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Polar phase of one-dimensional bosons with large spin

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Abstract. Spinor ultracold gases in one dimension (1D) represent an interesting example of strongly correlated quantum fluids. They have a rich phase diagram and exhibit a variety of quantum phase transitions. We consider a 1D spinor gas of bosons with a large spin S. A particular example is the gas of chromium atoms ($S = 3$), where the dipolar collisions efficiently change the magnetization and make the system sensitive to the linear Zeeman effect. We argue that in 1D the most interesting effects come from the pairing interaction. If this interaction is negative, it gives rise to a (quasi)condensate of singlet bosonic pairs with an algebraic order at zero temperature, and for $(2S + 1) \gg 1$ the saddle point approximation leads to physically transparent results. Since in 1D one needs a finite energy to destroy a pair, the spectrum of spin excitations has a gap. Hence, in the absence of a magnetic field, there is only one gapless mode corresponding to phase fluctuations of the pair quasicondensate. Once the magnetic field exceeds the gap, another condensate emerges, namely the quasicondensate of unpaired bosons with spins aligned along the magnetic field. The spectrum then contains two gapless modes corresponding to the singlet-paired and spin-aligned unpaired Bose condensed particles, respectively. At $T = 0$, the corresponding phase transition is of the commensurate–incommensurate type.

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Contents

1. Introduction

Spinor Bose gases attracted a great deal of attention in the last decade as they exhibit a much richer variety of macroscopic quantum phenomena than spinless bosons (see [1] for a review). The physics of three-dimensional (3D) spin-1 and spin-2 bosons is rather well investigated, both theoretically [2–9] and in experiments with Na and ^{87}Rb atoms [10–14]. The structure of the ground state strongly depends on the interactions, and in particular ferromagnetic, polar (singlet-paired) and cyclic phases have been analyzed on the mean-field level and beyond the mean field [1]. The spinor physics of 3D spin-3 bosons is described in [15] and, after successful experiments with Bose–Einstein condensates of ^{52}Cr atoms ($S = 3$) [16], experimental studies of the spinor physics in this system are expected in the near future.

The observation of non-ferromagnetic states requires very low and stable magnetic fields (well below 1 mG) at which the interaction energy per particle exceeds the Zeeman energy. Currently, the obtained stable field on the level of 0.1 mG is expected to reveal a transition between ferromagnetic and non-ferromagnetic states in chromium [17], and experiments using magnetic field shielding and aiming at even lower fields are under way [18].

It is important to emphasize that a change in magnetization of an atomic spinor gas under variations of the magnetic field requires spin–dipolar collisions, since the short-range atom–atom interaction does not change the total spin. In dilute gases of sodium and rubidium, the spin–dipolar collisions are very weak, and the magnetization does not feel a change in the magnetic field on the time scale of the experiment. In contrast, in a gas of chromium atoms that have a large magnetic moment of $6\mu_B$, the spin–dipolar collisions efficiently change the magnetization and the gas becomes sensitive to the linear Zeeman effect [19].

Spinor Bose gases in 1D are in many respects quite different from their 2D and 3D counterparts and represent an interesting example of strongly correlated quantum fluids. In this paper, having in mind the gas of chromium atoms ($S = 3$), we assume that the system is sensitive to the linear Zeeman effect. We consider a 1D spinor gas of bosons where the dominant interactions are the density–density and the attractive pairing interactions. This choice is justified by the fact that in 1D, only the latter interaction gives rise to a nontrivial quasi-long-range order. In contrast to 2D and 3D, in 1D, pairs with nonzero spin $\bar{S}$ do not condense. This is related to the fact that for $\bar{S} \neq 0$, the symmetry of the condensate order parameter is non-Abelian. It is well known that strong quantum fluctuations in 1D dynamically generate spectral gaps for non-Abelian Goldstone modes, which leads to exponential decay of the correlations (see e.g. [20]). As far as the polar phase (the condensate of $\bar{S} = 0$ pairs) is concerned, it can be formed because the symmetry of the order parameter is Abelian. However, in 1D, its

magnetic spectrum is quite different from that in 2D and 3D: in the absence of a magnetic field, the spin excitations have a gap. For a large spin S, the saddle point approximation gives a physically transparent description of the polar phase. A sufficiently large magnetic field closes the gap and leads to the transition from the singlet-paired (polar) phase to the ferromagnetic state. The presence of the spin gap strongly changes the physics of the 1D polar phase and the polar–ferromagnetic transition compared with the higher dimensions discussed for spin-3 bosons in [15]. We investigate the 1D polar phase and this quantum transition, and discuss prospects for observing them in chromium experiments.

2. The model

As the atom–atom short-range interaction conserves the total spin, the Hamiltonian of binary interactions for 1D bosons with spin S can be written as a sum of projection operators on the states with different even spins $\bar{S}$ of interacting pairs [1],

$$V = \frac{1}{2} \int dx \sum_{\bar{S}=0}^{2S} \gamma_{\bar{S}} \hat{\mathcal{P}}_{\bar{S}}(x), \tag{1}$$

where x is the coordinate. For the 1D regime obtained by tightly confining the motion of particles in two directions, the interaction constants $\gamma_{\bar{S}}$ are related to the 3D scattering lengths $a_{3D}(\bar{S})$ at a given spin $\bar{S}$ of the colliding pair. Omitting the confinement-induced resonance [21], we have

$$\gamma_{\bar{S}} = \frac{2\hbar^2}{M l_0^2} a_{3D}(\bar{S}), \tag{2}$$

where $l_0 = (\hbar/M\omega_0)^{1/2}$ is the confinement length, M is the atom mass and ω_0 is the confinement frequency.

Imagine that all $\gamma_{\bar{S}}$ are equal to each other ($\gamma_{\bar{S}} = \gamma > 0$), except for $\gamma_{\bar{S}}$ at $\bar{S} = q$. We then use the relation $\sum_{\bar{S}} \hat{\mathcal{P}}_{\bar{S}}(x) =: \hat{n}^2(x)$, where $\hat{n}$ is the density operator and the symbol :: denotes the normal ordering, and we reduce the interaction Hamiltonian to the form $V = (1/2) \int dx (\gamma : n^2(x) : +(\gamma_q - \gamma)\hat{\mathcal{P}}_q(x)$. For a positive value of $(\gamma_q - \gamma)$, the system is an ordinary Luttinger liquid, but when $(\gamma_q - \gamma) < 0$, the situation may change. In 3D, a negative value of $(\gamma_q - \gamma)$ would lead to spontaneous symmetry breaking with formation of the order parameter in the form of a condensate of pairs with total spin q. In 1D, only a quasi-long-range order is possible and only if $q = 0$ so that the symmetry is Abelian [20]. Therefore, interactions with negative coupling constants, which have q different from zero or from $2S$, will not produce quasi-long-range order. The case $q = 2S$ is exceptional because it corresponds to a ferromagnetic state where the order parameter (the total spin) commutes with the Hamiltonian. Therefore, at $T = 0$, this state can exist even in 1D. We do not discuss this interesting state, and the only possibility that remains is $q = 0$. Hence, in our model, we have a (repulsive) density–density interaction and the pairing interaction that gives rise to the formation of singlet pairs.

In realistic systems, the coupling constants $\gamma_{\bar{S}}$ are not equal to each other, although they are generally of the same order of magnitude. We thus have to single out the density–density interaction in a proper way and then deal with the rest. For example, the interaction Hamiltonian

(1) can be represented as a sum of squares of certain local operators, as is usually done in the theory of spinor Bose gases [1, 15],

$$V = \frac{1}{2} \int dx \left[c_0 : \hat{n}^2(x) + c_1 \hat{\mathbf{F}}^2(x) + c_2 \hat{\mathcal{P}}_0(x) + c_3 \operatorname{Tr} \hat{\mathcal{O}}^2(x) : + \cdots \right], \tag{3}$$

where $\mathbf{F} = \psi^+ \mathbf{S} \psi$, $\mathcal{O}_{ij} = \psi^+(S^i S^j)\psi$, the constants c_i are linear combinations of $\gamma_{\bar{S}}$, and the symbol ... stands for higher-order spin terms, which we do not write. The operators $: \hat{\mathbf{F}}^2(x) :$ and $: \hat{\mathcal{O}}^2(x) :$ are given by $\sum_{\bar{S}} [\bar{S}(\bar{S}+1)/2 - S(S+1)] \hat{\mathcal{P}}_{\bar{S}}(x)$ and $\sum_{\bar{S}} [\bar{S}(\bar{S}+1)/2 - S(S+1)]^2 \hat{\mathcal{P}}_{\bar{S}}(x)$, respectively, where the summation includes all values of $\bar{S}$ from zero to $2S$. We then move the $\bar{S} = 0$ part of these terms to the term $c_2 \hat{\mathcal{P}}_0(x)$ and carry out the same procedure with higher-order spin terms, which changes the constant c_2. The $: \hat{\mathbf{F}}^2(x) :$, $: \hat{\mathcal{O}}^2(x) :$, etc terms then no longer contain the interactions with $\bar{S} = 0$ and, hence, can only lead to renormalizations of the density–density and pairing interactions.

In the case of ^{52}Cr, we have $c_0 = 0.65\gamma_6$, and the 3D scattering length is $a_6 = 112 a_B$ [15], where a_B is the Bohr radius. The exact value of the 3D scattering length $a_{3D}(0)$ is not known and, hence, the constants γ_0 and c_2 are also unknown. In this paper, when discussing ^{52}Cr atoms, we omit the $: \hat{\mathbf{F}}^2(x) :$ and $: \hat{\mathcal{O}}^2(x) :$ (renormalized) terms, treat c_2 as a free parameter and focus on the case of $c_2 < 0$.

We then write down the following Hamiltonian density in terms of the bosonic field operators Ψ_j,

$$\mathcal{H} = \frac{1}{2M} \sum_j \partial_x \Psi_j^+ \partial_x \Psi_j + \frac{g}{2N} \left[\sum_j \Psi_j^+ \Psi_j \right]^2 - \frac{g_0}{2N} \left[\sum_j (-1)^j \Psi_j^+ \Psi_{-j}^+ \right] \left[\sum_j (-1)^j \Psi_j \Psi_{-j} \right],$$

$$\tag{4}$$

where the spin projection j ranges from $-S$ to S, the coupling constant g_0 is assumed to be positive, and we put $\hbar = 1$. The coupling constants g_0 and g are related to c_0 and c_2. For example, in the case of ^{52}Cr, we have $g = 7c_0 = 4.55\gamma_6 > 0$ and $g_0 = -c_2$.

3. Zero magnetic field: saddle point approximation

We now consider the case of $N = 2S + 1 \gg 1$ and apply the $1/N$-approximation to the model described by the Hamiltonian density (4). First, we decouple the pairing from the density–density interaction by the Hubbard–Stratonovich transformation [22],

$$-\frac{g_0}{2N} \left[\sum_j (-1)^j \Psi_j^+ \Psi_{-j}^+ \right] \left[\sum_j (-1)^j \Psi_j \Psi_{-j} \right] \rightarrow N|\Delta|^2/2g_0 + \left[\Delta \sum_j (-1)^m \Psi_j^+ \Psi_{-j}^+ + \text{h.c.} \right],$$

$$\frac{g}{2N} \left[\sum_j \Psi_j^+ \Psi_j \right]^2 \rightarrow N\lambda^2/2g + \mathrm{i}\lambda \sum_j \Psi_j^+ \Psi_j, \tag{5}$$

where $\Delta(\tau, x)$ and $\lambda(\tau, x)$ are auxiliary dynamical fields. At large N, the path integral is dominated by the field configurations in the vicinity of the saddle point $\Delta(\tau, x) = \Delta$, $\lambda(\tau, x) = \mathrm{i}\lambda_0$. The values of Δ and λ_0 are determined self-consistently from the minimization of the free energy. The stability of the saddle point is guaranteed by the fact that the integration over the Ψ, Ψ^+ fields yields a term proportional to N and therefore the entire action is $\sim N$. The

presence of large N in the exponent in the path integral suppresses fluctuations of the fields Δ and λ, thus making the saddle point stable.

The bosonic action at the saddle point is

$$\tilde{S} = \sum_{\omega,k,m} (\Psi^+_{\omega,m}, \Psi_{-\omega,-m}) \begin{pmatrix} i\omega - \epsilon & (-1)^m \Delta \\ (-1)^m \Delta^+ & -i\omega - \epsilon \end{pmatrix} \begin{pmatrix} \Psi_{\omega,m} \\ \Psi^+_{-\omega,-m} \end{pmatrix}, \tag{6}$$

where $\epsilon = k^2/2M - \mu$ and $\mu = \mu_0 - \lambda_0$, with μ_0 being the bare chemical potential. From equation (6), we find the mean-field spectrum of quasiparticles (we assume that $\mu < 0$),

$$E(k) = \frac{1}{2M}\sqrt{(k^2 + \kappa^2)(k^2 + k_0^2)}; \quad \kappa^2 = 2M(-\mu - \Delta), \quad k_0^2 = 2M(-\mu + \Delta). \tag{7}$$

The saddle point equations are

$$\frac{\Delta}{g_0} = \int \frac{d\omega \, dk}{(2\pi)^2} \frac{\Delta}{\omega^2 + E^2(k)}, \tag{8}$$

$$n = \int \frac{d\omega \, dk}{(2\pi)^2} \frac{\epsilon(k) - E(k)}{\omega^2 + E^2(k)}, \tag{9}$$

$$\mu = \mu_0 - gn, \tag{10}$$

where n is the density of one of the bosonic species.

The quasiparticles (spin modes) constitute a $(2S + 1)$-fold degenerate multiplet. As follows from equation (7), the quasiparticles have a nonzero spectral gap,

$$E(0) \equiv m = k_0 \kappa / 2M. \tag{11}$$

This result agrees with the one for $N = 3$ obtained in [24]. This is a special feature of 1D. In 2D and 3D, the integral in the saddle point equation (8) does not diverge at small ω, k for $\kappa \to 0$, and such a gap is not formed. Therefore, one has a gapless spectrum of spin modes, which for $S = 3, 2$ and 1 has been obtained in the studies of spinor Bose gases (see e.g. [1] and references therein). We would like to emphasize the fact that although equations (8), (9) and (10) resemble the equations for a superconductor, due to the bosonic nature of the problem, the order parameter amplitude Δ is not equal to the spectral gap, and the latter is related to the parameter κ.

After the integration in equations (8) and (9), we obtain the saddle point equations in the parametric form,

$$2\pi n = \frac{\kappa^2 + k_0^2}{2k_0} K\left(\sqrt{1 - \frac{\kappa^2}{k_0^2}}\right) - k_0 E\left(\sqrt{1 - \frac{\kappa^2}{k_0^2}}\right), \tag{12}$$

$$\frac{\pi k_0}{M g_0} = K\left(\sqrt{1 - \frac{\kappa^2}{k_0^2}}\right), \tag{13}$$

where $K(x)$ and $E(x)$ are elliptic functions[5]. From the form of these equations, it is clear that κ/k_0 is a function of the parameter

$$\eta = \frac{1}{2\pi}\left(\frac{Mg_0}{n}\right)^{1/2}, \tag{14}$$

and k_0 can be written in the form

$$k_0 = b(\eta)(nMg_0)^{1/2}. \tag{15}$$

Accordingly, equation (11) for the gap takes the form

$$m = \frac{ng_0}{2}b^2(\eta)\frac{\kappa}{k_0}(\eta), \tag{16}$$

so that the gap in the units of ng_0 depends only on the parameter η.

In the limit of weak interactions where $\eta \ll 1$, we obtain

$$\frac{\kappa}{k_0} \simeq \frac{4}{e}\exp\left(-\frac{1}{\eta}\right), \quad b \simeq 2, \tag{17}$$

and equation (11) gives an exponentially small gap,

$$m \simeq \frac{8ng_0}{e}\exp\left(-\frac{1}{\eta}\right). \tag{18}$$

For strong interactions, $\eta \gg 1$, we have

$$\frac{\kappa}{k_0} \simeq 1, \quad b \simeq \pi\eta, \tag{19}$$

and the gap is given by

$$m \simeq \frac{\pi^2\eta^2}{2}ng_0. \tag{20}$$

The numerically obtained dependence of κ/k_0 on η is displayed in figure 1, and the function $b(\eta)$ is shown in figure 2. The gap is presented in figure 3. The asymptotic formula (18) obtained in the limit of small η already works with 20% accuracy for $\eta = 0.05$. With the same accuracy, the large-η asymptotic formula (20) is already valid for $\eta = 1$.

In the limit of weak interactions, taking into account that $|\mu| \approx \Delta$ and using equations (7) and (15), we obtain $\mu \approx -ng_0/4M$. Substituting this relation into equation (10), we obtain

$$n = \frac{\mu_0}{g - g_0}. \tag{21}$$

Hence, the system is thermodynamically stable for $g > g_0$.

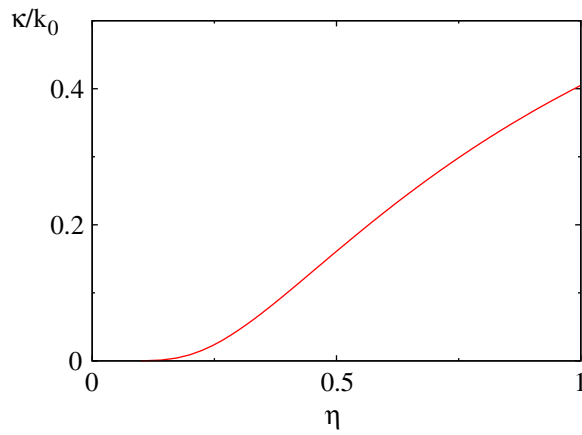[5] We adopt the definition of the elliptic functions given in [23].

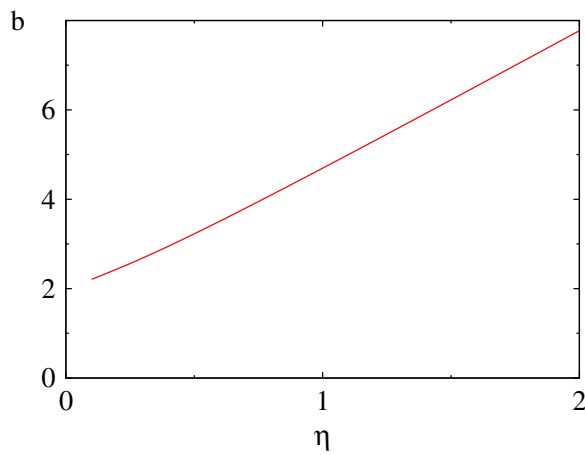**Figure 1.** The ratio κ/k_0 as a function of η.

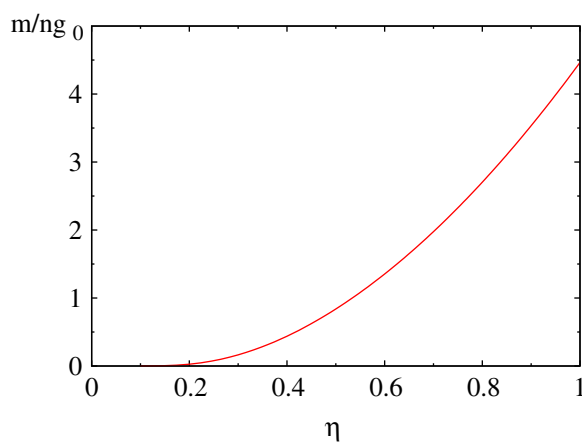**Figure 2.** The parameter b as a function of η.

Figure 3. The gap m in the units of ng_0 as a function of η.

The only gapless excitation of the system is the phase mode of the complex scalar field Δ. This excitation describes sound waves of the pair condensate. The effective Hamiltonian for the phase mode Φ is

$$H_{\text{phase}} = \frac{v}{2} \int dx [K_s \Pi^2 + K_s^{-1}(\partial_x \Phi)^2], \quad [\Pi(x), \Phi(y)] = -i\delta(x-y), \tag{22}$$

where Π is a canonically conjugate momentum. The velocity v and the Luttinger parameter K_s are extracted from the functional derivatives of the saddle point action and are given by the following equations,

$$
\begin{aligned}
(K_s v)^{-1} &= -N\Delta^2 \int \frac{d\omega\,dk}{2(2\pi)^2} \frac{\partial G(\omega,k)}{\partial \omega} \frac{\partial G(-\omega,-k)}{\partial \omega} \\
&= N\Delta^2 \int \frac{d\omega\,dk}{2(2\pi)^2} \frac{(\epsilon^2 + \omega^2 - \Delta^2)^2 + (2\Delta\omega)^2}{(\omega^2 + \epsilon^2 - \Delta^2)^4} \\
&= \frac{N\Delta^2}{8\pi} \int dk \frac{2E^2 + \Delta^2}{E^5} \\
&= \frac{N\Delta^2}{4\pi} \frac{M^3}{\kappa^5} f_1(k_0/\kappa),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
v/K_s &= -N\Delta^2 \int \frac{d\omega\,dk}{2(2\pi)^2} \frac{\partial G(\omega,k)}{\partial k} \frac{\partial G(-\omega,-k)}{\partial k} \\
&= N\Delta^2 \int \frac{d\omega\,dk}{(2\pi)^2} (k/M)^2 \frac{(\epsilon^2 + \omega^2 - \Delta^2)^2 + (2\Delta\epsilon)^2}{(\omega^2 + \epsilon^2 - \Delta^2)^4} \\
&= \frac{N\Delta^2}{8\pi} \int dk (k/M)^2 \frac{2E^4 + 5(\Delta\epsilon)^2}{E^7} \\
&= \frac{2N\Delta^2}{\pi} \frac{M}{\kappa^3} f_2(k_0/\kappa),
\end{aligned}
\tag{24}
$$

where $G(\omega, k)$ is the Green's function of the field Ψ_j, defined as $\langle\langle \Psi_j(\omega,k)\Psi_{j'}^+(\omega,k)\rangle\rangle = \delta_{jj'}G(\omega,k)$, and

$$f_1(x) = \int_0^\infty dy \frac{8(1+y^2)(y^2+x^2) + (1-x^2)^2}{[(y^2+1)(y^2+x^2)]^{5/2}},$$

$$f_2(x) = \int_0^\infty dy\, y^2 \frac{2(y^2+1)^2(y^2+x^2)^2 + (5/4)(y^2+x^2/2+1/2)^2(x^2-1)^2}{[(y^2+1)(y^2+x^2)]^{7/2}}.$$

The functions $f_1(k_0/\kappa)$ and $f_2(k_0/\kappa)$ can be expressed in terms of elliptic functions $E(\sqrt{1-\kappa^2/k_0^2})$ and $K(\sqrt{1-\kappa^2/k_0^2})$, but the expressions are cumbersome and we do not present them. In the limit of weak interactions, we have

$$f_1(k_0/\kappa) = \frac{2\kappa}{3k_0}, \quad f_2(k_0/\kappa) = \frac{k_0}{24\kappa}, \quad \eta \ll 1 \quad \text{and} \quad \kappa \ll k_0, \tag{25}$$
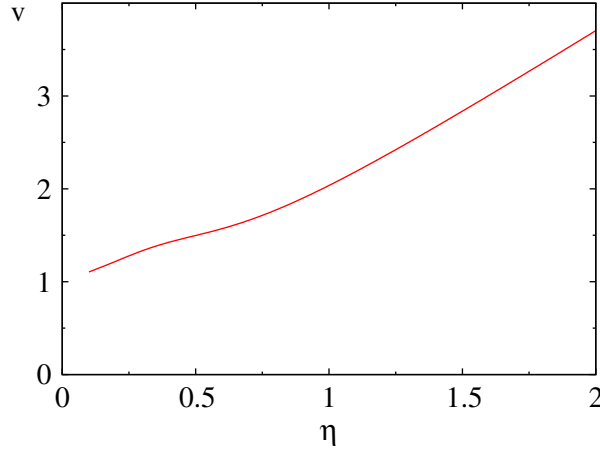
Figure 4. The velocity v in the units of $\sqrt{2ng_0/M}$ as a function of η.

whereas for strong interactions, these functions are given by

$$f_1(k_0/\kappa) = \frac{3\pi}{2}, \quad f_2(k_0/\kappa) = \frac{\pi}{8}, \quad \eta \gg 1 \quad \text{and} \quad \kappa \simeq k_0. \tag{26}$$

The velocity v is given by the relation

$$v = \sqrt{8}\frac{k_0}{M}\left\{\frac{\kappa}{k_0}\left[\frac{f_2(k_0/\kappa)}{f_1(k_0/\kappa)}\right]^{1/2}\right\}, \tag{27}$$

so that for weak interactions using equations (25), (15) and $b \simeq 2$, we have

$$v \simeq \sqrt{\frac{2ng_0}{M}}, \quad \eta \ll 1. \tag{28}$$

For strong interactions, equations (26), (15) and (19) lead to

$$v \simeq \sqrt{\frac{2ng_0}{3M}}\,\pi\eta, \quad \eta \gg 1. \tag{29}$$

The dependence of v on the parameter η is displayed in figure 4.

The Luttinger parameter K_s follows from the relation

$$K_s^{-1} = \frac{N}{16\pi}\left(\frac{k_0^2}{\kappa^2} - 1\right)^2\sqrt{f_1(k_0/\kappa)\,f_2(k_0/\kappa)}. \tag{30}$$

The scaling dimension of the Δ field is $d = K_s/4\pi$ and it decreases very rapidly with η, which indicates that the mean-field approximation works very well. In figure 5, we show the dependence of d on η for $N = 7$. In the regime of weak interactions, the scaling dimension is exponentially small and it remains significantly smaller than unity, even for $\eta \simeq 2$.

To conclude this part, we give a brief summary of the properties of the paired phase. With certain modifications, the properties for an arbitrarily large spin S are similar to the ones for $S = 1$ described in [24]. Namely, all single particle correlation functions decay exponentially.
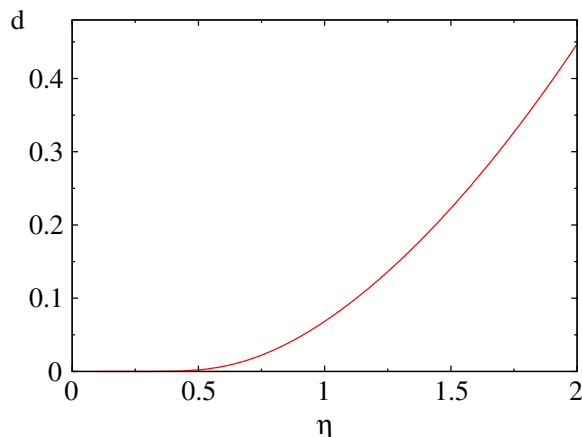
Figure 5. The scaling dimension d of the order parameter field Δ for $N = 7$ as a function of η.

This follows from the fact that the operator ψ_j^+ always emits a gapped vector excitation (Bogolyubov quasiparticle) from the $(2S + 1)$-fold degenerate multiplet (for $S = 1$ it is a gapped triplet). Two-particle correlation functions of the operators $\psi_j \psi_j$ and their Hermitian conjugates (no summation assumed) undergo a power law decay.

4. Magnetic field and the exact solution

In order to study the influence of the magnetic field on the properties of the singlet-paired phase, one can also use the saddle point approximation employed in the previous section. However, the saddle point equations become too involved. Therefore, we resort to non-perturbative methods.

As was demonstrated in [25], the model described by the Hamiltonian density (4) possesses $U(1) \times O(2S + 1)$ symmetry. Therefore, it is reasonable to suggest that the low-energy sector of this model is described by a combination of the $U(1)$ Gaussian theory and the $O(2S + 1)$ nonlinear sigma (NLσ) model. For $S = 1$, this was explicitly demonstrated in [24]. Both the $U(1)$ theory and the sigma model are integrable and the exact solution gives access to the low-energy sector, of the model. At a special ratio of the coupling constants, one can get even further, because it was demonstrated [25] that the entire model (4) is integrable at a particular ratio of g_0/g. Below we restrict our consideration to the low-energy sector, where we are not constrained to this particular ratio.

As we have said, the $O(N)$ NLσ model is exactly solvable. In the absence of a magnetic field, its excitations are massive particles transforming under the vector representation of the $O(N)$ group. This agrees with our result for model (4) based on the large N approximation. As is always the case for Lorentz invariant integrable models, all of the information about the thermodynamics is contained in the two-body S-matrix, which was found in [26]. Consider $N = 2S + 1$ (S integer) and physical particles that have a relativistic-like spectrum $\epsilon(\theta) = m \cosh \theta$, $p(\theta) = \tilde{v}^{-1} m \sinh \theta$, with mass (gap) m, velocity $\tilde{v}$ and the total energy

$$E = m \sum_i \cosh \theta_i. \tag{31}$$

For the particles confined in a box of length L with periodic boundary conditions, the Bethe Ansatz equations read

$$\exp[i\tilde{v}^{-1}mL\sinh\theta_i] = \prod_{i\neq j}^{n} S_0^{-1}(\theta_i-\theta_j) \prod_{a_1}^{m_1} \frac{\theta_i - \lambda_{a_1}^{(1)} - i\pi}{\theta_i - \lambda_{a_1}^{(1)} + i\pi},$$

$$\prod_{i=1}^{n} \frac{\lambda_{a_1}^{(1)} - \theta_i - i\pi}{\lambda_{a_1}^{(1)} - \theta_i + i\pi} \prod_{a_2=1}^{m_2} \frac{\lambda_{a_1}^{(1)} - \lambda_{a_2}^{(2)} - i\pi}{\lambda_{a_1}^{(1)} - \lambda_{a_2}^{(2)} + i\pi} = \prod_{b_1=1}^{m_1} \frac{\lambda_{a_1}^{(1)} - \lambda_{b_1}^{(1)} - 2i\pi}{\lambda_{a_1}^{(1)} - \lambda_{b_1}^{(1)} + 2i\pi},$$

$$\cdots$$

$$\prod_{a_{S-1}=1}^{m_{S-1}} \frac{\lambda_{a_S}^{(S)} - \lambda_{a_{S-1}}^{(S-1)} - i\pi}{\lambda_{a_1}^{(1)} - \lambda_{a_2}^{(2)} + i\pi} = \prod_{b_S=1}^{m_S} \frac{\lambda_{a_S}^{(S)} - \lambda_{b_S}^{(S)} - i\pi}{\lambda_{a_S}^{(S)} - \lambda_{b_S}^{(S)} + i\pi}, \tag{32}$$

where S_0 is related in the standard way to the integral kernel $K(\theta)$,

$$\frac{1}{2i\pi} \frac{d\ln S_0(\theta)}{d\theta} = \delta(\theta) - K(\theta)$$

and

$$K(\theta) = \int e^{i\omega\theta/\pi} K(\omega) d\omega/2\pi, \quad K(\omega) = \frac{1 - \exp[-2|\omega|/(N-2)]}{1 + \exp[-|\omega|]}.$$

The spectral gap m (the particle mass) and velocity $\tilde{v}$ are related to the bare parameters of the model (4). For $N \gg 1$, one can use equation (16) for the gap, and in the low-energy limit, the spectrum (7) of the spin modes becomes $E(k) = \sqrt{m^2 + \tilde{v}^2 k^2}$ with $\tilde{v} = (b/2)\sqrt{ng_0(1+\kappa^2/k_0^2)/M}$, so that in the limit of weak interactions, we have $\tilde{v} \approx \sqrt{ng_0/M}$.

Equations (32) constitute a system of $S+1$ coupled algebraic equations for the quantities $\theta_1, \ldots \theta_n, \lambda_1^{(1)}, \ldots \lambda_{m_1}^{(1)} \lambda_1^{(2)}, \ldots \lambda_{m_2}^{(2)}, \ldots \lambda_1^{(S)}, \ldots \lambda_{m_S}^{(S)}$. Integer numbers m_i are eigenvalues of the S Cartan generators of the group $O(2S+1)$. From equation (32) it is obvious that the total energy of the system depends on configurations of λs and through them it depends on the spin indices of constituent particles. Some of the Cartan generators for the problem under consideration were constructed in [25], where it was also shown that the projection of the total spin of the system is

$$S^z = Sn - \sum_i m_i. \tag{33}$$

The Cartan generators commute with the Hamiltonian, and m_is are integrals of motion. Therefore, the magnetic field that couples to m_i through S^z (33) does not violate integrability. Moreover, once the field is applied, the energies of all eigenstates with $m_i \neq 0$ go up. Thus, at sufficiently low temperature one may consider only eigenstates with no λ-rapidities, since their energies decrease in the field. When the magnetic field exceeds the spectral gap m, the θ-rapidities start to condense, creating a Fermi sea. In the ground state, the θ-rapidities are distributed over a finite interval $(-B, B)$. The distribution function $\rho(\theta)$ and magnetization S^z are determined by the following integral equations,

$$\int_{-B}^{B} K(\theta - \theta')\rho(\theta') \, d\theta' = \frac{m}{2\pi\tilde{v}} \cosh\theta, \tag{34}$$

$$\int_{-B}^{B} K(\theta - \theta')\epsilon(\theta')\,\mathrm{d}\theta' = m\cosh\theta - hS, \quad \epsilon(\pm B) = 0, \tag{35}$$

$$S^z/L = S\int_{B}^{B} \mathrm{d}\theta\,\rho(\theta), \tag{36}$$

where $h = g_L\mu_B\mathcal{B}$, with $\mathcal{B}$ being the magnetic field and g_L the Landee factor. There is obviously one transition at $h_c = m/S$. The magnetization is zero when $h < h_c$ and gradually increases with the field when $h > h_c$ (there is another transition in high magnetic fields corresponding to the saturation of the magnetization, but the low-energy theory cannot describe it). In order to find the magnetic field dependence of the magnetization near the transition, where $B \ll 1$, we approximate the kernel as

$$K(\theta) = \delta(\theta) - A + O(\theta^2), \quad A = \ln 4 + \psi(1/2 + 2/N - 2) - \psi(2/N - 2), \tag{37}$$

and look for the solution of equations (34) and (35) in the form

$$\epsilon(\theta) \approx a(\theta^2 - B^2), \quad \rho(\theta) \approx \text{const.} \tag{38}$$

Substituting $\epsilon(\theta)$ and $\rho(\theta)$ given by equation (38) into (34) and (35), we obtain

$$\frac{S^z}{L} = \frac{S}{\pi\tilde{v}}[2m(hS-m)]^{1/2}\{1+8A/3[2(hS-m)/m]^{1/2} + O((hS-m)/m)\}, \quad h > h_c. \tag{39}$$

Keeping only the leading term in equation (39) and restoring the dimensions, we have

$$\frac{S^z}{L} = \frac{\sqrt{2}Sm}{\pi\hbar\tilde{v}}\left(\frac{\mathcal{B} - \mathcal{B}_c}{\mathcal{B}_c}\right)^{1/2}, \quad \mathcal{B} > \mathcal{B}_c, \tag{40}$$

with the critical magnetic field given by

$$\mathcal{B}_c = \frac{m}{g_L\mu_B S}. \tag{41}$$

Equation (40) shows a typical field dependence of the magnetization for the quantum commensurate–incommensurate transition. This transition was first studied by Japaridze and Nersesyan [27] in the context of spin systems with a gap, where (as in our case) it is driven by the magnetic field. Later, Pokrovsky and Talapov [28] considered such a transition in the charge sector, where it is driven by a change in the chemical potential. The magnetization is exactly zero below the critical field and increases as $\sqrt{\mathcal{B} - \mathcal{B}_c}$ above the critical field near the transition. Note that this is quite different from the 3D case, where the magnetization decreases continuously with the magnetic field when the latter goes below the critical value (see e.g. [15]).

5. Conclusions

We found that the phase diagram and general properties of 1D bosons with a large spin S resemble the properties of spin-1 bosons. For the attractive pairing interaction ($g_0 > 0$), the bosons with opposite spin projections create pairs that Bose condense, giving rise to quasi-long-range order. The saddle point approximation based on the condition of large $N = 2S + 1$ gives a transparent picture of the emerging polar phase. The peculiarity of 1D is that all spin excitations

have a spectral gap. Hence, in the absence of a magnetic field, there is only one gapless mode corresponding to phase fluctuations of the pair quasicondensate. Once the magnetic field exceeds the gap, another quasicondensate emerges. This is the condensate of unpaired bosons with spins aligned along the magnetic field. The spectrum then acquires two gapless modes corresponding to the singlet-paired and spin-aligned unpaired bosons, respectively. At $T = 0$, the corresponding phase transition is of the commensurate–incommensurate type, which is qualitatively similar to what we have in the case of the O(3) NLσ model. There is a second transition at high magnetic fields corresponding to the saturation of the magnetization. However, it is not described by the low-energy theory and is beyond the scope of this paper. In the context of ultracold quantum gases, the commensurate–incommensurate transition has also been discussed for 1D spin-3/2 fermions in the presence of the quadratic Zeeman effect [29].

The observation of the commensurate–incommensurate phase transition in a 1D gas of ^{52}Cr atoms would require (aside from a positive value of g_0 and, hence, a negative c_2) fairly strong interactions corresponding to the parameter $M g_0/\bar{n} \sim 1$, where $\bar{n} = 7n$ is the total density, so that $\eta \sim 0.5$. Then the spin gap in the polar phase is of the order of ng_0 and (assuming that g_0 is by a factor of 3 smaller than g) can be made on the level of $100\,\text{nK}$ at 1D densities $\bar{n} \sim 10^5\,\text{cm}^{-1}$. Then the transition occurs at the critical field of the order of $0.2\,\text{mG}$ and can be observed at temperature $T \sim 20\,\text{nK}$. This, however, is likely to require the 1D regime with a rather strong confinement in the transverse directions (with a frequency of the order of $100\,\text{kHz}$, as in the ongoing chromium experiment in the 1D regime at Villetaneuse [17]).

Note added

After this work was completed, the Villetaneuse group reported the observation of the demagnetization transition for ^{52}Cr atoms in the 1D regime, under a decrease in the magnetic field to below $0.5\,\text{mG}$. However, the experiment was performed at temperatures $\sim 100\,\text{nK}$ and the state that is reached by decreasing the magnetic field does not necessarily reveal the nature of the ground state due to thermal excitations (and due to diabaticity at the transition in the experiment).

Acknowledgments

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