lambda n)(\lambda m)(\lambda n)} \sim \lambda C_{j(m+n)mn}, \quad \text{with } j \ll m, n, \quad (3.24)$$

$$C_{k(\lambda l)(\lambda m)n} \sim \lambda^2 C_{klmn}, \quad \text{with } l, m \gg k, n. \quad (3.25)$$

Note that the coefficients with two large indices are actually one power of λ higher than those with large three indices. This means that one can use either the λ scaling to analyze the double integral, or simply keep the boundary terms of

⁹Even though the sum over diagonal terms may appear to be logarithmically diverging, indicating some mild cutoff dependence, the n^2 factor still guarantees that it does not ruin phase coherence. This apparent divergence is mitigated or eliminated in the boundary gauge.

the x integral and use the λ^2 scaling. They lead to the same answer:

$$\begin{aligned}
 A_0^2 n^{5-2\alpha} \int_0^{\frac{N}{n}} dy \int_0^{y+1} dx 2C_{x(y+1-x)y} [x(y+1-x)y]^{-\alpha} \\
 \approx A_0^2 n^{5-2\alpha} \int_0^{\frac{N}{n}} dy 2C_{0(y+1)y} [(y+1)y]^{-\alpha} \\
 \approx A_0^2 n^{5-2\alpha} \left(\frac{N}{n}\right)^{3-2\alpha} \sim E_{\text{tot}} n^2. \quad (3.26)
 \end{aligned}$$

We can see that the $A_0^2 N^{3-2\alpha}$ combination correctly reduces to the finite total energy. This is a good assurance that our estimation is reasonable. The case with $\alpha = 3/2$ will produce a log in the second last step but also cancels out exactly to reach the same final answer.

Quite interestingly, the n^2 scaling, thus the phase-lock condition, is guaranteed by any divergent power law, thus provides an upper bound $\alpha \leq 3/2$. Extensive numerics has been done in $3+1$ dimensions, and the most up-to-date result seems to suggest $E_n \sim n^{-6/5}$ [25], namely $\alpha = 8/5$, which slightly exceeds our upper bound. Note that our bound requires an infinite power law, and any actual numerical study must have a UV cutoff. It is possible that such cutoff forbids the power law to be exactly achieved. In the future, one can try to check whether pushing to higher cutoff makes the value of α closer to $3/2$. If the current value of $\alpha = 6/5$ is confirmed, then one of our assumptions must be wrong. One obvious candidate is that it may be wrong to replace the sums by integrals.

IV. INITIAL PHASE COHERENCE

Note that the phase-coherent solutions are not guaranteed to be attractors. Even if the phase $B_n(\tau)$ is dominated by a term proportional to n at late times, we cannot just drop the subleading terms. The phases only matter mod 2π , thus any finite contribution matters. In fact, even the initial phases are relevant throughout the entire process. Here we will demonstrate that the two-mode initial data, an initial condition that has been frequently tested to lead to collapse, provide an appropriate initial condition leading to coherent phases.

The two-mode initial data is given by $A_0 \sim A_1 \sim \epsilon$ with arbitrary initial phases B_0 and B_1 . For $t \ll \epsilon^2$, namely $\tau \ll 1$, we can pretend that ϕ_0 and ϕ_1 stay as the free eigenstates, and solve higher modes in Eq. (3.3) as being resonantly driven, starting from zero amplitudes, by the lower ones. For example, ϕ_2 obeys the equation

$$\begin{aligned}
 \ddot{\phi}_2 + w_2^2 \phi_2 &= S_{1102} \phi_1^2 \phi_0 \\
 &\sim \epsilon^3 \cos[(2w_1 - w_0)t + (2B_1 - B_0)], \quad (4.1)
 \end{aligned}$$

where in the last equality we have only kept the source terms that are in resonance. This is solved by

$$\phi_2 \sim \epsilon^3 t \cos[(2w_1 - w_0)t + (2B_1 - B_0) - \pi/2]. \quad (4.2)$$

Again we have dropped some order-one factors. We only care about the powers of ϵ and t , and the phases. The above behavior for ϕ_2 is nothing but the well-known fact that a constant amplitude, resonant driving force will lead to a linear growth. It is actually a special case of a ‘‘polynomially driven’’ harmonic oscillator,

$$\ddot{f} + w^2 f = C t^j \cos(wt + \theta_j), \quad (4.3)$$

with the solution

$$f \sim t^{j+1} \cos(wt + \theta_j - \pi/2), \quad (4.4)$$

as we will show in Appendix B.

Using this general polynomial growth, one can show that higher modes, during the time $1 \ll t \ll \epsilon^{-2}$, are given by the following general form:

$$\phi_n \sim \epsilon(\epsilon^2 t)^{n-1} \cos[w_n t + (n-1)(B_1 - B_0 - \pi/2) + B_1]. \quad (4.5)$$

To establish this, note that Eq. (4.2) is not only the special case with $n = 2$, but also the first step for a proof of mathematical induction. The next step is to assume that Eq. (4.5) is true for all $2 \leq i < n$, and show that it holds for ϕ_{n+1} . We can show this as follows:

$$\begin{aligned}
 \ddot{\phi}_n + w_n \phi_n &= \sum_{0 \leq k, l, m < n}^{k+l=m+n} C_{klmn} \phi_k \phi_l \phi_m \\
 &\sim \sum_{1 \leq k, l, m < n}^{k+l=m+n} \epsilon^3 (\epsilon^2 t)^{k+l+m-3} \cos[w_n t + (k+l-m-1)(B_1 - B_0 - \pi/2) + B_1] \\
 &\quad + \sum_{1 \leq k, l < n}^{k+l=n} \epsilon^3 (\epsilon^2 t)^{k+l-2} \cos[w_n t + (k+l-2)(B_1 - B_0 - \pi/2) + 2B_1 - B_0] \\
 &\sim \epsilon^3 (\epsilon^2 t)^{n-2} \cos[w_n t + (n-2)(B_1 - B_0 - \pi/2) + 2B_1 - B_0]. \quad (4.6)
 \end{aligned}$$

The key point allowing for the simplification is that the sum is dominated by terms with $m = 0$, since it has the lowest power of ϵ . Note that these terms have the same phase, which will be true as long as the initial amplitudes of two modes are comparable and dominate over others. Thus, the last line in Eq. (4.6) has only one phase just like Eq. (4.3), with

$$\theta_n = (n-2)(B_1 - B_0 - \pi/2) + 2B_1 - B_0. \quad (4.7)$$

Thus the solution ϕ_n is given by Eq. (4.4), which indeed proves Eq. (4.5). We can now identify the phases B_n in this regime,

$$B_n = \theta_n - \pi/2 = n(B_1 - B_0 - \pi/2) + B_0 + \pi/2. \quad (4.8)$$

These phases are coherent in the sense of Eq. (3.12). Furthermore, since in the early stage the phases B_n already develop a linear n dependence, we think it is natural for the late time asymptotics to maintain such behavior, thus we should focus on the $\gamma \neq 0$ case in Eq. (3.13).

Note that every dominant term having the same phase in Eq. (4.6), independent of the initial amplitudes (as long as it is two-mode dominated), is a very special property. A three-mode initial data would have immediately undermined our simple analysis. Thus we can see that the two-mode initial data is particularly appropriate to provide initially coherent phases. It would be very interesting to extend this type of analysis to more general initial data. This could give insight into which initial data evolve into a coherent cascade.

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APPENDIX A: RANDOM PHASE ANSATZ

We can rewrite the two-time equation of motion for the amplitude A_n and phase B_n , Eq. (3.6) and (3.7), as one complex equation for $a_n = A_n e^{iB_n}$:

$$2i w_n \dot{a}_n = \sum_{klm} C_{klmn} a_k a_l a_m^*. \quad (A1)$$

Here the dot is the derivative with respect to the ‘‘long time’’ τ . The random phase ansatz assumes that the phase B_n is randomly distributed between 0 and 2π , with a constant weight, and every mode is independent from one another. Using this statistics property, we know the property of any two-mode correlator while averaging over an ensemble of random phases,

$$\langle a_m a_n \rangle = 0, \quad \langle a_m a_n^* \rangle = \frac{N_n}{w_n} \delta_{mn}. \quad (A2)$$

Here N_n is the expectation value of ‘‘particle number’’ in a mode as defined in [23]. We also know the behavior of any four-mode correlator since it factorizes:

$$\langle a_k a_l a_m^* a_n^* \rangle = N_m N_n (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}). \quad (A3)$$

It is then easy to show that

$$\begin{aligned} \dot{N}_n &= w_n (\langle \dot{a}_n a_n^* \rangle + \langle a_n \dot{a}_n^* \rangle) \\ &= \frac{w_n}{2} \left(-i \sum_{klm} C_{klmn} \langle a_k a_l a_m^* a_n^* \rangle + i \sum_{klm} C_{klmn} \langle a_k^* a_l^* a_m a_n \rangle \right) \\ &= 0. \end{aligned} \quad (A4)$$

In the last step, we simply plug in Eq. (A3).

This proves that if we combine two-time formalism with the random phase ansatz, we will get no dynamics at the leading order time scale of the two-time formalism.

APPENDIX B: POLYNOMIALLY DRIVEN HARMONIC OSCILLATOR

In the main text we needed the solution to a polynomially driven oscillator, satisfying the equation

$$\ddot{f} + w^2 f = C t^j \cos(wt + \theta_j). \quad (B1)$$

First we assume that the solution is

$$f = \sum_{i=0}^{j+1} c_i t^i \cos(wt + \xi_i). \quad (B2)$$

Taking derivatives and rearranging the sum, we get

$$\begin{aligned} \ddot{f} + w^2 f &= -2(j+1)t^j c_{j+1} w \sin(wt + \xi_{j+1}) \\ &+ \sum_{i=1}^j t^{i-1} [-2i c_i w \sin(wt + \xi_i) + i(i+1) c_{i+1} \\ &\times \cos(wt + \xi_{i+1})]. \end{aligned} \quad (B3)$$

This can be solved recursively as

$$c_{j+1} = \frac{C}{2w(j+1)}, \quad (B4)$$

$$c_i = \frac{(j+1)!c_{j+1}(2w)^i}{(2w)^{j+1}i!}, \quad (\text{B5})$$

$$\xi_{j+1} = \theta_j - \pi/2, \quad (\text{B6})$$

$$\xi_i = \xi_{i+1} + \pi/2. \quad (\text{B7})$$

Whenever $(wt) \gg 1$, the solution is dominated by the highest polynomial,

$$f(t) \approx \frac{C}{2w} \frac{t^{j+1}}{j+1} \cos(wt + \theta_j - \pi/2). \quad (\text{B8})$$

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