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Orthogonal rational functions and modified approximants

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Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in the open unit disk in the complex plane and let

$$\mathbb{B}_0 = 1 \quad \text{and} \quad \mathbb{B}_n(z) = \prod_{k=0}^{n} \frac{\overline{\alpha_k}}{|\alpha_k|} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z}, \quad n = 1, 2, \ldots,$$

($\overline{\alpha_k}/|\alpha_k| = -1$ when $\alpha_k = 0$). Let μ be a positive Borel measure on the unit circle, and let $\{\phi_n\}_{n=0}^{\infty}$ be the orthonormal sequence obtained by orthonormalization of the sequence $\{\mathbb{B}_n\}_{n=0}^{\infty}$ with respect to μ. Let $\{\psi_n\}_{n=0}^{\infty}$ be the sequence of associated rational functions. Using the functions ϕ_n, ψ_n and certain conjugates of them, we obtain modified Padé-type approximants to the function

$$F_\mu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z} d\mu(\theta), \quad (t = e^{i\theta}).$$

Keywords: Positive-definite, Hermitian inner product, orthogonal rational functions, associated functions, moment problem.

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1. Introduction

The purpose of this paper is to give certain modified rational approximants to the function

$$F_\mu(z) = \int_{-\pi}^{\pi} \frac{t+z}{t-z} d\mu(\theta), \quad (t = e^{i\theta})$$

where μ is a positive Borel measure on the unit circle in the complex plane. Let

$$T = \{z \in \mathbb{C} : |z| = 1\}, \quad D = \{z \in \mathbb{C} : |z| < 1\}, \quad E = \{z \in \mathbb{C} : |z| > 1\}$$

and let α_n, $n = 0, 1, 2, \ldots$ be given points in D with $\alpha_0 = 0$. The Blaschke factors ζ_n are given by

$$\zeta_n(z) = \frac{\overline{\alpha_n}}{|\alpha_n|} \cdot \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}, \quad n = 0, 1, 2, \ldots,$$

where by convention

$$\frac{\overline{\alpha_n}}{|\alpha_n|} = -1 \quad \text{when} \quad \alpha_n = 0.$$

The (finite) Blaschke products are

$$\mathbb{B}_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots \quad \text{and} \quad \mathbb{B}_0(z) = 1.$$

We define the linear spaces $\mathcal{L}_n$, $n = 0, 1, 2, \ldots$ and $\mathcal{L}$ by

$$\mathcal{L}_n = \text{span}\{\mathbb{B}_m : m = 0, 1, \ldots, n\} \quad \text{and} \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

Clearly $\mathcal{L}_n$ consists of the functions that may be written as

$$\frac{p_n(z)}{\pi_n(z)},$$

where

$$\pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k} z), \quad n = 1, 2, \ldots \quad \text{and} \quad \pi_0(z) = 1$$

and p_n belongs to Π_n, the set of polynomials of degree at most n. The substar conjugate f_* of a function f is defined as

$$f_*(z) = \overline{f(1/\bar{z})}.$$

For $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the superstar conjugate f^* will be

$$f^*(z) = \mathbb{B}_n(z) f_*(z).$$

If $f \in \mathcal{L}_0$, then $f^* = f_*$.

The linear spaces $\mathcal{L}_{n*}$, $n = 0, 1, 2, \ldots$, and $\mathcal{L}_*$ are defined as

$$\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\} \quad \text{and} \quad \mathcal{L}_* = \{f_* : f \in \mathcal{L}\}.$$

Then we have

$$\mathcal{L}_{n*} = \text{span}\left\{\frac{1}{\mathbb{B}_m} : m = 0, 1, \ldots, n\right\} = \text{span}\left\{\frac{1}{\omega_m} : m = 0, 1, \ldots, n\right\},$$

where

$$\omega_m(z) = \prod_{k=1}^{m}(z - \alpha_k), \quad \text{and} \quad \omega_0(z) = 1.$$

As in [1] we also put

$$\mathcal{L}_n(\alpha_n) = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}, \quad n = 1, 2, \ldots$$

and similarly

$$\mathcal{L}_{n*}(1/\overline{\alpha}_n) = \{f \in \mathcal{L}_{n*} : f(1/\overline{\alpha_n}) = 0\}, \quad n = 1, 2, \ldots.$$

Furthermore, we assume that M is a linear functional on $\mathcal{L} + \mathcal{L}_*$ such that for $f \in \mathcal{L}$ we have

$$M(f_*) = \overline{M(f)}, \quad \text{and} \quad M(ff_*) > 0 \quad \text{if} \quad f \neq 0.$$

Then this also holds for $f \in \mathcal{L} + \mathcal{L}_*$. The functional M induces an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$ by

$$\langle f, g \rangle = M(fg_*), \quad f, g \in \mathcal{L}.$$

Note that $\mathcal{L}\mathcal{L}_* = \mathcal{L} + \mathcal{L}_*$, as can be seen by partial fraction decomposition. Also for $f, g \in \mathcal{L}_*$ we may define $\langle f, g \rangle = M(fg_*)$. Then we get

$$\langle f, g \rangle = \langle g_*, f_* \rangle \quad \text{for} \quad f, g \in \mathcal{L}.$$

As $\overline{\langle g, f \rangle} = \overline{M(gf_*)} = M(fg_*) = \langle f, g \rangle$ for $f, g \in \mathcal{L}$ and $\langle f, f \rangle = M(ff_*) > 0$ for $f \in \mathcal{L}, f \neq 0$, the inner product is Hermitian and positive-definite on $\mathcal{L} \times \mathcal{L}$.

In this paper we assume that μ is a solution to the following "moment" problem:

Given the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L} \times \mathcal{L}$ (or the linear functional M on $\mathcal{L} + \mathcal{L}_*$), find a non-decreasing function μ on $[-\pi, \pi]$ (or a positive Borel measure μ on $(-\pi, \pi]$) such that

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta) \quad \text{for} \quad f, g \in \mathcal{L}$$

$$\left(\text{or} \quad M(f) = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta) \quad \text{for} \quad f \in \mathcal{L} + \mathcal{L}_*\right).$$

This moment problem always has a solution. Two non-decreasing functions which are solutions of the moment problem such that their difference is a constant at all the points at which it is continuous, are considered to be the same solution of the moment problem. We will give modified rational approximants to the function F_μ in terms of orthogonal rational functions and their associates. Besides, we obtain some results about related quadrature formulas.

2. Orthogonal rational functions

In our approach orthogonal rational functions will play an important rôle. Let the sequence $\{\phi_n\}_{n=0}^{\infty}$ in $\mathcal{L}$ be obtained by orthonormalization of the sequence $\{\mathbb{B}_n\}_{n=0}^{\infty}$ with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{L}\times\mathcal{L}$, i.e.

$$\phi_n \in \mathcal{L}_n \quad \text{and} \quad \langle\phi_n,\phi_n\rangle = 1, \quad n = 0,1,2,\ldots$$

and

$$\langle f,\phi_n\rangle = 0 \quad \text{for} \quad f \in \mathcal{L}_{n-1}, \quad n = 1,2,\ldots.$$

It follows easily that

$$\langle f,\phi_n^*\rangle = 0 \quad \text{for} \quad f \in \mathcal{L}_n(\alpha_n), \quad n = 1,2,\ldots,$$

because $\mathbb{B}_n f_* \in \mathcal{L}_{n-1}$ for such f. Each ϕ_n can be written as

$$\phi_n(z) = \sum_{k=0}^{n} b_k^{(n)} \mathbb{B}_k(z).$$

Here the non-zero number $b_n^{(n)}$ is called the leading coefficient of ϕ_n. We assume that the ϕ_n are chosen such that $b_n^{(n)} > 0$ and we write $\kappa_n = b_n^{(n)}$. It is easily shown that

$$\kappa_n = \overline{\phi_n^*(\alpha_n)} = \phi_n^*(\alpha_n).$$

Using the uniqueness of the reproducing kernel

$$\sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)}$$

for the inner product space $\mathcal{L}_n$ one can show (see for instance [1]) that the following Christoffel-Darboux formula holds

$$\sum_{k=0}^{n-1} \phi_k(z)\overline{\phi_k(w)} = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}, \tag{2.1}$$

and equivalently

$$\sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)} = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}}. \tag{2.2}$$

The ϕ_n and ϕ_n^* satisfy the recurrence relations

$$\phi_n(z) = \epsilon_n \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}(z) + \delta_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}^*(z), \quad n = 1,2,\ldots \tag{2.3}$$

and (superstar conjugation)

$$\phi_n^*(z) = -\frac{\overline{\alpha_n}}{|\alpha_n|}\,\overline{\delta_n}\,\frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}(z) - \frac{\overline{\alpha_n}}{|\alpha_n|}\,\overline{\epsilon_n}\,\frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z}\frac{\kappa_n}{\kappa_{n-1}}\phi_{n-1}^*(z),$$
$$n = 1,2,\ldots \tag{2.4}$$

with $\phi_0 = \phi_0^* = \kappa_0$. Here

$$\epsilon_n = -\frac{\overline{\alpha_n}}{|\alpha_n|}\,\frac{1-\overline{\alpha_{n-1}}\alpha_n}{1-|\alpha_{n-1}|^2}\,\frac{\overline{\phi_n^*(\alpha_{n-1})}}{\kappa_n}, \tag{2.5}$$

$$\delta_n = \frac{1-\alpha_{n-1}\overline{\alpha_n}}{1-|\alpha_{n-1}|^2}\,\frac{\phi_n(\alpha_{n-1})}{\kappa_n}. \tag{2.6}$$

It follows from the Christoffel-Darboux formula (2.1) with $z = w = \alpha_{n-1}$ that $\epsilon_n \neq 0$. A proof of (2.3) and (2.4) can be found in [1] or in [2], but (2.3) and (2.4) may also be derived from the superstar conjugates with respect to w and with repect to z and w of the Christoffel-Darboux formula. We mention another consequence of the Christoffel-Darboux formula. Taking the superstar conjugate of (2.1) with respect to z and w and writing

$$\mathbb{B}_{n\backslash k} = \mathbb{B}_n/\mathbb{B}_k, \quad k = 0, 1, \ldots, n; \quad n = 0, 1, \ldots,$$

we obtain

$$\frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}}{1-\zeta_n(z)\overline{\zeta_n(w)}} = \sum_{k=0}^{n-1} \mathbb{B}_{(n-1)\backslash k}(z)\overline{\mathbb{B}_{(n-1)\backslash k}(w)}\phi_k^*(z)\overline{\phi_k^*(w)}. \tag{2.7}$$

For $z = w = \alpha_{n-1}$ this gives

$$\begin{aligned} |\phi_n^*(\alpha_{n-1})|^2 - |\phi_n(\alpha_{n-1})|^2 &= |\phi_{n-1}^*(\alpha_{n-1})|^2\left[1-|\zeta_n(\alpha_{n-1})|^2\right] \\ &= \kappa_{n-1}^2\,\frac{(1-|\alpha_n|^2)(1-|\alpha_{n-1}|^2)}{|1-\overline{\alpha_n}\alpha_{n-1}|^2}. \end{aligned}$$

Together with (2.5) and (2.6) this leads to

$$|\epsilon_n|^2 - |\delta_n|^2 = \frac{\kappa_{n-1}^2}{\kappa_n^2}\,\frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2}. \tag{2.8}$$

In particular this implies

$$|\epsilon_n| > |\delta_n|. \tag{2.9}$$

A different proof of (2.8) can be found in [4].

3. Associated functions

Next to the orthogonal functions ϕ_n we consider the associated functions ψ_n defined by

$$\psi_0(z) = -\frac{1}{\kappa_0}, \quad (\psi_0(z) = -M(\phi_0)),$$

and

$$\psi_n(z) = M(D(t,z)[\phi_n(z) - \phi_n(t)]), \quad n = 1, 2, \ldots.$$

Here M is acting on t and

$$D(t,z)=\frac{t+z}{t-z}.$$

Obviously $\psi_n\in\mathcal{L}_n$ for $n=0,1,2,\ldots$. For $f\in\mathcal{L}_{(n-1)*}$ we may write

$$f(t)=\frac{a(t)}{\omega_{n-1}(t)}$$

with $a\in\Pi_{n-1}$, so, if $f\not\equiv 0$, then

$$\begin{aligned}D(t,z)\left[1-\frac{f(t)}{f(z)}\right]&=\frac{t+z}{t-z}\left[1-\frac{a(t)}{a(z)}\,\frac{\omega_{n-1}(z)}{\omega_{n-1}(t)}\right]\\&=\frac{(t+z)[a(z)\omega_{n-1}(t)-a(t)\omega_{n-1}(z)]}{(t-z)a(z)}\,\frac{1}{\omega_{n-1}(t)}\end{aligned}$$

is in $\mathcal{L}_{(n-1)*}$. Hence

$$M\left(D(t,z)\left[1-\frac{f(t)}{f(z)}\right]\phi_n(t)\right)=0\quad\text{for}\quad f\in\mathcal{L}_{(n-1)*},\quad f\not\equiv 0.$$

This gives immediately

$$\psi_n(z)=M\left(D(t,z)\left[\phi_n(z)-\frac{f(t)}{f(z)}\,\phi_n(t)\right]\right)\ \text{for}\ f\in\mathcal{L}_{(n-1)*},\quad f\not\equiv 0,\quad n=1,2,\ldots.\tag{3.1}$$

For the superstar conjugates of the ψ_n we have

$$\psi_0^*(z)=-\frac{1}{\kappa_0}$$

and, since $\psi_n\in\mathcal{L}_n$,

$$\begin{aligned}\psi_n^*(z)&=\mathbb{B}_n(z)\overline{M(D(t,1/\bar{z})[\phi_n(1/\bar{z})-\phi_n(t)])}\\&=\mathbb{B}_n(z)M(D(1/t,1/z)[\overline{\phi_n(1/\bar{z})}-\overline{\phi(1/\bar{t})}])\\&=-\mathbb{B}_n(z)M(D(t,z)[\phi_{n*}(z)-\phi_{n*}(t)])\\&=M\left(D(t,z)\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)}\,\phi_n^*(t)-\phi_n^*(z)\right]\right),\quad n=1,2,\ldots,\end{aligned}$$

so

$$\psi_n^*(z)=M\left(D(t,z)\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)}\,\phi_n^*(t)-\phi_n^*(z)\right]\right),\quad n=1,2,\ldots.\tag{3.2}$$

If $f\in\mathcal{L}_{n*}(1/\overline{\alpha_n})$, we may write

$$f(t)=\frac{(1-\overline{\alpha_n}t)b(t)}{\omega_n(t)}\quad\text{with}\quad b\in\Pi_{n-1}.$$

So, for $f \not\equiv 0$,

$$D(t,z)\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)} - \frac{f(t)}{f(z)}\right]$$
$$= \frac{\omega_n(z)}{\pi_n(z)(1-\overline{\alpha_n}z)b(z)} \; \frac{(t+z)[\pi_n(t)(1-\overline{\alpha_n}z)b(z) - \pi_n(z)(1-\overline{\alpha_n}t)b(t)]}{t-z} \; \frac{1}{\omega_n(t)}$$

belongs to $\mathcal{L}_{n*}(1/\overline{\alpha_n})$, and it follows that

$$M\left(D(t,z)\left[\frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)} - \frac{f(t)}{f(z)}\right]\phi_n^*(t)\right) = 0.$$

This gives

$$\psi_n^*(z) = M\left(D(t,z)\left[\frac{f(t)}{f(z)}\phi_n^*(t) - \phi_n^*(z)\right]\right) \text{ for } f \in \mathcal{L}_{n*}(1/\overline{\alpha_n}),\ f \not\equiv 0,\ n = 1,2,\ldots. \tag{3.3}$$

The functions ψ_n and ψ_n^* satisfy the recurrences

$$\psi_n(z) = \epsilon_n \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) - \delta_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}^*(z), \quad n = 1,2,\ldots \tag{3.4}$$

and (superstar conjugation)

$$\psi_n^*(z) = \frac{\overline{\alpha_n}}{|\alpha_n|} \overline{\delta_n} \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) - \frac{\overline{\alpha_n}}{|\alpha_n|} \overline{\epsilon_n} \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}^*(z),$$
$$n = 1,2,\ldots. \tag{3.5}$$

A proof of these recurrence formulas is given in [1], but they also follow easily from the above results. Indeed, writing

$$A_n(z) = \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z} \quad \text{and} \quad B_n(z) = \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z},$$

for $n \geq 2$ we have

$$\psi_n(z) - \epsilon_n A_n(z)\frac{\kappa_n}{\kappa_{n-1}}\psi_{n-1}(z)$$
$$= M\left(D(t,z)\left[\phi_n(z) - \frac{f(t)}{f(z)}\phi_n(t)\right]\right) - \epsilon_n A_n(z)\frac{\kappa_n}{\kappa_{n-1}} M(D(t,z)[\phi_{n-1}(z) - \phi_{n-1}(t)]),$$

where $f \in \mathcal{L}_{(n-1)*}, f \not\equiv 0$ such that $f(1/\overline{\alpha_n}) = 0$, so

$$f(t) = \frac{(1-\overline{\alpha_n}t)p(t)}{\omega_{n-1}(t)} \quad \text{with} \quad p \in \Pi_{n-2}.$$

Elementary calculations using (2.3) and (2.4) give

$$\psi_n(z) - \epsilon_n A_n(z) \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) = I_1 + I_2$$

with

$$\begin{aligned} I_1 &= \delta_n \frac{\kappa_n}{\kappa_{n-1}} B_n(z) M\left(D(t,z) \left[\phi^*_{n-1}(z) - \frac{f(t)}{f(z)} \frac{B_n(t)}{B_n(z)} \phi^*_{n-1}(t) \right] \right) \\ &= -\delta_n \frac{\kappa_n}{\kappa_{n-1}} B_n(z) \psi^*_{n-1}(z) \end{aligned}$$

since $f(t)B_n(t) \in \mathcal{L}_{(n-1)*}(1/\overline{\alpha_{n-1}})$, and

$$I_2 = \epsilon_n \frac{\kappa_n}{\kappa_{n-1}} A_n(z) M\left(D(t,z) \left[1 - \frac{f(t)}{f(z)} \frac{A_n(t)}{A_n(z)} \right] \phi_{n-1}(t) \right) = 0$$

since

$$f(t)A_n(t) = \frac{(1-\overline{\alpha_n}t)p(t)}{\omega_{n-1}(t)} \frac{t-\alpha_{n-1}}{1-\overline{\alpha_n}t} = \frac{p(t)}{\omega_{n-2}(t)} \in \mathcal{L}_{(n-2)*}.$$

Formula (3.5) follows by superstar conjugation. The case $n = 1$ is easily verified. Thus the pair $(\psi_n, -\psi_n^*)$ satisfies the same recurrence as the pair (ϕ_n, ϕ_n^*). The initial values are $(\phi_0, \phi_0^*) = \kappa_0(1,1)$ and $(\psi_0, -\psi_0^*) = (-1/\kappa_0)(1,-1)$.

4. Para-orthogonal functions, quadrature formulas and modified approximants

It follows easily from the Christoffel-Darboux formula (2.1) that the zeros of ϕ_n are in D and that the zeros of ϕ_n^* are in E. Moreover, we have $|\phi_n(z)| < |\phi_n^*(z)|$ for $z \in D$ and $|\phi_n(z)| > |\phi_n^*(z)|$ for $z \in E$. As we intend to give quadrature formulas with nodes in T we consider the functions

$$Q_n(z,w) = \phi_n(z) + w\phi_n^*(z), \quad n = 0, 1, 2, \ldots \tag{4.1}$$

with $w \in T$ arbitrary. Clearly the zeros $z_1, \ldots, z_n$ of $Q_n(z,w)$ are all in T and it is easy to show that they are simple. See [1]. Of course the zeros z_j depend on n and w. Since

$$Q_n(z,w) \perp \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n), \quad n = 1, 2, \ldots$$

and

$$\langle Q_n(z,w), 1 \rangle \neq 0 \quad \text{and} \quad \langle Q_n(z,w), \mathbb{B}_n(z) \rangle \neq 0, \quad n = 1, 2, \ldots,$$

where the inner product acts on z, the sequence is called para-orthogonal. As

$$Q_n^*(z,w) = \overline{w} Q_n(z,w),$$

superstar conjugation with respect to z, the Q_n are called $\overline{w}$-invariant. Notice that the above orthogonality remains valid if for each n we take for w a fixed w_n

in T. If

$$\Lambda_{n,i}(z)=\frac{1-\overline{\alpha_n}z}{1-\overline{\alpha_n}z_i}\,\frac{Q_n(z,w)}{(z-z_i)Q_n'(z_i,w)},\quad i=1,\dots,n, \tag{4.2}$$

where the prime means differentiation with respect to z, then $\Lambda_{n,i}\in\mathcal{L}_{n-1}$ and we have the quadrature formula (see [1])

$$M(R)=\sum_{j=1}^{n}\lambda_{n,j}R(z_j)\quad\text{for}\quad R\in\mathcal{L}_{(n-1)*}+\mathcal{L}_{n-1}, \tag{4.3}$$

with $\lambda_{n,j}=M(\Lambda_{n,j})>0$ for $j=1,\dots,n$.

Let us assume now that $z_j=e^{i\theta_j}$, $j=1,2,\dots,n$, with

$$-\pi\le\theta_1<\theta_2<\dots<\theta_n<\pi.$$

Then, using the functions μ_n given by

$$\mu_n(\theta)=\begin{cases}0 & \text{if}\quad -\pi\le\theta\le\theta_1,\\ \sum_{j=1}^{k}\lambda_{n,j} & \text{if}\quad \theta_k<\theta\le\theta_{k+1},\quad k=1,\dots,n-1,\\ M(1) & \text{if}\quad \theta_n<\theta\le\pi\end{cases}$$

(or using the measures $\mu_n=\sum_{j=1}^{n}\lambda_{n,j}\delta_{\theta_j}$, where δ_{θ_j} is the translated Dirac measure), it follows from Helly's theorems (or from the weak* compactness of the closed unit ball in the dual space of the Banach space $C(T)$), that the moment problem has a solution, say μ. So there is a non-decreasing function (or a positive Borel measure) μ such that

$$M(R)=\int_{-\pi}^{\pi}R(e^{i\theta})d\mu(\theta)\quad\text{for}\quad R\in\mathcal{L}_*+\mathcal{L}. \tag{4.4}$$

It follows from the fact that the inner product is positive definite that the solutions μ must have infinitely many points of increase (or must be measures with infinite support).

Now let

$$F_\mu(z)=\int_{-\pi}^{\pi}\frac{t+z}{t-z}d\mu(\theta),\quad (t=e^{i\theta}) \tag{4.5}$$

and

$$R_n(z,w)=\int_{-\pi}^{\pi}\frac{t+z}{t-z}d\mu_n(\theta)=\sum_{j=1}^{n}\lambda_{n,j}\frac{z_j+z}{z_j-z}.$$

Then $R_n(z,w)$ can be written as

$$R_n(z,w)=\frac{P_n(z,w)}{Q_n(z,w)}\quad\text{with}\quad P_n(z,w)\in\mathcal{L}_n.$$

We will show that

$$P_n(z,w)=\psi_n(z)-w\psi_n^*(z),\quad n=1,2,\dots. \tag{4.6}$$

Indeed, for $n \geq 2$ we have by the results of section 3

$$\begin{aligned}\psi_n(z) - w\psi_n^*(z) &= M\left(D(t,z)\left[\phi_n(z) - \frac{f(t)}{f(z)}\phi_n(t)\right]\right)\\ &\quad + wM\left(D(t,z)\left[\phi_n^*(z) - \frac{f(t)}{f(z)}\phi_n^*(t)\right]\right)\\ &= M\left(D(t,z)\left[\phi_n(z) + w\phi_n^*(z) - \frac{f(t)}{f(z)}(\phi_n(t) + w\phi_n^*(t))\right]\right)\\ &= M\left(D(t,z)\left[Q_n(z,w) - \frac{f(t)}{f(z)}Q_n(t,w)\right]\right)\end{aligned}$$

for $f \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_{n*}(1/\overline{\alpha_n}), f \not\equiv 0$. As

$$D(t,z)\left[Q_n(z,w) - \frac{f(t)}{f(z)}Q_n(t,w)\right] \in \mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1}$$

for such f, we have

$$\psi_n(z) - w\psi_n^*(z) = \sum_{j=1}^{n} \lambda_{n,j}\frac{z_j + z}{z_j - z}\ Q_n(z,w) = P_n(z,w).$$

The case $n = 1$ follows by direct verification, using $\lambda_{1,1} = M(1) = 1/\kappa_0^2$.

In [3] a formula like (4.6) could only be obtained in the "cyclic" situation, i.e. in the case of a finite number of points α_n repeated in cyclic order.

From the partial fraction decomposition

$$R_n(z,w) = \sum_{j=1}^{n} \lambda_{n,j}\frac{z_j + z}{z_j - z}$$

it follows that

$$(z - z_k)R_n(z,w) = \sum_{\substack{j=1\\ j\neq k}}^{n} \lambda_{n,j}\frac{(z_j + z)(z - z_k)}{z_j - z} - \lambda_{n,k}(z_k + z),$$

so the limit $z \to z_k$ gives

$$\frac{P_n(z_k,w)}{Q_n'(z_k,w)} = -2z_k\lambda_{n,k}, \quad k = 1,\ldots,n.$$

Hence

$$\lambda_{n,k} = -\frac{1}{2z_k}\frac{P_n(z_k,w)}{Q_n'(z_k,w)}, \quad k = 1,\ldots,n. \tag{4.7}$$

From the fact that

$$\int_{-\pi}^{\pi} g(e^{i\theta})d(\mu - \mu_n)(\theta) = 0 \quad \text{for} \quad g \in \mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1}$$

if μ is a solution of the moment problem and μ_n is a solution of the "truncated" moment problem as above, we get the general form of some results which were obtained in [3], (p.63, formula (3.43)), for the cyclic situation.

If $g \in \mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1}$, then g is of the form

$$g(t) = \frac{p(t)}{\omega_{n-1}(t)\pi_{n-1}(t)} \quad \text{with} \quad p \in \Pi_{2n-2},$$

so

$$D(t,z)[g(t) - g(z)] = \frac{(t+z)[\omega_{n-1}(z)\pi_{n-1}(z)p(t) - \omega_{n-1}(t)\pi_{n-1}(t)p(z)]}{(t-z)\omega_{n-1}(z)\pi_{n-1}(z)} \cdot \frac{1}{\omega_{n-1}(t)\pi_{n-1}(t)}$$

is in $\mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1}$, and

$$\int_{-\pi}^{\pi} D(t,z)[g(t) - g(z)]d(\mu - \mu_n)(\theta) = 0, \quad (t = e^{i\theta}).$$

As

$$F_\mu(z) - R_n(z,w) = \int_{-\pi}^{\pi} D(t,z)d(\mu - \mu_n)(\theta),$$

it follows now that

$$\begin{aligned} h(z) &= \int_{-\pi}^{\pi} D(t,z)g(t)d(\mu - \mu_n)(\theta) \\ &= g(z)\int_{-\pi}^{\pi} D(t,z)d(\mu - \mu_n)(\theta) = g(z)\big(F_\mu(z) - R_n(z,w)\big). \end{aligned}$$

Clearly, h is analytic in D and in E and $h(0) = 0$ and $h(\infty) = \lim_{z\to\infty} h(z) = 0$. For $g(z) = \mathbb{B}_{n-1}(z)$ we get

$$F_\mu(z) - R_n(z,w) = \frac{1}{\mathbb{B}_{n-1}(z)} h_\infty(z),$$

where h_∞ is analytic in $D \cup E$ and $h_\infty(\infty) = 0$ and for $g(z) = 1/\mathbb{B}_{n-1}(z)$ we obtain

$$F_\mu(z) - R_n(z,w) = \mathbb{B}_{n-1}(z)h_0(z),$$

where h_0 is analytic in $D \cup E$ and $h_0(0) = 0$. Thus $R_n(z,w)$ is a "modified" Padé-type approximant to F_μ. Notice that the functions h, h_0, h_∞ depend on the parameter w. Because the approximants $R_n(z,w)$ have the same structure relative to the orthogonal sequence $\{\phi_n\}_{n=0}^{\infty}$ and the sequence of the associated functions $\{\psi_n\}_{n=0}^{\infty}$ as in the case of so called modified approximants to HPC (=Hermitian Perron-Carathéodory) continued fractions (see [6]), the $R_n(z,w)$ are also called *modified* approximants in the present situation.

Since

$$P_n(z,w) = M\left(D(t,z)\left[Q_n(z,w) - \frac{f(t)}{f(z)}Q_n(t,w)\right]\right)$$

for $f \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_{n*}(1/\overline{\alpha_n}), f \not\equiv 0$, it follows that for the error $F_\mu(z) - R_n(z,w)$ we also have (with $t = e^{i\theta}$)

$$\begin{aligned} F_\mu(z) - R_n(z,w) &= \int_{-\pi}^{\pi} D(t,z)d\mu(\theta) \\ &\quad - \frac{1}{Q_n(z,w)}\int_{-\pi}^{\pi} D(t,z)\left[Q_n(z,w) - \frac{f(t)}{f(z)}Q_n(t,w)\right]d\mu(\theta) \\ &= \frac{1}{f(z)Q_n(z,w)}\int_{-\pi}^{\pi} D(t,z)\, f(t)\; Q_n(t,w)\; d\mu(\theta). \end{aligned}$$

See [5].

We conclude the paper with a remark on the quadrature weights. Recall that $z_1, \ldots, z_n$ are the zeros of Q_n and that $|z_j| = 1$, $j = 1, \ldots, n$. For $z = z_j$ we have $\phi_n(z_j) + w\phi_n^*(z_j) = 0$ and $\overline{\zeta_n(z_j)} = 1/\zeta_n(z_j)$, so by the Christoffel-Darboux formula

$$\zeta_n(z_j)\overline{\phi_n(z_j)}\;\frac{Q_n(z_j,w) - Q_n(t,w)}{z_j - t} = \frac{\zeta_n(z_j) - \zeta_n(t)}{z_j - t}\sum_{k=0}^{n-1}\overline{\phi_k(z_j)}\phi_k(t).$$

Using

$$\frac{\zeta_n'(z_j)}{\zeta_n(z_j)} = \frac{1 - |\alpha_n|^2}{(z_j - \alpha_n)(1 - \overline{\alpha_n}z_j)},$$

the limit $t \to z_j$ yields

$$Q_n'(z_j,w) = \frac{1}{\overline{\phi_n(z_j)}}\;\frac{1 - |\alpha_n|^2}{(z_j - \alpha_n)(1 - \overline{\alpha_n}z_j)}\;\sum_{k=0}^{n-1}|\phi_k(z_j)|^2.$$

Using (2.8) and the recurrence relations (2.3), (2.4), (3.4), (3.5) it follows that the functions ϕ_n and ψ_n and their superstar conjugates satisfy a determinant formula

$$\phi_n^*(z)\psi_n(z) + \phi_n(z)\psi_n^*(z) = \frac{1 - |\alpha_n|^2}{1 - \overline{\alpha_n}z}\;\frac{-2z\mathbb{B}_n(z)}{z - \alpha_n}.$$

As $\phi_n(z_j) + w\phi_n^*(z_j) = 0$, this gives

$$P_n(z_j,w) = \frac{1}{\phi_n^*(z_j)}\;\frac{-2z_j\mathbb{B}_n(z_j)(1 - |\alpha_n|^2)}{(z_j - \alpha_n)(1 - \overline{\alpha_n}z_j)}.$$

Using (4.7), we get

$$\lambda_{nj} = -\frac{1}{2z_j}\;\frac{P_n(z_j,w)}{Q_n'(z_j,w)} = \frac{\overline{\phi_n(z_j)}}{\phi_n^*(z_j)}\;\frac{\mathbb{B}_n(z_j)}{\sum_{k=0}^{n-1}|\phi_k(z_j)|^2}.$$

Since $|z_j| = 1$, we have

$$\phi_n^*(z_j) = \mathbb{B}_n(z_j)\overline{\phi_n(z_j)}$$

and we obtain

$$\lambda_{n,j} = \frac{1}{\sum_{k=0}^{n-1} |\phi_k(z_j)|^2}, \quad j = 1, \ldots, n; \quad n \in \mathbb{N}.$$

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