



UvA-DARE (Digital Academic Repository)

Evolution of density perturbations in a flat FRW universe

Miedema, P.G.; van Leeuwen, W.A.

DOI

[10.1103/PhysRevD.54.7237](https://doi.org/10.1103/PhysRevD.54.7237)

Publication date

1996

Published in

Physical Review D. Particles, Fields, Gravitation, and Cosmology

[Link to publication](#)

Citation for published version (APA):

Miedema, P. G., & van Leeuwen, W. A. (1996). Evolution of density perturbations in a flat FRW universe. *Physical Review D. Particles, Fields, Gravitation, and Cosmology*, 54(12), 7237-7242. <https://doi.org/10.1103/PhysRevD.54.7237>

General rights

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

Evolution of density perturbations in a flat FRW universe

P. G. Miedema*

Koninklijke Militaire Academie, Kasteelplein 10, NL-4811 XC Breda, The Netherlands

W. A. van Leeuwen

Universiteit van Amsterdam, Instituut voor Theoretische Fysica, Valckenierstraat 65, NL-1018 XE Amsterdam, The Netherlands

(Received 15 July 1996)

The gauge-invariant perturbation equations proposed in the preceding article are solved in this article in order to study the evolution of the energy density contrast in the radiation- and matter-dominated eras. The results are compared with earlier non-gauge-invariant as well as gauge-invariant treatments. In a number of cases the solutions are different. In the radiation-dominated era growing perturbations are found, where other treatments lead to constant or decaying energy density perturbations. [S0556-2821(96)06524-1]

PACS number(s): 98.80.Bp, 04.25.Nx, 98.65.Dx

I. INTRODUCTION

As has been discussed in the companion article [1], former attempts to formulate a gauge-invariant perturbation theory have led to equations that differ from ours, for physical quantities of which the definitions differ from ours. Examples of such physical quantities are the energy density perturbation $\epsilon_{(1)}$, its associated density contrast $\delta := \epsilon_{(1)}/\epsilon_{(0)}$, and the particle number density perturbation $\nu_{(1)}$. We do *not* claim, however, that the results of others are wrong. What we do claim is that our gauge-invariant approach yields the nonrelativistic limit in a more transparent and natural way than former gauge-invariant treatments.

In this article we will apply our method to determine explicit expressions for the gauge-invariant density contrast δ on the isotropic energy density background $\epsilon_{(0)}$ in two limiting cases, namely, the radiation-dominated and the matter-dominated eras of the expanding universe. After what has been said in the first phrase of this section, it will not come as a surprise that our final results often differ from those obtained earlier. An overview of the various results of the different approaches to the perturbation theory in question is given in Table I. The most striking difference is the occurrence of growing solutions, where other approaches lead to constant or even decaying solutions.

Formula numbers preceded by a roman numeral I refer to equation numbers in the companion article [1].

II. GAUGE-INDEPENDENT PERTURBATION EQUATIONS: RÉSUMÉ

It is our purpose to study the behavior of the density contrast, $\delta := \epsilon_{(1)}/\epsilon_{(0)}$, in an expanding universe. The quantities $\epsilon_{(1)}$ (the energy density perturbation) and $\nu_{(1)}$ (the particle number density perturbation) obey the manifestly gauge-invariant equations (I.70):

$$\dot{\omega}_{(1)} = -3H \left(1 - \frac{n_{(0)} p_n}{\epsilon_{(0)} + p_{(0)}} \right) \omega_{(1)}, \quad (1a)$$

$$\ddot{\epsilon}_{(1)} + \alpha \dot{\epsilon}_{(1)} + \beta \epsilon_{(1)} = \gamma \omega_{(1)}, \quad (1b)$$

where the quantity $\omega_{(1)}$ is defined as a combination (I.69) of $\epsilon_{(1)}$ and $\nu_{(1)}$:

$$\omega_{(1)} := \nu_{(1)} - \frac{n_{(0)}}{\epsilon_{(0)} + p_{(0)}} \epsilon_{(1)}. \quad (2)$$

The coefficients α , β , and γ are given by (I.63) with (I.64). The Hubble factor H , the zero-order energy density $\epsilon_{(0)}$, and the zero-order particle number density $n_{(0)}$ are solutions of the unperturbed Einstein equations (I.16) for a flat, unperturbed Friedmann-Robertson-Walker (FRW) universe:

$$\dot{H} = -3H^2 + \frac{1}{2} \kappa (\epsilon_{(0)} - p_{(0)}), \quad (3a)$$

$$H^2 = \frac{1}{3} \kappa \epsilon_{(0)}, \quad (3b)$$

$$\dot{\epsilon}_{(0)} = -3H(\epsilon_{(0)} + p_{(0)}), \quad (3c)$$

$$\dot{n}_{(0)} = -3Hn_{(0)}. \quad (3d)$$

These are three differential equations; Equation (3b) is a condition only on the initial values. The constant κ is equal to $8\pi G/c^4$ with G as Newton's gravitational constant and c as the speed of light. A dot denotes differentiation with respect to ct . The equation of state $p = p(n, \epsilon)$ is supposed to be a given function of n , the particle number density, and ϵ , the energy density. In order to solve Eqs. (1), we first have to solve the zero-order equations (3). We will limit the discussion to two extremal cases, namely, the radiation- and matter-dominated universe. This will be the subject of Secs. III and IV.

III. DENSITY CONTRAST IN THE RADIATION-DOMINATED ERA

Since in a radiation-dominated universe, the pressure is equal to one-third of the total energy density

$$p = \frac{1}{3} \epsilon, \quad (4)$$

*Electronic address: p.miedema@kma.nl

TABLE I. Growth rates of the density contrast obtained from different perturbation theories. We distinguish between large scale ($\lambda \rightarrow \infty$) and small scale ($\lambda \rightarrow 0$) perturbations of the radiation- and matter-dominated eras. Our results (MvL) are compared to those of Mukhanov, Feldman, and Brandenberger (MFB) and to those of the nongauge-invariant theory in synchronous coordinates (SC).

Scale	Perturbation theory	Radiation-dominated era		Matter-dominated era	
				$p \neq 0$	$p = 0$
$\lambda \rightarrow \infty$	MvL	$t, t^{1/2}$	$t^{2/3}, t^{-1}$	$t^{2/3}$	
	MFB	t^0	—	$t^0, t^{-5/3}$	
	SC	$t, t^{1/2}, t^{-1}$	—	$t^{2/3}, t^{-1}$	
$\lambda \rightarrow 0$	MvL	$t^{1/2}$	$t^{-1/6}$	$t^{2/3}$	
	MFB	$t^0, t^{-1/2}$	—	$t^{2/3}, t^{-5/3}$	
	SC	t^0	—	$t^{2/3}, t^{-1}$	

the set of unperturbed equations (3) reduces to

$$\dot{H} = -3H^2 + \frac{1}{3} \kappa \varepsilon_{(0)}, \quad (5a)$$

$$H^2 = \frac{1}{3} \kappa \varepsilon_{(0)}, \quad (5b)$$

$$\dot{\varepsilon}_{(0)} = -4H\varepsilon_{(0)}. \quad (5c)$$

We have omitted Eq. (3d) because we are not interested in the time behavior of the particle number density of the radiation-dominated era and because it does not occur in the remaining equations. The solutions of Eqs. (5) are

$$H(t) = \frac{1}{2} (ct)^{-1}, \quad t \geq t_0, \quad (6a)$$

$$\varepsilon_{(0)}(t) = \frac{3}{4\kappa} (ct)^{-2}, \quad t \geq t_0, \quad (6b)$$

where t_0 is the time at which the radiation-dominated era sets in. Using the relation between the Hubble parameter H and the scale factor a , $H = \dot{a}/a$, one finds, from Eq. (6a),

$$a(t) = a(t_0) \left(\frac{t}{t_0} \right)^{1/2}. \quad (7)$$

We will now rewrite Eq. (1b) as an equation for the density contrast. From Eq. (4) we have

$$p_\varepsilon = \frac{1}{3}, \quad p_n = 0, \quad (8)$$

as follows with Eq. (I.5). Upon substituting Eqs. (4) and (8) into the coefficients (I.63) with (I.64), one finds that Eq. (1b) reduces to

$$\ddot{\varepsilon}_{(1)} + 7H\dot{\varepsilon}_{(1)} + \frac{1}{3} \nabla^2 \varepsilon_{(1)} + 6H^2 \varepsilon_{(1)} = 0. \quad (9)$$

In view of Eq. (8), the coefficient γ in Eq. (1b) vanishes. Therefore Eq. (1a) need not be considered as far as Eq. (1b) is concerned. Moreover, we omit Eq. (1a) since we are not interested in the evolution of the ancillary quantity $\omega_{(1)}$.

We now can rewrite the evolution equation (9) in terms of the density contrast. As usual, we define the latter as the quotient of the energy density perturbation and the background energy density

$$\delta(t, x^i) := \frac{\varepsilon_{(1)}(t, x^i)}{\varepsilon_{(0)}(t)}. \quad (10)$$

However since $\varepsilon_{(1)}$ is the—newly defined—gauge-invariant quantity used in Ref. [1], the density contrast (10) differs from the density contrast defined elsewhere. Upon substituting Eq. (10) into Eq. (9) and using Eqs. (5), we arrive at a second-order homogeneous differential equation for our density contrast $\delta(t, x^i)$:

$$\ddot{\delta} - H\dot{\delta} + \frac{1}{3} \nabla^2 \delta + 2H^2 \delta = 0. \quad (11)$$

We may try to solve Eq. (11) by Fourier analysis of the function δ . Writing

$$\delta(t, x^i) = \delta(t, q) e^{i\mathbf{q} \cdot \mathbf{x}}, \quad q = |\mathbf{q}| = \frac{2\pi}{\lambda}, \quad (12)$$

where λ is the wavelength of the perturbation and $i^2 = -1$, the evolution equation (11) for the amplitude $\delta(t, q)$ reads

$$\ddot{\delta} - \frac{1}{2ct} \dot{\delta} + \frac{q^2}{3a^2(t_0)} \left(\frac{t}{t_0} \right)^{-1} \delta + \frac{1}{2(ct)^2} \delta = 0, \quad (13)$$

where we have used Eqs. (I.25), (6a), and (7). Let us rewrite this equation in such a way that the coefficients become dimensionless. To that end we take, as an independent variable,

$$\tau := \frac{t}{t_0}, \quad t \geq t_0. \quad (14)$$

From Eq. (14) we get

$$\frac{d^n}{c^n dt^n} = [2H(t_0)]^n \frac{d^n}{d\tau^n}, \quad n = 1, 2, \dots, \quad (15)$$

where we have used Eq. (6a). Using Eqs. (14), (6a), and (15), Eq. (13) for the density contrast $\delta(\tau, q)$ can be written as

$$\delta'' - \frac{1}{2\tau} \delta' + \left(\frac{1}{\tau} \frac{q^2}{12a^2(t_0)H^2(t_0)} + \frac{1}{2\tau^2} \right) \delta = 0, \quad (16)$$

where a prime denotes differentiation with respect to τ . This equation can be solved easily with a computer algebra program, e.g., MAPLE V [2]. The result is, transforming back from τ to t ,

$$\delta(t, q) = [C_1 \sin \sigma_1(t) + C_2 \cos \sigma_1(t)] \left(\frac{t}{t_0} \right)^{1/2}, \quad (17)$$

where the function $\sigma_1(t)$ is given by

$$\sigma_1(t) := \xi \left(\frac{t}{t_0} \right)^{1/2}, \quad (18)$$

with

$$\xi := \frac{q}{a(t_0)H(t_0)\sqrt{3}}, \quad (19)$$

and the constants C_1 and C_2 are given by

$$C_1 = \delta(t_0, q) \sin \xi - \frac{\cos \xi}{\xi} \left[\delta(t_0, q) - \frac{\dot{\delta}(t_0, q)}{H(t_0)} \right], \quad (20a)$$

$$C_2 = \delta(t_0, q) \cos \xi + \frac{\sin \xi}{\xi} \left[\delta(t_0, q) - \frac{\dot{\delta}(t_0, q)}{H(t_0)} \right], \quad (20b)$$

where we have used that

$$\delta(t_0, q) = \delta(\tau = 1, q), \quad \dot{\delta}(t_0, q) = 2H(t_0)\delta'(\tau = 1, q), \quad (21)$$

see Eq. (15). We now will consider our general result (17)–(20) for the radiation-dominated era in two limiting cases, namely for large and small wavelengths λ of the perturbation (12).

A. Large-scale perturbations

Large-scale perturbations are perturbations of large wavelengths. In the limit of large wavelengths, $\lambda \rightarrow \infty$, the magnitude of the wave vector $q = 2\pi/\lambda$ vanishes. Writing $\delta(t) \equiv \delta(t, q=0)$ and $\dot{\delta}(t) \equiv \dot{\delta}(t, q=0)$, we find, from Eqs. (17)–(20), for $t \geq t_0$,

$$\delta(t) = \left[2\delta(t_0) - \frac{\dot{\delta}(t_0)}{H(t_0)} \right] \left(\frac{t}{t_0} \right)^{1/2} - \left[\delta(t_0) - \frac{\dot{\delta}(t_0)}{H(t_0)} \right] \frac{t}{t_0}. \quad (22)$$

Consequently, in this limit, there is no oscillatory character any more.

The nongauge-invariant perturbation theory in synchronous coordinates [3–6] yields, next to the growth rates $t^{1/2}$ and t of Eq. (22), also a solution which is proportional to t^{-1} [see Ref. [5], Eq. (86.20), and Ref. [6], Eq. (9.121)]. The latter solution is a gauge solution [5,6], which in our gauge-invariant approach does not show up. Hence, our result and the classical result happen to be closely parallel. This is in contrast with what is found in the gauge-invariant treatment of Mukhanov, Feldman, and Brandenberger (MFB [7]), who find a mode of constant amplitude: see their Eq. (5.47).

B. Small-scale perturbations

In the limit $\lambda \rightarrow 0$ (or, equivalently, $q \rightarrow \infty$) we find, using Eqs. (17)–(20),

$$\delta(t, q) \approx \delta(t_0, q) \left(\frac{t}{t_0} \right)^{1/2} \cos \sigma_2(t), \quad (23)$$

where

$$\sigma_2(t) := \xi \left[1 - \left(\frac{t}{t_0} \right)^{1/2} \right], \quad (24)$$

with ξ given by Eq. (19). Hence, small-scale perturbations are seen to oscillate with an amplitude which increases proportional to $t^{1/2}$. The nongauge-invariant perturbation theory in synchronous coordinates as well as the gauge-invariant theory of MFB yield small-scale energy density perturbations oscillating with a constant amplitude: see Ref. [5], Eq. (88.12) and Ref. [7], Eq. (5.46), respectively.

IV. DENSITY CONTRAST IN THE MATTER-DOMINATED ERA

Once protons and electrons recombine to yield hydrogen at a temperature around 4000 K, the radiation pressure becomes negligible, and the equations of state reduce to those of a nonrelativistic monatomic perfect gas [Weinberg [3], Eqs. (15.8.20)–(15.8.21)]

$$\varepsilon(n, T) = nm_{\text{H}}c^2 + \frac{3}{2}nk_{\text{B}}T, \quad p(n, T) = nk_{\text{B}}T, \quad (25)$$

where k_{B} is Boltzmann's constant, m_{H} the proton mass, and T the temperature of the matter.

We first consider the zero-order Einstein equations (3). The maximum temperature in the matter-dominated era occurs around the time t_0 of the decoupling of matter and radiation: $T(t_0) \approx 4000$ K. Hence, from Eq. (25) it follows that the pressure is negligible with respect to the energy density: $p/\varepsilon \approx k_{\text{B}}T/m_{\text{H}}c^2 \leq 3.7 \times 10^{-10}$. This implies that, to a good approximation, $\varepsilon_{(0)} \pm p_{(0)} \approx \varepsilon_{(0)}$ and $\varepsilon_{(0)} \approx n_{(0)}m_{\text{H}}c^2$. Thus as is well known, in an unperturbed flat FRW universe, one can neglect the pressure with respect to the energy density. Using the above facts, we find that the Einstein equations (3) reduce to

$$\dot{H} = -3H^2 + \frac{1}{2}\kappa\varepsilon_{(0)}, \quad (26a)$$

$$H^2 = \frac{1}{3}\kappa\varepsilon_{(0)}, \quad (26b)$$

$$\dot{\varepsilon}_{(0)} = -3H\varepsilon_{(0)}. \quad (26c)$$

Note that in a matter-dominated universe, Eqs. (3c) and (3d) are identical. Thus the general solutions of the zero-order Einstein equations are [3]

$$H(t) = \frac{2}{3}(ct)^{-1}, \quad t \geq t_0, \quad (27a)$$

$$\varepsilon_{(0)}(t) = \frac{4}{3}\kappa(ct)^{-2}, \quad t \geq t_0, \quad (27b)$$

where t_0 is the time at which the matter-dominated era sets in. Using the definition of the Hubble parameter $H := \dot{a}/a$, one finds from Eq. (27a) for the scale factor

$$a(t) = a(t_0) \left(\frac{t}{t_0} \right)^{2/3}. \quad (28)$$

In the next two subsections we consider the perturbed flat FRW universe. We distinguish between the nonzero and the zero pressure cases.

A. Nonzero pressure

Since $k_B T/m_H c^2 \ll 1$ in the matter-dominated era, one may verify that, to a good approximation,

$$\omega_{(1)} \approx 0, \quad (29)$$

implying that the Einstein equation (1a) is identically satisfied. This implies, in turn, that Eq. (1b) is homogeneous. In order to calculate the coefficients α and β occurring in Eq. (1b), we need the equation of state $p = p(n, \varepsilon)$. Eliminating the absolute temperature T from Eqs. (25), one finds

$$p(n, \varepsilon) = \frac{2}{3} (\varepsilon - n m_H c^2). \quad (30)$$

From this equation it follows that

$$p_\varepsilon = \frac{2}{3}, \quad p_n = -\frac{2}{3} m_H c^2, \quad (31)$$

so that, to a good approximation,

$$p_\varepsilon + \frac{n_{(0)} p_n}{\varepsilon_{(0)} + p_{(0)}} \approx \frac{5}{3} \frac{k_B T}{m_H c^2}, \quad (32)$$

where we have used Eq. (25). From Eq. (I.64) and (32) it follows that the quantity ζ figuring in the coefficients α and β occurring in Eq. (1b) is a function of the matter temperature:

$$\zeta = \frac{\dot{T}}{T}. \quad (33)$$

In order to determine ζ as a function of the Hubble parameter H , we recall the following: After decoupling of matter and radiation at 4000 K, the temperature of the matter, T , is no longer locked to the photon temperature T_γ . However, when $T_\gamma > 2000$ K, the matter temperature still follows the photon temperature quite closely (see Weinberg, Chap. 15, Sec. 5). Since the photon temperature is, during the whole history of the universe, inversely proportional to the scale factor, we have, to a good approximation

$$T \approx T_\gamma \propto a^{-1}, \quad T_\gamma > 2000 \text{ K}. \quad (34)$$

Combining Eqs. (33) and (34), one finds

$$\zeta = -H, \quad T_\gamma > 2000 \text{ K}, \quad (35)$$

where we have used that $H := \dot{a}/a$. For temperatures $T_\gamma < 2000$ K, the thermal contact between matter and radiation is negligible, implying that $T \propto a^{-2}$ [see Weinberg, Eq.

(15.5.16)]. Moreover, for low temperatures the pressure is vanishingly small. This case will be considered in Sec. IV B.

Upon substituting Eqs. (29), (32), and (35) into Eq. (1b) with (I.63) and (I.64), one arrives at

$$\ddot{\varepsilon}_{(1)} + 8H\dot{\varepsilon}_{(1)} + \left(9H^2 + \frac{5}{3} \frac{k_B T}{m_H c^2} \nabla^2 \right) \varepsilon_{(1)} = 0. \quad (36)$$

Since, as noted before, $k_B T/m_H c^2 \ll 1$ in the matter-dominated era, Eq. (32) survives only as a coefficient of ∇^2 . Equation (36) can be rewritten as an equation for the density contrast $\delta(t, x^i)$:

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{5}{3} \frac{k_B T}{m_H c^2} \nabla^2 - \frac{4\pi G}{c^4} \varepsilon_{(0)} \right) \delta = 0, \quad (37)$$

where we have used the definition of the density contrast, Eq. (10), and the Einstein equations (26). Writing $\varepsilon_{(0)} = \rho_{(0)} c^2$, Eq. (37) becomes the familiar equation of the Newtonian theory of gravity

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{v_s^2}{c^2} \nabla^2 - \frac{4\pi G}{c^2} \rho_{(0)} \right) \delta = 0, \quad (38)$$

[Weinberg, Eq. (15.9.23)] where v_s is the speed of sound of a nonrelativistic gas, which for a monatomic gas is given by

$$v_s = \sqrt{\frac{5}{3} \frac{k_B T}{m_H}}. \quad (39)$$

We thus have shown that the purely relativistic equation (37) happens to be identical to the corresponding equation found from the Newtonian theory of gravity. Although Eqs. (37) and (38) have the same appearance, their derivation is different: see Weinberg, Chap 15, Sec. 9 for a derivation of Eq. (38) with the help of the Newtonian theory.

MFB, Sec. 5.3 did not arrive at Eq. (37), but obtained a different result. Equation (56) of Ref. [8] has been made gauge invariant by choosing the integration constant γ equal to zero. This procedure works well if the pressure vanishes.

We will come to Eq. (37). From the constraint equation (26b), it follows that Eq. (37) can be written

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{5}{3} \frac{k_B T}{m_H c^2} \nabla^2 - \frac{3}{2} H^2 \right) \delta = 0. \quad (40)$$

As usual, we solve Eq. (40) by Fourier analysis of the function δ . From Eqs. (I.25) and (12) it follows

$$\nabla^2 \delta(t, x^i) = \frac{q^2}{a^2(t)} \delta(t, q). \quad (41)$$

Furthermore, from Eqs. (27), (28), and (34) we find that $T/(a^2 H^2)$ is independent of the time. Hence, we have

$$T \nabla^2 = \frac{q^2}{a^2(t)} \frac{T(t)}{H^2(t)} H^2(t) = \frac{q^2}{a^2(t_p)} \frac{T(t_p)}{H^2(t_p)} H^2(t), \quad (42)$$

where q is the wave vector associated with the scale of a fluctuation and t_p is the present time. Using Eq. (42), Eq. (40) can be rewritten in the form

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\delta H^2(1-\mu) = 0, \quad (43)$$

where the constant μ is given by

$$\mu := \frac{10}{9} \frac{q^2}{a^2(t_p)} \frac{k_B T(t_p)}{m_H c^2} \frac{1}{H^2(t_p)}. \quad (44)$$

We now switch to a dimensionless time τ with the help of Eqs. (14) and (27a). We have

$$\frac{d^n}{c^n d\tau^n} = \left[\frac{3}{2}H(t_0)\right]^n \frac{d^n}{d\tau^n}, \quad n = 1, 2, \dots, \quad (45)$$

so that Eq. (43) can be written in the form of a Euler equation

$$\delta'' + \frac{4}{3\tau}\delta' - \frac{2}{3\tau^2}(1-\mu)\delta = 0, \quad (46)$$

where a prime denotes differentiation with respect to τ . Equation (46) happens to be identical to Eq. (15.9.44) of Weinberg, where it applies, however, to a different physical situation. Using that [see Eq. (45)]

$$\delta(t_0, q) = \delta(\tau = 1, q), \quad \dot{\delta}(t_0, q) = \frac{3}{2}H(t_0)\delta'(\tau = 1, q), \quad (47)$$

one finds, for $t \geq t_0$,

$$\delta(t, q) = C_+ \left(\frac{t}{t_0}\right)^{-1/6 + (1/6)\sqrt{25-24\mu}} + C_- \left(\frac{t}{t_0}\right)^{-1/6 - (1/6)\sqrt{25-24\mu}}, \quad \mu < \frac{25}{24}, \quad (48a)$$

$$\delta(t, q) = \left[\delta(t_0, q) + \left(\frac{1}{6}\delta(t_0, q) + \frac{2\dot{\delta}(t_0, q)}{3H(t_0)}\right) \ln\left(\frac{t}{t_0}\right) \right] \times \left(\frac{t}{t_0}\right)^{-1/6}, \quad \mu = \frac{25}{24}, \quad (48b)$$

$$\delta(t, q) = \left[\delta(t_0, q) \cos\sigma_3(t) + \left(\delta(t_0, q) + \frac{4\dot{\delta}(t_0, q)}{H(t_0)} \frac{\sin\sigma_3(t)}{\sqrt{24\mu-25}} \right) \left(\frac{t}{t_0}\right)^{-1/6} \right], \quad \mu > \frac{25}{24}, \quad (48c)$$

where the constants C_+ and C_- are given by

$$C_{\pm} := \pm \frac{[1 \pm \sqrt{25-24\mu}]\delta(t_0, q) + \frac{4\dot{\delta}(t_0, q)}{H(t_0)}}{2\sqrt{25-24\mu}}, \quad (49)$$

and the function $\sigma_3(t)$ is given by

$$\sigma_3(t) := \frac{1}{6}\sqrt{24\mu-25} \ln\left(\frac{t}{t_0}\right). \quad (50)$$

Note that Eq. (48b) can be obtained from Eqs. (48a) or (48c) by taking the limit $\mu \rightarrow \frac{25}{24}$. We consider the general result (48)–(50) for the matter-dominated era in two limiting cases, namely for large and small λ .

Large-scale perturbations

In the limit $\lambda \rightarrow \infty$ the magnitude of the wave vector q vanishes. Writing $\delta(t) \equiv \delta(t, q=0)$ and $\dot{\delta}(t) \equiv \dot{\delta}(t, q=0)$ we find, from Eqs. (44), (48a), and (49),

$$\delta(t) = \left[\frac{3}{5}\delta(t_0) + \frac{2\dot{\delta}(t_0)}{5H(t_0)} \right] \left(\frac{t}{t_0}\right)^{2/3} + \left[\frac{2}{5}\delta(t_0) - \frac{2\dot{\delta}(t_0)}{5H(t_0)} \right] \left(\frac{t}{t_0}\right)^{-1}. \quad (51)$$

Thus large-scale perturbations grow proportional to $t^{2/3}$.

Small-scale perturbations.

In the limit $\lambda \rightarrow 0$ the magnitude of the wave vector q becomes large. From Eqs. (44), (48c), and (50) we get

$$\delta(t, q) \approx \delta(t_0, q) \left(\frac{t}{t_0}\right)^{-1/6} \cos\sigma_3(t). \quad (52)$$

Thus small-scale perturbations decrease proportional to $t^{-1/6}$.

B. Zero pressure

When the photon temperature has dropped to $T_\gamma \approx 2000$ K, the pressure has become negligible. In Ref. [1], Sec. VIII, it has been shown that if the pressure vanishes, then $\epsilon_{(1)}$ obeys the first-order differential equation

$$\dot{\epsilon}_{(1)} + 2H\epsilon_{(1)} = 0, \quad (53)$$

as follows from Eqs. (I.78c) and (I.78d). Using Eqs. (10) and (26c), one finds that the density contrast δ obeys the first-order differential equation

$$\dot{\delta} - \delta H = 0. \quad (54)$$

The solution of Eq. (54) is

$$\delta(t, x^i) = \delta(t_0, x^i) \left(\frac{t}{t_0}\right)^{2/3}, \quad (55)$$

where we have used $H := \dot{a}/a$ and Eq. (28). The result (55) is independent of λ , i.e., of the scale of a perturbation.

The Newtonian perturbation theory [3–6] yields a second-order differential equation for the density contrast δ , namely Eq. (38) with $v_s = 0$ [see, e.g., Weinberg, Eq. (15.9.25) with solutions (15.9.29) and (15.9.30)].

MFB find that the evolution of a density perturbation does depend on the scale of that perturbation: see their Eq. (5.33). They conclude that small-scale perturbations increase as $t^{2/3}$ and that large-scale perturbations remain constant.

V. SUMMARY AND CONCLUSION

Starting out with a manifestly gauge-invariant perturbation scheme (I.70) of a companion article, we obtained the energy density contrast $\delta := \epsilon_{(1)}/\epsilon_{(0)}$. The results are collected in Table I. A most striking difference between our treatment and the perturbation theories of the literature is the $t^{1/2}$ growth rate of the energy density contrast in the radiation-dominated era in the limit of small wavelengths ($\lambda \rightarrow 0$).

In the nongauge-invariant perturbation theory in synchronous coordinates, [see, e.g., Eqs. (I.40)], the function $\epsilon_{(1)}$ that is used to describe an energy density perturbation is *gauge dependent*. This implies that $\epsilon_{(1)}$ has no physical significance. As a consequence, the function $\epsilon_{(1)}$ cannot be interpreted easily as an energy density perturbation.

Since only *gauge-invariant* quantities have an inherent physical meaning, a first step is to define a gauge-invariant quantity $\epsilon_{(1)}$, which may play the role of an energy density perturbation. Since there are many possibilities to define $\epsilon_{(1)}$, one has to choose this quantity in such a way that the perturbation theory based upon it has a Newtonian limit, i.e., one of the equations can be cast into the form of the Poisson

equation of the classical Newtonian theory of gravity with $\epsilon_{(1)}$ as source term.

MFB define a gauge-invariant quantity $\delta\epsilon^{(gi)}$ by their Eq. (5.11) and interpret it as an energy density perturbation. This interpretation is based upon the assumption that their Eq. (I.92) reduces to the Poisson equation in the Newtonian limit: see the text in Ref. [7] just below Eq. (5.19). However, the perturbation theory of MFB has not the usual Newtonian limit: since the Hubble parameter $H \neq 0$, Eq. (I.92) cannot be cast into the form of a Poisson equation with $\delta\epsilon^{(gi)}$ as source term. Therefore the gauge-invariant function $\delta\epsilon^{(gi)}$ cannot be interpreted as the perturbation on the energy density.

In Sec. V of a companion article we define a gauge-invariant quantity $\epsilon_{(1)}$ by Eq. (I.37), which is different from the definition of the quantity $\delta\epsilon^{(gi)}$ used by MFB. In Sec. VIII of Ref. [1] we consider the set of perturbation equations (I.45) in the limit that the pressure vanishes. We have shown that in this limit Eqs. (I.45) imply, for $H \neq 0$, the usual Newtonian limit, i.e., the Poisson equations (I.89)–(I.90) with $\epsilon_{(1)}$ as source term. This leads us to the conclusion that the quantity $\epsilon_{(1)}$ can indeed be interpreted as the perturbation on the energy density. Thus our gauge-invariant approach yields the usual nonrelativistic limit in a more transparent and natural way than the treatments of MFB and predecessors.

-
- [1] P. G. Miedema and W. A. van Leeuwen, preceding paper, Phys. Rev. D **54** 7227 (1996).
 [2] B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagan, and S. M. Watt, *Maple V, Library Reference Manual* (Springer-Verlag, Berlin, 1992).
 [3] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).
 [4] L. D. Landau and E. M. Lifshitz, *The Classical Theory of*

- Fields* (Pergamon Press, Oxford, 1975).
 [5] P. J. E. Peebles, *The Large-Scale Structure of the Universe*, Princeton Series in Physics (Princeton University Press, Princeton, NJ, 1980).
 [6] E. W. Kolb and M. S. Turner, *The Early Universe*, Frontiers in Physics (Addison-Wesley, Reading, MA, 1990).
 [7] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
 [8] P. G. Miedema, Phys. Rev. D **50**, 2431 (1994).