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Abstract. In this paper, we study charged spin-1/2 particles in two dimensions, subjected to a perpendicular non-Abelian magnetic field. Specializing to a choice of vector potential that is spatially constant but non-Abelian, we investigate the Landau level spectrum in planar and spherical geometry, paying particular attention to the role of the total angular momentum $\vec{J} = \vec{L} + \vec{S}$. Then, we show that the adiabatic insertion of non-Abelian flux in a spin-polarized quantum Hall state leads to the formation of charged spin textures, which in the simplest cases can be identified with quantum Hall Skyrmions.

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Contents

1. Introduction

It has long been known that the motion of charged particles in a plane and in the presence of a perpendicular magnetic field is highly special. In classical mechanics, particles exhibit cyclotron motion, while quantum mechanics leads to the Landau level band structure [1], with each level providing a macroscopic number of one-particle states that are strictly degenerate in energy. In this setting, free electrons can form integer quantum Hall states, with intricate topological properties resulting in a quantization of the Hall conductance. Adding interactions leads to a plethora of remarkable many-body states, collectively known as fractional quantum Hall states. Such states are known to admit excitations with fractional charge and fractional statistics. The possibility that specific quantum Hall states have excitations with non-Abelian braid statistics has opened the exciting prospect of applications in the realm of topological quantum computation [2].

     A setting that is physically very different from, but mathematically similar to, that of two-dimensional (2D) electrons in a perpendicular field is that of rapidly rotating cold atoms. In this analogy, the vorticity of a rotating liquid is akin to magnetic flux in the electron system. The limit of rapid rotation leads to a Landau level structure and one expects the formation of atomic quantum Hall states (see [3] for a review). The path towards the experimental realization of atomic quantum Hall states is extremely challenging, but the lowest Landau level (LLL) has been reached [4] and there are indications of the formation of incompressible quantum liquids in small clusters [5].

     The cold atom setting allows for yet another variation on the same theme: cold atomic gases subjected to external time-dependent potentials that are such that they mimic the effects of magnetic fields or, equivalently, of rotation. Recent proposals for implementing such *artificial gauge fields* can be found in [6]–[12]; a successful experimental realization is reported in [13].

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Very interestingly, the latter setup is flexible enough to allow for a generalization of regular, Abelian, magnetic fields, corresponding to the $U(1)$ gauge symmetry of Maxwell's theory, to gauge fields pertaining to non-Abelian symmetries. The simplest version of this idea is the case of spin-1/2 particles subjected to an external gauge potential for a non-Abelian $U(2)$ symmetry [14]–[17]; see [18] for the case of spin-1 particles. Remarkably, very similar if not identical Hamiltonians arise in the original setting of a 2D electron gas in a perpendicular magnetic field if the Rashba [19] and Dresselhaus [20] spin–orbit coupling terms are taken into account. This connection can be exploited to study the physics of the quantum spin Hall effect in the context of cold atoms [21, 22].

With this, there is ample reason to study non-Abelian external gauge potentials with special emphasis on what is new as compared to the Abelian case. For particles confined to a lattice geometry, the non-Abelian case leads to an interesting generalization of the 'Hofstadter butterfly' fractal one-particle spectrum, dubbed the 'Hofstadter moth' [15]. The Landau level problem and the possibility of realizing (fractional) quantum Hall states with particles subjected to non-Abelian gauge fields have already attracted much attention [16, 17, 23, 24].

In this paper, we analyze non-interacting spin-1/2 particles in an external non-Abelian field and reflect on some of the fundamental differences with the Abelian setup. We analyze and solve the Landau level problem in spherical geometry, highlighting the fundamental role of the *total* angular momentum $\vec{J} = \vec{L} + \vec{S}$, which commutes with the Hamiltonian. In this setup, the non-Abelian field penetrating the sphere agrees with the asymptotic (large radius) limit of the non-Abelian magnetic monopoles first written by 't Hooft and Polyakov [25]. One reason for focusing on spherical geometry is that this geometry is known to be particularly useful for the numerical study of many-body states arising on the addition of interactions to the Landau-level problem [26].

We then proceed to the process where a non-Abelian flux is inserted in the background of an otherwise Abelian flux. In our simplest case, $J_z = L_z + S_z$ remains a good quantum number during the flux insertion. This suggests a prominent role for transitions where particles flip their spin while at the same time changing their L_z by (plus or minus) one unit, meaning that they jump to an adjacent Landau level orbital of the Abelian problem. This is indeed what happens: starting from a spin-polarized integer quantum Hall state, inserting non-Abelian flux at the origin leads to a state where, depending on their distance from the origin, particles have an amplitude for changing their spin and moving radially in or out. The resulting state is a spin texture of unit electric and topological charge, which is easily identified with a quantum Hall Skyrmion. More general external fields lead to more intricate textures.

In the literature, thought experiments involving the insertion of (Abelian) flux are often invoked as a probe of the characteristics of the quantum phase of a many-body system. One spectacular example is the argument by Laughlin that the insertion of a unit flux through a gapped medium with Hall conductance $\sigma_H = \nu e^2/h$ leads to the nucleation of an excitation with fractional charge $e* = \nu e$ [27]. In the case of the quantum spin Hall state, the insertion of flux leads to spin-full excitations at the sample edges [28]. One motivation for the present study has been the wish to extend these considerations to the case of non-Abelian flux.

We will briefly comment on possible experimental realization of our ideas. In the case of electrons, the integer quantum Hall reference state is readily available, and the non-Abelian flux can in principle, in the presence of spin–orbit coupling, be generated via a perpendicular electric field (see section 3). Integer quantum Hall states for cold atomic fermions are expected to arise in rapidly rotating systems [29] or in laser-induced artificial gauge fields [30]. The paper [15]

has indicated how non-Abelian external gauge fields can be generated with the help of external lasers; if such laser pulses can be spatially focused, this will result in pulses of non-Abelian flux similar to the ones we propose here. Clearly these external fields will deviate from our expressions in the details; we expect, however, that the spin textures that we predict here are relatively robust. Reading out spin textures of effective spin-1/2 atomic states is possible; see, for example, [31], where a Skyrmion lattice for effective spin-1/2 bosons was imaged.

This paper is organized as follows. We start (section 2) with a general exposition on non-Abelian gauge fields. Next (section 3), we discuss the non-Abelian Landau problem, with particular emphasis on spherical geometry. Section 4 discusses the response of spin-full particles to the insertion of non-Abelian flux in an otherwise Abelian background. We end with some conclusions in section 5. Appendices A–D present further details and background material.

2. On non-Abelian gauge fields

Before we start to investigate the Landau problem, we start with a quick review of non-Abelian gauge fields, highlighting the differences with the Abelian case. We are interested in the quantum problem of a particle coupled to an external non-Abelian gauge field $\vec{A}$. In a cold atom experiment, these fields are controlled by external lasers. Therefore in the present paper, we are not concerned with the dynamics of these gauge fields, which are treated as control parameters. The Hamiltonian for a non-relativistic particle of mass m in an external magnetic field is

$$H = \frac{1}{2m}(\vec{p} - \vec{A})^2. \tag{1}$$

In the non-Abelian setup, each component of the vector potential $\vec{A}$ is a matrix,

$$\vec{A} = A_x\vec{u}_x + A_y\vec{u}_y + A_z\vec{u}_z = A_a\vec{u}_a. \tag{2}$$

The corresponding magnetic field (or curvature) $\vec{B}$ is

$$\vec{B} = \vec{\nabla} \times \vec{A} - \frac{i}{\hbar}\vec{A} \times \vec{A}, \tag{3}$$

or equivalently in components

$$B_a = \epsilon_{abc}\left(\partial_b A_c - \frac{i}{\hbar}A_b A_c\right). \tag{4}$$

The first term is the usual curl, while the second term $\vec{A} \times \vec{A}$ vanishes identically for Abelian gauge fields. However, it is non-zero in the generic non-Abelian case, as the components A_a of the potential may not commute. In contrast to the Abelian case, even a uniform potential $\vec{A}$ can produce a non-zero magnetic field. Introducing the covariant derivative is $\vec{D} = \vec{\nabla} - \frac{i}{\hbar}\vec{A} = \frac{i}{\hbar}(\vec{p} - \vec{A})$, the magnetic field takes the following compact form,

$$\vec{B} = \vec{D} \times \vec{A}. \tag{5}$$

A gauge transformation is simply a unitary transformation U and a change of $\vec{A}$, such that $\vec{D}$ transforms covariantly,

$$\vec{A} \rightarrow U\vec{A}U^\dagger + i\hbar U\vec{\nabla}U^\dagger. \tag{6}$$

Then the following quantities transform covariantly (note that the magnetic field is no longer gauge invariant),

$$\vec{D} \to U\vec{D}U^{\dagger}, \quad \vec{B} \to U\vec{B}U^{\dagger}, \tag{7}$$

and Hamiltonian (1) is clearly a covariant quantity,

$$H = \frac{1}{2m}(\vec{p} - \vec{A})^2 = -\frac{\hbar^2}{2m}\vec{D}^2. \tag{8}$$

In the Abelian case, two field configurations $\vec{A}$ yielding the same magnetic field are necessarily equivalent up to a gauge transformation (on a simply connected space). This is no longer the case for non-Abelian gauge groups. In particular, it is known from [32] that there are two gauge-inequivalent kinds of non-Abelian vector potentials that produce a uniform $\vec{B}$ field (although this statement is completely general, we illustrate this for $\vec{B} = 2\sigma_z\vec{u}_z$):

- A commuting field with linear potential $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r} = \sigma_z(-y, x, 0)$, for which only $\vec{\nabla} \times \vec{A}$ contributes to the field strength. This case is Abelian in nature, in the sense that all components of the potential vector commute with each other.
- A uniform non-commuting potential, for instance $\vec{A} = (-\sigma_y, \sigma_x, 0)$, such that $\vec{B} = -\frac{i}{\hbar}\vec{A} \times \vec{A}$. This is only possible in a non-Abelian gauge group.

Although these two kinds of potentials give rise to the same magnetic field, they lead to completely different physical properties. While the first kind induces a magnetic length, the Aharanov–Bohm effect and the Landau level discrete spectrum, the second one has no Aharanov–Bohm effect, and the spectrum of a particle in such an external field configuration is continuous.

3. The non-Abelian Landau problem: particles in a perpendicular uniform non-Abelian external field

In this paper, we focus on the simplest case of non-Abelian gauge fields, when A_a and B_a are 2×2 Hermitian matrices. The gauge group is then $U(2) = U(1) \times \mathrm{SU}(2)$ and decomposes into

- An Abelian $U(1)$ part, namely the fields proportional to the identity matrix $\mathbb{I}$;
- A non-Abelian $\mathrm{SU}(2)$ component, whose fields are linear combinations of the Pauli matrices σ_a.

The $U(2)$ case is a natural choice as it is the simplest case allowing non-Abelian gauge fields. However, there is a deeper reason for focusing on $U(2)$ gauge fields. Very similar physics can arise in a 2D electron gas when taking into account relativistic corrections in the Pauli–Schrödinger equation, such as the Thomas term,

$$H_{\mathrm{T}} = -\frac{q\hbar}{4m^2c^2}\vec{\sigma} \cdot (\vec{E} \times \vec{p}). \tag{9}$$

This spin–orbit coupling term plays a crucial role in spintronics [33], and it mimics the effect of a non-Abelian gauge potential,

$$\vec{A} \sim \vec{E} \times \vec{\sigma}. \tag{10}$$

In this section, we study the quantum problem of a non-relativistic particle confined to a 2D manifold in the background of a uniform perpendicular $U(2)$ magnetic field. We present the

spectra for two different geometries: the plane and the sphere. It turns out that this Hamiltonian can be mapped exactly to the one of a 2D electron in a perpendicular $U(1)$ magnetic field when the Thomas term is present and an additional perpendicular $U(1)$ electric field $\vec{E}$ is applied.

3.1. On the plane

To set this problem on the plane, we consider a perpendicular, uniform magnetic field $\vec{B} = B_z \vec{u}_z$, where B_z is a 2×2 Hermitian matrix. We can always choose a basis where the matrix B_z is diagonal,

$$B_z = B\mathbb{I} + \frac{2}{\hbar}\beta'^2 \sigma_z = B\left(\mathbb{I} + 2\beta^2 \sigma_z\right). \tag{11}$$

We introduced the pure number $\beta = \frac{\beta'}{l_m B}$ involving the magnetic length $l_m = \sqrt{\frac{\hbar}{B}}$ (for $B > 0$). From now on, we work with $\hbar = 1$. This magnetic field is a $U(2)$ matrix, and is a superposition of a $U(1)$ field B and an SU(2) field $2\beta^2 B\sigma_z$. In view of the previous discussion, there is ambiguity in the notion of a non-Abelian uniform magnetic field, and one has to specify the non-Abelian part of the potential $\vec{A}$. The first kind of potential $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$ boils down to an Abelian $U(1) \times U(1)$ gauge group, and the physics is simply that of two non-interacting species of particles coupled to different Abelian magnetic fields. The second kind, however, a constant and non-commutative potential, is much more interesting and leads to new physics [16, 17, 23, 24],

$$\vec{A} = \frac{B}{2}\begin{pmatrix} -y\mathbb{I} \\ x\mathbb{I} \\ 0 \end{pmatrix} + \beta'\begin{pmatrix} -a\sigma_y \\ a^{-1}\sigma_x \\ 0 \end{pmatrix}. \tag{12}$$

The Hamiltonian describing a particle confined to a plane in this non-Abelian background,

$$H = \frac{1}{2m}(\vec{p} - \vec{A})^2, \tag{13}$$

enjoys the translation symmetry of the plane. Since the non-Abelian part of $\vec{A}$ is uniform, the magnetic translation operators are insensitive to the non-Abelian part and have the usual expressions,

$$T_x = \left(-i\partial_x - \frac{y}{2l_m^2}\right), \quad T_y = \left(-i\partial_y + \frac{x}{2l_m^2}\right), \tag{14}$$

which implies immediately the Abelian Aharanov–Bohm effect,

$$[\vec{a} \cdot \vec{T}, \vec{b} \cdot \vec{T}] = -i\frac{(\vec{a} \times \vec{b}) \cdot \vec{u}_z}{l_m^2}\mathbb{I}. \tag{15}$$

The rhs is simply the flux of the Abelian part of the magnetic field $B\mathbb{I}$ through the parallelogram delimited by the vectors $\vec{a}$ and $\vec{b}$, and the (Abelian) magnetic length scale is l_m. Only the Abelian part of the magnetic field is quantized, and the number of states in a given Landau level will only depend on the Abelian field strength B.

It turns out that the problem of a particle in such a non-Abelian external field can be mapped exactly to the Hamiltonian of 2D electron in the presence of both the Rashba and the Dresselhaus spin–orbit interaction [24], and it was first solved in this context by Zhang [34].

Having in mind to solve this problem on the sphere in the next section, we demand rotational symmetry around the z-axis, and we focus on the symmetric gauge,

$$\vec{A} = \frac{B}{2} \begin{pmatrix} -y\mathbb{I} \\ x\mathbb{I} \\ 0 \end{pmatrix} + \beta' \begin{pmatrix} -\sigma_y \\ \sigma_x \\ 0 \end{pmatrix}. \tag{16}$$

This symmetric case corresponds to the absence of the Dresselhaus interaction in [34], and Hamiltonian (13) is much simpler to solve in this case. Moreover, it can be mapped to a Thomas term (9) in the presence of a perpendicular uniform electric field $\vec{E} \propto \beta' \vec{u}_z$. This Hamiltonian can be expanded as

$$H = \omega_c(a^\dagger a + \sqrt{2}\beta(a^\dagger \sigma_+ + a\sigma_-) + \tfrac{1}{2}(1 + 2\beta^2)), \tag{17}$$

where a and $a^\dagger$ are the usual annihilation and creation operators appearing in the Landau problem (see appendix A). Up to a change of spin basis $U = \sigma_x$, this is nothing but the celebrated Jaynes–Cummings Hamiltonian, and it is straightforward to obtain its spectrum,

$$E_0 = \omega_c(\tfrac{1}{2} + \beta^2), \tag{18}$$

$$E_n^\pm = \omega_c\left(n \pm \sqrt{2\beta^2 n + \tfrac{1}{4}} + \beta^2\right). \tag{19}$$

3.2. On the sphere

It can be rather instructive to solve such a problem on a sphere instead of the plane. The surface of the sphere being finite, the degeneracy of the Landau levels becomes finite too, which is very interesting for numerics. Moreover, the translation invariance of the plane is promoted to the rotational symmetry of the sphere, and the spectrum decomposes into SU(2) multiplets. In the Abelian case, this geometry was first solved in [35] and later used by Haldane [26] in the context of the quantum Hall effect. In order to fix our notations, we recall these main results in appendix B.

3.2.1. Field configuration. A uniform perpendicular magnetic field implies the presence of a magnetic monopole at the center of the sphere. In the Abelian case (see appendix B), the corresponding potential $\vec{A}_{Ab}$ must have a singularity (Dirac string) somewhere on the sphere, for instance at the south pole $\theta = \pi$,

$$\vec{A}_{Ab} = \frac{N_\Phi}{2} \frac{1 - \cos(\theta)}{r \sin(\theta)} \vec{u}_\phi. \tag{20}$$

When Dirac's quantization condition $N_\Phi \in \mathbb{Z}$ is satisfied, this singularity has no physical consequence as it can be moved around through gauge transformations [36]. This is the well-known quantization of the magnetic flux piercing the sphere, which must be an integer number N_Φ of flux quanta,

$$\int_S \vec{B} \cdot d\vec{S} = 2\pi N_\Phi, \quad \text{i.e.} \quad B_r = \frac{N_\Phi}{2r^2}. \tag{21}$$

To this U(1) potential (20), we add the following SU(2) component,

$$\vec{A}(\alpha) = \vec{A}_{Ab} + \alpha \frac{\vec{r} \times \vec{\sigma}}{r^2}. \tag{22}$$
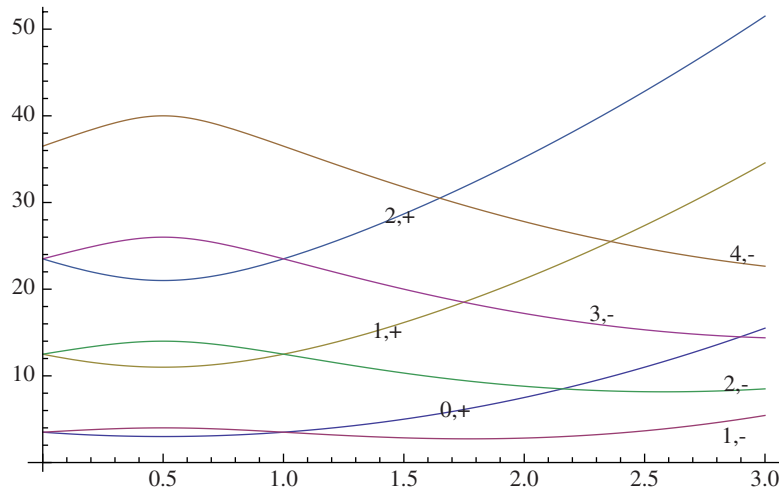
Figure 1. Band structure for the non-Abelian Landau problem on the sphere as a function of the non-Abelian field strength α. This is the case for $N_\Phi = 7$ Abelian flux quanta, and only the lowest part of the spectrum is shown.

Once again, this corresponds to a Thomas term (9) with a radial uniform electric field $\vec{E} \propto \alpha \vec{u}_r$, and this will turn out to be the correct extension of the symmetric gauge on the plane (16). The corresponding magnetic field is

$$B_r = \frac{N_\Phi}{2r^2} - 2\alpha(1-\alpha)\frac{(\vec{r}\cdot\vec{\sigma})}{r^3}. \tag{23}$$

It is quite remarkable that the strength α of the non-Abelian field need not be quantized, as the vector potential $\frac{\vec{r}\times\vec{\sigma}}{r^2}$ has no singularity on the sphere: α can be any real number. As the radial $U(1)$ field is created by a magnetic monopole, it is not very surprising that the SU(2) counterpart involves a non-Abelian monopole. Indeed, the potential $\vec{A} = \alpha\frac{\vec{r}\times\vec{\sigma}}{r^2}$ is nothing but the large distance asymptotic of a true non-Abelian monopole [25], and in this context it is well known that there is no need for a Dirac string, nor is there a singularity anywhere on the sphere.

3.2.2. The Hamiltonian and the spectrum. The details of the derivation of the spectrum of a particle confined to a sphere of radius r in this non-Abelian background can be found in appendix C. The Hamiltonian is the following,

$$H(\alpha) = \frac{1}{2mr^2}[\vec{r}\times(\vec{p} - \vec{A}(\alpha))]^2, \tag{24}$$

which is a scalar under global rotations generated by $\vec{J} = \vec{L} + \vec{S}$, where $\vec{L}$ is the angular momentum and $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the spin. Its eigenstates form SU(2) multiplets corresponding to the decomposition of the Hilbert space into irreducible representations of $\vec{J}$,

$$\mathcal{H} = (j_0) \oplus 2(j_1) \oplus 2(j_2) \oplus \cdots \oplus 2(j_n) \oplus \cdots, \tag{25}$$

where $j_n = \frac{N_\Phi - 1}{2} + n$. The corresponding eigenvalues are (see figure 1)

$$E_0(\alpha) = \frac{1}{2mr^2}\left(\frac{N_\Phi}{2} - 2\alpha(1-\alpha)\right), \tag{26}$$

$$E_n^{\pm}(\alpha) = \frac{1}{2mr^2}\left(n(N_{\Phi}+n) - 2\alpha(1-\alpha) \pm \sqrt{(2\alpha-1)^2\,n\,(N_{\Phi}+n) + \left(\frac{N_{\Phi}}{2}\right)^2}\right). \tag{27}$$

As can be seen in figure 1, multiple level crossings are observed. This also occurs in the planar case [23], which can be recovered from the sphere in the limit of infinite radius.

3.2.3. Recovering the plane. The non-Abelian $\vec{A}$ field we considered on the sphere is indeed the correct extension of the planar symmetric gauge (16). The planar problem is recovered by taking the sphere radius $r \to \infty$ while keeping constant the gauge field strength on the surface,

$$\frac{N_{\Phi}}{2r^2} \sim B, \quad \frac{\alpha}{r} \sim \beta' = \frac{\beta}{\sqrt{B}}. \tag{28}$$

The vector potential and the magnetic field become in this limit

$$\vec{A} \to \frac{B}{2}\begin{pmatrix} -y\mathbb{I} \\ x\mathbb{I} \\ 0 \end{pmatrix} + \beta'\begin{pmatrix} -\sigma_y \\ \sigma_x \\ 0 \end{pmatrix} \quad \vec{B} \to B(\mathbb{I} + 2\beta^2\sigma_z)\vec{u}_z. \tag{29}$$

It is straightforward to check that the eigenvalues of the Hamiltonian $E_n^{\pm}(\alpha)$ behave in the planar limit as

$$E_0(\alpha) \to \omega_c(\tfrac{1}{2} + \beta^2), \tag{30}$$

$$E_n^{\pm}(\alpha) \to \omega_c\left(n \pm \sqrt{2\beta^2 n + \tfrac{1}{4}} + \beta^2\right), \tag{31}$$

reproducing the planar spectrum. Moreover, one can expand the Hamiltonian on the sphere in terms of $\vec{L}$,

$$H = \frac{1}{2mr^2}\left[\vec{L}^2 - \left(\frac{N_{\Phi}}{2}\right)^2 + 2\alpha\left(\vec{L} + \frac{N_{\Phi}}{2}\frac{\vec{r}}{r}\right)\cdot\vec{\sigma} + 2\alpha^2\right], \tag{32}$$

and using the Holstein–Primakoff representation,

$$L_+ = b^{\dagger}\sqrt{N_{\Phi} + 2a^{\dagger}a - b^{\dagger}b}, \tag{33}$$

$$L_- = \sqrt{N_{\Phi} + 2a^{\dagger}a - b^{\dagger}b}\,b, \tag{34}$$

$$L_z = b^{\dagger}b - \frac{N_{\Phi}}{2} - a^{\dagger}a, \tag{35}$$

we recover the planar Hamiltonian

$$H \to \omega_c[(a^{\dagger}a + \tfrac{1}{2}) + \sqrt{2}\beta(a\sigma + a^{\dagger}\sigma_+) + \beta^2]. \tag{36}$$

This is not surprising in view of the mapping of this problem with a spin-1/2 electron under an effective non-Abelian potential coming from the Thomas term with a perpendicular electric

field. Indeed, the infinite radius limit of a sphere in a radial $\vec{E}$ and $\vec{B}$ field is clearly the plane under perpendicular $\vec{E}$ and $\vec{B}$ fields.

In summary, we analyzed the non-Abelian Landau problem on the sphere and obtained the spectrum. The degeneracy of Abelian Landau levels is preserved, and the number of states per area remains $1/l_{\mathrm{m}}^2$, as we expected from the absence of a non-Abelian Aharanov–Bohm effect on the plane. Indeed, the Abelian Aharanov–Bohm (15) effect enforces a minimum surface of l_{m}^2 for the wavefunctions.

4. Adiabatic insertion of non-Abelian flux

For the non-Abelian gauge group SU(2), there is no notion of a 'quantum of non-Abelian flux'. To appreciate this fact, we recall that in an Abelian gauge theory the number of flux quanta is nothing but the winding number of the potential around the flux tube, and this integer cannot be continuously deformed. The homotopy group of $U(1)$ is the set of integers $\mathbb{Z}$. However, the situation is different for the gauge group SU(2), which has a trivial homotopy group so that any flux insertion can be smoothly deformed away. This explains why there is no non-Abelian analog of the notion of flux quantum, in agreement with the absence of quantization of the non-Abelian field strength on the sphere.

Nonetheless, we wish to consider the insertion of non-Abelian flux in a quantum Hall (qH) fluid. In the Abelian case, the celebrated Laughlin argument shows that the insertion of a quantum of Abelian flux leads to the accumulation of electric charge $\pm \nu e$, with ν being the filling fraction of the qH liquid. Here we wish to start from an (integer) qH fluid and to insert, at given location $\mathcal{O}$, a non-Abelian field configuration $\delta \vec{A}$ centered at $\mathcal{O}$ and chosen such that (i) it generates no magnetic field away from $\mathcal{O}$ and (ii) it can be removed by a gauge transformation. After this process, we perform a gauge transformation on the evolved state, in such a way that it lives in the same Hilbert space as the state we started off with.

Before entering into the details, we briefly sketch the well-known argument for the Abelian case, as discussed by Laughlin [27] and Halperin [37]. This leans heavily on the existence of a gap separating the ground state of the qH liquid from its excited states. Take a qH droplet and adiabatically insert a flux quantum at the origin, $\delta A_\phi = \frac{-1}{r}$. This value of the flux is very special in the sense that it creates no Aharanov–Bohm effect and therefore it can be gauged away. This means that after this adiabatic insertion, one can map the system back to its original field configuration. The net effect of this process is a charge transport away from the origin leaving behind a quasihole, as single particle states in the LLL (see appendix A) evolve according to

$$|m\rangle \rightarrow |m+1\rangle. \tag{37}$$

The usual description of adiabatic processes involves Berry phases [38, 39], or more generally Berry matrices [40] in the presence of ground state degeneracies. Before we specify our system in detail, we first give a quick review of the derivation of the Berry matrix.

4.1. The Berry matrix

Considering an adiabatic process in the presence of energy degeneracies, the Berry phase should be generalized as was done in [40]. We will present the details of the Berry matrix that are needed for our purposes. Suppose that the Hamiltonian is a smooth function of a parameter $\{\lambda(t)\}$ and at every point in parameter space has a degenerate ground state energy separated

from higher levels by some finite gap. Starting from a state $|\alpha(0)\rangle$ belonging to the lowest energy subspace of the total Hilbert space $\mathcal{H}_{E(0)}$, the adiabatic theorem tells us that we end up in an eigenstate, which is again an element of the subspace of ground states at time t, $|\alpha(t)\rangle$. During this process, the eigenstates obey the Schrödinger equation $H(\lambda)|\alpha(\lambda)\rangle = E_\alpha(\lambda)|\alpha(\lambda)\rangle$. Since nothing forbids this state to pick up a phase or a unitary matrix, the final state can be written as

$$|\psi_\alpha(t)\rangle = \mathrm{e}^{-(\mathrm{i}/\hbar)\int_0^t E_0(t')\mathrm{d}t'} U_\mathrm{B}(t)|\alpha(t)\rangle, \tag{38}$$

where U_B is the *Berry matrix* generalizing the Berry phase. It is a unitary mapping $U_\mathrm{B}(t)$: $\mathcal{H}_{E_0(t)} \to \mathcal{H}_{E_0(t)}$ from the subspace of ground states to itself such that $U_B(0) = \mathbb{I}$. Such a map is known as a holonomy, cf [39]. The phase in front of the Berry matrix is the dynamical phase, depending on the evolution of the ground state energy, but it will not be of any importance for our consideration, so we will discard it. The Berry matrix can be written in terms of path ordered integrals,

$$U_\mathrm{B}(t) = \mathcal{P}\exp\left[i\int_0^t \mathcal{A}(t')\,\mathrm{d}t'\right] \tag{39}$$

$$= \mathbb{I} + \sum_{n=1}^\infty i^n \int_0^t \mathrm{d}t_n \int_0^{t_n} \mathrm{d}t_{n-1}\ldots\int_0^{t_2} \mathrm{d}t_1 \mathcal{A}(t_1)\ldots\mathcal{A}(t_n), \tag{40}$$

where $\mathcal{A}_{\alpha,\beta} \equiv \mathrm{i}\langle\alpha(t)|\frac{\mathrm{d}}{\mathrm{d}t}|\beta(t)\rangle$ is the Berry connection, which behaves under unitary transformations as a gauge potential.

4.2. Choosing the field configuration

We consider a non-relativistic spin-$1/2$ particle confined to the plane in a perpendicular magnetic field, $B_z = B\mathbb{I}$. Writing the vector potential, we use cylindrical coordinates and choose the symmetric gauge

$$A_\phi = \frac{Br}{2}\mathbb{I}. \tag{41}$$

The Hamiltonian of the system $H = \frac{1}{2m}(\vec{p} - q\vec{A})^2\mathbb{I}$ acts identically on both spin states, i.e. the Landau levels are doubly degenerate. We only consider the LLL and write for the eigenstates $|m,\epsilon\rangle$ where $\epsilon \in \{\uparrow,\downarrow\}$. For more details of these eigenstates, see appendix A. To derive the effect of inserting a field configuration $\delta\vec{A}(r)$ in this system, we will consider two particular field configurations labeled as $M = 0$ and $M = -1$, respectively. In appendix D, we present a generic configuration, labeled by an integer M, of which these two are specific cases. There we also give a more detailed derivation, to avoid any cumbersome equations in the main body of this paper and we explain how the label M can be interpreted. Mimicking the insertion of Abelian flux briefly mentioned at the start of this section, we will insert a gauge field in such a way that no additional magnetic field is created away from the origin. Furthermore, we choose a symmetric gauge and make the simplification $\partial_z(\delta\vec{A}) = \vec{0}$. The field now looks like a pure gauge,

$$\delta\vec{A} = \mathrm{i}U(\lambda)\nabla U^\dagger(\lambda), \tag{42}$$

for some unitary matrix $U(\lambda)$, which depends on a parameter λ controlling the adiabatic process. The evolved Hamiltonian is now easily found to be

$$H(\lambda) = U(\lambda)H(0)U^\dagger(\lambda). \tag{43}$$

Since this is just a gauge transformation, we automatically meet the requirement that there is a gap of $\hbar\omega_c$ separating the subspace of ground states from excited states at every point in parameter space. Also, the eigenstates of (43) are easily found to be $U(\lambda)|m, \epsilon\rangle$. This is all the information we need to construct the Berry matrix and find the evolved state. We separately present the results for the gauge configurations with $M = 0$ and $M = -1$.

4.2.1. The case $M = 0$. Our $M = 0$ non-Abelian field configuration reads as follows,

$$\delta A_r(\lambda) = \frac{-\lambda}{1 + (\lambda r)^2}\sigma_\phi,$$

$$\delta A_\phi(\lambda) = \frac{-\lambda^2 r}{1 + (\lambda r)^2}\sigma_z + \frac{\lambda}{1 + (\lambda r)^2}\sigma_r, \tag{44}$$

where we introduced cylindrical coordinates and $\sigma_r = \vec{\sigma} \cdot \vec{u}_r$ and $\sigma_\phi = \vec{\sigma} \cdot \vec{u}_\phi$. The reason why we label this field by $M = 0$ is stated in appendix D and will become especially clear for the case $M = -1$. Note that for $r \gg 1/\lambda$ this field configuration does not depend on λ and behaves as

$$\delta A_\phi \sim -\frac{1}{r}\sigma_z + O(1/r^2). \tag{45}$$

The structure of this field corresponds to shifting the orbital of a spin-$\uparrow$ (spin-$\downarrow$) particle by $+1$ (-1), precisely what would happen by inserting an Abelian flux quantum, where the sign depends on the spin of the particle. From this point onwards, we will refer to such a field as a σ_z-flux quantum, to explicitly distinguish it from the insertion of a purely Abelian flux.

Our starting point is a fully polarized integer qH state, represented as a product state $|\psi(0)\rangle = \bigotimes_{m=0}^{m_f} |m \uparrow\rangle$, which has the first $(m_f + 1)$ LLL orbitals filled with spin-$\uparrow$ particles. We adiabatically insert the non-Abelian flux (44) by slowly sweeping λ from $\lambda = 0$ to its final value. Just like in the Abelian case, we gauge this evolved state back to the initial situation. In this way, the final state lives in the same Hilbert space as the initial one, $\mathcal{H}_{E(0)}$. The resulting final state is as follows,

$$|\psi_0(\lambda)\rangle = \bigotimes_{m=0}^{m_f} (u_m(\lambda)|m \uparrow\rangle - v_m(\lambda)|m+1 \downarrow\rangle). \tag{46}$$

The mixing coefficients $\{u_m(\lambda), v_m(\lambda)\}$ depend on both the orbital m and the adiabatic parameter λ. They are given in (D.8) and are plotted in figure 2 as a function of m, for two values of λ. Around the orbital number $m \sim 1/(2\lambda^2)$, $u_m(\lambda)$ and $v_m(\lambda)$ cross, resulting in a vanishing of the z-component of the spin.

The asymptotic behavior of the mixing coefficients as $m \to \infty$ can be read off from (D.9). For large m, corresponding to a radius $r \gg 1/\lambda$, the adiabatic process boils down to $|m \uparrow\rangle \to |m+1 \downarrow\rangle$. This follows directly from the Berry matrix calculation, but it can be understood simply from the conservation of $J_z = L_z + \frac{1}{2}\sigma_z$. At a large distance, the flux we insert is essentially a σ_z-flux quantum (45), and it induces a charge transfer $|m\rangle \to |m+1\rangle$, since all
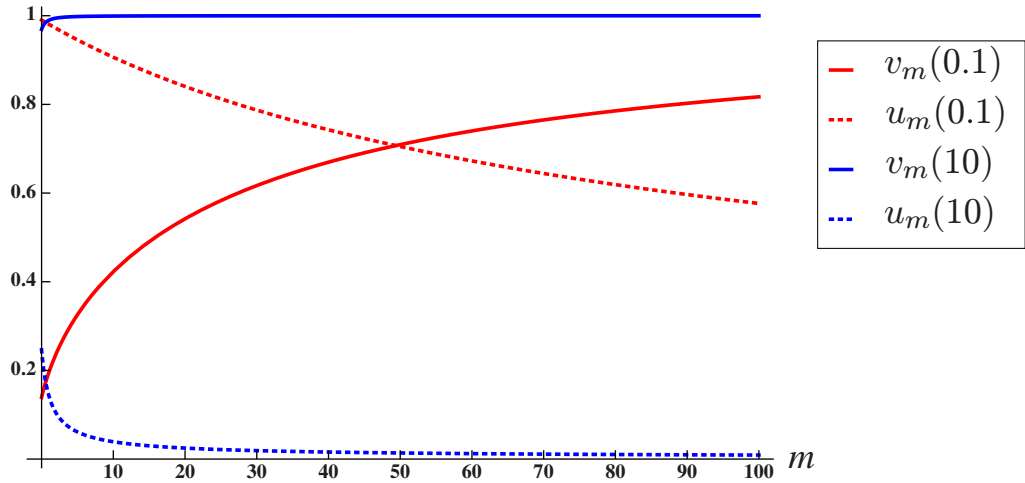
Figure 2. Mixing coefficients $u_m(\lambda)$ and $v_m(\lambda)$ for two different values of λ. Around $m \sim 1/(2\lambda^2)$, $u_m(\lambda)$ and $v_m(\lambda)$ are equal to each other.

particles are spin-$\uparrow$, changing the angular momentum L_z by one unit. Then the only way to accommodate the conservation of J_z is through an accompanying spin flip $|\uparrow\rangle \rightarrow |\downarrow\rangle$.

We can analyze the effect of this adiabatic insertion on the product state, by looking at the density and spin profile of the final state (46). The density is given by

$$\rho(r; \lambda) = \sum_{m=0}^{m_f} \frac{r^{2m} \, \mathrm{e}^{-r^2/2}}{2^m m! 2\pi} \left(u_m(\lambda)^2 + v_m(\lambda)^2 \frac{r^2}{2(m+1)} \right) \tag{47}$$

and is shown in figure 3 for four different values of λ. The solid line shows the droplet before insertion of the non-Abelian field configuration; this is a flat profile. Once we insert flux, charge is depleted from the origin and deposited at the edge of the droplet. This is exactly one unit of charge.

The expectation value of spin in the z-direction of these configurations is depicted in figure 4. The state before flux insertion is the blue solid line, which has trivial spin texture. On increasing λ, particles move one orbital out while flipping their spin. This motion starts at the outer edge of the sample and propagates towards the center. When the final value of λ has been reached, the particles constitute a spin texture of size $1/\lambda$ with spin-$\uparrow$ at the origin and spin-$\downarrow$ at the edge of the droplet. Figure 5 shows the spin field after inserting a flux parameterized by $\lambda = 1/3$. The right picture displays the (x, y)-components of the spin field, showing that the spins have in plane winding number 1.

The charged spin texture created by the insertion of non-Abelian flux is recognized as a quantum Hall Skyrmion of unit electric charge, $q = e$, and unit topological charge, $Q_{\text{top}} = 1$. The topological charge, given by the Pontryagin index Q_{top}, measures the winding of the spin vector around the system (see, for example, [41], chapter 7) and can easily be determined by looking at figure 5. The left figure shows that the spin in the z-direction points up in the origin and down at the edge, so the in plane winding cannot be deformed into a trivial texture.

4.2.2. The case $M = -1$. Our second non-Abelian field configuration corresponds to the $M = -1$ case of the generic flux presented in appendix D. We will insert it adiabatically into
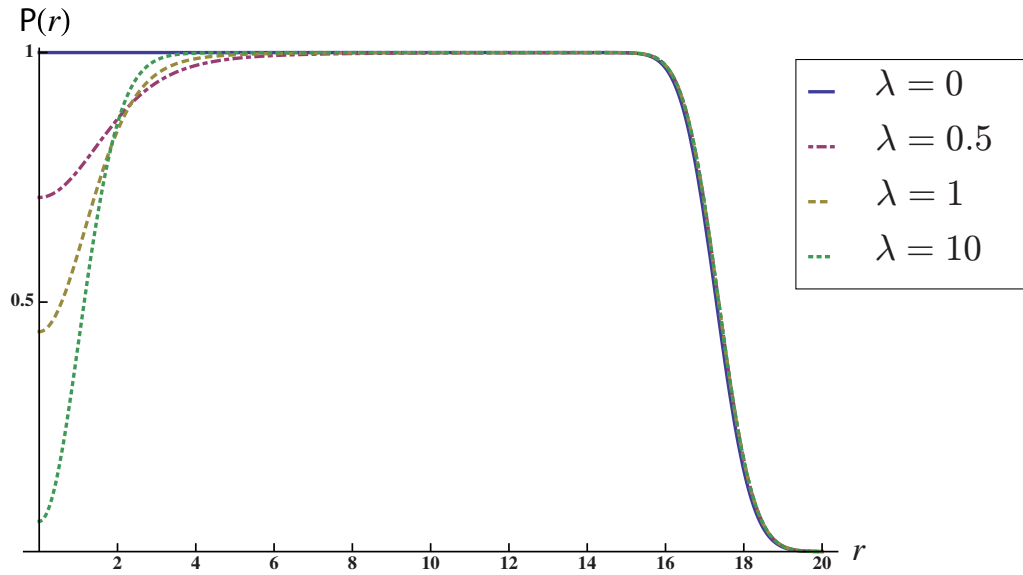
Figure 3. Density profile for different values of λ of a product state where the first 150 orbitals are filled with spin-$\uparrow$ particles. Before flux insertion ($\lambda = 0$), there is a flat profile, but for finite values of λ, a quasihole of unit charge is created around the origin.
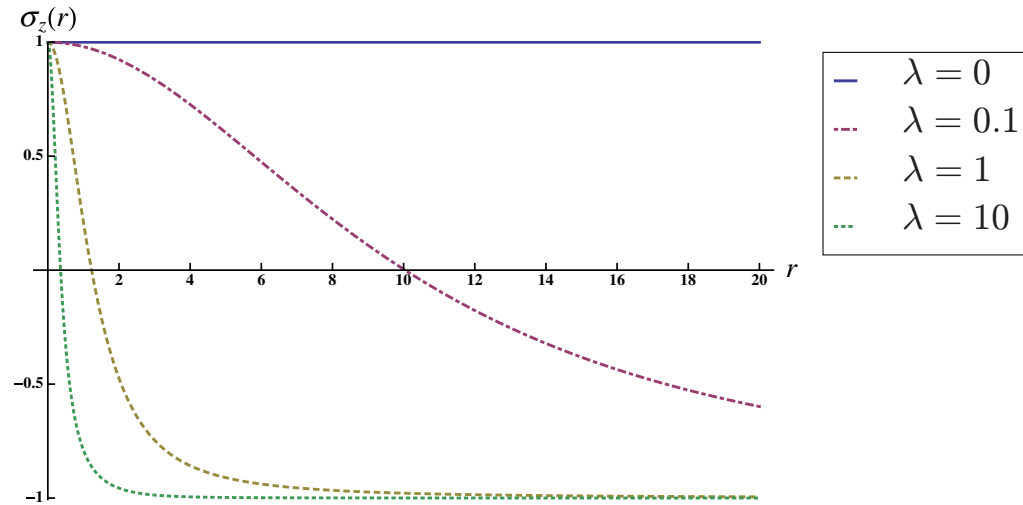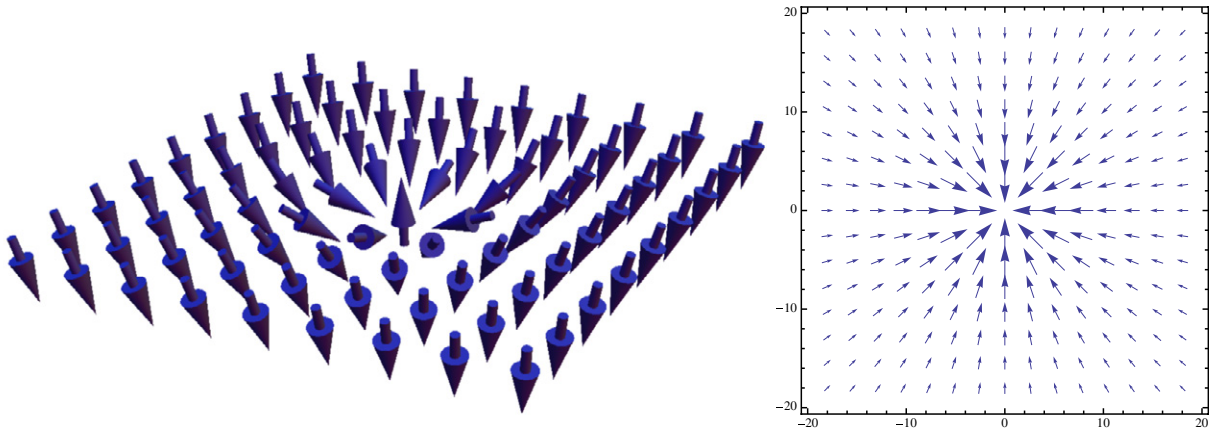

Figure 4. Expectation value of the z-component of the spin field in a product state where the initial state has its first 150 orbitals filled with spin-$\uparrow$ particles. These profiles are after insertion of non-Abelian flux, for different values of λ. An insertion of finite λ creates a non-trivial spin texture. The radius at which the expectation value of σ_z equals zero is around $r \sim 1/\lambda$.

the initial setting (41). The gauge potential reads

$$\delta A_r(\lambda) = \frac{\lambda}{\lambda^2 + r^2} \sigma_\phi,$$

$$\delta A_\phi(\lambda) = -\frac{r}{\lambda^2 + r^2} \sigma_z - \frac{\lambda}{\lambda^2 + r^2} \sigma_r. \tag{48}$$

Figure 5. Spin texture obtained after inserting non-Abelian flux with $\lambda = 1/3$. The three components of the spin field are shown in the left picture. At the origin, the spin points up and at the edge it points down. The figure at the right depicts the x- and y-components of the spin field. These two figures clearly show that a Skyrmion with in-plane winding number 1 is created.

There is a subtlety that did not arise in the previously discussed configuration and that will shed light on why we label the different fields by an integer M. This time, $\delta A_\phi(0) = \frac{-1}{r}\sigma_z \neq 0$, which is the insertion of a σ_z-flux quantum, resulting in a shift in orbital number depending on the spin of the particle,

$$|m \uparrow\rangle \rightarrow |m+1 \uparrow\rangle, \quad |m \downarrow\rangle \rightarrow |m-1 \downarrow\rangle. \tag{49}$$

The adiabatic process consists of two parts now. We start by adiabatically inserting a σ_z-flux quantum, leading to the configuration (48) at $\lambda = 0$. After that, we slowly sweep λ so as to reach its final value. Note that at every point of the adiabatic process we are able to find the eigenstates of the evolved Hamiltonian. Again starting from a product state of spin-$\uparrow$ particles and gauging back to the original Hamiltonian after the adiabatic process, we obtain a final state

$$|\psi_{-1}(\lambda)\rangle = \bigotimes_{m=0}^{m_f} (u_{m+1}(\lambda)|m+1 \uparrow\rangle - v_{m+1}(\lambda)|m \downarrow\rangle), \tag{50}$$

where the coefficients can be found in (D.8) and are plotted in figure 6. This time the scale at which the spins are flipped is set by $r \sim \lambda$.

The density of (50) is shown in figure 7, before flux insertion and for the values $\lambda = 0, 1, 10$. Again, charge is depleted from the origin, but this time the depth of the hole is largest for $\lambda = 0$, meaning after the insertion of a σ_z-flux quantum at $\lambda = 0$. On increasing λ, particles move inward while flipping their spin. This motion starts at the origin and moves out towards the edge of the sample. The resulting spin texture is depicted in figures 8 and 9. The electric charge ($q = e$) and topological charge $Q_{\text{top}} = 1$ agree with those of the Skyrmion found for $M = 0$.

5. Conclusion

In this paper, we have analyzed and solved the non-Abelian Landau problem on the sphere and analyzed the charge and spin dynamics induced by the insertion of non-Abelian flux in an otherwise Abelian background.
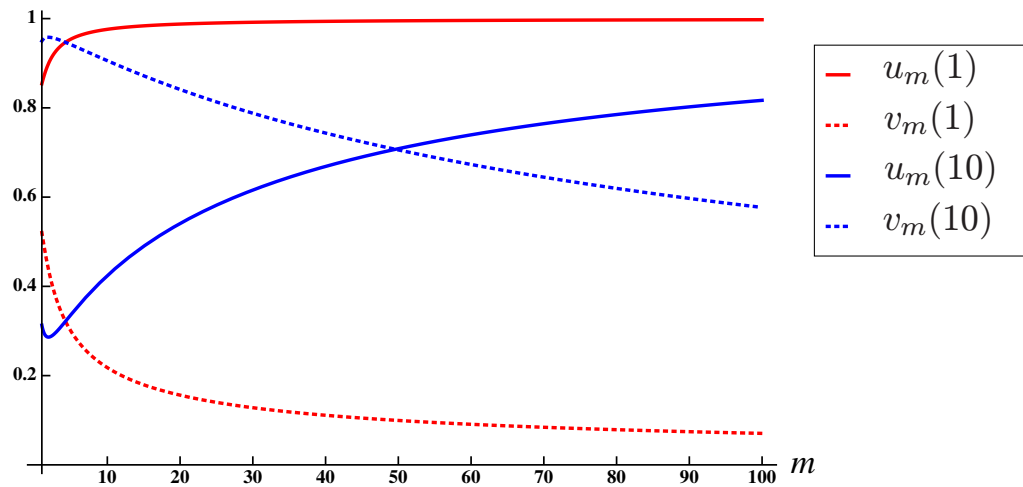
Figure 6. Mixing coefficients of (50) for two different values of the adiabatic parameter λ. The point where $u_m(\lambda) = v_m(\lambda)$ is around $m \sim \lambda^2/2$. The asymptotes of the coefficients are exactly the opposite of the $M = 0$ case.
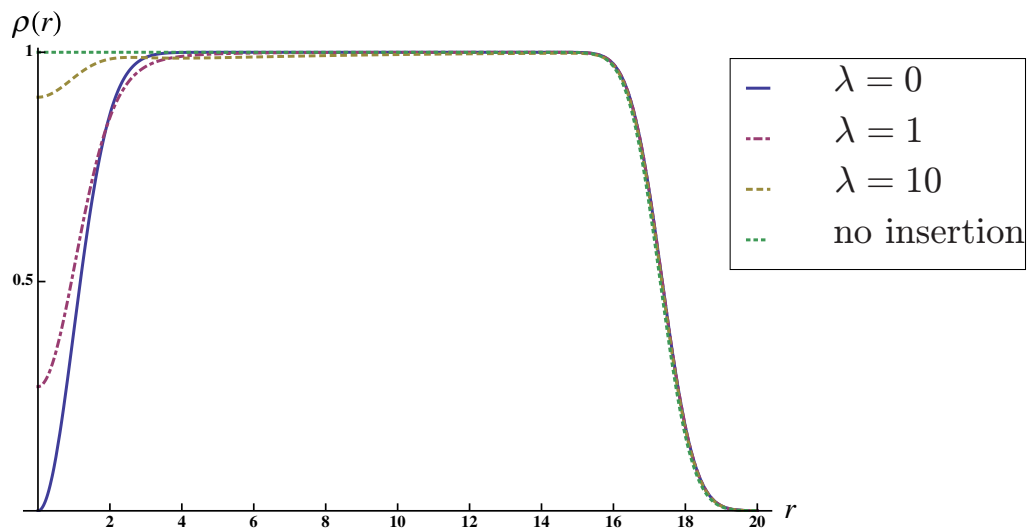

Figure 7. The density of a product state before flux insertion and after, for different values of λ. The flat profile depicts the initial product state. For $\lambda = 0$, a σ_z-flux quantum is inserted, creating a quasihole at the origin. At finite λ, there is still a density depletion around the origin, but it is less sharp.

We remark that, in the usual (Abelian) quantum Hall setting, quantum Hall Skyrmions arise due to a balance between the effects of the Zeeman energy, which favors single overturned spins, and the Coulomb interaction, which favors configurations with small spin gradients [42]. It is quite remarkable that our procedure for driving the non-interacting polarized electron gas with non-Abelian external flux leads to the very same Skyrmion configurations.

Repeating the non-Abelian flux insertion in the background of a $\nu = 2$ integer quantum Hall state, with both the spin-↑ and spin-↓ LLLs completely filled, has a very different effect.
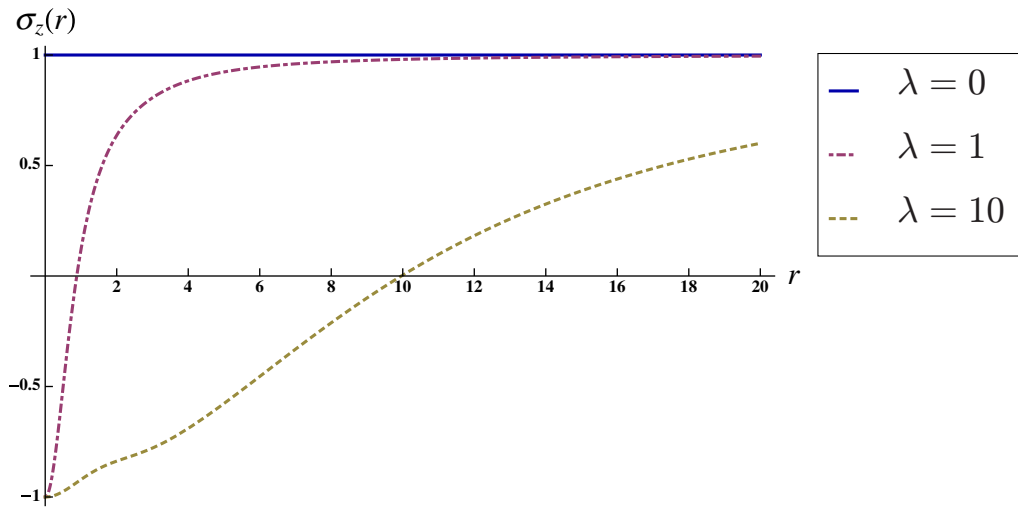
Figure 8. Expectation value of the z-component of the spin field, in a product state (50) for different values of λ. The radius at which the state is unpolarized is around $r \sim \lambda$.
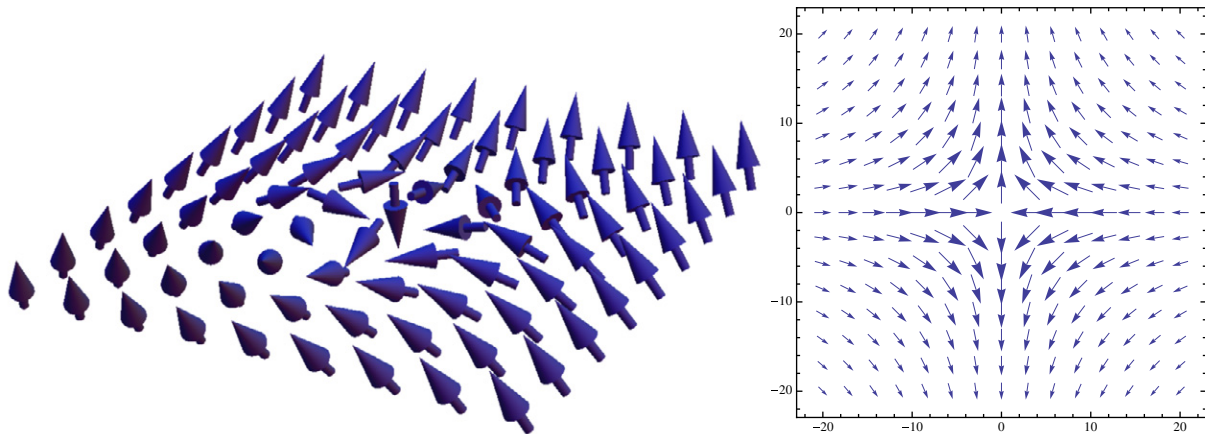
Figure 9. The left figure shows the spin field for a final state labeled by $\lambda = 5$. At the origin, the spin is pointing down; at the edge, it points up. The right figure shows the x- and y-components of the spin field, from which we see that this flux insertion created a Skyrmion with in-plane winding number -1.

In this case, the bulk state cannot accommodate any spin flips and the effects of the flux insertion are limited to the edges. Inserting non-Abelian flux through the central hole in a Corbino disc leads to neutral $S_z = \pm 1$ excitations at both the inner and the outer edge. This situation is in many ways reminiscent of a theoretical experiment, where a minimal amount of Abelian flux inserted into a 2D spin quantum Hall (SQH) topological phase acts as a spin pump, resulting in neutral $S_z = \pm 1/2$ excitations at the edges [28].

The details of the charge and spin dynamics associated with the insertion of non-Abelian flux depend on the specific form of our gauge potentials and on the way these depend on the sweep parameter λ. One expects that many features, in particular the topological quantum

numbers characterizing the resulting spin textures, will be robust against changes in the detailed shape of the external gauge potentials.

Acknowledgments

Appendix A. Landau levels on the plane

To make this paper self-contained and also to fix notations, we recall the main results about the Landau problem on the plane. We consider a particle of charge q and mass m confined to a plane, under an external perpendicular magnetic field $\vec{B} = B\vec{u}_z$ (with $qB > 0$). The standard choice for the vector potential $\vec{A}$ is

$$\vec{A} = \frac{B}{2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{Br}{2}\vec{u}_\phi. \tag{A.1}$$

This is called the symmetric gauge because it behaves as a vector under rotations around $\vec{u}_z$. The only scale of the classical problem is the cyclotron frequency ω_c,

$$\omega_c = \frac{qB}{m}. \tag{A.2}$$

The quantum mechanical problem has an additional scale, the magnetic length l_m,

$$l_m = \sqrt{\frac{\hbar}{qB}}. \tag{A.3}$$

The Hamiltonian reads in the symmetric gauge,

$$H = \frac{1}{2}\omega_c \left(\left(-il_m\partial_x + \frac{y}{2l_m} \right)^2 + \left(-il_m\partial_y - \frac{x}{2l_m} \right)^2 \right). \tag{A.4}$$

It is very convenient to go to complex coordinates (rescaled by the magnetic length), and to introduce two commuting families of creation and annihilation operators,
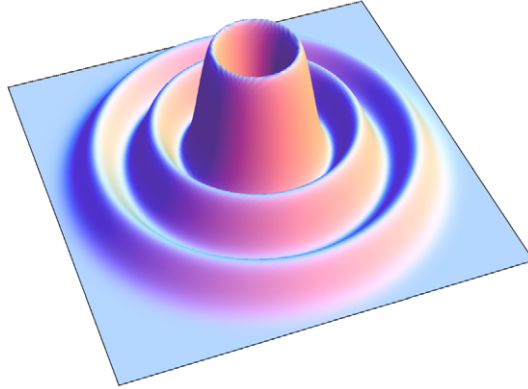
$$a = \sqrt{2}\left(\bar{\partial} + \frac{z}{4} \right), \quad a^\dagger = \sqrt{2}\left(-\partial + \frac{\bar{z}}{4} \right), \tag{A.5}$$

$$b = \sqrt{2}\left(\partial + \frac{\bar{z}}{4} \right), \quad b^\dagger = \sqrt{2}\left(-\bar{\partial} + \frac{z}{4} \right), \tag{A.6}$$

where $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. In these notations, the Hamiltonian and the angular momentum have a very simple expression,

$$H = \omega_c \left(a^\dagger a + \frac{1}{2} \right), \quad L_z = b^\dagger b - a^\dagger a, \tag{A.7}$$

from which the spectrum $E_n = \omega_c(n + 1/2)$ follows immediately. b and $b^\dagger$ are also the generators of magnetic translation. Since they commute with the Hamiltonian, all of these eigenvalues

Figure A.1. Density profile of a few orbitals in the LLL, namely $m = 2$, 12 and 30 LLL orbitals. The cylindrical symmetry is clearly visible.

are infinitely degenerate. The subspace of energy $E_n = \omega_c(n + 1/2)$ is called the nth Landau level (LL).

Denoting by n and m the eigenvalues of $a^\dagger a$ and $b^\dagger b$, respectively, the Hilbert space is spanned by the states $|n, m\rangle$ for $n, m \geqslant 0$. The additional quantum number m is related to the value of the angular momentum $L_z|n, m\rangle = (m - n)|n, m\rangle$.

The explicit form of their wave functions is known and involves a special class of functions called Hermite polynomials. In this appendix, we focus on the LLL $n = 0$: it is obtained by acting with $b^\dagger$ on the state $|0, 0\rangle$,

$$|0, m\rangle = \frac{\left(b^\dagger\right)^m}{\sqrt{m!}}|0, 0\rangle \;\rightarrow\; \langle z|0, n\rangle = \frac{1}{\sqrt{2\pi}}\frac{z^m}{\sqrt{2^m m!}}\exp(-z\bar{z}/4). \tag{A.8}$$

The orbital $|m\rangle$ has angular momentum m, and the support of the wave function (A.8) is a ring located at distance $\sqrt{2m}$ (in magnetic length scale) from the origin, as can be seen from figure A.1.

Appendix B. Landau levels on the sphere

In this section of the appendix, we set $q = \hbar = 1$.

B.1. Field configuration: magnetic monopole

On the sphere, a uniform perpendicular magnetic field $\vec{B} = \frac{N_\phi}{2r^2}\vec{u}_r$ implies the presence of a magnetic monopole at the center of the sphere, and the potential $\vec{A}$ must have a singularity (Dirac string) somewhere on the sphere. The gauge where the singularity lies at the south pole, i.e.

$$\vec{A} = \frac{N_\Phi}{2}\frac{1 - \cos(\theta)}{r\sin(\theta)}\vec{u}_\phi, \tag{B.1}$$

and the gauge where the singularity lies at the north pole,

$$\vec{A} = -\frac{N_\Phi}{2}\frac{1 + \cos(\theta)}{r\sin(\theta)}\vec{u}_\phi, \tag{B.2}$$

are related by the following unitary transformation $U = e^{iN_\phi \phi}$. This operator only makes sense on the sphere when N_Φ is an integer. This is Dirac's quantization condition [36].

B.2. The Hamiltonian and the spectrum

The Hamiltonian of a particle confined to the sphere of radius r in the background of such a magnetic monopole is

$$H = \frac{\vec{\Lambda}^2}{2mr^2} \quad \text{with } \vec{\Lambda} = \vec{r} \wedge (\vec{p} - \vec{A}). \tag{B.3}$$

The operators Λ_a have the following (gauge invariant) commutation relations,

$$[\Lambda_a, \Lambda_b] = i\epsilon_{abc}(\Lambda_c + (\vec{r} \cdot \vec{B})r_c), \tag{B.4}$$

and the generators of (magnetic) rotations have the form

$$\vec{L} = \vec{\Lambda} - (\vec{r} \cdot \vec{B})\vec{r} = \vec{L} - \frac{N_\Phi}{2}\vec{u}_r. \tag{B.5}$$

They generate an SU(2) symmetry,

$$[L_a, L_b] = i\epsilon_{abc}L_c, \tag{B.6}$$

and the Hamiltonian boils down to the Casimir L^2. Indeed, the relation

$$\Lambda^2 = L^2 - \left(\frac{N_\Phi}{2}\right)^2 \tag{B.7}$$

- ensures that all L_a commute with the Hamiltonian;
- gives the spectrum of the Hamiltonian: $\frac{1}{2mr^2}(l(l+1) - (\frac{N_\Phi}{2})^2)$.

The last statement simply comes from the SU(2) algebra obeyed by the L_as, which forces the eigenvalues of L^2 to be of the form $l(l+1)$, where $l \in \frac{1}{2}\mathbb{N}$. However not of all these values of l are possible. Using the explicit expression of $\vec{L}$, Wu and Yang [35] obtained the following decomposition of the Hilbert space into irreducible representations of the SU(2) algebra generated by $\vec{L}$,

$$\mathcal{H} = \left(\frac{N_\Phi}{2}\right) \oplus \left(\frac{N_\Phi}{2} + 1\right) \oplus \cdots \oplus \left(\frac{N_\Phi}{2} + n\right) \oplus \cdots \tag{B.8}$$

and the (Abelian) spectrum on the sphere finally reads

$$E_n = \frac{1}{2mr^2}\left(n(N_\phi + n + 1) + \frac{N_\Phi}{2}\right) \quad n \geqslant 0. \tag{B.9}$$

Appendix C. More details of the non-Abelian field on the sphere

In this section, we derive the spectrum of the Hamiltonian

$$H(\alpha) = \frac{1}{2mr^2}[\vec{r} \times (\vec{p} - \vec{A}(\alpha))]^2, \tag{C.1}$$

describing a particle confined to a sphere of radius r in the non-Abelian background potential

$$\vec{A}(\alpha) = \vec{A}_{\text{Ab}} + \alpha \frac{\vec{r} \times \vec{\sigma}}{r^2}, \tag{C.2}$$

where $\vec{A}_{\text{Ab}}$ stands for the $U(1)$ potential (B.1). Note that there is a gauge transformation mapping $\alpha \to 1 - \alpha$ implemented by the unitary transformation $U = \frac{\vec{r} \cdot \vec{\sigma}}{r} = \sigma_r$.

C.1. Rotational symmetry and decomposition of the Hilbert space

There are two sets of SU(2) generators in this problem:

- the usual (Abelian) action on the coordinates implemented by $\vec{L} = \vec{r} \times (\vec{p} - \vec{A}_{\text{Ab}}) - \frac{N_\Phi}{2} \frac{\vec{r}}{r}$ defined in (B.5),
- the rotations in spin space generated by $\vec{S} = \frac{1}{2}\vec{\sigma}$.

The Hamiltonian we are considering is not invariant under $\vec{L}$ and $\vec{S}$ separately. However, it is a scalar under global rotations generated by $\vec{J} = \vec{L} + \vec{S}$, as can be seen from the expansion in terms of $\vec{J}$,

$$H(\alpha) = \frac{1}{2mr^2}\left[J^2 + \frac{1}{4} - 2\alpha(1-\alpha) + (2\alpha - 1)\left(\vec{J}\cdot\vec{\sigma} - \frac{1}{2} + \frac{N_\Phi}{2}U \right) + \frac{N_\Phi}{2}U \right]. \tag{C.3}$$

Therefore, this Hamiltonian is block diagonal with respect to the decomposition of the Hilbert space into irreducible representations of $\vec{J}$. This decomposition follows directly from the Abelian one (B.8),

$$\mathcal{H} = \left(\frac{N_\Phi - 1}{2} \right) \oplus 2\left(\frac{N_\Phi + 1}{2} \right) \oplus 2\left(\frac{N_\Phi + 3}{2} \right) \oplus 2\left(\frac{N_\Phi + 5}{2} \right) \oplus \cdots. \tag{C.4}$$

C.2. The spectrum

Working in the subspace $J^2 = j(j+1)$, we simply need to diagonalize the term $X = (2\alpha - 1)(\vec{J}\cdot\vec{\sigma} - \frac{1}{2} + \frac{N_\Phi}{2}U) + \frac{N_\Phi}{2}U$. We first derive the following two relations,

$$\left\{ U, \left(\vec{J}\cdot\vec{\sigma} - \frac{1}{2} + \frac{N_\Phi}{2}U \right) \right\} = 0, \tag{C.5}$$

$$\left(\vec{J}\cdot\vec{\sigma} - \frac{1}{2} \right)^2 = \vec{J}^2 + \frac{1}{4}. \tag{C.6}$$

The first one is a consequence of the gauge equivalence $UH(\alpha)U = H(1-\alpha)$, and the second one can be checked using the explicit form of $\vec{J}$. From this, we deduce that X^2 is a constant,

$$X^2 = (2\alpha - 1)^2\left(\vec{J}^2 + \frac{1}{4} \right) + \alpha(1-\alpha)N_\Phi^2 \tag{C.7}$$

and we obtain the following spectrum for X,

$$\lambda_{(j)}^{\pm} = \pm \sqrt{(2\alpha - 1)^2\left(\left(j + \frac{1}{2} \right)^2 - \left(\frac{N_\Phi}{2} \right)^2 \right) + \left(\frac{N_\Phi}{2} \right)^2}. \tag{C.8}$$

As can be seen in (25), for $j \geqslant \frac{N_\Phi + 1}{2}$, there are two representations of spin (j), and from (C.5), both eigenvalues $\lambda_{(j)}^{\pm}$ belong to the spectrum. However, there is a unique representation of spin $j = \frac{N_\Phi - 1}{2}$. In this irrep. $U = 1$ and the corresponding eigenvalue is the positive one: $\frac{N_\Phi}{2}$. Rewriting $j = n + \frac{N_\Phi - 1}{2}$, we obtain the following spectrum for the Hamiltonian,

$$E_0(\alpha) = \frac{1}{2mr^2}\left(\frac{N_\Phi}{2} - 2\alpha(1-\alpha) \right), \tag{C.9}$$

$$E_n^{\pm}(\alpha) = \frac{1}{2mr^2}\left(n(N_\Phi + n) - 2\alpha(1-\alpha) \pm \sqrt{(2\alpha - 1)^2 \, n\,(N_\Phi + n) + (N_\Phi/2)^2} \right). \tag{C.10}$$

Appendix D. Generic non-Abelian field configuration

In this appendix, we present a detailed derivation of the final state obtained after an adiabatic insertion of non-Abelian flux. This is done for the generic case, two specific examples of which are discussed in section 4. The field configuration we insert is the following,

$$
\delta A_r(\lambda) = -\left(M + \frac{1}{2}\right)\frac{2\lambda r^{2M}}{1 + \lambda^2 r^{2+4M}}\sigma_\phi,
$$

$$
\delta A_\phi(\lambda) = \left(M + \frac{1}{2}\right)\frac{1 - \lambda^2 r^{2+4M}}{1 + \lambda^2 r^{2+4M}}\frac{1}{r}\sigma_z + \left(M + \frac{1}{2}\right)\frac{2\lambda r^{2M}}{1 + \lambda^2 r^{2+4M}}\sigma_r - \frac{1}{2r}\sigma_z,
$$

(D.1)

where M can be interpreted as the number of σ_z-flux quanta inserted, which will be explained below (D.3). Inserting such a field boils down to performing a gauge transformation on the system,

$$
U_M(\lambda) = \frac{1}{\sqrt{1 + \lambda^2 r^{2+4M}}}\begin{pmatrix} 1 & -\lambda \bar{z} r^{2M} \\ \lambda z r^{2M} & 1 \end{pmatrix}\exp(\mathrm{i}M\phi\sigma_z).
$$

(D.2)

So, at every point of the adiabatic process, we know the LLL eigenstates of the evolved Hamiltonian; they are given by

$$
|\alpha(\lambda)\rangle = U_M(\lambda)|m, \epsilon\rangle.
$$

(D.3)

Before we proceed with calculating the Berry matrix, an important subtlety needs to be considered. We wish to insert this field configuration into a background (41). But at $\lambda = 0$ and for $M \neq 0$, (D.1) is given by $\delta A_\phi(0) = (M/r)\sigma_z \neq 0$, which means that we have to start by adiabatically inserting M σ_z-flux quanta, resulting in a shift in the orbitals depending on the spin of the particle,

$$
|m \uparrow\rangle \to |m - M \uparrow\rangle, \quad |m \downarrow\rangle \to |m + M \downarrow\rangle.
$$

(D.4)

After the insertion of these σ_z-flux quanta, we slowly sweep λ from zero to some final value resulting into (D.1). Now we can use the eigenstates (D.3) to compute the Berry connection,

$$
\mathcal{A}_{\alpha,\beta} \equiv \mathrm{i}\langle\alpha(\lambda)|\frac{\mathrm{d}}{\mathrm{d}t}|\beta(\lambda)\rangle = \mathrm{i}\langle\alpha(0)|U_M^\dagger \dot{U}_M|\beta(0)\rangle,
$$

(D.5)

where

$$
\mathrm{i}U_M^\dagger \dot{U}_M = \frac{\mathrm{i}\dot{\lambda}}{1 + \lambda^2 r^{2+4M}}\begin{pmatrix} 0 & -\bar{z}^{2M+1} \\ z^{2M+1} & 0 \end{pmatrix}.
$$

(D.6)

The Berry connection only has non-zero elements between states of the form $\{U_M(\lambda)|m \uparrow\rangle, U_M(\lambda)|m + 2M + 1 \downarrow\rangle\}$. Written in this basis, for every m, the Berry matrix is a 2×2 matrix,

$$
U_B^m = \cos(\theta_m^{(M)}(\lambda))\mathbb{I} + \mathrm{i}\sin(\theta_m^{(M)}(\lambda))\sigma_y,
$$

(D.7)

where the angle is given by

$$
\theta_m^{(M)}(\lambda) = \int_0^\infty \mathrm{d}r \arctan(\lambda r^{1+2M})\frac{r^{2+2M+2m}\,\mathrm{e}^{-r^2/2}}{2^{m+M}\sqrt{2m!(m+2M+1)!}}.
$$

(D.8)

This angle has interesting asymptotes in two different limits,

$$
\lim_{m\to\infty}\theta_m^{(M)} = \arctan(\lambda(2m)^{M+1/2}),
$$

(D.9)

$$\lim_{\lambda\to\infty}\theta_m^{(M)}=\frac{\pi}{2}\frac{\Gamma(m+M+3/2)}{\sqrt{m!(m+2M+1)!}}. \tag{D.10}$$

After the adiabatic insertion of flux, we gauge the system back to the initial one. This cycle has the following effect on a single particle state $|m\uparrow\rangle$,

$$U_M^\dagger(\lambda)U_B^m(\lambda)U_M(\lambda)|m\uparrow\rangle=u_m^{(M)}(\lambda)|m\uparrow\rangle-v_m^{(M)}(\lambda)|m+2M+1\downarrow\rangle, \tag{D.11}$$

where the mixing coefficients are expressed in terms of (D.8),

$$u_m^{(M)}(\lambda)\equiv\cos\big(\theta_m^{(M)}(\lambda)\big),\quad v_m^{(M)}(\lambda)\equiv\sin\big(\theta_m^{(M)}(\lambda)\big). \tag{D.12}$$

That the equality in (D.11) holds can be seen by inserting unity $U_M(\lambda)|m',\epsilon\rangle\langle m',\epsilon|U_M^\dagger(\lambda)$ between $U_M^\dagger$ and U_B^m. After deducing the effect of the two stages of the adiabatic process, we can combine them to find the final state. Before we give the final state, one last remark needs to be made. Since we want to stay in the LLL, we have to put the state on a Corbino disc, meaning that we fill the orbitals of the initial product state with spin-$\uparrow$ particles starting from some initial orbital m_i up to a final orbital m_f. The two specific adiabatic flux insertions given in section 4 are actually the only two situations for which the Corbino disc is not a necessary geometry for staying in the LLL. Starting from a product state on a Corbino disc where the orbitals are filled with spin-$\uparrow$ particles, the final state, after first adiabatically inserting M σ_z-flux quanta, then cranking up the value of λ in (D.1) and finally gauging back to the initial configuration, is given by

$$|\psi_M(\lambda)\rangle=\bigotimes_{m=m_i}^{m_f}\Big(u_{m-M}^{(M)}(\lambda)|m-M\uparrow\rangle-v_{m-M}^{(M)}(\lambda)|m+1+M\downarrow\rangle\Big). \tag{D.13}$$

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