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Electron-quasihole duality and second-order differential equation for Read-Rezayi and Jack wave functions

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We consider the quasihole wave functions of the non-Abelian Read-Rezayi quantum-Hall states which are given by the conformal blocks of the minimal model $\mathrm{WA}_{k-1}(k+1,k+2)$ of the WA_{k-1} algebra. By studying the degenerate representations of this conformal field theories, we derive a second-order differential equation satisfied by a general many-quasihole wave function. We find a duality between the differential equations fixing the electron and quasihole wave functions. They both satisfy the Laplace-Beltrami equation. We use this equation to obtain an analytic expression for the generic wave function with one excess flux. These results also apply to the more general models $\mathrm{WA}_{k-1}(k+1,k+r)$ corresponding to the recently introduced Jack states.

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I. INTRODUCTION

Since the success of the Laughlin states,[1] the use of trial wave functions in the fractional quantum-Hall (FQH) effect has provided deep insights into the physics of these systems. Recent new interest has focused on non-Abelian fractional quantum-Hall states of matter. These are gapped phases of matter in which the adiabatic transport of one fundamental excitation around another implies a unitary transformation within a subspace of degenerate wave functions which differ from each other only globally.[2] Systems exhibiting non-Abelian statistics can store topologically protected qubits and are therefore interesting for topological quantum computation.[3]

Much of the comprehension of non-Abelian quantum-Hall states relies on the conformal field-theory (CFT) (Refs. 4 and 5) approach. In this approach, the analytic part of trial wave functions is given by the conformal blocks of a given CFT.[6,7] The basic idea is that the effects of an adiabatic exchange of excitations coincides with the monodromy properties of conformal blocks.[2,8] In this case, the universal properties characterizing a topological phase, such as the quantum numbers of the ground state or the statistics of the excitations, are directly related to the analytic properties of the conformal blocks in the corresponding CFT.

The Read-Rezayi (RR) states play a paradigmatic role in the physics of non-Abelian FQH states. The RR states are built from the conformal blocks of the $\mathbb{Z}_k$ parafermionic theory[9] and describe bosons (fermions) at Landau-level filling fractions $\nu=k/(Mk+2)$, where M is an even (odd) integer. For example, the $k=1$ RR states correspond to the simplest Laughlin states associated to the (Abelian) $c=1$ CFTs. The $k=2,\ M=1$ RR state corresponds to the Moore-Read state[6] which describes a (p-wave) pairing of electrons occurring at the first excited Landau level for filling fraction $\nu=5/2$. The non-Abelian nature of the Moore-Read state is well understood in terms of the $\mathbb{Z}_2$ parafermionic theory, which coincides with the Ising CFT.[4,5] The general k RR states[7] have been introduced as a natural generalization of the pairing $k=2$ structure of the Moore-Read states. The k RR states describe incompressible fluids made by clusters of k particles and can be fully characterized by specifying the manner in which these states vanish as $k+1$ particles come to the same point. This so-called k-clustering properties are inherited from the $\mathbb{Z}_k$ symmetry of the parafermionic theory and makes the k RR states to be the exact zero-energy states of a $k+1$-body interaction Hamiltonians.[7,10] It is now established that the RR sequence could be physically relevant: the $k=3,4,\ M=1$ RR state, for instance, may capture the physics of the FQH plateau observed at $\nu=12/5$ and $\nu=13/5$ (Ref. 11) while bosonic RR states may also be realizable in rotating Bose gases.[12] Moreover, for topological quantum computation purposes, the $k>2$ RR states are particularly interesting as their non-Abelian braiding is sufficiently rich to carry out universal quantum computation.[13]

Since their introduction, many properties of the RR states, such as the degeneracy of the quasihole excitations and the associated braiding properties, have been studied in detail. Generally, these properties can be derived from the parafermionic fusion rules which in turn can be directly read out, together with the values of the conformal dimension of the fields, from the representation theory of the $\mathbb{Z}_k$ chiral algebra. On the other hand, the computation of multipoint correlation functions or, in other words, of the explicit expressions of the wave functions, is much more difficult. Clearly, the knowledge of the exact expression of the wave functions provides information which goes well beyond the quasihole braiding properties, as been discussed in Refs. 14 and 15. For the RR states,[7] the form of the ground-state wave function is known explicitly for any number of particles as being a Jack polynomial. Concerning the excited wave functions, an explicit expression is not known. An implicit construction of the linearly independent quasihole polynomials is given in Ref. 16 but the explicit expression for the state with one flux added in the general $\mathbb{Z}_k$ theory is still missing. The simplest excited wave functions, i.e., with the smallest number of quasiholes exhibiting non-Abelian statistics have been evaluated in Ref. 14 for the $k=2$ Moore-Read case and in Ref. 15 for general k. Still, the explicit expressions for general quasihole wave functions and for general k is not known.

It is the purpose of this paper to find the differential operator which annihilates a general quasihole wave function of the RR series and use it to obtain a general expression for the excited one flux-added excitations of all the $\mathbb{Z}_k$ RR states. During the process, we find that the Read-Rezayi pinned quasihole states (and in fact all the Jack polynomial states) satisfy a previously unknown electron-quasihole duality. We find that the differential operator that diagonalizes the pinned quasihole wave function is second order in *both* electron and quasihole coordinates. Moreover, the form of the operator in electron and quasihole coordinates is of Laplace Beltrami form but with dual coupling strength. We use this to obtain the expression for the one-flux-added quasihole wave function for all the $\mathbb{Z}_k$ Read-Rezayi states. Due to the dual nature of the electron and quasihole states, the wave function for pinned quasiholes decomposes into a sum over products of Jack polynomials in electron coordinates and dual Jack polynomials in quasihole coordinates.

A possible way to derive differential equations for RR states would be to use the $SU(2)_k/U(1)$ coset realization of the $\mathbb{Z}_k$ parafermionic theory. Indeed, the correlation functions of the $\mathbb{Z}_k$ parafermionic theory can be directly related to the ones of the $SU(2)_k$ primaries of a Wess-Zumino-Witten model[17] which satisfy a system of linear differential equations, the so called Knizhnik-Zamolodchikov equations.[18] However, this method is increasingly impractical as the size of the system of differential equations grows with the number of quasiholes and particles. While in small number of cases (such as the $\mathbb{Z}_3$ and $\mathbb{Z}_2$ parafermions) the differential equation satisfied by the quasiholes can be obtained through identifying the theory with a minimal model, it is generally of high order in derivatives (third order for the $\mathbb{Z}_3$ RR state). The differential equation we obtain contains only second-order derivatives for any k. The lowering of derivative order is important—the fact that the Laughlin or Halperin wave functions are annihilated by first-order differential operators is crucial to the mathematical proof of the quasihole braiding.

Our starting point will be instead to consider the description of the $\mathbb{Z}_k$ parafermionic theory in terms of a family of CFTs with W extended symmetry, the WA_{k-1} CFTs.[19,20] The importance of the W symmetry for understanding the RR sequence was pointed out and used in Ref. 21. We will show in this paper that, by exploiting the W symmetry, we can derive a second-order differential equation satisfied by a general k RR quasihole wave function. Moreover, our approach can be straightforwardly applied to a more general family of trial wave functions which can be written in term of single Jack polynomial[22] and therefore named Jack wave functions.[23,24]

The series of minimal models $\mathrm{WA}_{k-1}(p,q)$ associated to the WA_{k-1} algebras are indexed by two coprime integers p and q.[19,20] The $\mathbb{Z}_k$ parafermionic theory coincides with the theory $\mathrm{WA}_{k-1}(k+1,k+2)$. The Jack wave functions represent a possible generalization of the Read-Rezayi states based on the $\mathrm{WA}_{k-1}(k+1,k+r)$ theory, with $r>2$.[23–27] The states constructed from the $\mathrm{WA}_{k-1}(k+1,k+r)$ conformal block have been shown to satisfy the so called (k,r) clustering conditions:[23,24,28,29] they vanish with power r when the $k+1^{\text{th}}$ particle approaches a cluster of k particles. Here, it is important to stress that the $\mathrm{WA}_{k-1}(k+1,k+2)$ are nonunitary for $r>2$. Although there is increasingly strong evidence that only rational unitary CFTs can describe a gapped phase of matter,[30–32] these nonunitary states may still describe interesting critical points between gapped phases and therefore be worth studying.[23,24,33–35] Unitary CFTs generating (k,r) clustering polynomials have been discussed in Ref. 36 and their physical relevance further discussed in Ref. 37.

In Ref. 38, two of us considered the ground-state wave functions for general $\mathrm{WA}_{k-1}(k+1,k+r)$ states. In this respect we studied the multipoints correlation functions of certain W primary fields which are identified as particle operators (defined below). By exploiting the representations of the WA_{k-1} symmetry, and in particular, the degeneracy properties of these particle fields, we showed that their N-point correlation functions satisfy a second-order differential equation. Moreover we showed that this equation can to be transformed into a Calogero Hamiltonian with negative rational coupling. This completed the proof of the fact that the ground states $\mathrm{WA}_{k-1}(k+1,k+r)$ theories can be written as a single Jack polynomial.[23,24] In this paper we show that the approach followed in Ref. 38 can be extended to the study of general quasihole wave functions for the Read-Rezayi states and, more generally, to the quasihole wave functions of the $\mathrm{WA}_{k-1}(k+1,k+r)$ theories. Moreover, we show how to apply the second-order differential equation to rigorously generalize an ansatz found in earlier works[24] for the computation of certain quasihole wave functions. In the process we find a previously unknown electron-quasihole duality.

In Sec. II we define the FQH trial wave functions which are the object of our study and we make explicit the FQH-CFT connection. In Sec. III, we characterize completely the electron and quasihole operators by specifying their WA_{k-1} quantum numbers. In Sec. IV we derive a partial differential equation (PDE) satisfied by the quasihole wave functions. In Sec. V we present the main results of the paper: we derive the PDE for generic quasihole wave functions describing the non-Abelian Jack quantum-Hall states. This PDE turns out to be surprisingly simple and exhibits an interesting duality between electrons and quasiholes. In Sec. VI we solve the PDE for the case of one flux-added wave functions and obtain them as a nice expansion of products of Jack polynomials in electron and quasihole coordinates. Section VII gives our conclusions; an Appendix gives a wave-function example.

II. GENERAL FORM OF READ-REZAYI AND JACK WAVE FUNCTIONS

In this section we briefly define the FQH trial wave functions which are the object of our study and we make explicit the FQH-CFT connection. It is convenient to consider a system of N particles on a sphere of radius R with a uniform radial magnetic field with total flux N_ϕ.[39] The position of the ith particle on the sphere can be represented as its stereographic projection z_i, which is a complex variable. Each particle in the lowest Landau level has orbital angular momentum $N_\phi/2$ and the single-particle basis states have the form $z^m\mu(z,\bar{z})$, $m=0,\ldots,N_\Phi$. The L_z momentum quantum number is $m-\frac{N_\Phi}{2}$ and $\mu(z,\bar{z})$ is the measure on the sphere, $\mu(z,\bar{z})$

$=1/[1+(|z|/2R)^2]^{1+N_\phi/2}$. Therefore a many-body wave function $\tilde{\Psi}$ (Ref. 1) describing N particle states in the lowest Landau levels takes the form

$$\tilde{\Psi}(z_1,\bar{z}_1,\ldots,z_N,\bar{z}_N)=P_N(z_1,\ldots,z_n)\prod_{i=1}^{N}\mu(z_i,\bar{z}_i), \qquad (1)$$

where $P_N(\{z_i\})$ is a polynomial in the N variables z_i, (anti) symmetric for bosons (fermions). In the following we drop the measure factors and focus on the analytic part of the many-body wave function.

A. Ground-state wave function

On the sphere, a quantum-Hall ground-state wave function is rotationally invariant. In projective coordinates, the corresponding polynomial $P_N(\{z_i\})$ is translationally invariant and homogeneous with a total degree $\frac{1}{2}NN_\Phi$, with N_ϕ being the highest power in each variable z_i. The total flux N_ϕ and the number of particles N are related by the linear identity

$$N_\phi=\nu^{-1}N-\delta, \qquad (2)$$

where ν is the filling fraction and δ the so-called shift. Let $P_N^{(k,r)}(z_1,\ldots,z_n)$ denote a symmetric polynomial satisfying the (k,r) clustering properties defined by

$$P_N^{(k,r)}(\underbrace{z_1=\cdots=z_k}_{=Z},z_{k+1},z_{k+2},\ldots,z_N)=\prod_{i=k+1}^{N}(Z-z_i)^r P_{N-k}^{(k,r)} \times(z_{k+1},z_{k+2},\ldots,z_N). \qquad (3)$$

Because of the above properties, the state $P_N^{(k,r)}(\{z_i\})$ is a zero energy eigenstates of the projection operator $\mathcal{P}_k^r$ (Ref. 10) which annihilates states where any cluster of $k+1$ particles has relative angular momentum less than r. In other words no more than k particles can occupy r consecutive orbitals. A (partial) classification of the symmetric polynomial with (k,r) clustering conditions has been discussed in Refs. 40–42.

1. (k,2) clustering states: The Read-Rezayi sequence

The RR sequence describes particles at filling fraction

$$\nu=k/(2+kM) \qquad \delta=2+M. \qquad (4)$$

Hereafter, without any loss of generality, we focus on the bosonic $M=0$ RR states. The RR states with general M are related to the case $M=0$ by a simple Jastrow factor.

The k RR ground state is uniquely defined by the $(k,2)$ clustering conditions in Eq. (3). The polynomial $P_N^{(k,2)}(\{z_i\})$ can be constructed in the following way:

$$P_N^{(k,2)}(z_1,\ldots,z_N)=\langle\Psi_1(z_1),\ldots,\Psi_1(z_N)\rangle\prod_{i<j}(z_i-z_j)^{2/k}, \qquad (5)$$

where $\langle\Psi_1(z_1),\ldots,\Psi_1(z_N)\rangle$ is the N-points conformal correlation function of the parafermionic field Ψ_1. The field Ψ_1 has been identified as the particle operator. This identification traces back to the fact that the field Ψ_1 is the fundamental conserved current associated to the extended $\mathbb{Z}_k$ symmetry. The associated chiral algebra is formed by the set of chiral currents $\{\Psi_0(z),\Psi_1(z),\cdots\Psi_{k-1}(z)\}$, $\Psi_0=I$ being the identity operator. These currents have conformal dimensions

$$\Delta_q=\frac{q(k-q)}{k} \qquad (6)$$

and satisfy the following operator product expansions (OPEs):

$$\Psi_q(z)\Psi_{q'}(w)=\frac{\gamma_{q,q'}}{(z-w)^{\Delta_q+\Delta_{q'}-\Delta_{q+q'}}}[\Psi_{q+q'}(w)], \qquad (7)$$

$$\Psi_q(z)\Psi_{k-q}(w)=\frac{1}{(z-w)^{2\Delta_q}}\left[1+(z-w)^2\frac{2\Delta_q}{c}T(w)+\cdots\right], \qquad (8)$$

where the sums $q+q'$ are defined modulo k, $[\Psi]$ indicates the operator Ψ and its descendants, while the $\gamma_{q,q'}$ are the algebra coupling constants. Notice that the fields Ψ_1 and its conjugate Ψ_{k-1} have the same conformal dimension and fuse to the identity, $\Psi_1\times\Psi_{k-1}=1$. The field $\Psi_{k-1}(z)$ represents a cluster of $k-1$ particles at the position z.

The $(k,2)$ clustering properties in Eq. (3) are a direct consequence of the fusion rules in Eqs. (7) and (8). Moreover the relations in Eqs. (5) and (6) imply $N_\phi=2(N-k)/k$, which corresponds to Eq. (4) with $M=0$.

2. (k,r) clustering states: Jack wave functions

The OPEs in Eqs. (7) and (8) are the general form for a candidate chiral algebra realizing the $\mathbb{Z}_k$ symmetry. When considering arbitrary values of the conformal dimensions Δ_q, the principal difficulty is to define completely the form of the corresponding OPEs, i.e., all the singular terms, in order to obtain an associative algebra. Other associative solutions besides the ones in Eq. (6) have been found for the following values:[43–47]

$$\Delta_q=\frac{r}{2}\frac{q(k-q)}{k}, \qquad (9)$$

where r is an integer $(r=2,3,\ldots)$. In the following we denote this algebra as $\mathbb{Z}_k^{(r)}$.

The $WA_{k-1}(k+1,k+r)$ conformal field theory is a particular realization of this algebra. It contains a set of primaries Ψ_q of dimension in Eq. (9) that forms a parafermionic algebra in Eqs. (7) and (8).[19] Let $\langle\Psi_1(z_1)\cdots\Psi_1(z_N)\rangle^{(r)}$ denote the correlator of N parafermionic fields Ψ_1 of the $WA_{k-1}(k+1,k+r)$ theory. The OPEs in Eqs. (7) and (8) together with the conformal dimensions in Eq. (9) imply that the polynomial

$$P_N^{(k,r)}(z_1,\ldots,z_N)\equiv\langle\Psi_1(z_1),\ldots,\Psi_1(z_N)\rangle^{(r)}\prod_{i<j}(z_i-z_j)^{r/k} \qquad (10)$$

satisfies the (k,r) clustering properties. Moreover, these clustering states describe particles at filling fraction[23,24]

$$\nu^{(r)} = k/r \quad \text{and} \quad \delta^{(r)} = r. \tag{11}$$

Clearly, for $r=2$, one finds the quasihole wave functions of the RR sequence. In Ref. 38 it was proven that the ground-state wave functions $P_N^{(k,r)}$ defined in Eq. (10) can be written in terms of single Jack polynomials (see Sec. IV). The aim of this paper is to extend this analysis to quasihole states.

B. Quasihole wave functions

One type of excitations over the quantum-Hall ground state are obtained by injecting quasiholes, i.e., defect of charge, at fixed positions w_j. The elementary quasihole operator is represented generally by a field σ which has to be identified among the primaries of the correspondent CFT theory. In order to identify the quasihole operator, it is required that particles and quasihole be mutually local.

We consider in the following a general $WA_{k-1}(k+1,k+r)$ theory, recovering the RR states by setting $r=2$. The field representing the elementary quasihole operator for the RR states and more generally for (k,r) clustering Jack states is a field σ with dimension h

$$h = \Delta_\sigma = \frac{(k-1)[1+k(2-r)]}{2k(r+k)}. \tag{12}$$

The conformal dimension h has been extracted in Ref. 35 from the clustering properties and the assumption that the electronic wave functions can be described purely in terms of (k,r) admissible Jack polynomials.

The OPEs between the fields Ψ_1 and Ψ_{k-1} and σ have the form

$$\Psi_1(z)\sigma(w) = \frac{1}{(z-w)^{1/k}}\phi(w) + \cdots,$$

$$\Psi_{k-1}(z)\sigma(w) = \frac{1}{(z-w)^{(k-1)/k}}\phi'(w) + \cdots,$$

where ϕ and ϕ' are others $WA_{k-1}(k+1,k+r)$ primaries. In the Hilbert space of the $WA_{k-1}(k+1,k+r)$ one finds the field conjugate to σ. This field σ' has the same conformal dimension $\Delta_{\sigma'}=h$, see Eq. (12), and has a fusion channel with σ into the identity

$$\sigma(z)\sigma'(w) = \frac{1}{(z-w)^{2h}} + \cdots. \tag{13}$$

The field σ' represents then the object which is precisely the fusion of $k-1$ quasiholes. The fusion rules of the field σ' with the fields Ψ_1 and Ψ_{k-1} have the form

$$\Psi_1(z)\sigma'(w) = \frac{1}{(z-w)^{(k-1)/k}}\tilde{\phi}'(w) + \cdots,$$

$$\Psi_{k-1}(z)\sigma'(w) = \frac{1}{(z-w)^{1/k}}\tilde{\phi}(w) + \cdots.$$

Again the fields $\tilde{\phi}$ and $\tilde{\phi}'$ are W primaries with the same dimension, respectively, of the fields ϕ and ϕ' appearing in Eq. (13). The degenerate subspace of wave functions with M quasiholes is then built of the different conformal blocks of the correlator

$$\begin{aligned} P_{(a)}^{(k,r)}(w_1,w_2,\ldots,w_M|z_1,\ldots,z_N) &= \langle \sigma(w_1)\sigma(w_2)\cdots\sigma(w_M)\Psi(z_1)\cdots\Psi(z_N)\rangle_{(a)}^{(r)} \\ &\times \prod_{i<j}^{N}(z_i-z_j)^{r/k}\prod_{i=1}^{N}\prod_{j=1}^{n}(z_i-w_j)^{1/k}\prod_{i<j}^{n}(w_i-w_j)^{1/2rk}. \end{aligned} \tag{14}$$

In the above expression the index r indicates that we are considering a correlation function of the $WA_{k-1}(k+1,k+r)$ theory while the index a runs over the possible conformal blocks.

III. WA_{k-1} THEORIES: CHARACTERIZATION OF THE ELECTRON AND QUASIHOLE OPERATORS

In this section we characterize completely the electron and quasihole operators by specifying their WA_{k-1} quantum numbers.

A. Representation theory of the WA_{k-1} algebras: Main results

A complete construction of W algebras and their representation theories can be found in Refs. 19 and 20. A brief review of the main results about W theories can be found in Ref. 38. In the following we just report the main definitions about W theories and we fix our notations.

The chiral fields $W^{(s)}(z)$ $(s=3,\ldots,k-1)$ classify, together with $T(z)$, all the operators of the model in terms of primaries and descendants of the chiral algebra $\mathcal{W}_k$. The behavior of a primary field Φ under the action of the symmetry generators $W^{(s)}(z)$ is encoded in the OPEs

$$T(z)\Phi(w) = \sum_{n=-\infty}^{\infty}\frac{L_n\Phi(w)}{(z-w)^{n+2}} = \frac{\Delta\Phi(w)}{(z-w)^2} + \frac{\partial_w\Phi(w)}{(z-w)} + \cdots, \tag{15}$$

$$W^{(s)}(z)\Phi(w) = \sum_{n=-\infty}^{\infty}\frac{W_n^{(s)}\Phi(w)}{(z-w)^{n+s}} = \frac{\omega^{(s)}\Phi(w)}{(z-w)^s} + \frac{W_{-1}^{(s)}\Phi(w)}{(z-w)^{s-1}} + \frac{W_{-2}^{(s)}\Phi(w)}{(z-w)^{s-2}} + \cdots. \tag{16}$$

The conformal dimension Δ and the $k-2$ quantum numbers $\omega^{(s)}$ are, respectively, the zero mode eigenvalues of the chiral fields $T(z)$ and $W^{(s)}(z)$

$$L_0\Phi = \Delta\Phi \quad W_0^{(s)}\Phi = \omega^{(s)}\Phi \quad s=3,\ldots,k-1. \tag{17}$$

They characterize each representation Φ. Notice also that, from the above definitions, $L_{-1}\Phi(z)=\partial_z\Phi(z)$.

The minimal theories $WA_{k-1}(p,p')$ (p and p' coprimes) are characterized by a finite number of primary fields. Each primary operator $\Phi_{(n_1,\ldots,n_{k-1}|n'_1,\ldots,n'_{k-1})}$ is then labeled by a set of integers n_a, n'_a $(a=1,\ldots,k-1)$. The principal domain of the Kac table contains the set of primary operators which form a closed fusion algebra and is delimited as follows:

$$n_a \geq 1; \quad n'_a \geq 1, \tag{18}$$

$$\sum_a n_a < p'; \quad \sum_a n'_a < p. \tag{19}$$

The allowed values of these integers are defined by the condition of complete degeneracy of the modules with respect to the chiral algebra.[19]

Introducing $n_0 \equiv p' - \Sigma_{a=1}^{k-1} n_a$ and $n'_0 \equiv p - \Sigma_{a=1}^{k-1} n'_a$, the Kac table is simply delimited by $n_a, n'_a \geq 1$ $(a=0,\ldots,k-1)$. Moreover, positions in the Kac table are identified modulo cyclic permutations of the indices

$$n_a \to n_{a+1} \quad \text{and} \quad n'_a \to n'_{a+1} \quad \text{with} \quad n_k \equiv n_0. \tag{20}$$

This means that the fields $\Phi_{(n_1,\ldots,n_{k-1}|n'_1,\ldots,n'_{k-1})}$ and $\Phi_{(n_2,\ldots,n_{k-1},n_0|n'_2,\ldots,n'_{k-1},n'_0)}$ represent the same primary field, and implies that they have the same quantum numbers $\omega^{(s)}$. In particular the conformal dimension is invariant

$$\Delta_{(n_1,\ldots,n_{k-1}|n'_1\ldots n'_{k-1})} = \Delta_{(n_2,\ldots,n_{k-1},n_0|n'_2,\ldots,n'_{k-1},n'_0)}. \tag{21}$$

The representation corresponding to the primary field $\Phi_{(\vec{n}|\vec{n}')}$ exhibits k null vectors χ_a $(a=0,\ldots,k-1)$ at level $N_a = n_a n'_a$. This directly generalizes the well-known case of the degenerate representations of the Virasoro algebra (i.e., the WA_1 algebra).[4] Another remark is that the primary fields $\Phi_{(\vec{n}|\vec{n}')}$ and $\Phi_{(\vec{n}'|\vec{n})}$ are dual from one another, in the sense that the corresponding modules have the same degeneracies. This will translate into the FQH effect language as an electron-quasihole duality, and will play a major role in the discussion of quasihole wave functions.

B. WA_{k-1} quantum numbers of electron and quasihole operators

Of particular interest is the model $WA_{k-1}(k+1,k+r)$, where $p=k+1$ and $p'=k+r$. By using the fusion rules of this theory,[25,38] one can verify that the set of operators

$$\Psi_i = \Phi^{(1,\ldots,1|1,\ldots,\underset{\substack{\uparrow\\ i\text{th}}}{1,2},1,\ldots,1)} \qquad i=1,\ldots,k-1, \quad \Delta_i = \frac{r}{2}\frac{i(k-i)}{k} \tag{22}$$

form a parafermionic algebra in Eqs. (7) and (8), namely, $\Psi_i \times \Psi_j = \Psi_{i+j \bmod k}$. An important remark is that these fusion rules are only valid when $p=k+1$. In general the fusion rules of these fields are multi channeled but for $p=k+1$ the identifications in Eq. (20) gives

$$\Psi_i = \Phi^{(1,\ldots,1|1,\ldots,\underset{\substack{\uparrow\\ i^{\text{th}}}}{1,2},1,\ldots,1)} = \Phi^{(1,\ldots,\underset{\substack{\uparrow\\ k-i^{\text{th}}}}{1,r+1},1,\ldots,1|1,\ldots,1)} \tag{23}$$

and the usual fusion rules are truncated accordingly, becoming effectively single channeled.

Notice that the correlation function of Ψ_i operators are symmetric under the conjugation of charge $i \to k-i$. This means that one could take Ψ_{k-1} as the electron operator instead. In particular, the conformal dimension is insensitive to this charge conjugation. However the eigenvalue of $W_0^{(3)}$ changes sign under $i \to k-i$, and two conjugate fields have opposite value of the quantum number $\omega^{(3)}$, as can be derived from the Ward identities of the two-point function.

The eigenvalues associated to the Ψ_1 and Ψ_{k-1} representations are fixed to

$$(\omega^{(3)}_{\Psi_1})^2 = (w^{(3)}_{\Psi_{k-1}})^2 = \frac{(k-1)^2(k-2)}{18k^3}\frac{r^2[r(k+2)+k]}{(3k+2)-rk}. \tag{24}$$

From the above equations, one can see that the signs of $\omega^{(3)}_{\Psi_1}$ and $\omega^{(3)}_{\Psi_{k-1}}$ are not fully determined. There is a global ambiguity in the sign for the quantum number $\omega^{(3)}$, as one is free to change the sign of the current $W^{(3)}(z)$ without changing the OPEs of the WA_{k-1} algebra. This is not true for the conformal dimension as the sign of the stress energy tensor $T(z)$ is set by the OPE

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \cdots. \tag{25}$$

In the following we choose a sign for $\omega^{(3)}_{\Psi_1}$ and all the remaining eigenvalues are then fully determined. We use quite often the symbol Ψ for the electron field Ψ_1, Δ for its conformal dimension

$$\Delta = \frac{r}{2}\frac{(k-1)}{k} \tag{26}$$

and $\omega^{(3)}_{\Psi}$ for the associated eigenvalue of $W_0^{(3)}$. We set then

$$\omega^{(3)}_{\Psi} = -\omega^{(3)}_{\Psi_{k-1}} = \sqrt{\frac{(k-1)^2(k-2)}{18k^3}\frac{r^2[r(k+2)+k]}{(3k+2)-rk}}. \tag{27}$$

The quasihole operators σ and σ', see Sec. II B, correspond to the degenerate representations

$$\sigma = \Phi_{(2,1,\ldots,1|1,1,\ldots,1)} \qquad \sigma' = \Phi_{(1,,1,\ldots,2|1,1,\ldots,1)} \tag{28}$$

whose dimensions is given by Eq. (12). The $W_0^{(3)}$ eigenvalues have been fixed to

$$\omega^{(3)}_{\sigma} = -\omega^{(3)}_{\sigma'} = \sqrt{\frac{(k-1)^2(k-2)}{18k^3}\frac{(2k+1-kr)^2(3k+2-kr)}{(k+r)^2[r(k+2)+k]}}. \tag{29}$$

We would like to stress that the two sets of fields (Ψ_1,Ψ_{k-1}) and (σ,σ') are dual from one another under the transformation

$$WA(p,p') \to WA(p',p). \tag{30}$$

For the minimal model $WA_1(p,q)$ this duality is usually expressed in terms of an electric-magnetic duality transformation, also called T duality in the literature.[48,49] Under this transformation the value of r changes into its dual $\tilde{r}$

$$(k+r)(k+\tilde{r})=(k+1)^2. \tag{31}$$

In particular one can check that the quantum numbers $[\Delta,(w_\Psi^{(3)})^2]$ and $[h,(w_\sigma^{(3)})^2]$ are exchanged under this transformation.

IV. SECOND-ORDER DIFFERENTIAL EQUATIONS FOR GROUND-STATE AND QUASIHOLE WAVE FUNCTIONS

In Ref. 38 the N-points correlation function $\langle\Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle^{(r)}$ of the $\mathrm{WA}_{k-1}(k+1,k+r)$ theory has been considered. This correlation function gives the ground state of the corresponding (k,r) Jack FQH state. By using the Ward identities associated to the spin 3 current $W^{(3)}(z)$ and the degeneracy properties of the Ψ_1 and Ψ_{k-1} representations, it was showed that their N-points correlation functions satisfies a second-order differential equation. In particular, this equation can be transformed into a Calogero-Sutherland Hamiltonian with negative rational coupling $\alpha=-(k+1)/(r-1)$. This completed the proof of the conjecture[24,26,28,29,35] which states that the N-points correlation functions of Ψ_1 (Ψ_{k-1}) can be written in term of a single Jack polynomial. In this section we extend this analysis to general quasihole wave functions and we derive a partial differential equation satisfied by the correlator $\langle\sigma(w_1)\sigma(w_2)\cdots\sigma(w_M)\Psi(z_1)\Psi(z_2)\cdots\Psi(z_N)\rangle$.

A. General relations from Ward identities

The WA_{k-1} symmetry manifests itself at the quantum level in the Ward identities associated to the symmetry currents $T(z)$ and $W^{(s)}(z)$, $s=3,\ldots,k$. In Ref. 38, we have written down these relations for the specific case of the N-points Ψ correlation function. These relations are very general and apply to any primary fields correlation function of the WA_{k-1} theory.

These identities can be easily obtained from Eq. (16). Consider a set of W primaries ϕ_i of dimension Δ_{ϕ_i} and $W_0^{(3)}$ eigenvalues $\omega_{\phi_i}^{(3)}$. The Ward identities associated to the stress energy tensor $T(z)$ and $W^{(3)}(z)$ take the form

$$\langle T(z)\phi_1(z_1),\ldots,\phi_N(z_N)\rangle=\sum_{j=1}^{N}\left[\frac{\Delta_{\phi_j}}{(z-z_j)^2}+\frac{\partial_j}{(z-z_j)}\right]\times\langle\phi_1(z_1),\ldots,\phi_N(z_N)\rangle, \tag{32}$$

$$\begin{aligned}\langle W^{(3)}(z)\phi_1(z_1),\ldots,\phi_N(z_N)\rangle &=\sum_{j=1}^{N}\left[\frac{\omega_{\phi_j}^{(3)}}{(z-z_j)^3}\langle\phi_1(z_1),\ldots,\phi_N(z_N)\rangle\right.\\&+\left\langle\phi_1(z_1)\cdots\frac{W_{-1}^{(3)}\phi_j(z_j)}{(z-z_j)^2}\cdots\phi_N(z_N)\right\rangle\\&\left.+\left\langle\phi_1(z_1)\cdots\frac{W_{-2}^{(3)}\phi_j(z_j)}{(z-z_j)}\cdots\phi_N(z_N)\right\rangle\right].\end{aligned} \tag{33}$$

The asymptotics of the functions $\langle T(z)\cdots\rangle$ and $\langle W^{(3)}(z)\cdots\rangle$ have, respectively, the form

$$T(z)\sim\frac{1}{z^4}\quad\text{and}\quad W^{(3)}(z)\sim\frac{1}{z^6}\quad\text{as } z\to\infty. \tag{34}$$

These asymptotic behaviors, together with the Ward identities in Eqs. (32) and (33), imply a set of relations satisfied by the correlation functions $\langle\phi_1,\ldots,\phi_N\rangle$. For instance, plugging the decomposition in Eq. (32) into Eq. (34), one finds simple differential equations which impose the invariance of the correlation function $\langle\phi_1(z_1),\ldots,\phi_N(z_N)\rangle$ under global conformal transformations.[4] Analogously to the case of the conformal symmetry, one can derive a set of relations associated to the symmetry generated by the spin 3 current $W^{(3)}(z)$.[38,50] In particular, we will use the following relation:

$$\sum_{j=1}^{N}\langle\phi_1(z_1),\ldots,(z_j^2W_{-2}^{(3)}+2z_jW_{-1}^{(3)}+\omega_{\phi_j}^{(3)})\phi_j(z_j),\ldots,\phi_N(z_N)\rangle=0. \tag{35}$$

B. Second-level null vector conditions

In Ref. 38, it was showed that, for the first two levels, all the W descendants of the fields Ψ_1 and Ψ_{k-1} can be expressed in terms of Virasoro modes. As already mentioned, the fields σ and σ' are dual to the fields Ψ_1 and Ψ_{k-1} in the sense of Eq. (31). This means also that, apart from different values of their quantum numbers, the associated representation modules have the same structure, and in particular, the same degeneracies. These four primary fields have null vectors of the form

$$W_{-1}^{(3)}\Phi_i=a_iL_{-1}\Phi_i,$$

$$W_{-2}^{(3)}\Phi_i=(\mu_iL_{-1}^2+\nu_iL_{-2})\Phi_i, \tag{36}$$

where the exact coefficients a_i,μ_i and ν_i depend on the quantum numbers of the field under consideration

$$a_i=\frac{3\omega_i^{(3)}}{2\Delta_i},$$

$$\mu_i=a_i\frac{2(2\Delta_i+c)}{(-10\Delta_i+16\Delta_i^2+2c\Delta_i+c)},$$

$$\nu_i=\mu_i\frac{8\Delta_i(\Delta_i-1)}{(2\Delta_i+c)}. \tag{37}$$

C. From the Ward identity to differential equations for conformal blocks

Following Refs. 38, 50, and 51, we derive a second-order differential equation satisfied by all the conformal blocks $\langle\sigma(w_1)\cdots\sigma(w_M)\Psi(z_1)\cdots\Psi(z_N)\rangle$. In order to write any of the relations in Eq. (35) in a differential form, we use the null-vector conditions in Eq. (36) to express the action of the modes $W_{-2}^{(3)}$ and $W_{-1}^{(3)}$ in terms of the Virasoro modes L_{-2} and $L_{-1}(=\partial)$. Notice that one could start from another Ward identity instead of Eq. (35) and obtain a different differential equation. As suggested by the results known for the Jack polynomials, all these differential equations are not independent and can be obtained form one another.

We consider the most general conformal block describing a M quasiholes wave function in Eq. (14)

$$\mathcal{F}^{(r)}_{(a)}(\{w_i\},\{z_i\}) = \langle \sigma(w_1)\cdots\sigma(w_M)\Psi_1(z_1)\cdots\Psi_1(z_N)\rangle^{(r)}_{(a)}. \tag{38}$$

By using Eq. (36) in Eq. (35), we obtain the following PDE for $\mathcal{F}^{(r)}_{(a)}$:

$$\mathcal{H}(k,r)\mathcal{F}^{(r)}_{(a)}(\{w_i\},\{z_i\}) = 0, \tag{39}$$

where the differential operator $\mathcal{H}(k,r)$ is a second-order differential operator depending on the two integers k,r parametrizing the theory $\mathrm{WA}_{k-1}(k+1,k+r)$. Defining

$$L^{(x_i)}_{-2}\langle\phi_1(x_1)\cdots\phi_N(x_N)\rangle = \left[\sum_{j\neq i}\frac{\Delta_j}{(x_i-x_j)^2} + \frac{\partial_{x_j}}{(x_i-x_j)}\right] \times\langle\phi_1(x_1)\cdots\phi_N(x_N)\rangle \tag{40}$$

the differential operator $\mathcal{H}(k,r)$ has the following form:

$$\mathcal{H}(k,r) = \mu_\Psi\sum_{i=1}^{N}\left[z_i^2\left(\partial_i^2 + \frac{\nu_\Psi}{\mu_\Psi}L^{(z_i)}_{-2}\right) + \frac{a_\Psi}{\mu_\Psi}\left(2z_i\partial_{z_i} + \frac{2}{3}\Delta\right)\right] + \mu_\sigma\sum_{l=1}^{M}\left[w_l^2\left(\partial^2_{w_l} + \frac{\nu_\sigma}{\mu_\sigma}L^{(w_l)}_{-2}\right) + \frac{a_\sigma}{\mu_\sigma}\left(2w_l\partial_{w_l} + \frac{2}{3}h\right)\right]. \tag{41}$$

The coefficients $\mu_{\Psi,\sigma}$ and $\nu_{\Psi,\sigma}$, which depends on k and r, are defined in Eq. (37).

If we perform the addition of an arbitrary number of unit of flux on a Jack FQH ground state, we obtain, in general, a degenerate space spanned by all the quasihole conformal blocks. In the CFT context, it is clear that all these different conformal blocks obey the same PDE and it is manifestly the case in Eq. (41). As such this PDE is quite complicated and not very practical. In particular, the operators $L^{(z_i)}_{-2}$ and $L^{(\omega_l)}_{-2}$ mixes electron and quasihole coordinates.

However this PDE acts on CFT conformal blocks and not the actual FQH effect wave functions, which differ by a $U(1)$ charge term. In the next section we show that taking into account this additional charge term greatly simplifies this differential equation.

V. PDE FOR QUASIHOLE WAVE FUNCTIONS AND QUASIHOLE-ELECTRON DUALITY

In this section the main results of this paper are presented. We derive the PDE for generic quasihole wave functions describing the non-Abelian Jack quantum-Hall states. This PDE turns out to be surprisingly simple and exhibits an interesting duality between electrons and quasiholes. Although electrons and quasiholes have very different physical properties, the differential equation is invariant under the exchange

$$\Psi\leftrightarrow\sigma,\quad r\leftrightarrow\tilde{r} = \frac{2k+1-kr}{k+r}.$$

We then use this PDE to compute the most generic wave function with one extra flux quantum. We recall that, by setting $r=2$ in the above equations, one finds the second-order differential equations satisfied by the k Read-Rezayi quasihole wave functions.

PDE and Calogero-Sutherland Hamiltonians

We consider the (k,r) Jack wave function for M quasiholes and N electrons, as defined through the conformal block

$$F(\{\omega\},\{z\}) \equiv \langle\sigma(\omega_1)\cdots\sigma(\omega_M)\Psi(z_1)\cdots\Psi(z_N)\rangle\prod_{i<j}(z_i - z_j)^{r/k}\prod_{l<m}(\omega_l-\omega_m)^{\tilde{r}/k}\prod_{i,l}(z_i-\omega_l)^{1/k}. \tag{42}$$

Note that this is not exactly the usual wave function, as the quasihole $U(1)$ term has been taken to be $\prod_{k<l}(\omega_l-\omega_m)^{\tilde{r}/k}$ rather than $\prod_{k<l}(\omega_l-\omega_m)^{1/2rk}$. This choice of normalization was made in order to emphasize the quasihole-electron duality. Taking into account the additional $U(1)$ term in Eq. (42), the PDE in Eq. (41) takes the strikingly simple form

$$\alpha\mathfrak{H}^{(\alpha)}(z)F(\{\omega\},\{z\}) = \tilde{\alpha}\mathfrak{H}^{(\tilde{\alpha})}(\omega)F(\{\omega\},\{z\}), \tag{43}$$

where $\mathfrak{H}^{(\alpha)}(z)$ and $\mathfrak{H}^{(\tilde{\alpha})}(\omega)$ are order 2 differential operators acting purely on electron and quasihole coordinates, respectively,

$$\mathfrak{H}^{(\alpha)}(z) \equiv [\mathcal{H}^{(\alpha)}_2(z) - \epsilon^{(0)}(r,N)] + \left(\frac{N-k}{\alpha}-1\right)\left[\mathcal{H}_1(z) - \frac{1}{2}NN^{(0)}_\Phi(r,N)\right] - \frac{MN}{k^2}(M-k), \tag{44}$$

$$\mathfrak{H}^{(\tilde{\alpha})}(\omega) \equiv [\mathcal{H}^{(\tilde{\alpha})}_2(\omega) - \epsilon^{(0)}(\tilde{r},M)] + \left(\frac{M-k}{\tilde{\alpha}}-1\right)\left[\mathcal{H}_1(w) - \frac{1}{2}MN^{(0)}_\Phi(\tilde{r},M)\right] - \frac{MN}{k^2}(N-k). \tag{45}$$

Moreover they both belong to the Calogero-Sutherland family of commuting Hamiltonians

$$\mathcal{H}^{(\alpha)}_2(z) = \sum_{i=1}^{N}(z_i\partial_{z_i})^2 + \frac{1}{\alpha}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(z_i\partial_{z_i} - z_j\partial_{z_j}), \tag{46}$$

$$\mathcal{H}_1 = \sum_{i=1}^{N} z_i\partial_{z_i} \tag{47}$$

but for different coupling constants

$$\alpha = -\frac{k+1}{r-1}\qquad \tilde{\alpha} = 1-\alpha = \frac{k+r}{r-1}. \tag{48}$$

For the sake of compactness we introduced the following quantities: (a) the ground-state number of flux $N^{(0)}_\phi(r,N)$

$\equiv r\frac{N-k}{k}$, (b) the Calogero-Sutherland eigenvalue $\epsilon_\lambda^{(0)}(r,N) \equiv \frac{Nr(N-k)[2Nr+k^2(1-2r)+k(N-r+Nr)]}{6k^2(k+1)}$ corresponding to the ground-state partition $\lambda^{(0)}=[k0^{r-1}k0^{r-1}k\cdots 0^{r-1}k]$ and the coupling constant $\alpha=-\frac{k+1}{r-1}$.

The form (43) of the PDE is extremely simple as it does not mix electron and quasihole coordinates. Moreover it extends the proof relating $\mathrm{WA}_{k-1}(k+1,k+r)$ theories to (k,r) admissible Jack polynomials to the generic case of quasihole wave functions. Even more interestingly, this PDE treats electrons and quasihole equally, and is manifestly invariant under the transformation

$$\Psi \leftrightarrow \sigma, \quad \alpha \leftrightarrow 1-\alpha \quad \text{i.e.} \quad r \leftrightarrow \tilde{r} = \frac{2k+1-kr}{k+r}.$$

This electron-quasihole duality at the PDE level is induced by the CFT duality in Eq. (31). However we insist that this mathematical artifact of the underlying CFT is in no way a symmetry of the quantum-Hall state since electrons and quasiholes have very different physical properties.

VI. FULL EXPANSION OF THE k-QUASIHOLE WAVE FUNCTION

We have directly verified that all the known RR wave functions proposed in earlier works satisfy Eq. (43). These differential equations provide a rigorous proof of the validity of the ansatz used in Ref. 15 to compute the four quasihole wave functions for the general k Read-Rezayi states.

As an explicit application of this differential equation, we compute in the following the full expansion of the one-flux added k quasihole wave functions for any Jack state, including the Read-Rezayi sequence. The full expansion of these wave functions were previously unknown and illustrates the electron-quasihole duality. It also provides a proof of the form of the k-quasihole Jack wave function with $k-1$ quasiholes at the north pole proposed in Ref. 24, which was obtained assuming that the electronic part was spanned by (k,r) admissible Jack polynomials.

In the Appendix we provide the expansions of all other k-quasihole wave functions with n quasiholes at the north pole and $(k-n)-1$ quasiholes at the south pole, as obtained from the clustering condition. All these expressions are recovered by an appropriate specialization of the full expansion derived in this section.

A. Structure of the k-quasihole wave function

In the presence of a single extra flux, k elementary quasiholes are present in the quantum-Hall liquid. For $M=k$ quasiholes and N electrons, the CFT correlator in Eq. (38) is single channeled and the wave function $F(\{w\},\{z\})$ as defined in Eq. (42) is simply a polynomial in $\{w,z\}$. The quasihole part of the wave function can be expanded into the eigenbasis of $\mathcal{H}_{CS}^{(\tilde{\alpha})}$, namely, Jack polynomials $J_\mu^{(\tilde{\alpha})}(\{w\})$

$$F(\{w\},\{z\}) = \sum_\mu a_\mu J_\mu^{\tilde{\alpha}}(\{w\}) P_\mu(\{z\}). \tag{49}$$

The sum over μ means the sum over all $\times k$ partitions $\frac{N}{k} \geq \mu_1 \geq \mu_2 \cdots \mu_k \geq 0$. The corresponding Jack polynomials $J_\mu^{\tilde{\alpha}}(\{w\})$ span the entire quasihole space (k bosons in $\frac{N}{k}$ orbitals) and are always well defined as $\tilde{\alpha}=\frac{k+r}{r-1}$ is positive. Plugging this expression in the PDE in Eq. (43), we obtain that $P_\mu(\{z\})$ is an eigenvector of $\mathcal{H}_{CS}^{(\alpha)}$. Neglecting the accidental spectrum degeneracies of $\mathcal{H}_{CS}^{(\alpha)}$, $P_\mu(\{z\})$ is then a Jack polynomial $J_{\lambda(\mu)}^{(\alpha)}$ for some partition $\lambda(\mu)$ depending on μ

$$F(\{w\},\{z\}) = \sum_\mu a_\mu J_\mu^{\tilde{\alpha}}(w) J_{\lambda(\mu)}^{\alpha}(z). \tag{50}$$

To any partition μ corresponds a (k,r) admissible partition $\lambda(\mu)$. The correspondence $\mu \to \lambda$ is determined by plugging the function $J_\lambda^\alpha(z) J_\mu^{\tilde{\alpha}}(w)$ in the PDE in Eq. (43), which leads to the following equations between eigenvalues:

$$\epsilon_\lambda(\alpha) - \epsilon_\lambda^{(0)}(\alpha) + \frac{k+r}{k+1}\left[\epsilon_\mu(\tilde{\alpha}) - \frac{N^2}{k}\right] + \frac{r-1}{k+1}(N-k-1) \times \left(\sum_{i=1}^k \mu_i - N\right) = 0. \tag{51}$$

The inverse mapping $\lambda \to \mu$ is the following: to the (k,r) admissible partition λ corresponds the quasihole partition μ defined as

$$\mu_i = \sum_{\substack{n=1 \\ n=k+1-i \bmod k}}^{N} (1-l_n) \quad \text{for} \quad i=1,\ldots,k, \tag{52}$$

where we introduced for compactness $l_i \equiv \lambda_i - \lambda_i^{(0)}$ for $i=1,\ldots,N$ [$\lambda^{(0)}$ is the densest (k,r) admissible partition]. Note that $l_i \in \{0,1\}$ when there is a single extra flux and this is why this case is quite simple (there is no trace of the non-Abelian braiding yet and the counting of (k,r) admissible partitions is trivial).

Equivalently the direct mapping is

$$(l_i, l_{i+k}, l_{i+2k}, \ldots, l_{i+N-k}) = (\underbrace{1,1,1,\ldots 1}_{N/k-\mu_{k+1-i}}, \underbrace{0,\ldots,0}_{\mu_{k+1-i}}). \tag{53}$$

It is straightforward to check that this mapping ensures the relation in Eq. (51).

B. Global translation invariance

The explicit expression of the coefficients a_μ's in Eq. (50) can be obtained by demanding global translation invariance of the quasihole wave function and using Lassale's formula for Jack polynomials

$$\left(\sum_{i=1}^N \partial_{z_i} + \sum_{l=1}^k \partial_{\omega_l}\right)\left[\sum_\mu a_\mu J_\mu^{(\tilde{\alpha})}(w) J_\lambda^{(\alpha)}(z)\right] = 0. \tag{54}$$

When acting with $L_- = \Sigma_i \partial_{z_i}$ on a Jack polynomial $J_\lambda^{(\alpha)}$ we get

$$L_- J_\lambda^{(\alpha)} = \sum_i A(\lambda,\lambda(i),\alpha) J_{\lambda(i)}^{(\alpha)}, \tag{55}$$

where the partition $\lambda(i)$ is obtained from λ by removing a box in the ith row and the coefficient $A[\lambda,\lambda(i),\alpha]$ is given by Lassale's formula[52]

$$A[\lambda,\lambda(i),\alpha]=\frac{N-i+\lambda_i\alpha}{\alpha}\prod_{j=i+1}^{N}\frac{[\alpha(\lambda_i-\lambda_j-1)+j-i+1][\alpha(\lambda_i-\lambda_j)+j-i-1]}{[\alpha(\lambda_i-\lambda_j)+j-i][\alpha(\lambda_i-\lambda_j-1)+j-i]}. \tag{56}$$

When acting with $L_{-}(\{w\})+L_{-}(\{z\})=\Sigma_l\partial_{w_l}+\Sigma_i\partial_{z_i}$ on

$$F(\{w\},\{z\})=\sum_{\mu}a_\mu J_\mu^{\tilde{\alpha}}(\{w\})J_{\lambda(\mu)}^{\alpha}(\{z\}) \tag{57}$$

we obtain a polynomial of the form

$$\left(\sum_{i=1}^{N}\partial_{z_i}+\sum_{l=1}^{k}\partial_{\omega_l}\right)F[(w),(z)]=\sum_{\mu}\sum_{i}\{a_{\mu(i)}A[\mu,\mu(i),\tilde{\alpha}] + a_\mu A(\lambda^{(i)},\lambda,\alpha)\}J_{\mu(i)}^{\tilde{\alpha}}(w)J_{\lambda(\mu)}^{\alpha}(z), \tag{58}$$

where $\mu(i)$ is the partition obtained from μ and removing a box in the ith row, $\lambda=\lambda(\mu)$ and $\lambda^{(i)}\equiv\lambda[\mu(i)]$ is the partition corresponding to $\mu(i)$ through the mapping in Eq. (52). Explicitly the partition $\lambda^{(i)}$ is obtained from λ by adding a box in the $(N+k+1-i-k\mu_i)$th row

$$\lambda_j^{(i)}=\lambda_j+\delta_{j,N+(k+1-i)-k\mu_i}. \tag{59}$$

Translation invariance boils down to the set of relations

$$a_{\mu(i)}=-a_\mu\frac{A(\lambda^{(i)},\lambda,\alpha)}{A[\mu,\mu(i),\tilde{\alpha}]}. \tag{60}$$

The generic expressions for $A[\mu,\mu(i),\tilde{\alpha}]$ and $A(\lambda^{(i)},\lambda,\alpha)$ are quite complicated. However their ratio turns out to be extremely simple

$$\frac{A[\mu,\mu(i),\tilde{\alpha}]}{A(\lambda^{(i)},\lambda,\alpha)}=\frac{k+1-i}{i}. \tag{61}$$

Up to a global multiplicative constant, we find

$$a_\mu=\prod_{i=1}^{k}\left(-\frac{k+1-i}{i}\right)^{\mu_i}. \tag{62}$$

This is quite remarkably independent of r: these coefficients are the same for the RR states and their $r>2$ generalizations. To summarize we obtained the most general k quasiholes wave function for the (k,r) Jack states to be

$$F(\{w\},\{z\})=\sum_{\mu}a_\mu J_\mu^{\tilde{\alpha}}(\{w\})J_{\lambda(\mu)}^{\alpha}(\{z\}) \quad a_\mu=\prod_{i=1}^{k}\left(-\frac{k+1-i}{i}\right)^{\mu_i} \tag{63}$$

with $\alpha=1-\tilde{\alpha}=-\frac{k+1}{r-1}$.

This expression generalizes the expansions obtained in Ref. 24, which correspond to the specific case of $k-1$ quasiholes located at the north pole (see Appendix).

VII. CONCLUSION

In this paper we have derived a second-order PDE, written in Eq. (41), satisfied by the most general Read-Rezayi and Jack quasihole wave functions. In order to achieve this result, we have generalized the results of Ref. 38, in which the ground state wave functions were considered, to the quasihole excited wave functions. We showed that this is possible because the WA_{k-1} representation modules corresponding to the quasihole and the particle operators share the same degeneracy structure. The W symmetry is completely manifest in these differential equations, thus completing the observations done in Ref. 21, which explicitly depend on the values of the $W_0^{(3)}$ eigenvalues.

The differential equation decomposes as the sum of two Hamiltonians of Calogero-Sutherland-type acting on the electron and quasihole coordinates, unravelling a remarkable electron-quasihole duality at the PDE level. It would be interesting to investigate the relation with the K-matrix duality obtained in Ref. 53.

We used these results to characterize the quasihole wave functions whose explicit expression is in general not known for these theories. We proved the validity of some ansatz used in Refs. 15 and 35 to compute certain quasihole wave functions. Finally we provided the full expansion of the one-flux added quasihole wave functions. On the basis of this PDE, it would be interesting to understand the quasihole wave functions with more than one flux added, and their non-Abelian monodromy properties, in terms of nonpolynomial solutions of Calogero-Sutherland Hamiltonians. This is, at our knowledge, a quite unexplored mathematical problem.

As a last remark, we point out that the question of the vanishing of the Berry connection of the Read-Rezayi states has not been completely proven, even if general conditions for it to hold has been discussed in Ref. 8. In this respect, some strong arguments for the Moore-Read states have been provided by using the Coulomb gas approach.[54] For Halperin's Abelian spin-singlet state, it was shown in Ref. 55 that the Berry connection vanishes by using the Knizhnik-Zamolodchikov equations. It should be interesting then to explore the possibilities to compute the Berry connection for the Read-Rezayi states by using the WA_{k-1} Coulomb gas representation or by using the differential equations we derived in this paper.[56]

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APPENDIX: EXPANSIONS OF SOME SPECIALIZED k-QUASIHOLE WAVE FUNCTIONS

The expansion of the k-quasihole wave function with $k-1$ quasiholes bunched up at the north pole in terms of Jack polynomials has been proposed in Ref. 24. The derivation was based on the specific clustering properties of this wave function and relied on the assumption that only (k,r) Jack polynomials appear in the expansion. Using a similar approach, we derive in this section the expansion of the wave functions with n quasiholes at the north pole and $(k-n)-1$ quasiparticles at the south pole in terms of powers of the remaining quasihole. On the plane, this corresponds to the wave function with n quasiholes at zero and $(k-n)-1$ at infinity. We then compare the expression found by the clustering method with the special cases of the general expressions obtained in the algebraic calculations in this paper.

The physical operation we can perform on a FQH ground state is the addition of one unit of flux, which corresponds, in this case, to the addition of k charge $\frac{1}{k}$ quasiholes, which then further fractionalize and become distinct. We then take n quasiholes to the north pole (origin in the disk geometry), $(k-n)-1$ at the south pole (infinity in the disk geometry) and the k'th non-Abelian quasihole at w. This fractionalized quasihole wave function $\psi(w;z_1,\ldots,z_N)$ is defined by the following clustering property:

$$\psi_n(w;z_1,\ldots,z_N)|_{z_1=\cdots=z_k=w}=0, \tag{A1}$$

which destroys the wave function if k particles reach the fractionalized quasihole w. We impose two further conditions: (a) $\psi_n(w;z_1,\ldots,z_N)|_{z_1=\cdots=z_{k-n+1}=\text{North Pole}}=0$ pins n quasiholes at the North pole And (b) $\psi_n(w;z_1,\ldots,z_N)|_{z_1=\cdots=z_{n+2}=\text{South Pole}}=0$ pins $(k-n)-1$ quasiparticles at the South pole

When zero quasiholes are either at the North or South pole, the two conditions become identical to the electron clustering properties of a (k,r) FQH state. We can most easily express this wave function as a function of the difference between the number of quasiholes at the north and south pole: $\Delta n=n-[(k-n)-1]=2n-k+1$, which we take without any loss of generality to be ≥ 0. The ≤ 0 part of Δn is just a mirror reflection of the $\Delta n\geq 0$. We have

$$\psi_n(w;z_1,\ldots,z_N)=\sum_{i=0}^{N/k}(-\beta_n w)^i J^{\alpha}_{\lambda_i^{(n)}}(\{z\}), \tag{A2}$$

where

$$\beta_n=\frac{k+1-\Delta n}{k+1+\Delta n}=\frac{k-n}{n+1} \tag{A3}$$

and the polynomials $J^{\alpha}_{\lambda_i^{(n)}}(\{z\})$ are Jack polynomials with partition $\lambda_i^{(n)}$ given by, in orbital occupation notation

$$\lambda_0^{(n)}=|a_nb_n0,\ldots,0a_nb_n0,\ldots,0a_nb_n0,\ldots,0a_nb_n\rangle,$$

$$\lambda_1^{(n)}=|a_n+1b_n-1\ \ 0,\ldots,0a_nb_n0,\ldots,a_nb_n0,\ldots,0a_nb_n\rangle,$$

$$\lambda_2^{(n)}=|a_n+1b_n-1\ \ 0,\ldots,0a_n+1b_n-1\ \ 0,\ldots,0a_nb_n0,\ldots,0a_nb_n\rangle,$$

$$\lambda_3^{(n)}=|a_n+1b_n-1\ \ 0,\ldots,0a_n+1b_n-1\ \ 0,\ldots,0a_n+1b_n-1\ \ 0,\ldots,0a_nb_n\rangle,$$

$$\cdots$$

$$\lambda_{N/k}^{(n)}=|a_n+1b_n-1\ \ 0,\ldots,0a_n+1b_n-1\ \ 0,\ldots,0a_n+1b_n-1\ \ 0,\ldots,0a_n+1b_n-1\rangle, \tag{A4}$$

where

$$a_n=\frac{k-1-\Delta n}{2}=k-n-1 \quad b_n=n+1 \tag{A5}$$

and where the number of zeroes between one b_n and the next a_n to the right of it is $r-2$. Of course, the kets above are admissible Jack polynomials at $-(k+1)/(r-1)$; the partitions are obviously (k,r) admissible.

The wave functions obtained above are all special cases of the generic k-quasiholes wave function in Eq. (63) and they can be recovered by putting n quasihole at 0 and $k-n-1$ at infinity

$$\psi_n(w;z_1,\ldots,z_N)=F(w_1=\cdots=w_n=0,w_{n+1}=w,w_{n+2}=\cdots=w_k=\infty;z_1,\ldots,z_N). \tag{A6}$$

This specialization kills all the $J^{\tilde{\alpha}}_{\mu}(\{w\})$, except for the partitions

$$\mu_0^{(n)}=\left[\overbrace{\frac{N}{k}\cdots\frac{N}{k}0}^{k-n-1}\ \overbrace{0\cdots0}^{n}\right],$$

$$\mu_1^{(n)}=\left[\frac{N}{k}\cdots\frac{N}{k}1\ \ 0\cdots0\right],$$

$$\mu_2^{(n)}=\left[\frac{N}{k}\cdots\frac{N}{k}2\ \ 0\cdots0\right]$$

$$\cdots$$

$$\mu_{N/k}^{(n)}=\left[\frac{N}{k}\cdots\frac{N}{k}\frac{N}{k}0\cdots0\right] \tag{A7}$$

for which the quasihole Jack polynomial becomes

$$J^{\tilde{\alpha}}_{\mu_i^{(n)}}(w_1=\cdots=w_n=0;w_{n+1}=w,w_{n+2}=\cdots=w_k=\infty)=w^i. \tag{A8}$$

According to the mapping in Eq. (53), the corresponding electronic partitions $\lambda(\mu_i^{(n)})$ are precisely the partitions in Eq. (A4)

$$\lambda(\mu_i^{(n)})=\lambda_i^{(n)} \quad i=0,\ldots,\frac{N}{k} \tag{A9}$$

and we also recover the value of β_n from the general value of a_μ in Eq. (62)

$$a_{\mu_i^{(n)}}\propto\left(-\frac{k-n}{n+1}\right)^i\Rightarrow\beta_n=\frac{k-n}{n+1}. \tag{A10}$$

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