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Hommes, C.H.; Nusse, H.E.

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"Period Three to Period Two" Bifurcation for Piecewise Linear Models

By

Cars H. Hommes, Groningen, The Netherlands, and
Helena E. Nusse, Groningen, The Netherlands, and
College Park, ML, USA*

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In Hommes, Nusse, and Simonovits (1990) the dynamics of a simple economic model was studied. Although this piecewise linear model is quite simple, its dynamics shows different kinds of behavior such as periodic, quasi-periodic, and chaotic behavior. In particular, a new kind of bifurcation, namely a period three to period two bifurcation, was observed numerically. This paper deals with this new bifurcation phenomenon and we show that the "period three to period two" bifurcation occurs and is a structurally stable phenomenon in a class of two-dimensional continuous, piecewise linear systems. In particular, the "period three to period two" bifurcation is a structurally stable phenomenon in economic models with Hicksian nonlinearities.

1. Introduction

Studying low dimensional dynamical systems that depend on one or more parameters, bifurcation diagrams provide some information on the evolution of attractors when a parameter is varied. Familiar bifurcation phenomena include saddle node, period doubling, and Hopf bifurcation. In the literature dealing with bifurcation theory, it is frequently assumed that the map corresponding to the dynamical system is differentiable,

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see for example Guckenheimer and Holmes (1983). In Hommes, Nusse, and Simonovits (1990) the dynamics of a simple piecewise linear economic model was studied, and a "period three to period two" bifurcation was observed numerically.

The purpose of this paper is to study the occurrence of this bifurcation phenomenon for two-dimensional continuous, piecewise linear maps generated by two linear maps only. We show the significance of this bifurcation phenomenon for economic models with Hicksian nonlinearities, that is, models that are linear with ceilings and floors. To illustrate the phenomenon, consider the one-parameter family of maps g_μ $(-0.1 < \mu < 0.1)$ from the plane to itself, defined by

$$g_\mu(x,y) = \begin{cases} (-1.25x - 5.25y + \mu, 0) & \text{if } x \leq 0 \ , \\ (-2x - 5.25y + \mu, 0.5x) & \text{if } x > 0 \ . \end{cases}$$

For $\mu > 0$ there is a stable period three orbit which shrinks to a point E_0 as $\mu \rightarrow 0$, and for $\mu < 0$ there is a stable period two orbit which similarly shrinks to E_0 as $\mu \rightarrow 0$. Hence, one observes the following stable periodic orbits: a stable period 3 orbit collapses to a point and is reborn as a stable period 2 orbit. The bifurcation diagram exhibiting the "period three to period two" bifurcation is presented in figure 1. [The computer assisted pictures (figures 1 and 4) were made by using the DYNAMICS program (Yorke, 1990).]

We show that a "period three to period two" bifurcation occurs for a class of one-parameter families of two-dimensional continuous, piecewise linear maps of which one of the linear maps involved has a Jacobian matrix with an eigenvalue zero.

This paper is organized as follows. The main result is stated in section 2. In section 3 an application for a simple macromodel with Hicksian nonlinearities is presented. Section 4 contains the proof of the main result, and in section 5 we make some concluding remarks.

2. Statement of the Result

Let a, b, c, and d denote nonnegative real numbers. Define the linear maps F_L and F_R from the plane to itself by

$$F_L(x,y) = (-a\,x - d\,y, 0), \quad F_R(x,y) = (-b\,x - d\,y, c\,x) \ .$$

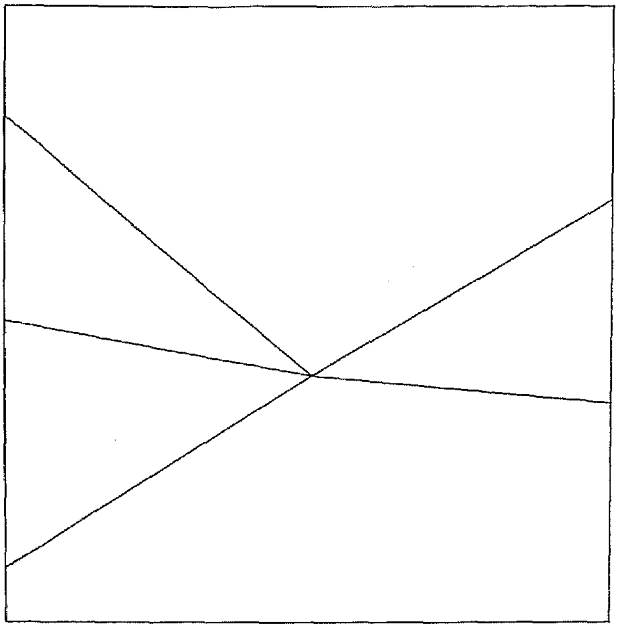

Figure 1. Bifurcation diagram exhibiting the "period three to period two" bifurcation of the map $g_\mu(x, y) = (-1.25x - 5.25y + \mu, 0)$ if $x \leq 0$, and $g_\mu(x, y) = (-2x - 5.25y + \mu, 0.5x)$ if $x > 0$. The parameter μ is horizontally varied from 0.1 (left) to -0.1 (right), and x is plotted vertically from -0.3 (top) to 0.2 (bottom).

Consider the one-parameter family F_μ from the plane to itself, defined by

$$F_\mu(x, y) = \begin{cases} F_L(x, y) + (\mu, 0) & \text{if } x \leq 0 , \\ F_R(x, y) + (\mu, 0) & \text{if } x > 0 , \end{cases}$$

where μ is in an open interval I including zero. The family F_μ is a family of continuous, piecewise linear maps.

We write TT-BIFMAP for the set of all one-parameter families of maps F_μ such that (a, b, c, d) satisfies the following four conditions:

(a1) $a > 1$, $b > 1$, $c > 0$, and $d > 0$;

(a2) $c < b < a\,b < b + 1$;

(a3) $4c\,d > b^2$;

(a4) $a^2 b < a\,c\,d < a^2 b + 1$.

Notice that the family g_μ in section 1 (that is, F_μ with $a = 1.25$, $b = 2.0$, $c = 0.5$, and $d = 5.25$) is in TT-BIFMAP.

Theorem. At $\mu = 0$, every family F_μ in TT-BIFMAP has a "period three to period two" bifurcation at $(0, 0)$.

The proof is given in section 4. Restricting for a moment the attention to the family g_μ, the idea of the proof is the following. For $\mu > 0$, each point p on the X-axis is mapped to a point p' on the X-axis after three iterates, so $g_\mu^3(p) = p'$. The graph of the corresponding return map H of the X-axis is given in figure 2. The map H has an unstable fixed point $p_u = \frac{4}{27} \cdot \mu > 0$ and two stable fixed points $q_s = -\frac{4}{9} \cdot \mu < 0$ and $p_s = \frac{14}{9} \cdot \mu > 0$. The point $(p_u, 0)$ is one of the three points of an unstable period 3 orbit of the map g_μ; the two points $(q_s, 0)$ and $(p_s, 0)$ are both points of the same stable period 3 orbit of the map g_μ.

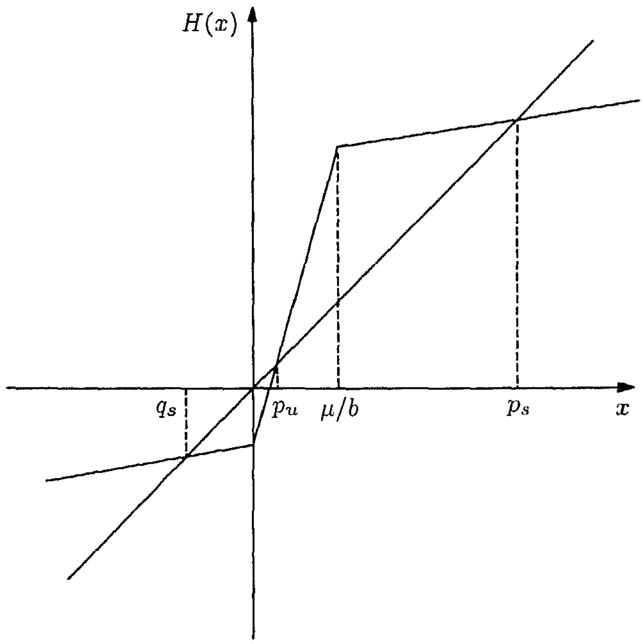

Figure 2. The return map H of the map g_μ [defined by $g_\mu(x, y) = (-1.25x - 5.25y + \mu, 0)$ if $x \leq 0$, and $g_\mu(x, y) = (-2x - 5.25y + \mu, 0.5x)$ if $x > 0$] on the X-axis when $\mu > 0$. The map H, defined by $H(x) = g_\mu^3(x, 0)$, has an unstable fixed point $p_u = \frac{4}{27} \cdot \mu > 0$ and two stable fixed points $q_s = -\frac{4}{9} \cdot \mu < 0$ and $p_s = \frac{14}{9} \cdot \mu > 0$.

For $\mu < 0$, write W_μ for the interval $[\frac{1}{a}\cdot\mu, \infty) = [0.8\cdot\mu, \infty)$ on the X-axis. We have (1) the image $g_\mu(p)$ of each point p on the X-axis but not in W_μ is in W_μ, and (2) each point p in W_μ is mapped to a point p' on the X-axis after two iterates, so $g_\mu^2(p) = p'$. In figure 3, the graph of the corresponding return map G on W_μ is given. G has an unstable fixed point $p_u = \frac{4}{9}\cdot\mu < 0$ and a stable fixed point $p_s = -\frac{2}{9}\cdot\mu > 0$. The point $(p_u, 0)$ is the unstable equilibrium point of the map g_μ; the point $(p_s, 0)$ is a point of a stable period 2 orbit of the map g_μ.

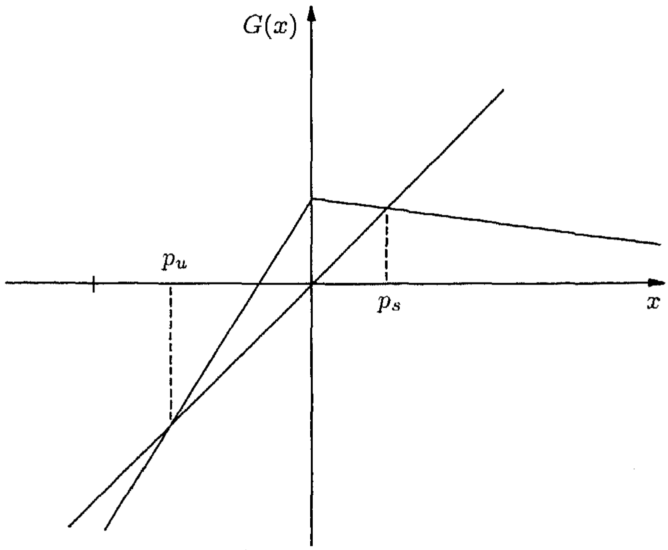

Figure 3. The return map G of the map g_μ [defined by $g_\mu(x,y) = (-1.25x - 5.25y + \mu, 0)$ if $x \le 0$, and $g_\mu(x,y) = (-2x - 5.25y + \mu, 0.5x)$ if $x > 0$] on the interval $[\frac{1}{a}\cdot\mu, \infty) = [0.8\cdot\mu, \infty)$ on the X-axis when $\mu < 0$. The map G, defined by $G(x) = g_\mu^2(x, 0)$, has an unstable fixed point $p_u = \frac{4}{9}\cdot\mu < 0$ and a stable fixed point $p_s = -\frac{2}{9}\cdot\mu > 0$.

3. Application: A Simple Macro-economic Model

As an application, we consider a simple macro-economic model of a socialist economy. This model was extensively studied in Hommes, Nusse, and Simonovits (1990). In fact, the piecewise linear model is a control system with two state variables and two control variables which interact: the changes in the state variables depend on the control variables, while the control variables react to the lagged state variables. For many parameter values, the equilibrium of the associated linear system is unstable. However, we have truncated control, that is, there are lower and upper bounds for the control variables. In the resulting piecewise linear system all the time paths are bounded.

In order to illustrate the theorem, we recall the equations of the piecewise linear model; for more details, see Hommes, Nusse, and Simonovits (1990). All the variables are GDP ratios: investment ratio i, net import ratio b, start ratio s, and commitment ratio k. The minimal value of the commitment ratio is denoted by k^*, and the minimal value of the net import ratio is denoted by b^*. The GDP grows at a constant rate, denoted by $\Gamma - 1$, where $\Gamma > 1$. The internal tension e and external tension a are defined as the deviations of the commitment and the net import from their minimal values, respectively.

An important characteristic of the model is that the control variables start and investment are truncated. The upper and lower bounds of the start and investment ratios are time invariant and are denoted by s^u, i^u, s^ℓ, and i^ℓ, respectively. The model is given by the following eight equations:

Commitment: $$k_t = f\,k_{t-1} + \sigma_s s_t - i_t\ ; \tag{E1}$$

Internal tension: $$e_t = k_t - k^*\ ; \tag{E2}$$

Net import: $$b_t = -\beta + \beta_i i_t\ ; \tag{E3}$$

External tension: $$a_t = b_t - b^*\ ; \tag{E4}$$

Intended start: $$s_t^p = \sigma - \sigma_e e_{t-1} - \sigma_a a_{t-1}\ ; \tag{E5}$$

Start: $$s_t = \begin{cases} s^\ell & \text{if } s_t^p \le s^\ell\ , \\ s_t^p & \text{if } s^\ell < s_t^p < s^u\ , \\ s^u & \text{if } s_t^p \ge s^u\ ; \end{cases} \tag{E6}$$

Intended investment: $$i_t = \iota + \iota_e e_{t-1}\ ; \tag{E7}$$

Investment: $$i_t = \begin{cases} i^\ell & \text{if } i_t^p \le i^\ell\ , \\ i_t^p & \text{if } i^\ell < i_t^p < i^u\ , \\ i^u & \text{if } i_t^p \ge i^u\ , \end{cases} \tag{E8}$$

where $f = 1/\Gamma < 1 \le \sigma_s$, $\beta > 0$, $\beta_i \ge 1$, $\sigma > 0$, $\sigma_e > 0$, $\sigma_a > 0$, $\iota > 0$, and $\iota_e > 0$. From the equations (E1)–(E8) we can eliminate four variables resulting into the following two equations:

$$e_t = f\, e_{t-1} + \sigma_s s_t - i_t - (1-f)k^* \ , \tag{E9a}$$

$$a_t = \beta_i i_t - (\beta + b^*) \ , \tag{E9b}$$

where s_t and i_t are defined in (E6) and (E8), respectively.

The dynamics of the model can be analyzed, using the equations (E9a) and (E9b) only, since the future values of the variables start, investment, net import, and commitment are obtained by substitution in (E1)–(E8). We choose the parameter values: $\sigma_s = 1.2$, $\beta_i = 1$, $\beta = 0.2$, $b^* = 0$, $k^* = 0.4$, $\Gamma = \frac{20}{19}$, $\sigma = 0.4$, and $\iota = 0.2$, and the bounds are $s^\ell = 0.18$, $s^u = 0.29$, $i^\ell = 0.23$, $i^u = 0.28$. The reaction coefficients σ_e, σ_a, and ι_e will be specified later on. Substituting these values in Eqs. (E9a) and (E9b) yields

$$e_t = 0.95 e_{t-1} + 1.2 s_t - i_t - 0.02 \ , \tag{E10a}$$

$$a_t = i_t - 0.2 \ , \tag{E10b}$$

with s_t and i_t defined as in (E6) and (E8), respectively.

Selecting the parameter values $\sigma_e = 1.75$ and $\iota_e = 0.6$, the bifurcation diagram in which the parameter σ_a is the parameter to be varied, exhibits a "period three to period two" bifurcation at $\sigma_a \approx 3.4028$, see figure 4.

The theorem presented in section 2 of this paper can be applied using a simple linear coordinate transformation.

4. Proof of the Theorem

In this section, we present a proof of the theorem. The geometrical proof might give insight whether or not other bifurcations (for example, period 5 to period 2 bifurcation) may occur in models fitting in the Hicksian tradition, that is, linear models with ceilings and floors.

Let F_μ be a one-parameter family in TT-BIFMAP. We write $p_0 = (x_0, y_0)$ for an initial condition and $p_n = (x_n, y_n)$ for its n-th iterate, that is, $p_n = F_\mu^n(p_0)$ for any parameter μ. For the particular initial value $(0,0)$, we write $A_0 = (0,0)$, $A_1 = F_\mu(A_0)$, $A_2 = F_\mu(A_1)$, $A_3 = F_\mu(A_2)$, and $A_4 = F_\mu(A_3)$.

The map F_μ has a unique fixed point denoted E_μ, that is, $F_\mu(E_\mu) = E_\mu$. Then $E_\mu = (\frac{1}{1+a} \cdot \mu, 0)$ if $\mu \leq 0$ and $E_\mu = (\frac{1}{1+b+c\,d} \cdot \mu, \frac{c}{1+b+c\,d} \cdot \mu)$ if $\mu > 0$.

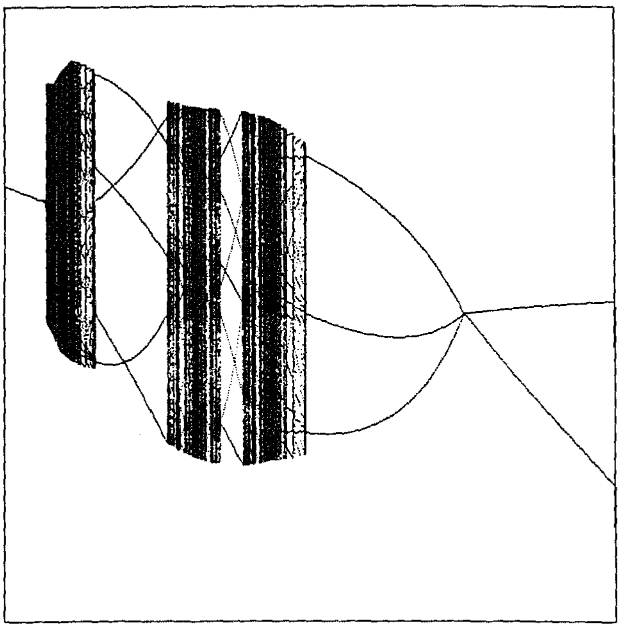

Figure 4. Bifurcation diagram exhibiting the "period three to period two" bifurcation of the map corresponding to the macro-economic model described by the equations (E1)–(E8). For $\sigma_e = 1.75$ and $\iota_e = 0.6$, the bifurcation diagram in which the parameter σ_a is varied horizontally between 1.2 (left) and 4 (right), exhibits a "period three to period two" bifurcation at $\sigma_a \approx 3.4028$. The coordinate e is plotted vertically, $0 \leq e \leq 0.1$.

Let $M_L = \begin{bmatrix} -a & -d \\ 0 & 0 \end{bmatrix}$ and $M_R = \begin{bmatrix} -b & -d \\ 0 & 0 \end{bmatrix}$ be the matrices that correspond to the linear maps F_L and F_R, respectively.

The eigenvalues of M_L are 0 and $-a$, so if $\mu < 0$ then the fixed point E_μ is unstable since $a > 1$. The eigenvalues of M_R $(-0.5b \pm 0.5\sqrt{b^2 - 4cd})$ are complex, since $4cd > b^2$. For $\mu > 0$ the fixed point E_μ is unstable (repelling), since the product cd of the eigenvalues of M_R exceeds 1 [this follows from the assumptions (a1) and (a4)].

For each initial value $p_0 = (x_0, y_0)$ we have if $x_0 \leq 0$ then $y_1 = 0$, and if $x_0 > 0$ then $y_1 = cx_0 > 0$. Hence, it is sufficient to consider initial values in the upper half plane.

Assume first, $\mu < 0$. Assume $p_0 = (x_0, y_0)$ is any initial value with $y_0 \geq 0$, then we have if $x_0 \leq 0$ then $y_1 = 0$, and if $x_0 > 0$, then $x_1 = -b\,x_0 - d\,y_0 + \mu < 0$ and so $y_2 = 0$. Therefore, it is sufficient to consider points on the X-axis, and we will do so.

Computation of the iterates of $p_0 = (0,0) = A_0$ yields $A_1 = (\mu, 0)$, $A_2 = ((1-a)\mu, 0)$, $A_3 = ((b(a-1)+1)\mu, c(1-a)\mu)$, and $A_4 = ((a-1)(c\,d - a\,b - 1)\mu, 0)$. The assumption $a^2b < a\,c\,d < a^2b+1$ implies $0 < x_4 < x_2$. Notice that $|\, x_3 \,| > y_3$, since $0 < c < b$. Hence, A_1 is on the X-axis to the left of A_0, A_3 is above and to the left of A_1, and both A_2 and A_4 are on the X-axis to the right of A_0 and A_4 is between A_0 and A_2.

Let $p_0 = (x_0, 0)$ be any point. If $x_0 > 0$ then $p_1 = (-b\,x_0 + \mu, c\,x_0)$, and if $x_0 \leq 0$ then $p_1 = (-a\,x_0 + \mu, 0)$. Therefore, the image of the right half of the X-axis with end point A_0 is the half line through A_3 with end point $A_1 = F_\mu(A_0)$, and the image of the left half of the X-axis with end point A_0 is the half line on the X-axis to the right of A_1 with end point A_1.

Define $Q = (\frac{1}{a} \cdot \mu, 0) = (x_Q, 0)$ and $R = (\frac{a^2}{(a+1)\,(a\,b - c\,d)} \cdot \mu, 0) = (x_R, 0)$. The point Q is mapped to A_0 iterating F_μ once, that is, $F_\mu(Q) = A_0$, and Q is on the X-axis between A_1 and E_μ since $A_1 = (\mu, 0)$, $E_\mu = (\frac{1}{1+a} \cdot \mu, 0)$ and $a > 1$. The point R is on the X-axis to the right of A_0, and R is mapped to E_μ iterating F_μ twice, that is, $F_\mu^2(R) = E_\mu$.

Let $p_0 = (x_0, 0)$ be any point. Straightforward computation gives the following. If $x_0 > 0$ (that is, p_0 is on the X-axis to the right of A_0) then $p_1 = (-b\,x_0 + \mu, c\,x_0)$ and $p_2 = ((a\,b - c\,d)x_0 + (1-a)\mu, 0)$, so p_2 is on the X-axis. If $x_0 = 0$ (that is, $p_0 = A_0$) then $p_1 = (\mu, 0)$ and $p_2 = ((1-a)\mu, 0)$, so p_2 is on the X-axis to the right of A_0. If $\frac{1}{a} \cdot \mu \leq x_0 < 0$ (that is, $p_0 = Q$ or p_0 is on the X-axis between Q and A_0) then $p_1 = (-a\,x_0 + \mu, 0)$ and $p_2 = (-a(-a\,x_0 + \mu) + \mu, 0)$, so p_2 is on the X-axis. If $x_0 < \frac{1}{a} \cdot \mu$ (that is, p_0 is to the left of Q) then $p_1 = (-a\,x_0 + \mu, 0)$ and $p_2 = (-b(-a\,x_0 + \mu) + \mu, c(-a\,x_0 + \mu))$ and $p_3 = ((-a^2b + a\,c\,d)x_0 + (a\,b - a - c\,d + 1)\mu, 0)$, and so p_3 is on the X-axis while p_2 is not. Summarizing, for each point p_0 on the X-axis to the right of Q we have $p_2 = F_\mu^2(p_0)$ is on the X-axis. Therefore, we have a return map on the interval consisting of the points on the X-axis to the right of Q.

Let G denote the return map of F_μ on $[Q, \infty)$, so $G(x) = F_\mu^2(x, 0)$ for each $x \geq x_Q$. The above results imply $G(x) = a^2x + (1-a)\mu$ for $\frac{1}{a} \cdot \mu \leq x \leq 0$, and $G(x) = (a\,b - c\,d)x + (1-a)\mu$ for $x \geq 0$. The graph of G is similar to figure 3. The map G has two fixed points, namely $p_u = \frac{1}{1+a} \cdot \mu$, and $p_s = \frac{a-1}{a\,b-c\,d-1} \cdot \mu$, and $p_u < 0 < p_s$. The fixed point p_u is unstable since the slope of G in p_u is $a^2 > 1$, and the fixed point p_s is stable since the slope of G at p_s is $a\,b - c\,d$ for which $-1 < a\,b - c\,d < 0$. Furthermore, for all x with $p_u < x < x_R$ we have $\lim_{n\to\infty} G^n(x) = p_s$. The properties (1) $x_Q < \frac{1}{1+a} \cdot \mu < 0$, (2) G has slope $a^2 > 1$ if $x_Q < x < 0$, (3) G has slope $-1 < a\,b - c\,d < 0$ for $x > 0$, and (4) $G(0) > 0$, imply that F_μ has a stable 2-cycle consisting of the points $P_1 = (\frac{a-1}{a\,b-c\,d-1} \cdot \mu, 0)$ and $P_2 = F_\mu(P_1) = (\frac{b-c\,d-1}{a\,b-c\,d-1} \cdot \mu, \frac{(a-1)c}{a\,b-c\,d-1} \cdot \mu)$. Notice that the norms of both these points converge to zero as μ goes to zero, that is, both $\| P_1 \| \to 0$ and $\| P_2 \| \to 0$ as $\mu \to 0$.

Now assume $\mu = 0$. Assume $p_0 = (x_0, y_0)$ is any initial value with $y_0 \geq 0$, then $x_0 \leq 0$ implies $y_1 = 0$, and $x_0 > 0$ implies $x_1 = -b\,x_0$ yielding $y_2 = 0$. Hence, it is sufficient to consider points on the X-axis. Let $p_0 = (x_0, 0)$ be given. If $x_0 < 0$ then $p_1 = (-a\,x_0, 0)$ which is on the positive X-axis. If $x_0 = 0$ then $p_1 = (-a\,x_0, 0)$ and so p_0 is the fixed point of F_0. If $x_0 > 0$, then $p_1 = (-b\,x_0, c\,x_0)$, and $p_2 = ((a\,b - c\,d)x_0, 0)$. Consequently the point $A_0 = (0,0)$ is a globally stable fixed point of F_0, since $-1 < a\,b - c\,d < 0$.

Now assume $\mu > 0$. The fixed point $E_\mu = (\frac{1}{1+b+c\,d} \cdot \mu, \frac{c}{1+b+c\,d} \cdot \mu)$ is unstable with complex eigenvalues since it was assumed $c\,d > 1$ and $4c\,d > b^2$. Assume $p_0 = (x_0, y_0)$ is any initial value. Then $x_0 \leq 0$ implies $y_1 = 0$. Now assume that $x_0 > 0$. Since the equilibrium $E_\mu = (\frac{1}{1+b+c\,d} \cdot \mu, \frac{c}{1+b+c\,d} \cdot \mu)$ is repelling with complex eigenvalues, there exists a smallest positive integer N such that $x_N \leq 0$. Hence, it, follows that $y_{N+1} = 0$. Therefore, it is sufficient to consider initial values on the X-axis.

Let $p_0 = (x_0, y_0) = (x_0, 0)$ be any point on the X-axis. If $x_0 \leq 0$ then $p_1 = F_\mu(p_0) = (-a\,x_0 + \mu, 0) = (x_1, y_1)$, so $x_1 > 0$. Every point $q_0 = (w_0, 0)$ such that $w_0 < x_0 \leq 0$ satisfies $q_1 = F_\mu(q_0) = (-a\,w_0 + \mu, 0) = (w_1, z_1)$, so $w_1 > x_1 > 0$. The conclusion is that points on the X-axis to the left of $A_0 = (0,0)$ will be mapped monotonically into the X-axis to the right of $(\mu, 0)$.

Let $p_0 = (0,0)$. A simple computation shows $p_1 = (\mu, 0)$, $p_2 = ((1-b)\mu, c\,\mu)$, $p_3 = ((a\,b - a - c\,d + 1)\mu, 0)$, and $p_4 = (-a\,x_3 + \mu, 0)$. Notice $x_3 < 0$, hence $x_4 > \mu = x_1$. Recall that $p_0 = A_0$, $p_1 = A_1$, $p_2 = A_2$, $p_3 = A_3$, and $p_4 = A_4$. The conclusion is that A_0, A_1, A_3, and A_4 are on the X-axis, and A_3 is to the left of A_0, and both A_1 and A_4 are to the right of A_0 with A_1 between A_0 and A_4.

Let $p_0 = (x_0, 0)$ be any point on the X-axis for which $x_0 > 0$.

Then $p_1 = (-b\,x_0 + \mu, c\,x_0)$. Notice that if $x_0 = \frac{1}{b} \cdot \mu$ then $x_1 = 0$ and $y_1 = \frac{c}{b} \cdot \mu$. Write $B_0 = (\frac{1}{b} \cdot \mu, 0)$, $B_1 = F_\mu(B_0)$, $B_2 = F_\mu(B_1)$, and $B_3 = F_\mu(B_2)$. Then $B_1 = (0, \frac{c}{b} \cdot \mu)$, $B_2 = ((1 - \frac{c\,d}{b})\mu, 0)$, and $B_3 = ((a(\frac{c\,d}{b} - 1) + 1)\mu, 0)$. Notice that B_1 denotes the point on the Y-axis at which the line segment $[A_1, A_2]$ intersects the Y-axis, and that B_2 is a point on the X-axis to the left of A_0. The assumption $c\,d > a\,b$ implies that $A_3 = ((a\,b - a - c\,d + 1)\mu, 0)$ is on the X-axis to the left of B_2.

The image of the half line $[A_1, \infty)$ through A_2 under the map F_μ is the kinked half line $[A_2, B_2] \cup [B_2, \infty)$ through A_3. The image of this kinked half line $[A_2, B_2] \cup [B_2, \infty)$ is on the X-axis. In particular, the image of the line $[B_2, \infty)$ through A_3 is $[B_3, \infty)$ on the X-axis to the right of $A_1 = (\mu, 0)$ and the image of the line segment $[A_2, B_2]$ is $[A_3, B_3]$.

Let $p_0 = (x_0, 0)$ be any point on the X-axis. Straightforward computation shows the following. If $x_0 \geq \frac{1}{b} \cdot \mu$ (that is, p_0 is to the right of B_0) then $p_1 = (-b\,x_0 + \mu, c\,x_0)$, $p_2 = ([a\,b - c\,d]x_0 + (1 - a)\mu, 0)$, and $p_3 = (-a[a\,b - c\,d]x_0 + \mu + a(a - 1)\mu, 0)$. Hence, the points p_2 and p_3 are on the X-axis for $x_0 \geq \frac{1}{b} \cdot \mu$. If $0 \leq x_0 \leq \frac{1}{b} \cdot \mu$ (that is, p_0 is on the X-axis between A_0 and B_0) then $p_1 = (-b\,x_0 + \mu, c\,x_0)$, $p_2 = ([b^2 - c\,d]x_0 + (1 - b)\mu, -b\,c\,x_0 + c\,\mu)$, and $p_3 = (\{b(c\,d - a\,b) + a\,c\,d\}x_0 + \{1 + a\,b - a - c\,d\}\mu, 0)$, so the point p_3 is on the X-axis. If $x_0 < 0$ then $p_1 = (-a\,x_0 + \mu, 0)$, $p_2 = (a\,b\,x_0 + (1 - b)\mu, -a\,c\,x_0 + c\,\mu)$, and $p_3 = (-a\{a\,b - c\,d\}x_0 + \{1 + a(b - 1) - c\,d\}\mu, 0)$, so the point p_3 is on the X-axis. The conclusion is that for each point $p_0 = (x_0, 0)$ on the X-axis, the third iterate of p_0 is also on the X-axis, that is, $F_\mu^3(p_0) = (x_3, 0)$. Hence, a return map exists on the X-axis.

Let H denote the return map of F_μ on the real line, so $H(x) = F_\mu^3(x, 0)$. The above results imply $H(x) = a \cdot (c\,d - a\,b)x + (1 + a\,b - a - c\,d) \cdot \mu$ for $x < 0$, $H(x) = (b\,c\,d - a\,b^2 + a\,c\,d)x + (1 + a\,b - a - c\,d) \cdot \mu$ for $0 \leq x \leq \frac{1}{b} \cdot \mu$, and $H(x) = a \cdot (c\,d - a\,b)x + (a^2 - a + 1) \cdot \mu$ for $x \geq \frac{1}{b} \cdot \mu$. The graph of H is similar to figure 2. The map H has three fixed points, namely $q_s = \frac{a\,b - c\,d + 1 - a}{1 + a(a\,b - c\,d)} \cdot \mu < 0$, $p_u = \frac{c\,d - a\,b + a - 1}{b(c\,d - a\,b) + a\,c\,d - 1} \cdot \mu$, and $p_s = \frac{1 + a(a - 1)}{1 + a(a\,b - c\,d)} \cdot \mu > 0$. The fixed point p_u is unstable since the slope of H in p_u is bigger than 1, and the two fixed points q_s and p_s are stable since the slope of H at both q_s and p_s is between 0 and 1. Furthermore, for all x with $x < p_u$ we have $\lim_{n \to \infty} H^n(x) = q_s$, and for all x with $x > p_u$ we have $\lim_{n \to \infty} H^n(x) = p_s$. The properties (1) H has slope between 0 and 1 for $x < 0$, (2) H has slope bigger than 1 for $0 < x < \frac{1}{b} \cdot \mu$, (3) H has slope between 0 and 1 for $x > \frac{1}{b} \cdot \mu$, and (4) $H(0) < 0$

and $H(\frac{1}{b} \cdot \mu) > \frac{1}{b} \cdot \mu$, imply F_μ has a stable 3-cycle consisting of the points $S_1 = (\frac{a\,b-c\,d+1-a}{1+a(a\,b-c\,d)} \cdot \mu, 0)$, $S_2 = (\frac{a(a-1)+1}{1+a(a\,b-c\,d)} \cdot \mu, 0)$, and $S_3 = (\{\frac{-b(a(a-1)+1)}{1+a(a\,b-c\,d)} + 1\} \cdot \mu, \frac{a\,c(a-1)+c}{1+a(a\,b-c\,d)} \cdot \mu)$. Notice that the norms of all three points converge to zero as μ goes to zero, that is, all three $\| S_1 \| \to 0$, $\| S_2 \| \to 0$, and $\| S_3 \| \to 0$ as $\mu \to 0$.

5. Concluding Remarks

We have shown the occurrence of a new bifurcation phenomenon, namely a period three to period two bifurcation. This bifurcation phenomenon is structurally stable in the class TT-BIFMAP, that is, for any F_μ in TT-BIFMAP, there exists an open neighborhood U of F_μ in TT-BIFMAP such that for each G_μ in U the family G_μ has a period three to period two bifurcation at $\mu = 0$. The geometrical proof presented above might give insight whether or not other bifurcations (for example, period 5 to period 2 bifurcation) may occur in similar piecewise linear models.

The theorem guarantees that if F_μ is in TT-BIFMAP, then at $\mu = 0$, the family F_μ exhibits a period three to period two bifurcation at $(0, 0)$. It is not true that the property "F_μ has a period three to period two bifurcation at $\mu = 0$" implies that the map F_μ is in TT-BIFMAP. To illustrate this, the family F_μ corresponding to $a = 2$, $b = 2$, $c = 1$, and $d = 4$ is not in TT-BIFMAP, while F_μ has a period three to period two bifurcation at $\mu = 0$.

We emphasize that the "period three to period two" bifurcation phenomenon (and related bifurcation phenomena) can be expected to occur in many economic models being of Hicksian type, that is, linear models with ceilings and floors. For models with Hicksian nonlinearities there are generally regions in the phase space in which one of the variables assumes its lower or upper bound. In such a region the Jacobian matrix of the corresponding linear map has an eigenvalue zero. Therefore models with Hicksian nonlinearities are a natural application of models exhibiting these bifurcation phenomena.

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Addresses of authors: Mr. Cars H. Hommes, Rijksuniversiteit Groningen, Faculteit der Economische Wetenschappen, Vakgroep Econometrie, Postbus 800, NL-9700 AV Groningen, The Netherlands; Mrs. Helena E. Nusse, Rijksuniversiteit Groningen, Faculteit der Economische Wetenschappen, Vakgroep Econometrie, Postbus 800, NL-9700 AV Groningen, The Netherlands, and University of Maryland, Institute for Physical Science and Technology, College Park, MD 20742, USA.