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On Stein's equation, Vandermonde matrices and Fisher's information matrix of time series processes. Part II: The ARMAX process

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Abstract

An approach is presented to get interconnections between the Fisher information matrix of an ARMAX process and a corresponding solution of a Stein equation. The case of algebraic multiplicity greater than one and the case of distinct eigenvalues are addressed. Appropriate links are constructed for these two cases by applying a factorization both for the Fisher information matrix and for a corresponding solution of a Stein equation. These factored forms are nonsquare linear systems of equations $Ax = b$, the kernels of the appropriate coefficient matrices are described. These are of fundamental importance for the solutions of the obtained linear systems. The structured coefficient matrix associated with the factored form of the Fisher information matrix is composed by basis vectors associated with an ARMAX polynomial, whereas the coefficient matrix obtained through the solution of a Stein equation consists of resolvant matrices associated with a companion matrix used in a corresponding Stein equation. The presence of Vandermonde matrices in right inverses of coefficient matrices of the obtained linear systems is investigated. Links between coefficient matrices which originate both from the Fisher information matrix and a corresponding solution of the Stein equation are derived. An example is provided for illustrating a solution of a Stein equation in terms of the Fisher information matrix as well as for describing the kernels of the appropriate coefficient matrices.

AMS classification: 15A06

Keywords: Fisher information matrix; Stein equation; Linear systems; Kernel ; Coefficient matrix; ARMAX process.

1. Introduction

In a previous paper, part I- [12], interconnections between the asymptotic Fisher information matrix of an ARMA process (autoregressive and moving average) and solutions of corresponding Stein equations have been investigated. The subject of this paper is concerned with the development of comparable links for ARMAX processes but with an alternative and more generalized approach when compared with [12]. The ARMAX processes are of common use in signal processing, control and system theory, statistics and econometrics, see e.g. [1], [18], [2], [3]. The concept of the Fisher information plays a vital role in estimation theory, various algorithms have been developed for computing the information matrix, e.g. [6], [9] and [20]. In [9] two algorithms have been proposed for a fast computation of the Fisher information matrix of a SISO process, detailed comments of these type of processes can be found in [18], a SISO process is a generalized version of the ARMAX process which is considered in this paper. Since more recently there is also an increasing interest for the Fisher information in physics taking into account that "all things physical are information-theoretic in origin and this a participatory universe... Observer participancy gives rise to information and information gives rise to physics" a quote of the famous physicist John Archibald Wheeler. A comprehensive study of the role of the Fisher information in physics can be found in [5] where the author points out that the information in question, referring to physical information, is surprisingly, not Shannon or Boltzman entropy but, rather, Fisher information. However, it can be shown that the statistical information expressed by the Fisher information can be locally interconnected with the Shannon entropy. In [19] the authors argue that additional information measures like Kullback-Leibler and Rényi can be interconnected through the second order Taylor series approximation and they can be similarly interconnected with the Fisher information matrix for zero-mean stationary Gaussian univariate time series processes, which is the case of the ARMAX process considered in this paper. Consequently, the results obtained in this paper can also be used to express a solution of a Stein equation in terms of other information measures like Shannon entropy, Kullback-Leibler and Rényi information measures.

The aim of this paper is to express statistical information concepts in terms of linear algebraic concepts, in other words to build a bridge between the Fisher information of time series processes and linear algebra and the algebra of polynomials. The latter can be directly applied since the ARMA-X processes are described by polynomials, a treatment of the algebra of polynomials can be found in [4]. In the next section a definition of the ARMAX polynomials will be given. In [12] interconnections between the Fisher information matrix of an ARMA process and solutions to Stein equations are set forth by means of left inverses of appropriate matrices and these left inverses contain several forms of Vandermonde matrices. In this paper interconnections are constructed by solving linear systems of equations $Ax = b$. These forms are obtained by applying appropriate factorizations to both the Fisher information matrix and to the corresponding solution of a Stein equation. Heinig and Rost [8] have developed recursive fast algorithms for the solution of such linear systems for the cases when the coefficient matrix A is square and nonsquare, the coefficient matrix A consists of a Cauchy and a Vandermonde part. In the present paper we only have situations with nonsquare linear systems or with a nonsquare coefficient matrix A. The structure of the coefficient matrix which is obtained after factorization of the Fisher information matrix is composed by basis vectors associated with the ARMAX polynomial. Whereas the coefficient matrix present in the factored form of the corresponding solution to Stein's equation consists of resolvant matrices which are associated with a companion matrix used in the corresponding Stein equation. In order to find a nontrivial solution to the presented problem, the kernels of the appropriate coefficient matrices are studied considering that the number of rows is much less than the number of columns for both coefficient matrices. Since the specific structure of the Fisher information matrix of an ARMAX process is characterized by polynomials which contain the polynomial basis vectors $\left(1,\ z,\ z^2, \dots,\ z^{x-1}\right)^\top$ and $\left(z^{x-1},\ z^{x-2}, \dots,\ z,\ 1\right)^\top$, it then leads to the possibility for constructing an implementable algorithm for the appropriate stuctured kernels . This algorithm is also used for describing the kernel of the coefficient matrix associated with the factored form of the Stein solution. This can be realized by using an equality which interconnects elements of the coefficient matrices associated with the Fisher information matrix and given by the polynomial basis vectors mentioned above and the corresponding solution to the Stein equation given by adj $(zI - C_h)$ with C_h being a companion matrix.Two situations are considered, algebraic multiplicity of the roots respective eigenvalues are greater than one and a situation with distinct roots, the eigenvalues originate from the chosen companion matrix in the Stein equation and the roots of the ARMAX polynomials are considered. Interconnections are established for a particular block of the Fisher information matrix which contains information of the input process. The Fisher information matrix as a whole and not decomposed is also presented for establishing appropriate interconnections, it is also expressed in terms of Sylvester resultants. The presence of Vandermonde matrices in right inverses of coefficient matrices of the obtained linear systems is studied, the emphasis is put on the coefficient matrices associated with the factored form of the solution of a Stein equation and some result for the factored Fisher information coefficient matrix is also set forth. In [12], a right inverse is evaluated for the appropriate matrix containing adj $(zI - C_h)$ and which is associated with the solution of a Stein equation, whereas in this paper more elaborate and explicit results are derived, they are used to establish interconnections between the coefficient matrices extracted from the factored forms of the Fisher information matrix and a corresponding solution to a Stein equation. An example illustrates the structure of a solution of a Stein equation in terms of the Fisher information matrix. An appropriate construction of the kernel of the coefficient matrices associated with the factored forms of the Fisher information matrix and a Stein solution is also set forth through the example. The obtained results can also be useful from a numerical point of view, solutions of Stein equations can be computed by means of the Fisher information matrix and the Fisher information matrix can be computed by means of a corresponding Stein solution, for the latter singular value decomposition should be applied for obtaining the desired pseudo inverse.

The paper is organized as follows. First we present the definitions which are followed by interconnections of a Fisher information matrix block and a solution to a Stein equation, this is done for the algebraic multiplicity of the roots respective eigenvalues greater than one as well as for distinct roots. In section 3 a description of the kernels of the coefficient matrices associated with the linear systems of equations obtained in section 2 is provided. In section 4 Vandermonde matrices present in some right inverses of appropriate coefficient matrices is investigated. These results are used to construct interconnections between the coefficient matrices associated with the Fisher information matrix and a solution of an appropriate Stein equation. In section 5 interconnections are established between the Fisher information matrix containing all the parameter blocks and corresponding solutions to Stein equations and is followed by an example which is illustrated in section 6.

2. Link solution Stein's equation-Fisher's information: The parameter-block approach

2.1. General case

In this section the Fisher information matrix of an ARMAX process will be formulated where the parameter blocks are considered, whereas in section 5 the global form will be studied.

Depending on the situation, both cases have their importance and this is the reason why the two cases are treated separately.

Consider the ARMAX process y specified as the solution of

$$a^*(L)y = b^*(L)x + c^*(L)\varepsilon \tag{2.1}$$

with L the lag operator, x is the input process which is independent of the white noise sequence ε which has σ^2 as its variance.We make the assumptions that a, b and c have zeros inside the unit disc, a, b and c are the following monic polynomials

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$$
$$b(z) = z^q + b_1 z^{q-1} + \cdots + b_q$$
$$c(z) = z^r + c_1 z^{r-1} + \cdots + c_r.$$

By a^*, b^* and c^* we denote the reciprocal polynomials, $a^*(z) = z^p a(z^{-1})$, $b^*(z) = z^q b(z^{-1})$ and $c^*(z) = z^r c(z^{-1})$. The input process x is usually described by an ARMA process with spectral density $(2\pi)^{-1} R_x(z)$ where $R_x(z) = \sigma_\eta^2 \left(d(z)d(z^{-1})/h(z)h(z^{-1})\right)$. For simplicity we assume $\sigma^2 = 1$ and $\sigma_\eta^2 = 1$, the latter represents the variance of the white noise sequence η which generates the process x, and we further assume ε and η independent. For simplicity we further assume the process x to be autoregressive so that $d(z)d(z^{-1}) = 1$ and the order of the monic polynomial $h(z)$ is v. Define the vectors

$$\begin{aligned} u_k(z) &= \left(1, z, \ldots, z^{k-1}\right)^\top, \; u_k^*(z) = \left(z^{k-1}, z^{k-1}, \ldots, 1\right)^\top \\ \theta &= \left(a_1, a_2, \ldots, a_p, b_1, b_2, ..., b_q, c_1, c_2, \ldots, c_r\right)^\top. \end{aligned}$$

The order and roots of the ARMAX polynomials $a(z), b(z)$ and $c(z)$ are presented as well as the poles resulting from the spectral density $R_x(z)$. We assume the polynomial $a(z)$ having p_0 distinct roots, α_1, α_2,..., α_{p_0} , with algebraic multiplicity $n_1+1, n_2+1, ..., n_{p_0}+1$ respectively and $\sum_{i=1}^{p_0}(n_i+1) = p$, $b(z)$ has q_0 distinct roots, β_1, β_2,..., β_{q_0} , with algebraic multiplicity $m_1 + 1, m_2 + 1, ..., m_{q_0} + 1$ respectively and $\sum_{i=1}^{q_0}(m_i+1) = q$ and polynomial $c(z)$ has r_0 distinct roots γ_1, γ_2,..., γ_{r_0} with algebraic multiplicity $s_1 + 1, s_2 + 1, ..., s_{r_0} + 1$ respectively and $\sum_{i=1}^{r_0}(s_i + 1) = r$. (As can be seen from the Fisher information matrix blocks which contain $R_x(z)$, poles are also resulting from $R_x(z)$ and therefore have to be taken into account for computing the corresponding circular integral). The roots resulting from polynomial $h(z)$ are described as follows, it is assumed to have v_0 distinct roots τ_1, τ_2,..., τ_{v_0} with algebraic multiplicity $\ell_1 + 1, \ell_2 + 2, ..., \ell_{v_0} + 1$ respectively and $\sum_{i=1}^{v_0}(\ell_i + 1) = v$.

It is known, see [10], that Fisher's information matrix of (2.1) is $F(\theta) = \left(1/\sigma^2\right) G(\theta)$ with the following block decomposition for $G(\theta)$.

$$G(\theta) = \begin{pmatrix} G_{aa} & G_{ab} & G_{ac} \\ G_{ab}^T & G_{bb} & G_{bc} \\ G_{ac}^T & G_{bc}^T & G_{cc} \end{pmatrix}. \tag{2.2}$$

The matrices appearing in (2.2) can be expressed as.

$$G_{aa} = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{b(z)b(z^{-1})R_x(z)u_p(z)u_p^\top(z^{-1})}{a(z)a(z^{-1})c(z)c(z^{-1})} \frac{dz}{z} \tag{2.3}$$

$$+\frac{\sigma^2}{2\pi i} \oint\limits_{|z|=1} \frac{u_p(z)u_p^\top(z^{-1})}{a(z)a(z^{-1})} \frac{dz}{z} \tag{2.4}$$

$$G_{ab} = -\frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{b(z)R_x(z)u_p(z)u_q^\top(z^{-1})}{a(z)c(z)c(z^{-1})} \frac{dz}{z} \tag{2.5}$$

$$G_{ac} = -\frac{\sigma^2}{2\pi i} \oint\limits_{|z|=1} \frac{u_p(z)u_r^\top(z^{-1})}{a(z)c(z^{-1})} \frac{dz}{z} \tag{2.6}$$

$$G_{bb} = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{R_x(z)u_q(z)u_q^\top(z^{-1})}{c(z)c(z^{-1})} \frac{dz}{z} \tag{2.7}$$

$$G_{bc} = 0 \tag{2.8}$$

$$G_{cc} = \frac{\sigma^2}{2\pi i} \oint\limits_{|z|=1} \frac{u_r(z)u_r^\top(z^{-1})}{c(z)c(z^{-1})} \frac{dz}{z}\ . \tag{2.9}$$

As can be seen from blocks (2.3)-(2.9) which constitute $G(\theta)$, the blocks (2.4), (2.6) and (2.9) have forms which are the corresponding ARMA part of $G(\theta)$, whereas the remaing blocks contain information of the input process x. In part I [12] it was sufficient to consider an appropriate left inverse (which contains Vandermonde structures) for interconnecting Stein's solution with the Fisher information matrix, an interconnection derived in [12] has the following form for the ARMA (a, a) block

$$S_{aa} = M_n(\alpha) \left\{ \left(W_n(\alpha)_L^- \ \mathrm{vec}\ F_{aa}\right) \otimes I_p \right\}$$

where S_{aa} and F_{aa} are respectively the Stein solution and the Fisher information matrix, $(\cdot)_L^-$ is a left inverse, vec A is the vector formed by stacking the columns of A into one long vector, $M_n(\alpha)$ is extracted from a corresponding solution to Stein's equation for a vector of eigenvalues denoted by α and $W_n(\alpha)$ is associated with vec F_{aa}.

The purpose of this paper is to use a different and more generalized approach for establishing comparable links for ARMAX processes, we therefore consider solutions of linear systems of equations. The coefficient matrices associated with the obtained linear systems of equations consist of the polynomial basis vectors $u_k(z)$ and $u_j^*(z)$ for arbitrary k and j and a combination of adj $(zI - C_h)$ with adj $(I - zC_h)$ where C_h is a chosen companion matrix. These coefficient matrices are obtained by applying a factorization accordingly both for the Fisher information matrix and a corresponding solution to a Stein equation. The structured kernels of the derived coefficient matrices will be explicitely described so that the equations which interconnect the Fisher information matrix and a corresponding solution to a Stein equation will be established in a different way than in [12]. In this paper the interconnections consist of the Fisher information matrix and not its vectorized form. This will be done both for the algebraic multiplicity of the appropriate eigenvalues greater than one and for distinct roots. We will first focus on the general case, algebraic multiplicity greater or equal to one, followed by the special case when all the eigenvalues are distinct. The link between the Fisher information matrix and a solution of a Stein equation for the (b, b)-block is deduced, consequently interconnections for the remaining blocks can be established in a similar manner.

Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $\Gamma \in \mathbb{C}^{n\times m}$ and we consider the Stein equation.

$$S - BSA^{\top} = \Gamma, \tag{2.10}$$

it has a unique solution iff $\lambda\mu \neq 1$ for any $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$. From [16] we take

Theorem 2.1 *Let A and B be such that there is a single closed contour C with $\sigma(B)$ inside C and for each nonzero $w \in \sigma(A)$, w^{-1} is outside C. Then the Stein equation (2.10) has a unique solution S*

$$S = \frac{1}{2\pi i}\oint_C (\lambda I - B)^{-1}\,\Gamma\,(I - \lambda A)^{-\top}\,d\lambda.$$

In order to apply this theorem for linking (2.7) with a corresponding Stein solution we apply (2.10) with $B = C_h$ and $A = C_h$ in

$$S_{bb} - C_h S_{bb} C_h^{\top} = \Gamma \tag{2.11}$$

where the companion matrix C_h

$$C_h = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -e_{r+v} & -e_{r+v-1} & \cdots & -e_1 \end{pmatrix}$$

and the entries e_i are given by $z^{r+v} + \sum_{i=1}^{r+v} e_i z^{r+v-i} = c(z)h(z) = e(z)$. The condition for uniqueness of the solution of Stein's equation is verified for this choice of the companion matrix. Inserting companion matrices in (2.10) for the coefficients A and B is also suggested in [15].

Block $G_{bb}(\theta)$ given in (2.7) becomes

$$G_{bb}(\theta) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{u_q(z)u_q^{*\top}(z)}{h(z)c(z)h^*(z)c^*(z)z^{l+1}}dz. \tag{2.12}$$

Here $l + 1 = q - v - r$ is introduced and the cases $l + 1 > 0$, $l + 1 = 0$ and $l + 1 < 0$ shall be discussed. For typographical brevity we introduce the following notation. Given a polynomial $p(\cdot)$ and assuming that for some natural number j $(z - \beta)^j$ is a factor of $p(\cdot)$, and β has multiplicity $j \geq 1$ we define the polynomial $p_j(.\,;\beta)$ by $p_j(z\,;\beta) = \frac{p(z)}{(z-\beta)^j}$. Applying Cauchy's theorem to (2.12) for $l + 1 > 0$ yields

$G_{bb}(\theta) = g_1(\gamma_1) + g_2(\gamma_2) + \cdots + g_{r_0}(\gamma_{r_0}) + k_1(\tau_1) + k_2(\tau_2) + \cdots + k_{v_0}(\tau_{v_0}) + f(0).$

Where the terms are

$$
\begin{aligned}
g_i(\gamma_i) &= \frac{1}{s_i!}\left(\frac{\partial^{s_i}}{\partial z^{s_i}}\frac{u_q(z)u_q^{*\top}(z)}{c_{s_i+1}(z;\gamma_i)h(z)h^*(z)c^*(z)z^{l+1}}\right)_{z=\gamma_i} && i=1,\dots,r_0\\
k_j(\tau_j) &= \frac{1}{\ell_j!}\left(\frac{\partial^{\ell_j}}{\partial z^{\ell_j}}\frac{u_q(z)u_q^{*\top}(z)}{c(z)h_{\ell_j+1}(z;\tau_j)h^*(z)c^*(z)z^{l+1}}\right)_{z=\tau_j} && j=1,\dots,v_0\\
f(0) &= \frac{1}{l!}\left(\frac{\partial^{l}}{\partial z^{l}}\frac{u_q(z)u_q^{*\top}(z)}{c(z)h(z)h^*(z)c^*(z)}\right)_{z=0}.
\end{aligned}
$$

A useful factorization of $G_{bb}(\theta)$ can be obtained by applying Leibnitz rule to j-fold differentiation of a product of two functions to have

$$
G_{bb}(\theta) = \left(\overline{\mathcal{U}}_r(\gamma)\;\overline{\mathcal{U}}_v(\tau)\;\overline{\mathcal{U}}_l(0)\right)(\vartheta\otimes I_q) \tag{2.13}
$$

with

$$
\begin{aligned}
\overline{\mathcal{U}}_r(\gamma) &= \begin{pmatrix} \overline{\mathcal{U}}_{s_1}(\gamma_1), & \overline{\mathcal{U}}_{s_2}(\gamma_2), & \dots, & \overline{\mathcal{U}}_{s_{r_0}}(\gamma_{r_0}) \end{pmatrix}\\
\overline{\mathcal{U}}_v(\tau) &= \begin{pmatrix} \overline{\mathcal{U}}_{\ell_1}(\tau_1), & \overline{\mathcal{U}}_{\ell_2}(\tau_2), & \dots, & \overline{\mathcal{U}}_{\ell_{v_0}}(\tau_{v_0}) \end{pmatrix}\\
\overline{\mathcal{U}}_l(0) &= \begin{pmatrix} \overline{\mathcal{U}}_l^{(l)}(0), & \overline{\mathcal{U}}_l^{(l-1)}(0), & \dots, & \overline{\mathcal{U}}_l^{(0)}(0) \end{pmatrix}.
\end{aligned}
$$

The matrices $\overline{\mathcal{U}}_{s_i}(\gamma_i)$ and $\overline{\mathcal{U}}_{\ell_j}(\tau_j)$ are composed as follows

$$
\overline{\mathcal{U}}_{s_i}(\gamma_i) = \begin{pmatrix} \overline{\mathcal{U}}_{s_i}^{(s_i)}(z), & \overline{\mathcal{U}}_{s_i}^{(s_i-1)}(z), & \dots & ,\overline{\mathcal{U}}_{s_i}^{(0)}(z) \end{pmatrix}_{z=\gamma_i} \quad i=1,\dots,r_0
$$

each block being

$$
\overline{\mathcal{U}}_{s_i}^{(s_i-j)}(\gamma_i) = \left(\frac{\partial^{s_i-j}}{\partial z^{s_i-j}}\left(u_q(z)u_q^{*\top}(z)\right)\right)_{z=\gamma_i} \qquad j=0,\dots,s_i.
$$

Whereas $\overline{\mathcal{U}}_{\ell_j}(\tau_j)$ has a similar functional form to $\overline{\mathcal{U}}_{s_i}(\gamma_i)$ but for $z=\tau_j$ and $j=1,\dots,v_0$.
The term $\overline{\mathcal{U}}_l(0)$ counts only one pole and each block can be expressed as

$$
\overline{\mathcal{U}}_l^{(l-k)}(0) = \left(\frac{\partial^{l-k}}{\partial z^{l-k}}\left(u_q(z)u_q^{*\top}(z)\right)\right)_{z=0} \qquad k=0,\dots,l.
$$

The ϑ in the second term of (2.13) is

$$
\vartheta = \left(\mu_{s_1}^\top(\gamma_1),\mu_{s_2}^\top(\gamma_2),\dots,\mu_{s_{r_0}}^\top(\gamma_{r_0}),\zeta_{\ell_1}^\top(\tau_1),\zeta_{\ell_2}^\top(\tau_2),\dots,\zeta_{\ell_{v_0}}^\top(\tau_{v_0}),\xi_0^\top(0)\right)^\top \tag{2.14}
$$

with

$$
\mu_{s_i}(\gamma_i) = \frac{1}{s_i!}\left(\mu_{s_i}^{(0)}(z),\mu_{s_i}^{(1)}(z),\dots,\mu_{s_i}^{(s_i)}(z)\right)^\top_{z=\gamma_i} \qquad i=1,\dots,r_0
$$

and each component being

$$
\mu_{s_i}^{(s_i-j)}(\gamma_i) = \begin{pmatrix} s_i\\ s_i-j\end{pmatrix}\left(\frac{\partial^{s_i-j}}{\partial z^{s_i-j}}\mu_i(z)\right)_{z=\gamma_i} \qquad j=s_i,\ s_i-1,\dots,0.
$$

Analogously

$$
\zeta_{\ell_j}(\tau_j) = \frac{1}{\ell_j!}\left(\zeta_{\ell_j}^{(0)}(z),\zeta_{\ell_j}^{(1)}(z),\dots,\zeta_{\ell_j}^{(\ell_j)}(z)\right)^\top_{z=\tau_j} \qquad j=1,\dots,v_0
$$

with elements

$$
\zeta_{\ell_j}^{(\ell_j-k)}(\tau_j) = \begin{pmatrix} \ell_j\\ \ell_j-k\end{pmatrix}\left(\frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}}\;\zeta_j(z)\right)_{z=\tau_j} \qquad k=\ell_j,\ \ell_j-1,\dots,0.
$$

The last term of ϑ becomes

$$
\xi_0(0) = \frac{1}{l!}\left(\xi_0^{(0)}(z),\xi_0^{(1)}(z),\dots,\xi_0^{(l)}(z)\right)^\top_{z=0}
$$

with

$$\xi_0^{(l-k)}(0) = \begin{pmatrix} l \\ l-k \end{pmatrix} \left(\frac{\partial^{l-k}}{\partial z^{l-k}} \; \xi(z) \right)_{z=0} \qquad\qquad k = l, \, l-1, \dots, 0.$$

The individual components are

$$\mu_i(z) = \left(\frac{1}{c_{s_i+1}(z;\gamma_i)h(z)h^*(z)c^*(z)z^{l+1}} \right)$$
$$\zeta_j(z) = \left(\frac{1}{c(z)h_{\ell_j+1}(z;\tau_j)h^*(z)c^*(z)z^{l+1}} \right)$$
$$\xi(z) = \left(\frac{1}{c(z)h(z)h^*(z)c^*(z)} \right).$$

We can return to Stein's equation (2.10) with the appropriate insertion in (2.11).

Its solution is

$$\begin{aligned} S_{bb} &= \frac{1}{2\pi i} \oint_{|z|=1} (zI - C_h)^{-1} \Gamma \, (I - zC_h)^{-\top} \, dz \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{\mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top}{h(z)c(z)h^*(z)c^*(z)} dz. \end{aligned}$$

In order to extract ϑ from S_{bb} we first trivially rewrite the solution as

$$S_{bb} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{\mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top \, z^{l+1}}{h(z)c(z)h^*(z)c^*(z)z^{l+1}} dz. \tag{2.15}$$

Then, applying Cauchy's residue theorem to (2.15) yields,

$$S_{bb} = \mathcal{G}_1(\gamma_1) + \mathcal{G}_2(\gamma_2) + \cdots + \mathcal{G}_{r_0}(\gamma_{r_0}) + \mathcal{K}_1(\tau_1) + \mathcal{K}_2(\tau_2) + \cdots + \mathcal{K}_{v_0}(\tau_{v_0}) + \mathcal{F}(0)$$
$$\mathcal{G}_i(\gamma_i) = \frac{1}{s_i!} \left(\frac{\partial^{s_i}}{\partial z^{s_i}} \frac{\mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top \, z^{l+1}}{c_{s_i+1}(z;\gamma_i)h(z)h^*(z)c^*(z)z^{l+1}} \right)_{z=\gamma_i}$$
$$\mathcal{K}_j(\tau_j) = \frac{1}{\ell_j!} \left(\frac{\partial^{\ell_j}}{\partial z^{\ell_j}} \frac{\mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top \, z^{l+1}}{c(z)h_{\ell_j+1}(z;\tau_j)h^*(z)c^*(z)z^{l+1}} \right)_{z=\tau_j}$$
$$\mathcal{F}(0) = \frac{1}{l!} \left(\frac{\partial^{l}}{\partial z^{l}} \frac{\mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top \, z^{l+1}}{c(z)h(z)h^*(z)c^*(z)} \right)_{z=0}.$$

Since the eigenvalues of C_h are within the unit disc, the conditions for a unique solution of Stein's equation is fulfilled. A similar factorization as in (2.13) is applied to (2.15) to obtain

$$S_{bb} = \left(\overline{\mathcal{M}}_r(\gamma) \; \overline{\mathcal{M}}_v(\tau) \; \overline{\mathcal{M}}_l(0) \right) (\vartheta \otimes I_{r+v}) \tag{2.16}$$

with ϑ as in (2.14) and

$$\overline{\mathcal{M}}_r(\gamma) = \begin{pmatrix} \overline{\mathcal{M}}_{s_1}(\gamma_1), & \overline{\mathcal{M}}_{s_2}(\gamma_2), & ..., & \overline{\mathcal{M}}_{s_{r_0}}(\gamma_{r_0}) \end{pmatrix}$$
$$\overline{\mathcal{M}}_v(\tau) = \begin{pmatrix} \overline{\mathcal{M}}_{\ell_1}(\tau_1), & \overline{\mathcal{M}}_{\ell_2}(\tau_2), & ..., & \overline{\mathcal{M}}_{\ell_{v_0}}(\tau_{v_0}) \end{pmatrix}$$
$$\overline{\mathcal{M}}_l(0) = \begin{pmatrix} \overline{\mathcal{M}}_l^{(l)}(0), & \overline{\mathcal{M}}_l^{(l-1)}(0), & ..., & \overline{\mathcal{M}}_l^{(0)}(0) \end{pmatrix}.$$

The blocks which form $\overline{\mathcal{M}}_{s_i}(\gamma_i)$ and $\overline{\mathcal{M}}_{\ell_j}(\tau_j)$ are

$$\overline{\mathcal{M}}_{s_i}(\gamma_i) = \begin{pmatrix} \overline{\mathcal{M}}_{s_i}^{(s_i)}(z), & \overline{\mathcal{M}}_{s_i}^{(s_i-1)}(z), & ... & , \overline{\mathcal{M}}_{s_i}^{(0)}(z) \end{pmatrix}_{z=\gamma_i} \qquad i = 1, \dots, r_0$$

each block being

$$\overline{\mathcal{M}}_{s_i}^{(s_i-j)}(\gamma_i) = \left(\frac{\partial^{s_i-j}}{\partial z^{s_i-j}} \mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top \, z^{l+1} \right)_{z=\gamma_i} \qquad j = 0, \dots, s_i.$$

Block $\overline{\mathcal{M}}_{\ell_j}(\tau_j)$ has the same functional form as $\overline{\mathcal{M}}_{s_i}(\gamma_i)$ but with $z = \tau_j$ and $j = 1, \dots, v_0$.

Note that the term $\overline{\mathcal{M}}_l(0)$ counts only one pole and each block can be expressed as

$$\overline{\mathcal{M}}_l^{(l-k)}(0) = \left(\frac{\partial^{l-k}}{\partial z^{l-k}} \mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top z^{l+1} \right)_{z=0} \qquad k = 0, \ldots, l.$$

It is clear that $\overline{\mathcal{M}}_l^{(l-k)}(0) = 0$ for $k = 0, \ldots, l$ so that $\overline{\mathcal{M}}_l(0) = 0$.

We can now proceed constructing a possible interconnection between $G_{bb}(\theta)$ and S_{bb} by solving $(\vartheta \otimes I_q)$ and $(\vartheta \otimes I_{r+v})$ from (2.13) and (2.16) respectively. This will happen according to the solution of the linear system $AX = B$ where A, B and X are matrices of appropriate dimension. The matrix A will be represented by the corresponding coefficient matrices $\left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\right)$ and $\left(\overline{\mathcal{M}}_r(\gamma)\,\overline{\mathcal{M}}_v(\tau)\,\overline{\mathcal{M}}_l(0)\right)$ in (2.13) and (2.16) respectively. It is clear that the kernels of the appropriate coefficient matrices in both cases will be different from zero and have dimensions which are substantially larger than the number of columns of X or B (q and $r+v$ in (2.13) and (2.16) respectively). This argument holds for all the situations considered, $l+1 > 0$, $l+1 = 0$ and $l+1 < 0$. The linear system $AX = B$ has a solution if and only if $B \in \mathrm{Im}\ (A)$, a solution of the linear system is given by $X = X_0 + \mathcal{A}$ where X_0 is a particular solution of the matrix equation $AX = B$ and $\mathcal{A} \in$ Ker (A), the kernel of A. The matrix $X_0 = A^+B$ is the best approximate solution of the linear system of equations $AX = B$ where A^+ is the Moore-Penrose inverse of A. In general, the solution set is a manifold of matrices obtained by a shift of Ker (A). In (2.13) and (2.16) it can be seen that $B \in$ Im (A). Applying this to the linear systems (2.13) and (2.16) in order to obtain an interconnection or equation involving the Fisher information matrix and a solution to Stein equation we consider the particular solution of the linear systems (2.13) and (2.16) to obtain

$$(\vartheta \otimes I_q) = \left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\right)^+ G_{bb}(\theta) + \mathcal{A} \tag{2.17}$$

where $\mathcal{A} \in \mathrm{Ker}\ \left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\right)$.

Similarly for Stein's solution

$$(\vartheta \otimes I_{r+v}) = \left(\overline{\mathcal{M}}_r(\gamma)\,\overline{\mathcal{M}}_v(\tau)\,\overline{\mathcal{M}}_l(0)\right)^+ S_{bb} + \mathcal{B} \tag{2.18}$$

with $\mathcal{B} \in \mathrm{Ker}\ \left(\overline{\mathcal{M}}_r(\gamma)\,\overline{\mathcal{M}}_v(\tau)\,\overline{\mathcal{M}}_l(0)\right)$.

Taking into consideration the fact that $l+1 > 0$ or $q > r+v$ leads to the following property

$$I_q = \begin{pmatrix} I_{r+v} & 0 \\ 0 & I_{q-(r+v)} \end{pmatrix} \quad \text{and} \quad I_q \otimes \vartheta = \begin{pmatrix} I_{r+v} \otimes \vartheta & 0 \\ 0 & I_{q-(r+v)} \otimes \vartheta \end{pmatrix}.$$

In order to obtain the forms $(\vartheta \otimes I_{r+v})$ and $(\vartheta \otimes I_q)$ we use the following property of the Kronecker product of two matrices. Let A be an $m \times n$ matrix and B a $p \times q$ matrix. Then there exist universal permutation matrices $\mathcal{R}_{pm}$ and $\mathcal{R}_{nq}$ such that $\forall\ A, B : \mathcal{R}_{pm}\,(A \otimes B)\,\mathcal{R}_{nq} = B \otimes A$ for a permutation matrix $\mathcal{R}_{xy}$ independent of A and B, see e.g. [17]. Double application of this rule to $A = \vartheta$ and $B = I_q$ and with $B = I_{r+v}$ results in an equation which involves the Fisher information matrix and a Stein solution. In the next theorem we summarize the results in a formula which involves a solution of a Stein equation S_{bb} with the Fisher information block $G_{bb}(\theta)$ for the case $l+1 > 0$.

Theorem 2.2 *The following equality holds true for $l+1 > 0$*

$$\mathcal{R}_{q(r+v+l+1)} \left\{ \left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\right)^+ G_{bb}(\theta) + \mathcal{A} \right\} \mathcal{R}_q =$$

$$\begin{pmatrix} \mathcal{R}_{(r+v)(r+v+l+1)} \left\{ \left(\overline{\mathcal{M}}_r(\gamma)\,\overline{\mathcal{M}}_v(\tau)\,\overline{\mathcal{M}}_l(0)\right)^+ S_{bb} + \mathcal{B} \right\} \mathcal{R}_{r+v} & 0 \\ 0 & I_{q-(r+v)} \otimes \vartheta \end{pmatrix}$$

where $\mathcal{A} \in$ *Ker* $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\, \overline{\mathcal{U}}_l(0)\right)$ *is such that equation (2.17) holds and likewise* $\mathcal{B} \in$ *Ker* $\left(\overline{\mathcal{M}}_r(\gamma)\, \overline{\mathcal{M}}_v(\tau)\, \overline{\mathcal{M}}_l(0)\right)$ *is such that (2.18) holds.*

In the next section a detailed description of Ker $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\, \overline{\mathcal{U}}_l(0)\right)$ will be given as well as some aspects of Ker $\left(\overline{\mathcal{M}}_r(\gamma)\, \overline{\mathcal{M}}_v(\tau)\, \overline{\mathcal{M}}_l(0)\right)$.

The case $l+1=0$ or $q=v+r$ is such that equivalent forms of (2.13) and (2.16) become

$$G_{bb}(\theta) = \left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)(\varphi \otimes I_q) \tag{2.19}$$

$$S_{bb} = \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\, \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)(\varphi \otimes I_{r+v}) = \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\, \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)(\varphi \otimes I_q). \tag{2.20}$$

The vector φ has the same form as ϑ in (2.14) but without $\xi_0(0)$, the terms $\mu_i(z)$ and $\zeta_j(z)$ do not contain z^{l+1} since $l+1=0$. The corresponding blocks composing $\widetilde{\overline{\mathcal{M}}}_r(\gamma)$ and $\widetilde{\overline{\mathcal{M}}}_v(\tau)$ are

$$\widetilde{\overline{\mathcal{M}}}_{s_i}^{(s_i-j)}(z) = \frac{\partial^{s_i-j}}{\partial z^{s_i-j}}\left(\text{adj }(zI-C_h)\,\Gamma\text{adj }(I-zC_h)^\top\right)$$

and

$$\widetilde{\overline{\mathcal{M}}}_{\ell_j}^{(\ell_j-k)}(z) = \frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}}\left(\text{adj }(zI-C_h)\,\Gamma\text{adj }(I-zC_h)^\top\right)$$

respectively. From theorem 2.2 as well as from (2.19) and (2.20) can be seen that for a particular solution of the linear systems (2.19) and (2.20), the particular solution of both linear systems is associated with $(\varphi \otimes I_q)$, the following equality holds true

$$\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)^+ G_{bb}(\theta) + \mathcal{Q} = \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\, \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)^+ S_{bb} + \mathcal{T} \tag{2.21}$$

where $\mathcal{Q} \in$ Ker $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)$ and $\mathcal{T} \in$ Ker $\left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\, \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)$.

In this case the interconnection between $G_{bb}(\theta)$ and S_{bb} can be set forth in both directions where the Fisher information matrix can be explained in terms of the Stein solution and vice versa. This can be realized when the $(q \times q^2)$ matrices $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)$ and $\left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\, \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)$ are surjective or have full row rank. For that purpose it remains to show that the rank of the corresponding matrices is q. This will be proved in the next propositions.

Proposition 2.3 *For the coefficient matrix in (2.19) we have that*
dim Im $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right) = q$.

Proof. The case $l+1=0$ or $q=r+v$ implies that the algebraic multiplicity of a root of an ARMAX polynomial will always be smaller than q. Matrix $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)$ is partitioned as mentioned above

$$\overline{\mathcal{U}}_r(\gamma) = \left(\ \overline{\mathcal{U}}_{s_1}(\gamma_1),\ \ \overline{\mathcal{U}}_{s_2}(\gamma_2),\ \ ...,\ \ \overline{\mathcal{U}}_{s_{r_0}}(\gamma_{r_0})\ \right)$$

$$\overline{\mathcal{U}}_v(\tau) = \left(\ \overline{\mathcal{U}}_{\ell_1}(\tau_1),\ \ \overline{\mathcal{U}}_{\ell_2}(\tau_2),\ \ ...,\ \ \overline{\mathcal{U}}_{\ell_{v_0}}(\tau_{v_0})\ \right).$$

Since the functional form of each of the blocks which build $\overline{\mathcal{U}}_r(\gamma)$ and $\overline{\mathcal{U}}_v(\tau)$ is the same, a description of a single subspace is therefore sufficient. In section 3 such a description is provided and it can be seen that dim Ker $\overline{\mathcal{U}}_{s_i}(\gamma_i) = (q-1)(s_i+1)$ for $i=1,\dots,r_0$ when $s_i+1<q$. Consequently, dim Ker $\overline{\mathcal{U}}_r(\gamma) = (q-1)\sum_{i=1}^{r_0}(s_i+1) = (q-1)r$. Analogously for $\overline{\mathcal{U}}_v(\tau)$ where dim Ker $\overline{\mathcal{U}}_v(\tau) = (q-1)\sum_{i=1}^{v_0}(\ell_i+1) = (q-1)v$, can be concluded that dim Ker $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right) = (q-1)(r+v) = (q-1)q$. Since the matrix $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)$ is $q \times q^2$ it then follows that dim Im $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right) = q$. ■

Before proving the full row rankness property of the matrix $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)$ we have the following lemma to consider. To that end we introduce some notation.

Consider the matrix $A \in \mathbb{R}^{n\times n}$ in the following companion form.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -a_n & & & -a_2 & -a_1 \end{pmatrix}. \tag{2.22}$$

Let $a^\top = (a_1, \ldots, a_n)$, and we redefine $u(z)^\top = (1, z, \ldots, z^{n-1})$ and $u^*(z)^\top = (z^{n-1}, \ldots, 1)$ (where $^\top$ denotes transposition). Define (Hörner recursion) recursively the polynomials $a_k(z)$ by $a_0(z) = 1$ and $a_k(z) = za_{k-1}(z) + a_k$. Notice that $a_n(z)$ is the characteristic polynomial of A. We will denote it by $\pi(z)$.

Write $a(z)$ for the n-vector $(a_0(z), \ldots, a_{n-1}(z))^\top$. Furthermore S will denote the shift matrix, so $S_{ij} = \delta_{i,j+1}$ and J the *backward* or *antidiagonal* identity matrix.

Lemma 2.4 *Let A be an $n\times n$ companion matrix as in (2.22). Let $P_k(z) = (adj(z-A), \frac{d}{dz} adj(z-A), \ldots, \frac{d^{k-1}}{dz^{k-1}} adj(z-A))$ and $P = (P_{k_1}(\lambda_1), \ldots, P_{k_s}(\lambda_s)) \in \mathbb{R}^{n\times n^2}$, where the λ_j are all the different eigenvalues of A, with multiplicities k_j, so $\sum_{j=1}^s k_j = n$. Then P has rank n.*

Proof. We will use proposition 3.2 of [14], which says that the adjoint of $z - A$, with A a companion matrix, is

$$\operatorname{adj}(z-A) = u(z)a(z)^\top J - \pi(z)\sum_{j=0}^{n-1} z^j S^{j+1}. \tag{2.23}$$

If we evaluate this expression in z equal to an eigenvalue, the second term at the RHS vanishes, and the same holds true if we consider multiple eigenvalues and compute the $(k-1)$-th derivative of $\operatorname{adj}(z-A)$ in an eigenvalue with multiplicity at least equal to k.

Let then λ be an eigenvalue of multiplicity k. Clearly $\operatorname{Im}\operatorname{adj}(\lambda - A)$ is spanned by $u(\lambda)$, $\operatorname{Im}\frac{d}{dz}\operatorname{adj}(z-A)|_{z=\lambda}$ is spanned by $u(\lambda)$ and $u'(\lambda)$, etc. up to $\operatorname{Im}\frac{d^{k-1}}{dz^{k-1}}\operatorname{adj}(z-A)|_{z=\lambda}$ which is spanned by $u(\lambda)$ up to $u^{(k-1)}(\lambda)$. As a conclusion we get for such a λ that $\operatorname{Im}(\operatorname{adj}(z-A), \frac{d}{dz}\operatorname{adj}(z-A), \ldots, \frac{d^{k-1}}{dz^{k-1}}\operatorname{adj}(z-A))|_{z=\lambda}$ is also spanned by $u(\lambda)$ up to $u^{(k-1)}(\lambda)$.

It now follows from the above that $\operatorname{Im} P$ is spanned by all the columns of a non-singular confluent Vandermonde matrix. Therefore P has maximal (row) rank and is thus surjective. ■ In the next proposition we use a symmetrizer associated with a polynomial. For a given polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ of degree n we write $S(p)$ to denote the $n\times n$ matrix

$$S(p) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 1 & 0 \\ a_{n-1} & & & a_1 & 1 \end{pmatrix}. \tag{2.24}$$

Proposition 2.5 *Let* $\mathbf{V}$ *be the confluent Vandermonde matrix associated with all the eigenvalues of* C_h *and let* $S(e)$ *be the symmetrizer associated to the coefficients of the characteristic polynomial of* C_h*. Assume that* Γ *is such that none of the rows of* $V^\top S(e)\Gamma$ *is the null vector. Then* $R = \left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)$ *has rank* q*.*

Proof. The proof consists of a more elaborate reasoning as in the proof of lemma **2.4.** We now have to consider all relevant derivatives of $\mathrm{adj}(zI - C_h)\Gamma\mathrm{adj}(I - zC_h)^\top$ evaluated at the different eigenvalues γ_i and τ_i, call them λ_i, with their multiplicities k_i. It is easy to see (by computing these derivatives and inserting the eigenvalues) that the range of R is the same as the range of R^0 which is row block matrix with blocks R_i^0 defined by $R_i^0 = (u(\lambda_i), u'(\lambda_i), \ldots, u^{(k_i-1)}(\lambda_i))a(\lambda_i)^\top J\Gamma$.
Since the vectors $u(\lambda_i), u'(\lambda_i), \ldots, u^{(k_i-1)}(\lambda_i)$ with varying i are independent, the only case in which R^0 has full row rank is obtained by having all $a(\lambda_i)^\top J\Gamma$ not equal to the null vector. ■

Remark. The condition of this proposition can alternatively be described as follows. Let $e_k(\lambda)$ be the $k-th$ Hörner polynomial associated with the coefficients of C_h evaluated at an eigenvalue λ. Put then $\tilde{e}(\lambda) = (e_{q-1}(\lambda), \ldots, e_0(\lambda))$. Then none of the rows of $V^\top S(e)\Gamma$ is the null vector iff none of the vectors $\tilde{e}(\lambda)$ belongs to the left kernel of Γ. This condition is satisfied if one chooses Γ such that the resulting solution of the Stein equation is the Fisher information matrix.

Since the matrix $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)$ is $\left(r + v \times (r+v)^2\right)$, it can be concluded in virtue of proposition 2.5 that dim Ker $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right) = q^2 - q$.

In virtue of these propositions the properties
$\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)^+ = I_q$ and $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)^+ = I_q$ hold, these are the orthogonal projections onto Im $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ and Im $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)$ respectively. The following interconnections can now be summarized in the next lemma.

Corollary 2.6 *For* $l+1=0$ *the following interconnections hold true*

$$S_{bb} = \left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)\left\{\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)^+ G_{bb}(\theta) + \mathcal{Q}\right\}$$

$$G_{bb}(\theta) = \left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)\left\{\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)^+ S_{bb} + \mathcal{T}\right\}$$

where $\mathcal{Q} \in$ *Ker* $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\ \right)$ *is such that equation (2.21) holds and likewise for* $\mathcal{T} \in$ *Ker* $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\ \right)$*.*

A new form for a solution of a Stein equation is now provided in terms of the Fisher information matrix. However, it is can be seen that for $l+1=0$ $G_{bb}(\theta)$ is satisfying a Stein equation (this is also pinpointed in [11] and [12]),

$$G_{bb}(\theta) - C_h G_{bb}(\theta) C_h^\top = \Gamma$$

and this for $\Gamma = e_{r+v}e_{r+v}^\top$, where e_{r+v} is the last basis vector of $\mathbb{R}^{r+v}$.

The case $l+1<0$ or $q<v+r$ yields the following equations

$$G_{bb}(\theta) = \left(\widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{U}}}_v(\tau)\right)(\varphi \otimes I_q) \qquad (2.25)$$

$$S_{bb} = \left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)(\varphi \otimes I_{r+v}). \qquad (2.26)$$

The block components of $\widetilde{\overline{\mathcal{U}}}_r(\gamma)$ and $\widetilde{\overline{\mathcal{U}}}_v(\tau)$ are composed by $\widetilde{\overline{\mathcal{U}}}_{s_i}^{(s_i-j)}(z) = \frac{\partial^{s_i-j}}{\partial z^{s_i-j}} u_q(z) u_q^{*\top}(z)\ z^{l+1}$ and $\widetilde{\overline{\mathcal{U}}}_{\ell_j}^{(\ell_j-k)}(z) = \frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}} u_q(z) u_q^{*\top}(z)\ z^{l+1}$ respectively and evaluated at $z=\gamma_j$ and $z=\tau_j$. By solving the linear system described above, the desired particular solution of the systems (2.25) and (2.26) yields

$$(\varphi \otimes I_q) = \left(\widetilde{\overline{\mathcal{U}}}_r(\gamma)\ \widetilde{\overline{\mathcal{U}}}_v(\tau)\right)^+ G_{bb}(\theta) + \mathcal{D} \tag{2.27}$$

$$\text{where } \mathcal{D} \in \text{Ker } \left(\widetilde{\overline{\mathcal{U}}}_r(\gamma)\ \widetilde{\overline{\mathcal{U}}}_v(\tau)\right)$$

and

$$(\varphi \otimes I_{r+v}) = \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\ \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)^+ S_{bb} + \mathcal{E} \tag{2.28}$$

$$\text{where } \mathcal{E} \in \text{Ker } \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\ \widetilde{\overline{\mathcal{M}}}_v(\tau)\right).$$

An appropriate link is given in the next corollary.

Corollary 2.7 *An equation which involves S_{bb} and $G_{bb}(\theta)$ is for $l+1<0$*

$$\mathcal{R}_{(r+v)(r+v)} \left\{ \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\ \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)^+ S_{bb} + \mathcal{E} \right\} \mathcal{R}_{r+v} =$$

$$\begin{pmatrix} \mathcal{R}_{q(r+v)} \left\{ \left(\widetilde{\overline{\mathcal{U}}}_r(\gamma)\ \widetilde{\overline{\mathcal{U}}}_v(\tau)\right)^+ G_{bb}(\theta) + \mathcal{D} \right\} \mathcal{R}_q & 0 \\ 0 & I_{r+v-q} \otimes \varphi \end{pmatrix}$$

where $\mathcal{E} \in Ker \left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\ \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)$ is such that equation (2.28) holds and likewise $\mathcal{D} \in Ker \left(\widetilde{\overline{\mathcal{U}}}_r(\gamma)\ \widetilde{\overline{\mathcal{U}}}_v(\tau)\right)$ is such that equation (2.27) holds and $\mathcal{R}$ is a permutation matrix.

Equations which involve the remaining blocks of $G(\theta)$ and a corresponding solution to a Stein equation can be obtained similarly.

2.2. Special case

The approach used to derive the equations and interconnections as presented in the previous section can be similarly applied for the case of distinct roots. In this subsection we present the coefficient matrices of the factored versions of both the Fisher information matrix and a corresponding Stein solution for $l+1>0$, $l+1=0$ and $l+1<0$. These coefficient matrices can be used to formulate similar interconnections as presented in the general case, we therefore will not reformulate the equivalent of theorem 2.2 and the remaining corollaries.

For $l+1>0$ we have

$$G_{bb}(\theta) = \left(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau)\ \overline{\mathcal{U}}_l(0)\right)(\overline{\vartheta} \otimes I_q) \tag{2.29}$$

$$\mathcal{U}_r(\gamma) = \begin{pmatrix} \mathcal{U}_1(\gamma_1), & \mathcal{U}_2(\gamma_2), \dots & \mathcal{U}_r(\gamma_r) \end{pmatrix}$$
$$\mathcal{U}_v(\tau) = \begin{pmatrix} \mathcal{U}_1(\tau_1), & \mathcal{U}_2(\tau_2), \dots & \mathcal{U}_v(\tau_v) \end{pmatrix}.$$

The blocks being $\mathcal{U}_i(\gamma_i) = (\left(u_q(z)u_q^{*\top}(z)\right)_{z=\gamma_i}$ $i=1,\dots,r$ and $\mathcal{U}_j(\tau_j) = (\left(u_q(z)u_q^{*\top}(z)\right)_{z=\tau_j}$ $j=1,\dots v$ and $\overline{\mathcal{U}}_l(0)$ can be found in (2.13).

The $\overline{\vartheta}$ in the second term of (2.29) is

$$\overline{\vartheta} = \left(\overline{\mu}_1(\gamma_1), \overline{\mu}_2(\gamma_2), \ldots, \overline{\mu}_r(\gamma_r), \overline{\zeta}_1(\tau_1), \overline{\zeta}_2(\tau_2), \ldots, \overline{\zeta}_v(\tau_v), \xi_0^T(0)\right)^\top$$

the components $\overline{\mu}_i(z)$ and $\overline{\zeta}_j(z)$ have the same form as $\mu_i(z)$ and $\zeta_j(z)$ but with $c_{s_i+1}(z;\gamma_i)$ and $h_{\ell_j+1}(z;\tau_j)$ being replaced by $c_1(z;\gamma_i)$ and $h_1(z;\tau_j)$ respectively as can be seen in $\overline{g}_i(z)$ and $\overline{k}_j(z)$.

Factorization of the corresponding Stein solution leads to the form

$$S_{bb} = \left(\mathcal{M}_r(\gamma)\ \mathcal{M}_v(\tau)\ \overline{\mathcal{M}}_l(0)\right)\left(\overline{\vartheta} \otimes I_{r+v}\right) \tag{2.30}$$

with

$$\mathcal{M}_r(\gamma) = \begin{pmatrix} \mathcal{M}_1(\gamma_1), & \mathcal{M}_2(\gamma_2), \ldots, & \mathcal{M}_r(\gamma_r) \end{pmatrix}$$

$$\mathcal{M}_v(\tau) = \begin{pmatrix} \mathcal{M}_1(\tau_1), & \mathcal{M}_2(\tau_2), \ldots, & \mathcal{M}_v(\tau_v) \end{pmatrix}.$$

The blocks which form $\mathcal{M}_r(\gamma)$ and $\mathcal{M}_v(\tau)$ have the following structure

$$\mathcal{M}_i(\gamma_i) = \left(\text{adj}\,(zI - C_h)\,\Gamma \text{adj}\,(I - zC_h)^\top z^{l+1}\right)_{z=\gamma_i} \qquad i = 1, \ldots, r$$

$$\mathcal{M}_j(\tau_j) = \left(\text{adj}\,(zI - C_h)\,\Gamma \text{adj}\,(I - zC_h)^\top z^{l+1}\right)_{z=\tau_j} \qquad j = 1, \ldots, v.$$

The blocks $\mathcal{F}(0)$ and $\overline{\mathcal{M}}_l(0)$ are also to be found in (2.15) and (2.16) respectively. An equality involving $G_{bb}(\theta)$ and S_{bb} given in (2.29) and (2.30) respectively can then be realized through an equivalent form of theorem 2.2.

The case $l + 1 = 0$ yields

$$G_{bb}(\theta) = \left(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau)\right)\left(\overline{\varphi} \otimes I_q\right) \tag{2.31}$$

$$S_{bb} = \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)\left(\overline{\varphi} \otimes I_q\right) \tag{2.32}$$

with $\overline{\varphi}$ having the same form as $\overline{\vartheta}$ without $\xi_0(0)$, the elements $\overline{\mu}_i(z)$ and $\overline{\zeta}_j(z)$ do not contain z^{l+1}, the matrices appearing in (2.32) are $\widetilde{\mathcal{M}}_r(z) = z^{-(l+1)}\mathcal{M}_r(z)$ and $\widetilde{\mathcal{M}}_v(z) = z^{-(l+1)}\mathcal{M}_v(z)$. The following equality can be verified by taking the particular solution of the linear systems (2.31) and (2.32) into account with $(\overline{\varphi} \otimes I_q)$ as a common factor to both linear systems, to obtain

$$\left(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau)\right)^+ G_{bb}(\theta) + \mathcal{S} = \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)^+ S_{bb} + \mathcal{C}$$

where the appropriate matrices $\mathcal{S}$ and $\mathcal{C}$

$$\mathcal{S} \in \text{Ker}\ \left(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau)\right) \text{ and } \mathcal{C} \in \text{Ker}\ \left(\widetilde{\mathcal{M}}_r(\gamma)\widetilde{\mathcal{M}}_v(\tau)\right).$$

An interconnection between the Fisher information matrix and a solution to a Stein equation can be set forth under the condition that the matrices $(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau))$ and $\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ have full row rank. For guaranteeing the surjectiveness of the coefficient matrices for the case $l + 1 = 0$ proposition 2.3 can directy be applied for the coefficient matrix $(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau))$ whereas for the coefficient matrix $\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ the following proposition is proved.

Proposition 2.8 *For an appropriate choice of* Γ *in the corresponding Stein equation, the coefficient matrix in (2.32) has the rank given by*
dim Im $\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right) = q.$

Proof. A generalization of the factorization used in (3.11) yields for the $(r + v) \times (r + v)^2$ matrix

$$\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right) = \left(\widetilde{\mathcal{M}}_r^{(1)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(1)}(\tau)\right)\left\{\text{diag}\left(\widetilde{\mathcal{M}}_r^{(2)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(2)}(\tau)\right)\right\}.$$

To show full row rankness of $\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ we use the following property;

$\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ has full row rankness $\Leftrightarrow$ $\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ has a right inverse.

The matrix $\left\{\mathrm{diag}\left(\widetilde{\mathcal{M}}_r^{(2)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(2)}(\tau)\right)\right\}$ is square invertible as is commented in section 3.2 and in [12] a structure for a right inverse of $\left(\widetilde{\mathcal{M}}_r^{(1)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(1)}(\tau)\right)$ is set forth, to obtain

$$\left(\widetilde{\mathcal{M}}_r^{(1)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(1)}(\tau)\right)\ (I_{r+v} \otimes e_{r+v}) = V_{\gamma\tau}$$

and

$$\left(\widetilde{\mathcal{M}}_r^{(1)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(1)}(\tau)\right)\ \left(V_{\gamma\tau}^{-1} \otimes e_{r+v}\right) = I_{r+v}.$$

A right inverse is then

$$\left(\widetilde{\mathcal{M}}_r^{(1)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(1)}(\tau)\right)_R^{-} = \left(V_{\gamma\tau}^{-1} \otimes e_{r+v}\right)$$

with e_{r+v} being the last standard basis vector in $\mathbb{R}^{r+v}$ and the Vandermonde matrix

$$V_{\gamma\tau} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \vdots & \gamma_r & \tau_1 & \tau_2 & \vdots & \tau_v \\ \gamma_1^2 & \gamma_2^2 & \vdots & \gamma_r^2 & \tau_1^2 & \tau_2^2 & \vdots & \tau_v^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{r+v-1} & \gamma_2^{r+v-1} & \cdots & \gamma_r^{r+v-1} & \tau_1^{r+v-1} & \tau_2^{r+v-1} & \cdots & \tau_v^{r+v-1} \end{pmatrix}.$$

An appropriate right inverse is now set forth

$$\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)_R^{-} = \left\{\mathrm{diag}\left(\widetilde{\mathcal{M}}_r^{(2)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(2)}(\tau)\right)\right\}^{-1} \left(V_{\gamma\tau}^{-1} \otimes e_{r+v}\right).$$

Consequently, the full row rankness of $\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ is assured and

$$\dim \mathrm{Im} \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right) = r + v.$$

For $l + 1 = 0$ or $r + v = q$ the proof is completed. ■

Taking the dimension of $\mathrm{Im}\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)$ into account yields $\dim \mathrm{Ker} \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right) = (r+v)^2 - (r+v)$ this is in agreement with the dimension rule. Note that the matrices $(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau))$ and $\left(\widetilde{\mathcal{M}}_r^{(1)}(\gamma)\ \widetilde{\mathcal{M}}_v^{(1)}(\tau)\right)$ have an equivalent right inverse, see also [12].

These properties entail

$$(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau))\,(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau))^{+} = I_q \text{ and } \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)\left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)^{+} = I_q.$$

The case $l+1 < 0$ allows an equivalent of corollary 2.7 taking into account the particular solutions of the respective linear systems to obtain

$$(\overline{\varphi} \otimes I_{r+v}) = \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right)^{+} S_{bb} + \mathcal{O} \ \text{ with } \mathcal{O} \in \mathrm{Ker} \left(\widetilde{\mathcal{M}}_r(\gamma)\ \widetilde{\mathcal{M}}_v(\tau)\right) \text{ and}$$

$$(\overline{\varphi} \otimes I_q) = \left(\widetilde{\mathcal{U}}_r(\gamma)\ \widetilde{\mathcal{U}}_v(\tau)\right)^{+} G_{bb}(\theta) + \mathcal{I} \ \text{ with } \mathcal{I} \in \mathrm{Ker} \left(\widetilde{\mathcal{U}}_r(\gamma)\ \widetilde{\mathcal{U}}_v(\tau)\right),$$

with the matrix blocks $\widetilde{\mathcal{U}}_r(z) = z^{l+1}\mathcal{U}_r(z)$ and $\widetilde{\mathcal{U}}_v(z) = z^{l+1}\mathcal{U}_v(z)$.

3. Kernel description

In this section we will describe explicit formulas which are valid for the kernels introduced in the previous chapter. For the kernel of $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\ \overline{\mathcal{U}}_l(0)\right)$ a description is provided as well as for the subspace $\mathrm{Ker}\ \left(\overline{\mathcal{M}}_r(\gamma)\ \overline{\mathcal{M}}_v(\tau)\ \overline{\mathcal{M}}_l(0)\right)$.

3.1. General case

In this subsection a description is provided for the null spaces involved in the established inteconnections between the Fisher information matrix and a corresponding Stein solution. We first focus on the null space appearing in theorem 2.2, namely Ker $\big(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\big)$, since the matrix blocks which constitute $\overline{\mathcal{U}}_r(\gamma)$, $\overline{\mathcal{U}}_v(\tau)$ and $\overline{\mathcal{U}}_l(0)$ are evaluated at distinct roots, we then have the property that Im $\big(\overline{\mathcal{U}}_\nu(\sigma)\big) \cap$ Im $\big(\overline{\mathcal{U}}_\mu(\rho)\big) = \{0\}$ for all the different eigenvalues σ and ρ (with corresponding algebraic multiplicity ν and μ) which appear in Im $\big(\overline{\mathcal{U}}_r(\gamma)\big)$, Im $\big(\overline{\mathcal{U}}_v(\tau)\big)$ and Im $\big(\overline{\mathcal{U}}_l(0)\big)$. Consequently, the subspace Ker $\big(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\big)$ can be decomposed into a direct sum Ker $\big(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\,\overline{\mathcal{U}}_l(0)\big) =$ Ker $\big(\overline{\mathcal{U}}_r(\gamma)\big) \oplus$ Ker $\big(\overline{\mathcal{U}}_v(\tau)\big) \oplus$ Ker $\big(\overline{\mathcal{U}}_l(0)\big)$, a similar decomposition can also be applied to the subspaces on the right-hand side to obtain Ker $\big(\overline{\mathcal{U}}_r(\gamma)\big) = \bigoplus_{i=1}^{r_0}$ Ker $\big(\overline{\mathcal{U}}_{s_i}(\gamma_i)\big)$, Ker $\big(\overline{\mathcal{U}}_v(\tau)\big) = \bigoplus_{j=1}^{v_0}$ Ker$\big(\overline{\mathcal{U}}_{\ell_j}(\tau_j)\big)$. This property follows from the next lemma.

Lemma 3.1 *Consider two matrices A and B with appropriate dimensions, then*

$$Im\ A \cap Im\ B = \{0\} \text{ iff } Ker\ (A\ B) = \begin{pmatrix} Ker\ A \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ Ker\ B \end{pmatrix}.$$

Proof. When moving from right to left it is clear that it holds true. From left to right, we assume $\begin{pmatrix} x \\ y \end{pmatrix} \in$ Ker $(\ A\ B)$, this implies $Ax + By = 0$ and since Im $A \cap$ Im $B = \{0\}$ we have $Ax = 0$ and $By = 0$. ■

Since the functional form of the individual null spaces have the same structure, it therefore suffices to specify the null space evaluated at one single root and we therefore represent a root by σ with algebraic multiplicity $\nu + 1$. In the next subsections an algorithm for Ker $\big(\overline{\mathcal{U}}_\nu(\sigma)\big)$ is set forth and is followed by some properties of Ker $\big(\ \overline{\mathcal{M}}_\nu(\sigma)\big)$.

3.1.1. An algorithm for Ker $\big(\overline{\mathcal{U}}_\nu(\sigma)\big)$

In this section we shall adapt the notations used in the previous section accordingly. Consider $u_q(z) = (1, z, \dots, z^{q-1})^\top$ and $v_p(z) = z^{p-1}u_p(z^{-1})^\top$. Define the $q \times p(n+1)$ matrix $U_{nqp}(z) = (U^n_{qp}, \cdots, U^0_{qp})$ by

$$U^k_{qp}(z) = \left(\frac{d}{dz}\right)^k u_q(z)v_p(z).$$

We will give an expression for Ker $U_{nqp}(z)$. Let x be vector belonging to this kernel and decompose x as $x^\top = (x_0^\top, \dots, x_n^\top)$, with the $x_k \in \mathbb{R}^p$. Then

$$\begin{aligned} U_{nqp}(z)x &= \sum_{k=0}^{n}(\frac{d}{dz})^k u_q(z)v_p(z)x_{n-k} \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}u_q^{(j)}(z)v_p^{(k-j)}(z)x_{n-k} \\ &= \sum_{j=0}^{n}u_q^{(j)}(z)\sum_{k=j}^{n}\binom{k}{j}v_p^{(k-j)}(z)x_{n-k} \\ &= \sum_{j=0}^{n\wedge(q-1)}u_q^{(j)}(z)\sum_{k=j}^{n}\binom{k}{j}v_p^{(k-j)}(z)x_{n-k}. \end{aligned}$$

Since the vectors $u_q^{(j)}(z)$ are independent as long as $j \le q-1$, we see that $U_{nqp}(z)x = 0$ iff for all

$j \leq (q-1) \wedge n$ we have

$$\sum_{k=j}^{n} \binom{k}{j} v_p^{(k-j)}(z) x_{n-k} = 0. \tag{3.1}$$

(Notice that in this summation we only have non-zero contributions for $k \leq (j+p-1) \wedge n$.)
Thus we consider a system of $(q-1) \wedge n+1$ equations of type (3.1). Clearly, this system is triangular, which leads to a recursive solution procedure.
We introduce some more notation. Let $K_p(z)$ be a $p \times (p-1)$ matrix whose columns span Ker $v_p(z)$ (later on we will specify a certain choice for $K_p(z)$). We proceed in steps.

First we consider the case in which $n < q$, so we have a system of $n+1$ equations.
Set $j = n$. Then the corresponding equation becomes $v_p(z)x_0 = 0$. Hence $x_0 = K_p(z)\gamma_0$ for an arbitrary vector $\gamma_0 \in \mathbb{R}^{p-1}$.
Consider now (with x_0 given above) the equation for $j = n-1$:

$$v_p(z)x_1 + nv_p'(z)x_0 = 0.$$

A particular solution of this equation is $x_1 = -nl_p v_p'(z)x_0$, with l_p the last standard basis vector of $\mathbb{R}^p$ and hence the general solution is given by $x_1 = -nl_p v_p'(z)x_0 + K_p(z)\gamma_1$ with arbitrary γ_1, so $x_1 = K_p(z)\gamma_1 - nl_p v_p'(z)K_p(z)\gamma_0$.
Continuing this way, we look at the equation for $j = n-2$. It is

$$v_p(z)x_2 + (n-1)v_p'(z)x_1 + \frac{1}{2}n(n-1)v_p''(z)x_0 = 0.$$

A particular solution is given by

$$\begin{aligned} x_2 &= -l_p\left((n-1)v_p'(z)x_1 + \frac{1}{2}n(n-1)v_p''(z)x_0\right) \\ &= -l_p\left((n-1)v_p'(z)(-nl_p v_p'(z)K_p(z)\gamma_0 + K_p(z)\gamma_1) + \frac{1}{2}n(n-1)v_p''(z)x_0\right) \\ &= -l_p\left((n-1)v_p'(z)K_p(z)\gamma_1 + \frac{1}{2}n(n-1)v_p''(z)K_p(z)\gamma_0\right), \end{aligned}$$

where we used in the last equality that $v_p'(z)l_p = 0$. The general solution now becomes

$$x_2 = K_p(z)\gamma_2 - l_q\left((n-1)v_q'(z)K_p(z)\gamma_1 + \frac{1}{2}n(n-1)v_q''(z)K_p(z)\gamma_0\right).$$

Proceeding in this way, one obtains the following recursion for the x_k and then its explicit form.

$$x_{k+1} = K_p(z)\gamma_{k+1} - \sum_{j=1}^{k} \binom{n-k+j}{j} l_p v_p^{(j)}(z) x_{k+1-j} \tag{3.2}$$

$$x_k = K_p(z)\gamma_k - \sum_{j=1}^{k} \binom{n-k+j}{j} l_p v_p^{(j)}(z) K_p(z)\gamma_{k-j}. \tag{3.3}$$

If we put all the x_k underneath each other, we get

$$x = L_n(z)(I_{n+1} \otimes K_p(z))\gamma, \tag{3.4}$$

with $L_n(z) \in \mathbb{R}^{(n+1)p\times(n+1)p}$ the lower triangular matrix

$$\begin{pmatrix} I_p & 0 & & & & \\ -\binom{n}{1} l_p v_p^{(1)}(z) & I_p & 0 & & & \\ -\binom{n}{2} l_p v_p^{(2)}(z) & -\binom{n-1}{1} l_p v_p^{(1)}(z) & I_p & 0 & & \\ \vdots & & & & & \\ -l_p v_p^{(n)}(z) & & & & -l_p v_p^{(1)}(z) & I_p \end{pmatrix}. \tag{3.5}$$

Clearly dim Ker $U_{nqp}(z) = (n+1)(p-1)$.
Since obviously, the image space of $U_{nqp}(z)$ is spanned by the vectors $u_q^{(j)}(z)$, for $j = 0, \dots, n$ (recall that $n < q$), it has dimension $n+1$. This is in agreement with the dimension rule: $(n+1)(p-1)+n+1$ is the number of coulmns of $U_{nqp}(z)$.

A convenient choice of $K_p(z)$ is

$$\begin{pmatrix} -1 & 0 & & 0 \\ z & -1 & & \\ 0 & z & \ddots & 0 \\ & \ddots & \ddots & -1 \\ 0 & & 0 & z \end{pmatrix}.$$

In particular the computation of the products $v_p^{(j)}(z)K_p(z)$ now becomes easy. Differentiate $v_p(z)K_p(z)$ j times. Since K has zero derivatives of order greater than 1 and since $v_p(z)K_p(z) = 0$, we get $v_p^{(j)}(z)K_p(z) = -jv_p^{(j-1)}(z)K'(z)$. But this is nothing else than the vector $-jv_p^{(j-1)}(z)$ without its first element.

For the case in which $n \geq q$, a similar procedure as above has to be followed. The prime difference is that we now consider the set of q equations (3.1), for $j = 0, \dots, q-1$. Consider first the equation for $j = q-1$:

$$\sum_{k=q-1}^{n} \binom{k}{q-1} v_p^{(k-q+1)}(z) x_{n-k} = 0.$$

To get a solution we choose the $x_0, \dots, x_{n-q}$ completely free, say $x_k = \beta_k$ with $\beta_k \in \mathbb{R}^p$. Then we get for x_{n-q+1} the general solution

$$x_{n-q+1} = -l_p \sum_{k=q}^{n} \binom{k}{q-1} v_p^{(k-q+1)}(z)\beta_{n-k} + K_p(z)\gamma_{n-q+1},$$

with γ_{n-q+1} an arbitrary vector in $\mathbb{R}^{p-1}$. Continuing this way as in the case with $n < q$ we now get the solution x given by

$$x = \begin{pmatrix} I_{(n-q+1)p} & 0_{(n-q+1)p\times qp} \\ M(z) & L_q(z)(I_q \otimes K_p(z)) \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \tag{3.6}$$

with $L_q(z) \in \mathbb{R}^{qp\times q(p-1)}$ like the matrix $L_n(z)$ above, $M(z) \in \mathbb{R}^{qp\times(n-q+1)p}$ defined by

$$M(z) = \begin{pmatrix} -\binom{n}{q-1} l_p v_p^{(n-q+1)}(z) & \cdots & \binom{q}{q-1} l_p v_p^{(1)}(z) \\ \vdots & & \vdots \\ -\binom{n}{0} l_p v_p^{(n)})(z) & \cdots & \binom{q}{0} l_p v_p^{(q)}(z) \end{pmatrix}$$

and $\beta = (\beta_0^\top, \dots, \beta_{n-q}^\top)^\top$, $\gamma = (\gamma_{n-q+1}^\top, \dots, \gamma_n^\top)^\top$.

Since the image of $U_{nqp}(z)$ is now spanned by the vectors $u_q^{(j)}(z)$, for $j = 0, \dots, q-1$ (recall that $n \geq q$), it has dimension q. For the kernel we now have that its dimension is $(n-q+1)p$ (from the first components) plus $q(p-1)$ (from the other other components), $np + p - q$ in total. Notice again that this is in agreement with the dimension rule.

To compute Ker $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\, \overline{\mathcal{U}}_l(0)\right)$ one proceeds as follows. It follows from the definition of $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\, \overline{\mathcal{U}}_l(0)\right)$ that this kernel is the direct sum of the kernels of the $\overline{\mathcal{U}}_{\nu_i}(\sigma_i)$ for all the distinct eigenvalues σ_i and hence it's dimension is the sum of the dimensions of the summands.

3.1.2. Some property of Ker $\left(\overline{\mathcal{M}}_\nu(\sigma)\right)$

Attention is also paid to the second null space appearing in theorem 2.2, namely Ker $\left(\overline{\mathcal{M}}_r(\gamma)\, \overline{\mathcal{M}}_v(\tau)\, \overline{\mathcal{M}}_l(0)\right)$, the subspaces Im $\left(\overline{\mathcal{M}}_r(\gamma)\right)$, Im $\left(\overline{\mathcal{M}}_v(\tau)\right)$ and Im $\left(\overline{\mathcal{M}}_l(0)\right)$ have the property formulated in lemma 3.1 and this can be justified because the matrix blocks which form $\overline{\mathcal{M}}_r(\gamma)$, $\overline{\mathcal{M}}_v(\tau)$ and $\overline{\mathcal{M}}_l(0)$ are evaluated at dinstinct roots. We therefore have Ker $\left(\overline{\mathcal{M}}_r(\gamma)\, \overline{\mathcal{M}}_v(\tau)\, \overline{\mathcal{M}}_l(0)\right)$ = Ker $\left(\overline{\mathcal{M}}_r(\gamma)\right) \oplus$ Ker $\left(\overline{\mathcal{M}}_v(\tau)\right) \oplus$ Ker $\left(\overline{\mathcal{M}}_l(0)\right)$ with Ker $\left(\overline{\mathcal{M}}_r(\gamma)\right) = \bigoplus_{i=1}^{r_0}$ Ker $\left(\overline{\mathcal{M}}_{s_i}(\gamma_i)\right)$, Ker $\left(\overline{\mathcal{M}}_v(\tau)\right) = \bigoplus_{j=1}^{v_0}$ Ker$\left(\overline{\mathcal{M}}_{\ell_j}(\tau_j)\right)$ and since $\overline{\mathcal{M}}_l(0) = 0$ we have that Ker $\overline{\mathcal{M}}_l(0) = \mathbb{C}^{r+v} \oplus \mathbb{C}^{r+v} \oplus \cdots \oplus \mathbb{C}^{r+v}$. Since the functional forms of the individual null spaces have the same structure, it is then sufficient to consider the null space evaluated at one single root. We denote a root by σ with algebraic multiplicity $\nu + 1$ to represent a general form for the subspace Ker $\left(\overline{\mathcal{M}}_\nu(\sigma)\right)$. The appropriate structure of the matrix $\overline{\mathcal{M}}_\nu(\sigma)$ can be rewritten as

$$\overline{\mathcal{M}}_\nu(\sigma) = \left(\overline{\mathcal{M}}_\nu^{(\nu)}(z), \quad \overline{\mathcal{M}}_\nu^{(\nu-1)}(z), \quad \dots \quad , \overline{\mathcal{M}}_\nu^{(0)}(z) \right)_{z=\sigma}$$

each block being

$$\overline{\mathcal{M}}_\nu^{(\nu-j)}(\sigma) = \left(\frac{\partial^{\nu-j}}{\partial z^{\nu-j}} \mathrm{adj}\,(zI - C_h)\,\Gamma \mathrm{adj}\,(I - zC_h)^\top z^{l+1} \right)_{z=\sigma} \qquad j = 0, \dots, \nu.$$

In order to express the appropriate subspace in a useful and practical form, an additional factorization is applied on each block $\overline{\mathcal{M}}_\nu^{(\nu-j)}(\sigma)$, this is realized through the application of Leibnitz rule to j-fold differentiation of a product of two functions, this yields

$$\overline{\mathcal{M}}_\nu^{(\nu-j)}(\sigma) = \overline{\mathcal{M}}_\nu^{(\nu-j)(1)}(\sigma) \overline{\mathcal{M}}_\nu^{(\nu-j)(2)}(\sigma)$$

where for $j = 0, 1, \dots, \nu$ the $(r+v) \times (r+v)(\nu - j + 1)$ matrix

$$\overline{\mathcal{M}}_\nu^{(\nu-j)(1)}(\sigma) = \left(\frac{\partial^{\nu-j}}{\partial z^{\nu-j}} \mathrm{adj}\;(zI - C_h)\,, \frac{\partial^{\nu-j-1}}{\partial z^{\nu-j-1}} \mathrm{adj}\;(zI - C_h)\,, \dots ,\; \mathrm{adj}\;(zI - C_h) \right)_{z=\sigma}$$

and the $(r+v)(\nu - j + 1) \times (r+v)$ matrix

$$\overline{\mathcal{M}}_\nu^{(\nu-j)(2)}(\sigma) = \begin{pmatrix} \binom{\nu-j}{0} \Gamma \mathrm{adj}\,(I - zC_h)^\top z^{l+1} \\ \binom{\nu-j}{1} \Gamma \frac{\partial}{\partial z} \mathrm{adj}\,(I - zC_h)^\top z^{l+1} \\ \vdots \\ \binom{\nu-j}{\nu-j} \Gamma \frac{\partial^{\nu-j}}{\partial z^{\nu-j}} \mathrm{adj}\,(I - zC_h)^\top z^{l+1} \end{pmatrix}_{z=\sigma} .$$

To obtain

$$\overline{\mathcal{M}}_\nu(\sigma) = \overline{\mathcal{M}}_\nu^{(1)}(\sigma) \overline{\mathcal{M}}_\nu^{(2)}(\sigma) \tag{3.7}$$

where the $(r+v) \times (r+v)(\nu+1)(\nu+2)/2$ first matrix has the following form

$$\overline{\mathcal{M}}_{\nu}^{(1)}(\sigma) = \left(\overline{\mathcal{M}}_{\nu}^{(\nu)(1)}(z), \ \overline{\mathcal{M}}_{\nu}^{(\nu-1)(1)}(z), \ \ldots \ , \overline{\mathcal{M}}_{\nu}^{(0)(1)}(z) \right)_{z=\sigma}$$

and the $(r+v)(\nu+1)(\nu+2)/2 \times (r+v)(\nu+1)$ second matrix is

$$\overline{\mathcal{M}}_{\nu}^{(2)}(\sigma) = \begin{pmatrix} \overline{\mathcal{M}}_{\nu}^{(\nu)(2)}(z) & 0 & \ldots & 0 \\ 0 & \overline{\mathcal{M}}_{\nu}^{(\nu-1)(2)}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \overline{\mathcal{M}}_{\nu}^{(0)(2)}(z) \end{pmatrix}_{z=\sigma}.$$

The description of the subspace Ker $(\mathcal{M}_{\nu}(\sigma))$ is reduced to the description of a subspace associated with a product of two matrices. It will be shown in lemma 3.2 that the kernels of all the matrices $\overline{\mathcal{M}}_{\nu}^{(\nu-j)(2)}(\sigma)$ are zero so that Ker $\overline{\mathcal{M}}_{\nu}^{(2)}(\sigma) = \{0\}$. This property will be used for formulating the kernel of the product AB in terms of A and B taking into account Ker $B = \{0\}$.

Lemma 3.2 *The following property holds true for an appropriate choice of* Γ

$$Ker \begin{pmatrix} \overline{\mathcal{M}}_{\nu}^{(\nu)(2)}(z) & 0 & \ldots & 0 \\ 0 & \overline{\mathcal{M}}_{\nu}^{(\nu-1)(2)}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \overline{\mathcal{M}}_{\nu}^{(0)(2)}(z) \end{pmatrix}_{z=\sigma} = \{0\}.$$

Proof. The kernels of all the block elements of $\mathcal{M}_{\nu}^{(\nu)(2)}(\sigma)$ are

$$\text{Ker}\left(\overline{\mathcal{M}}_{\nu}^{(\nu-j)(2)}(\sigma)\right) = \text{Ker} \begin{pmatrix} \begin{pmatrix} \nu-j \\ 0 \end{pmatrix} \Gamma \text{adj} \left(I - zC_h\right)^{\top} z^{l+1} \\ \begin{pmatrix} \nu-j \\ 1 \end{pmatrix} \Gamma \frac{\partial}{\partial z} \text{adj} \left(I - zC_h\right)^{\top} z^{l+1} \\ \vdots \\ \begin{pmatrix} \nu-j \\ \nu-j \end{pmatrix} \Gamma \frac{\partial^{\nu-j}}{\partial z^{\nu-j}} \text{adj} \left(I - zC_h\right)^{\top} z^{l+1} \end{pmatrix}_{z=\sigma} =$$

$$\text{Ker} \left(\Gamma \text{adj} \ (I - zC_h)^{\top} z^{l+1}\right)_{z=\sigma} \cap \text{Ker} \left(\begin{pmatrix} \nu-j \\ 1 \end{pmatrix} \frac{\partial}{\partial z} \Gamma \text{adj} \ (I - zC_h)^{\top} z^{l+1}\right)_{z=\sigma} \cap \ldots \cap \text{Ker} \left(\frac{\partial^{\nu-j}}{\partial z^{\nu-j}} \Gamma \text{adj} \ (I - zC_h)^{\top} z^{l+1}\right)_{z=\sigma} = \{0\}.$$

The last result can easily be justified by taking into account the fact that the companion matrix in the Stein equation is such that the product of the eigenvalues of $(zI - C_h)$ and $(I - zC_h)$ are different from one and this implies that the eigenvalues of $(I - zC_h)$ will never be σ^{-1}. This property combined with an appropriate choice of Γ results in $\left(\Gamma \text{adj} \ (I - zC_h)^{\top} z^{l+1}\right)_{z=\sigma}$ being an invertible matrix. ■

Therefore there exists a left inverse of $\begin{pmatrix} \overline{\mathcal{M}}_{\nu}^{(\nu)(2)}(z) & 0 & \ldots & 0 \\ 0 & \overline{\mathcal{M}}_{\nu}^{(\nu-1)(2)}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \overline{\mathcal{M}}_{\nu}^{(0)(2)}(z) \end{pmatrix}_{z=\sigma}$, i.e. a matrix K such that

$$K \begin{pmatrix} \overline{\mathcal{M}}_\nu^{(\nu)(2)}(z) & 0 & \dots & 0 \\ 0 & \overline{\mathcal{M}}_\nu^{(\nu-1)(2)}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \overline{\mathcal{M}}_\nu^{(0)(2)}(z) \end{pmatrix}_{z=\sigma} = I_{(r+v)(\nu+1)}.$$

We have now the property given in the next lemma.

Lemma 3.3 *Let A and B be complex matrices of sizes $m \times n$ and $n \times p$ respectively and assume Ker $B = \{0\}$. Then*

$$Ker\ (AB) \simeq (Im\ B \cap Ker\ A).$$

Proof. Assume $x \in \text{Ker}\ (AB)$ then $AB\ x = 0$ and since $B\ x \neq 0$ implies

$$B\ x \in (\text{Im}\ B \cap \text{Ker}\ A).$$

If $y \in (\text{Im}\ B \cap \text{Ker}\ A)$ then there exists a x such that $y = Bx$ and $Ay = 0$ and consequently $AB\ x = 0$ or $x \in \text{Ker}\ (AB)$. ■

This property can be used for expressing the rank of AB see e.g. [7] and consequently the dimension of the kernel of AB can then be specified, to obtain

$$\dim \text{Im}\ (AB) + \dim\ (\text{Im}\ B \cap \text{Ker}\ A) = \dim \text{Im}\ B. \tag{3.8}$$

A derivation of Ker $\overline{\mathcal{M}}_\nu^{(1)}(\sigma)$ is now presented where

$$\overline{\mathcal{M}}_\nu^{(1)}(\sigma) = \left(\overline{\mathcal{M}}_\nu^{(\nu)(1)}(z),\ \overline{\mathcal{M}}_\nu^{(\nu-1)(1)}(z),\ \dots\ , \overline{\mathcal{M}}_\nu^{(0)(1)}(z) \right)_{z=\sigma}.$$

We will show that we can essentially reduce this problem to the computation of the kernel of a matrix like $U_{nqp}(z)$ which we treated in section 3.1.1. We consider the block-row decomposition of this matrix in terms of the $\overline{\mathcal{M}}_\nu^{(\nu-j)(1)}(\sigma)$. One easily sees that

$$\overline{\mathcal{M}}_\nu^{(\nu-j)(1)}(\sigma) = U_{\nu-j,p,p}(\sigma)(I_{\nu-j+1} \otimes S(c)), \tag{3.9}$$

with $p = r + v$. Indeed, observe first that $c(z)^\top J = u^*(z)^\top S(c)$, with notation similar to what we used before lemma 2.4. For instance, the vector $c(z)$ consists of the Hörner polynomials asociated with the characteristic polynomial π of C_h. Hence we get, like equation (2.23) that $\text{adj}(z - C_h) = u(z)u^*(z)^\top S(c) - \pi(z) \sum z^j S^{j+1}$. But then, by computing the relevant k-th order derivatives in any of the eigenvalues σ we get the simple relations $\left(\frac{d}{dz}\right)^k \text{adj}(z - C_h)|_{z=\sigma} = U_{pp}^k(\sigma)S(c)$, in the notation of section 3.1.1. Doing this for the different σ's under consideration results in equation (3.9).
Let $y \in \text{Ker}\ \overline{\mathcal{M}}_\nu^{(1)}(\sigma)$ and decompose y as $y^\top = \left(y_0^\top, \dots, y_\nu^\top\right)$, with $y_j \in \mathbb{R}^{p(\nu+1-j)}$. Let $z_j = (I_{\nu+1-j} \otimes S(c))y_j$ and $z^\top = \left(z_0^\top, \dots, z_\nu^\top\right)$. Then we have $y \in \text{Ker}\overline{\mathcal{M}}_\nu^{(1)}(\sigma)$ iff

$$z \in \text{Ker}\ \left(U_{\nu,p,p}(\sigma), U_{\nu-1,p,p}(\sigma), \dots, U_{0,p,p}(\sigma)\right). \tag{3.10}$$

Decompose the z_j further according to $z_j^\top = \left(z_{j,0}^\top, \dots, z_{j,\nu-j}^\top\right)$, with the $z_{ji} \in \mathbb{R}^p$.
For z to be in the kernel of $(U_{\nu,p,p}(\sigma), U_{\nu-1,p,p}(\sigma) \dots,, U_{0,p,p}(\sigma))$ it must satisfy the equation

$$\sum_{j=0}^{\nu} U_{\nu-j,p,p}(\sigma) z_j = 0.$$

Using the $z_{j,i}$ and the matrices $U^k_{pp}(\sigma)$ that we introduced before (so $U_{\nu-j,p,p}(\sigma) = \big(U^{\nu-j}_{pp}(\sigma), \dots, U^0_{pp}(\sigma)\big)$), we obtain that the kernel equation (3.10) is equivalent to

$$\sum_{j=0}^{\nu}\sum_{i=0}^{\nu-j} U^{\nu-i-j}_{pp}(\sigma) z_{j,i} = 0.$$

A little bit of rearranging terms in this equation leads to

$$\sum_{l=0}^{\nu} U^{\nu-l}_{pp}(\sigma) \sum_{j=0}^{l} z_{j,l-j} = 0.$$

Denote now by x_l the sum $\sum_{j=0}^{l} z_{j,l-j}$. Then we see that the vector x given by $x^\top = \big(x_0^\top, \dots, x_\nu^\top\big)$ belongs to the kernel of $U_{\nu,p,p}(\sigma)$. But from section 3.1.1 we know how the x_l look. It follows that for each l the vector $\tilde{z}_l = \Big(z^\top_{0,l}, \dots, z^\top_{l,0}\Big)^\top$ belongs to a subspace of dimension $lp + s$, where s is the dimension of the subspace where x_l lives. Moreover the $z_{j,l-j}$ that constitute $\tilde{z}_l$ are all independent from the $z_{j,l'-j}$ that constitute $\tilde{z}_{l'}$ for $l \neq l'$.
Working back through this procedure, one will be able to explicitly describe the space where the z and y live. We omit the uninteresting details. However, it can be concluded that dim Ker $\overline{\mathcal{M}}^{(1)}_\nu(\sigma)$ is $p[(\nu+1)(\nu+2)/2 \; -1]$ if $p \leq \nu$ and equal to $p(\nu+1)(\nu+2)/2 \; -\nu - 1$ if $p \geq \nu + 1$.

We shall now formulate an expression for $\big(\text{Ker } (zI - C_h)^*\big)^\perp_{z=\sigma}$ which shall be summarized in the next lemma and for that purpose we introduce the following polynomial

$$\pi_x(z) = z^x + e_1\, z^{x-1} + e_2\, z^{x-2} + \dots + e_x.$$

Lemma 3.4 *Given the companion matrix C_h used in (2.15) yields*

$$Ker\ (zI - C_h)^*_{z=\sigma} = \begin{pmatrix} e_{(r+v)-1} + z\ \pi_{(r+v)-2}(z) \\ e_{(r+v)-2} + z\ \pi_{(r+v)-3}(z) \\ \vdots \\ \pi_1(z) \\ \pi_0(z) \end{pmatrix}_{z=\sigma}$$

$$\text{with } \big(Ker\ (zI - C_h)^*_{z=\sigma}\big)^\perp = \begin{pmatrix} \xi \\ I_{(r+v)-1} \end{pmatrix}$$

$$\text{where} \qquad \xi = -\ \big(e_{(r+v)-1} + z\ \pi_{(r+v)-2}(z)\big)^{-1}_{z=\sigma} \times$$

$$\big(\ e_{(r+v)-2} + z\ \pi_{(r+v)-3}(z), e_{(r+v)-3} + z\ \pi_{(r+v)-4}(z), \dots, \pi_1(z), \pi_0(z)\big)_{z=\sigma} \text{ and } \pi_0(\sigma) = 1.$$

$$\text{and dim } \big(Ker\ (zI - C_h)^*_{z=\sigma}\big)^\perp = r + v - 1.$$

Where X^* is the complex conjugate transpose of X and $Y^\perp$ is the orthogonal complement of Y. Taking into consideration the property dim $\big(\text{Ker } (zI - C_h)^*\big)^\perp_{z=\sigma}$ = dim Ker adj $(zI - C_h)_{z=\sigma}$ = $r + v - 1$ (as can be seen from lemma 2.4 and lemma 3.4) leads to the following result.

Corollary 3.5 *The following holds true*

$$\big(Ker\ (zI - C_h)^*_{z=\sigma}\big)^\perp = Ker\ adj\ (zI - C_h)_{z=\sigma}.$$

Proof. We use the following approach by rewriting the adjoint of $(zI - C_h)_{z=\sigma}$ as

$$\text{adj } (zI - C_h)\ (zI - C_h) = \det\ (zI - C_h)\ I_{r+v}$$

so that

$$\text{adj }(zI - C_h)_{z=\sigma}\ (zI - C_h)_{z=\sigma} = \det\ (zI - C_h)_{z=\sigma}\ I_{r+v} = 0.$$

Consequently,

$$\text{Ker adj }(zI - C_h)_{z=\sigma} = \text{Im }(zI - C_h)_{z=\sigma} = \left(\text{Ker }\ (zI - C_h)^*_{z=\sigma}\right)^{\perp}. \qquad \blacksquare$$

However, it can be seen from equation (2.23) that an alternative representation for Ker adj $(zI - C_h)_{z=\sigma}$ can be derived.

An example is now provided for $r + v = 4$. Consider the companion form used in (2.15) to have

$$\text{adj }(zI - C_h)_{z=\sigma} = \begin{pmatrix} e_3 + e_2\ z + e_1\ z^2 + z^3 & e_2 + e_1\ z + z^2 & e_1 +\ z & 1 \\ -\ e_4 & e_2\ z + e_1\ z^2 + z^3 & e_1\ z + z^2 & z \\ -\ e_4\ z & -\ e_4 - e_3\ z & e_1\ z^2 + z^3 & z^2 \\ -\ e_4\ z^2 & -\ e_4\ z - e_3\ z^2 & -e_4 - e_3\ z - e_2\ z^2 & z^3 \end{pmatrix}_{z=\sigma}$$

an appropriate choice of a matrix $\mathcal{A}$ with columns in Ker(adj $(zI - C_h)_{z=\sigma}$) is then

$$\mathcal{A} = \left\{ -\frac{1}{e_3+e_2\ z+e_1\ z^2+z^3} \begin{pmatrix} e_2 + e_1\ z + z^2 & e_1 +\ z & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}_{z=\sigma}.$$

3.2. Special case

The kernels involved in the interconnections when the roots or eigenvalues are distinct shall now be given. Attention is paid to the kernels of the coefficient matrices appearing in (2.29) and (2.30), to have

$$\text{Ker }\left(\mathcal{U}_r(\gamma)\ \mathcal{U}_v(\tau)\ \overline{\mathcal{U}}_l(0)\right) = \text{Ker }(\mathcal{U}_r(\gamma)) \oplus \text{Ker }(\mathcal{U}_v(\tau)) \oplus \text{Ker }\left(\overline{\mathcal{U}}_l(0)\right)$$

with

$$\text{Ker }(\mathcal{U}_r(\gamma)) = \bigoplus_{i=1}^{r} \text{Ker }(\mathcal{U}_i(\gamma_i)),\ \text{Ker }(\mathcal{U}_v(\tau)) = \bigoplus_{j=1}^{v} \text{Ker }(\mathcal{U}_j(\tau_j)).$$

It is sufficient to represent one case since the argument of linear independence holds

$$\text{Ker }(\mathcal{U}_i(\gamma_i)) = \begin{pmatrix} -z^{-(q-1)}u_{q-1}^{\top}(z) \\ J_{q-1} \end{pmatrix}_{z=\gamma_i} \quad \text{and} \quad \dim \text{Ker }(\mathcal{U}_i(\gamma_i)) = q - 1$$

and an analogous representation holds for Ker $(\mathcal{U}_j(\tau_j))$ we further have

$$\text{Ker }\left(\overline{\mathcal{U}}_\delta(0)\right) = \text{Ker }\ \left(\frac{\partial^\delta}{\partial z^\delta}\left(u_q(z)u_q^{*\top}(z)\right)\right)_{z=0} = \begin{cases} \begin{pmatrix} J_{q-1-\delta} \\ 0_{1+\delta} \end{pmatrix} & \text{for } \delta = 0,\ 1, \ldots, q-1 \\ \begin{pmatrix} 0_{2q-1-\delta} \\ J_{\delta-(q-1)} \end{pmatrix} & \text{for } \delta = q,\ q+1, \ldots, 2q-2 \end{cases}$$

$$\dim \text{Ker }\ \left(\frac{\partial^\delta}{\partial z^\delta}\left(u_q(z)u_q^{*\top}(z)\right)\right)_{z=0} = \begin{cases} (q-1) - \delta & \text{for } \delta = 0, 1, \ldots, q-1 \\ \delta - (q-1) & \text{for } \delta = q, q+1, \ldots, 2q-2. \end{cases}$$

The null spaces constituting Ker $\left(\mathcal{M}_r(\gamma)\ \mathcal{M}_v(\tau)\ \overline{\mathcal{M}}_l(0)\right)$ are obtained according to

$$\text{Ker }\left(\mathcal{M}_r(\gamma)\ \mathcal{M}_v(\tau)\ \overline{\mathcal{M}}_l(0)\right) = \text{Ker }(\mathcal{M}_r(\gamma)) \oplus \text{Ker }(\mathcal{M}_v(\tau)) \oplus \text{Ker }\left(\overline{\mathcal{M}}_l(0)\right)$$

with

$$\text{Ker }(\mathcal{M}_r(\gamma)) = \bigoplus_{i=1}^{r} \text{Ker }(\mathcal{M}_i(\gamma_i)),\ \text{Ker }(\mathcal{M}_v(\tau)) = \bigoplus_{j=1}^{v} \text{Ker}(\mathcal{M}_j(\tau_j))$$

and as in the general case Ker $\overline{\mathcal{M}}_l(0) = \mathbb{C}^{r+v} \oplus \mathbb{C}^{r+v} \oplus \cdots \oplus \mathbb{C}^{r+v}$. Since there is an equivalent functional form for all the subspaces, it suffices to consider for example

$\text{Ker}\ (\mathcal{M}_r(\gamma)) = \bigoplus_{i=1}^{r} \text{Ker}\ (\mathcal{M}_i(\gamma_i))$ and the following factorization is applied

$$\mathcal{M}_r(\gamma) = \mathcal{M}_r^{(1)}(\gamma)\mathcal{M}_r^{(2)}(\gamma) \tag{3.11}$$

where

$$\mathcal{M}_r^{(1)}(\gamma) = \left(\text{adj }\ (zI - C_h)_{z=\gamma_1}, \text{adj }\ (zI - C_h)_{z=\gamma_2}, \dots, \text{adj }\ (zI - C_h)_{z=\gamma_r}\right)$$

and

$$\mathcal{M}_r^{(2)}(\gamma) = \begin{pmatrix} \Gamma\left(\text{adj}\,(I - zC_h)^\top z^{l+1}\right)_{z=\gamma_1} & 0 & \dots & 0 \\ 0 & \Gamma\left(\text{adj}\,(I - zC_h)^\top z^{l+1}\right)_{z=\gamma_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \Gamma\left(\text{adj}\,(I - zC_h)^\top z^{l+1}\right)_{z=\gamma_r} \end{pmatrix}.$$

Since the blocks composing $\mathcal{M}_r^{(2)}(\gamma)$ are square invertible matrices, we then have

$$\text{Ker}\ (\mathcal{M}_r(\gamma)) = \left(\mathcal{M}_r^{(2)}(\gamma)\right)^{-1} \text{Ker}\ \left(\mathcal{M}_r^{(1)}(\gamma)\right)$$

and for a similar argument as mentioned in lemma 3.1 we obtain

$$\text{Ker}\left(\mathcal{M}_r^{(1)}(\gamma)\right) = \text{Ker}\left(\text{adj }\ (zI - C_h)_{z=\gamma_1}\right) \oplus \text{Ker}\left(\text{adj }\ (zI - C_h)_{z=\gamma_2}\right) \oplus \dots \oplus \text{Ker}\left(\text{adj }\ (zI - C_h)_{z=\gamma_r}\right).$$

An appropriate choice of a matrix with columns in Ker $\left(\text{adj }\ (zI - C_h)_{z=\gamma_i}\right)$ ($i = 1, 2, \dots, r$) can be deduced from lemma 3.4 and corollary 3.5 and from which can also be concluded that dim Ker $\left(\mathcal{M}_r^{(1)}(\gamma)\right) = r\ (r + v - 1)$, a similar argument holds for the remaining blocks. It is clear that these results can be used for the three cases considered, $l + 1 > 0$, $l + 1 = 0$ and $l + 1 < 0$. The interconnection between the Fisher information matrix and a corresponding Stein solution takes place for the case $l + 1 = 0$ and this implies dim Ker $(\mathcal{M}_r(\gamma)\ \mathcal{M}_v(\tau)) = (r + v)\ (r + v - 1)$ and dim Im $(\mathcal{M}_r(\gamma)\ \mathcal{M}_v(\tau)) = (r + v)$ as it is also pointed out in section 2.2 but here the results are based on lemma 3.4 and corollary 3.5.

4. Right inverses and Vandermonde matrices

In this section some appropriate right inverses which appear in the interconnection between a Stein solution and Fisher's information shall be given. The presence of Vandermonde matrices will be set forth as well as some equalities involving the matrices $\overline{\mathcal{U}}_\nu(\sigma)$ and $\widetilde{\overline{\mathcal{M}}}_\nu(\sigma)$, these are derived from equations which contain Vandermonde matrices.

A right inverse of $\left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\right)$ will be provided for $r + v = q$. First the following $(q \times q)$ generalized Vandermonde matrix is introduced

$$\mathcal{W}_{r,v}(\gamma, \tau) = \left(\ \mathcal{W}_{s_1}(\gamma_1), \ \ \mathcal{W}_{s_2}(\gamma_2), \ \ ..., \ \ \mathcal{W}_{s_{r_0}}(\gamma_{r_0}), \ \ \mathcal{V}_{\ell_1}(\tau_1), \ \ \mathcal{V}_{\ell_2}(\tau_2), \ \ ..., \ \ \mathcal{V}_{\ell_{v_0}}(\tau_{v_0})\ \right)$$

where

$$\mathcal{W}_{s_i}(\gamma_i) = \left(\ \mathcal{W}_{s_i}^{(s_i)}(z), \ \ \mathcal{W}_{s_i}^{(s_i-1)}(z), \ \ ... \ \ , \mathcal{W}_{s_i}^{(0)}(z)\ \right)_{z=\gamma_i}$$

and

$$\mathcal{W}_{s_i}^{(s_i-k)}(\gamma_i) = \left(\frac{\partial^{s_i-k}}{\partial z^{s_i-k}} u_q(z)\right)_{z=\gamma_i} \qquad k = 0, 1, \dots, s_i.$$

The matrix

$$\mathcal{V}_{\ell_j}(\tau_j) = \Big(\; \mathcal{V}_{\ell_j}^{(\ell_j)}(z), \quad \mathcal{V}_{\ell_j}^{(\ell_j-1)}(z), \quad \ldots \quad , \mathcal{V}_{\ell_j}^{(0)}(z) \; \Big)_{z=\tau_j}$$

with

$$\mathcal{V}_{\ell_j}^{(\ell_j-k)}(\tau_j) = \left(\frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}} u_q(z) \right)_{z=\tau_j} \qquad k = 0, 1, \ldots, \ell_j.$$

To obtain

$$\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right) \; (I_q \otimes e_q) \;=\; \mathcal{W}_{r,\,v}(\gamma, \tau) \tag{4.1}$$

$$\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right) \left((\mathcal{W}_{r,\,v}(\gamma, \tau))^{-1} \otimes e_q \right) \;=\; I_q. \tag{4.2}$$

Consequently, an appropriate right inverse is $\left(\overline{\mathcal{U}}_r(\gamma)\, \overline{\mathcal{U}}_v(\tau)\right)_R^- = \left((\mathcal{W}_{r,\,v}(\gamma, \tau))^{-1} \otimes e_q \right)$ where e_q is the last standard basis vector in $\mathbb{R}^q$ (which consists of one on the last position and zeros for the remaining elements).

In [12] Vandermonde structures have been detected in the matrices of the form $\widetilde{\widetilde{\mathcal{M}}}_\nu(\sigma)$ whereas in this study Vandermonde matrices are associated with the coefficient matrix $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)$, for the case $\ell + 1 = 0$ or $q = r + v$. As in [12] an appropriate factorization is set forth in order to separate the terms involving adj $(zI - C_h)$, Γ and $\mathrm{adj}(I - zC_h)^\top$. For achieving such a factorization Leibnitz rule to j- fold differentiation of a product of two functions is applied. However, the equations set forth in this paper are more general than the equivalent ones in [12]. The right inverses which are derived in this study are also used to establish equations which interconnect the coefficient matrices associated with the factored forms of the Fisher information matrix and a corresponding solution to a Stein equation. The suggested factorization has the following form

$$\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right) = \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{r,\,v}(\gamma, \tau)\, \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{r,\,v}\, \widetilde{\widetilde{\mathcal{M}}}^{(2)}_{r,\,v}(\gamma, \tau) \tag{4.3}$$

where the $q \times q\ (\delta_1 + \delta_2)$ matrix $\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{r,\,v}(\gamma, \tau)$ is composed as

$$\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{r,\,v}(\gamma, \tau) = \left(\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{s_1}(\gamma_1)\, \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{s_2}(\gamma_2) \ldots \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{s_{r_0}}(\gamma_{r_0})\, \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{\ell_1}(\tau_1)\, \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{\ell_2}(\tau_2) \ldots \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{\ell_{v_0}}(\tau_{v_0}) \right)$$

with

$$\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{s_i}(\gamma_i) = \left(\widetilde{\widetilde{\mathcal{M}}}^{(s_i)\ (1)}_{s_i}(z)\, \widetilde{\widetilde{\mathcal{M}}}^{(s_i-1)\ (1)}_{s_i}(z) \ldots \widetilde{\widetilde{\mathcal{M}}}^{(0)\ (1)}_{s_i}(z) \right)_{z=\gamma_i}$$

and

$$\widetilde{\widetilde{\mathcal{M}}}^{(s_i-j)\ (1)}_{s_i}(\gamma_i) = \left(\frac{\partial^{s_i-j}}{\partial z^{s_i-j}} \mathrm{adj}\,(zI - C_h) \right)_{z=\gamma_i} \quad j = 0, 1, \ldots, s_i$$

we further have

$$\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{\ell_j}(\tau_j) = \left(\widetilde{\widetilde{\mathcal{M}}}^{(\ell_j)\ (1)}_{\ell_j}(z)\, \widetilde{\widetilde{\mathcal{M}}}^{(\ell_j-1)\ (1)}_{\ell_j}(z) \ldots \widetilde{\widetilde{\mathcal{M}}}^{(0)\ (1)}_{\ell_j}(z) \right)_{z=\tau_j}$$

with

$$\widetilde{\widetilde{\mathcal{M}}}^{(\ell_j-k)\ (1)}_{\ell_j}(\tau_j) = \left(\frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}} \mathrm{adj}\,(zI - C_h) \right)_{z=\tau_j} \quad k = 0,\, 1, \ldots, \ell_j.$$

Note that

$$\delta_1 = \sum_{i=1}^{r_0} \frac{(s_i + 1)\,(s_i + 2)}{2} \text{ and } \delta_2 = \sum_{j=1}^{v_0} \frac{(\ell_j + 1)\,(\ell_j + 2)}{2}.$$

The $q\ (\delta_1 + \delta_2) \times q\ (\delta_1 + \delta_2)$ block diagonal matrix $\widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{r,v}$ has the following form

$$\widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{r,\,v} = \mathrm{diag}\left\{ \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{s_1}\, \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{s_2} \ldots \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{s_{r_0}}\, \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{\ell_1}\, \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{\ell_2} \ldots \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{\ell_{v_0}} \right\}$$

with

$$\widetilde{\overline{\mathcal{M}}}_{s_i}^{\Gamma} = \begin{pmatrix} \Gamma^{(s_i)} & 0 & \cdots & 0 \\ 0 & \Gamma^{(s_i-1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma^{(0)} \end{pmatrix} \text{ where } \Gamma^{(s_i-j)} = \begin{pmatrix} \binom{s_i-j}{0}\Gamma & 0 & \cdots & 0 \\ 0 & \binom{s_i-j}{1}\Gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \binom{s_i-j}{s_i-j}\Gamma \end{pmatrix}$$

and

$$\widetilde{\overline{\mathcal{M}}}_{\ell_j}^{\Gamma} = \begin{pmatrix} \Gamma^{(\ell_j)} & 0 & \cdots & 0 \\ 0 & \Gamma^{(\ell_j-1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma^{(0)} \end{pmatrix} \text{ where } \Gamma^{(\ell_j-k)} = \begin{pmatrix} \binom{\ell_j-k}{0}\Gamma & 0 & \cdots & 0 \\ 0 & \binom{\ell_j-k}{1}\Gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \binom{\ell_j-k}{\ell_j-k}\Gamma \end{pmatrix}$$

the matrix $\widetilde{\overline{\mathcal{M}}}_{r,\ v}^{\Gamma}$ is invertible for an appropriate choice of Γ (Γ is invertible) and $j = 0, 1, \ldots, s_i$ and $k = 0, 1, \ldots, \ell_j$.

The $q\ (\delta_1 + \delta_2) \times q\ (r + v)$ block diagonal matrix $\widetilde{\overline{\mathcal{M}}}_{r,\ v}^{(2)}(\gamma, \tau)$ has the form given by
$\widetilde{\overline{\mathcal{M}}}_{r,\ v}^{(2)}(\gamma, \tau) = \operatorname{diag}\left\{ \widetilde{\overline{\mathcal{M}}}_{s_1}^{(2)}(\gamma_1)\ \widetilde{\overline{\mathcal{M}}}_{s_2}^{(2)}(\gamma_2) \ldots \widetilde{\overline{\mathcal{M}}}_{s_{r_0}}^{(2)}(\gamma_{r_0})\ \widetilde{\overline{\mathcal{M}}}_{\ell_1}^{(2)}(\tau_1)\ \widetilde{\overline{\mathcal{M}}}_{\ell_2}^{(2)}(\tau_2) \ldots \widetilde{\overline{\mathcal{M}}}_{\ell_{v_0}}^{(2)}(\tau_{v_0}) \right\}$
where

$$\widetilde{\overline{\mathcal{M}}}_{s_i}^{(2)}(\gamma_i) = \begin{pmatrix} \widetilde{\overline{\mathcal{M}}}^{(s_i)\ (2)}(z) & 0 & \cdots & 0 \\ 0 & \widetilde{\overline{\mathcal{M}}}^{(s_i-1)\ (2)}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widetilde{\overline{\mathcal{M}}}^{(0)\ (2)}(z) \end{pmatrix}_{z=\gamma_i}$$

$$\text{with } \widetilde{\overline{\mathcal{M}}}^{(s_i-j)\ (2)}(\gamma_i) = \begin{pmatrix} \operatorname{adj}\ (I - zC_h)^{\top} \\ \frac{\partial}{\partial z}\operatorname{adj}\ (I - zC_h)^{\top} \\ \vdots \\ \frac{\partial^{s_i-j}}{\partial z^{s_i-j}}\operatorname{adj}\ (I - zC_h)^{\top} \end{pmatrix}_{z=\gamma_i} \quad \text{for } \ j = 0, 1, \ldots, s_i$$

$$\widetilde{\overline{\mathcal{M}}}_{\ell_j}^{(2)}(\tau_j) = \begin{pmatrix} \widetilde{\overline{\mathcal{M}}}^{(\ell_j)\ (2)}(z) & 0 & \cdots & 0 \\ 0 & \widetilde{\overline{\mathcal{M}}}^{(\ell_j-1)\ (2)}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widetilde{\overline{\mathcal{M}}}^{(0)\ (2)}(z) \end{pmatrix}_{z=\tau_j}$$

$$\text{with } \widetilde{\overline{\mathcal{M}}}^{(\ell_j-k)\ (2)}(\tau_j) = \begin{pmatrix} \operatorname{adj}\ (I - zC_h)^{\top} \\ \frac{\partial}{\partial z}\operatorname{adj}\ (I - zC_h)^{\top} \\ \vdots \\ \frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}}\operatorname{adj}\ (I - zC_h)^{\top} \end{pmatrix}_{z=\tau_j} \quad \text{for } k = 0, 1, \ldots, \ell_j.$$

Before deriving equations which involve Vandermonde matrices in (4.3) we first set forth appropriate equalities for one single block case, i.e. one block extracted from $\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)$ and $\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{r,\ v}(\gamma,\tau)$ respectively. The purpose of doing so is to ease the transition to the generalizations applied to the matrices $\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)$ and $\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{r,\ v}(\gamma,\tau)$. Like in section 3 we denote a root by σ with corresponding algebraic multiplicity $\nu+1$. A first result is set forth

$$\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q^\top\right)\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{\nu}(\sigma)=\mathcal{W}_q^*(\sigma) \tag{4.4}$$

where the $(\nu+1)(\nu+2)/2\times(\nu+1)\ q$ matrix $\mathcal{W}_q^*(\sigma)$ has the form

$$\mathcal{W}_q^*(\sigma)=\begin{pmatrix}\mathcal{W}^*_{\nu,q}(z) & 0 & \cdots & 0\\ 0 & \mathcal{W}^*_{\nu-1,q}(z) & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \mathcal{W}^*_{0,q}(z)\end{pmatrix}$$

and the $(\nu-j+1)\times q$ generalized Vandermonde matrix is given by

$$\mathcal{W}^*_{\nu-j,\ q}(z)=\begin{pmatrix}u_q^{*\top}(z)\\ \dfrac{\partial}{\partial z}u_q^{*\top}(z)\\ \vdots\\ \dfrac{\partial^{\upsilon-j}}{\partial z^{\nu-j}}u_q^{*\top}(z)\end{pmatrix}\quad\text{with } j=0,1,\ldots,\nu.$$

Since the case $\ell+1=0$ is considered, we then have that $\nu+1<q$, consequently a right inverse of the matrix $\mathcal{W}^*_{\nu-j,\ q}(z)$ can be constructed.

$$\mathcal{W}^*_{\nu-j,\ q}(z)\begin{pmatrix}I_{\nu-j+1}\\ 0_{q-(\nu-j+1)\times\nu-j+1}\end{pmatrix}=\mathcal{W}^*_{\nu-j,\ \nu-j+1}(z)$$

where the $(\nu-j+1)\times(\nu-j+1)$ generalized Vandermonde matrix $\mathcal{W}^*_{\nu-j,\ \nu-j+1}(z)$ is composed of the first $(\nu-j+1)$ columns (from left to right) of $\mathcal{W}^*_{\nu-j,\ q}(z)$, to obtain

$$\mathcal{W}^*_{\nu-j,\ q}(z)\begin{pmatrix}\mathcal{W}^{-*}_{\nu-j,\ \nu-j+1}(z)\\ 0_{q-(\nu-j+1)\times\nu-j+1}\end{pmatrix}=I_{\nu-j+1}.$$

where $\mathcal{W}^{-*}_{\nu-j,\ \nu-j+1}(z)=\left(\mathcal{W}^*_{\nu-j,\ \nu-j+1}(z)\right)^{-1}$.

Consequently, a right inverse of $\mathcal{W}^*_{\nu-j,\ q}(\sigma)$ is given by

$$\mathcal{W}^{-*}_{R,\ \nu-j,\ q}(\sigma)=\begin{pmatrix}\mathcal{W}^{-*}_{\nu-j,\ \nu-j+1}(z)\\ 0_{q-(\nu-j+1)\times\nu-j+1}\end{pmatrix}_{z=\sigma}.$$

We then have $\mathcal{W}^*_q(\sigma)\ \mathcal{W}^{-*}_{R,\ q}(\sigma)=I_{(\nu+1)(\nu+2)/2}$ or fully written as

$$\begin{pmatrix}\mathcal{W}^*_{\nu,\ q}(z) & 0 & \cdots & 0\\ 0 & \mathcal{W}^*_{\nu-1,\ q}(z) & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \mathcal{W}^*_{0,\ q}(z)\end{pmatrix}\begin{pmatrix}\mathcal{W}^{-*}_{R,\ \nu,\ q}(z) & 0 & \cdots & 0\\ 0 & \mathcal{W}^{-*}_{R,\ \nu-1,\ q}(z) & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \mathcal{W}^{-*}_{R,\ 0,\ q}(z)\end{pmatrix}=$$
$$I_{(\nu+1)(\nu+2)/2}.$$

The following equation can now be summarized in the following proposition.

Proposition 4.1 *The following equality holds true*

$$\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q^\top\right)\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{\nu}(\sigma)\mathcal{W}^{-*}_{R,\ q}(\sigma)=I_{(\nu+1)(\nu+2)/2}. \tag{4.5}$$

This equation is a variant of lemma 3.10 in [12]. We now proceed with setting forth a first step for constructing a right inverse of the matrix $\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)$, to obtain

$$\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q\right)=\mathcal{W}_{\nu}(\sigma) \tag{4.6}$$

where the $q\times(\nu+1)(\nu+2)/2$ generalized Vandermonde matrix

$$\mathcal{W}_{\nu}(\sigma)=\left(\mathcal{W}^{(\nu)}(z)\ \mathcal{W}^{(\nu-1)}(z)\ldots\mathcal{W}^{(0)}(z)\right)_{z=\sigma}$$

with

$$\mathcal{W}^{(\nu-j)}(\sigma)=\left(\frac{\partial^{\nu-j}}{\partial z^{\nu-j}}u_q(z)\right)_{z=\gamma_i}\qquad j=0,\,1,\ldots,\nu.$$

Note that

$$\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q^{\top}\right)\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q\right)=I_{(\nu+1)(\nu+2)/2}$$

since $e_q^{\top}e_q=1$, consequently a right and a left inverse of the appropriate matrices in (4.5) and (4.6) are respectively given by their corresponding transpose.

An equation involving the matrices $\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)$, $\widetilde{\overline{\mathcal{M}}}_{\nu}^{(2)}(\sigma)$ and appropriate Vandermonde matrices is obtained by combining (4.5) and (4.6) according to

$$\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q\right)\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q^{\top}\right)\widetilde{\overline{\mathcal{M}}}_{\nu}^{(2)}(\sigma)\mathcal{W}_{R,\,q}^{-*}(\sigma)=$$
$$\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q\right)I_{(\nu+1)(\nu+2)/2}=\mathcal{W}_{\nu}(\sigma)$$

or

$$\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)\left(I_{(\nu+1)(\nu+2)/2}\otimes e_q e_q^{\top}\right)\widetilde{\overline{\mathcal{M}}}_{\nu}^{(2)}(\sigma)\mathcal{W}_{R,\,q}^{-*}(\sigma)=\mathcal{W}_{\nu}(\sigma). \tag{4.7}$$

Although equations (4.6) and (4.7) give no explicit form for a right inverse of $\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)$, these equations shall be used for the general case (consisting of the matrices $\widetilde{\overline{\mathcal{M}}}_{r,\,v}^{(1)}(\gamma,\tau)$ and $\widetilde{\overline{\mathcal{M}}}_{r,\,v}^{(2)}(\gamma,\tau)$, and for $\ell+1=0$ or $q=r+v$), where a right inverse of $\widetilde{\overline{\mathcal{M}}}_{r,\,v}^{(1)}(\gamma,\tau)$ is given. But first some attention is paid for equations which involve $\widetilde{\overline{\mathcal{M}}}_{\nu}^{(1)}(\sigma)$, $\widetilde{\overline{\mathcal{M}}}_{\nu}^{(2)}(\sigma)$ and $\overline{\mathcal{U}}_{\nu}(\sigma)$. From the $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ structure can be seen for the one block case, with a root-eigenvalue equal to σ and algebraic multiplicity $\nu+1$, that the following holds true

$$\overline{\mathcal{U}}_{\nu}(\sigma)\left(I_{\nu+1}\otimes e_q\right)=\mathcal{W}_{\nu}{}^{q}(\sigma) \tag{4.8}$$

where the $q\times(\nu+1)$ generalized Vandermonde matrix $\mathcal{W}_{\nu}{}^{q}(\sigma)$ has the following structure

$$\mathcal{W}_{\nu}{}^{q}(\sigma)=\left(\frac{\partial^{\nu}}{\partial z^{\nu}}u_q(z)\ \frac{\partial^{\nu-1}}{\partial z^{\nu-1}}u_q(z)\ldots u_q(z)\right)_{z=\sigma}.$$

$\mathcal{W}_{\nu}(\sigma)$ which is the right-hand side of equation (4.6), fulfills the following property

$$\mathcal{W}_{\nu}(\sigma)\ \mathcal{E}_{\nu+1}=\mathcal{W}_{\nu}{}^{q}(\sigma)$$

where the $(\nu+1)(\nu+2)/2\times(\nu+1)$ matrix $\mathcal{E}_{\nu+1}$ has the following form

$$\mathcal{E}_{\nu+1}=\begin{pmatrix} e_1^{\nu+1}\left(e_1^{\nu+1}\right)^{\top} & & & \\ 0_{\nu} & e_1^{\nu}\left(e_1^{\nu}\right)^{\top} & & \\ 0_{\nu-1} & 0_{\nu-1} & e_1^{\nu-1}\left(e_1^{\nu-1}\right)^{\top} & \\ \vdots & & \vdots & \ddots \\ 0_1 & \cdots & 0_1 & e_1^{1}\left(e_1^{1}\right)^{\top} \end{pmatrix}$$

where the $0_i'$s are column vectors consisting of zeros ($i = 1, 2, \ldots, \nu$) and the elements e_1^j are the first standard basis vectors in $\mathbb{R}^j$ ($j = 1, 2, \ldots, \nu+1$). An alternative transformation can be set forth as

$$\mathcal{W}_\nu(\sigma) \begin{pmatrix} I_{\nu+1} \\ 0_{\nu(\nu+1)/2\times(\nu+1)} \end{pmatrix} = \mathcal{W}_\nu{}^q(\sigma).$$

Note that the matrices $\mathcal{E}_{\nu+1}$ and $\begin{pmatrix} I_{\nu+1} \\ 0_{\nu(\nu+1)/2\times(\nu+1)} \end{pmatrix}$ have a zero kernel and consequently have left inverses which are equal to their own transpose considering their respective structure or

$$\mathcal{E}^-_{\nu+1,L} = \mathcal{E}^\top_{\nu+1} \text{ and } \begin{pmatrix} I_{\nu+1} \\ 0_{\nu(\nu+1)/2\times(\nu+1)} \end{pmatrix}^-_L = \begin{pmatrix} I_{\nu+1} \\ 0_{\nu(\nu+1)/2\times(\nu+1)} \end{pmatrix}^\top.$$

We can now transform (4.6) according to

$$\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma) \left(I_{(\nu+1)(\nu+2)/2} \otimes e_q\right) \mathcal{E}_{\nu+1} = \mathcal{W}_\nu{}^q(\sigma). \tag{4.9}$$

Combining (4.8) and (4.9) allows to present an interconnection between $\overline{\mathcal{U}}_\nu(\sigma)$ and $\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma)$.

Proposition 4.2 *An interconnection between* $\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma)$ *and* $\overline{\mathcal{U}}_\nu(\sigma)$ *is verified through the following equality*

$$\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma) \left(I_{(\nu+1)(\nu+2)/2} \otimes e_q\right) \mathcal{E}_{\nu+1} = \overline{\mathcal{U}}_\nu(\sigma) \left(I_{\nu+1} \otimes e_q\right).$$

A equivalent equation can be verified through equation (4.7) by rewriting it as

$$\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma) \left(I_{(\nu+1)(\nu+2)/2} \otimes e_q e_q^\top\right) \overline{\widetilde{\mathcal{M}}}_\nu^{(2)}(\sigma)\ \mathcal{W}^{-*}_{R,\ q}(\sigma)\ \mathcal{E}_{\nu+1} = \mathcal{W}\,{}^q_\nu(\sigma)$$

to obtain

Proposition 4.3 *An alternative interconnection between* $\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma)$ *and* $\overline{\mathcal{U}}_\nu(\sigma)$ *is verified through the following equation*

$$\overline{\widetilde{\mathcal{M}}}_\nu^{(1)}(\sigma) \left(I_{(\nu+1)(\nu+2)/2} \otimes e_q e_q^\top\right) \overline{\widetilde{\mathcal{M}}}_\nu^{(2)}(\sigma)\ \mathcal{W}^{-*}_{R,\ q}(\sigma)\ \mathcal{E}_{\nu+1} = \overline{\mathcal{U}}_\nu(\sigma)\ \left(I_{\nu+1} \otimes e_q\right).$$

We shall now extend the obtained results to the matrices $\overline{\widetilde{\mathcal{M}}}_{r,\ v}^{(1)}(\gamma, \tau)$ and $\overline{\widetilde{\mathcal{M}}}_{r,\ v}^{(2)}(\gamma, \tau)$ by presenting a generalization of equation (4.6), this is given by the following equality

$$\overline{\widetilde{\mathcal{M}}}_{r,\ v}^{(1)}(\gamma, \tau)\ \left(I_{\delta_1+\delta_2} \otimes e_q\right) = \mathcal{W}_{r,\ v}^{\ \delta}(\gamma, \tau) \tag{4.10}$$

where the $q \times (\delta_1 + \delta_2)$ generalized Vandermonde matrix $\mathcal{W}_{r,\ v}^{\ \delta}(\gamma, \tau)$ is

$$\mathcal{W}_{r,\ v}^{\ \delta}(\gamma, \tau) = \Big(\mathcal{W}_{s_1}^{\ \delta}(\gamma_1)\ \mathcal{W}_{s_2}^{\ \delta}(\gamma_2) \ldots\ \mathcal{W}_{s_{r_0}}^{\ \delta}(\gamma_{r_0})\ \mathcal{W}_{\ell_1}^{\ \delta}(\tau_1)\ \mathcal{W}_{\ell_2}^{\ \delta}(\tau_2) \ldots\ \mathcal{W}_{\ell_{v_0}}^{\ \delta}(\tau_{v_0})\Big)$$

with

$$\mathcal{W}_{s_i}^{\ \delta}(\gamma_i) = \Big(\mathcal{W}_{s_i}^{(s_i)\ \delta}(z)\ \mathcal{W}_{s_i}^{(s_i-1)\ \delta}(z)\ \ldots\ \mathcal{W}_{s_i}^{(0)\ \delta}(z)\Big)_{z=\gamma_i}$$

with

$$\mathcal{W}_{s_i}^{(s_i-j)\ \delta}(\gamma_i) = \left(\frac{\partial^{s_i\ -\ j}}{\partial z^{s_i\ -\ j}} u_q(z)\right)_{z=\gamma_i} \qquad j = 0, 1, \ldots, s_i$$

and

$$\mathcal{W}_{\ell_j}^{\ \delta}(\tau_j) = \Big(\mathcal{W}_{\ell_j}^{(\ell_j)\ \delta}(z)\ \mathcal{W}_{\ell_j}^{(\ell_j-1)\ \delta}(z) \ldots\ \mathcal{W}_{\ell_j}^{(0)\ \delta}(z)\Big)_{z=\tau_j}$$

with

$$\mathcal{W}_{\ell_j}^{(\ell_j-k)\ \delta}(\tau_j) = \left(\frac{\partial^{\ell_j-k}}{\partial z^{\ell_j-k}} u_q(z)\right)_{z=\tau_j} \qquad k = 0, 1, \ldots, \ell_j.$$

In order to successfully construct a right inverse of $\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,v}(\gamma,\tau)$, the following equality is set forth

$$\mathcal{W}_{r,\ v}^{\ \delta}(\gamma,\tau)\ \mathcal{I}_{\delta,\ r,\ v} = \mathcal{W}_{r,\ v}(\gamma,\tau) \tag{4.11}$$

where the $(r+v)\times(r+v)$ or $q\times q$ generalized Vandermonde matrix $\mathcal{W}_{r,\ v}(\gamma,\tau)$ is given in the $\big(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\big)$ case (equations (4.1) and (4.2)).

The $(\delta_1+\delta_2)\times(r+v)$ matrix $\mathcal{I}_{\delta,\ r,\ v}$ has the following form

$$\mathcal{I}_{\delta,\ r,\ v} = \begin{pmatrix} \mathcal{I}_{\delta_1,\ r} & 0 \\ 0 & \mathcal{I}_{\delta_2,\ v} \end{pmatrix}$$

where

$$\mathcal{I}_{\delta_1,\ r} = \begin{pmatrix} I_{s_1+1} & 0 & \ldots & 0 \\ 0_{((s_1+1)\ s_1\ /\ 2\ )\times\ (s_1+1)} & & & \\ 0 & I_{s_2+1} & & \vdots \\ & 0_{((s_2+1)\ s_2\ /\ 2\ )\times\ (s_2+1)} & & \\ \vdots & & \ddots & 0 \\ & & & I_{s_{r_0}+1} \\ 0 & \ldots & 0 & 0_{\left(\left(s_{r_0}+1\right)\ s_{r_0}\ /\ 2\right)\ \times\ \left(s_{r_0}+1\right)} \end{pmatrix}$$

and

$$\mathcal{I}_{\delta_2,\ v} = \begin{pmatrix} I_{\ell_1+1} & 0 & \ldots & 0 \\ 0_{((\ell_1+1)\ \ell_1\ /\ 2)\ \times\ (\ell_1+1)} & & & \\ 0 & I_{\ell_2+1} & & \vdots \\ & 0_{((\ell_2+1)\ \ell_2\ /\ 2)\ \times\ (\ell_2+1)} & & \\ \vdots & & \ddots & 0 \\ & & & I_{\ell_{v_0}+1} \\ 0 & \ldots & 0 & 0_{\left(\left(\ell_{v_0}+1\right)\ \ell_{v_0}\ /\ 2\right)\ \times\ \left(\ell_{v_0}+1\right)} \end{pmatrix}.$$

Multiplying both the left and right-hand side of equation (4.10) with $\mathcal{I}_{\delta,\ r,\ v}$ results in the $q\times(r+v)$ or $(r+v)\times(r+v)$ square generalized Vandermonde matrix $\mathcal{W}_{r,\ v}(\gamma,\tau)$ on the right-hand side of the newly obtained equation according to

$$\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)\ (I_{\delta_1+\delta_2}\otimes e_q)\ \mathcal{I}_{\delta,\ r,\ v} = \mathcal{W}_{r,\ v}(\gamma,\tau) \tag{4.12}$$

so that the following proposition can be formulated

Proposition 4.4 *The equation can be verified as*
$\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)\ (I_{\delta_1+\delta_2}\otimes e_q)\ \mathcal{I}_{\delta,\ r,\ v}\ (\mathcal{W}_{r,\ v}(\gamma,\tau))^{-1} = I_{r+v}$.

Thus an appropriate right inverse is given by

$$\left(\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)\right)^{-}_{R} = (I_{\delta_1+\delta_2}\otimes e_q)\ \mathcal{I}_{\delta,\ r,\ v}\ (\mathcal{W}_{r,\ v}(\gamma,\tau))^{-1},$$

note that this result is a variant of lemma 5.3 in [12].

Since the generalized Vandermonde matrix $\mathcal{W}_{r,\, v}(\gamma,\tau)$ also appears in the right inverse of $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ expressed in the equations (4.1) and (4.2), a combination of these equations with (4.12) results in the next equality

Proposition 4.5 *An interconnection between* $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ *and* $\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)$ *has the form*

$$\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)\ (I_q \otimes e_q) = \overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)\ (I_{\delta_1+\delta_2} \otimes e_q)\ \mathcal{I}_{\delta,\, r,\, v}.$$

A generalization of equation (4.10) shall now be set forth and for that purpose the following equation is considered

$$(I_{\delta_1+\delta_2} \otimes e_q^\top)\ \overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau) = \mathcal{W}_q{}^*(\gamma,\tau) \tag{4.13}$$

where the $(\delta_1+\delta_2) \times q\ (r+v)$ block diagonal matrix $\mathcal{W}_q^*(\gamma,\tau)$ has the following structure

$$\mathcal{W}_q{}^*(\gamma,\tau) = \mathrm{diag}\left\{\mathcal{W}_q{}^*(\gamma_1)\ \mathcal{W}_q{}^*(\gamma_2)\ldots\mathcal{W}_q{}^*(\gamma_{r_0})\ \mathcal{W}_q{}^*(\tau_1)\ \mathcal{W}_q{}^*(\tau_2)\ldots\mathcal{W}_q{}^*(\tau_{v_0})\right\}.$$

The generalized Vandermonde matrices constituting $\mathcal{W}_q{}^*(\gamma,\tau)$ have a similar functional form to the right-hand side of (4.4). A right inverse of $\mathcal{W}_q{}^*(\gamma,\tau)$ can be constructed accordingly to result in the $q\ (r+v)\times\ (\delta_1+\delta_2)$ block diagonal matrix having the following form

$$\mathcal{W}^{-*}_{R,\, q}(\gamma,\tau) = \mathrm{diag}\left\{\mathcal{W}^{-*}_{R,\, q}(\gamma_1)\ \mathcal{W}^{-*}_{R,\, q}(\gamma_2)\ldots\mathcal{W}^{-*}_{R,\, q}(\gamma_{r_0})\ \mathcal{W}^{-*}_{R,\, q}(\tau_1)\ \mathcal{W}^{-*}_{R,\, q}(\tau_2)\ldots\mathcal{W}^{-*}_{R,\, q}(\tau_{v_0})\right\}$$

where $\mathcal{W}^{-*}_{R,\, q}(\gamma_1)\ \mathcal{W}^{-*}_{R,\, q}(\gamma_2)\ldots\mathcal{W}^{-*}_{R,\, q}(\gamma_{r_0})\ \mathcal{W}^{-*}_{R,\, q}(\tau_1)\ \mathcal{W}^{-*}_{R,\, q}(\tau_2)\ldots\mathcal{W}^{-*}_{R,\, q}(\tau_{v_0})$ are the right inverses of $\mathcal{W}_q{}^*(\gamma_1)\ \mathcal{W}_q{}^*(\gamma_2)\ldots\mathcal{W}_q{}^*(\gamma_{r_0})\ \mathcal{W}_q{}^*(\tau_1)\ \mathcal{W}_q{}^*(\tau_2)\ldots\mathcal{W}_q{}^*(\tau_{v_0})$ respectively and have an equivalent functional form to $\mathcal{W}^{-*}_{R,\, q}(\sigma)$ in (4.5) and (4.7), consequently

$$\mathcal{W}_q{}^*(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau) = I_{\delta_1+\delta_2}.$$

Transforming equation (4.13) accordingly results in

$$(I_{\delta_1+\delta_2} \otimes e_q^\top)\ \overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau) = I_{\delta_1+\delta_2}.$$

Combining $\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)$ and $\overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)$ as given in (4.10) and (4.13) respectively yields

$$\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)\ (I_{\delta_1+\delta_2} \otimes e_q)\ (I_{\delta_1+\delta_2} \otimes e_q^\top)\overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau) = \mathcal{W}^{\,\delta}_{r,\, v}(\gamma,\tau)$$

in virtue of equation (4.11) we obtain

$$\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)\ (I_{\delta_1+\delta_2} \otimes e_q e_q^\top)\ \overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau)\ \mathcal{I}_{\delta,\, r,\, v} = \mathcal{W}_{r,\, v}(\gamma,\tau)$$

taking the inversion of $\mathcal{W}_{r,\, v}(\gamma,\tau)$ into account leads to the next proposition.

Proposition 4.6 *The following equation holds true*

$$\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)\ (I_{\delta_1+\delta_2} \otimes e_q e_q^\top)\ \overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau)\ \mathcal{I}_{\delta,\, r,\, v}\ \left(\mathcal{W}_{r,\, v}(\gamma,\tau)\right)^{-1} = I_{r+v}.$$

A choice for a right inverse of $\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)$ is then

$$\left(\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)\right)^{-}_{R} = (I_{\delta_1+\delta_2} \otimes e_q e_q^\top)\ \overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau)\ \mathcal{I}_{\delta,\, r,\, v}\ \left(\mathcal{W}_{r,\, v}(\gamma,\tau)\right)^{-1}.$$

Exploiting some of these results gives the possibility to set forth an equality involving $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$, $\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)$ and $\overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)$ according to the next proposition.

Proposition 4.7 *An interconnection between* $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ *and the blocks* $\overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)$ *and* $\overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)$ *has the form*

$$\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)\ (I_q \otimes e_q) = \overline{\widetilde{\mathcal{M}}}{}^{(1)}_{r,\, v}(\gamma,\tau)\ (I_{\delta_1+\delta_2} \otimes e_q e_q^\top)\ \overline{\widetilde{\mathcal{M}}}{}^{(2)}_{r,\, v}(\gamma,\tau)\ \mathcal{W}^{-*}_{R,\, q}(\gamma,\tau)\ \mathcal{I}_{\delta,\, r,\, v}.$$

From this section can be seen that a right inverse necessary for interconnecting the Fisher information matrix and a corresponding Stein solution is derived. A right inverse of $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ is given, it has a structured form expressed in terms of a generalized Vandermonde matrix. Whereas for the coefficient matrix $\left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\ \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)$ two expressions for a right inverse of $\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)$ are derived as well as links involving the blocks $\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)$ and $\widetilde{\overline{\mathcal{M}}}^{(2)}_{r,\ v}(\gamma,\tau)$ (obtained after factorization of $\left(\widetilde{\overline{\mathcal{M}}}_r(\gamma)\ \widetilde{\overline{\mathcal{M}}}_v(\tau)\right)$). Equations interconnecting the coefficient matrices extracted from the Fisher information matrix and a Stein solution respectively are provided, some of these results are summarized in propositions 4.2 and 4.5 which also enable us to compute Ker $\widetilde{\overline{\mathcal{M}}}^{(1)}_{\nu}(\sigma)$ and Ker $\widetilde{\overline{\mathcal{M}}}^{(1)}_{r,\ v}(\gamma,\tau)$ next to the derivation proposed in section 3.1.2.

5. Link solution Stein equation-Fisher information: The global approach

5.1. General case

In this section an extension of previous sections is implemented by presenting interconnections where the entire Fisher information matrix, not decomposed, is taken as one block. Fisher's information matrix will be interconnected not only with the corresponding Stein solution but also with Sylvester's resultant. Since the functional form of Fisher's information matrix and the corresponding solutions to Stein's equation are similar to the structures studied for the block-(b,b), we will not derive interconnections but will present them more or less under their final forms. The same holds for the appropriate kernels which result from the solutions of the linear systems and allow the interconnections to take its present form.

In Klein and Spreij [10] the following interconnection between Fisher's information matrix of an ARMAX process and Sylvester resultant matrices is verified,

$$G(\theta) = \begin{pmatrix} -S_p(b) \\ S_q(a) \\ 0 \end{pmatrix} Q(\theta) \begin{pmatrix} -S_p(b) \\ S_q(a) \\ 0 \end{pmatrix}^{\top} + \begin{pmatrix} -S_p(c) \\ 0 \\ S_r(a) \end{pmatrix} P(\theta) \begin{pmatrix} -S_p(c) \\ 0 \\ S_r(a) \end{pmatrix}^{\top} \tag{5.1}$$

where

$$Q(\theta) = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{R_x(z) u_{p+q}(z) u^{\top}_{p+q}(z^{-1})}{a(z)a(z^{-1})c(z)c(z^{-1})} \frac{dz}{z} \tag{5.2}$$

$$P(\theta) = \frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{u_{p+r}(z) u^{\top}_{p+r}(z^{-1})}{a(z)c(z)a(z^{-1})c(z^{-1})} \frac{dz}{z} \tag{5.3}$$

and $S_p(b)$ and $S_q(a)$ are blocks of the Sylvester resultant matrices $S(-b,a)$

$$S(-b,a) = \begin{pmatrix} -S_p(b) \\ S_q(a) \end{pmatrix} \text{ and } S(-c,a) = \begin{pmatrix} -S_p(c) \\ S_r(a) \end{pmatrix}.$$

Where $S_p(b)$ is formed by the top p rows of $S(-b,a)$ and similarly for the remaining blocks. The Sylvester resultrant $S(c,-a)$ is the $(p+r)\times(p+r)$ matrix defined as

$$S(a,c) = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_p & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \ddots & \\ 0 & & 1 & a_1 & a_2 & \cdots & a_p \\ 1 & c_1 & c_2 & \cdots & c_r & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \ddots & \\ 0 & & 1 & c_1 & c_2 & \cdots & c_r \end{pmatrix}.$$

The interconnection between a Stein solution and $P(\theta)$ has been derived in [12], this will be reformulated in a new form which involves appropriate kernels and a similar representation will be set forth for $Q(\theta)$. In [12] an interconnection is constructed by means of a left inverse which involves Vandermonde matrices whereas in this paper the general solution of the linear system $AX = B$ involving Ker (A) is used. The assumption is made that the polynomials involved in the Fisher information matrix have no common roots. Application of the same approach as in section 2 results in the following structure,

$$P(\theta) = \left(\widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\ \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\right)(\Phi\otimes I_{p+r}) \tag{5.4}$$

where Φ is an appropriate equivalent form of φ in (2.19)

$$\widetilde{\widetilde{\mathcal{U}}}_p(\alpha) = \left(\ \widetilde{\widetilde{\mathcal{U}}}_{n_1}(\alpha_1),\ \ \widetilde{\widetilde{\mathcal{U}}}_{n_2}(\alpha_2),\ \ ...,\ \ \widetilde{\widetilde{\mathcal{U}}}_{n_{p_0}}(\alpha_{p_0})\ \right)$$

and

$$\widetilde{\widetilde{\mathcal{U}}}_r(\gamma) = \left(\ \widetilde{\widetilde{\mathcal{U}}}_{s_1}(\gamma_1),\ \ \widetilde{\widetilde{\mathcal{U}}}_{s_2}(\gamma_2),\ \ ...,\ \ \widetilde{\widetilde{\mathcal{U}}}_{s_{r_0}}(\gamma_{r_0})\ \right).$$

With

$$\widetilde{\widetilde{\mathcal{U}}}_{n_i}(\alpha_i) = \left(\ \widetilde{\widetilde{\mathcal{U}}}_{n_i}^{(n_i)}(z),\ \ \widetilde{\widetilde{\mathcal{U}}}_{n_i}^{(n_i-1)}(z),\ \ ...\ \ ,\widetilde{\widetilde{\mathcal{U}}}_{n_i}^{(0)}(z)\ \right)_{z=\alpha_i} \qquad i=1,\dots,p_0$$

and each block has the following form

$$\widetilde{\widetilde{\mathcal{U}}}_{n_i}^{(n_i-k)}(\alpha_i) = \left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}\left(u_{p+r}(z)u_{p+r}^{*\top}(z)\right)\right)_{z=\alpha_i} \qquad k=0,\dots,n_i.$$

Whereas

$$\widetilde{\widetilde{\mathcal{U}}}_{s_j}(\gamma_j) = \left(\ \widetilde{\widetilde{\mathcal{U}}}_{s_j}^{(s_j)}(z),\ \ \widetilde{\widetilde{\mathcal{U}}}_{s_j}^{(s_j-1)}(z),\ \ ...\ \ ,\widetilde{\widetilde{\mathcal{U}}}_{s_j}^{(0)}(z)\ \right)_{z=\gamma_j} \qquad j=1,\dots,r_0$$

and each block being

$$\widetilde{\widetilde{\mathcal{U}}}_{s_j}^{(s_j-l)}(\gamma_j) = \left(\frac{\partial^{s_j-l}}{\partial z^{s_j-l}}\left(u_{p+r}(z)u_{p+r}^{*\top}(z)\right)\right)_{z=\gamma_j} \qquad l=0,\dots,s_j.$$

A Stein equation and its corresponding solution are envisaged and for that purpose the following $(p+r)\times(p+r)$ companion matrix is introduced

$$A^P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \\ -g_{p+r} & -g_{p+r-1} & \cdots & -g_1 \end{pmatrix}$$

where the entries g_i are given by $z^{p+r} + \sum_{i=1}^{p+r} g_i z^{p+r-i} = a(z)c(z) = g(z)$. The condition for uniqueness of the solution of Stein's equation is verified. Stein's equation and its solution are respectively,

$$S^P - A^P S^P \left(A^P\right)^\top = \Gamma^P$$

$$
\begin{aligned}
S^P &= \frac{1}{2\pi i} \oint_{|z|=1} \left(zI - A^P\right)^{-1} \Gamma^P \left(I - zA^P\right)^{-\top} dz \\
&= \frac{1}{2\pi i} \oint_{|z|=1} \frac{\operatorname{adj}\left(zI - A^P\right) \Gamma^P \operatorname{adj}\left(I - zA^P\right)^{\top}}{a(z)c(z)a^*(z)c^*(z)} dz.
\end{aligned}
$$

An appropriate factorization similar to the one applied in section 2 yields

$$
S^P = \left(\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\right) (\Phi \otimes I_{p+r}) \tag{5.5}
$$

where

$$
\widetilde{\widetilde{\mathcal{M}}}_p(\alpha) = \left(\widetilde{\widetilde{\mathcal{M}}}_{n_1}(\alpha_1), \quad \widetilde{\widetilde{\mathcal{M}}}_{n_2}(\alpha_2), \quad ..., \quad \widetilde{\widetilde{\mathcal{M}}}_{n_{p_0}}(\alpha_{p_0}) \right)
$$

and

$$
\widetilde{\widetilde{\mathcal{M}}}_r(\gamma) = \left(\widetilde{\widetilde{\mathcal{M}}}_{s_1}(\gamma_1), \quad \widetilde{\widetilde{\mathcal{M}}}_{s_2}(\gamma_2), \quad ..., \quad \widetilde{\widetilde{\mathcal{M}}}_{s_{r_0}}(\gamma_{r_0}) \right).
$$

With the blocks building $\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)$ and $\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)$ given by

$$
\widetilde{\widetilde{\mathcal{M}}}_{n_i}(\alpha_i) = \left(\widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(n_i)}(z), \quad \widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(n_i-1)}(z), \quad ... \quad , \widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(0)}(z) \right)_{z=\alpha_i} \qquad i = 1, \ldots, p_0
$$

each block being

$$
\widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(n_i-j)}(\alpha_i) = \left(\frac{\partial^{n_i-j}}{\partial z^{n_i-j}} \operatorname{adj}\left(zI - A^P\right) \Gamma^P \operatorname{adj}\left(I - zA^P\right)^{\top} \right)_{z=\alpha_i} \qquad j = 0, \ldots, n_i.
$$

The second block consists of the following forms

$$
\widetilde{\widetilde{\mathcal{M}}}_{s_j}(\gamma_j) = \left(\widetilde{\widetilde{\mathcal{M}}}_{s_j}^{(s_j)}(z), \quad \widetilde{\widetilde{\mathcal{M}}}_{s_j}^{(s_j-1)}(z), \quad ... \quad , \widetilde{\widetilde{\mathcal{M}}}_{s_j}^{(0)}(z) \right)_{z=\gamma_j} \qquad j = 1, \ldots, r_0
$$

each block being

$$
\widetilde{\widetilde{\mathcal{M}}}_{s_j}^{(s_j-k)}(\gamma_j) = \left(\frac{\partial^{s_j-k}}{\partial z^{s_j-k}} \operatorname{adj}\left(zI - A^P\right) \Gamma^P \operatorname{adj}\left(I - zA^P\right)^{\top} \right)_{z=\gamma_j} \qquad k = 0, \ldots, s_j.
$$

Combining $P(\theta)$ and S^P and taking into consideration the property of full row rankness of the coefficient matrices $\left(\widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\right)$ and $\left(\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\right)$, proofs similar to the ones given in propositions 2.3 and 2.5 can be formulated for the new coefficient matrices, results in interconnections which are summarized in the following lemma

Lemma 5.1 *The following interconnections hold true*

$$
P(\theta) = \left(\widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\right) \left\{ \left(\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\right)^{+} S^P + \mathcal{V} \right\}
$$

where $\mathcal{V} \in Ker \left(\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\right)$ *is associated with the appropriate particular solution of (5.5) and likewise*

$$
S^P = \left(\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\right) \left\{ \left(\widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\right)^{+} P(\theta) + \mathcal{W} \right\}
$$

where $\mathcal{W} \in Ker \left(\widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\right)$ *is associated with the appropriate particular solution of (5.4).*

For describing the subspace Ker $\left(\widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\right)$ the algorithm developed in section 3 can be used and is limited to the case where the algebraic multiplicity of the chosen root respective eigenvalue is smaller than the length of the basis vector $u_x(z)$ for an appropriate x, this is clearly the situation here. It is a matter of fact, for the diagonal ARMA blocks of Fisher's information matrix $G(\theta)$, the matrix blocks which do not involve the input process expressed through $R_x(z)$, are such that the algebraic multiplicity of an appropriate root can never exceed the length of the corresponding basis vector $u_x(z)$. Consequently, when an interconnection between a block expressed in (2.4) or (2.9) and a corresponding Stein solution is envisaged, the nullspaces involved in the links can be described by the form given in section 3 for the cases $\nu + 1 < q$ and $\nu + 1 = q$. It is worth reminding that in [11] and [12] we showed that $P(\theta)$ satisfies a Stein equation,

$$P(\theta) - A^P P(\theta) \left(A^P\right)^\top = \Gamma^P$$

and this for $\Gamma^P = e_{p+r} e_{p+r}^\top$, where e_{p+r} is the last basis vector of $\mathbb{R}^{p+r}$.

Matrix $Q(\theta)$ has a similar form as G_{bb}, it can be rewritten as

$$Q(\theta) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{u_{p+q}(z) u_{p+q}^{*\top}(z)}{h(z)h^*(z)a(z)a^*(z)c(z)c^*(z)z^{l+1}} dz$$

where $l+1 = q-v-r$ and an efficient factorization yields

$$Q(\theta) = \left(\widetilde{\widetilde{\mathcal{U}}}_v(\tau)\, \widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{U}}}_l(0)\right) (\Omega \otimes I_{p+q}).$$

With Ω being an appropriate equivalent form of ϑ in (2.13) and (2.14) for $l+1>0$, the functional form of each term constituting the matrix blocks $\widetilde{\widetilde{\mathcal{U}}}_v(z), \widetilde{\widetilde{\mathcal{U}}}_p(z), \widetilde{\widetilde{\mathcal{U}}}_r(z)$ and $\widetilde{\widetilde{\mathcal{U}}}_l(z)$ is $\frac{\partial^j}{\partial z^j}\left(u_{p+q}(z)u_{p+q}^{*\top}(z)\right)$ for a given j.

The following $(p+r+v)\times(p+r+v)$ companion matrix is proposed for Stein's equation

$$A^Q = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \\ -f_{p+r+v} & -f_{p+r+v-1} & \cdots & -f_1 \end{pmatrix}$$

where the entries f_i are given by $z^{p+r+v} + \sum_{i=1}^{p+r+v} f_i z^{p+r+v-i} = a(z)c(z)h(z) = f(z)$.

Stein's equation and its solution are now given,

$$S^Q - A^Q S^Q \left(A^Q\right)^\top = \Gamma^Q$$

$$\begin{aligned} S^Q &= \frac{1}{2\pi i} \oint_{|z|=1} \left(zI - A^Q\right)^{-1} \Gamma^Q \left(I - zA^Q\right)^{-\top} dz \\ &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{\operatorname{adj}\left(zI - A^Q\right)\, \Gamma^Q \operatorname{adj}\left(I - zA^Q\right)^\top z^{l+1}}{h(z)a(z)c(z)h^*(z)a^*(z)c^*(z)z^{l+1}} dz. \end{aligned}$$

A factored version yields

$$S^Q = \left(\widetilde{\widetilde{\mathcal{M}}}_v(\tau)\, \widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{M}}}_l(0)\right) (\Omega \otimes I_{p+r+v})$$

where

$$\widetilde{\widetilde{\mathcal{M}}}_p(\alpha) = \left(\ \widetilde{\widetilde{\mathcal{M}}}_{n_1}(\alpha_1),\quad \widetilde{\widetilde{\mathcal{M}}}_{n_2}(\alpha_2),\quad ...,\quad \widetilde{\widetilde{\mathcal{M}}}_{n_{p_0}}(\alpha_{p_0})\ \right)$$

and each block of $\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)$ is given by

$$\widetilde{\widetilde{\mathcal{M}}}_{n_i}(\alpha_i) = \left(\ \widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(n_i)}(z),\quad \widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(n_i-1)}(z),\quad ...\quad ,\widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(0)}(z)\ \right)_{z=\alpha_i} \qquad i = 1, \ldots, p_0.$$

The blocks constituting the matrices $\widetilde{\widetilde{\mathcal{M}}}_v(z)$, $\widetilde{\widetilde{\mathcal{M}}}_r(z)$, $\widetilde{\widetilde{\mathcal{M}}}_l(z)$ and $\widetilde{\widetilde{\mathcal{M}}}_p(z)$ have the same functional form and it is illustrated for the blocks which form $\widetilde{\widetilde{\mathcal{M}}}_p(\alpha)$,

$$\widetilde{\widetilde{\mathcal{M}}}_{n_i}^{(n_i-j)}(\alpha_i) = \left(\frac{\partial^{n_i-j}}{\partial z^{n_i-j}} \operatorname{adj}\left(zI - A^Q\right)\, \Gamma^Q \operatorname{adj}\left(I - zA^Q\right)^\top z^{l+1}\right)_{z=\alpha_i} \qquad j = 0, \ldots, n_i.$$

A particular solution of the appropriate linear systems can be written as

$$(\Omega \otimes I_{p+q}) = \left(\widetilde{\widetilde{\mathcal{U}}}_v(\tau)\, \widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{U}}}_l(0)\right)^+ Q(\theta) + \mathcal{F} \tag{5.6}$$

$$\mathcal{F} \in \operatorname{Ker}\left(\widetilde{\widetilde{\mathcal{U}}}_v(\tau)\, \widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{U}}}_l(0)\right)$$

and

$$(\Omega \otimes I_{p+r+v}) = \left(\widetilde{\widetilde{\mathcal{M}}}_v(\tau)\, \widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{M}}}_l(0)\right)^+ S^Q + \mathcal{G} \tag{5.7}$$

$$\mathcal{G} \in \operatorname{Ker}\left(\widetilde{\widetilde{\mathcal{M}}}_v(\tau)\, \widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\, \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\, \widetilde{\widetilde{\mathcal{M}}}_l(0)\right).$$

The case $l+1>0$ is considered, it implies $p+q>p+r+v$ and results in an equation which involves $Q(\theta)$ and S^Q, this is summarized in the next lemma.

Lemma 5.2 *The following holds true for $l+1>0$*

$$\mathcal{R}_{(p+q)(p+r+v+l+1)}\left\{\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\,\tilde{\tilde{\mathcal{U}}}_l(0)\right)^+ Q(\theta)+\mathcal{F}\right\}\mathcal{R}_{p+q}=$$

$$\begin{pmatrix} \mathcal{R}_{(p+r+v)(p+r+v+l+1)}\left\{\left(\tilde{\tilde{\mathcal{M}}}_v(\tau)\,\tilde{\tilde{\mathcal{M}}}_p(\alpha)\,\tilde{\tilde{\mathcal{M}}}_r(\gamma)\,\tilde{\tilde{\mathcal{M}}}_l(0)\right)^+ S^Q+\mathcal{G}\right\}\mathcal{R}_{p+r+v} & 0 \\ 0 & I_{q-(r+v)}\otimes\Omega \end{pmatrix}.$$

where $\mathcal{F}\in$ Ker $\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\,\tilde{\tilde{\mathcal{U}}}_l(0)\right)$ is such that equation (5.6) holds and likewise $\mathcal{G}\in$ Ker $\left(\tilde{\tilde{\mathcal{M}}}_v(\tau)\,\tilde{\tilde{\mathcal{M}}}_p(\alpha)\,\tilde{\tilde{\mathcal{M}}}_r(\gamma)\,\tilde{\tilde{\mathcal{M}}}_l(0)\right)$ is such that equation (5.7) holds and $\mathcal{R}$ is a permutation matrix.

When $l+1=0$ or $p+r+v=p+q$ we have the following form for $Q(\theta)$ and S^Q

$$Q(\theta)=\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\right)(\Theta\otimes I_{p+q}) \tag{5.8}$$

and

$$S^Q=\left(\tilde{\tilde{\tilde{\mathcal{M}}}}_v(\tau)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_p(\alpha)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_r(\gamma)\right)(\Theta\otimes I_{p+q}) \tag{5.9}$$

where Θ is the appropriate equivalent form of φ in (2.19). The functional form of all the terms which build the blocks $\tilde{\tilde{\tilde{\mathcal{M}}}}_r(z)$, $\tilde{\tilde{\tilde{\mathcal{M}}}}_v(z)$ and $\tilde{\tilde{\tilde{\mathcal{M}}}}_p(z)$ is $\frac{\partial^j}{\partial z^j}\mathrm{adj}\big(zI-A^Q\big)\,\Gamma^Q\,\mathrm{adj}\big(I-zA^Q\big)^\top$ for a given j. Equivalently with propositions 2.3 and 2.5 it can be shown that

$$\dim\,\mathrm{Im}\,\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\right)=p+q$$

and

$$\dim\,\mathrm{Im}\,\left(\tilde{\tilde{\tilde{\mathcal{M}}}}_v(\tau)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_p(\alpha)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_r(\gamma)\right)=p+q,$$

so that full row rankness of the appropriate matrices is guaranteed. Interconnections between the Fisher information matrix and a corresponding solution to Stein's equation can then be summarized in the following lemma.

Lemma 5.3 *For $l+1=0$ the following equations are verified*

$$S^Q=\left(\tilde{\tilde{\tilde{\mathcal{M}}}}_v(\tau)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_p(\alpha)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_r(\gamma)\right)\left\{\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\right)^+ Q(\theta)+\mathcal{E}\right\}$$

where $\mathcal{E}\in$ Ker $\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\right)$ is associated with the appropriate particular solution of equation (5.8)

and

$$Q(\theta)=\left(\tilde{\tilde{\mathcal{U}}}_v(\tau)\,\tilde{\tilde{\mathcal{U}}}_p(\alpha)\,\tilde{\tilde{\mathcal{U}}}_r(\gamma)\right)\left\{\left(\tilde{\tilde{\tilde{\mathcal{M}}}}_v(\tau)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_p(\alpha)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_r(\gamma)\right)^+ S^Q+\mathcal{D}\right\}$$

with $\mathcal{D}\in$ Ker $\left(\tilde{\tilde{\tilde{\mathcal{M}}}}_v(\tau)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_p(\alpha)\,\tilde{\tilde{\tilde{\mathcal{M}}}}_r(\gamma)\right)$ being associated with the appropriate particular solution of equation (5.9).

It can be seen that for $l+1=0$ $Q(\theta)$ verifies a Stein equation given by

$$Q(\theta)-A^Q Q(\theta)\left(A^Q\right)^\top=\Gamma^Q$$

where $\Gamma^Q=e_{p+q}e_{p+q}^\top$ and e_{p+q} is the last standard basis vector in $\mathbb{R}^{p+q}$.

The case $l+1<0$ results in the equations

$$Q(\theta) = \left(\tilde{\tilde{\overline{\mathcal{U}}}}_v(\tau)\, \tilde{\tilde{\overline{\mathcal{U}}}}_p(\alpha)\, \tilde{\tilde{\overline{\mathcal{U}}}}_r(\gamma)\right)(\Theta \otimes I_{p+q})$$

and

$$S^Q = \left(\widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_v(\tau)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_p(\alpha)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_r(\gamma)\right)(\Theta \otimes I_{p+r+v}).$$

The functional form of all the terms of $\tilde{\tilde{\overline{\mathcal{U}}}}_v(z)$, $\tilde{\tilde{\overline{\mathcal{U}}}}_p(z)$ and $\tilde{\tilde{\overline{\mathcal{U}}}}_r(z)$ is $\frac{\partial^j}{\partial z^j}\left(u_{p+q}(z)u^{*\top}_{p+q}(z)z^{l+1}\right)$ for some j and a particular solution of the linear systems yields

$$(\Theta \otimes I_{p+q}) = \left(\tilde{\tilde{\overline{\mathcal{U}}}}_v(\tau)\, \tilde{\tilde{\overline{\mathcal{U}}}}_p(\alpha)\, \tilde{\tilde{\overline{\mathcal{U}}}}_r(\gamma)\right)^+ Q(\theta) + \mathcal{X} \tag{5.10}$$

$$\mathcal{X} \in \operatorname{Ker}\left(\tilde{\tilde{\overline{\mathcal{U}}}}_v(\tau)\, \tilde{\tilde{\overline{\mathcal{U}}}}_p(\alpha)\, \tilde{\tilde{\overline{\mathcal{U}}}}_r(\gamma)\right)$$

and

$$(\Theta \otimes I_{p+r+v}) = \left(\widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_v(\tau)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_p(\alpha)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_r(\gamma)\right)^+ S^Q + \mathcal{Y} \tag{5.11}$$

$$\mathcal{Y} \in \operatorname{Ker}\left(\widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_v(\tau)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_p(\alpha)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_r(\gamma)\right).$$

An equation which involves S^Q and $Q(\theta)$ is given in the next lemma.

Lemma 5.4 *The case $l+1<0$ results in the following equality*

$$\mathcal{R}_{(p+r+v)(p+r+v)}\left\{\left(\widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_v(\tau)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_p(\alpha)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_r(\gamma)\right)^+ S^Q + \mathcal{Y}\right\}\mathcal{R}_{p+r+v} =$$

$$\begin{pmatrix} \mathcal{R}_{(p+q)(p+r+v)}\left\{\left(\tilde{\tilde{\overline{\mathcal{U}}}}_v(\tau)\, \tilde{\tilde{\overline{\mathcal{U}}}}_p(\alpha)\, \tilde{\tilde{\overline{\mathcal{U}}}}_r(\gamma)\right)^+ Q(\theta) + \mathcal{X}\right\}\mathcal{R}_{p+q} & 0 \\ 0 & I_{r+v-q} \otimes \Theta \end{pmatrix}$$

where $\mathcal{Y} \in$ Ker $\left(\widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_v(\tau)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_p(\alpha)\, \widetilde{\widetilde{\widetilde{\overline{\mathcal{M}}}}}_r(\gamma)\right)$ is such that equation (5.11) holds and likewise $\mathcal{X} \in$ Ker $\left(\tilde{\tilde{\overline{\mathcal{U}}}}_v(\tau)\, \tilde{\tilde{\overline{\mathcal{U}}}}_p(\alpha)\, \tilde{\tilde{\overline{\mathcal{U}}}}_r(\gamma)\right)$ is such that (5.10) holds and $\mathcal{R}$ is a permutation matrix.

By inserting $P(\theta)$ and $Q(\theta)$, given in lemma 5.1 and lemma 5.3 respectively, in equation (5.1) results in an equality where the global Fisher information matrix $G(\theta)$ is expressed in terms of Sylvester resultant matrices and the solutions to Stein equations given by S^P and S^Q.

The kernels involved in all the interconnections summarized in this section can be evaluated according to the results derived in section 3.

5.2. Special case

As in section 2.2 we present the coefficient matrices related with the Fisher information matrix and a corresponding Stein solution for the case of distinct roots. The equations involving the Fisher information and Stein solution, are such that their structure can be found in the previous section but with appropriate coefficient matrices. These equations will therefore not be provided and we limit ourselves to the presentation of the coefficient matrices obtained in the present case.

A factorized form of $P(\theta)$ also formulated in [12] but not exploited for an interconnection between Fisher's information matrix and a corresponding Stein solution is

$$P(\theta) = \left(\widetilde{\mathcal{U}}_p(\alpha)\, \widetilde{\mathcal{U}}_r(\gamma)\right)(\Lambda \otimes I_{p+r})$$

with

$$\widetilde{\mathcal{U}}_p(\alpha) = \begin{pmatrix} \widetilde{\mathcal{U}}_1(\alpha_1), & \widetilde{\mathcal{U}}_2(\alpha_2), & ..., & \widetilde{\mathcal{U}}_p(\alpha_p) \end{pmatrix}$$
$$\widetilde{\mathcal{U}}_r(\gamma) = \begin{pmatrix} \widetilde{\mathcal{U}}_1(\gamma_1), & \widetilde{\mathcal{U}}_2(\gamma_2), & ..., & \widetilde{\mathcal{U}}_r(\gamma_r) \end{pmatrix}.$$

The blocks are $\widetilde{\mathcal{U}}_i(\alpha_i) = ((u_{p+r}(z)u^{*\top}_{p+r}(z))_{z=\alpha_i}$ for $i = 1, 2, \dots, p$ and $\widetilde{\mathcal{U}}_j(\gamma_j) = ((u_{p+r}(z)u^{*\top}_{p+r}(z))_{z=\gamma_j}$ for $j = 1, 2, \dots, r$, and Λ is an appropriate equivalent form of $\overline{\varphi}$ in (2.31). A Stein solution can now be formulated as

$$S^P = \left(\widetilde{\mathcal{M}}_p(\alpha)\ \widetilde{\mathcal{M}}_r(\gamma)\right)(\Lambda \otimes I_{p+r})$$

where

$$\widetilde{\mathcal{M}}_p(\alpha) = \begin{pmatrix} \widetilde{\mathcal{M}}_1(\alpha_1), & \widetilde{\mathcal{M}}_2(\alpha_2), & ..., & \widetilde{\mathcal{M}}_p(\alpha_p) \end{pmatrix}$$

and

$$\widetilde{\mathcal{M}}_r(\gamma) = \begin{pmatrix} \widetilde{\mathcal{M}}_1(\gamma_1), & \widetilde{\mathcal{M}}_2(\gamma_2), & ..., & \widetilde{\mathcal{M}}_r(\gamma_r) \end{pmatrix}.$$

Each block being

$$\widetilde{\mathcal{M}}_i(\alpha_i) = \left(\mathrm{adj}\left(zI - A^P\right)\ \Gamma^P\ \mathrm{adj}\left(I - zA^P\right)^\top\right)_{z=\alpha_i}$$

and

$$\widetilde{\mathcal{M}}_j(\gamma_j) = \left(\mathrm{adj}\left(zI - A^P\right)\ \Gamma^P\ \mathrm{adj}\left(I - zA^P\right)^\top\right)_{z=\gamma_j}.$$

Full row rankness or surjectiveness of the coefficient matrices $\left(\widetilde{\mathcal{U}}_p(\alpha)\ \widetilde{\mathcal{U}}_r(\gamma)\right)$ and $\left(\widetilde{\mathcal{M}}_p(\alpha)\ \widetilde{\mathcal{M}}_r(\gamma)\right)$ can be proved similarly to the block case.

Note that in [12] a right inverse of $\left(\widetilde{\mathcal{U}}_p(\alpha)\ \widetilde{\mathcal{U}}_r(\gamma)\right)$ has been set forth (the distinct root equivalent of (4.2)), according to

$$\left(\widetilde{\mathcal{U}}_p(\alpha)\ \widetilde{\mathcal{U}}_r(\gamma)\right)(V^{-1}_{\alpha\gamma} \otimes e_{p+r}) = I_{p+r}$$

with e_{p+r} being the last standard basis vector in $\mathbb{R}^{p+r}$ and $V_{\alpha\gamma}$ is the following Vandermonde matrix

$$V_{\alpha\gamma} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \vdots & \alpha_p & \gamma_1 & \gamma_2 & \vdots & \gamma_r \\ \alpha_1^2 & \alpha_2^2 & \vdots & \alpha_p^2 & \gamma_1^2 & \gamma_2^2 & \vdots & \gamma_r^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{p+r-1} & \alpha_2^{p+r-1} & \cdots & \alpha_p^{p+r-1} & \gamma_1^{p+r-1} & \gamma_2^{p+r-1} & \cdots & \gamma_r^{p+r-1} \end{pmatrix}.$$

We proceed with formulating an equality involving $Q(\theta)$ and S^Q. A factored form yields

$$Q(\theta) = \left(\widetilde{\widetilde{\mathcal{U}}}_v(\tau)\ \widetilde{\widetilde{\mathcal{U}}}_p(\alpha)\ \widetilde{\widetilde{\mathcal{U}}}_r(\gamma)\ \overline{\overline{\mathcal{U}}}_l(0)\right)(\Upsilon \otimes I_{p+q})$$

where Υ is an appropriate equivalent of $\overline{\vartheta}$ in (2.29) for $l + 1 > 0$ and the functional form of each term building the matrix blocks $\widetilde{\widetilde{\mathcal{U}}}_v(z), \widetilde{\widetilde{\mathcal{U}}}_p(z)$ and $\widetilde{\widetilde{\mathcal{U}}}_r(z)$ is $\left(u_{p+q}(z)u^{*\top}_{p+q}(z)\right)$, whereas the blocks constituting $\overline{\overline{\mathcal{U}}}_l(z)$ have the same functional form as in the general case namely $\frac{\partial^k}{\partial z^k}\left(u_{p+q}(z)u^{*\top}_{p+q}(z)\right)$ for a given k. A factored version of a suitable Stein solution is

$$S^Q = \left(\widetilde{\widetilde{\mathcal{M}}}_v(\tau)\ \widetilde{\widetilde{\mathcal{M}}}_p(\alpha)\ \widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \overline{\overline{\mathcal{M}}}_l(0)\right)(\Upsilon \otimes I_{p+r+v}),$$

where each block of $\overline{\overline{\mathcal{M}}}_l(z)$ has the same functional form as in the general case, $\dfrac{\partial^j}{\partial z^j}\left(\mathrm{adj}\ \left(zI - A^Q\right)\Gamma^Q\mathrm{adj}\ \left(I - zA^Q\right)^\top z^{l+1}\right)$ for a given j, whereas the block elements of $\widetilde{\widetilde{\mathcal{M}}}_v(z)$, $\widetilde{\widetilde{\mathcal{M}}}_p(z)$ and $\widetilde{\widetilde{\mathcal{M}}}_r(z)$ have the functional form given by $\left(\mathrm{adj}\ \left(zI - A^Q\right)\Gamma^Q\mathrm{adj}\ \left(I - zA^Q\right)^\top z^{l+1}\right)$.

When $l + 1 = 0$ the following linear systems hold true

$$Q(\theta) = \left(\tilde{\tilde{\mathcal{U}}}_v(\tau) \; \tilde{\tilde{\mathcal{U}}}_p(\alpha) \; \tilde{\tilde{\mathcal{U}}}_r(\gamma) \right) (\Xi \otimes I_{p+q})$$

and

$$S^Q = \left(\tilde{\tilde{\mathcal{N}}}_v(\tau) \; \tilde{\tilde{\mathcal{N}}}_p(\alpha) \; \tilde{\tilde{\mathcal{N}}}_r(\gamma) \right) (\Xi \otimes I_{p+q})$$

where Ξ is the appropriate equivalent form of $\overline{\varphi}$ in (2.31). The functional form of all the terms which build the blocks $\tilde{\tilde{\mathcal{N}}}_r(z)$, $\tilde{\tilde{\mathcal{N}}}_v(z)$ and $\tilde{\tilde{\mathcal{N}}}_p(z)$ is $\left(\text{adj } \left(zI - A^Q\right) \Gamma^Q \text{adj } \left(I - zA^Q\right)^\top\right)$. As in the block case the property of full row rankness can be shown for the matrices $\left(\tilde{\tilde{\mathcal{N}}}_v(\tau) \; \tilde{\tilde{\mathcal{N}}}_p(\alpha) \; \tilde{\tilde{\mathcal{N}}}_r(\gamma) \right)$ and $\left(\tilde{\tilde{\mathcal{U}}}_v(\tau) \; \tilde{\tilde{\mathcal{U}}}_p(\alpha) \; \tilde{\tilde{\mathcal{U}}}_r(\gamma) \right)$.

When $l + 1 < 0$ the following forms are verified

$$Q(\theta) = \left(\widetilde{\widetilde{\mathcal{W}}}_v(\tau) \; \widetilde{\widetilde{\mathcal{W}}}_p(\alpha) \; \widetilde{\widetilde{\mathcal{W}}}_r(\gamma) \right) (\Xi \otimes I_{p+q})$$

and

$$S^Q = \left(\tilde{\tilde{\mathcal{N}}}_v(\tau) \; \tilde{\tilde{\mathcal{N}}}_p(\alpha) \; \tilde{\tilde{\mathcal{N}}}_r(\gamma) \right) (\Xi \otimes I_{p+r+v}).$$

The functional form of the terms building $\widetilde{\widetilde{\mathcal{W}}}_v(z)$, $\widetilde{\widetilde{\mathcal{W}}}_p(z)$ and $\widetilde{\widetilde{\mathcal{W}}}_r(z)$ is $\left(u_{p+q}(z) \; u^{*\top}_{p+q}(z) z^{l+1}\right)$.

The kernels involved in the interconnections which are introduced in this section can be deduced according to the development set forth in section 3.

From section 2 through section 5 can be concluded that under appropriate conditions the solution of a Stein equation can be expressed in terms of the Fisher information matrix and vice versa. In (5.1) $P(\theta)$ and $Q(\theta)$ can be replaced by equations which are expressed by corresponding solutions of Stein equations so the Fisher information matrix $G(\theta)$ is explained by these solutions as well as by Sylvester resultants (in [10] it is also shown through equation (5.1) that the Fisher information matrix has resultant properties). This enables us to evaluate numerically a Stein solution since an algorithm of the Fisher information matrix of a SISO process (which is a generalized form of the ARMAX process considered in this paper), is developed in [13]. By further taking the kernels described in section 3 into account as well as some appropriate right inverses of corresponding coefficient matrices set forth in section 4 should allow a numerical approach to be satisfactory. This can be a subject for further study.

6. Example

In this section the $G_{bb}(\theta)$ block is illustrated for $p = q = 3$, $r = 2$ and $v = 1$. The polynomial basis vectors and appropriate polynomials involved are $u_3(z) = (1, z, z^2)^\top$, $u_3^*(z) = (z^2, z, 1)^\top$, $c(z) = (z - \gamma)^2$ and $h(z) = (z - \tau)$ so the Fisher information matrix block $G_{bb}(\theta)$ admits the following form

$$G_{bb}(\theta) = \frac{1}{2\pi i} \oint_{|z|=1} \begin{pmatrix} z^2 & z & 1 \\ z^3 & z^2 & z \\ z^4 & z^3 & z^2 \end{pmatrix} \frac{dz}{(z-\tau)(z-\gamma)^2 (1 - z\,\tau)\,(1 - z\,\gamma)^2}$$

the components of the Toeplitz and symmetric matrix $G_{bb}(\theta)$ are given by means of Cauchy's formula according to

$$\frac{\tau^j}{(\tau-\gamma)^2 (1 - \tau^2)\,(1 - \tau\,\gamma)^2} + \left(\frac{\partial}{\partial z} \frac{z^j}{(z-\tau)(1 - z\,\tau)\,(1 - z\,\gamma)^2} \right)_{z=\gamma}$$ for $j = 0, 1, 2, 3, 4$, to obtain

$$G_{bb}(\theta) = \frac{1}{(\gamma^2-1)^3\ (\gamma\ \tau-1)^2\ (\tau^2-1)} \begin{pmatrix} G_{bb}^{11}(\theta) & G_{bb}^{12}(\theta) & G_{bb}^{13}(\theta) \\ G_{bb}^{21}(\theta) & G_{bb}^{22}(\theta) & G_{bb}^{23}(\theta) \\ G_{bb}^{31}(\theta) & G_{bb}^{32}(\theta) & G_{bb}^{33}(\theta) \end{pmatrix}$$

$G_{bb}^{11}(\theta) = G_{bb}^{22}(\theta) = G_{bb}^{33}(\theta) = 1 + 2\ \gamma\ \tau - 2\ \gamma^3\ \tau - \gamma^4\ \tau^2 - \gamma^2\ (\tau^2 - 1)$

$G_{bb}^{12}(\theta) = G_{bb}^{23}(\theta) = G_{bb}^{21}(\theta) = G_{bb}^{32}(\theta) = 2\ \gamma + \tau - \gamma^4\ \tau - 2\ \gamma^3\ \tau^2$

$G_{bb}^{13}(\theta) = G_{bb}^{31}(\theta) = -\gamma^4 + 2\ \gamma\ \tau - 2\ \gamma^3\ \tau + \tau^2 - 3\ \gamma^2\ (\tau^2 - 1).$

The matrix $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ has the following structure

$$\overline{\mathcal{U}}_r(\gamma) = \left(\frac{\partial}{\partial z}\begin{pmatrix} z^2 & z & 1 \\ z^3 & z^2 & z \\ z^4 & z^3 & z^2 \end{pmatrix}, \begin{pmatrix} z^2 & z & 1 \\ z^3 & z^2 & z \\ z^4 & z^3 & z^2 \end{pmatrix}\right)_{z=\gamma} \quad \text{and } \overline{\mathcal{U}}_v(\tau) = \begin{pmatrix} z^2 & z & 1 \\ z^3 & z^2 & z \\ z^4 & z^3 & z^2 \end{pmatrix}_{z=\tau}.$$

An appropriate choice for a right inverse of $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ is

$$\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)_R^- = \left((\mathcal{W}_{r,\ v}(\gamma,\tau))^{-1} \otimes e_3\right)$$

where

$$e_3 = (0,0,1)^\top \quad \text{and} \quad (\mathcal{W}_{r,\ v}(\gamma,\tau)) = \left\{\left(\frac{\partial}{\partial z}u_3(z)\ u_3(z)\right)_{z=\gamma} (u_3(z))_{z=\tau}\right\} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & \gamma & \tau \\ 2\ \gamma & \gamma^2 & \tau^2 \end{pmatrix}.$$

We have $(\mathcal{W}_{r,\ v}(\gamma,\tau))^{-1} = \frac{1}{(\gamma-\tau)^2}\begin{pmatrix} \gamma\ \tau\ (\gamma-\tau) & -\left(\gamma^2-\tau^2\right) & (\gamma-\tau) \\ (\gamma-\tau)^2\ -\gamma^2 & 2\ \gamma & -1 \\ \gamma^2 & -2\ \gamma & 1 \end{pmatrix}$

so that

$$\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)_R^- = \frac{1}{(\gamma-\tau)^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma\ \tau\ (\gamma-\tau) & -\left(\gamma^2-\tau^2\right) & (\gamma-\tau) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\gamma-\tau)^2\ -\gamma^2 & 2\ \gamma & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma^2 & -2\ \gamma & 1 \end{pmatrix}.$$

Consider the subspace Ker $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right) = $ Ker $\overline{\mathcal{U}}_r(\gamma) \oplus$ Ker $\overline{\mathcal{U}}_v(\tau)$ where

$$\text{Ker } \overline{\mathcal{U}}_v(\tau) = \begin{pmatrix} -z^2\ u_2^\top(z) \\ J_2 \end{pmatrix}_{z=\tau} = \begin{pmatrix} -z^{-2} & -z^{-1} \\ 0 & 1 \\ 1 & 0 \end{pmatrix}_{z=\tau}.$$

The following parameters which are of paramount importance for constructing the null space Ker $\overline{\mathcal{U}}_r(\gamma)$ are $n = k = 1$ and $p = q = 3$ (conforming the notations used in section 3). This results in the following equations which belong to the subspace Ker $\overline{\mathcal{U}}_r(\gamma)$, to obtain

$$x_0 = K_3(z)\ \gamma_0$$
$$x_1 = -l_3\ v_3^{\cdot}(z)\ K_3(z)\ \gamma_0 + K_3(z)\ \gamma_1$$

or

$$x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} I_3 & 0 \\ -l_3 v_3^{\cdot}(z) & I_3 \end{pmatrix} (I_2 \otimes K_3(z)) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}$$

where

$K_3(z) = \begin{pmatrix} -1 & 0 \\ z & -1 \\ 0 & z \end{pmatrix}$, $v_3(z) = (z^2, z, 1)$, $l_3 = (0,0,1)^\top$ and arbitrary γ_0 and γ_1. Let us denote the components of γ_0 and γ_1 by $\left(\gamma_0^1\ \gamma_0^2\right)^\top$ and $\left(\gamma_1^1\ \gamma_1^2\right)^\top$ respectively, so that the vector belonging to the subspace Ker $\overline{\mathcal{U}}_r(\gamma)$ can be expressed as

$$x = \begin{pmatrix} -1 & 0 & 0 & 0 \\ z & -1 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & z & -1 \\ z & 1 & 0 & z \end{pmatrix}_{z=\gamma} \begin{pmatrix} \gamma_0^1 \\ \gamma_0^2 \\ \gamma_1^1 \\ \gamma_1^2 \end{pmatrix} = \begin{pmatrix} -\gamma_0^1 \\ \gamma_0^1\ z - \gamma_0^2 \\ \gamma_0^2\ z \\ -\gamma_1^1 \\ \gamma_1^1\ z - \gamma_1^2 \\ \left(\gamma_0^1 + \gamma_1^2\right)\ z + \gamma_0^2 \end{pmatrix}_{z=\gamma}.$$

A form for Ker $\overline{\mathcal{U}}_v(\tau)$ is given in section 3.2 and from section 3.1.1 and section 3.2 can be concluded that dim Ker $\overline{\mathcal{U}}_r(\gamma) = 4$ and dim Ker $\overline{\mathcal{U}}_v(\tau) = 2$, consequently dim Ker $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right) = 6$. It is then clear that the matrix $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ is surjective since dim Im $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right) = 3$, a confirmation of proposition 2.3.

Since γ_0 and γ_1 are arbitrary we take for example $\gamma_0 = (1,1)^\top$ and $\gamma_1 = (2,3)^\top$ so that a choice for a 9×3 matrix $\mathcal{Q}$ such that $\mathcal{Q} \in$ Ker $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right))$ can be expressed as

$$\mathcal{Q} = \begin{pmatrix} -1 & 0 & 0 \\ \gamma - 1 & 0 & 0 \\ \gamma & 0 & 0 \\ -2 & 0 & 0 \\ 2\,\gamma - 3 & 0 & 0 \\ 4\,\gamma + 1 & 0 & 0 \\ 0 & -\tau^{-2} & -\tau^{-1} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

However, as specified in the corresponding corollaries and lemmas which set forth interconnections between the Fisher information matrix and a solution to Stein's equation, the appropriate matrix to be chosen is such that it is associated with the particular solution common to both linear systems (2.19) and (2.20). This results in $(\varphi \otimes I_q)$ being the common solution of both systems (2.19) and (2.20). Consequently the choice of the matrix, name it $\mathcal{A}$, contained in the subspace Ker $\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)$ can then be evaluated according to

$$\mathcal{A} = (\varphi \otimes I_q) - \left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)^+ G_{bb}(\theta).$$

This yields the following matrices which are derived according to the notations used in (2.14) for $l + 1 = 0$, to obtain

$$(\varphi \otimes I_3) =$$

$$\begin{pmatrix} \frac{1}{(1-\gamma^2)^2(\gamma-\tau)(1-\gamma\tau)} & 0 & 0 \\ 0 & \frac{1}{(1-\gamma^2)^2(\gamma-\tau)(1-\gamma\tau)} & 0 \\ 0 & 0 & \frac{1}{(1-\gamma^2)^2(\gamma-\tau)(1-\gamma\tau)} \\ \frac{1+4\gamma^3\tau+\tau^2-3\gamma^2(1+\tau^2)}{(-1+\gamma^2)^3(-\gamma+\tau)^2(-1+\gamma\tau)^2} & 0 & 0 \\ 0 & \frac{1+4\gamma^3\tau+\tau^2-3\gamma^2(1+\tau^2)}{(-1+\gamma^2)^3(-\gamma+\tau)^2(-1+\gamma\tau)^2} & 0 \\ 0 & 0 & \frac{1+4\gamma^3\tau+\tau^2-3\gamma^2(1+\tau^2)}{(-1+\gamma^2)^3(-\gamma+\tau)^2(-1+\gamma\tau)^2} \\ \frac{1}{(-\gamma+\tau)^2(1-\tau^2)(1-\gamma\tau)^2} & 0 & 0 \\ 0 & \frac{1}{(-\gamma+\tau)^2(1-\tau^2)(1-\gamma\tau)^2} & 0 \\ 0 & 0 & \frac{1}{(-\gamma+\tau)^2(1-\tau^2)(1-\gamma\tau)^2} \end{pmatrix}$$

and

$$\left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\right)^+ G_{bb}(\theta) =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\gamma^2}{(-1+\gamma^2)^2(\gamma-\tau)(-1+\gamma\tau)} & -\frac{\gamma}{(-1+\gamma^2)^2(\gamma-\tau)(-1+\gamma\tau)} & -\frac{1}{(-1+\gamma^2)^2(\gamma-\tau)(-1+\gamma\tau)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\gamma(2\tau+2\gamma^4\tau-\gamma(1+\tau^2)-\gamma^3(1+\tau^2))}{(-1+\gamma^2)^3(\gamma-\tau)^2(-1+\gamma\tau)^2} & \frac{\tau+3\gamma^4\tau-2\gamma^3(1+\tau^2)}{(-1+\gamma^2)^3(\gamma-\tau)^2(-1+\gamma\tau)^2} & \frac{1+4\gamma^3\tau+\tau^2-3\gamma^2(1+\tau^2)}{(-1+\gamma^2)^3(\gamma-\tau)^2(-1+\gamma\tau)^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\tau^2}{(\gamma-\tau)^2(-1+\gamma\tau)^2(-1+\tau^2)} & -\frac{\tau}{(\gamma-\tau)^2(-1+\gamma\tau)^2(-1+\tau^2)} & -\frac{1}{(\gamma-\tau)^2(-1+\gamma\tau)^2(-1+\tau^2)} \end{pmatrix}$$

with $\left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\right)^-_R$ as an appropriate choice for $\left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\right)^+$.
The desired matrix $\mathcal{A}$ is then

$$\mathcal{A} = \frac{1}{(-1+\gamma^2)^3(\gamma-\tau)^2(-1+\gamma\tau)^2(-1+\tau^2)} \times$$

$$\begin{pmatrix} -(-1+\gamma^2)(\gamma-\tau)(-1+\gamma\tau)(-1+\tau^2) & 0 & 0 \\ 0 & -(-1+\gamma^2)(\gamma-\tau)(-1+\gamma\tau)(-1+\tau^2) & 0 \\ \gamma^2(-1+\gamma^2)(\gamma-\tau)(-1+\gamma\tau)(-1+\tau^2) & \gamma(-1+\gamma^2)(\gamma-\tau)(-1+\gamma\tau)(-1+\tau^2) & 0 \\ (-1+\tau^2)(1+4\gamma^3\tau+\tau^2-3\gamma^2(1+\tau^2)) & 0 & 0 \\ 0 & (-1+\tau^2)(1+4\gamma^3\tau+\tau^2-3\gamma^2(1+\tau^2)) & 0 \\ \gamma(-1+\tau^2)(-2\tau-2\gamma^4\tau+\gamma(1+\tau^2)+\gamma^3(1+\tau^2)) & -(-1+\tau^2)(\tau+3\gamma^4\tau-2\gamma^3(1+\tau^2)) & 0 \\ -(-1+\gamma^2)^3 & 0 & 0 \\ 0 & -(-1+\gamma^2)^3 & 0 \\ \tau^2(-1+\gamma^2)^3 & \tau(-1+\gamma^2)^3 & 0 \end{pmatrix},$$

the property $\mathcal{A} \in \mathrm{Ker}\,\left(\overline{\mathcal{U}}_r(\gamma)\,\overline{\mathcal{U}}_v(\tau)\right)$ holds.
The companion matrix used in the corresponding Stein equation is

$$C_h = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -e_3 & -e_2 & -e_1 \end{pmatrix} \text{ where } e_1 = -(2\,\gamma + \tau),\ e_2 = \gamma\,(\gamma + 2\,\tau) \text{ and } e_3 = -\tau\,\gamma^2.$$

With $r = 2$ and $v = 1$ equation 4.3 becomes

$$\left(\widetilde{\overline{\mathcal{M}}}_2(\gamma)\ \widetilde{\overline{\mathcal{M}}}_1(\tau)\right) = \widetilde{\overline{\mathcal{M}}}^{(1)}_{2,\,1}(\gamma,\tau)\ \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{2,\,1}\ \widetilde{\overline{\mathcal{M}}}^{(2)}_{2,\,1}(\gamma,\tau)$$

the $3{\times}12$ block $\widetilde{\overline{\mathcal{M}}}^{(1)}_{2,\,1}(\gamma,\tau) = \left(\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 1}(\gamma)\ \ \widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 0}(\tau)\right)$ with $\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 1}(\gamma) = \left(\widetilde{\overline{\mathcal{M}}}^{(1)\ (1)}_{\ 1}(z)\ \ \widetilde{\overline{\mathcal{M}}}^{(0)\ (1)}_{\ 1}(z)\right)_{z=\gamma}$,

the blocks constituting $\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 1}(\gamma)$ are

$\widetilde{\overline{\mathcal{M}}}^{(1)\ (1)}_{\ 1}(\gamma) = \left(\dfrac{\partial}{\partial z}\text{adj }(zI - C_h)\ \text{adj }(zI - C_h)\right)_{z=\gamma}$, $\widetilde{\overline{\mathcal{M}}}^{(0)\ (1)}_{\ 1}(\gamma) = (\text{adj }(zI - C_h))_{z=\gamma}$ respectively and $\widetilde{\overline{\mathcal{M}}}^{\ (1)}_{\ 0}(\tau) = (\text{adj }(zI - C_h))_{z=\tau}$. The appropriate adjoint matrices are

$$\text{adj }(zI - C_h) = \begin{pmatrix} z^2 + e_1\,z + e_2 & z + e_1 & 1 \\ -e_3 & z^2 + e_1\,z & z \\ -e_3\,z & -e_2\,z - e_3 & z^2 \end{pmatrix}$$

$$\text{adj }(I - zC_h) = \begin{pmatrix} 1 + e_1\,z + e_2\,z^2 & z + e_1\,z^2 & z^2 \\ -e_3\,z^2 & 1 + e_1\,z & z \\ -e_3\,z & -e_2\,z - e_3\,z^2 & 1 \end{pmatrix}$$

with the coefficients e_1, e_2 and e_3 of the companion matrix specified above. We then obtain

$$\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 1}(\gamma) = \begin{pmatrix} -\tau & 1 & 0 & \tau\gamma & -\gamma-\tau & 1 & \tau\gamma & -\gamma-\tau & 1 \\ 0 & -\tau & 1 & \tau\gamma^2 & -\gamma^2-\tau\gamma & \gamma & \tau\gamma^2 & -\gamma^2-\tau\gamma & \gamma \\ \tau\gamma^2 & -\gamma^2-2\tau\gamma & 2\gamma & \tau\gamma^3 & -\gamma^3-\tau\gamma^2 & \gamma^2 & \tau\gamma^3 & -\gamma^3-\tau\gamma^2 & \gamma^2 \end{pmatrix}$$

and

$$\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 0}(\tau) = \begin{pmatrix} \gamma^2 & -2\gamma & 1 \\ \tau\gamma^2 & -2\tau\gamma & \tau \\ \tau^2\gamma^2 & -2\tau^2\gamma & \tau^2 \end{pmatrix}.$$

The approach outlined in section 3.1.2 is illustrated for describing the kernel of $\widetilde{\overline{\mathcal{M}}}^{(1)}_{2,\,1}(\gamma,\tau)$. The vector $x = \left(x_0^\top\ x_1^\top\right)^\top$ is belonging to the kernel of $\overline{\mathcal{U}}_r(\gamma)$ and its derived structure is described above, for typographical clarity we use the following notations $\chi_0 = \left(\chi_0^1\ \chi_0^2\right)^\top$ and $\chi_1 = \left(\chi_1^1\ \chi_1^2\right)^\top$ for specifying the arbitrary terms of the vector x. To successfully derive the kernel of $\widetilde{\overline{\mathcal{M}}}^{(1)}_{2,\,1}(\gamma,\tau)$, we first apply the algorithm as proposed in section 3.1.2 to $\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 1}(\gamma)$ since the eigenvalue-root γ has an algebraic multiplicity equal to 2. This is followed by a kernel description of $\widetilde{\overline{\mathcal{M}}}^{(1)}_{\ 0}(\tau)$. We start from the sum $\sum_{j=0}^{l} z_{j,l-j} = x_l$ for $l = 0, 1, \ldots, \nu$ with algebraic multiplicity $\nu + 1 = 2$ to obtain $x_0 = z_{\,0,0}$ and $x_1 = z_{\,0,1} + z_{\,1,0}$. Taking into consideration the result obtained for the vector x yields

$$z_{\,0,0} = \begin{pmatrix} -\chi_0^1 \\ \chi_0^1\,z - \chi_0^2 \\ \chi_0^2\,z \end{pmatrix}_{z=\gamma} \quad \text{and } z_{\,0,1} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}$$

with the elements ζ_1, ζ_2 and ζ_3 being arbitrary, consequently $z_{\,1,0}$ is

$$z_{\,1,0} = \begin{pmatrix} \chi_1^1 - \zeta_1 \\ \chi_1^1\,z - \chi_1^2 - \zeta_2 \\ \left(\chi_0^1 + \chi_1^2\right)\,z + \chi_0^2 - \zeta_3 \end{pmatrix}_{z=\gamma}.$$

The vector $z = \left(z_0^\top\ z_1^\top\right)^\top$ is constituted according to $z_0 = \left(z^\top_{\,0,0}\ z^\top_{\,0,1}\right)^\top$ and $z_1 = z_{\,1,0}$. The desired vector $y = \begin{pmatrix} y_0^\top & y_1^\top \end{pmatrix}^\top$ with the property $y \in \text{Ker}\ \widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)$ can now be set forth according to the following equations

$$y_0 = (I_2 \otimes S(c))^{-1}\ z_0 \text{ and } y_1 = (I_1 \otimes S(c))^{-1}\ z_1$$

where the symmetrizer $S(c)$ introduced in (2.24) is for the case under study given by the following matrix

$$S(c) = \begin{pmatrix} 1 & 0 & 0 \\ e_1 & 1 & 0 \\ e_2 & e_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2\gamma - \tau & 1 & 0 \\ \gamma^2 + 2\gamma\tau & -2\gamma - \tau & 1 \end{pmatrix}.$$

The coefficients e_1, e_2 and e_3 of the Hörner polynomial are associated with the characteristic polynomial of the companion matrix C_h. A general form for the vector y such that $y \in \text{Ker}\ \widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)$ can now be presented as

$$y = \begin{pmatrix} -\chi_0^1 \\ -\chi_0^2 + \chi_0^1\ \gamma - \chi_0^1\ (2\gamma + \tau) \\ \chi_0^2\ \gamma + (\chi_0^1\ \gamma - \chi_0^2)(2\gamma + \tau) - \chi_0^1(3\gamma^2 + 2\gamma\tau + \tau^2) \\ \zeta_1 \\ \zeta_2 + \zeta_1(2\gamma + \tau) \\ \zeta_3 + \zeta_2(2\gamma + \tau) + \zeta_1(3\gamma^2 + 2\gamma\tau + \tau^2) \\ -\chi_1^1 - \zeta_1 \\ -\chi_1^2 + \chi_1^1\ \gamma - \zeta_2 - (\chi_1^1 + \zeta_1)(2\gamma + \tau) \\ \chi_0^2 + \chi_0^1\ \gamma + \chi_1^2\ \gamma - \zeta_3 + (\chi_1^1\ \gamma - \chi_1^2 - \zeta_2)(2\gamma + \tau) - (\chi_1^1 + \zeta_1)(3\gamma^2 + 2\gamma\tau + \tau^2) \end{pmatrix}.$$

As can be seen from section 3.1.2 that for the case under study dim Ker $\widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma) = 7$. Consequently a choice of a matrix $\mathcal{B}$ such that $\mathcal{B} \in \text{Ker}\left(\widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau)\right)$ with $\text{Ker}\left(\widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau)\right) = \text{Ker}\ \widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma) \oplus \text{Ker}\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau)$, the latter holds since $\text{Im}\ \widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)\ \cap\ \text{Im}\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau) = \{0\}$ for $\gamma \neq \tau$, can now be introduced. However, note that the subspace $\text{Ker}\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau) = \text{Ker adj}\ (zI - C_h)_{z=\tau}$ can be evaluated according to lemma 3.4 and corollary 3.5 and its dimension is also provided, to get dim Ker adj $(zI - C_h)_{z=\tau} = 2$, to obtain for the following chosen values, $\chi_0^1 = 1$, $\chi_0^2 = 2$, $\chi_1^1 = 3$, $\chi_1^2 = 5$, $\zeta_1 = 7$, $\zeta_2 = -4$ and $\zeta_3 = 10$ the next form of the matrix $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} -1 & 0 & 0 \\ -2 - \gamma - \tau & 0 & 0 \\ -2\gamma - \gamma^2 - 2\tau - \gamma\tau - \tau^2 & 0 & 0 \\ 7 & 0 & 0 \\ -4 + 14\gamma + 7\tau & 0 & 0 \\ 10 - 8\gamma + 21\gamma^2 - 4\tau + 14\gamma\tau + 7\tau^2 & 0 & 0 \\ -10 & 0 & 0 \\ -1 - 17\gamma - 10\tau & 0 & 0 \\ -8 + 4\gamma - 24\gamma^2 - \tau - 17\gamma\tau - 10\tau^2 & 0 & 0 \\ 0 & 2/\gamma & -1/\gamma^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear now that $\text{dim Ker}\left(\widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau)\right) = 9$ so that $\text{dim Im}\left(\widetilde{\widetilde{\mathcal{M}}}_1^{(1)}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_0^{(1)}(\tau)\right) = 3$, consequently the surjectiveness of the matrix $\left(\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{1}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{0}(\tau)\right)$ is confirmed, a related result is formulated in propositions 4.4 and 4.6 where a right inverse of $\left(\widetilde{\widetilde{\mathcal{M}}}^{(1)}_{1}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}^{(1)}_{0}(\tau)\right)$ is given.

The 12×12 matrix $\widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{2,\ 1}$ is $\widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{2,\ 1} = \text{diag}\ \{\ \Gamma, \Gamma, \Gamma, \Gamma\}$ whereas the 12×9 matrix $\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{2,\ 1}(\gamma, \tau)$ has the form described as $\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{2,\ 1}(\gamma, \tau) = \text{diag}\left\{\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{1}(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}^{(2)}_{0}(\tau)\right\}$ with

$$\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{1}(\gamma) = \begin{pmatrix} \widetilde{\widetilde{\mathcal{M}}}^{(1)\ (2)}(z) & 0 \\ 0 & \widetilde{\widetilde{\mathcal{M}}}^{(0)\ (2)}(z) \end{pmatrix}_{z=\gamma} \qquad \widetilde{\widetilde{\mathcal{M}}}^{(0)\ (2)}(\tau) = \widetilde{\widetilde{\mathcal{M}}}^{(2)}_{0}(\tau) = \left(\ \text{adj}\ (I - zC_h)^{\top}\right)_{z=\tau}$$

$$\widetilde{\widetilde{\mathcal{M}}}^{(1)\ (2)}(\gamma) = \begin{pmatrix} \text{adj}\ (I - zC_h)^{\top} \\ \dfrac{\partial}{\partial z}\text{adj}\ (I - zC_h)^{\top} \end{pmatrix}_{z=\gamma} \qquad \widetilde{\widetilde{\mathcal{M}}}^{(0)\ (2)}(\gamma) = \left(\ \text{adj}\ (I - zC_h)^{\top}\right)_{z=\gamma}$$

, an explicit form is

$$\widetilde{\widetilde{\mathcal{M}}}^{(1)\ (2)}(\gamma) = \begin{pmatrix} 1 - 2\gamma^2 + \gamma^4 - \gamma\tau + 2\gamma^3\tau & \gamma^4\tau & \gamma^3\tau \\ \gamma - 2\gamma^3 - \gamma^2\tau & 1 - 2\gamma^2 - \gamma\tau & -\gamma^3 - 2\gamma^2\tau + \gamma^4\tau \\ \gamma^2 & \gamma & 1 \\ -2\gamma + 2\gamma^3 - \tau + 4\gamma^2\tau & 2\gamma^3\tau & \gamma^2\tau \\ 1 - 4\gamma^2 - 2\gamma\tau & -2\gamma - \tau & -\gamma^2 - 2\gamma\tau + 2\gamma^3\tau \\ 2\gamma & 1 & 0 \end{pmatrix}$$

and

$$\widetilde{\widetilde{\mathcal{M}}}^{(0)\ (2)}(\gamma) = \begin{pmatrix} 1 - 2\gamma^2 + \gamma^4 - \gamma\tau + 2\gamma^3\tau & \gamma^4\tau & \gamma^3\tau \\ \gamma - 2\gamma^3 - \gamma^2\tau & 1 - 2\gamma^2 - \gamma\tau & -\gamma^3 - 2\gamma^2\tau + \gamma^4\tau \\ \gamma^2 & \gamma & 1 \end{pmatrix}$$

$$\widetilde{\widetilde{\mathcal{M}}}^{(2)}_{0}(\tau) = \begin{pmatrix} 1 - 2\gamma\tau - \tau^2 + \gamma^2\tau^2 + 2\gamma\tau^3 & \gamma^2\tau^3 & \gamma^2\tau^2 \\ \tau - 2\gamma\tau^2 - \tau^3 & 1 - 2\gamma\tau - \tau^2 & -\gamma^2\tau - 2\gamma\tau^2 + \gamma^2\tau^3 \\ \tau^2 & \tau & 1 \end{pmatrix}.$$

For this example one chooses the matrix $\Gamma = I_3$ as the identity matrix, when Γ is constituted of arbitrary elements Γ_{ij}, the results become too long to be presented.

Inserting the matrices $\left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right), \left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)^{+} = \left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)^{-}_{R}$, $G_{bb}(\theta)$ and $\mathcal{A}$, with $r = 2$ and $v = 1$, in the equation

$$S_{bb} = \left(\widetilde{\widetilde{\mathcal{M}}}_r(\gamma)\ \widetilde{\widetilde{\mathcal{M}}}_v(\tau)\right)\left\{\left(\overline{\mathcal{U}}_r(\gamma)\ \overline{\mathcal{U}}_v(\tau)\right)^{+} G_{bb}(\theta) + \mathcal{A}\right\}$$

gives the following solution to the Stein equation when it is expressed in terms of the Fisher information matrix, to obtain

$$S_{bb} = \frac{1}{(\gamma^2 - 1)^3(\gamma - \tau)^2(\tau^2 - 1)(-1 + \gamma\tau)^2} \begin{pmatrix} S^{11}_{bb} & S^{12}_{bb} & S^{13}_{bb} \\ S^{21}_{bb} & S^{22}_{bb} & S^{23}_{bb} \\ S^{31}_{bb} & S^{32}_{bb} & S^{33}_{bb} \end{pmatrix}$$

with

$S^{11}_{bb} = -(\gamma - \tau)^2(-3 + 5\gamma^2 - 7\gamma^4 + \gamma^6 + 2\gamma\tau - 14\gamma^3\tau + 10\gamma^5\tau - 2\gamma^7\tau + 2\tau^2 - 11\gamma^2\tau^2 + 9\gamma^4\tau^2 - \gamma^6\tau^2 + \gamma^8\tau^2 - 4\gamma\tau^3 + 8\gamma^3\tau^3 - 4\gamma^5\tau^3 + 4\gamma^7\tau^3 + 2\gamma^2\tau^4 + 4\gamma^6\tau^4)$

$S^{12}_{bb} = -(\gamma - \tau)^2(-2\gamma - 2\gamma^5 - \tau - 7\gamma^4\tau + 4\gamma^6\tau - 8\gamma^3\tau^2 + 8\gamma^5\tau^2 - 4\gamma^2\tau^3 + 7\gamma^4\tau^3 + \gamma^8\tau^3 + 2\gamma^3\tau^4 + 2\gamma^7\tau^4)$

$S^{13}_{bb} = -(-2\gamma^4 - 2\gamma^6 + 4\gamma^3\tau - 2\gamma^5\tau + 2\gamma^9\tau - 3\gamma^2\tau^2 + 6\gamma^4\tau^2 + 6\gamma^6\tau^2 - 2\gamma^8\tau^2 + \gamma^{10}\tau^2 + 2\gamma\tau^3 -$

$4\gamma^3\tau^3 - 4\gamma^7\tau^3 - 2\gamma^9\tau^3 - \tau^4 + 2\gamma^2\tau^4 - 8\gamma^6\tau^4 + 3\gamma^8\tau^4 - 2\gamma\tau^5 + 2\gamma^5\tau^5 + 4\gamma^7\tau^5 + 4\gamma^4\tau^6 - 4\gamma^6\tau^6)$

$S_{bb}^{21} = -(\gamma-\tau)^2(-2\gamma-2\gamma^5-\tau-7\gamma^4\tau+4\gamma^6\tau-8\gamma^3\tau^2+8\gamma^5\tau^2-4\gamma^2\tau^3+7\gamma^4\tau^3+\gamma^8\tau^3+2\gamma^3\tau^4+2\gamma^7\tau^4)$

$S_{bb}^{22} = -(\gamma - \tau)^2(-2 + 2\gamma^2 - 4\gamma^4 - 8\gamma^3\tau + 4\gamma^5\tau + \tau^2 - 7\gamma^2\tau^2 + 3\gamma^4\tau^2 + 3\gamma^6\tau^2 - 2\gamma\tau^3 + 2\gamma^3\tau^3 +$

$2\gamma^5\tau^3 + 2\gamma^7\tau^3 + \gamma^2\tau^4 + \gamma^4\tau^4 + \gamma^6\tau^4 + \gamma^8\tau^4)$

$S_{bb}^{23} = -(-2\gamma^3 - 2\gamma^7 + 3\gamma^2\tau - 3\gamma^6\tau + 4\gamma^8\tau + 2\gamma^5\tau^2 + 6\gamma^7\tau^2 - \tau^3 + 5\gamma^4\tau^3 - 5\gamma^6\tau^3 - 8\gamma^8\tau^3 + \gamma^{10}\tau^3 -$
$4\gamma^5\tau^4 - 4\gamma^2\tau^5 + 3\gamma^4\tau^5 + 5\gamma^8\tau^5 + 2\gamma^3\tau^6 + 2\gamma^5\tau^6 - 4\gamma^7\tau^6)$

$S_{bb}^{31} = -(\gamma-\tau)^2(-2\gamma^2-2\gamma^4-6\gamma^3\tau+2\gamma^7\tau-\tau^2-2\gamma^2\tau^2+2\gamma^6\tau^2+\gamma^8\tau^2-2\gamma\tau^3+6\gamma^5\tau^3+2\gamma^4\tau^4+2\gamma^6\tau^4)$
$S_{bb}^{32} = S_{bb}^{12}$

$S_{bb}^{33} = -(-\gamma^2 - \gamma^4 - \gamma^6 - \gamma^8 + 2\gamma\tau + 2\gamma^9\tau - \tau^2 + 3\gamma^2\tau^2 - 2\gamma^6\tau^2 + 9\gamma^8\tau^2 - \gamma^{10}\tau^2 - 2\gamma\tau^3 + 4\gamma^3\tau^3 -$
$4\gamma^7\tau^3 - 6\gamma^9\tau^3 - 3\gamma^2\tau^4 + 5\gamma^4\tau^4 - 5\gamma^6\tau^4 - 3\gamma^8\tau^4 + 2\gamma^{10}\tau^4 - 4\gamma^3\tau^5 + 4\gamma^7\tau^5 + 4\gamma^9\tau^5 + 4\gamma^4\tau^6 - 4\gamma^8\tau^6)$.

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