Regularity properties and definability in the real number continuum: idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy

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Chapter 4

Hausdorff gaps

Now we are going to depart somewhat from the theme of regularity properties, instead focusing on the definability of special objects. We mentioned some of these in Section 1.3: a Bernstein set, an ultrafilter, a maximal almost disjoint (mad) family, etc. In this chapter, we look at *Hausdorff gaps*, another kind of combinatorial object known since the beginning of the 20th century. Hausdorff's construction [Hau36] of an (ω_1, ω_1) gap in $\mathscr{P}(\omega)/\text{fin}$ was widely celebrated as an early success of the techniques and methods of set theory in mathematics. Many aspects of Hausdorff gaps, and other kinds of gaps, have been studied since then, such as extensions to higher cardinals, more general algebras, and the way forcing can destroy or create gaps. Surprisingly enough, the definability question for Hausdorff gaps has only been considered recently, in the work of Stevo Todorčević [Tod96] who showed, among other things, that there are no analytic Hausdorff gaps in $\mathscr{P}(\omega)/\text{fin}$. We shall continue this line of research, investigating what happens on higher projective levels, as well as the Solovay model, and under suitable axioms of determinacy.

4.1 Introduction

The underlying space in this chapter will be $[\omega]^\omega$, the collection of infinite subsets of ω. The notations $=^*$ and $\subseteq^*$ will be used throughout to represent the equality or subset relation between two elements of $[\omega]^\omega$ modulo finite. The following terminology has been established in the parlance of Hausdorff gaps: two sets $a, b \in [\omega]^\omega$ are *orthogonal* (notation $a \perp b$) if $a \cap b$ is finite. If B is a set, then a is *orthogonal to* B (notation $a \perp B$) if $a \perp b$ for every $b \in B$. Finally, $A, B \subseteq [\omega]^\omega$ are *orthogonal* (notation $A \perp B$) if $a \perp b$ holds for every $a \in A$ and every $b \in B$.

A pair (A, B) of orthogonal subsets of $[\omega]^\omega$ is called a *pre-gap*. There is one very simple way of constructing a pre-gap: take any infinite, co-infinite set $c \in [\omega]^\omega$ and pick any A and B so that $\forall a \in A : a \subseteq^* c$ and $\forall b \in B : c \cap b =^* \varnothing$. Such a set c is said to *separate*, or *interpolate*, the pre-gap (A, B). Of course, the interesting object is a pre-gap (A, B) which is *not* constructed in this trivial fashion.

Definition 4.1.1. *A pre-gap (A, B) is called a* gap *if there is no c which separates A from B.*

An early result of Hadamard [Had94] already established that there cannot be a gap (A, B) if both A and B are countable, although this is most widely known from [Hau36]. On the other hand, a gap (A, B) where $|A| = |B| = 2^{\aleph_0}$ can be explicitly constructed. For example, in [Tod96, p 56–57] Todorčević gives a very simple construction of a gap (A, B) where A and B are perfect sets: for $x \in 2^\omega$, define $a_x := \{x{\upharpoonright}n \mid x(n) = 0\}$ and $b_x := \{x{\upharpoonright}n \mid x(n) = 1\}$. Identifying $2^{<\omega}$ with ω, it is not hard to see that $(\{a_x \mid x \in 2^\omega\}, \{b_x \mid x \in 2^\omega\})$ is a gap.

Hausdorff's classical construction [Hau36] was very different. His gap (A, B) was such that $|A| = |B| = \aleph_1$, regardless of the size of the continuum; moreover, A and B were *σ-directed.*

Definition 4.1.2.

1. *A set $A \subseteq [\omega]^\omega$ is* σ-directed *if for every countable collection $\{a_n \in A \mid n \in \omega\}$, there exists $a \in A$ such that $a_n \subseteq^* a$ for all n.*
2. *A pair (A, B) is called a* Hausdorff gap *if it is a gap and both A and B are σ-directed.*

In the literature, the definition of a Hausdorff gap usually requires that A and B are well-ordered by $\subseteq^*$, as the original construction from [Hau36] in fact was, but for our purposes σ-directedness is sufficient.

That the perfect gap created by Todorčević in [Tod96, p 56–57] cannot be a Hausdorff gap follows from the following result of the same paper:

Theorem 4.1.3 (Todorčević)**.** *Let (A, B) be a pre-gap such that both A and B are σ-directed and A is analytic. Then (A, B) is not a gap.*

Proof. See [Tod96, Corollary 1]. □

This is, to our knowledge, the only result that deals with Hausdorff gaps from the definable point of view. We are interested in extending Todorčević's result in several directions and looking at Hausdorff gaps on definability levels beyond the analytic. We shall use the following notation: if $\mathbf{\Gamma}$ is a projective pointclass, we say that (A, B) is a $(\mathbf{\Gamma}, \mathbf{\Gamma})$*-Hausdorff gap* if both A and B are in $\mathbf{\Gamma}$, and a $(\mathbf{\Gamma}, \cdot)$*-Hausdorff gap* if $A \in \mathbf{\Gamma}$ and B is arbitrary. The theorem above says that there are no $(\mathbf{\Sigma}^1_1, \cdot)$-Hausdorff gaps. Our main result from Sections 4.2 and 4.3 (Corollary 4.3.10) will show that the following are equivalent:

1. there is no $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap,
2. there is no $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$-Hausdorff gap,
3. there is no $(\mathbf{\Pi}^1_1, \cdot)$-Hausdorff gap,
4. there is no $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap,
5. $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$.

The implications (1) ⇒ (2) ⇒ (4) and (1) ⇒ (3) ⇒ (4) are trivial; (5) ⇒ (1) will be proved in the next section, using a variation of the argument from [Tod96]; and (4) ⇒ (5) will be proved in Section 4.3 using the method of Miller [Mil89] for the inductive construction of $\mathbf{\Pi}^1_1$ sets in L.

In Section 4.4 we show that in the Solovay model, there are no Hausdorff gaps whatsoever, and in Section 4.5 we show that the same is true under $\mathsf{AD}_{\mathbb{R}}$ (the axiom of real determinacy). In Section 4.6 we briefly look at non-Hausdorff gaps and generalize a dichotomy proved in [Tod96, Theorem 2].

4.2 Hausdorff gaps on the second level

Because of the equivalence mentioned above, the statement "there is no $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap" has large cardinal strength, so it cannot be obtained by iterated forcing over L. For the same reason, we cannot hope to have a forcing-style proof of the implication "$\forall r\ (\aleph_1^{L[r]} < \aleph_1) \implies \nexists(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gap", as we did, say, in Corollary 2.2.7. Indeed, Todorčević's proof that there are no $(\mathbf{\Sigma}^1_1, \cdot)$-Hausdorff gaps was forcing-free, relying instead on a classical construction similar to the Cantor-Bendixson method. We will extend this method to prove the result about $(\mathbf{\Sigma}^1_2, \cdot)$-Hausdorff gaps. The way our proof is derived from Todorčević's original proof is similar to the way the Mansfield-Solovay theorem (Theorem 1.3.14) is derived from the theorem that all analytic sets satisfy the perfect set property (compare, e.g., [Jec03, Theorem 25.23] and [Jec03, Theorem 11.17 (iii)]).

Here and in the future, it will be useful to look at the space $\omega^{\uparrow\omega}$ of strictly increasing functions from ω to ω and, as usual, to identify elements of $[\omega]^\omega$ with their increasing enumerations. For the proof we need several definitions.

Definition 4.2.1. *Let (A, B) be a pre-gap (not necessarily σ-directed).*

1. *Let C be a set. We say that A and B are C-separated if $C \perp B$ and for every $a \in A$ there is $c \in C$ such that $a \subseteq^* c$.*
2. *We say that A and B are* σ-separated *if they are C-separated by some countable C.*

3. *Let S be a tree on $\omega^{\uparrow\omega}$. We call S an* (A,B)-tree *if*

   (a) $\forall \sigma \in S : \{i \in \omega \mid \sigma^\frown \langle i \rangle \in S\}$ *has infinite intersection with some* $b \in B$, *and*

   (b) $\forall x \in [S]$, $\mathrm{ran}(x) \subseteq^* a$ *for some* $a \in A$.

If (A,B) is not a gap, then it is σ-separated, but the converse need not be true in general. It is, however, true whenever A is σ-directed. On the other hand, the existence of an (A,B)-tree contradicts B being σ-directed.

Lemma 4.2.2. *Let (A,B) be a pre-gap. If B is σ-directed, then there is no (A,B)-tree.*

Proof. Suppose, towards contradiction, that S is an (A,B)-tree. For each $\sigma \in S$, fix some $b_\sigma \in B$ such that $\{i \mid \sigma^\frown \langle i \rangle \in S\} \cap b_\sigma$ is infinite. By σ-directedness, there is a $b \in B$ which almost contains every b_σ. In particular, for each σ, the set $\{i \mid \sigma^\frown \langle i \rangle \in S\} \cap b$ is infinite. Therefore we can inductively pick $i_0, i_1, i_2 \in b$ in such a way that $\langle i_0, i_1, i_2, \dots \rangle$ is a branch through S. Then by definition of an (A,B)-tree $\{i_0, i_1, i_2 \dots\} \subseteq^* a$ for some $a \in A$. But that implies that $a \cap b$ is infinite, contradicting the orthogonality of A and B. □

Todorčević's proof in fact shows the following dichotomy: if (A,B) is a pre-gap and A is analytic, then either A and B are σ-separated or there exists an (A,B)-tree. We prove a similar dichotomy for $\boldsymbol{\Sigma}^1_2$ sets, with separation by a subset of $L[r]$ replacing σ-separation.

Theorem 4.2.3. *Let (A,B) be a pre-gap such that A is $\Sigma^1_2(r)$. Then:*

1. *either there is a $C \subseteq L[r]$ which separates A from B, or*

2. *there exists an (A,B)-tree.*

Proof. Let $A^* \subseteq \omega^{\uparrow\omega}$ be such that $x \in A^*$ iff $\mathrm{ran}(x) \in A$. Let T be a tree on $\omega \times \omega_1$, increasing in the first coordinate, such that $A^* = p[T]$ and $T \in L[r]$. Define an operation on such trees T as follows

- for $(s,h) \in T$, let

$$c_{(s,h)} := \{i > \max(\mathrm{ran}(s)) \mid \exists (s',h') \in T \text{ extending } (s,h) \text{ s.t. } i \in \mathrm{ran}(s')\}$$

- let $T' := \{(s,h) \in T \mid c_{(s,h)}$ has infinite intersection with some $b \in B\}$.

Now let $T_0 := T$, $T_{\alpha+1} := T'_\alpha$ and $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$ for limit λ. Note that this definition is absolute for $L[r]$ so all the trees T_α are in $L[r]$.

Let α be least such that $T_\alpha = T_{\alpha+1}$. We distinguish two cases:

- **Case 1:** $T_\alpha = \varnothing$. Let $x \in A^*$ be given. Let $f \in \omega_1^\omega$ be such that $(x, f) \in [T_0]$. Let $\gamma < \alpha$ be such that $(x, f) \in [T_\gamma] \setminus [T_{\gamma+1}]$, and let $(s, h) \subseteq (x, f)$ be such that $(s, h) \in T_\gamma \setminus T_{\gamma+1}$. Now let $c_x := c_{(s,h)}$ and note that this set is in $L[r]$ since it is constructible from T_γ and (s, h) both of which are in $L[r]$. By assumption $c_x \perp B$, and it is also clear that $\operatorname{ran}(x) \subseteq^* c_x$. It follows that the collection $C := \{c_x \mid x \in A^*\}$, with each c_x defined as above, forms a subset of $L[r]$ which separates A from B.

- **Case 2:** $T_\alpha \neq \varnothing$. In this case we will use the tree T_α to construct an (A, B)-tree S. By induction, we will construct S and to each $\sigma \in S$ associate $(s_\sigma, h_\sigma) \in T_\alpha$, satisfying the following conditions:

  - $\sigma \subseteq \tau \implies (s_\sigma, h_\sigma) \subseteq (s_\tau, h_\tau)$, and
  - $\operatorname{ran}(\sigma) \subseteq \operatorname{ran}(s_\sigma)$.

  First $\varnothing \in S$, and we associate to it $(s_\varnothing, h_\varnothing) := (\varnothing, \varnothing)$. Next, suppose $\sigma \in S$ has already been defined and $(s_\sigma, h_\sigma) \in T_\alpha$ associated to it. By assumption, $(s_\sigma, h_\sigma) \in T'_\alpha$, so $c_{(s_\sigma,h_\sigma)}$ has infinite intersection with some $b \in B$. For each $i \in c_{(s_\sigma,h_\sigma)}$ we add $\sigma^\frown \langle i \rangle$ to S. Moreover, by assumption, for each $i \in c_{(s_\sigma,h_\sigma)}$ there exists $(s', h') \in T_\alpha$ extending (s, h) such that $i \in \operatorname{ran}(s')$. Now associate precisely these (s', h') to $\sigma^\frown \langle i \rangle$, i.e., let $s_{\sigma^\frown\langle i \rangle} := s'$ and $h_{\sigma^\frown\langle i \rangle} := h'$. By induction, it follows that the condition $\operatorname{ran}(\sigma^\frown \langle i \rangle) \subseteq \operatorname{ran}(s_{\sigma^\frown\langle i \rangle})$ is satisfied.

  Now we have a tree S on $\omega^{\uparrow\omega}$. By definition, for every $\sigma \in S$ the set of its successors $c_{(s_\sigma,h_\sigma)}$ has infinite intersection with some $b \in B$. Now let $x \in [S]$. By construction, $\bigcup\{(s_\sigma, h_\sigma) \mid \sigma \subseteq x\}$ forms an infinite branch through T_α, whose projection $a := \bigcup\{s_\sigma \mid \sigma \subseteq x\}$ is a member of $p[T_\alpha] \subseteq p[T_0] = A^*$. Since by assumption $\operatorname{ran}(\sigma) \subseteq \operatorname{ran}(s_\sigma)$ holds for all $\sigma \subseteq x$, it follows that $\operatorname{ran}(x) \subseteq \operatorname{ran}(a)$. This proves that S is an (A, B)-tree. □

Corollary 4.2.4. *If* $\forall r\ (\aleph_1^{L[r]} < \aleph_1)$ *then there is no* $(\boldsymbol{\Sigma}^1_2, \cdot)$*-Hausdorff gap.*

Proof. Let (A, B) be a pre-gap such that A and B are σ-directed and A is $\Sigma^1_2(r)$. By Lemma 4.2.2, the second alternative of Theorem 4.2.3 is impossible, hence there is a $C \subseteq L[r]$ which separates A from B. Since the reals of $L[r]$ are countable, C is countable, so A and B are σ-separated. Since A is also σ-directed, (A, B) cannot be a gap. □

4.3 Inaccessibility by reals

It was already mentioned in [Tod96] that if $V = L$ then there exists a $(\boldsymbol{\Pi}^1_1, \boldsymbol{\Pi}^1_1)$-Hausdorff gap, though a proof of this fact was not provided. In this section we give a proof of this result and, moreover, prove the stronger statement that the non-existence of a $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap implies that $\aleph_1$ is inaccessible in L. This complements the result of the previous section, i.e., Corollary 4.2.4.

Since the argument will involve a modification of Hausdorff's original construction, let us briefly review it.

The (ω_1, ω_1)-gap Hausdorff constructed in [Hau36] had the form (A, B) where $A = \{a_\alpha \mid \alpha < \aleph_1\}$, $B = \{b_\alpha \mid \alpha < \aleph_1\}$, both A and B are well-ordered by $\subseteq^*$, and, additionally, the following condition is satisfied:

$$\forall \alpha < \aleph_1 \, \forall k \in \omega \, (\{\gamma < \alpha \mid a_\alpha \cap b_\gamma \subseteq k\} \text{ is finite}). \qquad \text{(HC)}$$

We refer to this as *Hausdorff's condition* (HC).

Lemma 4.3.1 (Hausdorff)**.** *Any pre-gap (A, B) satisfying HC is a gap.*

Proof. Towards contradiction, suppose c separates A from B. For each $\alpha < \aleph_1$, let n_α be such that $a_\alpha \setminus c \subseteq n_\alpha$ and $b_\alpha \cap c \subseteq n_\alpha$. The values n_α must be constant on some uncountable set $X \subseteq \omega_1$, i.e., there is n such that $n_\alpha = n$ for all $\alpha \in X$. Pick any $\alpha \in X$ such that there are infinitely many γ below α in X. For all of these γ, we have $a_\alpha \cap b_\gamma \subseteq (a_\alpha \setminus c) \cup (b_\gamma \cap c) \subseteq n$, contradicting HC. □

The point of Hausdorff's condition is that it provides an *absolute* way to prove that (A, B) cannot be separated. In general, the notion of a gap is not absolute, i.e., a gap existing in some model could become a non-gap if a real c separating A from B is generically added to the model. However, if the original gap satisfies Hausdorff's condition, then this cannot happen as long as $\aleph_1$ is preserved.

Lemma 4.3.2. *Let (A, B) be a pre-gap in V, satisfying HC. Let W be a larger model with $\aleph_1^W = \aleph_1^V$. Then in W, (A, B) is still a gap.*

Proof. Apply the same argument as before. □

In particular, if (A, B) is a pre-gap in $L[r]$ and $\aleph_1^{L[r]} = \aleph_1$, then (A, B) is still a gap in V. Therefore our goal is to construct a pre-gap (A, B) in $L[r]$ satisfying HC, with both A and B being $\mathbf{\Pi}^1_1$, and in such a way that the same $\mathbf{\Pi}^1_1$ definition can work in V, too.

For starters, let us see how to construct a $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$-gap satisfying HC in L. The $\mathbf{\Pi}^1_1$ construction will then be a subtle modification of it using a method developed by Arnold Miller in [Mil89]. Hausdorff constructed his gap by induction on $\alpha < \aleph_1$, using the following instrumental Lemma at each induction step.

Lemma 4.3.3 (Hausdorff)**.** *Let α be some countable ordinal, and let $(\{a_\gamma \mid \gamma < \alpha\}, \{b_\gamma \mid \gamma < \alpha\})$ be a pre-gap well-ordered by $\subseteq^*$ and satisfying HC. Then there exist sets c, d such that $(\{a_\gamma \mid \gamma < \alpha\} \cup \{c\}, \{b_\gamma \mid \gamma < \alpha\} \cup \{d\})$ is still a pre-gap, is well-ordered by $\subseteq^*$, and satisfies HC.*

Proof. See [Hau36] or [Sch93, Theorem 10]. □

An (ω_1, ω_1)-gap satisfying HC can now inductively be constructed using this lemma. And just as we have already seen many times (cf. Fact 1.2.11, Fact 1.3.8, Theorem 3.2.1 etc.), in L this construction can be modified to produce a $\boldsymbol{\Sigma}^1_2$ definable gap. So, at step α, instead of just picking an arbitrary pair (c, d) given by Lemma 4.3.3, pick the $<_L$-least such. Let $A = \{a_\alpha \mid \alpha < \aleph_1\}$ and $B = \{a_\alpha \mid \alpha < \aleph_1\}$ be the resulting sets. Now, as before, we may write $a \in A$ iff $\exists L_\delta \ (a \in L_\delta \ \wedge \ L_\delta \models a \in A)$, or equivalently: there is $E \subseteq \omega \times \omega$ such that

1. E is well-founded,
2. $(\omega, E) \models \Theta$,
3. $\exists n \ (a = \pi_E(n)$ and $(\omega, E) \models n \in \pi_E^{-1}[A])$.

This statement is $\boldsymbol{\Sigma}^1_2$. Clearly, the same can be done for the set B. Notice also that the $\boldsymbol{\Sigma}^1_2$ definitions of A and B define the same sets in any larger model, i.e., even when $V \neq L$, the set of all a satisfying the sentence $\exists L_\delta \ (a \in L_\delta \ \wedge \ L_\delta \models a \in A)$ defines the same subset of L, and the same holds for B. It is also clear that we can replace L with an arbitrary $L[r]$ in this argument. We have now already proved the following:

Proposition 4.3.4. *If there is no $(\boldsymbol{\Sigma}^1_2, \boldsymbol{\Sigma}^1_2)$-Hausdorff gap, then $\forall r \ (\aleph_1^{L[r]} < \aleph_1)$.*

Proof. Assume r is such that $\aleph_1^{L[r]} = \aleph_1$. In $L[r]$, construct a $(\Sigma^1_2(r), \Sigma^1_2(r))$-Hausdorff gap, satisfying HC, as described above. By Lemma 4.3.2, it is still a gap in V. □

In [Mil89], Miller introduced a method by which many inductive constructions in L, like the one above, could be rendered not only $\boldsymbol{\Sigma}^1_2$ definable, but $\boldsymbol{\Pi}^1_1$ definable. The idea is to eliminate the existential quantifier in the sentence "$\exists L_\delta \dots$", or "$\exists E \subseteq \omega \times \omega \dots$", by coding E directly into the real a constructed at each stage. This would allow us to write "$a \in A \iff e(a)$ is well-founded, etc.", where e is a recursive "decoding" function recovering the relation $E \subseteq \omega \times \omega$ from a. Quoting Miller:

> "The general principle is that if a transfinite construction can be done so that at each stage an arbitrary real can be encoded into the real constructed at that stage then the set being constructed will be Π^1_1. The reason is basically that then each element of the set can encode the entire construction up to that point at which it itself is constructed."
>
> [Mil89, p. 194]

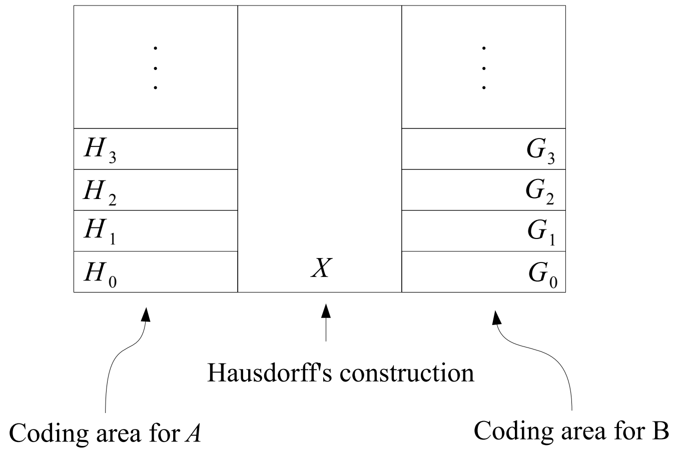

Figure 4.1: Partition of ω.

Miller himself applied this principle to show that in L there is a $\mathbf{\Pi}^1_1$ subset of $\mathbb{R}^2$ meeting every line in exactly two points, a $\mathbf{\Pi}^1_1$ mad family, and a $\mathbf{\Pi}^1_1$ Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. In [KSZ08, Theorem 3.1] the authors used the same method to show that in L there is a $\mathbf{\Pi}^1_1$ ω-mad family. Other applications exist in the literature, for instance the recent [FT10] showing that in L there is a $\mathbf{\Pi}^1_1$ maximal set of orthogonal measures on Cantor space.

To apply Miller's method, we need to prove a Coding Lemma: a stronger version of Lemma 4.3.3 stating that the c and d constructed at each induction step can encode an arbitrary relation $E \subseteq \omega \times \omega$. First, we recursively partition ω into three infinite sets: H, X and G. Further, we recursively partition H into infinitely many infinite sets H_n, and G into infinitely many infinite sets G_n. All the essential properties of Hausdorff's construction will take place within X, while the areas H and G will be used for coding purposes only. The plan is to encode an arbitrary real $z \in 2^\omega$ into a set $a \in [\omega]^\omega$ by making sure that $|H_n \cap a|$ is even if $z(n) = 1$ and odd if $z(n) = 0$, and the same for b and the G_n. A relation $E \subseteq \omega \times \omega$ can easily be encoded into a real $z \in 2^\omega$ (using a recursive bijection between ω and $\omega \times \omega$).

Lemma 4.3.5 (Coding Lemma)**.** *Let α be some countable ordinal and let $(\{a_\gamma \mid \gamma < \alpha\}, \{b_\gamma \mid \gamma < \alpha\})$ be a pre-gap with $a_\gamma \subseteq H \cup X$ and $b_\gamma \subseteq X \cup G$, which is well-ordered by $\subseteq^*$, satisfies HC, and also satisfies the following condition:*

$$(*) \qquad \forall n \in \omega\, \forall \gamma < \alpha\, (|a_\gamma \cap H_n| < \omega \text{ and } |b_\gamma \cap G_n| < \omega).$$

Let $E \subseteq \omega \times \omega$ be an arbitrary relation. Then there exist infinite sets c, d, with $c \subseteq H \cup X$, $d \subseteq X \cup G$, such that $(\{a_\gamma \mid \gamma < \alpha\} \cup \{c\}, \{b_\gamma \mid \gamma < \alpha\} \cup \{d\})$ is still a pre-gap, well-ordered by $\subseteq^*$, satisfies HC, satisfies condition $(*)$, and moreover both c and d recursively encode E.

Proof. First, we consider the restriction of the pre-gap to X: $(\{a_\gamma \cap X \mid \gamma < \alpha\}, \{b_\gamma \cap X \mid \gamma < \alpha\})$. Note that this is also a pre-gap well-ordered by $\subseteq^*$. Moreover, since for all γ, γ' we know that a_γ is disjoint from $b_{\gamma'}$ everywhere outside of X, the restricted pre-gap must satisfy HC, too. Using a bijection between X and ω we can apply Hausdorff's original Lemma 4.3.3 to the restricted pre-gap, and get new sets $c', d' \subseteq X$, such that $(\{a_\gamma \cap X \mid \gamma < \alpha\} \cup \{c'\}, \{b_\gamma \cap X \mid \gamma < \alpha\} \cup \{d'\})$ is a pre-gap, is well-ordered by $\subseteq^*$, and satisfies HC.

We now describe what happens inside H and G. Let $\{a'_n \mid n < \omega\}$ and $\{b'_n \mid n < \omega\}$ be a re-enumeration of the countable sets $\{a_\gamma \mid \gamma < \alpha\}$ and $\{b_\gamma \mid \gamma < \alpha\}$. Let $z \in 2^\omega$ be a real recursively coding the relation E. Now pick $c_n \subseteq H_n$ and $d_n \subseteq G_n$ such that

1. c_n and d_n are finite,
2. $\bigcup_{m \leq n}(a'_m \cap H_n) \subseteq c_n$,
3. $\bigcup_{m \leq n}(b'_m \cap G_n) \subseteq d_n$, and
4. $|c_n|$ and $|d_n|$ are even if $z(n) = 1$, and odd if $z(n) = 0$.

That this can always be done follows from condition $(*)$ of the induction hypothesis.

Now we set $c := c' \cup \bigcup_n c_n$ and $d := d' \cup \bigcup_n d_n$, and claim that the new pair of sequences $(\{a_\gamma \mid \gamma < \alpha\} \cup \{c\}, \{b_\gamma \mid \gamma < \alpha\} \cup \{d\})$ satisfies all the requirements of the lemma. It is obvious that it is a pre-gap and satisfies HC. Condition $(*)$ is also clear, since $c \cap H_n = c_n$ and $d \cap G_n = d_n$ and we defined these to be finite. To show that it is well-ordered by $\subseteq^*$, pick any a_γ. We must show that $a_\gamma \subseteq^* c$, and for that, we need to show that $a_\gamma \cap X \subseteq^* c \cap X$, and $a_\gamma \cap H \subseteq^* c \cap H$. The former is clear, because on X we have applied Lemma 4.3.3. For the latter, suppose $a_\gamma = a'_n$ in the re-enumeration used. For $k \geq n$, the definition implies that $a'_n \cap H_k \subseteq c_k$. Moreover, $\bigcup_{k<n}(a'_n \cap H_k)$ is finite by property $(*)$ of the induction hypothesis. This shows that indeed $a'_n \cap H \subseteq^* \bigcup_k c_k$ as had to be shown. Analogously, we can show $b_\gamma \subseteq^* d$.

Finally, it is clear that c and d recursively encode the relation E. $\square$

Now define the two decoding functions $e_0, e_1 : [\omega]^\omega \to 2^\omega$ by

$$e_0(a)(n) := \begin{cases} 1 & \text{if } |H_n \cap a| \text{ is even} \\ 0 & \text{if } |H_n \cap a| \text{ is odd} \end{cases}$$

$$e_1(b)(n) := \begin{cases} 1 & \text{if } |G_n \cap b| \text{ is even} \\ 0 & \text{if } |G_n \cap b| \text{ is odd} \end{cases}$$

and by identifying z with the relation $E \subseteq \omega \times \omega$ that it recursively codes, we consider e_0 and e_1 as functions from $[\omega]^\omega$ to $\mathscr{P}(\omega \times \omega)$.

In order to use Miller's method, some special properties of the constructible hierarchy are needed, which we now present as black box results.

Definition 4.3.6 (Miller). *For a countable limit ordinal α, an L_α is called* point-definable *if there exists $E \in L_{\alpha+\omega}$ such that $(\omega, E) \cong (L_\alpha, \in)$.*

Fact 4.3.7 (Miller).

1. *There are unboundedly many $\alpha < \aleph_1$ such that L_α is point-definable.*
2. *Suppose L_α is point-definable. For any $\beta \leq \alpha$, if L_β is point-definable then $L_{\alpha+\omega} \models$ "L_β is point-definable".*
3. *If L_α is point-definable and E is such that $(\omega, E) \cong (L_\alpha, \in)$, then there is a recursive function mapping E to another relation, $E^{+\omega}$, such that $(\omega, E^{+\omega}) \cong (L_{\alpha+\omega}, \in)$.*

Proof. The point-definable L_α's are those levels of the constructible hierarchy whose closure under the definable Skolem functions of L is isomorphic to itself. For a detailed proof, see [KSZ08], specifically Lemmas 3.4, 3.5 and 3.6, and the relevant comments regarding absoluteness of the definitions. □

Fix an enumeration $\{\xi_\alpha \mid \alpha < \aleph_1\}$ of those countable limit ordinals for which L_{ξ_α} is point-definable. We may assume without loss of generality that $\xi_\alpha + \omega < \xi_{\alpha+1}$ for all α. By induction on $\alpha < \aleph_1$, we can now build our sets $\{a_\alpha \mid \alpha < \aleph_1\}$ and $\{b_\alpha \mid \alpha < \aleph_1\}$, with an induction hypothesis guaranteeing that a_α and b_α are members of $L_{\xi_\alpha+\omega}$.

Suppose $\{a_\gamma \mid \gamma < \alpha\}$ and $\{b_\gamma \mid \gamma < \alpha\}$ has been constructed and satisfies all the relevant conditions, i.e., is a pre-gap, is well-ordered by $\subseteq^*$, satisfies HC, and satisfies condition $(*)$ from the Coding Lemma (Lemma 4.3.5). Also, assume $a_\gamma, b_\gamma \in L_{\xi_\gamma+\omega}$ for each $\gamma < \alpha$. Then in fact $a_\gamma, b_\gamma \in L_{\xi_{\gamma+1}} \subseteq L_{\xi_\alpha}$, therefore the sets $\{a_\gamma \mid \gamma < \alpha\}$ and $\{b_\gamma \mid \gamma < \alpha\}$ are in $L_{\xi_\alpha+1}$. Moreover, since $L_{\xi_\alpha+\omega}$ contains an E satisfying $(\omega, E) \cong (L_{\xi_\alpha}, \in)$, it follows that $L_{\xi_\alpha+\omega} \models \alpha$ is countable. In particular, the two initial segments of the Hausdorff gap are countable in $L_{\xi_\alpha+\omega}$, so we can apply the Coding Lemma inside $L_{\xi_\alpha+\omega}$ to get two sets c, d in $L_{\xi_\alpha+\omega}$ which both recursively encode E. We choose a_α and b_α to be the $<_L$-least (or $<_{L_{\xi_\alpha+\omega}}$-least) such c and d. The Coding Lemma guarantees that all the requirements to proceed with the induction are satisfied by the extended initial segments $\{a_\gamma \mid \gamma \leq \alpha\}$ and $\{b_\gamma \mid \gamma \leq \alpha\}$.

Let $A := \{a_\alpha \mid \alpha < \aleph_1\}$ and $B := \{b_\alpha \mid \alpha < \aleph_1\}$ be the sets thus constructed. It is clear that (A, B) is a Hausdorff gap satisfying HC. Now recall the decoding functions e_0 and e_1. By Fact 4.3.7 (3), there are also recursive functions $e_0^{+\omega}$ and $e_1^{+\omega}$ such that if $(\omega, e_i(a)) \cong (L_{\xi_\alpha}, \in)$ for some ξ_α, then $(\omega, e_i^{+\omega}(a)) \cong (L_{\xi_\alpha+\omega}, \in)$. Now it only remains to prove the following:

Claim 4.3.8.

1. *For all* $a \in [\omega]^\omega$, $a \in A \iff$
   (a) $e_0(a)$ *is well-founded,*
   (b) $(\omega, e_0(a)) \models \Theta$,
   (c) $\exists n \in \omega\, (a = \pi_{e_0^{+\omega}(a)}(n)$ *and* $(\omega, e_0^{+\omega}(a)) \models n \in \pi^{-1}_{e_0^{+\omega}(a)}[A])$.
2. *For all* $b \in [\omega]^\omega$, $b \in B \iff$
   (a) $e_1(b)$ *is well-founded,*
   (b) $(\omega, e_1(b)) \models \Theta$,
   (c) $\exists n \in \omega\, (b = \pi_{e_1^{+\omega}(b)}(n)$ *and* $(\omega, e_1^{+\omega}(b)) \models n \in \pi^{-1}_{e_1^{+\omega}(b)}[B])$.

Proof. The two parts are obviously analogous so let us check the first one. If $a \in A$, then $a = a_\alpha$ for some α. Let $E := e_0(a)$. Then by construction $(\omega, E) \cong (L_{\xi_\alpha}, \in)$, so points (a) and (b) are satisfied. Moreover, note that the way we picked a_α in $L_{\xi_\alpha+\omega}$ using Lemma 4.3.5 was absolute between $L_{\xi_\alpha+\omega}$ and L because the relevant initial segment of the construction was in $L_{\xi_\alpha+\omega}$ and we picked the $<_L$-least such a_α. Therefore $L_{\xi_\alpha+\omega} \models a \in A$, so point (c) is satisfied.

Conversely, suppose a satisfies points (a), (b) and (c). Let $E := e_0(a)$. Then $(\omega, E) \cong (L_\delta, \in)$ for some countable limit ordinal δ, $a \in L_{\delta+\omega}$ and $L_{\delta+\omega} \models a \in A$. Then $L_{\delta+\omega} \models (\omega, e_0(a)) \cong (L_{\xi_\alpha}, \in)$ for some $\xi_\alpha < \delta + \omega$. But since this isomorphism must be absolute, in fact $\xi_\alpha = \delta$, so $L_{\xi_\alpha+\omega} \models a \in A$. Then the absoluteness of the definition of A implies that $a \in A$ holds in L, too (in fact $a = a_\alpha$). □

The claim gives us a $\mathbf{\Pi}^1_1$ definition of both A and B, since part (a) is a $\mathbf{\Pi}^1_1$ statement and the others are arithmetical. As before, it is also clear that in any larger model V, the sets A and B defined as in the claim are still exactly the same subsets of L. Also, L can be replaced by an arbitrary $L[r]$ in all the above arguments. As a result, we have shown the following:

Theorem 4.3.9. *If there is no* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-Hausdorff gap, then* $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$.

Combining this result with what we proved in the last section, we get, as promised, the following corollary:

Corollary 4.3.10. *The following are equivalent:*

1. *there is no* $(\mathbf{\Sigma}^1_2, \cdot)$*-Hausdorff gap,*
2. *there is no* $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$*-Hausdorff gap,*
3. *there is no* $(\mathbf{\Pi}^1_1, \cdot)$*-Hausdorff gap,*

4. *there is no* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-Hausdorff gap,*

5. $\forall r\, (\aleph_1^{L[r]} < \aleph_1)$.

Proof. The direction (5) ⇒ (1) is Corollary 4.2.4 and (4) ⇒ (5) is Theorem 4.3.9. The other implications are obvious. □

4.4 Solovay model

We now turn our attention to the Solovay model, and the question whether Hausdorff gaps have to exist at all, not assuming AC.

Theorem 4.4.1. *Let* V *be a model with an inaccessible cardinal* κ *and* $V[G]$ *the Lévy collapse of* κ*. In* $V[G]$*, let* (A, B) *be a pre-gap with* A *and* B *definable from a countable sequence of ordinals. Then*

1. *either* A *and* B *are* σ*-separated, or*

2. *there exists an* (A, B)*-tree.*

Proof. In $V[G]$, let $s \in \mathrm{Ord}^\omega$ be such that A is definable by $\varphi(s, x)$ and B is definable by $\psi(s, x)$. By standard properties of the Lévy collapse (Lemma 1.2.20), there are formulas $\tilde{\varphi}$ and $\tilde{\psi}$ such that for all x, $V[G] \models \varphi(s, x)$ iff $V[s][x] \models \tilde{\varphi}(s, x)$, and $V[G] \models \psi(s, x)$ iff $V[s][x] \models \tilde{\psi}(s, x)$.

Assume that A and B are not σ-separated. Since κ is inaccessible in $V[s]$, there are countably many reals in $V[s]$, so A and B are not $V[s]$-separated (in the sense of Definition 4.2.1 (1)). Hence, there exists an $a \in A$ such that for all $c \in V[s]$, if $a \subseteq^* c$ then $c \not\perp B$. Let $x \in \omega^{\uparrow\omega}$ be the increasing enumeration of a. The sentence

$$\Phi(x) \;\equiv\; \forall c \in V[s]\; (\mathrm{ran}(x) \subseteq^* c \;\rightarrow\; \exists b\; (V[s][b] \models \tilde{\psi}(s, b) \wedge |c \cap b| = \aleph_0))$$

is true in $V[G]$. By another standard property of the Lévy collapse, there is a generic H such that $V[s][H] = V[G]$, and moreover there is a partial suborder Q of the Lévy collapse (depending on x), such that $Q \in V[s]$, $|Q| < \kappa$ and $x \in V[s][Q \cap H]$. Then in $V[s]$ there is a name $\dot{x}$ and a condition $p \in Q$ such that

$$p \Vdash \Phi(\dot{x}) \wedge V[s][\dot{x}] \models \tilde{\varphi}(s, \dot{x}).$$

Let $\{D_i \mid i \in \omega\}$ enumerate all the Q-dense sets in $V[s]$ (there are only countably many because κ is inaccessible in $V[s]$).

Now we inductively construct a tree $S \subseteq \omega^{<\uparrow\omega}$, and for every $t \in S$ a condition $p_t \leq p$ and an infinite set $c_t \in V[s]$, such that the following conditions are satisfied:

1. $s \subseteq t \iff p_t \leq p_s$,

2. for every t, $p_t \in D_{|t|}$,
3. for every t, $p_t \Vdash \mathrm{ran}(t) \subseteq \mathrm{ran}(\dot{x})$, and
4. for every t, $p_t \Vdash \mathrm{ran}(\dot{x}) \subseteq^* \check{c}_t$.

Let $p_\varnothing \leq p$ be any condition in D_0. Clearly conditions (2) and (3) are satisfied. Assume p_t is already defined for $t \in S$, and satisfies condition (3). Let

$$c_t := \{i \mid i > \max(t) \text{ and } \exists q \leq p_t \; (q \Vdash i \in \mathrm{ran}(\dot{x}))\}.$$

Then c_t is in $V[s]$ and condition (4) is satisfied (at stage t). For every $i \in c_t$, let $t^\frown \langle i \rangle$ also be an element of the tree S, and let $p_{t^\frown \langle i \rangle} \leq p$ be a condition such that $p_{t^\frown \langle i \rangle} \Vdash i \in \mathrm{ran}(\dot{x})$ and $p_{t^\frown \langle i \rangle} \in D_{|t|+1}$. Now each such $p_{t^\frown \langle i \rangle}$ also satisfies condition (3) (at stage $t^\frown \langle i \rangle$), completing the induction step.

Thus we have constructed the tree S, and now we claim that it is an (A, B)-tree. For every $t \in S$, c_t is the set of immediate successors of t in S. By condition (4), $p_t \Vdash \mathrm{ran}(\dot{x}) \subseteq^* \check{c}_t$, and since $p_t \Vdash \Phi(\dot{x})$ and obviously $p_t \Vdash \check{c}_t \in V[s]$, it follows that some $q \leq p_t$ forces the consequent of $\Phi(\dot{x})$, i.e., the statement "$\exists b \; (V[s][b] \models \tilde{\psi}(s, b) \wedge |b \cap c_t| = \aleph_0)$". Then for some H' Q-generic over $V[s]$ containing q, this statement holds in $V[s][H']$, and therefore there exists a b such that $V[s][b] \models \tilde{\psi}(s, b)$ and $|b \cap c_t| = \aleph_0$, i.e., there exists $b \in B$ such that $|b \cap c_t| = \aleph_0$. Since this holds for every c_t, one part of the definition of an (A, B)-tree is fulfilled.

It remains to prove that every branch through S is contained in an element of A. Let $z \in [S]$, and let H_z be the filter over Q generated by $\{p_t \mid t \subseteq z\}$. By construction, H_z is Q-generic over $V[S]$. Since all p_t force the statement "$V[s][\dot{x}] \models \tilde{\varphi}(s, \dot{x})$", we get that $V[s][\dot{x}_{H_z}] \models \tilde{\varphi}(s, \dot{x}_{H_z})$, and therefore $\dot{x}_{H_z} \in A$ (here we have identified $\dot{x}_{H_z}$ with its range, but it should be clear that this is fine). Moreover, since by condition (3) we have, for every $t \subseteq z$, that $p_t \Vdash \mathrm{ran}(t) \subseteq \mathrm{ran}(\dot{x})$, it follows that $\mathrm{ran}(t) \subseteq \mathrm{ran}(\dot{x}_{H_z})$ holds for every $t \subseteq z$, and therefore $\mathrm{ran}(z) \subseteq \mathrm{ran}(\dot{x}_{H_z}) \in A$. This is what we wanted to show. □

Corollary 4.4.2. *Let V be a model with an inaccessible cardinal κ and $V[G]$ the Lévy collapse. If (A, B) is a pre-gap in $V[G]$ such that A and B are definable from a countable sequence of ordinals, and moreover A and B are σ-directed, then (A, B) is not a gap.*

Proof. As before, if B is σ-directed then there cannot be an (A, B)-tree by Lemma 4.2.2, and if A is also σ-directed then alternative 1 from Theorem 4.4.1 implies that A and B are separated. □

Corollary 4.4.3. Con(ZFC+ *"there are no projective Hausdorff gaps"*) *and* Con($\mathsf{ZF} + \mathsf{DC}$+ *"there are no Hausdorff gaps"*).

4.5 Axiom of real determinacy

The determinacy of infinite games is often used as a tool to prove regularity properties. The strongest result we could hope to prove in this setting is that AD implies that there are no Hausdorff gaps. We were not able to prove this, but, as already discussed in Question 2.6.4 and Question 2.6.5, $\mathsf{AD}_{\mathbb{R}}$ may be a more appropriate axiom in this case. So, we will take $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ as the ambient theory in this section, and construct a game with real moves whose determinacy proves the non-existence of Hausdorff gaps.

Definition 4.5.1. *Let (A, B) be a pre-gap. The game $G_{\mathrm{H}}(A, B)$ is played as follows:*

$$\begin{array}{c|cccccccc} \mathrm{I}: & c_0 & & (s_1, c_1) & & (s_2, c_2) & & \dots & \\ \hline \mathrm{II}: & & i_0 & & i_1 & & i_2 & & \dots \end{array}$$

where $s_n \in \omega^{<\omega}$, $c_n \in [\omega]^\omega$ and $i_n \in \omega$. The conditions for player I are that

1. $\min(s_n) > \max(s_{n-1})$ *for all* $n \geq 1$,
2. $\min(c_n) > \max(s_n)$,
3. $c_n \not\perp B$ *for all* n, *and*
4. $i_n \in \mathrm{ran}(s_{n+1})$ *for all* n.

Conditions for player II are that

5. $i_n \in c_n$ *for all* n.

If all five conditions are satisfied, let $s^* := s_1{}^\frown s_2{}^\frown \dots$ be an infinite increasing sequence formed by the play of the game. Player I wins iff $\mathrm{ran}(s^*) \in A$.

Theorem 4.5.2.

1. *If player I has a winning strategy in $G_{\mathrm{H}}(A, B)$ then there exists an (A, B)-tree.*
2. *If player II has a winning strategy in $G_{\mathrm{H}}(A, B)$ then A and B are σ-separated.*

Proof. 1. Let σ be a winning strategy for player I and let T_σ be the tree of partial positions according to σ. If $p \in T_\sigma$ is a position of the form $p = \langle c_0, i_0, (s_1, c_1), i_1, \dots, (s_n, c_n)\rangle$, we use the notation $p^* := s_1{}^\frown \dots {}^\frown s_n$.

Now we use T_σ to inductively construct the tree S. To each $s \in S$ we associate a $p_s \in T_\sigma$ (of odd length), such that

1. $s \subseteq t$ iff $p_s \subseteq p_t$, and

2. $\text{ran}(s) \subseteq \text{ran}(p_s^*)$.

First $\varnothing \in S$ and $p_\varnothing = \varnothing$. Suppose $s \in S$ and p_s are already defined and $\text{ran}(s) \subseteq \text{ran}(p_s^*)$ holds. Assume $p_s = \langle \dots, (s_n, c_n) \rangle$. For every $i_n \in c_n$, let (s_{n+1}, c_{n+1}) be the response of the strategy σ to $p_s{}^\frown \langle i_n \rangle$. Let $s^\frown \langle i_n \rangle$ be in S and associate to it $p_{s^\frown \langle i_n \rangle} := p_s{}^\frown \langle i_n \rangle^\frown \langle (s_{n+1}, c_{n+1}) \rangle$. Since for each $i_n \in c_n$ we know that $i_n \in \text{ran}(s_{n+1})$, it follows that $\text{ran}(s^\frown \langle i_n \rangle) \subseteq \text{ran}(p^*_{s^\frown \langle i_n \rangle})$, completing the induction step.

Now it is clear that the tree S has exactly the c_n's as the branching-points, which all have infinite intersection with some $b \in B$ by assumption. Moreover, if x is a branch through S, then by construction $z := \bigcup \{p_s \mid s \subseteq x\}$ forms a branch through T_σ satisfying $\text{ran}(x) \subseteq \text{ran}(z^*)$. Since z is an infinite play of the game according to the winning strategy σ, it follows that $\text{ran}(z^*) \in A$, so S is an (A, B)-tree.

2. Now let τ be a winning strategy for player II, and let T_τ be the tree of partial plays according to τ. Our method will be similar to the proof of the standard Banach-Mazur theorem, but the problem is that the tree T_τ has uncountable branching. Therefore we first thin it out to another tree $\tilde{T}_\tau$, as follows: for every node of even length $p = \langle \dots, (s_n, c_n), i_n \rangle \in T_\tau$, fix s and i and consider the collection $\text{Succ}_{T_\tau}(p, s, i) := \{(s, c) \mid p^\frown \langle (s, c) \rangle^\frown \langle i \rangle \in T_\tau\}$. In other words, this is the collection of all valid moves by player I following position p, such that the first component of the move is s, and such that II's next move according to τ is i. If this collection is non-empty, throw away all members of $\text{Succ}_{T_\tau}(p, s, i)$, and their generated subtrees, except for one, so that $\text{Succ}_{T_\tau}(p, s, i)$ becomes a singleton. Notice that this construction is justified because, since we are working under $\mathsf{AD}_\mathbb{R}$, we have to our disposal the fragment of the Axiom of Choice allowing us to choose from collections indexed by real numbers. Therefore, we can perform this "pruning" operation for every $s \in \omega^{<\omega}$ and every $i \in \omega$, and inductively form the new tree $\tilde{T}_\tau$—this is also going to be a tree of positions according to τ, but it will be a countably branching tree. Now we can use a Banach-Mazur-style argument on $\tilde{T}_\tau$.

For every $p \in \tilde{T}_\tau$ and $x \in \omega^{\uparrow\omega}$, where $p = \langle \dots (s_n, c_n), i_n \rangle$, we say that p *is compatible with* x if $p^* \subseteq x$ and $i_n \in \text{ran}(x)$. We say that p *rejects* x if it is compatible with x and maximally so with respect to $\tilde{T}_\tau$, i.e., if for every (s, c) such that $p^\frown \langle (s, c) \rangle \in \tilde{T}_\tau$ and $p^{*\frown} s \subseteq x$, $i := \tau(p^\frown \langle (s, c) \rangle) \notin \text{ran}(x)$.

It is clear that for every x with $\text{ran}(x) \in A$ there is a $p \in \tilde{T}_\tau$ which rejects x—otherwise we could inductively find an infinite branch z through $\tilde{T}_\tau$ such that $z^* = x$, implying that $\text{ran}(x) \notin A$ since z is a play according to a strategy that was winning for player II. For each $p \in \tilde{T}_\tau$ let $K_p := \{x \mid p \text{ rejects } x\}$. Also, write $K_p^* := \{\text{ran}(x) \mid p \text{ rejects } x\}$. Since $A \subseteq \bigcup_{p \in \tilde{T}_\tau} K_p^*$ and $\tilde{T}_\tau$ is countable, the result will follow if we can prove that each K_p^* is σ-separated from B.

For this, fix some $p = \langle \dots (s_n, c_n), i_n \rangle$, and for every $s \in \omega^{<\omega}$ such that $i_n \in \text{ran}(s)$ and $\min(s) > \max(p^*)$, consider the set

$$a_s := \bigcup \{\text{ran}(x) \mid x \in K_p \text{ and } p^{*\frown} s \subseteq x\}.$$

We claim that the collection $\{a_s \mid i_n \in \text{ran}(s) \text{ and } \min(s) > \max(p^*)\}$ σ-separates K_p^* from B. First, clearly if $x \in K_p$ then there exists some s, satisfying the conditions, such that $p^{*\frown} s \subseteq x$, so that $\text{ran}(x) \subseteq a_s$. Secondly, suppose that there is some s, with $i_n \in \text{ran}(s)$ and $\min(s) > \max(p^*)$, such that a_s has infinite intersection with some $b \in B$. Let $a'_s := a_s \setminus \max(s)$. According to the rules of the game, player I is then allowed to play the move "(s, a'_s)" after position p. The only problem is that $p^\frown \langle (s, a'_s) \rangle$ might not be in $\tilde{T}_\tau$. However, by construction there is some c such that $i := \tau(p^\frown \langle (s, c) \rangle) = \tau(p^\frown \langle (s, a'_s) \rangle)$ and $p^\frown \langle (s, c) \rangle \in \tilde{T}_\tau$. But then we must have $i \in a'_s$, so by definition there is some $x \in K_p$ such that $p^{*\frown} s \subseteq x$ and $i \in \text{ran}(x)$. But then $p^\frown \langle (s, c) \rangle^\frown \langle i \rangle$ is still compatible with x and hence p does not reject x, contradicting $x \in K_p$.

So we must have $a_s \perp B$ for all s, and this completes the proof. □

Corollary 4.5.3. $\mathsf{AD}_\mathbb{R}$ *implies that every pre-gap* (A, B) *is either* σ*-separated or there exists an* (A, B)*-tree.*

Corollary 4.5.4. $\mathsf{AD}_\mathbb{R}$ *implies that there are no Hausdorff gaps.*

4.6 Other gaps

In the last section, we briefly consider non-Hausdorff gaps, i.e., gaps (A, B) in which A and B are not necessarily σ-directed, and extend the second main theorem of [Tod96], by combining its proof with results from [Fen93].

As we know, such gaps can be quite explicitly defined. Let us recall the example we mentioned in the introduction: $A := \{a_x \mid x \in 2^\omega\}$ and $B := \{b_x \mid x \in 2^\omega\}$, where $a_x := \{x{\restriction}n \mid x(n) = 0\}$ and $b_x := \{x{\restriction}n \mid x(n) = 1\}$ for every $x \in 2^\omega$. In [Tod96, p 57], Todorčević isolated the main ingredient of this construction and defined a concept that he called a *perfect Luzin gap.*

Definition 4.6.1. (A, B) *is called a* perfect Luzin gap *if* A *can be written as* $\{a_x \mid x \in 2^\omega\}$ *and* B *can be written as* $\{b_x \mid x \in 2^\omega\}$*, such that the functions* $x \mapsto a_x$ *and* $x \mapsto b_x$ *are continuous, and so that the following condition* (Luzin's condition) *is satisfied: there exists some* $n \in \omega$ *such that*

1. *for every* $x \in 2^\omega$*,* $a_x \cap b_x \subseteq n$*, and*

2. *for every* $x \neq y$*, either* $a_x \cap b_y \nsubseteq n$ *or* $a_y \cap b_x \nsubseteq n$*.*

Any pre-gap (A, B) satisfying Luzin's condition can be shown to be a gap (see, e.g., [Sch93] for details), and moreover, A and B are perfect subsets of $[\omega]^\omega$. The second main result of Todorčević [Tod96, Theorem 2] shows that a perfect Luzin gap is essentially the only type of analytic gap. First we need a weaker notion of separation.

Definition 4.6.2. *A pre-gap (A, B) is* weakly σ-separated *if there is a countable set C such that for every $a \in A$ and $b \in B$, there is a $c \in C$ such that $a \subseteq^* c$ and $c \cap b$ is finite.*

If (A, B) are σ-separated then they are also weakly σ-separated, though the converse need not be true. Of course, in the case of a Hausdorff gap, the two notions are the same, and are equivalent to (A, B) being separated, but in general we should be more careful.

Definition 4.6.3. *We say that a pre-gap (A, B) satisfies the* perfect Luzin dichotomy *if either*

1. *(A, B) is weakly σ-separated, or*
2. *there is a perfect Luzin sub-gap (A', B') of (A, B) (i.e., $A' \subseteq A$ and $B' \subseteq B$).*

Theorem 4.6.4 (Todorčević)**.** *Every $(\mathbf{\Sigma}^1_1, \mathbf{\Sigma}^1_1)$-pre-gap satisfies the perfect Luzin dichotomy.*

Proof. See [Tod96, Theorem 2] □

Can we extend this theorem, and prove results similar to the ones we proved about Hausdorff gaps? We will show that this is indeed the case, and, in fact, it follows by putting together several existing results. First, we note that the main ingredient of the proof of [Tod96, Theorem 2] is a perfect set version of the Open Colouring Axiom studied by Qi Feng in [Fen93], itself being a variant of the original Open Colouring Axiom (OCA) introduced by Todorčević in [Tod89].

Definition 4.6.5 (Feng)**.** *A set A satisfies OCA_P if for every partition $[A]^2 = K_0 \cup K_1$, where K_0 is open in the relative topology of A, one of the following holds:*

1. *there exists a perfect set $P \subseteq A$ such that $[P]^2 \subseteq K_0$, or*
2. *$A = \bigcup_n A_n$, for some A_n satisfying $[A_n]^2 \subseteq K_1$.*

We write $\mathbf{\Gamma}(\mathsf{OCA}_P)$ to mean that every set in $\mathbf{\Gamma}$ satisfies OCA_P.

Theorem 4.6.6 (Feng)**.**

1. *$\mathbf{\Sigma}^1_1(\mathsf{OCA}_P)$,*

2. *the following are equivalent:*
   (a) $\mathbf{\Sigma}^1_2(\mathsf{OCA}_P)$,
   (b) $\mathbf{\Pi}^1_1(\mathsf{OCA}_P)$,
   (c) $\forall r\,(\aleph_1^{L[r]} < \aleph_1)$.
3. *in the Solovay model, all sets satisfy* OCA_P, *and*
4. $\mathsf{AD} \implies \mathsf{OCA}_P$.

Proof. See Theorem 1.1, Corollary 2.2, Theorem 4.1 and Theorem 3.3 of [Fen93], respectively. □

The proof of [Tod96, Theorem 2] in fact shows the following stronger result:

Theorem 4.6.7 (Todorčević). *For any pointclass* $\mathbf{\Gamma}$, *if* $\mathbf{\Gamma}(\mathsf{OCA}_P)$ *holds then every* $(\mathbf{\Gamma}, \mathbf{\Gamma})$*-pre-gap satisfies the perfect Luzin dichotomy.*

Combining this with Theorem 4.6.6, we immediately get:

Corollary 4.6.8.

1. *The following are equivalent:*
   (a) *Every* $(\mathbf{\Sigma}^1_2, \mathbf{\Sigma}^1_2)$*-pre-gap satisfies the perfect Luzin dichotomy,*
   (b) *Every* $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$*-pre-gap satisfies the perfect Luzin dichotomy,*
   (c) $\forall r\,(\aleph_1^{L[r]} < \aleph_1)$.
2. *in the Solovay model, all pre-gaps satisfy the perfect Luzin dichotomy, and*
3. AD *implies that all pre-gaps satisfy the perfect Luzin dichotomy.*

Proof. The only non-trivial direction is $(b) \Rightarrow (c)$ from part (1). For this, simply use the construction from Section 4.3, i.e., the $(\mathbf{\Pi}^1_1, \mathbf{\Pi}^1_1)$-Hausdorff gap satisfying HC in L. Clearly, it is not weakly σ-separated. On the other hand, if it would contain a perfect Luzin sub-gap (A', B'), then (A', B') would be a Hausdorff gap with a perfect set A', contradicting Theorem 4.1.3. □

Note that here we have an implication from AD rather than just $\mathsf{AD}_{\mathbb{R}}$. Whether the same could be done for Corollary 4.5.4 is still open.

Question 4.6.9. *Does* AD *imply that there are no Hausdorff gaps?*