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# Some results on Vandermonde matrices with an application to time series analysis

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# Some results on Vandermonde matrices with an application to time series analysis

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## Abstract

In this paper we study Stein equations where the coefficient matrices are in companion form. Solutions to such equations are relatively easy to compute as soon as one knows how to invert a Vandermonde matrix (in the generic case where all eigenvalues have multiplicity one) or a confluent Vandermonde matrix (in the general case). As an application we present a way to compute the Fisher information matrix of an ARMA process. The computation is based on the fact that this matrix can be decomposed into blocks where each block satisfies a certain Stein equation.

*Keywords* ARMA process, Fisher information matrix, Stein's equation, Vandermonde matrix, confluent Vandermonde matrix

*AMS classification* 15A09, 15A24, 62F10, 62M10, 65F05, 65F10, 93B25

# 1 Introduction

In this paper we investigate some properties of (confluent) Vandermonde and some related matrices aimed at and motivated by application of them to a problem in time series analysis. Specifically we show how to apply results on these matrices to obtain a simpler representation of the (asymptotic) Fisher information matrix of an ARMA process. The Fisher information matrix prominently features in the asymptotic analysis of estimators and in asymptotic testing theory, e.g. in the classical Cramer-Rao bound on the variance of unbiased estimators. See [10] for general results and for time series models, see for example [2]. However, the Fisher information matrix has also attracted considerable attention in the signal processing literature, e.g. [6], [19] and [12]. We have previously shown, see [14], that the Fisher information matrix of an ARMA process is the solution of a so-called Lyapunov equation. Let us be a little bit more precise, although we don't go into details on ARMA processes until Section 5. The Fisher information matrix in this case can be decomposed into blocks that are solutions of equations like

$$X + MXN^T = R.$$

The coefficients  $M$  and  $N$  in this equation turn out to be in companion form in the given context of time series analysis and the right hand side  $R$  is another given matrix.

The plan of attack that we follow to solve such an equation is to break up the solution procedure into a number of steps that are each relatively easy to perform. First we replace by a basis transformation the coefficient matrices with their Jordan forms, thereby also changing the variable matrix  $X$  and the right hand side  $R$ . Since a basis of (generalized) eigenvectors of companion matrices can be represented as the columns of a (confluent) Vandermonde matrix, the basis transformation needed for this can be expressed in terms of the just mentioned Vandermonde matrices. Performing the basis transformation requires knowledge on how to compute inverses of confluent Vandermonde matrices. One of the aims of our paper is to derive rather simple, but explicit, representations for these inverses. Of course this whole procedure would be meaningless if the equation in the new coordinate system is more complex than the original one. In Section 4 we will see that fortunately the resulting equation is much easier to solve than the original one, especially in a generic case, where the solution becomes almost trivial. By applying the developed procedure to the computation of the Fisher information matrix for an ARMA process, we reach our goal of giving an alternative way to represent this Fisher information matrix. This application motivates also from a statistical perspective the interest of analyzing (confluent) Vandermonde matrices.

The remainder of the paper is organized as follows. In Section 2 we introduce the basic notations that we use throughout the paper. Section 3 is devoted to technical results on companion matrices and confluent Vandermonde matrices, the main results concerning inversion of confluent Vandermonde matrices. In Section 4 we apply these results to describe solutions to Stein equations where

the coefficient matrices are in companion form. Finally in Section 5 we investigate the special case where the solutions to certain Stein equations are given by blocks of the Fisher information matrix of an ARMA process.

## 2 Notation and preliminaries

Consider the matrix  $A \in \mathbb{R}^{n \times n}$  in the following companion form.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -a_n & & & -a_2 & -a_1 \end{pmatrix} \quad (1)$$

Let  $a^\top = (a_1, \dots, a_n)$ ,  $u(z)^\top = (1, z, \dots, z^{n-1})$  and  $u^*(z)^\top = (z^{n-1}, \dots, 1)$  (where  $\top$  denotes transposition). Define recursively the Hörner polynomials  $a_k(\cdot)$  by  $a_0(z) = 1$  and  $a_k(z) = za_{k-1}(z) + a_k$ . Notice that  $a_n(z)$  is the characteristic polynomial of  $A$ . We will denote it by  $\pi(z)$ , occasionally by  $\pi_A(z)$  if we want to emphasize the role of the  $A$ -matrix.

Write  $a(z)$  for the  $n$ -vector  $(a_0(z), \dots, a_{n-1}(z))^\top$ . Furthermore  $S$  will denote the shift matrix, so  $S_{ij} = \delta_{i,j+1}$  and  $P$  the *backward* or *antidiagonal* identity matrix, so  $P_{ij} = \delta_{i+j,n+1}$  (assuming that  $P \in \mathbb{R}^{n \times n}$ ). As an example we have  $Pu(z) = u^*(z)$ . The matrix  $P$  has the following property: if  $M$  is a Toeplitz matrix, then  $PMP = M^\top$ , in particular  $P^2 = I$ , the identity matrix.

We associate with the vector  $a$  the matrix  $T_a \in \mathbb{R}^{n \times n}$  given by

$$T_a = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \\ a_{n-1} & \cdots & a_1 & 1 \end{pmatrix}.$$

Notice that the matrices  $T_a$  and  $S$  commute and that  $a(z) = T_a u(z)$ .

Denoting the  $k$ -th basis vector in  $\mathbb{R}^n$  by  $e_k$ , we can write

$$A = -e_n a^\top P + S^\top. \quad (2)$$

If  $q(\cdot)$  is a polynomial and if for some natural number  $k$  the term  $(z - \alpha)^k$  is a factor of  $q(z)$  (which happens if  $\alpha$  is a zero of  $q(\cdot)$  with multiplicity greater than or equal to  $k$ ), then we define the polynomial  $q_k(\cdot; \alpha)$  by  $q_k(z; \alpha) = \frac{q(z)}{(z - \alpha)^k}$ .

Notice the following identity:  $q_k(\alpha; \alpha) = q_k^{(k)}(\alpha)/k!$ . In the sequel we will often use  $D$  for differentiation (w.r.t.  $z$ ). For instance, instead of  $\frac{d}{dz} q_k(z; \alpha)$  we then write  $Dq_k(z; \alpha)$  and  $Dq_k(z; \alpha)$  in  $z = \alpha$  is denoted by  $Dq_k(\alpha; \alpha)$ . Notice also the formula

$$\pi(z) - \pi(\alpha) = (z - \alpha)u^*(z)^\top a(\alpha), \quad (3)$$

which follows from the definition of the Hörner polynomials by a direct computation.

We also need some results on Lagrange and Hermite interpolation problems. Assume to be given  $s$  pairwise different complex numbers  $\alpha_1, \dots, \alpha_s$  (so  $\alpha_i \neq \alpha_j$  iff  $i \neq j$ ) and we want to find  $n$  polynomials  $p_1, \dots, p_n$  of degree at most  $n-1$  such that the  $p_j(\alpha_i)$  take on certain given values. Notice that we have  $n^2$  unknown parameters to determine, but only  $ns$  conditions. Therefore we add constraints by prescribing certain values of the derivatives  $p_j^{(k)}(\alpha_i)$  for  $k = 1, \dots, m_i - 1$ , where the  $m_i$  are such that  $\sum_{i=1}^s m_i = n$ . In this way we obtain  $n^2$  constraints. The total set of prescribed values of the polynomials  $p_j$  and their derivatives that we consider are given by the following equations:

$$\frac{p_j^{(k-1)}(\alpha_i)}{(k-1)!} = \delta_{\sum_{l=1}^{i-1} m_l + k, j},$$

where  $j = 1, \dots, n$ ,  $i = 1, \dots, s$ ,  $k = 1, \dots, m_i$  and  $\delta$  denotes the Kronecker symbol. Notice that in the case where  $s = n$  all  $m_i$  are equal to 1, and we only require  $p_j(\alpha_i) = \delta_{ij}$ .

In order to give the solution to this interpolation problem an elegant form we present the conditions as described below. We need some notation. First we denote by  $p(z)$  the column vector  $(p_1(z), \dots, p_n(z))^T$ . For each  $i$  we denote by  $\Pi(i)$  the  $n \times m_i$  matrix with columns  $\Pi(i)_k = \frac{p^{(k-1)}(\alpha_i)}{(k-1)!}$ , with  $k = 1, \dots, m_i$ . The constraints are now given in compact form by the equality  $(\Pi(1), \dots, \Pi(s)) = I$ , where  $I$  is the  $n \times n$  identity matrix.

Write  $\pi(z) = \prod_{i=1}^s (z - \alpha_i)^{m_i} = \sum_{j=0}^n a_j z^{n-j}$  and let  $A$  be the associated companion matrix of equation (1), so that  $\pi$  is its characteristic polynomial. Let  $U_i(z)$  be the  $n \times m_i$  matrix with  $k$ -th column equal to  $\frac{1}{(k-1)!} u^{(k-1)}(z)$  and write  $U_i = U_i(\alpha_i)$ . We define the  $n \times n$  matrix  $V$  (often called the *confluent* Vandermonde matrix associated with the eigenvalues of  $A$ ) by  $V = (U_1, \dots, U_s)$ . Similar interpolation problems involving one polynomial only are known to have a unique solution, see e.g. [17, p. 306] or [5, p. 37]. Here the situation is similar and as an almost straightforward result from the current setup we have

**Proposition 2.1** *The unique solution to the interpolation problem is  $p(z) = V^{-1}u(z)$ .*

Write  $p^*(z) = z^{n-1}p(\frac{1}{z})$  and notice that we use multiplication with the same power of  $z$  for all entries of  $p(\frac{1}{z})$ . Let  $\Pi^*$  be defined by  $\Pi^* = V^{-1}PV$ . Then the matrix  $\Pi^*$  is involutive, i.e.  $(\Pi^*)^2 = I$ .

**Proposition 2.2** *The polynomials  $p$  and  $p^*$  are related by*

$$p^*(z) = V^{-1}PVp(z) = \Pi^*p(z). \quad (4)$$

*In particular,  $p^*(0) = V^{-1}e_n$ .*

**Proof.** This follows from

$$p^*(z) = z^{n-1}V^{-1}u\left(\frac{1}{z}\right) = V^{-1}Pu(z) = V^{-1}PVp(z).$$

### 3 Confluent Vandermonde matrices

The main topic of this section is to give some formulas for the inverse of a confluent Vandermonde matrix. We need some auxiliary results. First we give an expression for  $\text{adj}(z - A)$ , where  $A$  is a companion matrix of the form (2). The next proposition is an alternative to formula (31) on [7, page 84].

**Proposition 3.1** *Let  $A$  be a companion matrix with  $\pi$  as its characteristic polynomial. The following equation holds true.*

$$\text{adj}(z - A) = u(z)a(z)^\top P - \pi(z) \sum_{j=0}^{n-1} z^j S^{j+1}. \quad (5)$$

**Proof.** First we show that

$$a(z)^\top P(z - A) = \pi(z)e_1^\top. \quad (6)$$

Using (2), we have

$$\begin{aligned} a(z)^\top P(z - A) &= a(z)^\top P(z - S^\top + e_n a^\top P) \\ &= a(z)^\top (z - S + P e_n a^\top) P \\ &= (\pi(z)e_n^\top - a^\top + a(z)^\top P e_n a^\top) P \\ &= \pi(z)e_n^\top P, \end{aligned}$$

which gives (6). Multiply the right hand side of (5) by  $(z - A)$ . First we consider  $a(z)^\top P(z - A)$ . In view of (6), this is just

$$\pi(z)e_1^\top. \quad (7)$$

Then we consider  $\sum_{j=0}^{n-1} z^j S^{j+1}(z - A) = \sum_{j=0}^{n-1} z^{j+1} S^{j+1} + \sum_{j=0}^{n-1} z^j S^{j+1}(-S^\top + e_n a^\top P)$ . Since  $S e_n = 0$ , this reduces to  $\sum_{j=0}^{n-1} z^{j+1} S^{j+1} - \sum_{j=0}^{n-1} z^j S^{j+1} S^\top$ . Use now the equality  $SS^\top = I - e_1 e_1^\top$  to rewrite this as  $\sum_{j=0}^{n-1} z^j S^j (zS - I + e_1 e_1^\top)$ , which equals  $\sum_{j=0}^{n-1} z^j S^j (zS - I) + \sum_{j=0}^{n-1} z^j e_{j+1} e_1^\top$ . But this is equal to  $-I + u(z)e_1^\top$ , because the first summation is just  $-I$  and the latter one equals  $u(z)e_1^\top$ . Hence

$$\sum_{j=0}^{n-1} z^j S^{j+1}(z - A) = -I + u(z)e_1^\top. \quad (8)$$

So we obtain from equations (7) and (8) that the right hand side of (5) multiplied by  $z - A$  is equal to

$$u(z)\pi(z)e_1^\top + \pi(z)(I - u(z)e_1^\top),$$

which is  $\pi(z)I$ , precisely what we have to prove.  $\square$

For the application to time series that we have in mind, as explained in the introduction, we need the inverse of a (confluent) Vandermonde matrix. Especially in the 1970's this was a popular topic and many papers appeared on the subject. Quite often attention has been paid to the finding of efficient procedures to carry out the inversion numerically. Recently, there has been a renewed interest to a related subject, the inversion of Cauchy-Vandermonde matrices. These appear in rational interpolation problems and are beyond the scope of the present paper.

Further down we provide inversion formulae for confluent Vandermonde matrices. Some of these can be found in the older literature, but the derivation below is different. Of the many possible references we mention [11], [4] that give results for the relatively simple case of a genuine Vandermonde matrix, or in the spirit of our Proposition 3.3 (but obtained by different methods) for a confluent Vandermonde matrix and [20] who has elementwise expressions. Related results of a different nature can be found in e.g. [9], [3] or [18].

We need the Jordan decomposition of  $A$ . We use the notation  $S_{m_i}$  to denote the shift matrix of size  $m_i \times m_i$ . Recall that the confluent Vandermonde matrix as we defined is such that the columns are independent eigenvectors of  $A$ . The Jordan form of  $A$  is determined by the relation  $V^{-1}AV = J_A$  and  $J_A$  is block diagonal with  $i$ -th block given by  $\alpha_i I_{m_i} + S_{m_i}^\top$ . As a first step towards expressions for the inverse of a Vandermonde matrix we will use

**Proposition 3.2** *Let  $J_A$  be the Jordan form of the companion matrix  $A$ . Then*

$$\text{adj}(z - J_A) = p(z)a(z)^\top PV - \pi(z)V^{-1} \sum_{j=0}^{n-1} z^j S^{j+1}V. \quad (9)$$

*In particular*

$$\text{adj}(\alpha_k - J_A) = \pi(\alpha_k)a(\alpha_k)^\top PV. \quad (10)$$

**Proof.** This follows from Propositions 3.1 and 2.1.  $\square$

Next we proceed with some results of general nature. Let  $M$  be the block diagonal matrix with  $s$  blocks  $M(i)$  of size  $m_i \times m_i$  specified by

$$M(i)_{kl} = \begin{cases} \frac{1}{(k+l-m_i-1)!} D^{k+l-m_i-1} \pi_{m_i}(\alpha_i; \alpha_i) & \text{if } k+l-m_i-1 \geq 0 \\ 0 & \text{else.} \end{cases} \quad (11)$$

Notice that the  $M(i)$  are symmetric Hankel matrices and that the  $M(i)_{kl}$  are zero for  $k+l \leq m_i$ . We have for the matrices  $M(i)$  the following alternative expression:

$$M(i) = \sum_{l=0}^{m_i-1} \delta_l S^l P, \text{ where } \delta_l = \frac{1}{l!} D^l \pi_{m_i}(\alpha_i; \alpha_i).$$



Here we denoted by  $S$  the  $m_i \times m_i$  shift matrix and by  $P$  the  $m_i \times m_i$  backward identity matrix.

The computation of the inverse of an  $M(i)$  is simple because of its triangular structure and the fact that it is Hankel. Indeed, it is sufficient to know the first row of  $M(i)^{-1}$ , call it  $r_1$ , since all rows  $r_j$  are of the form  $r_1 S^{j-1}$ . As a matter of fact the inverses of the matrices  $M(i)$  have a particular simple structure. To enlighten this we introduce for a given  $m - 1$  times continuously differentiable real function  $f$  the matrix valued function  $L^f(z)$  of size  $m \times m$  be defined by

$$L_{kl}^f(z) = \begin{cases} \frac{1}{(k-l)!} D^{k-l} f(z) & \text{if } k \geq l \\ 0 & \text{else.} \end{cases}$$

Notice that the matrices  $L^f(z)$  are lower triangular and Toeplitz. One readily verifies that  $(L^f(z))^{-1} = L^{\frac{1}{f}}(z)$  in the points  $z$  where  $f$  doesn't vanish. In particular, the last row of  $(L^f(z))^{-1}$  is given by

$$\left( \frac{1}{f(z)}, \dots, \frac{1}{(m-1)!} D^{m-1} \left( \frac{1}{f(z)} \right) \right) P,$$

where  $P$  is as above of size  $m \times m$ .

Now we apply this result to  $f(z) = \pi_{m_i}(z; \alpha_i)$  and  $m = m_i$  to get the inverse of  $M(i)$ . We then have for this choice of  $f$  that  $M(i) = L^f(\alpha_i)P$ . The first row of  $M(i)^{-1}$  is then seen to be

$$\left( \frac{1}{\pi_{m_i}(\alpha_i; \alpha_i)}, \dots, \frac{1}{(m_i-1)!} D^{m_i-1} \left( \frac{1}{\pi_{m_i}(\alpha_i; \alpha_i)} \right) \right) P. \quad (12)$$

Next we define a matrix  $N$ , consisting of blocks  $N(ij)$  of size  $m_i \times m_j$ . To do so we need some additional notation. We write  $\pi^*(z) = z^n \pi(\frac{1}{z})$  and  $\pi_k^*(z; \alpha) = z^{n-1} \pi_k(\frac{1}{z}; \alpha)$ . Then we define the entries of the  $N(ij)$  by

$$N(ij)_{kl} = \frac{1}{(k-1)!} D^{k-1} \pi_l^*(\alpha_i; \alpha_j).$$

Unfortunately, the matrix  $N$  doesn't share the nice properties (block diagonal, block Hankel, block symmetric) with the matrix  $M$  above.

**Proposition 3.3** *The following equalities hold.*

$$u^*(z)^\top T_a = a(z)^\top P \quad (13)$$

$$u^*(z)^\top T_a V = (\pi_1(z; \alpha_1), \dots, \pi_{m_1}(z; \alpha_1), \dots, \pi_1(z; \alpha_s), \dots, \pi_{m_s}(z; \alpha_s)) \quad (14)$$

$$V^\top P T_a V = M \quad (15)$$

$$u(z)^\top T_a V = (\pi_1^*(z; \alpha_1), \dots, \pi_{m_1}^*(z; \alpha_1), \dots, \pi_1^*(z; \alpha_s), \dots, \pi_{m_s}^*(z; \alpha_s)) \quad (16)$$

$$V^\top T_a V = N \quad (17)$$

$$V^{-1} = M^{-1} V^\top P T_a = M^{-1} (T_a V)^\top P \quad (18)$$

**Proof.** The equality (13) is the result of the string  $u^*(z)^\top T_a = u(z)^\top P T_a = u(z)^\top T_a^\top P = a(z)^\top P$ .

We continue with showing (14). Consider (3) and differentiate  $k$  times w.r.t.  $\alpha$ . We obtain  $-D^k \pi(\alpha) = u^*(z)^\top ((z - \alpha) D^k a(\alpha) - k D^{k-1} a(\alpha))$ .

If  $\alpha$  is a zero with multiplicity  $m$ , then  $D^k \pi(\alpha) = 0$  for  $k \leq m - 1$ . So we get the system of equations  $0 = u^*(z)^\top ((z - \alpha) D^k a(\alpha) - k D^{k-1} a(\alpha))$  for  $1 \leq k \leq m - 1$  and  $\pi(z) = (z - \alpha) u^*(z)^\top a(\alpha)$ . Write now  $q_k(z) = u^*(z)^\top D^k a(\alpha)$ , then  $q_0(z) = \frac{\pi(z)}{z - \alpha}$  and we have the recursive system of equations  $0 = (z - \alpha) q_k(z) - k q_{k-1}(z)$  for  $k = 1, \dots, m - 1$ . Solving this system yields  $q_k(z) = k! \frac{\pi(z)}{(z - \alpha)^{k+1}} = k! \pi_{k+1}(z; \alpha)$ . In other words, we find

$$u^*(z)^\top D^k a(\alpha) = k! \pi_{k+1}(z; \alpha). \quad (19)$$

Consider now  $a(w) = T_a u(w) = T_a V p(w)$ , where  $p$  is the interpolation polynomial. Then we also have  $u^*(z)^\top a(w) = u^*(z)^\top T_a V p(w)$ . Take in this equation derivatives w.r.t.  $w$ , substitute the  $\alpha_i$  for  $w$  and use the definition of the interpolation polynomial to get

$$u^*(z)^\top T_a V = (a(\alpha_1), \dots, \frac{D^{m_1-1} a(\alpha_1)}{(m_1 - 1)!}, \dots, a(\alpha_s), \dots, \frac{D^{m_s-1} a(\alpha_s)}{(m_s - 1)!}). \quad (20)$$

Combination of equations (19) and (20) yields (14).

To prove (15) we start from equation (14). Take the appropriate  $j$ -th order derivatives, divide by  $j!$  and substitute the  $\alpha_i$  in the resulting expression. Doing so results in a block-diagonal matrix, with on the diagonal exactly the  $M(i)$  as above.

Equation (16) immediately follows from (14) by definition of the polynomials  $\pi_k^*(z; \alpha)$ .

The proof of (17) completely parallels that of (15) and is therefore omitted. Now we turn to (18). First we observe that all the matrices  $M(i)$  are invertible, because of their triangular structure and the non-zero elements  $\pi_{m_i}(\alpha_i; \alpha_i)$  ( $\alpha_i$  had multiplicity  $m_i$ ) on the antidiagonal. Therefore also  $M$  is invertible and taking inverses in equation (15) yields the first equality of (18). The second one then follows from  $P T_a = T_a^\top P$ .  $\square$

**Remark 3.4** The most important formula of Proposition 3.3 is Equation (18) that gives an expression for the inverse of the confluent Vandermonde matrix. We see that the only inversion that has to be carried out is that of  $M$ . For that we have (12) at our disposal.

**Corollary 3.5** *The matrices  $M$  and  $N$  are related through the following two identities*

$$M = N^\top \Pi^* \quad (21)$$

$$N = (\Pi^*)^\top M. \quad (22)$$

Moreover  $N M^{-1} = M N^{-1}$  and thus  $N M^{-1}$  is involutive.

**Proof.** From equation (17) we get  $V^{-\top} = T_a V N^{-1}$  and hence  $V^{\top} P V^{-\top} N = V^{\top} P T_a V$  and in view of equation (15) this equals  $M$ . Now  $\Pi^*$  was defined as  $\Pi^* = V^{-1} P V$ , so we get  $(\Pi^*)^{\top} N = M$  and since  $M$  is symmetric we obtain (21). But then also  $N = (\Pi^*)^{-\top} M = (\Pi^*)^{\top} M$ , since  $\Pi^*$  is involutive. For the same reason the final assertion of the corollary follows.  $\square$

In the next proposition we present integral representations for the matrices  $M$  and  $M^{-1}$ . Below we use the notations  $u_{m_i}(z)^{\top} = (1, z, \dots, z^{m_i-1})$  and  $u_{m_i}^*(z)^{\top} = (z^{m_i-1}, \dots, z, 1)$  and the  $\Gamma_{\alpha_i}$  are sufficiently small contours around the  $\alpha_i$ .

**Proposition 3.6** *The following integral representations for the matrices  $M(i)$  and  $M(i)^{-1}$  are valid.*

$$M(i) = \frac{1}{2\pi i} \oint_{\Gamma_{\alpha_i}} u_{m_i}^*(z - \alpha_i) u_{m_i}^*(z - \alpha_i)^{\top} \frac{\pi(z)}{(z - \alpha_i)^{2m_i}} dz \quad (23)$$

$$M(i)^{-1} = \frac{1}{2\pi i} \oint_{\Gamma_{\alpha_i}} u_{m_i}(z - \alpha_i) u_{m_i}(z - \alpha_i)^{\top} \frac{1}{\pi(z)} dz. \quad (24)$$

As we have noticed before,  $M(i)^{-1}$  is completely determined by its first row (or column). From Proposition 3.6 we get, using Cauchy's theorem, that this first row is given by

$$\frac{1}{2\pi i} \oint_{\Gamma_{\alpha_i}} u_{m_i}(z - \alpha_i)^{\top} \frac{1}{\pi(z)} dz = \left( \frac{1}{\pi_{m_i}(\alpha_i; \alpha_i)}, \dots, \frac{1}{(m_i - 1)!} D^{m_i-1} \frac{1}{\pi_{m_i}(\alpha_i; \alpha_i)} \right) P,$$

in agreement with what we already found in (12).

## 4 Application to Stein equations

The goal of the present section is to obtain a way to compute the solution of Stein's equation, where the coefficients are matrices in companion form. Apart from its interest this is chiefly motivated by the computation of Fisher's information matrix of an ARMA process. As we wrote in the introduction, the blocks of Fisher's information matrix are solutions to such a Stein equation, see [14]. We postpone the application to ARMA processes to Section 5.

Let  $A$  be a complex matrix of size  $n \times n$  (not necessarily in companion form). If  $f$  is a  $\mathbb{C}^{n \times l}$  valued analytic function, then we define  $f(A)$  as  $\sum_{k=0}^{\infty} \frac{1}{k!} A^k f^{(k)}(0)$ . We use the following known result (see for instance [17, Section 9.9, Theorem 2]).

**Lemma 4.1** *Let  $A$  be a complex matrix ( $n \times n$ ) whose eigenvalues lie strictly inside the unit disk  $\Gamma$ . Then for a  $\mathbb{C}^{n \times l}$  valued analytic function  $f$  one has*

$$\frac{1}{2\pi i} \oint_{\Gamma} (z - A)^{-1} f(z) dz = f(A).$$

As an application of Lemma 4.1 we solve the Stein equation. Given matrices  $A$ ,  $C$  and  $H$  of appropriate dimensions (we also assume that the eigenvalues of both  $A$  and  $C$  lie inside the unit disk), we are looking for the solution for  $S$  of

$$S - ASC^\top = H. \quad (25)$$

This equation is of interest in matrix and operator theory (e.g. the operator that takes  $S$  to  $S - ASC$  is called a displacement operator, see [8]). In [15] we study this equation further and relate solutions of various Stein equations to a certain Fisher information matrix.

The solution to (25), see [16], is given by  $\frac{1}{2\pi i} \oint_{\Gamma} (z - A)^{-1} f(z) dz$ , with  $f(z) = H(I - zC)^{-\top}$  and hence equal to  $\sum_{k=0}^{\infty} A^k H (C^\top)^k$ .

We continue with presenting an alternative way to obtaining a solution for the special case where both the matrices  $A$  and  $C$  are in companion form. Let  $V_A$  be the Vandermonde matrix associated with  $A$  and let  $V_C$  be likewise associated with  $C$ . Let  $\hat{S} = V_A^{-1} S V_C^{-\top}$  and  $\hat{H} = V_A^{-1} H V_C^{-\top}$ . The results of Section 3 on inverses of confluent Vandermonde matrices enable us to compute  $\hat{H}$ .

Premultiplication of (25) with  $V_A^{-1}$  together with postmultiplication with  $V_C^{-\top}$  results in

$$\hat{S} - J_A \hat{S} J_C^\top = \hat{H}, \quad (26)$$

where  $J_A$  and  $J_C$  are the Jordan forms of  $A$  and  $B$  respectively.

Let  $v = \text{vec}(\hat{S})$  and  $h = \text{vec}(\hat{H})$ . Then it is known (see [16]) that  $v$  is given by  $v = (I - J_C \otimes J_A)^{-1} h$  under the assumption that no product of an eigenvalue of  $A$  and an eigenvalue of  $C$  equals 1. This assumption is typically fulfilled in the context of stationary and invertible ARMA processes, where these eigenvalues are the zeros of both the AR and the MA-polynomial and thus lie inside the unit circle, see Section 5.

The computation of the inverse of the matrix  $I - J_C \otimes J_A$  can now be done in an efficient way. Let  $J_{A,i}$  be the Jordan block of  $J_A$  associated with the eigenvalue  $\alpha_i$  and  $J_{C,j}$  be the Jordan block of  $J_C$  associated with the eigenvalue  $\gamma_j$ . Then  $I - J_C \otimes J_A$  is block diagonal with diagonal blocks  $I - J_{C,j} \otimes J_{A,i}$ . Moreover, these blocks are upper triangular and even almost block diagonal. On the diagonal we find the blocks  $I - \gamma_j J_{A,i}$  and on the subdiagonal just above it the blocks  $-J_{A,i}$ . Therefore,  $(I - J_{C,j} \otimes J_{A,i})^{-1}$  is again upper triangular with on the diagonal the blocks  $(I - \gamma_j J_{A,i})^{-1}$  and on the  $k$ -th subdiagonal above it ( $k \leq m_j - 1$  with  $m_j$  the multiplicity of  $\gamma_j$ ) one finds the blocks  $(I - \gamma_j J_{A,i})^{-k-1} J_{A,i}^k$ . Finally the inverses of the  $I - \gamma_j J_{A,i}$  are upper triangular Toeplitz matrices with  $kl$ -element given by  $\gamma_j^{k-l} (1 - \alpha_i \gamma_j)^{-k+l-1}$  for  $k \geq l$ .

The generic case is that where all the eigenvalues of  $A$  have multiplicity 1 and likewise for the eigenvalues of  $C$ . Consequently the matrices  $J_A$  and  $J_C$  are diagonal. In this case equation (26) has a very simple solution:  $\hat{S}$  has elements  $\hat{S}_{ij} = \frac{1}{1 - \alpha_i \gamma_j} \hat{H}_{ij}$ .

## 5 Application to ARMA processes

Consider an ARMA( $p, q$ ) process  $y$ , a stationary discrete time stochastic process that satisfies

$$y_t + a_1 y_{t-1} + \dots + a_p y_{t-p} = \varepsilon_t + c_1 \varepsilon_{t-1} + \dots + c_q \varepsilon_{t-q}, \quad (27)$$

where  $\varepsilon$  is a Gaussian white noise sequence with unit variance. The real constants  $a_1, \dots, a_p$  and  $c_1, \dots, c_q$  will be fixed throughout the rest of this section. Introduce the monic polynomials  $a(z) = \sum_{i=0}^p a_{p-i} z^i$  and  $c(z) = \sum_{i=0}^q c_{q-i} z^i$  and let  $a^*$  and  $c^*$  be the corresponding reciprocal polynomials, so  $a^*(z) = \sum_{i=0}^n a_i z^i$  and  $c^*(z) = \sum_{i=0}^q c_i z^i$ . We make the common assumption that the ARMA process is causal and invertible, meaning that  $a$  and  $c$  have their zeros strictly inside the unit circle [2, chapter 3].

Write  $\theta = (a_1, \dots, a_p, c_1, \dots, c_q)^\top$ . Notice that the observations  $y$  (given random variables or their realized values) of course don't depend on the parameter  $\theta$ , but then the noise sequence  $\varepsilon$  does. The Fisher information matrix  $F_n(\theta)$  for  $n$  observations is defined (see [1]) as the covariance matrix of the score function and, because of the assumed Gaussian distribution of  $\varepsilon$ , it is asymptotically equal to  $n$  times the stationary Fisher information matrix

$$F(\theta) = \mathbb{E}_\theta \frac{\partial \varepsilon}{\partial \theta} \frac{\partial \varepsilon}{\partial \theta}^\top,$$

where  $\mathbb{E}_\theta$  denotes expectation under the parameter  $\theta$ . Knowledge of the Fisher information matrix is crucial for asymptotic statistical analysis. For instance, it is known (see e.g. [2]) that maximum likelihood estimators of the parameters of an ARMA process are consistent and have (using  $n$  observations) an asymptotic covariance matrix that is  $n^{-1}$  times the inverse (provided that it exists) of the stationary Fisher information matrix. The inverse exists if the polynomials  $a$  and  $c$  have no common zeros, see [13].

The matrix  $F(\theta)$  has a representation in the spectral domain given by the following block decomposition

$$F(\theta) = \begin{pmatrix} F_{aa} & F_{ac} \\ F_{ac}^\top & F_{cc} \end{pmatrix} \quad (28)$$

where the matrices appearing here have the following elements

$$\begin{aligned} F_{aa}^{jk} &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{j-k+p-1}}{a(z)a^*(z)} dz, \quad (j, k = 1, \dots, p) \\ F_{ac}^{jk} &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{j-k+q-1}}{c(z)a^*(z)} dz, \quad (j = 1, \dots, p, k = 1, \dots, q) \\ F_{cc}^{jk} &= \frac{1}{2\pi i} \oint_{|z|=1} \frac{z^{j-k+q-1}}{c(z)c^*(z)} dz, \quad (j, k = 1, \dots, q). \end{aligned}$$

With  $k(z) = a(z)a^*(z)c(z)c^*(z)$ ,  $u_p(z) = (1, \dots, z^{p-1})^\top$ ,  $u_q(z)$  likewise and  $u_p^*$  and  $u_q^*$  their reciprocal polynomials we have the following compact expression for the whole Fisher information matrix.

$$F(\theta) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{k(z)} \begin{pmatrix} c^*(z)u_p(z) \\ -a^*(z)u_q(z) \end{pmatrix} (c(z)u_p^*(z)^\top \quad -a(z)u_q^*(z)^\top) dz. \quad (29)$$

As in Section 2 we let  $A \in \mathbb{R}^{p \times p}$  be the companion matrix associated with the polynomial  $a(\cdot)$  (its precise form is given by equation (1) for  $n = p$ ). The matrix  $C \in \mathbb{R}^{q \times q}$  associated with the polynomial  $c(\cdot)$  has an analogous form. Let the matrix  $\tilde{A} \in \mathbb{R}^{(p+q) \times (p+q)}$  be given by

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}.$$

In [14] we showed that the Fisher information matrix  $F(\theta)$  is the solution of the following Stein equation

$$F(\theta) - \tilde{A}F(\theta)\tilde{A}^\top = ee^\top, \quad (30)$$

where  $e^\top = (e_{pp}^\top, e_{qq}^\top)$  with  $e_{pp}$  the  $p$ -th standard basis vector in  $\mathbb{R}^p$  and  $e_{qq}$  the  $q$ -th standard basis vector in  $\mathbb{R}^q$ . Using for  $F(\theta)$  the block decomposition (28), we see that each of the blocks involved satisfies a Stein equation with appropriate coefficients. For instance for  $F_{ac} \in \mathbb{R}^{p \times q}$  we have

$$F_{ac} - AF_{ac}C^\top = H_{ac}, \quad (31)$$

with  $H_{ac} = e_{pp}e_{qq}^\top$ . As we already announced in the introduction, equation (31) as well as the analogous equation for the other blocks of Fisher's information matrix motivated the study of solutions to Stein's equation, when the coefficient matrices are in companion form.

We apply the results of the previous sections as follows. Let  $V_A$  be a matrix whose columns are the generalized eigenvectors of  $A$ , and  $V_C$  the corresponding matrix for  $C$ . As we have seen before these matrices are confluent Vandermonde matrices. By  $J_A$  and  $J_C$  we denote the Jordan forms of  $A$  and  $C$  respectively. Let also  $\hat{F}_{ac} = V_A^{-1}F_{ac}V_C^{-\top}$  and  $\hat{H}_{ac} = V_A^{-1}H_{ac}V_C^{-\top}$ . Then we can replace (31) with the equivalent equation

$$\hat{F}_{ac} - J_A\hat{F}_{ac}J_C^\top = \hat{H}_{ac}. \quad (32)$$

A little more can be said. The matrix  $\hat{H}_{ac}$  here becomes  $V_A^{-1}e_{pp}(V_C^{-1}e_{qq})^\top$  and we observe that both  $V_A^{-1}e_{pp}$  and  $V_C^{-1}e_{qq}$  are the last columns of the inverse of a Vandermonde matrix. We have already seen in Section 2 how these columns are related to interpolation polynomials. We have for instance that  $V_A^{-1}e_{pp}$  is equal to  $p_A^*(0)$ , where  $p_A^*(z) = z^{p-1}p_A(\frac{1}{z})$  and  $p_A$  the interpolation polynomial related to the eigenvalues of  $A$  as described in Proposition 2.2. Likewise  $V_C^{-1}e_{qq} = p_C^*(0)$ .

Let us finish with considering the generic case of Fisher's information matrix, i.e. we assume that  $A$  and  $C$  only have eigenvalues of multiplicity 1. It then follows that  $\hat{F}_{ac}$  has as it's  $ij$ -th element

$$\frac{p_A^*(0)_i p_C^*(0)_j}{1 - \alpha_i \gamma_j}.$$

Now it is easy to compute  $F_{ac} = V_A \hat{F}_{ac} V_C^\top$ . To the other blocks of the Fisher information matrix the same procedure applies.

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