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Δ_1-completions of a Poset

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Abstract A join-completion of a poset is a completion for which each element is obtainable as a supremum, or join, of elements from the original poset. It is well known that the join-completions of a poset are in one-to-one correspondence with the closure systems on the lattice of up-sets of the poset. A Δ_1-completion of a poset is a completion for which, simultaneously, each element is obtainable as a join of meets of elements of the original poset and as a meet of joins of elements from the original poset. We show that Δ_1-completions are in one-to-one correspondence with certain triples consisting of a closure system of down-sets of the poset, a closure system of up-sets of the poset, and a binary relation between these two systems. Certain Δ_1-completions, which we call compact, may be described just by a collection of filters and a collection of ideals, taken as parameters. The compact Δ_1-completions of a poset include its MacNeille completion and all its join- and all

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its meet-completions. These completions also include the canonical extension of the given poset, a completion that encodes the topological dual of the poset when it has one. Finally, we use our parametric description of Δ_1-completions to compare the canonical extension to other compact Δ_1-completions identifying its relative merits.

Keywords Completions of a poset · Canonical extensions

1 Introduction

Embedding partially ordered sets in complete lattices is central to the solution of many problems. This is most famously exemplified by the construction of the real numbers from the rational numbers [6]. Dedekind's construction, generalised to arbitrary posets by MacNeille [24], yields a completion for any poset P. The MacNeille completion of a poset is minimal in some sense but it does not have good algebraic preservation properties, e.g. it does not preserve homomorphisms, equational properties, and the like. For this purpose one would rather want a greatest completion, or in other words, a free completion. One-sided free completions exist and have indeed been used to great success, e.g. in domain theory. However, two-sided free completions (that is, completions that are free with respect to complete lattice homomorphisms rather than just with respect to complete join-morphisms or complete meet-morphisms) are well known not to exist—see e.g. [20, Section 4.7].

In [21, 22] Bjarni Jónsson and Alfred Tarski introduced a two-sided completion for Boolean algebras, called canonical extension, which is the free completion within the class of completely distributive (or equivalently, atomic) complete Boolean algebras. The canonical extension is closely related to duality theory and provides an algebraic approach to topological duality [15, 16]. As a completion, it has the virtue of preserving more equational properties than the MacNeille completion [13] and it has played a substantial role, in a more or less hidden way, in the semantic study of many logics including modal and intuitionistic logic [4, 7]. More recently, canonical extension has been generalised to encompass distributive lattices [14], lattices [12], and even posets [8] and the theory has been applied in logic and algebra, e.g. [1, 8, 15, 16]. Canonical extensions have the drawback that they do not preserve existing infinite meets and joins and, in the other direction, they do not add finite joins and meets in a free way. Depending on the application, e.g. fixed-point logics versus very weak fragments of logic [18], different completions may be appropriate.

In this paper we study a class of two-sided completions encompassing both the MacNeille completion and the canonical extension. The identifying property of these completions is their placement in a meet-join complexity hierarchy for completions. This hierarchy was first brought to the attention of the first author by Keith Kearnes in a private communication and was subsequently studied in relation to canonical extension in [17]. The completions that we focus on here are called Δ_1-completions and figure in the hierarchy as the completions for which each element is reachable by joins of meets *and* by meets of joins of elements from the original poset. Included among the Δ_1-completions are the Σ_0-completions which are the well-known join-completions as well as the Π_0-completions which are the meet-completions. The canonical extension of a poset is an example of a Δ_1-completion which, in general, is neither a Σ_0-completion nor a Π_0-completion. Any Δ_1-completion can be built from a collection of up-sets and a collection of down-sets of the original poset. However, possibly somewhat surprisingly, specifying collections of up-sets and down-sets is not in general sufficient for specifying a Δ_1-completion.

Our main result (Theorem 3.3) is a complete classification of the Δ_1-completions of a poset P in terms of certain polarities $(\mathcal{F}, \mathcal{I}, R)$ where $\mathcal{F}$ is a closure system of up-sets of P and $\mathcal{I}$ is a closure system of down-sets of P and R is a relation from $\mathcal{F}$ to $\mathcal{I}$ satisfying four simple conditions. The relation essentially specifies which meets of down-sets are below which joins of up-sets in the completion.

Given a poset P, and closure systems $\mathcal{F}$ and $\mathcal{I}$ of up-sets and down-sets of P, respectively, there is always a smallest relation R from $\mathcal{F}$ to $\mathcal{I}$ satisfying the four conditions of Theorem 3.3. It is the relation of non-empty intersection, that is, for every $F \in \mathcal{F}$ and every $I \in \mathcal{I}$,

$$FRI \text{ iff } F \cap I \neq \varnothing. \tag{1}$$

In fact, even for collections of up-sets and down-sets of P which are not closure systems, one may consider the polarity obtained by this relation of non-empty intersection and thus reach a larger class of Δ_1-completions using only this one relation. This is the idea behind what we call compact Δ_1-completions and leads to notions of compactness and denseness which are parametric in collections $\mathcal{F}$ and $\mathcal{I}$ of up-sets and down-sets, respectively. We show that, for each choice of $\mathcal{F}$ and $\mathcal{I}$ containing the principal up-sets and the principal down-sets, respectively, there is (up to isomorphism) a unique completion of P which is $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense. Thus parametric compactness and denseness allow a study of Δ_1-completions in which the relational component may be omitted in the sense that it is fixed to be the relation of non-intersection as given by Eq. 1.

The class of parametrically compact and dense completions includes well-known completions such as the MacNeille completion, the canonical extension, and all join- and meet-completions of a given poset. However, we provide examples of posets for which it is not possible to describe all Δ_1-completions as compact and dense with respect to some $\mathcal{F}$ and some $\mathcal{I}$. Thus the relational component of our classification result is in general necessary. With the purpose of comparing canonical extension as defined in [8] to other completions which are $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense for some choice of $\mathcal{F}$ and $\mathcal{I}$, we give conditions on $\mathcal{F}$ and $\mathcal{I}$ corresponding to various properties of the $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense completion of P, concentrating on properties which are central to applications in logic.

The paper is organised as follows. In Section 2 we introduce Δ_1-completions and show that MacNeille completions and Galois connections are central to the study of these completions. In Section 3 we briefly review the necessary preliminaries on Galois connections including the abstract characterisation of the lattice of Galois closed sets of a polarity as well as its construction as the MacNeille completion of a certain 'intermediate' structure. These tools then allow us to give a complete classification of all Δ_1-completions of a given poset. In Section 4 we introduce what we call Δ_1-polarities which are based on collections of up-sets and down-sets that are not necessarily closure systems. In Section 5 we study parametric compactness and in Section 6 we identify the properties required of $\mathcal{F}$ and $\mathcal{I}$ in order that the corresponding completion have various desirable properties.

2 Δ_1-completions

We fix a poset P. A completion of P is an embedding of P in a complete lattice. To be precise, an *order embedding* is a map $e\colon P \to Q$ from a poset P into a poset Q so that, for every $x, y \in P$, we have $x \leq y$ if and only if $e(x) \leq e(y)$. A *poset extension* of P is a pair (e, Q), where Q is a partially ordered set and $e\colon P \to Q$ is an order embedding. A *completion* of P is a poset extension (e, C) of P such that C is a complete lattice. In order to lighten the notation we will generally identify P with $e[P]$ so that e is the inclusion map, and thus we will talk about Q or C being an extension or a completion of P.

Completion is a self-dual concept. However, in general, order properties and order theoretic results come in pairs of dual statements, since $\leq$ being an order implies that the converse relation, $\geq$, is an order as well. Throughout the present paper, we will take for granted that each definition and each result also yields a dual definition or result, indicating only the additional notation as needed.

A subset X of P is a *down-set* of P provided that for every $x \in X$ and $y \in P$ we have that $y \leq x$ implies $y \in X$. For $x \in P$ we denote by $\downarrow x$ the down-set $\{ y \in P \mid y \leq x \}$ and the least down-set that includes $X \subseteq P$ will be denoted by $\downarrow X$. We denote the complete lattice of all down-sets of P with the inclusion order by $\mathcal{D}(P)$. Likewise, the up-set of $x \in P$ will be denoted by $\uparrow x$, the least up-set containing $X \subseteq P$ will be denoted by $\uparrow X$, and the complete lattice of all up-sets of P will be denoted by $\mathcal{U}(P)$. The restriction of a down- or up-set to a subposet Q will be denoted by a subscript on the arrow, e.g. $\downarrow_Q X = \downarrow X \cap Q$.

Given a completion C of P, we have the following fundamental associated Galois connection:

$$\begin{array}{ccccccc} \bigvee_C & : & \mathcal{D}(P) & \leftrightarrows & C & : & \downarrow_P \\ & & I & \mapsto & \bigvee_C I & & \\ & & \downarrow x \cap P & \leftarrow\!\shortmid & x. & & \end{array}$$

The order dual Galois connection is denoted $\bigwedge_C\colon \mathcal{U}(P) \leftrightarrows C : \uparrow_P$. The fact that these are indeed Galois connections is easy to check. In the case of the second pair of maps, this means that for each up-set $F \in \mathcal{U}(P)$ and each element $x \in C$ we have that

$$x \leq \bigwedge\nolimits_C F \quad \text{if and only if} \quad F \subseteq \uparrow_P(x)$$

which is essentially the content of the definition of meet. Note that the same assignments above define Galois connections on the full power set of P, but the image of the maps $\uparrow_P$ and $\downarrow_P$ will always lie in $\mathcal{U}(P)$ and $\mathcal{D}(P)$ so that any completion of P gives rise to closure operators (given by the composition of the two Galois maps in each case) on $\mathcal{U}(P)$ and $\mathcal{D}(P)$, respectively.

The Galois closed sets of the first Galois connection are:

$$\begin{aligned} \mathcal{I}_C &= \{ I \in \mathcal{D}(P) \mid \downarrow_P(\textstyle\bigvee_C I) = I \} = \{ I \in \mathcal{D}(P) \mid \exists x \in C \quad \downarrow_P(x) = I \} \\ \mathbb{I}(C) &= \{ y \in C \mid \textstyle\bigvee_C \downarrow_P(y) = y \} = \{ y \in C \mid \exists I \in \mathcal{D}(P) \quad \bigvee_C I = y \}. \end{aligned}$$

The corresponding notation for the second Galois connection is $\mathcal{F}_C$ and $\mathbb{F}(C)$, respectively. We refer to the elements of $\mathcal{I}_C$ ($\mathcal{F}_C$) as the *C-normal ideals* (*C-normal* *filters*) of P relative to the completion C. The elements of $\mathbb{I}(C)$ ($\mathbb{F}(C)$) will be referred to as the *ideal elements* (*filter elements*) of the completion C. We also call $\mathbb{I}(C)$ ($\mathbb{F}(C)$) the join-closure (meet-closure) of P in C.

An extension Q of a poset P is *join-dense* provided each element of Q is the join of a collection of elements from P and a *join-completion* of P is a join-dense completion of P. It is clear that a completion C of P is a join-completion if and only if $C = \mathbb{I}(C)$ and that, for any completion C, the extension $\mathbb{I}(C)$ of P is a join-completion of P. The correspondence $C \mapsto \mathcal{I}_C$ is in fact the well-known correspondence between closure systems on $\mathcal{D}(P)$ that contain the principal down-sets and join-completions of P [2]. Following Erné, [9], we will call a collection $\mathcal{I}$ of down-sets of P *standard* provided $\downarrow x \in \mathcal{I}$ for every $x \in P$, and dually for systems of up-sets. Thus the join-completions of a poset P are in one-to-one correspondence with the standard closure systems of down-sets of P and meet-completions are in one-to-one correspondence with standard closure systems of up-sets of P.

An extension Q of P is called *doubly dense* provided it is both join- and meet-dense. It is well known that the MacNeille completion $\mathcal{N}(P)$ of a poset P is, up to isomorphism, the only completion of P that is doubly dense [3]. In this case the elements of the corresponding closure systems $\mathcal{I}_{\mathcal{N}(P)}$ and $\mathcal{F}_{\mathcal{N}(P)}$ are exactly what is usually known in the literature as the normal ideals and the normal filters of P.

We may think of the completions of a poset as lying in a complexity hierarchy according to the (possibly transfinite) number of alternations of joins and meets one has to apply to generate the completion [17]. Thus the Σ_0-completions of a poset are its join-completions, the Π_0-completions are its meet-completions, the Σ_1-completions are those completions of P, every element of which is obtainable as a join of meets of elements from P, and so on. A completion is called a Δ_n-completion if it is both a Σ_n- and a Π_n-completion. Thus the MacNeille completion of a poset is its unique Δ_0-completion. The Δ_1-completions of P are those completions, each element of which may be obtained both as a meet of joins of elements of P and as a join of meets of elements of P. In the canonical extension literature, these completions have been called dense completions [12] and the canonical extension of any poset is such a completion [8].

Our purpose in this paper is to study the Δ_1-completions of a poset. In this section we begin with a simple observation which allows us to see any Δ_1-completion as the MacNeille completion of an extension that is the union of a join-completion and a meet-completion of the original poset.

Proposition 2.1 *Let P be a poset. A completion C of P is a Δ_1-completion of P if and only if C is the MacNeille completion of the poset $\mathbb{F}(C) \cup \mathbb{I}(C)$.*

Proof This follows immediately from the fact that the MacNeille completion of a poset is uniquely determined as the doubly dense completion of the poset, along with the fact that the poset $\mathbb{F}(C) \cup \mathbb{I}(C)$ is join-dense in C if and only if $\mathbb{F}(C)$ is join-dense in C (since $\mathbb{F}(C)$ is always join-dense in $\mathbb{I}(C)$ because it contains P) and the order dual fact about meet-density of $\mathbb{F}(C) \cup \mathbb{I}(C)$ and $\mathbb{I}(C)$. □

Let C be a Δ_1-completion of P. Then $\mathbb{F}(C)$ is a meet-completion of P and it corresponds to the standard closure system $\mathcal{F}_C$ of up-sets of P. Dually, $\mathbb{I}(C)$ is a join-completion of P and corresponds to the standard closure system $\mathcal{I}_C$ of down-sets of P. It is clear that isomorphic completions yield identical closure systems $\mathcal{F}$ and $\mathcal{I}$. However, as the following example shows, it may happen that non-isomorphic Δ_1-completions C and C' of a given poset P satisfy $\mathcal{F}_C = \mathcal{F}_{C'}$ and $\mathcal{I}_C = \mathcal{I}_{C'}$. Consequently, knowing $\mathcal{F}_C$ and $\mathcal{I}_C$ is not always sufficient for determining C.

Example 2.2 Consider the poset $P = \omega \oplus \omega^\partial$, that is, the poset given by a countably infinite ascending chain under a countably infinite descending chain. The MacNeille completion of P is obtained by adjoining a point in between the chains, and the canonical extension of P is obtained by adjoining the two-element chain between the two infinite chains. So $C_1 = \omega \oplus \mathbf{1} \oplus \omega^\partial$ is the MacNeille completion and $C_2 = \omega \oplus \mathbf{2} \oplus \omega^\partial$ is the canonical extension of P. Let us find the closure systems corresponding to these two Δ_1-completions. It should be clear that in both cases the meet-closure of P adjoins one point to P. For C_1 this is the unique point in the middle, while it is the top of the two-element chain for C_2. However, in both cases the corresponding closure system $\mathcal{F}$ of up-sets is the set of all non-empty up-sets, which contains just one non-principal up-set, namely ω^∂. Similarly, the closure system $\mathcal{I}$ of all non-empty down-sets of P is the closure system of down-sets corresponding to the meet-closures of P in both C_1 and C_2.

3 A Classification of the Δ_1-completions of a Poset

It is clear from Example 2.2 that more information is needed to specify a Δ_1-completion of a poset than the associated standard closure systems of up-sets and of down-sets of P. In fact, what is needed is to know how $\mathcal{I}$ and $\mathcal{F}$ are glued together to form the poset $\mathbb{F}(C) \cup \mathbb{I}(C)$ in Proposition 2.1. The most convenient way to present this information is to switch from MacNeille completions to Galois closed sets of polarities.

Our approach to polarities will closely follow the approach taken in [11], and we will use the notation used there. For further details and background on polarities, Galois connections, and their relation to MacNeille completion see also [5, Chapters 3 and 7] and [10].

A polarity is a triple (X, Y, R) where X and Y are non-empty sets and R is a binary relation from X to Y. Such a polarity gives rise to a Galois connection given by

$$\begin{array}{rcccl} (\ \ )^R & : & \mathcal{P}(X) & \leftrightarrows & \mathcal{P}(Y) \quad : \quad {}^R(\ \ ) \\ & & A & \mapsto & \{\, y \mid \forall x\ (x \in A \Rightarrow xRy)\,\} \\ \{\, x \mid \forall y\ (y \in B \Rightarrow xRy)\,\} & & & \leftarrow\!\shortmid & B. \end{array}$$

The Galois closed subsets of X and of Y are, respectively,

$$\mathcal{G}(X, Y, R) = \left\{\, A \subseteq X \mid A = {}^R(A^R) \,\right\} = \left\{\, {}^R B \mid B \subseteq Y \,\right\},$$

$$\mathcal{G}(X, Y, R)^R = \left\{\, B \subseteq Y \mid B = ({}^R B)^R) \,\right\} = \left\{\, A^R \mid A \subseteq X \,\right\}.$$

These are both topped intersection structures and the maps $(\ \ )^R$ and ${}^R(\ \ )$ restrict to mutually inverse dual order-isomorphisms between these complete lattices.

There are natural maps from X and Y whose images are join- and meet-generating subsets of $\mathcal{G}(X, Y, R)$, respectively, [11, Proposition 2.10]. These maps are given by

$$\begin{array}{rl} \Xi : & X \to \mathcal{G}(X, Y, R) \\ & x \mapsto {}^{R}(\{x\}^{R}) \end{array} \qquad \begin{array}{rl} \Upsilon : & Y \to \mathcal{G}(X, Y, R) \\ & y \mapsto {}^{R}\{y\}. \end{array}$$

In particular, for every polarity (X, Y, R), the set $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ is doubly dense in $\mathcal{G}(X, Y, R)$. As a consequence, $\mathcal{G}(X, Y, R)$ is (up to isomorphism) the MacNeille completion of $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$.

Therefore, the ordered set $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ is particularly important for understanding the lattice $\mathcal{G}(X, Y, R)$. We will call the poset $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ the *intermediate structure* of $\mathcal{G}(X, Y, R)$ and denote it by $Int(X, Y, R)$. We can build the intermediate structure directly from the polarity (X, Y, R). To this end, we first take the disjoint union of X and Y and equip it with the pullback $\leq$ of the order on $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ (inherited from $\mathcal{G}(X, Y, R)$) along the map $\Xi \uplus \Upsilon$, thus obtaining a pre-ordered set *pre-*$Int(X, Y, R)$. We then take the quotient of this pre-ordered set by $\leq \cap \geq$ to obtain the poset $Int(X, Y, R)$. That is, for $z_1, z_2 \in$ *pre-*$Int(X, Y, R) := X \uplus Y$, we let

$$z_1 \leq z_2 \text{ iff } \Xi \uplus \Upsilon(z_1) \subseteq \Xi \uplus \Upsilon(z_2).$$

The resulting pre-order is described in the following proposition [11, Proposition 2.7]:

Proposition 3.1 *Let (X, Y, R) be a polarity and $z_1, z_2 \in$ pre-$Int(X, Y, R)$.*

(1) *if $z_1, z_2 \in X$, then $z_1 \leq z_2$ if and only if $(\forall y \in Y)(z_2 R y \Rightarrow z_1 R y)$;*
(2) *if $z_1, z_2 \in Y$, then $z_1 \leq z_2$ if and only if $(\forall x \in X)(x R z_1 \Rightarrow x R z_2)$;*
(3) *if $z_1 \in X$ and $z_2 \in Y$, then $z_1 \leq z_2$ if and only if $z_1 R z_2$;*
(4) *if $z_1 \in Y$ and $z_2 \in X$, then $z_1 \leq z_2$ if and only if*

$$(\forall x \in X)(\forall y \in Y)[(x R z_1 \;\&\; z_2 R y) \Rightarrow x R y].$$

Note that if $x \in X \cap Y$, then there are two copies of x in the disjoint union $X \uplus Y$: one, call it x', as element of X, and another, x'', as element of Y. Then, $x' \leq x''$ if and only if $\Xi(x) \subseteq \Upsilon(x)$ if and only if $x R x$. Moreover, $x'' \leq x'$ if and only if $\Upsilon(x) \subseteq \Xi(x)$.

The MacNeille completion may be treated axiomatically as the unique doubly dense completion of a poset. Such an axiomatic treatment is also possible for lattices of Galois closed sets of polarities. This fact was central to the paper [11] and is spelled out in the following theorem.

Theorem 3.2 *Let (X, Y, R) be a polarity. Then $\mathcal{G}(X, Y, R)$ is the unique (up to isomorphism) complete lattice equipped with mappings*

$$\Xi : X \to \mathcal{G}(X, Y, R) \text{ and } \Upsilon : Y \to \mathcal{G}(X, Y, R)$$

so that the following properties hold:

(1) *For $x \in X$ and $y \in Y$*

$$\Xi(x) \leq \Upsilon(y) \text{ if and only if } x R y;$$

(2) *$\mathcal{G}(X, Y, R)$ is join-generated by $\mathrm{Im}(\Xi)$;*
(3) *$\mathcal{G}(X, Y, R)$ is meet-generated by $\mathrm{Im}(\Upsilon)$.*

Proof As outlined above, $\mathcal{G}(X, Y, R)$ with Ξ and Υ satisfies the three conditions of the theorem. We need to prove uniqueness. To this end, suppose C is a complete lattice equipped with maps $\Xi\colon X \to C$ and $\Upsilon\colon Y \to C$ satisfying (1)–(3) with $\mathcal{G}(X, Y, R)$ replaced by C. Let $Int(C) = \mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ and consider the map $\Xi \uplus \Upsilon\colon X \uplus Y \to Int(C)$. Denoting the pre-order on $pre\text{-}Int(X, Y, R)$ by $\leq$ and the order on C by $\leq_C$, it is now straightforward to show that for every $z_1, z_2 \in X \uplus Y$,

$$z_1 \leq z_2 \quad \Leftrightarrow \quad \Xi \uplus \Upsilon(z_1) \leq_C \Xi \uplus \Upsilon(z_2).$$

For example, for $z_1, z_2 \in X$ we have

$$\begin{aligned} z_1 \leq z_2 \quad &\Longleftrightarrow \quad \forall y \in Y\ (z_2 R y \Rightarrow z_1 R y) \\ &\Longleftrightarrow \quad \forall y \in Y\ (\Xi(z_2) \leq_C \Upsilon(y) \ \Rightarrow\ \Xi(z_1) \leq_C \Upsilon(y)) \\ &\Longleftrightarrow \quad \Xi(z_1) \leq_C \Xi(z_2) \\ &\Longleftrightarrow \quad \Xi \uplus \Upsilon(z_1) \leq_C \Xi \uplus \Upsilon(z_2). \end{aligned}$$

Note that the third equivalence above is due to the assumption that property (3) holds for C. It follows that $Int(X, Y, R)$ is isomorphic to $Int(C)$, which implies that $\mathcal{G}(X, Y, R)$ is isomorphic to C since they are the MacNeille completions of order-isomorphic posets. □

Now we can state the following Galois connection variant of Proposition 2.1.

Corollary 3.3 *Let P be a poset. A completion C of P is a Δ_1-completion of P if and only if C is (isomorphic to) the lattice of Galois closed sets of the polarity $(\mathbb{F}(C), \mathbb{I}(C), \leq_{\mathbb{F}\times\mathbb{I}})$ where $\leq_{\mathbb{F}\times\mathbb{I}}$ denotes the order of C restricted to $\mathbb{F}(C) \times \mathbb{I}(C)$.*

Proof Suppose C is a Δ_1-completion of P. The inclusion maps of $\mathbb{F}(C)$ and $\mathbb{I}(C)$ into C satisfy the conditions of the above theorem for the polarity $(\mathbb{F}(C), \mathbb{I}(C), \leq_{\mathbb{F}\times\mathbb{I}})$, where $x \leq_{\mathbb{F}\times\mathbb{I}} y$ if and only if $x \leq y$ in C for $x \in \mathbb{F}(C)$ and $y \in \mathbb{I}(C)$. Thus $C \cong \mathcal{G}(\mathbb{F}(C), \mathbb{I}(C), \leq_{\mathbb{F}\times\mathbb{I}})$.

For the converse, a completion of the form $\mathcal{G}(\mathbb{F}(C), \mathbb{I}(C), \leq_{\mathbb{F}\times\mathbb{I}})$ is clearly a Δ_1-completion, since P is join-dense in $\mathbb{I}(C)$ and meet-dense in $\mathbb{F}(C)$ and their images in $\mathcal{G}(\mathbb{F}(C), \mathbb{I}(C), \leq_{\mathbb{F}\times\mathbb{I}})$ are, respectively, meet- and join-dense. □

We shall next obtain a description of all the Δ_1-completions of a poset P in terms of polarities $(\mathcal{F}, \mathcal{I}, R)$ where $\mathcal{F}$ is a standard closure system of up-sets and $\mathcal{I}$ a standard closure system of down-sets of P.

Theorem 3.4 *Let P be a poset. There is a one-to-one correspondence between Δ_1-completions of P and polarities $(\mathcal{F}, \mathcal{I}, R)$ where*

(1) *$\mathcal{F}$ is a standard closure system of up-sets of P;*
(2) *$\mathcal{I}$ is a standard closure system of down-sets of P;*
(3) *the relation $R \subseteq \mathcal{F} \times \mathcal{I}$ satisfies the following four conditions:*

(Pol 1) $\forall p \in P, x \in \mathcal{F}\ (p \in x \iff xRp)$;
(Pol 2) $\forall p \in P, y \in \mathcal{I}\ (p \in y \iff pRy)$;
(Pol 3) $\forall x, x' \in \mathcal{F}, y \in \mathcal{I}\ (x \supseteq x' R y \ \Rightarrow\ xRy)$;

(Pol 4) $\forall x \in \mathcal{F}, y, y' \in \mathcal{I}\ (xRy \subseteq y' \Rightarrow xRy')$.

Here $\uparrow p$ *and* $\downarrow p$ *and* p *are identified for every* $p \in P$.

Proof We have stated this theorem in terms of collections of up- and down-sets of P in order to make it clear that Δ_1-completions are built from the original poset P. However, since standard closure systems $\mathcal{F}$ of up-sets of P are in one-to-one correspondence with meet-completions of P, and standard closure systems $\mathcal{I}$ of down-sets of P are in one-to-one correspondence with join-completions of P, we may as well work abstractly with polarities (K, O, R) such that

(1) K is a meet-completion of P;
(2) O is a join-completion of P;
(3) the relation $R \subseteq K \times O$ satisfies the following four conditions:

(Pol 1) $\forall p \in P, x \in K\ (x \leq_K p \iff xRp)$;
(Pol 2) $\forall p \in P, y \in O\ (p \leq_O y \iff pRy)$;
(Pol 3) $\forall x, x' \in K\ \forall y \in O\ (x \leq_K x'Ry \Rightarrow xRy)$;
(Pol 4) $\forall x \in K\ \forall y, y' \in O\ (xRy \leq_O y' \Rightarrow xRy')$.

Given a Δ_1-completion C of P, we let $K = \mathbb{F}(C)$, $O = \mathbb{I}(C)$, and $R = \leq_C \cap (K \times O)$. Then it is clear that all the required properties hold and that $C \cong \mathcal{G}(K, O, R)$.

Conversely, given a polarity (K, O, R) satisfying the conditions (1)–(3) above, we let $C = \mathcal{G}(K, O, R)$. It suffices to show that $\Xi(p) = \Upsilon(p) = p$ for each $p \in P$, that $\mathbb{I}(C) \cong K$ via $\Xi\colon K \to C$, that $\mathbb{F}(C) \cong O$ via $\Upsilon\colon O \to C$, and that $R = \leq_C \cap (K \times O)$.

Let $p \in P$. Then $p \in K$ and $p \in O$. In order not to confuse these two copies of p (before we have shown that it is justified to do so), we will call the former p_K and the latter p_O. Since $p_K \leq_K p_K$, it follows that $p_K R p_O$ by property (Pol 1), and since the order from K to O in $C = \mathcal{G}(K, O, R)$ is R, we get $p_K \leq_C p_O$. In order to show $p_O \leq_C p_K$, we use the join-density of K in C. Let $x \in K$ with $x \leq_C p_O$. Then xRp_O, and thus $x \leq_K p_K$ by property (Pol 1) again. From this, using the fact that K is join-dense in C, we get $p_O \leq_C p_K$. This finishes the proof that $p_K = p_O$. From now on, we do not distinguish between the elements of P as they sit in K and as they sit in O.

Let $x, x' \in K$ with $x \leq_K x'$, and let $y \in O$. Then, by property (Pol 3), $x'Ry$ implies xRy. On the other hand, if $\forall y \in O\ (x'Ry \Rightarrow xRy)$, then, since $P \subseteq O$ and by property (Pol 1), $\forall p \in P\ (x' \leq_K p \Rightarrow x \leq_K p)$, and since K is a meet-dense completion of P, it follows that $x \leq_K x'$. Thus we have:

$$\begin{aligned} x \leq_K x' \quad &\iff \quad \forall y \in O(x'Ry \Rightarrow xRy) \\ &\iff \quad x \leq_C x'. \end{aligned}$$

The second equivalence holds because O is meet-dense in C. This proves that K embeds in C. We now show that $K = \mathbb{F}(C)$. Let $S \subseteq P$, and $x = \bigwedge_K S$. We show that $x = \bigwedge_C S$. First note that x is a lower bound for S in C by property (Pol 1). Also, if $z \in C$ is a lower bound of S, then, since K is join-dense in C, we have $z = \bigvee_C \{x' \in K \mid x' \leq z\}$. But if $x' \in K$ and $x' \leq_C z$, then $x' \leq_C p$ for every $p \in S$ and thus $x' \leq_K p$ for each $p \in S$. Therefore $x' \leq_K \bigwedge_K S = x$ and then $x' \leq_C x$. It follows that $z = \bigvee_C \{x' \in K \mid x' \leq z\} \leq x$ and thus $x = \bigwedge_C S$. By the order dual argument it follows that O embeds in C and that $O = \mathbb{I}(C)$. Since in $\mathcal{G}(K, O, R)$, it is always true that $R = \leq_C \cap (\mathbb{F}(C) \times \mathbb{I}(C))$, we also get that $R = \leq_C \cap (K \times O)$. □

4 Δ_1-polarities

In this section, we will consider polarities for which the corresponding collections $\mathcal{F}$ and $\mathcal{I}$ aren't necessarily closure systems. In this way we are able to reach more Δ_1-completions with a uniform choice of the relation in the polarity. We begin with some motivation.

Given a poset P, a standard closure system $\mathcal{F}$ of up-sets of P and a standard closure system $\mathcal{I}$ of down-sets of P, it is easy to see that the set of relations $R \subseteq \mathcal{F} \times \mathcal{I}$ satisfying conditions (Pol 1)–(Pol 4) in (3) of Theorem 3.4 is non-empty and closed under arbitrary non-empty intersections. Therefore, the least relation $R_l \subseteq \mathcal{F} \times \mathcal{I}$ satisfying these conditions exists. This least relation is in fact the "non-empty intersection relation", that is,

$$x R_l y \quad \text{iff} \quad x \cap y \neq \emptyset.$$

It is straightforward to check that R_l satisfies (Pol 1)–(Pol 4). To see that it is the smallest relation that satisfies the above mentioned conditions, let $S \subseteq \mathcal{F} \times \mathcal{I}$ be any such relation. Let $x \in \mathcal{F}$ and $y \in \mathcal{I}$ be such that $x R_l y$, and let $p \in x \cap y$. Hence $p \in y$, which implies pSy by condition (Pol 2), and since $x \supseteq \uparrow p$, condition (Pol 3) implies that xSy.

The canonical extension of a bounded lattice as defined in [12] is an example of a Δ_1-completion which corresponds to a polarity endowed with this smallest relation: for every bounded lattice L, the canonical extension of L is the Δ_1-completion of Galois closed sets of the polarity $(\mathcal{F}, \mathcal{I}, R_l)$ where $\mathcal{F}$ is the closure system of the lattice filters of L and $\mathcal{I}$ the closure system of the lattice ideals of L.

The MacNeille completion of a poset P corresponds, via Theorem 3.4, to the polarity $(\mathcal{F}_\mathcal{N}, \mathcal{I}_\mathcal{N}, R_\mathcal{N})$ where $\mathcal{F}_\mathcal{N}$ consists of the standard closure system of all filters of P that are normal and $\mathcal{I}_\mathcal{N}$ consists of the standard closure system of all normal ideals of P. Here normal filters are the sets $X \subseteq P$ that are equal to the set of upper bounds of the set of lower bounds of X, and order dually for ideals. In Example 2.2, we saw a lattice for which the MacNeille completion and the canonical extension were different but corresponded to the same closure systems of down-sets and up-sets. According to Theorem 3.4 then, in that example, the relation $R_\mathcal{N}$ cannot be the relation of non-empty intersection. Indeed, $\omega^\partial R_\mathcal{N} \omega$ even though $\omega^\partial \cap \omega = \emptyset$.

The MacNeille completion is however traditionally built from a different polarity, namely, it is the Δ_1-completion $\mathcal{G}(\mathcal{F}_l, \mathcal{I}_l, R_l)$ of Galois closed sets of the polarity $(\mathcal{F}_l, \mathcal{I}_l, R_l)$ where $\mathcal{F}_l = \{\uparrow p \mid p \in P\}$ is the set of principal filters and thus the least standard collection of filters, $\mathcal{I}_l = \{\downarrow p \mid p \in P\}$ is the set of all principal ideals, and R_l is the relation $\{\langle \uparrow p, \downarrow q\rangle \mid \uparrow p \cap \downarrow q \neq \emptyset\}$ of non-empty intersection. Note that this relation R_l between these (generally) non-closure systems satisfies (Pol 1)–(Pol 4). This example shows that it is important to consider completions of a poset P obtained as lattices of Galois closed sets of polarities $(\mathcal{F}, \mathcal{I}, R)$ such that $\mathcal{F}$ is a standard collection of up-sets of P and $\mathcal{I}$ is a standard collection of down-sets of P, and R satisfies (Pol 1)–(Pol 4), even in cases where $\mathcal{F}$ and $\mathcal{I}$ are not necessarily closure systems. We call these polarities *Δ_1-polarities over P*.

Recall that, as with every polarity, for a Δ_1-polarity $(\mathcal{F}, \mathcal{I}, R)$, the lattice $C = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ of Galois closed sets is isomorphic to the MacNeille completion of $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$. Since $\mathrm{Im}(\Xi) \cup \mathrm{Im}(\Upsilon)$ is isomorphic to $\mathit{Int}(\mathcal{F}, \mathcal{I}, R)$, C is the MacNeille completion of $\mathit{Int}(\mathcal{F}, \mathcal{I}, R)$.

We unravel the content of Proposition 3.1 in order to describe the pre-order of the intermediate structure $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R) = \langle \mathcal{F} \uplus \mathcal{I}, \leq \rangle$ of a Δ_1-polarity $(\mathcal{F}, \mathcal{I}, R)$.

Proposition 4.1 *Let P be a poset and $(\mathcal{F}, \mathcal{I}, R)$ a Δ_1-polarity over P. The following properties hold for the pre-order $\leq$ of $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$:*

(1) *if $x, x' \in \mathcal{F}$ then $x \leq x'$ if and only if $x' \subseteq x$;*
(2) *if $y, y' \in \mathcal{I}$ then $y \leq y'$ if and only if $y \subseteq y'$;*
(3) *if $x \in \mathcal{F}$ and $y \in \mathcal{I}$ then $x \leq y$ if and only if xRy;*
(4) *if $x \in \mathcal{F}$ and $y \in \mathcal{I}$ then $y \leq x$ if and only if*

$$\forall p, q \in P\, [(p \in y \text{ and } x \ni q) \text{ implies } p \leq q].$$

Proof We use Proposition 3.1 and the properties (Pol 1)–(Pol 4) of Δ_1-polarities. Condition (3) of the current proposition is precisely Proposition 3.1(3) and is thus true. To prove (1), note that by Proposition 3.1(1) we have $x \leq x'$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$ if and only if, for every $y \in \mathcal{I}$, if $x'Ry$ then xRy. If $x \supseteq x'$ and $y \in \mathcal{I}$ with $x'Ry$ then xRy by property (Pol 3) so that indeed $x \leq x'$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$. Conversely, if $x \leq x'$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$ and $p \in x'$ then by property (Pol 1) $x'Rp$ and thus $x' \leq p$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$. By transitivity of the pre-order on $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$, we obtain $x \leq p$, and thus xRp; therefore $p \in x$ by the reverse implication in property (Pol 1). The proof of (2) follows by order duality.

In order to prove (4), first suppose that $y \in \mathcal{I}$ and $x \in \mathcal{F}$ with $y \leq x$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$. If $p, q \in P$ with $p \in y$ and $x \ni q$ then pRy and xRq by conditions (Pol 1) and (Pol 2) and thus, by (3), $p \leq y \leq x \leq q$. Hence by transitivity, $p \leq q$. For the converse, suppose $y \nleq x$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$. Since $\mathcal{F}$ is join-dense in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$ there is $x' \in \mathcal{F}$ with $x' \leq y$ but $x' \nleq x$. Now by condition (1) of the present proposition, there is $q \in P$ with $q \in x$ but $q \notin x'$. Now it follows by property (Pol 1) and (3) that $x' \nleq q$ in $\textit{pre-Int}(\mathcal{F}, \mathcal{I}, R)$. In combination with $x' \leq y$ we obtain $y \nleq q$ and thus by (2) $y \nsubseteq {\downarrow} q$. That is, there is $p \in y$ with $p \nleq q$. □

Corollary 4.2 *Let P be a poset and $(\mathcal{F}, \mathcal{I}, R)$ a Δ_1-polarity over P. Then the following properties are satisfied:*

(1) *for each $p \in P$, we have $\Xi({\uparrow} p) = \Upsilon({\downarrow} p)$;*
(2) *$\Xi : \mathcal{F} \to \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ is an embedding of $\mathcal{F}$ with the reverse inclusion order into the filter elements of $\mathcal{G}(\mathcal{F}, \mathcal{I}, R)$;*
(3) *$\Upsilon : \mathcal{I} \to \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ is an embedding of $\mathcal{I}$ into ideal elements of $\mathcal{G}(\mathcal{F}, \mathcal{I}, R)$;*
(4) *the map $P \to \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ given by $p \mapsto \Xi({\uparrow} p) = \Upsilon({\downarrow} p)$ is a Δ_1-completion of P.*

Proof For property (1) we have $\Xi({\uparrow} p) \leq \Upsilon({\downarrow} p)$ by (3) of Proposition 4.1 combined with the fact that property (Pol 1) (or (Pol 2)) implies that pRp. We have $\Xi({\uparrow} p) \geq \Upsilon({\downarrow} p)$ by (4) of the above proposition. Properties (2) and (3) of the corollary are equivalent to (1) and (2) of the above proposition. Also by (4) in the above proposition we have $x = \bigwedge\{ p \in P \mid p \in x \}$ and $y = \bigvee\{ p \in P \mid p \in y \}$ in $\mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ for each $x \in \mathcal{F}$ and $y \in \mathcal{I}$ so that the map $P \to \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ given by $p \mapsto \Xi({\uparrow} p) = \Upsilon({\downarrow} p)$ yields a Δ_1-completion of P. □

As we saw in Example 2.2 a non-principal filter and a non-principal ideal may be identified by the equivalence relation $\leq \cap \geq$ coming from the pre-ordered set $\langle \mathcal{F} \uplus \mathcal{I}, \leq \rangle$. All that can be said is that such identifications will happen when both (3) and (4) of Proposition 4.1 are satisfied for non-principal x and y. However, it will not happen for the filters and ideals of a Δ_1-polarity based on the relation of non-empty intersection.

Corollary 4.3 *Let P be a poset, $\mathcal{F}$ be a standard collection of filters of P, and $\mathcal{I}$ be a standard collection of ideals of P, and consider the Δ_1-polarity $(\mathcal{F}, \mathcal{I}, R_l)$ with $R_l = \{ \langle x, y \rangle \in \mathcal{F} \times \mathcal{I} \mid x \cap y \neq \emptyset \}$. Then, for every $x \in \mathcal{F}$ and for every $y \in \mathcal{I}$,*

$$y \leq x \ \& \ x \leq y \quad \Longleftrightarrow \quad (\exists p \in P)(x = \uparrow p \ \& \ y = \downarrow p).$$

Among the Δ_1-polarities of a poset P, we have the ones with the additional property that $\mathcal{F}$ and $\mathcal{I}$ are closure systems. We will call these *full* Δ_1-polarities. According to Theorem 3.4, every Δ_1-polarity $(\mathcal{F}, \mathcal{I}, R)$ of a poset P corresponds to a unique full Δ_1-polarity of P, which we will denote $(\mathcal{F}^s, \mathcal{I}^s, R^s)$. The following proposition specifies how to obtain this full polarity from the original polarity.

Proposition 4.4 *Let P be a poset and $(\mathcal{F}, \mathcal{I}, R)$ be a Δ_1-polarity over P. Then*

$$\mathcal{G}(\mathcal{F}, \mathcal{I}, R) \cong \mathcal{G}(\mathcal{F}^s, \mathcal{I}^s, R^s)$$

where $\mathcal{F}^s$ is the closure under arbitrary intersections of $\mathcal{F}$ in $\mathcal{U}(P)$ and $\mathcal{I}^s$ is the closure under arbitrary intersections of $\mathcal{I}$ in $\mathcal{D}(P)$, and R^s is given by

$$xR^s y \quad \Leftrightarrow \quad (\forall x' \in \mathcal{F})\ (\forall y' \in \mathcal{I})\ \ (x' \supseteq x \ \& \ y \subseteq y' \ \Rightarrow \ x' R y').$$

Proof We will identify $\mathcal{F}$ with $\Xi[\mathcal{F}]$ and $\mathcal{I}$ with $\Upsilon[\mathcal{I}]$. To prove the statement, we just need to check that $\mathcal{F}^s$ and $\mathcal{I}^s$ are the closure systems corresponding to the meet-closure and the join-closure of $\mathcal{F}$ and $\mathcal{I}$ in $C = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$, respectively. The fact that R^s is given as above follows directly from the fact that $\mathcal{F}$ and $\mathcal{I}$ are join- and meet-dense in C, respectively.

We have seen in the proof of Corollary 4.2 that $x = \bigwedge_C \{ p \in P \mid p \in x \}$ for each $x \in \mathcal{F}$, and $x \leq p$ if and only $p \in x$. Thus, in the correspondence between elements of the meet-closure in a completion C and C-normal filters of P, each element of $\mathcal{F}$ corresponds to itself. Further, since, by definition, $\mathcal{F}^s$ is the closure system generated by $\mathcal{F}$, it is clear that the closure system of up-sets corresponding to the completion C contains $\mathcal{F}^s$. For the converse, let $u \in C$ be in the meet-closure of P and let $F = \uparrow u \cap P$ be the corresponding C-normal filter of P. Then, for each $p \in P$ with $p \notin F$ we have $u \nleq p$. Since $\mathcal{F}$ is join-dense in C, there is $x \in \mathcal{F}$ with $x \leq u$ and $x \nleq p$ in C. Now x and u are both filter elements of C, so $x \leq u$ is equivalent to $F \subseteq x$. Moreover, $x \nleq p$ corresponds to $p \notin x$, and thus $F = \bigcap \{ x \in \mathcal{F} \mid F \subseteq x \}$ as required.

The proof for $\mathcal{I}$ is order dual. □

Note that, for a Δ_1-polarity $(\mathcal{F}, \mathcal{I}, R_l)$ where R_l is the relation of non-empty intersection, the relation R_l^s in the corresponding full Δ_1-polarity $(\mathcal{F}^s, \mathcal{I}^s, R_l^s)$ does not need to be the relation of non-empty intersection, as is illustrated in the following example.

Example 4.5 We consider the poset $P = \omega \oplus \omega^\partial$ and its MacNeille $C = \omega \oplus \mathbf{1} \oplus \omega^\partial$ as in Example 2.2. Then $C = \mathcal{G}(\mathcal{F}_l, \mathcal{I}_l, R_l)$ where $\mathcal{F}_l$ is the set of principal filters of P and $\mathcal{I}_l$ is the set of principal ideals of P, and R_l is the relation of non-empty intersection. However, recall, from Example 2.2, that the closure systems of normal filters and normal ideals corresponding to the completion C are $\{\uparrow p \mid p \in P\} \cup \{\omega^\partial\}$ and $\{\downarrow p \mid p \in P\} \cup \{\omega\}$, respectively. Thus these are the closure systems $\mathcal{F}^s$ and $\mathcal{I}^s$. We also see that in this case R_l^s is different from the non-empty intersection relation from $\mathcal{F}^s$ to $\mathcal{I}^s$ as $(\omega^\partial, \omega) \in R_l^s$ and $\omega^\partial \cap \omega = \emptyset$.

5 Parametric Compactness

In Section 4, we saw that for standard closure systems $\mathcal{F}$ and $\mathcal{I}$ of up-sets and of down-sets, respectively, of a poset P there always exists a least relation R so that $(\mathcal{F}, \mathcal{I}, R)$ is a Δ_1-polarity, namely, the relation R_l of non-empty intersection. It is not hard to see that, in general, for $\mathcal{F}$ and $\mathcal{I}$ standard collections of up-sets and down-sets, respectively, $(\mathcal{F}, \mathcal{I}, R_l)$ is a Δ_1-polarity and that R_l is the least relation $R \subseteq \mathcal{F} \times \mathcal{I}$ for which $(\mathcal{F}, \mathcal{I}, R)$ is a Δ_1-polarity.

Note that by Proposition 4.1(3) and Corollary 4.2(4), we have that a Δ_1-completion $C = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ satisfies $R = R_l$ if and only if C satisfies the following compactness property:

$$(\forall x \in \mathcal{F})(\forall y \in \mathcal{I})(\textstyle\bigwedge_C x \leq \bigvee_C y \iff x \cap y \neq \emptyset). \qquad ((\mathcal{F}, \mathcal{I})\text{-Comp})$$

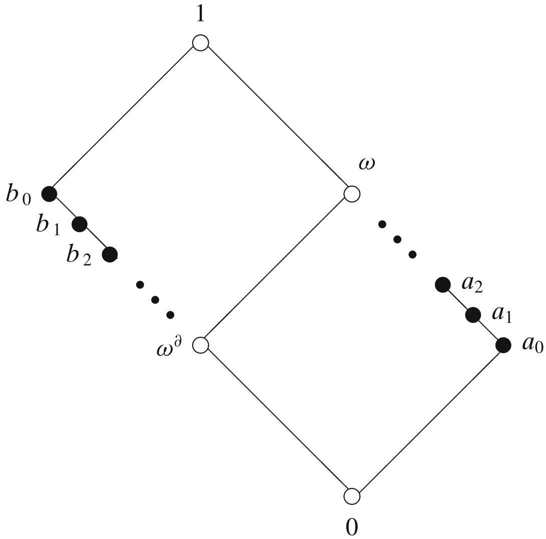
As illustrated in Example 4.5, a completion C may be obtained both from a Δ_1-polarity whose relational component is the relation of non-empty intersection and from one for which the relation isn't that of non-empty intersection. This leads to the following notion of parametric compactness.

Definition 5.1 Let P be a poset and let $\mathcal{F}$ and $\mathcal{I}$ be standard collections of up-sets and of down-sets of P, respectively. A completion C of P is $(\mathcal{F}, \mathcal{I})$-compact provided $((\mathcal{F}, \mathcal{I})\text{-Comp})$ holds.

While many Δ_1-completions can be described as $(\mathcal{F}, \mathcal{I})$-compact for some choice of $\mathcal{F}$ and $\mathcal{I}$, there are posets that have Δ_1-completions that are not $(\mathcal{F}, \mathcal{I})$-compact for any choice of $\mathcal{F}$ and $\mathcal{I}$. This is illustrated by the following example.

Example 5.2 Let P be the poset consisting of the disjoint union of a countably infinite increasing chain and a countably infinite decreasing chain. Let us denote by $\langle \omega, \leq \rangle$ the increasing chain and by $\langle \omega^\partial, \leq^\partial \rangle$ the decreasing chain. Note that ω is a non-principal down-set and ω^∂ is a non-principal up-set in P. Let $\mathcal{F} = \{\uparrow p \mid p \in P\} \cup \{\omega^\partial\}$ and $\mathcal{I} = \{\downarrow p \mid p \in P\} \cup \{\omega\}$. Consider the Δ_1-completion C as specified in Fig. 1. Note that ω^∂ is a completely join-irreducible element and ω is a completely meet-irreducible element in C. Recall that taking the MacNeille completion of a poset never creates new completely join- or completely meet-irreducible elements. Therefore, it must be true that for any choice of $\mathcal{F}$ and $\mathcal{I}$ for which C is an $(\mathcal{F}, \mathcal{I})$-completion, $\omega^\partial \in \mathcal{F}$ and $\omega \in \mathcal{I}$. Since $\bigwedge \omega^\partial = \omega^\partial \leq \omega = \bigvee \omega$, and $\omega^\partial \cap \omega = \emptyset$, it follows that C is not $(\mathcal{F}, \mathcal{I})$-compact for any choice of $\mathcal{F}$ and $\mathcal{I}$ that generate C as a Δ_1-completion.

Fig. 1 A Δ_1-completion that is not compact for any choice of $\mathcal{F}$ and $\mathcal{I}$



Next, we will show, under an appropriate denseness condition, that the notion of $(\mathcal{F}, \mathcal{I})$-compact completion of P captures the completion $\mathcal{G}(\mathcal{F}, \mathcal{I}, R_l)$. Since this involves arguments about non-complete poset extensions such as the intermediate structure $Int(\mathcal{F}, \mathcal{I}, R_l)$, we work in the more general setting of poset extensions as defined in Section 2. Accordingly, let P be a poset and Q be an extension of P that is not necessarily complete. We may consider the restrictions to Q of the Galois connections from Section 2, whose Galois closed sets are:

$$\mathbb{F}(Q) := \left\{ a \in Q \mid a = \bigwedge\nolimits_Q F \text{ for some } F \in \mathcal{U}(P) \right\},$$

$$\mathbb{I}(Q) := \left\{ b \in Q \mid b = \bigvee\nolimits_Q I \text{ for some } I \in \mathcal{D}(P) \right\}.$$

The elements of these sets will be called *filter* and *ideal elements* of Q, respectively. Further, the elements of

$$\mathcal{F}_Q := \{ \uparrow a \cap P \mid a \in \mathbb{F}(Q) \},$$

$$\mathcal{I}_Q := \{ \downarrow b \cap P \mid b \in \mathbb{I}(Q) \}$$

are the *Q-normal filters* and the *Q-normal ideals* of P, respectively. The latter filters and ideals are of course respectively in one-to-one correspondence with the filter elements of Q and the ideal elements of Q. Moreover, let $\mathcal{F}$ be a standard collection of up-sets of P, and $\mathcal{I}$ be a standard collection of down-sets of P. Then we call *$\mathcal{F}$-filter elements* of Q and *$\mathcal{I}$-ideal elements* of Q the elements in the following two sets, respectively:

$$\mathbb{F}^{\mathcal{F}}(Q) := \left\{ a \in Q \mid a = \bigwedge\nolimits_Q x \text{ for some } x \in \mathcal{F} \right\},$$

$$\mathbb{I}^{\mathcal{I}}(Q) := \left\{ b \in Q \mid b = \bigvee\nolimits_Q y \text{ for some } y \in \mathcal{I} \right\}.$$

Clearly $\mathbb{F}^{\mathcal{F}}(Q) \subseteq \mathbb{F}(Q)$ and $\mathbb{I}^{\mathcal{I}}(Q) \subseteq \mathbb{I}(Q)$ for any $\mathcal{F}$ and $\mathcal{I}$.

Definition 5.3 Let P be a poset, Q an extension of P and $\mathcal{F}$ a standard collection of up-sets of P and $\mathcal{I}$ a standard collection of down-sets of P. We say that Q is $(\mathcal{F}, \mathcal{I})$-*compatible* provided $\mathcal{F} \subseteq \mathcal{F}_Q$ and $\mathcal{I} \subseteq \mathcal{I}_Q$. This is equivalent to saying that

$$\begin{array}{ccccc} \bigwedge_Q & : & \mathcal{F} & \leftrightarrows & \mathbb{F}^{\mathcal{F}}(Q) : \quad \uparrow_P \\ & & x & \mapsto & \bigwedge_Q x \\ & & \uparrow a \cap P & \leftarrow\!\shortmid & a \end{array}$$

are well-defined, mutually inverse, isomorphisms and similarly for

$$\bigvee_Q \;:\; \mathcal{I} \quad \leftrightarrows \quad \mathbb{I}^{\mathcal{I}}(Q) : \quad \downarrow_P .$$

Further, if $\bigwedge_Q x$ exists for each $x \in \mathcal{F}$ and $\bigvee_Q y$ exists for each $y \in \mathcal{I}$, then we say that Q is $(\mathcal{F}, \mathcal{I})$-*compact* provided ($(\mathcal{F}, \mathcal{I})$-Comp) holds with C replaced by Q.

Proposition 5.4 *Let $\mathcal{F}$ be a standard collection of up-sets and $\mathcal{I}$ be a standard collection of down-sets of a poset P. If $\bigwedge_Q x$ exists for each $x \in \mathcal{F}$ and $\bigvee_Q y$ exists for each $y \in \mathcal{I}$, and Q is an $(\mathcal{F}, \mathcal{I})$-compact extension of P, then Q is $(\mathcal{F}, \mathcal{I})$-compatible.*

Proof Let $x \in \mathcal{F}$ and $p \in \uparrow(\bigwedge_Q x) \cap P$. Then $\bigwedge_Q x \leq \bigvee \downarrow p$ in Q and thus, by $(\mathcal{F}, \mathcal{I})$-compactness, it follows that $x \cap \downarrow p \neq \emptyset$, hence $p \in x$. Therefore, $x = \uparrow(\bigwedge_Q x) \cap P$ and so $x \in \mathcal{F}_Q$. Similarly for $y \in \mathcal{I}$. □

The next proposition provides a useful characterisation of the $(\mathcal{F}, \mathcal{I})$-compatible extensions that are $(\mathcal{F}, \mathcal{I})$-compact.

Proposition 5.5 *Let Q be a $(\mathcal{F}, \mathcal{I})$-compatible extension of P. The following are equivalent:*

(1) *Q is $(\mathcal{F}, \mathcal{I})$-compact;*
(2) *for every $a \in \mathbb{F}^{\mathcal{F}}(Q)$ and every $b \in \mathbb{I}^{\mathcal{I}}(Q)$ if $a \leq b$ then $a \leq p \leq b$ for some $p \in P$.*

Proof Assume (1) is true. Let $a \in \mathbb{F}^{\mathcal{F}}(Q)$ and $b \in \mathbb{I}^{\mathcal{I}}(Q)$ with $a \leq b$. Then there are $x \in \mathcal{F}$ and $y \in \mathcal{I}$ such that $a = \bigwedge_Q x$ and $b = \bigvee_Q y$. Consequently $\bigwedge_Q x = a \leq b = \bigvee_Q y$ and by compactness there exists $p \in x \cap y$. Hence $a = \bigwedge_Q x \leq p \leq \bigvee_Q y = b$ and we have shown that (1) implies (2).

Now assume that (2) is true. Let $x \in \mathcal{F}$ and $y \in \mathcal{I}$. Then by $(\mathcal{F}, \mathcal{I})$-compatibility $a = \bigwedge_Q x$ and $b = \bigvee_Q y$ exist. Suppose that $\bigwedge_Q x \leq \bigvee_Q y$. By (2), there is $p \in P$ with $a \leq p \leq b$. Since Q is a $(\mathcal{F}, \mathcal{I})$-compatible extension of P, we have $x = \uparrow_P a$ and $y = \downarrow_P b$. Therefore, $p \in x$ and $p \in y$ and $x \cap y \neq \emptyset$ as required. □

Corollary 5.6 *Let P be a poset, and Q an $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-compatible extension of P. Then the sub-poset $\mathbb{F}^{\mathcal{F}}(Q) \cup \mathbb{I}^{\mathcal{I}}(Q)$ of Q is isomorphic to $Int(\mathcal{F}, \mathcal{I}, R_I)$, where $x R_I y$ if and only if $x \cap y \neq \emptyset$.*

Proof It follows from the definition of $(\mathcal{F}, \mathcal{I})$-compatible extension that the set of $\mathcal{F}$-filter elements of Q is isomorphic to $\mathcal{F}$ with the reverse inclusion order and dually for $\mathcal{I}$. By Proposition 5.5, it follows that for any $\mathcal{F}$-filter element a and any

$\mathcal{I}$-ideal element b, we have $a \leq b$ if and only if $a \leq p \leq b$ for some $p \in P$. Finally, for $x \in \mathcal{F}$ and $y \in \mathcal{I}$, by the definition of join and meet, we have $\bigvee y \leq \bigwedge x$ if and only if $p \leq q$ for every $p \in y$ and every $q \in x$. All in all, we see that the order on $\mathbb{F}^{\mathcal{F}} \cup \mathbb{I}^{\mathcal{I}}$ is precisely the order of $Int(\mathcal{F}, \mathcal{I}, R_l)$ as described in Proposition 4.1 via the corresponding pre-order on $\mathcal{F} \uplus \mathcal{I}$. □

Let P be a poset and $(\mathcal{F}, \mathcal{I}, R)$ be a Δ_1-polarity. In the Δ_1-completion $C = \mathcal{G}(\mathcal{F}, \mathcal{I}, R)$ the set $\mathbb{F}^{\mathcal{F}}(C)$ is join-dense and the set $\mathbb{I}^{\mathcal{I}}(C)$ is meet-dense. This property will be called $(\mathcal{F}, \mathcal{I})$-denseness, but we define it in the broader context of poset extensions.

Definition 5.7 Let P be a poset and let $\mathcal{F}$ and $\mathcal{I}$ respectively be a standard collection of up-sets and a standard collection of down-sets of P. We say that a poset extension Q of P is $(\mathcal{F}, \mathcal{I})$-*dense* if $\mathbb{F}^{\mathcal{F}}(Q)$ is join-dense in Q and $\mathbb{I}^{\mathcal{I}}(Q)$ is meet-dense in Q.

If Q is a poset extension of P which is an $(\mathcal{F}, \mathcal{I})$-compatible extension, then the properties of $(\mathcal{F}, \mathcal{I})$-compactness and $(\mathcal{F}, \mathcal{I})$-denseness lift to the MacNeille completion of Q. This is because the MacNeille completion of any poset Q preserves all existing joins and all existing meets and because Q is both join- and meet-dense in its MacNeille completion. For every extension Q of a poset P, we write $\overline{Q}$ for the completion of P obtained by taking the MacNeille completion $\mathcal{N}(Q)$ of Q and composing the embedding from P to Q with the MacNeille embedding of Q into $\mathcal{N}(Q)$.

Proposition 5.8 *For every poset P and every $(\mathcal{F}, \mathcal{I})$-compatible extension Q of P:*

(1) *$\overline{Q}$ is a $(\mathcal{F}, \mathcal{I})$-compatible completion of P;*
(2) *if Q is $(\mathcal{F}, \mathcal{I})$-compact then so is $\overline{Q}$;*
(3) *if Q is $(\mathcal{F}, \mathcal{I})$-dense then so is $\overline{Q}$.*

Proof Statement (1) is true because $\mathbb{F}^{\mathcal{F}}(\overline{Q}) = \mathbb{F}^{\mathcal{F}}(Q)$ and $\mathbb{I}^{\mathcal{I}}(\overline{Q}) = \mathbb{I}^{\mathcal{I}}(Q)$, since existing meets and joins are preserved by the MacNeille completion.

Statement (2) also follows from the fact that meets and joins from Q are preserved by MacNeille completion: let $x \in \mathcal{F}$ and $y \in \mathcal{I}$ be such that $\bigwedge_{\overline{Q}} x \leq \bigvee_{\overline{Q}} y$. As Q is $(\mathcal{F}, \mathcal{I})$-compatible, the meet and the join are taken in Q, and thus, by compactness of Q, $x \cap y \neq \emptyset$.

In order to prove statement (3), let $u \in \overline{Q}$. Since Q is join- and meet-dense in $\overline{Q}$, there are $Y, Z \subseteq Q$ such that $u = \bigvee Y = \bigwedge Z$. Further, since Q is an $(\mathcal{F}, \mathcal{I})$-dense extension of P, we have, for every $y \in Y$, some $A_y \subseteq \mathbb{I}^{\mathcal{I}}(Q)$ and, for every $z \in Z$, some $B_z \subseteq \mathbb{F}^{\mathcal{F}}(Q)$ with $y = \bigvee_Q A_y$ and $z = \bigwedge_Q B_z$. Since $\bigvee_Q A_y = \bigvee_{\overline{Q}} A_y$ and $\bigwedge_Q B_z = \bigwedge_{\overline{Q}} B_z$, it follows that $x = \bigvee A = \bigwedge B$ in $\overline{Q}$ where $A = \bigcup_{y \in Y} A_y$ and $B = \bigcup_{z \in Z} B_z$. □

Definition 5.9 Let P be a poset and let $\mathcal{F}$ and $\mathcal{I}$ respectively be standard collections of up-sets and of down-sets of P. We say that a completion C of P is an $(\mathcal{F}, \mathcal{I})$-*completion* provided it is $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense.

Note that all $(\mathcal{F}, \mathcal{I})$-completions are necessarily $(\mathcal{F}, \mathcal{I})$-compatible by Proposition 5.4.

Theorem 5.10 *For every poset P, any standard collection $\mathcal{F}$ of up-sets of P, and any standard collection $\mathcal{I}$ of down-sets of P, the completion $\mathcal{G}(\mathcal{F}, \mathcal{I}, R_l)$ is (up to isomorphism) the unique $(\mathcal{F}, \mathcal{I})$-completion of P.*

Proof Let C be an $(\mathcal{F}, \mathcal{I})$-completion of P. By Corollary 5.6, we have that $\mathbb{F}^{\mathcal{F}}(C) \cup \mathbb{I}^{\mathcal{I}}(C)$ is isomorphic to $Int(\mathcal{F}, \mathcal{I}, R_l)$. Further, by the $(\mathcal{F}, \mathcal{I})$-denseness, any $(\mathcal{F}, \mathcal{I})$-completion of P must be the MacNeille completion of $\mathbb{F}^{\mathcal{F}} \cup \mathbb{I}^{\mathcal{I}}$. □

Corollary 5.11 *The $(\mathcal{F}, \mathcal{I})$-compatible extensions Q of P that are both $(\mathcal{F}, \mathcal{I})$-dense and $(\mathcal{F}, \mathcal{I})$-compact are, up to isomorphism, the posets satisfying $Int(\mathcal{F}, \mathcal{I}, R_l) \subseteq Q \subseteq \mathcal{G}(\mathcal{F}, \mathcal{I}, R_l)$.*

Proof This corollary follows immediately from Corollary 5.6 and $(\mathcal{F}, \mathcal{I})$-denseness. □

We end the section with a discussion of the relation between our parametric notion of compactness and other notions of compactness in the literature. The standard non-parametric notion of compact completion of a poset is the following: A completion C of a poset P is *compact* if and only if for every $X, Y \subseteq P$, if $\bigwedge_C X \leq \bigvee_C Y$ then there are finite sets $X' \subseteq X$ and $Y' \subseteq Y$ such that $\bigwedge_C X' \leq \bigvee_C Y'$. As we will see, under suitable assumptions on the parameters $\mathcal{F}$ and $\mathcal{I}$, the parametric notion of compactness is equivalent to (non-parametric) compactness.

In the sequel we will use the following notation. We will respectively use F and I to refer to elements of $\mathcal{F}$ and $\mathcal{I}$ and X, Y to refer to arbitrary subsets of the poset.

We say that a completion C of P is *weakly $(\mathcal{F}, \mathcal{I})$-compact* provided for every $F \in \mathcal{F}$ and every $I \in \mathcal{I}$, if $\bigwedge_C F \leq \bigvee_C I$ then there are finite sets $X \subseteq F$ and $Y \subseteq I$ such that $\bigwedge_C X \leq \bigvee_C Y$. Obviously if C is an $(\mathcal{F}, \mathcal{I})$-compact completion of P then it is weakly $(\mathcal{F}, \mathcal{I})$-compact. The converse holds if $\mathcal{F}$ is in addition a family of down-directed sets and $\mathcal{I}$ a family of up-directed sets.

Proposition 5.12 *For every standard collection $\mathcal{F}$ of down-directed up-sets of P and every standard collection $\mathcal{I}$ of up-directed down-sets of P, if C is a completion of P, then C is $(\mathcal{F}, \mathcal{I})$-compact if and only if C is weakly $(\mathcal{F}, \mathcal{I})$-compact.*

Proof We only need to prove the implication from right to left. To this end assume C is a weakly $(\mathcal{F}, \mathcal{I})$-compact completion of P and that $F \in \mathcal{F}$ and $I \in \mathcal{I}$ satisfy $\bigwedge_C F \leq \bigvee_C I$. Since C is a weakly $(\mathcal{F}, \mathcal{I})$-compact extension, there are finite sets $X \subseteq F$ and $Y \subseteq I$ with $\bigwedge_C X \leq \bigvee_C Y$. The down-directness of F implies that there is $p \in F$ with $p \leq \bigwedge X$ and the up-directedness of I that there is $q \in I$ such that $\bigvee_C Y \leq q$. It follows that $p \leq q$. Hence, $q \in F \cap I$ and so $F \cap I \neq \emptyset$. □

Lemma 5.13 *Let $\mathcal{F}$ be a standard collection of up-sets of a poset P and let $\mathcal{I}$ be a standard collection of down-sets of P. If C is an $(\mathcal{F}, \mathcal{I})$-compatible and $(\mathcal{F}, \mathcal{I})$-dense completion of P, then for every $X \subseteq P$,*

(1) $\bigwedge_C X = \bigwedge_C \bigcap\{ F \in \mathcal{F} \mid X \subseteq F \}$;
(2) $\bigvee_C X = \bigvee_C \bigcap\{ I \in \mathcal{I} \mid X \subseteq I \}$.

Proof We only prove (1), since (2) follows by order duality. The inclusion $\bigwedge_C \bigcap\{ F \in \mathcal{F} \mid X \subseteq F\} \leq \bigwedge_C X$ immediately follows from the fact that

$$X \subseteq \bigcap\{ F \in \mathcal{F} \mid X \subseteq F\}.$$

In order to prove the reverse inequality, let $z \leq \bigwedge_C X$ and $q \in \bigcap\{ F \in \mathcal{F} \mid X \subseteq F\}$. We will show that $z \leq q$. Since C is $(\mathcal{F}, \mathcal{I})$-dense, $z = \bigvee_C b_\alpha$ for some set $\{ b_\alpha \mid \alpha \in \kappa\} \subseteq \mathbb{F}^{\mathcal{F}}(C)$. Then, since C is $(\mathcal{F}, \mathcal{I})$-compatible, for every $\alpha \in \kappa$ there exists $F_\alpha \in \mathcal{F}$ such that $\bigwedge_C F_\alpha = b_\alpha$ and $F_\alpha = \uparrow_P b_\alpha$. Note that then for every $p \in X$ and every $\alpha \in \kappa$, we have $b_\alpha = \bigwedge_C F_\alpha \leq p$. Therefore, since $X \subseteq P$, we obtain that $p \in \uparrow_P b_\alpha = F_\alpha$ is true for every $\alpha \in \kappa$ and every $p \in X$. Hence, $X \subseteq F_\alpha$, and consequently $q \in F_\alpha = \uparrow_P b_\alpha$, for every $\alpha \in \kappa$. Therefore $z = \bigvee_C b_\alpha \leq q$, as required. □

Proposition 5.14 *Let P be a poset, and let $\mathcal{F}$ be a standard collection of up-sets of P and $\mathcal{I}$ a standard collection of down-sets of P. If $\mathcal{F}$ and $\mathcal{I}$ are algebraic closure systems, then for every $(\mathcal{F}, \mathcal{I})$-compatible and $(\mathcal{F}, \mathcal{I})$-dense completion C of P, we have that C is weakly $(\mathcal{F}, \mathcal{I})$-compact if and only if C is compact.*

Proof Suppose C is a $(\mathcal{F}, \mathcal{I})$-compatible and $(\mathcal{F}, \mathcal{I})$-dense completion of P. Assume that C is weakly $(\mathcal{F}, \mathcal{I})$-compact. In order to show that C is compact, let $X, Y \subseteq P$ be such that $\bigwedge_C X \leq \bigvee_C Y$. We need to find finite sets $X' \subseteq X$ and $Y' \subseteq Y$ with $\bigwedge_C X' \leq \bigvee_C Y'$. To this end we consider the sets $F_X = \bigcap\{ F \in \mathcal{F} \mid X \subseteq F\}$ and $I_Y = \bigcap\{ I \in \mathcal{I} \mid Y \subseteq I\}$. Since $\mathcal{F}$ and $\mathcal{I}$ are closure systems, we have $F_X \in \mathcal{F}$ and $I_Y \in \mathcal{I}$. Moreover, applying Lemma 5.13 we obtain $\bigwedge_C F_X = \bigwedge_C X \leq \bigvee_C Y = \bigvee_C I_Y$. Then, since C is weakly $(\mathcal{F}, \mathcal{I})$-compact, there are finite $Z \subseteq F_X$ and $W \subseteq I_Y$ such that $\bigwedge_C Z \leq \bigvee_C W$. Since $\mathcal{F}$ and $\mathcal{I}$ are algebraic, there are finite $X' \subseteq X$ and $Y' \subseteq Y$ such that $Z \subseteq F_{X'} = \bigcap\{ F \in \mathcal{F} \mid X' \subseteq F\}$ and $W \subseteq I_{Y'} = \bigcap\{ I \in \mathcal{I} \mid Y' \subseteq I\}$. It follows that $\bigwedge_C X' = \bigwedge_C F_{X'} \leq \bigwedge_C Z \leq \bigvee_C W \leq \bigvee_C I_{Y'} = \bigvee_C Y'$, as required. That compactness implies weak $(\mathcal{F}, \mathcal{I})$-compactness is obvious. □

Propositions 5.12 and 5.14 imply that if $\mathcal{F}$ and $\mathcal{I}$ are algebraic closure systems of down-directed and up-directed sets respectively, then the notions of $(\mathcal{F}, \mathcal{I})$-compactness, weak $(\mathcal{F}, \mathcal{I})$-compactness and compactness are coextensive when they are applied to completions C of P which are $(\mathcal{F}, \mathcal{I})$-compatible and $(\mathcal{F}, \mathcal{I})$-dense.

6 Properties of $(\mathcal{F}, \mathcal{I})$-completions

The canonical extension of a poset P [8] is the $(\mathcal{F}, \mathcal{I})$-completion of P where $\mathcal{F}$ is the collection of all down-directed up-sets and $\mathcal{I}$ is the collection of all up-directed down-sets. From the perspective of Δ_1-completions, this choice for $\mathcal{F}$ and $\mathcal{I}$ is just one among many possible choices. In this section, we identify the properties of $\mathcal{F}$ and $\mathcal{I}$ that are needed in order to obtain a completion with various properties that are crucial in making canonical extensions work the way they do. The key properties we will consider are: restricted distributive laws, having enough completely join- and meet-irreducibles, preserving existing finite meets and joins, and commuting with formation of order dual as well as products.

Throughout this section we assume P is a poset, $\mathcal{F}$ is a standard collection of up-sets of P, and $\mathcal{I}$ is a standard collection of down-sets of P. We let P^* denote the (unique up to isomorphism) $(\mathcal{F},\mathcal{I})$-completion of P, and we assume that $P \subseteq P^*$. That is, P^* is the unique $(\mathcal{F},\mathcal{I})$-compact and $(\mathcal{F},\mathcal{I})$-dense completion of P.

6.1 Restricted Distributive Laws

Definition 6.1 A completion C of P is said to satisfy the $(\mathcal{F},\mathcal{I})$-*restricted distributive laws* provided that for every collection $\mathcal{X}$ of non-empty down-directed sets of $\mathcal{F}$-filter elements of C and for every collection $\mathcal{Y}$ of non-empty up-directed sets of $\mathcal{I}$-ideal elements of C we have

(1) $\bigvee\{\bigwedge A \mid A \in \mathcal{X}\} = \bigwedge\{\bigvee f[\mathcal{X}] \mid f \in \Phi(\mathcal{X})\}$,
(2) $\bigwedge\{\bigvee B \mid B \in \mathcal{Y}\} = \bigwedge\{\bigvee f[\mathcal{Y}] \mid f \in \Phi(\mathcal{Y})\}$,

where $\Phi(\mathcal{X})$ is the set of all *choice functions* of $\mathcal{X}$, that is maps $f\colon \mathcal{X} \to \bigcup \mathcal{X}$ such that $f(A) \in A$ for every $A \in \mathcal{X}$.

We will provide a condition on $\mathcal{F}$ and I that implies that P^* satisfies the $(\mathcal{F},\mathcal{I})$-restricted distributive laws. We need a lemma.

Lemma 6.2 *Let A be a down-directed set of $\mathcal{F}$-filter elements and B an up-directed set of $\mathcal{I}$-ideal elements of P. If $F = \bigcup\{\uparrow_P p \mid p \in A\} \in \mathcal{F}$ and $I = \bigcup\{\downarrow_P q \mid q \in B\} \in \mathcal{I}$, then*

$$\bigwedge A \leq \bigvee B \quad \Longrightarrow \quad \exists p \in A\, \exists q \in B \quad p \leq q$$

holds in P^*.

Proof Since $A \subseteq \mathbb{F}^{\mathcal{F}}(P^*)$, for every $p \in A$, we have $p = \bigwedge \uparrow_P p$. Therefore,

$$\bigwedge A = \bigwedge \bigcup\{\uparrow_P p \mid p \in A\} = \bigwedge F.$$

Order dually, for every $q \in B$,

$$\bigvee B = \bigvee \bigcup\{\downarrow_P q \mid q \in B\} = \bigvee I.$$

Now assume that $\bigwedge A \leq \bigvee B$. Therefore $\bigwedge F \leq \bigvee I$. Since P^* is $(\mathcal{F},\mathcal{I})$-compact, there is $r \in F \cap I$. But then, there are $p \in A$ and $q \in B$ with $p \leq r \leq q$. □

Proposition 6.3 *Suppose $\mathcal{F}$ and $\mathcal{I}$ are closed under non-empty directed unions. Then P^* satisfies the $(\mathcal{F},\mathcal{I})$-restricted distributive laws.*

Proof We just prove one of the laws since the other follows by order duality. Let $\mathcal{X}$ be a collection of non-empty down-directed sets of $\mathcal{F}$-filter elements of P^*. Since every $f \in \Phi(\mathcal{X})$ is a choice function for $\mathcal{X}$ it follows that $\bigwedge A \leq f(A) \in f[\mathcal{X}]$ for every $A \in \mathcal{X}$. Hence, for every $A \in \mathcal{X}$, we have $\bigwedge A \leq \bigvee f[\mathcal{X}]$. This implies that

$$\bigvee\left\{\bigwedge A \mid A \in \mathcal{X}\right\} \leq \bigwedge\left\{\bigvee f[\mathcal{X}] \mid f \in \Phi(\mathcal{X})\right\}.$$

To prove the converse inequality, let $x = \bigvee\{\bigwedge A \mid A \in \mathcal{X}\}$. Since P^* is $(\mathcal{F},\mathcal{I})$-dense, we have

$$x = \bigwedge\left\{b \in \mathbb{I}^{\mathcal{I}}(P^*) \mid x \leq b\right\}.$$

Therefore, in order to show that $\bigwedge\{\bigvee f[\mathcal{X}] \mid f \in \Phi(\mathcal{X})\} \leq x$, it is enough to prove that for every $b \in \mathbb{I}^{\mathcal{I}}(P^*)$ with $x \leq b$ there exists $f \in \Phi(\mathcal{X})$ satisfying that $\bigvee f[\mathcal{X}] \leq b$. Let $b \in \mathbb{I}^{\mathcal{I}}(P^*)$ be such that

$$\bigvee\left\{\bigwedge A \mid A \in \mathcal{X}\right\} = x \leq b.$$

Then $\bigwedge A \leq b$ for every $A \in \mathcal{X}$. Since every $A \in \mathcal{X}$ is down-directed and

$$\bigcup\{\uparrow_P p \mid p \in A\} \in \mathcal{F},$$

by Lemma 6.2, we have that for every $A \in \mathcal{X}$ there exists $p_b \in A$ with $p_b \leq b$. Then the assignment given by $A \mapsto p_b$ defines a map $f_b \in \Phi(\mathcal{X})$ with the property that $\bigvee f_b[\mathcal{X}] \leq b$. □

6.2 Completely Irreducible Elements

Recall that an element j of a complete lattice C is *completely join-irreducible* provided that, for every $A \subseteq C$, if $j = \bigvee A$ then $j \in A$. Order dually, an element m of a complete lattice C is *completely meet-irreducible* provided that, for every $B \subseteq C$, if $m = \bigwedge B$ then $m \in B$. Let $J^\infty(C)$ denote the set of all completely join-irreducible elements of C and $M^\infty(C)$ the set of all completely meet-irreducible elements of C. A complete lattice is *perfect* provided $J^\infty(C)$ is join-dense in C and $M^\infty(C)$ is meet-dense in C.

Proposition 6.4

(1) $J^\infty(P^*) \subseteq \mathbb{F}^{\mathcal{F}}(P^*)$,
(2) $M^\infty(P^*) \subseteq \mathbb{I}^{\mathcal{I}}(P^*)$.

Proof These two claims follow immediately from the assumption that C is $(\mathcal{F}, \mathcal{I})$-dense. Indeed, if $x \in J^\infty(P^*)$, then $(\mathcal{F}, \mathcal{I})$-denseness implies that $x = \bigvee A$ for some set $A \subseteq \mathbb{F}^{\mathcal{F}}(P^*)$, and since x is completely join-irreducible, it follows that $x \in A$; hence $x \in \mathbb{F}^{\mathcal{F}}(P^*)$. The second statement follows by order duality. □

Proposition 6.5 *If $\mathcal{F}$ and $\mathcal{I}$ are closed under unions of chains, then P^* is a perfect lattice.*

Proof In order to prove that every element of P^* is a join of completely join-irreducibles, it suffices to prove that, if $a \in \mathbb{F}^{\mathcal{F}}(P^*)$ and $b \in \mathbb{I}^{\mathcal{I}}(P^*)$ are such that $a \not\leq b$, then $j \leq a$ and $j \not\leq b$ for some $j \in J^\infty(P^*)$. Let $F_a \in \mathcal{F}$ and $I_b \in \mathcal{I}$ with $a = \bigwedge F_a$ and $b = \bigvee I_b$. So, $\bigwedge F_a \not\leq \bigvee I_b$. Therefore, $F_a \cap I_b = \emptyset$. Since $\mathcal{F}$ is closed under unions of $\subseteq$-chains, by Zorn's lemma, there exists some $F \in \mathcal{F}$ that is maximal among the elements in $\mathcal{F}$ that are disjoint from I_b. Let $j = \bigwedge F$. From $F \cap I_b = \emptyset$ follows that $j \not\leq b$, for if $\bigwedge F = j \leq b = \bigvee I_b$ then, by $(\mathcal{F}, \mathcal{I})$-compactness, $F \cap I_b \neq \emptyset$. We claim that $F = \uparrow_P j = \{p \in P \mid j \leq p\}$. Since $j = \bigwedge F$, we have $F \subseteq \uparrow_P j$. If the inclusion is proper, then, by maximality, $\uparrow_P j \cap I_b \neq \emptyset$, and so there exists some $p \in P$ with $j \leq p \leq b$, a contradiction. Consequently, $F = \uparrow_P j$.

To finish the proof, we show that j is completely join-irreducible. By $(\mathcal{F}, \mathcal{I})$-denseness, it is enough to prove that if $A \subseteq \mathbb{F}^{\mathcal{F}}(P^*)$ is such that $j = \bigvee A$, then $j \in A$. Suppose that $j \neq a'$ for every $a' \in A$. Hence, since $j = \bigvee A$ implies that

$a' \le j$ for every $a' \in A$, we obtain $a' < j$, and so $F = {\uparrow_P} j \subsetneq F_{a'}$, which implies, by maximality, that $F_{a'} \cap I_b \neq \emptyset$. Hence for every $a' \in A$ there exists some $p_{a'} \in I_b$ such that $a' \le p_{a'}$. Therefore $\bigwedge F = a = \bigvee A \le \bigvee I_b$ and so, by $(\mathcal{F}, \mathcal{I})$-compactness, $F \cap I_b \neq \emptyset$, a contradiction. □

We provide a characterisation of the completely join-irreducible elements when $\mathcal{F}$ and $\mathcal{I}$ are closed under unions of chains. The following concepts generalise notions introduced by Urquhart [25] for filters and ideals in lattices and used by Hartung [23]. The term optimal comes from [19].

Definition 6.6 A filter $F \in \mathcal{F}$ is said to be $(\mathcal{F}, \mathcal{I})$-*optimal* if there exists some $I \in \mathcal{I}$ such that F is maximal in the set $\{ G \in \mathcal{F} \mid G \cap I = \emptyset \}$, and similarly we say that $I \in \mathcal{I}$ is $(\mathcal{F}, \mathcal{I})$-*optimal* if there exists some $G \in \mathcal{F}$ such that I is maximal in the set $\{ J \in \mathcal{I} \mid J \cap G = \emptyset \}$.

Proposition 6.7 *Assume that $\mathcal{F}$ and $\mathcal{I}$ are closed under unions of chains. For every $F \in \mathcal{F}$ and every $I \in \mathcal{I}$ with $F \cap I = \emptyset$,*

(1) *there is some $(\mathcal{F}, \mathcal{I})$-optimal $G \in \mathcal{F}$ with $F \subseteq G$ and $G \cap I = \emptyset$;*
(2) *there is some $(\mathcal{F}, \mathcal{I})$-optimal $J \in \mathcal{I}$ with $I \subseteq J$ and $F \cap J = \emptyset$.*

Proof To prove (1), one argues as in the proof of the previous proposition to obtain a maximal element G in the set $\{ G' \in \mathcal{F} \mid F \subseteq G', G' \cap I = \emptyset \}$. Then G is also maximal in $\{ G' \in \mathcal{F} \mid G' \cap I = \emptyset \}$. Therefore G is $(\mathcal{F}, \mathcal{I})$-optimal. The statement (2) is proved order dually. □

Proposition 6.8 *Assume that $\mathcal{F}$ and $\mathcal{I}$ are closed under unions of chains. For every $p \in P$, every $F \in \mathcal{F}$, and every $I \in I$,*

(1) *if $p \notin F$, there is an $(\mathcal{F}, \mathcal{I})$-optimal $G \in \mathcal{F}$ with $F \subseteq G$ and $p \notin G$;*
(2) *if $p \notin I$, there is an $(\mathcal{F}, \mathcal{I})$-optimal $J \in \mathcal{I}$ with $I \subseteq J$ and $p \notin J$.*

Proof To prove (1), it is enough to consider the element ${\downarrow} p$ of $\mathcal{I}$ which is disjoint from F, and apply Proposition 6.7, and to prove (2), to consider the element ${\uparrow} p$ of $\mathcal{F}$. □

Proposition 6.9 *An $F \in \mathcal{F}$ is $(\mathcal{F}, \mathcal{I})$-optimal if and only if $\bigwedge F$ is completely join-irreducible in P^*. Dually, $I \in \mathcal{I}$ is $(\mathcal{F}, \mathcal{I})$-optimal if and only if $\bigvee I$ is completely meet-irreducible in P^*.*

Proof Assume that $F \in \mathcal{F}$ is optimal. So, let $I \in \mathcal{I}$ be such that F is maximal in $\{ G \in \mathcal{F} \mid G \cap I = \emptyset \}$ and let $x = \bigwedge F \in \mathbb{F}^{\mathcal{F}}(P^*)$. In order to prove that x is completely join-irreducible, suppose that $x = \bigvee X$ for $X \subseteq P^*$. Since $\mathbb{F}^{\mathcal{F}}(P^*)$ is join-dense in P^*, we may assume that $X \subseteq \mathbb{F}^{\mathcal{F}}(P^*)$. Now suppose that $x \notin X$. Then for every $x' \in X$ we have $x' < x$. Thus, $F \subset {\uparrow_P} x'$ for every $x' \in X$. From the maximality of F in $\{ G \in \mathcal{F} \mid G \cap I = \emptyset \}$ follows that ${\uparrow_P} x' \cap I \neq \emptyset$. So, we choose for every $x' \in X$ an element $p_{x'} \in {\uparrow_P} x' \cap I$. Then $\bigwedge F = x = \bigvee X \le \bigvee p_{x'} \le \bigvee I$. Therefore, $F \cap I \neq \emptyset$, contrary to our assumption. Thus, $x \in X$ and we conclude that $\bigwedge F$ is completely meet-irreducible.

Suppose F is not $(\mathcal{F},\mathcal{I})$-optimal. Then, for each $I\in\mathcal{I}$ with $F\cap I=\emptyset$ there is a $G_I\in\mathcal{F}$ with $F\subsetneq G_I$ and $G_I\cap I=\emptyset$. Let $u=\bigvee\{\bigwedge G_I\mid I\in\mathcal{I}\text{ with }F\cap I=\emptyset\}$. We will show that $u=\bigwedge F$. Clearly $u\leq\bigwedge F$ since each G_I contains F. In order to show that $\bigwedge F\leq u$, by $(\mathcal{F},\mathcal{I})$-denseness, it suffices to show that $u\leq\bigvee I$ implies $\bigwedge F\leq\bigvee I$ for each $I\in\mathcal{I}$. By contraposition, suppose that $\bigwedge F\nleq\bigvee I$.Then, certainly $F\cap I=\emptyset$. Thus we have $\mathcal{F}\ni G_I\cap I=\emptyset$ and by $(\mathcal{F},\mathcal{I})$-compactness it follows that $\bigwedge G_I\nleq\bigvee I$. Since $\bigwedge G_I\leq u$ by definition of u we conclude that $u\nleq\bigvee I$. That is, $\bigwedge F=u=\bigvee\{\bigwedge G_I\mid I\in\mathcal{I}\text{ with }F\cap I=\emptyset\}$. If $\bigwedge F$ is completely join irreducible this would mean that $\bigwedge F=\bigwedge G_I$ for some I, which is a contradiction of $(\mathcal{F},\mathcal{I})$-compatibility since $F\neq G_I$ by assumption.

The statement for completely meet-irreducibles follows by order duality. □

6.3 Preservation of Finite Meets and Joins

Proposition 6.10

(1) *The finite joins existing in P are preserved in P^* iff every $I\in\mathcal{I}$ is closed under existing finite joins.*
(2) *The finite meets existing in P are preserved in P^* iff every $F\in\mathcal{F}$ is closed under existing finite meets.*

Proof We just prove (1) as (2) follows by order duality. Let $p,q\in P$ and suppose that $p\vee q$ exists in P. To say that $p\vee q$ is the supremum of p and q in P^* is, by $(\mathcal{F},\mathcal{I})$-denseness, equivalent to saying that, for every $y\in\mathbb{I}^{\mathcal{I}}(P^*)$, $p\vee q\leq y$ if and only if y is a common upper bound of p and q. But $p\leq y$ if and only if $p\in I$ where $I\in\mathcal{I}$ is the ideal corresponding to y and similarly for q and $p\vee q$. Thus we have that $p\vee q$ is the join of p and q in P^* if and only if

$$p\in I\text{ and }q\in I\quad\Longleftrightarrow\quad p\vee q\in I,$$

is true for every $I\in I$, which is precisely the statement that every $I\in\mathcal{I}$ must be closed under all existing joins. □

6.4 Dual Partial Orders

Let P be a poset, $\mathcal{F}$ be a standard collection of up-sets of P and $\mathcal{I}$ be a standard collection of down-sets of P. Let $P^\partial=(P,\geq)$ be the dual poset of P. We set $\mathcal{F}(P^\partial):=\mathcal{I}$ and $\mathcal{I}(P^\partial):=\mathcal{F}$. Then $\mathcal{F}(P^\partial)$ is a standard collection of up-sets of P^∂ and $\mathcal{I}(P^\partial)$ is a standard collection of down-sets of P^∂. Let $(P^\partial)^*$ denote the (unique up to isomorphism) $(\mathcal{F}(P^\partial),\mathcal{I}(P^\partial))$-completion of P^∂.

Proposition 6.11

(1) *$(P^*)^\partial$ is (up to isomorphism) the $(\mathcal{F}(P^\partial),\mathcal{I}(P^\partial))$-completion $(P^\partial)^*$.*
(2) $\mathbb{F}^{\mathcal{F}(P^\partial)}((P^\partial)^*)\cong\mathbb{I}^{\mathcal{I}}(P^*)$;
(3) $\mathbb{I}^{\mathcal{I}(P^\partial)}((P^\partial)^*)\cong\mathbb{F}^{\mathcal{F}}(P^*)$.

Proof The order dual of $(\mathcal{F},\mathcal{I})$-denseness is clearly $(\mathcal{F}(P^\partial),\mathcal{I}(P^\partial))$-denseness and the order dual of $(\mathcal{F},\mathcal{I})$-compactness is $(\mathcal{F}(P^\partial),\mathcal{I}(P^\partial))$-compactness. Thus (1) follows by uniqueness.

The properties (2) and (3) follow immediately from (1). We just prove (2). By (1) we can take $P^\partial \to (P^*)^\partial$ as the $(\mathcal{F}(P^\partial), \mathcal{I}(P^\partial))$-completion. For any $a \in (P^\partial)^*$, we have that $a \in \mathbb{F}^{\mathcal{F}(P^\partial)}((P^\partial)^*)$ if and only if

$$a = \bigwedge\nolimits_{(P^\partial)^*} F = \bigwedge\nolimits_{(P^*)^\partial} F = \bigvee\nolimits_{P^*} F$$

for some $F \in \mathcal{F}(P^\partial) = \mathcal{I}$. Conversely, for every $b \in \mathbb{I}^{\mathcal{I}}(P^*)$, we have

$$b = \bigvee\nolimits_{P^*} I = \bigwedge\nolimits_{(P^\partial)^*} I$$

for some $I \in \mathcal{I} = \mathcal{F}(P^\partial)$, so $b \in \mathbb{F}^{\mathcal{F}(P^\partial)}((P^\partial)^*)$. □

6.5 Products

Let P_1 and P_2 be bounded posets. That is, both posets have a least element, which we will denote by 0, and a largest element, which we will denote by 1. Further, let $\mathcal{F}_i$ and $\mathcal{I}_i$ be a standard collection of up-sets and a standard collection of down-sets of P_i, respectively, for $i = 1, 2$. Let $\mathcal{F} = \mathcal{F}(P_1 \times P_2)$ and $\mathcal{I} = \mathcal{I}(P_1 \times P_2)$ be a standard collection of up-sets and a standard collection of down-sets of the poset $P_1 \times P_2$, respectively. Recall that $\mathcal{N}(P)$ denotes the MacNeille completion of P.

Proposition 6.12 *If*

(i) $\mathcal{F} = \{F_1 \times F_2 \mid F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2\}$, *and*
(ii) $\mathcal{I} = \{I_1 \times I_2 \mid I_1 \in \mathcal{I}_1 \text{ and } I_2 \in \mathcal{I}_2\}$,

then

(1) $P_1^* \times P_2^*$ *is (up to isomorphism) the* $(\mathcal{F}, \mathcal{I})$*-completion* $(P_1 \times P_2)^*$.
(2) $\mathbb{F}^{\mathcal{F}}((P_1 \times P_2)^*) \cong \mathbb{F}^{\mathcal{F}_1}(P_1^*) \times \mathbb{F}^{\mathcal{F}_2}(P_2^*)$,
(3) $\mathbb{I}^{\mathcal{I}}((P_1 \times P_2)^*) \cong \mathbb{I}^{\mathcal{I}_1}(P_1^*) \times \mathbb{I}^{\mathcal{I}_2}(P_2^*)$.

Proof (1) By the uniqueness of the $(\mathcal{F}, \mathcal{I})$-completion, it is enough to show that the product embedding $P_1 \times P_2 \to P_1^* \times P_2^*$ is an $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense extension. Let us abbreviate $P_1 \times P_2$ as Q. For $(\mathcal{F}, \mathcal{I})$-denseness, let $(u, v) \in P_1^* \times P_2^*$; since P_i^* is an $(\mathcal{F}_i, \mathcal{I}_i)$-dense extension of P_i for $i = 1, 2$, we get that $u = \bigvee A_1$ and $v = \bigvee A_2$ for some $A_i \subseteq \mathbb{F}^{\mathcal{F}_i}(P_i^*)$. Notice that the existence of the 0 in P_i and the standardness of $\mathcal{F}_i$ guarantee that we can take $A_i \neq \emptyset$. Let $A = \{(a_1, a_2) \mid a_i \in A_i\}$. Clearly, $(u, v) = \bigvee A$. So in order to complete this part of the proof, it is enough to show that $A \subseteq \mathbb{F}^{\mathcal{F}}(Q)$. From $a_i \in A_i \subseteq \mathbb{F}^{\mathcal{F}_i}(P_i^*)$ it follows that $a_i = \bigwedge F_i$ for some $F_i \in \mathcal{F}_i$. Then consider $F = F_1 \times F_2$. By assumption, $F \in \mathcal{F}$. Moreover, $\bigwedge_Q F = (\bigwedge_{P_1^*} \pi_1[F], \bigwedge_{P_2^*} \pi_2[F]) = (\bigwedge_{P_1^*} F_1, \bigwedge_{P_2^*} F_2) = (a_1, a_2)$. The second part of denseness follows by order duality.

For $(\mathcal{F},\mathcal{I})$-compactness, let $F\in\mathcal{F}$ and $I\in\mathcal{I}$, then we have:

$$\begin{aligned}\bigwedge\nolimits_Q F\leq\bigvee\nolimits_Q I &\Leftrightarrow \left(\bigwedge\nolimits_{P_1^*}\pi_1[F],\bigwedge\nolimits_{P_2^*}\pi_2[F]\right)\leq\left(\bigvee\nolimits_{P_1^*}\pi_1[I],\bigvee\nolimits_{P_2^*}\pi_2[I]\right)\\ &\Leftrightarrow \bigwedge\nolimits_{P_i^*}\pi_i[F]\leq\bigvee\nolimits_{P_i^*}\pi_i[I],\ i=1,2\\ &\Leftrightarrow \pi_i[F]\cap\pi_i[I]\neq\emptyset,\ i=1,2 \qquad (\star)\\ &\Leftrightarrow (\exists u_i)(u_i\in\pi_i[F]\cap\pi_i[I]),\ i=1,2\\ &\Leftrightarrow (\exists(u_1,u_2))((u_1,u_2)\in F\cap I) \qquad (\star\star)\\ &\Leftrightarrow F\cap I\neq\emptyset\end{aligned}$$

The equivalence marked with $(\star)$ is true because P_i^* is an $(\mathcal{F}_i,\mathcal{I}_i)$-compact extension of P_i, and by our assumptions on $\mathcal{F}$ and $\mathcal{I}$. The equivalence marked with $(\star\star)$ is true because of our assumptions on $\mathcal{F}$ and $\mathcal{I}$.

The properties (2) and (3) follow immediately from (1). We just prove (2). By (1) we can take the product embedding $P_1\times P_2\to P_1^*\times P_2^*$ as the $(\mathcal{F},\mathcal{I})$-completion. For every $a\in(P_1\times P_2)^*$, we have that $a\in\mathbb{F}^{\mathcal{F}}((P_1\times P_2)^*)$ iff $a=\bigwedge_{(P_1\times P_2)^*}F=\bigwedge_{P_1^*\times P_2^*}F_1\times F_2$ for some $F=F_1\times F_2\in\mathcal{F}$, and some $F_i\in\mathcal{F}_i$, iff $a=(a_1,a_2)$ for $a_i=\bigwedge_{P_i^*}F_i\in\mathbb{F}^{\mathcal{F}_i}(P_i^*)$. □

Notice that the hypothesis of boundedness in the proposition above is needed: for instance, if P_1 is the 2-element chain, P_2 is the 2-element antichain, and the collections $\mathcal{F}$ (resp. $\mathcal{I}$) are the principal up-sets (resp. down-sets) in the three posets, then conditions (i) and (ii) are clearly verified; however, $(P_1\times P_2)^*=\mathcal{N}(P_1\times P_2)$ is the lattice obtained by adding a top and a bottom to the disjoint union of two 2-element chains, whereas $P_1^*\times P_2^*=\mathcal{N}(P_1)\times\mathcal{N}(P_2)$ is the Boolean algebra with three atoms.

Let us now further assume that, for $i=1,2$, the collections $\mathcal{F}_i$ and $\mathcal{I}_i$ respectively consist of down-directed up-sets and of up-directed down-sets.

Proposition 6.13 *If, for $i=1,2$:*

(a) *$\pi_i[F]\in\mathcal{F}_i$ for every $F\in\mathcal{F}$, and $F_1\times F_2\in\mathcal{F}$ for every $F_i\in\mathcal{F}_i$;*
(b) *$\pi_i[I]\in\mathcal{I}_i$ for every $I\in\mathcal{I}$, and $I_1\times I_2\in\mathcal{I}$ for every $I_i\in\mathcal{I}_i$;*

then,

(i) *$\mathcal{F}=\{F_1\times F_2\mid F_1\in\mathcal{F}_1\ \text{and}\ F_2\in\mathcal{F}_2\}$, and*
(ii) *$\mathcal{I}=\{I_1\times I_2\mid I_1\in\mathcal{I}_1\ \text{and}\ I_2\in\mathcal{I}_2\}$.*

Proof We only prove (i). The right–to–left inclusion immediately follows by the assumption.

To prove the converse inclusion, let $F\in\mathcal{F}$; it is enough to show that $F=\pi_1[F]\times\pi_2[F]$. Clearly, F is included in the product set. If $(u,v)\in\pi_1[F]\times\pi_2[F]$, then $(u,x_2),(x_1,v)\in F$ for some $x_i\in P_i$; the down-directedness of $\pi_i[F_i]$ implies that F is down-directed; hence, there exists some $(u',v')\in F$ which is less than or equal to both (u,x_2) and (x_1,v). Hence, $(u',v')\leq(u,v)$, which proves that $(u,v)\in F$. □

6.6 Choosing the Collections $\mathcal{F}$ and $\mathcal{I}$

The above analysis shows that the key properties for $\mathcal{F}$ and $\mathcal{I}$ are: being standard; closure under the relevant lattice operations; closure under directed unions. For the uniform assignment $P \mapsto (\mathcal{F}(P), \mathcal{I}(P))$, the key properties are its compatibility with order duality and with products as specified in Propositions 6.11 and 6.13.

Being standard guarantees that the original poset embeds in the completion. We have assumed that this is what we want in the present paper because we are talking about completions of posets, but in applications where P plays the role of some set of formal generators, as e.g. in presentations in domain theory, this may actually not be desirable. However, modulo taking a quotient of P, the results here should go through to the setting of domain theory. Closure under the lattice operations may in fact be desirable in some cases but not in others. In the setting of lattices, we want lattice completions, so the closure under the lattice operations is key to this setting. In some other situations we may want to add finite joins or meets freely. This was for example the case in our work [18] where we used the results from the present paper to develop a notion of canonical extension for logics, as these are treated in Abstract Algebraic Logic. Having enough completely join- and completely meet-irreducible elements is essential if the purpose is to develop relational semantics on the basis of the extensions. Applications to logic crucially involve operations that reverse order in some coordinates. This is why having extensions that commute with order reversal is important. Finally, it is crucial in the theory of canonical extensions that the canonical extension of a Boolean product is the full product of canonical extensions. This latter compatibility certainly includes compatibility with respect to finite products, which is necessary to being able to treat n-ary operations uniformly. There are further reasons for wanting compatibility with products: in the lattice setting, the powerful algebraic version of the Fine-van Benthem–Goldblatt theorem, which guarantees *inter alia* the canonicity of finitely generated varieties, relies heavily on the compatibility of extensions with Boolean products. Note that, starting from the principal up-sets and down-sets and closing under directed unions gives precisely the usual notions of filters and ideals for posets, namely down-directed up-sets and up-directed down-sets, respectively. This minimal choice with respect to these two properties also has all the other properties listed, and thus it is, also from the point of view of the present analysis, a natural uniform choice for $\mathcal{F}$ and $\mathcal{I}$. However, the benefit of the classification theorem in Section 3 and of the results on compact completions is that they provide a parametric class of completions that may contain the completions best suited in many different situations.

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