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Ranking judgments in Arrow's setting

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Abstract In this paper, I investigate the relationship between preference and judgment aggregation, using the notion of ranking judgment introduced in List and Pettit (Synthese 140(1–2):207–235, 2004). Ranking judgments were introduced in order to state the logical connections between the impossibility theorem of aggregating sets of judgments proved in List and Pettit (Economics and Philosophy 18:89–110, 2002) and Arrow's theorem (Arrow, Social choice and individual values, 1963). I present a proof of the theorem concerning ranking judgments as a corollary of Arrow's theorem, extending the translation between preferences and judgments defined in List and Pettit (Synthese 140(1–2):207–235, 2004) to the conditions on the aggregation procedure.

Keywords Arrow's theorem · Conodorcet's paradox · Discursive dilemma · Aggregation of ranking judgments · First order logic

1 Introduction

Doctrinal paradoxes show how the task of producing a rational collective judgment out of individual rational judgments, see Kornhauser and Sager (1986), may lead to inconsistency. This type of paradoxical outcomes was generalized in a theorem concerning the impossibility of aggregating sets of logically connected judgments in List and Pettit (2002). Since then, the relationship between Arrow's theorem and this new result has become an interesting theme in the debate on the possibility of collective choices.

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The result in List and Pettit (2002) states that it is not possible to define a procedure that aggregates rational sets of judgments to produce a rational collective outcome and satisfies some reasonable democratic features.

As this theorem resembles Arrow's result, it is interesting to see whether it is simply a version of Arrow's theorem or rather a different result, dealing with a different kind of aggregation dynamic.

In Dietrich and List (2007), a proof of Arrow's theorem is proposed within the framework of judgments; here, we compare the two impossibility results from the opposite direction: namely, we investigate whether we can prove properties of judgment aggregation using Arrow's theorem.

We will consider a particular type of judgment aggregation problem in order to show that it can be described precisely by the properties involved in Arrow's theorem.

List and Pettit (2004) use the concept of *ranking judgments* to embed the framework of preferences into the framework of judgments, and investigate the logical relationship between Arrow's theorem and the impossibility theorem on judgments.

They prove an impossibility result for ranking judgments, which demonstrates the logical connection with Arrow's reasoning.

The structure of the paper is as follows. In Sect. 2, I recall the definitions involved in Arrow's theorem. Section 3 contains an example of a doctrinal paradox, and the impossibility theorem on aggregating judgments. In the fourth section, I discuss the proposal of List and Pettit (2004) and the notion of ranking judgments. Section 5 contains the proof of the impossibility theorem for aggregating sets of ranking judgments as a corollary of Arrow's theorem. Finally, I point to some applications of the connection between logic and preference theory that are suggested by the translation via ranking judgments.

2 Arrow's theorem

We briefly recall the notions involved in Arrow's theorem, see Arrow (1963). We use the notation $\leq$ and $<$ for weak and strong preference relations, respectively.

Assume there is a finite set N of *individuals*, and a finite set X of *alternatives*. A *preference order* of an individual $i \in N$ over X, is a relation $\leq_i$ on X that satisfies the following properties: (1) *reflexivity*: for any x in $X x \leq_i x$; (2) *transitivity*: for any x, y and z in X, if $x \leq_i y$ and $y \leq_i z$, then $x \leq_i z$; (3) *connectedness*: for any x, for any y, $x \leq_i y$ or $y \leq_i x$.

A *profile of preference orders* is a function form N to the set of all strict preference orders on X; we denote it $\{\leq_1, \ldots, \leq_n\}$ or $\{\leq_i\}_{i \in N}$.

As in Arrow (1963), we can define a *strong* preference order, by considering equivalence classes of indifferent alternatives, namely those alternatives x and y such that $x \leq y$ and $y \leq x$ hold. We can then define the relation "strictly preferred" on equivalence classes; we denote the strong preference $<_i$.

A strict preference order can also be defined directly by the properties of: *irreflexivity* ($x < x$ does not hold for any x), *transitivity*, and *completeness* (for all x for all y, if $x \neq y$, then $x < y$ or $y < x$).

Arrow's theorem is concerned with the following problem: is it possible to aggregate individual preference orders into a collective preference order that keeps track of the individuals in a suitable way?

The process of aggregation is formally represented by a function F (social welfare function) that takes a profile of preference orders as input and produces as output a single preference order, considered the collective one, satisfying some conditions intended to describe the concept of a democratic choice.

These conditions are formally expressed by the following properties.[1]

We present Arrow's theorem for strict preferences. Let $<_i$ denote the individual preference orders in a profile in $\{<_i\}_{i \in N}$ and $<_c$ the collective preference order resulting from the application of F:

- *Universal domain* (U): the domain of F is the set of all preference profiles $\{<_i\}_{i \in N}$ over alternatives in X.
- *Weak Pareto principle* (P): Given a profile in $\{<_i\}_{i \in N}$, if for every i in N, $x <_i y$, then $x <_c y$.
- *Independence of irrelevant alternatives* (I): If $\{<_i\}_{i \in N}$ and $\left\{<'_i\right\}_{i \in N}$ are two profiles of preference orders, and x, y are two alternatives in X such that, for all individuals i in N, $x <_i y$ if and only if $x <'_i y$, then $x <_c y$ if and only if $x <'_c y$.
- *Non-Dictatorship* (D): There does not exist an individual i in N such that, for all profiles of preference orders, if $x <_i y$, then $x <_c y$.

If we assume that the set N contains at least two individuals and if the set X contains at least three alternatives, then Arrow's theorem (Arrow 1963) states that there is no function that satisfies the above properties.

Theorem 2.1 *There is no function F satisfying (U), (P), (I) and (D), that generates a collective preference order.*

3 Discursive dilemma

The theorem proved in List and Pettit (2002) concerns the impossibility of aggregating individual sets of judgments in a coherent and democratic way.

As Arrow's theorem can be considered as a generalization of Condorcet's paradox to any aggregation procedure satisfying some democratic conditions, the theorem about judgments can be seen as a generalization of situations known as *doctrinal paradox* or *discursive dilemma*. This type of voting paradoxes have actually emerged in the deliberative practice of collegial courts,[2] and they constitute a challenge for those theories of democracy that stress the deliberative features of making collective decisions.

1 For a discussion of these properties, see Arrow (1963). See also List and Pettit (2004) for a comparison of these properties and those involved in judgments framework.

2 See Kornhauser (1992).

A discursive dilemma can be stated as follows. Suppose there are three individuals labelled 1, 2 and 3, and there are propositions α, β and $\alpha \wedge \beta$. Suppose that the individuals judge the propositions involved in the following way.

	α	β	$\alpha \wedge \beta$
1:	true	true	true
2:	false	true	false
3:	true	false	false

Each individual holds a coherent set of judgments, but if we aggregate the three sets of judgments by majority rule, we would obtain that α is collectively judged to be true, β is collectively true, but $\alpha \wedge \beta$ is judged to be false, which contradicts the meaning of conjunction.

Generalizing this type of configurations leads to the formulation of an impossibility theorem on the aggregation of sets of judgments.

For a precise discussion, see List and Pettit (2002, 2004); here we simply state the result.

Suppose we have a set N of individuals and a set Ξ^J of propositions that contains at least two atomic propositions p and q and at least the conjunction $p \wedge q$ and its negation $\neg(p \wedge q)$.

Given an individual i in N, an individual set of judgments Φ_i^J is assumed to satisfy the following three conditions: (1) *completeness*: for all propositions ϕ in Ξ^J, either ϕ or $\neg\phi$ is in Φ_i^J; (2) *consistency*: there is no proposition ϕ in Ξ^J such that ϕ and $\neg\phi$ are in Φ_i^J; (3) *deductive closure*: if Φ_i^J entails a proposition ψ in Ξ^J, then ψ is also contained in Φ_i^J.

A *profile of sets of judgments* is a function form N to the set of all sets of judgements over Ξ^J, containing one set of individual judgments for each individual in N. We denote it $\{\Phi_1^J, \ldots, \Phi_n^J\}$, or also $\{\Phi_i^J\}_{i \in N}$.

The conditions on the judgment aggregation function F^J, which takes as input profiles of individual judgments and gives as output a single set of judgments representing the collective set, are as follows:

Universal domain $\left(U^J\right)$: The domain of F^J is the set of all profiles of complete, consistent and deductively closed individual sets of judgments.
Anonymity (A^J): For any $\{\Phi_i^J\}_{i \in N}$ in the domain of F^J and any permutation $\sigma : N \rightarrow N$, $F(\{\Phi_i^J\}_{i \in N}) = F(\{\Phi_i^J\}_{\sigma(i) \in N})$.
Systematicity (S^J): There exists a function $g : \{0, 1\}^n \rightarrow \{0, 1\}$ such that, for any $\{\Phi_i^J\}_{i \in N}$ in the domain of F^J, $F^J(\{\Phi_i^J\}_{i \in N}) = \{\phi \in \Xi^J : g(\delta_1(\phi), \ldots, \delta_n(\phi)) = 1\}$ where, for each $i \in N$ and each $\phi \in \Xi^J$, $\delta_i(\phi) = 1$ if $\phi \in \Phi_i^J$ and $\delta_i(\phi) = 0$ if $\phi \notin \Phi_i^J$.[3]

We can now state the impossibility result on the aggregation of judgments (List and Pettit 2002):

[3] The systematicity condition intuitively means that the aggregation procedure depends only on the pattern of *yes* and *no* expressed by individuals on a given judgment.

Theorem 3.1 *There exists no judgment aggregation function F_J generating complete consistent and deductively closed collective sets of judgments, which satisfies* (U^J), (A^J) *and* (S^J).

4 The impossibility theorem on the aggregation of ranking judgments

An interesting issue that emerged in the debate on deliberative democracy after the work of List and Pettit is the problem of stating the relationship between the two impossibility results we recalled. The theory of deliberative democracy may also be considered as a way to deal with the inconsistencies underlined by Arrow's impossibility result, introducing deliberative aspects that be considered more apt for discussion and revision, but also agreement.[4]

Therefore, the question whether there is a difference between the aggregation of preferences and the aggregation of judgments is itself relevant to deliberative democracy, since it is important to find out whether we are considering the *same* kind of aggregation problem or not.[5]

In order to compare Arrow's theorem with Theorem 3.1, List and Pettit[6] introduce the concept of *ranking judgment* to represent preference orders by means of sets of sentences in first order logic.

Consider the following example of Condorcet's paradox, with three alternatives, a, b and c, and three voters 1, 2 and 3:

$$\begin{aligned} &1\colon a < b < c\\ &2\colon b < c < a\\ &3\colon c < a < b \end{aligned}$$

Majority rule leads us to accept $a < b$ and $b < c$, so by transitivity we conclude $a < c$ too; however, by majority, we also have $c < a$, so by transitivity again, from $a < c$ and $c < a$, we get $a < a$, which contradicts the irreflexivity of strong preference.

In order to define ranking judgments from individual orders, we need to represent the relation $<_i$ using a logical relational constant, say P; we can translate a profile of preference order into a set of judgments in the following way.

Given the set of alternatives $X = \{a, b, c\}$, we can represent these alternatives using *constants* of the language of first order logic, and denote them by $\bar{a}, \bar{b}, \bar{c}$, respectively. We can translate the preference orders of individuals 1, 2, and 3 into the following three sets of judgments:

$$\begin{aligned} &1\colon a < b < c \qquad \{\bar{a}P\bar{b}, \bar{b}P\bar{c}, \bar{a}P\bar{c}\} = \Phi_1\\ &2\colon b < c < a \qquad \{\bar{b}P\bar{c}, \bar{c}P\bar{a}, \bar{b}P\bar{a}\} = \Phi_2\\ &3\colon c < a < b \qquad \{\bar{c}P\bar{a}, \bar{a}P\bar{b}, \bar{c}P\bar{b}\} = \Phi_3 \end{aligned}$$

4 See Dryzek and List (2003).

5 For a discussion of the relevance of the discursive dilemma in deliberative democracy, see List and Pettit (2002) and also Ottonelli (2005, 2009).

6 List and Pettit (2004).

Following (List and Pettit 2004), we call this type of judgments *ranking judgments*.

It is important to remark that, if we aggregate such sets of judgments by majority voting, as we did before with preferences, we obtain no contradiction: by majority we get $\bar{a}P\bar{b}$, $\bar{b}P\bar{c}$ and $\bar{c}P\bar{a}$, which from a logical point of view are just atomic sentences; in order to show that such set is coherent, we only need to define a model in which they are true.

Therefore, to represent Condorcet's paradox, we need to assume that the relational constant P satisfies the properties describing strong preference order.[7]

The properties required can be stated as follows: *irreflexivity*: (Irr): $\forall x \neg xPx$, *transitivity*: (Tra): $\forall x \forall y \forall z (xPy \wedge yPz \rightarrow xPz)$, *completeness*: (Com): $\forall x \forall y (xPy \vee yPx)$. Assuming these properties we get the following sets:

$$
\begin{array}{lcl}
1: \{\bar{a}P\bar{b}, \bar{b}P\bar{c}, \bar{a}P\bar{c}\} & \cup & \{(\text{Irr}), (\text{Tra}), (\text{Com})\} \\
2: \{\bar{b}P\bar{c}, \bar{c}P\bar{a}, \bar{b}P\bar{a}\} & \cup & \{(\text{Irr}), (\text{Tra}), (\text{Com})\} \\
3: \{\bar{c}P\bar{a}, \bar{a}P\bar{b}, \bar{c}P\bar{b}\} & \cup & \{(\text{Irr}), (\text{Tra}), (\text{Com})\}
\end{array}
$$

In this way it is possible to represent an instance of Condorcet's paradox in the form of a doctrinal paradox, obtaining a contradiction between *judgments*, for example, between (Irr) and $\bar{a}P\bar{a}$.

By generalizing the above argument on the relationship between Condorcet's paradox and discursive dilemmas, List and Pettit (2004) present an impossibility theorem on the aggregation of ranking judgments.

Let N be a set of individuals, with at least two members, and let X be a set of alternatives with at least three elements. Every individual $i \in N$ holds a $strong$ preference order $<_i$ over X, where $<_i$ satisfies $irreflexivity$, $transitivity$ and $completeness$. Moreover the collective order $<_c$ is also assumed to be a strong preference order.

We define a theory for ranking judgements, in order to state precisely the relationship with preference orders.

We define the alphabet for ranking judgments, as a subset of the alphabet of first order logic. It contains a finite number of individual constants $Const^{rj} = \{a_1, \ldots, a_n\}$ and variables $Var^{rj} = \{x_1, \ldots, x_n\}$ and a single predicative constant P.

The language of ranking judgments L^{rj} is defined as a (finite) fragment of the language of first order logic containing atomic sentences of the form $a_i P a_j$, where a_i and a_j belong to $Const^{rj}$, and it is closed by the usual definition of first order well formed sentences.[8]

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7 This is required if we want to represent orders by means of first order logic. Moreover it also seems to be assumed in List and Pettit (2004, pp. 9–10), when they translate an instance of Condorcet's paradox using sets of sentences of first order logic.

8 The set of ranking judgments was defined in List and Pettit (2004) as follows: $\Xi^{rj} = \{xPy : x, y \in X, x \neq y\}$. We remark that in that case we are simply stating the fact that x and y are different variables, so there is some ambiguity on the use of variables: it doesn't mean that we can exclude judgments of the form aPa by simply stating that, in xPy, we assume two different names for options. In the definition we propose, the fact that individual sets of judgments contain no sentences of the form aPa will be a consequence of the logical conditions on those sets. Moreover, in List and Pettit (2004), individual judgments are simply of the form aPb, while here we assume also that the set of issues of the aggregation problem represented by L^{rj} may contain complex proposition. In order to state the theorem on ranking judgments exactly as it was presented in List and Pettit (2004), we could define the language of ranking judgments as follows

The relationship between preference orders and ranking judgments is stated in the following definition.

Definition 4.1 Let $(X, <)$ be a preference order, where $<$ may be $<_i$ or the collective $<_c$, and $< \subset X \times X$ is irreflexive, transitive and complete.

1. We define an injective function $t_c : X \longrightarrow Const_{rj}$ assigning a name to each option in X;
2. We define a function $t_R : \{<\} \longrightarrow \{P\}$ associating to each preference relation $<$ the relational constant P;
3. We assume that $(x, y) \in <_i$ iff $t_c(x)\ P\ t_c(y) \in \Phi_i$.
4. Given $<$, we define a set of judgments

$$t(<) = \{t_c(a)\ P\ t_c(b) : (a, b) \in <\}.$$

An individual i accepts a judgment xPy if and only if i ranks x below y in his preference order.

In this framework, it is possible to reinterpret the conditions of irreflexivity, completeness and transitivity of ranking judgments as conditions on sets of judgments, as they were stated in Theorem 3.1.

More precisely, we assume that the sets of ranking judgments are rational in the following sense.[9] We assume three axioms: (Irr), (Tra), (Com) and we can use as an inference system any reasoning method for first order logic.

Let Φ^{rj} be an individual set of ranking judgments or the collective set. Assume the following properties:

– *Consistency*: there is no $\phi \in L^{rj}$ such that both $\phi \in \Phi^{rj}$ and $\neg\phi \in \Phi^{rj}$,
– *Completeness*: for every $\phi \in L^{rj}$, either $\phi \in \Phi^{rj}$ or $\neg\phi \in \Phi^{rj}$,
– *Deductive closure*: if $\phi \in L^{rj}$ and $\Phi^{rj} \cup \{(\mathrm{Irr}), (\mathrm{Tra}), (\mathrm{Com})\} \models \phi$, then $\phi \in \Phi^{rj}$.

So we can define Φ_i^{rj} and Φ^{rj} as the deductive closures of the sets $t(<_i)$ and $t(<_c)$ (with respect to the language L^{rj}).

Given a set of ranking judgments Φ^{rj}, we can define a set equipped with a preference relation as a *model* of the set Φ^{rj} :[10] we can associate an order structure $(X,<)$, interpreting the constants of Φ^{rj} as elements of X and interpreting P as $<$.

Footnote 8 continued
$\Xi^{rj} = \{\alpha : \alpha \in L^{rj}$ and α is atomic$\}$ and restrict deductive closure to propositions of that form. However it seems interesting to consider also propositions stating properties of preference relations as a matter of deliberative discussion; in this way the judgments model can be considered a genuine generalization of the descriptive features of the preference model.

9 In List and Pettit (2004), no particular deductive system is specified; here we can assume it is first order logic. Actually we need less, namely we need only to apply instances of transitivity, so it would be enough to assume rules for the elimination of universal quantifier and *modus ponens*.

10 We may here consider finite structures, since the set X of alternatives is a finite set. Therefore, we can describe models of first order sentences up to isomorphism. Actually, we would need to add the first order sentence stating the cardinality of the structure as an axiom.

So we can define a function m from sets Φ^{rj} to their model $(X, <)$.[11]

As in Arrow's theorem, a profile of strong preference orders $\{<_1, \ldots, <_n\}$ is a function from N to the set of strict orders on X, so we can obtain a *profile of sets of ranking judgments* by taking the corresponding sets of judgments, say $\{\Phi_1^{rj}, \ldots, \Phi_n^{rj}\}$.

A *ranking judgments aggregation function* is a function f^{rj} that takes profiles of ranking judgments as input and produces as output a single set of ranking judgments.

We can restate the conditions on f^{rj} in List and Pettit (2004) as follows:[12]

- *Universal domain* (u^{rj}): the domain of f^{rj} is the set of all possible profiles of sets of ranking judgments $\{\Phi_i^{rj}\}_{i \in N}$.
- *Anonymity* (a^{rj}): For any $\{\Phi_i^{rj}\}_{i \in N}$ in the domain of f^{rj}, and any permutation $\sigma : N \longrightarrow N$, $f^{rj}(\{\Phi_i^{rj}\}_{i \in N}) = f^{rj}(\{\Phi_{\sigma(i)}^{rj}\}_{i \in N})$.
- *Independence of non-welfare characteristics* (inw^{rj}): Let $\{\Phi_i^{rj}\}_{i \in N}$ and $\{\Phi_i'^{rj}\}_{i \in N}$ be two profiles of ranking judgments in the domain of f^{rj}, and let x_1 and y_1 and x_2, y_2 be two pairs of alternatives in X such that, for all individuals $i \in N$, $x_1 P y_1 \in \Phi_i^{rj}$ if and only if $x_2 P y_2 \in \Phi_i'^{rj}$. Then $x_1 P y_1 \in f^{rj}(\{\Phi_i^{rj}\}_{i \in N})$ if and only if $x_2 P y_2 \in f^{rj}(\{\Phi_i'^{rj}\}_{i \in N})$.[13]

We can now present the theorem on the impossibility of aggregating ranking judgments.

Theorem 4.2 *There exists no ranking judgments aggregating function* f^{rj} *that satisfies* (u^{rj}), (a^{rj}) *and* (inw^{rj}).

5 Ranking judgments in Arrow's setting

In this section we present a proof of Theorem 4.2 as a corollary of Arrow's theorem. The proof proposed here shows how the properties we assume to prove the theorem on ranking judgments are related to the properties in Arrow's result.

Before comparing the conditions at issue, we need to find a way to compare two apparently different frameworks, namely the strict orders framework and the ranking judgments framework.

In order to prove Theorem 4.2 as a corollary of Arrow's theorem, we will show that if the function f^{rj} described in Theorem 4.2 exists, then we can define a function

[11] Actually, m gives a representative of the equivalence class of models of Φ^{rj} modulo isomorphism of order structures. The function is well defined since every finite structure is first order definable up to isomorphism.

[12] We remark that in List and Pettit (2004), the function f^{rj} takes a strong preference order as input and gives as output a strong preference order. Anyway, the proof of the theorem uses properties of sets of judgments, in order to show how the mechanism of Theorem 3.1 can be applied to preference aggregation. Therefore it seems more direct to define the function f^{rj} directly on profiles of ranking judgments.

[13] Here we didn't require that $x_i \neq y_i$, since it is enough to assume the irreflexivity axiom and to assume that each set of ranking judgments is consistent, complete and deductively closed in order to obtain that aPa doesn't belong to each set. Moreover, we do not need to assume that yPx is the negation of xPy, since if they both belong to a set of ranking judgments, it entails a contradiction.

$h(f^{rj})$ that satisfies the conditions of Arrow's theorem, and so Arrow's social welfare function would exist.

Let (inw), (a) and (u) be the translations of the conditions on f^{rj} into the language of preference orders.

Consider a function f that takes a profile of strong preference orders and returns a collective strong preference order, which satisfies (inw), (a), (u).

The relationship between preference aggregation functions and ranking judgments aggregation functions can be stated as follows.

We can define a function f by means of f^{rj} and translations t and m defined above.

Given a function

$$f : (<_1, \ldots, <_n) \longmapsto <$$

we can define

$$f^{rj} : (\Phi_1^{rj}, \ldots, \Phi_n^{rj}) \longmapsto \Phi$$

as the composition:

$$\left(\Phi_1^{rj}, \ldots, \Phi_n^{rj}\right) \overset{m\times\ldots\times m}{\longmapsto} (<_1, \ldots, <_n) \overset{f}{\longmapsto} < \overset{t}{\longmapsto} \Phi^{rj}.$$

Moreover, given f^{rj}, we can define a function f in an analogous way. We can prove the following proposition.

Proposition 5.1 *A function f satisfies (inw) (respectively, (a), (u)) if and only if f^{rj} satisfies (inw^{rj}) (respectively, (a^{rj}), (u^{rj})). Moreover, if (I^{rj}), (D^{rj}) and (P^{rj}) are the translations of (I), (D), and (P) into the language of ranking judgments, then a function f satisfies $(I)((D), (P))$ if and only if f^{rj} satisfies (I^{rj}) $((D^{rj}), (P^{rj})$ respectively).*

Proof Consider for example (inw). We prove that if f satisfies (inw), then f^{rj} satisfies (inw^{rj}).

Assume that $x_1 P y_1 \in \Phi_i^{rj}$ if and only if $x_2 P y_2 \in \Phi_i^{\prime rj}$. Then by means of the translation m, we have $x_1 <_i y_1$ if and only if $x_2 <_i' y_2$. Since f satisfies (inw), we have $(x_1, y_1) \in f(\{<_i\}_i)$ if and only if $(x_2, y_2) \in f(\{<_i'\}_i)$. Applying the translation t, we obtain $x_1 P y_1 \in f^{rj}(\{\Phi_i^{rj}\})$ if and only if $x_2 P y_2 \in f^{rj}(\{\Phi_i^{\prime rj}\})$. □

As recalled in List and Pettit (2004), we have:

1. if f satisfies (inw), then f satisfies (I).
2. if f satisfies (a), then it satisfies (D).[14]

We prove that f^{rj} satisfies (P^{rj}), which is the translation of (P) into the language of ranking judgments:

[14] See also Pauly and van Hees (2006).

Weak Pareto principle (P^{rj}): if for all individuals $i \in N$, $xPy \in \Phi_i^{rj}$, then xPy belongs to the collective set of ranking judgments $f^{rj}\left(\{\Phi_i\}_{i\in N}\right)$.

We use the following property, proved in List and Pettit (2002, 2004), which holds for any function satisfying (inw^{rj}) and (a^{rj}).[15]

Proposition 5.2 *Let N_ϕ denote the set of individuals accepting a judgment $\phi \in L^{rj}$, $N_\phi = \{i : \phi \in \Phi_i^{rj}\}$ and $|N_\phi|$ the cardinality of N_ϕ. If $|N_\phi| = |N_\psi|$, then $\phi \in f^{rj}(\{\Phi_i^{rj}\}_{i\in N})$ if and only if $\psi \in f^{rj}(\{\Phi_i^{rj}\}_{i\in N})$.*

We can now prove the following lemma.

Lemma 5.3 *If f^{rj} satisfies (inw^{rj}) and (a^{rj}), then f^{rj} satisfies (P^{rj}).*

Proof Assume xPy belongs to each set of judgments Φ_i^{rj}. Then $|N_{xPy}| = |N|$, i.e., everyone holds xPy. Since each set is deductively closed, we have that (Tra) belongs to each set of judgments Φ_i^{rj} (since (Tra) is an axiom). Thus, we have

$$|N_{\forall x\forall y\forall z(xPy\wedge yPz\rightarrow xPz)}| = |N|$$

Therefore, we obtain

$$|N_{xPy}| = |N_{\forall x\forall y\forall z(xPy\wedge yPz\rightarrow xPz)}|$$

Using Proposition 5.2, we infer that xPy belongs to the collective set of judgments Φ^{rj} *if and only if* $\forall x\forall y\forall z(xPy \wedge yPz \rightarrow xPz)$ belongs to the collective set of judgments Φ^{rj}.

Since the collective set Φ^{rj} is deductively closed, $(Tra) \in \Phi^{rj}$. Therefore xPy is in Φ^{rj}. Thus (P^{rj}) holds.[16] □

We can now prove Theorem 4.2 as a corollary of Arrow's theorem.

Theorem 5.4 *There exists no ranking judgments aggregating function f^{rj} that satisfies (u^{rj}), (a^{rj}) and (inw^{rj}).*

Proof We show that if such an f^{rj} exists, then we could build an aggregation function $h(f^{rj})$ contradicting Arrow's theorem.

Suppose by contradiction that f^{rj} exists. So f^{rj} is a function that takes profiles of sets of ranking judgments and produces a collective set of judgments. Moreover, f^{rj} satisfies (u^{rj}), (a^{rj}) and (inw^{rj}). Define $h(f^{rj}) : (<_1, \ldots, < n) \longmapsto <$ as the composition $m \circ f^{rj} \circ t$. Then by Proposition 5.1, $h(f^{rj})$ satisfies (inw), (a) and (u).

Since $h(f^{rj})$ is a function form preference profiles to preference orders, we have that:

15 Actually, in List and Pettit (2002) the property we recalled is proved using (A^J) and (S^J), and it holds for sets of judgments of propositional logic. However, it is easy to adapt the proof in the case of (inw^{rj}) and the sets of judgements we are considering.

16 We can adapt this proof to the case in which individuals express their judgments only on propositions of the form xPy, in order to match exactly the notion of ranking judgments in List and Pettit (2004). It is enough to consider their deductive closure and prove the proposition on those sets.

1. If $h(f^{rj})$ satisfies (inw), then $h(f^{rj})$ satisfies (I)
2. If $h(f^{rj})$ satisfies (a), then it satisfies (D).

By Lemma 5.3, we also have that if f^{rj} satisfies (inw^{rj}) and (a^{rj}), then f^{rj} satisfies (P^{rj}).

Then we have that $h(f^{rj})$ satisfies (P).

So we have a function $h(f_{rj})$ from preference profiles to preference orders satisfying the conditions (I), (D), (U) and (P), which contradicts Arrow's theorem. □

So we can consider the theorem on ranking judgments as a particular case of Arrow's theorem: ranking judgments describe in a logical framework the aggregation procedure for strong preference relations.

The aggregation inconsistencies for ranking judgments then seem to be the same kind of problems shown by Condorcet's paradox and Arrow's theorem, with the main difference being the logical interpretation. In order to state the relationship between Theorem 4.2 and Arrow's theorem, it is required to use the notion of ranking judgment to define the (first order) theory of ordered sets of preferences.[17]

This approach points to two kinds of interesting applications, which may be investigated in future work. On the one hand, we can apply preference theory results to judgment aggregation: for example, by considering models of sets of ranking judgments satisfying the well known sufficient conditions avoiding cycles (such as single peakedness or value restriction;[18]) since in the language L^{rj} is possible to express for example properties like "the option *a* is not ranked first", the logical interpretation of these conditions would allow us to define a reasoning method to model situations in which the propositions stating the properties of the preference relations are matter of deliberation among voters.

Moreover, it is also possible to consider which kind of relations are likely to lead to inconsistency, when we try to aggregate judgments over them: we considered the language of ranking judgments as a fragment of first order logic, so we may ask whether the impossibility result holds for wider fragments, namely for relations with different constraints.

This would constitute a step towards a sort of generalization of the impossibility theorem of aggregating judgments from propositional logic to predicate logic.

On the other hand, by means of the translation in List and Pettit (2004) we can represent a Condorcet's paradox using the language of ranking judgments; therefore, we can investigate preference aggregation dynamics by means of logical notions: for example, we can apply the procedure of *merging* sentences, which has been described in Pigozzi (2006) for judgment aggregation, to preference aggregation.

[17] It is also possible to prove Theorem 4.2 as a corollary of Arrow's theorem by working directly on ordered sets and show that (INW) and (A) entail (P) and (I). We can adapt the proof of Lemma 5.3 for preference orders completing each preference order with a top ⊤; considering a biconditional of the form $(x, y) \in <_c$ iff $(z, \top) \in <_c$, we can apply Proposition 5.2. However, I preferred to state the connection between Arrow's theorem and ranking judgements theorem as above for the following reason. This approach shows that in the framework of ranking judgements it is required that sentences stating the constraints on preference relations are common knowledge between individuals. I defined the set of ranking judgements as if there were unanimity on transitivity, completeness and irreflexivity, but it is possible to consider partial agreement on rationality constraints.

[18] See List and Elsholtz (2005), for a compact presentation of the conditions avoiding cycles.

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