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# Genesis of indifference thresholds in discrete time optimal control problems: with applications to the lake pollution problem

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This article investigates the indifference-attractor bifurcation of infinite horizon discrete-time optimal control problems with a single state variable. We show that these bifurcations are linked to a heteroclinic bifurcation scenario of the state-costate equations, and we analyse the consequences for the optimal solutions. In particular, we can characterise the bifurcation value at which indifference thresholds appear by a geometric condition, and we find that for certain parameter values, there are countably infinitely many indifference points. We apply our results to a modified version of the lake pollution management problem.

*JEL classification:* C61, Q57

*Keywords:* Indifference points, lake problem, Discrete time dynamic optimisation, Bifurcation theory

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Dechert and Nishimura (1983) investigated one-sector optimal growth models for which the technology is not convex. They showed that the time sequence of capital stocks is necessarily monotonic. Moreover, they showed that depending on the discount rate three situations can occur: the capital stock converges for all initial values to some positive steady state value, or for all initial values it converges to zero, or finally it depends on the initial state whether the capital stock converges to a positive steady state or to zero. In the last situation, there can be a initial state for which there are two optimal solutions, one tending to the positive steady state, the other tending to zero.

Subsequently, indeterminate states of this type have been called ‘Skiba’, ‘Dechert-Nishimura-Skiba’ (DNS) or ‘Dechert-Nishimura-Sethi-Skiba’ (DNSS) states (see Grass et al., 2008), recognising the contributions of Skiba (1978) and Sethi (1977). We prefer the designation *indifference state* for a state from which several different optimal solutions originate, possibly converging to the same long-run steady state or long-run dynamics, and the designation *indifference threshold* for an indifference state for which the originating optimal solutions converge to different long-run steady states or long-run dynamics.

In the present article we study the genesis of indifference thresholds for a class of single-state discrete time dynamic optimisation problems, as a system parameter changes. The class under consideration contains a wide range of economic models, like the optimal growth models studied by Dechert and Nishimura, but also the discrete time version of the lake pollution models introduced by Mäler et al. (2003). We consider state-costate — or *phase* — orbits that are associated to optimal state orbits, making use of the fact that these have to be on the stable manifolds of saddle fixed points of the phase system. We find, as in Wagener (2003), that the genesis of indifference thresholds is linked to the occurrence of heteroclinic orbits in the phase space, that is, orbits that are forward asymptotic to one saddle fixed point and backward asymptotic to another. In particular, we show that if the phase system goes through a so-called heteroclinic bifurcation scenario, an indifference threshold and a locally optimal steady state are generated in an indifference-attractor bifurcation.

An essential feature of discrete time planar dynamical systems is that the existence of a heteroclinic orbit, that is an orbit which is forward asymptotic to one saddle fixed point and backward asymptotic to another, does not force the associated invariant manifolds to coincide. On the contrary, generically they will form a so-called ‘heteroclinic tangle’. This geometric fact has consequences for the structure of the totality of optimal solutions: we show that if the system is at an indifference-attractor bifurcation, then generically there are an infinity of indifference states. We illustrate our findings by computing the indifference threshold and some of the indifference points in a version of the lake pollution problem that is slightly modified in order to make the heteroclinic tangle visible in the simulations.

Methodologically, we contribute to the geometrical analysis of phase systems deriving from dynamic optimisation problems. In particular, we make extensive use of differential forms and geometric integration. Contrary to the continuous time setting, phase space methods are not particularly popular in the discrete time setting. There are several probable reasons for this unpopularity: the omnipresence of the Bellman equation, which is well-understood, easy to generalise to stochastic problems, and which has a elegant theory of existence and uniqueness of solutions.

Moreover, some powerful instruments of the continuous time theory are not readily available: for instance, in a continuous time problem with a one-dimensional state space, knowledge of the isoclines allows to reconstruct the geometry of the phase trajectories to a great extent. In the discrete time setting, there are backward and forward isoclines, and their knowledge does not allow to reconstruct orbits of the phase system as easily. In the continuous time setting we can evaluate the value function in terms of the initial state and costate values of an optimal orbit; this is an immediate corollary to the Hamilton-Jacobi equation. In the discrete time setting, there is no such direct way to find the value function, though in proposition B.8 we have obtained a partial replacement.

We expect that similar results to those obtained here hold in the general case of  $n$ -dimensional phase spaces; indeed, the only place where our assumption of one-dimensionality of the state space is at the moment essential, is in the proof of proposition 4.3. To obtain the results in the general case, this has to be replaced by a study of the relative position of the stable and unstable manifolds of a saddle fixed point with respect to the vertical  $n$ -dimensional subspace through the fixed point.

The study of indifference thresholds in dynamics optimisation problems, initiated in the late 1970's, took off only comparatively recently. Especially in the context of environmental economics, where natural systems often feature nonconvexities, indifference thresholds have been studied by several authors; see for instance Tahvonen (1995); Brock and Starrett (2003); Mäler et al. (2003); other studies include Grüne and Semmler (2004); Steindl and Feichtinger (2004); Dawid and Deissenberg (2005). The genesis of indifference thresholds and its link to a heteroclinic bifurcation in the phase space was noticed in Wagener (2003). Subsequently work on the bifurcations of optimal vector fields has been done by Steindl and Feichtinger (2004); Caulkins et al. (2007); Kiseleva and Wagener (2010). Except for Brock and Dechert (2008) there seems comparatively little work to have been done on indifference thresholds in the deterministic discrete time setting; see however Dechert and O'Donnell (2006) for the stochastic case.

## 1 Heteroclinic orbits for one-dimensional state spaces

This section introduces the class of optimisation problems to be studied; the class is characterised in terms of the phase map, which is the discrete time analogue of the state-costate vector field of continuous time problems. In particular, attention is restricted to the situation that the optimisation problem, and therefore also the phase map, is defined on a one-dimensional state space, and that it depends on a system parameter  $\mu$ . The main assumption is that the phase map goes through a heteroclinic bifurcation scenario. We shall find that this abstract mathematical condition has a number of powerful implications for the structure of the set of optimising trajectories.

**1.1 The optimisation problem.** In the following we state a string of assumptions. Their main function is to delineate the simplest configuration for which our results hold; all of them can be checked, at least numerically, for a given system. Moreover, they hold true for a large class of problems of practical interest.

We consider the problem to maximise an objective

$$J = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t}, \quad (1)$$

where  $\rho > 0$ , under the side condition that for all  $t \geq 1$  we have

$$x_t = f(x_{t-1}, u_t). \quad (2)$$

The state  $x_{t-1}$  and the control  $u_t$  take values in open intervals  $\mathcal{X}$  and  $\mathcal{U}$  respectively. Moreover, the initial state  $x_0$  is assumed to be given. Furthermore, we shall assume that  $f$  takes its values in  $\mathcal{X}$ ; this implies that there are no binding state constraints.

In appendix A, we present the facts from discrete-time optimal control theory that we need below. Introduce the *discrete present-value Pontryagin function*

$$P(x, y, u) = g(x, u) + yf(x, u).$$

If the sequences  $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$  and  $\mathbf{u} = \{u_t\}_{t=1}^{\infty}$  optimise  $J$  subject to (2), given the initial state  $x_0$ , then necessarily there is a sequence  $\mathbf{y} = \{y_t\}_{t=0}^{\infty}$  such that we have for every  $t \geq 1$  that

$$P_u = 0, \quad x_t = P_y, \quad e^{\rho} y_{t-1} = P_x, \quad (3)$$

where the argument of  $P$  is  $(x_{t-1}, y_t, u_t)$ . Moreover, under the conditions of proposition A.4, also the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\rho t} y_t = 0 \quad (4)$$

has to hold true. Note that we only consider interior solutions, which is reflected by our choice of  $\mathcal{X}$  and  $\mathcal{U}$  as open sets.

The necessary conditions can be reformulated in terms of a *Hamilton function* as follows. Assuming that  $f_u \neq 0$  everywhere, we first note that the equation  $P_u = 0$  implies that

$$y = -\frac{g_u(x, u)}{f_u(x, u)} = Y(x, u);$$

introduce  $\mathcal{Y} = Y(\mathcal{X}, \mathcal{U})$ ; consequently, assuming  $P_{uu} < 0$  it follows that  $P_u = 0$  can be solved for  $u$  as  $u = U(x, y)$ , where  $U : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{U}$ . Then the *discrete present-value Hamilton function* reads as

$$H(x, y) = P(x, y, U(x, y)),$$

and the necessary conditions now read as

$$x_t = H_y, \quad e^{\rho} y_{t-1} = H_x, \quad (5)$$

where the argument of  $H$  is  $(x_{t-1}, y_t)$ . Together with the condition that the state orbit starts at  $x_0$  and the transversality condition (4) they constitute a boundary value problem for optimal orbits.

**1.2 The phase map.** Introduce the *state-costate* or *phase space*  $\mathcal{M} = \mathcal{X} \times \mathcal{Y} \subset \mathbb{R}^2$ . Note that equations (5) define a dynamical system on  $\mathcal{M}$  if  $H_{xy} \neq 0$ ; for then we can solve the equation  $e^\rho y_{t-1} = H_x$  for  $y_t$  as  $y_t = \varphi_2(x_{t-1}, y_{t-1})$ ; substitution into  $x_t = H_y$  yields then  $x_t = \varphi_1(x_{t-1}, y_{t-1})$ . Thus have we obtained the *phase map*  $\varphi$ , which determines the state-costate dynamics by

$$z_t = \varphi(z_{t-1}). \quad (6)$$

Sometimes, we shall use the coordinate representation

$$(x_t, y_t) = \left( \varphi_1(x_{t-1}, y_{t-1}), \varphi_2(x_{t-1}, y_{t-1}) \right)$$

of this relation. Our first assumption implies that the phase map actually exists.

**Assumption 1.** *The discrete Hamilton function  $H = H(x, y)$  satisfies  $H_{xy} > 0$  and  $H_{yy} > 0$ .*

The stronger assumption  $H_{xy} > 0$  is needed below.

Phase maps originating from optimisation problems of the type given by (13) and (2) have, like their continuous time counterparts, special geometrical properties: they are called conformally symplectic maps, and, in the case  $\rho = 0$ , they are even symplectic. Symplecticity is an abstract mathematical concept, related to integrability theory; some of its implications are worked out in appendix B. Here we note only the property that symplectic maps in the plane are area-preserving, and that the conformally symplectic maps that arise if  $\rho > 0$  multiply the areas of regions in phase space uniformly with  $e^\rho$ . As our arguments are heavily based on these concepts, we recall these quickly. For a fuller treatment of this material, especially of differential forms, we refer the reader to the excellent expositions of Spivak (1965) or Arnol'd (1989).

Let

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In  $\mathbb{R}^2$ , the *standard symplectic 2-form* is the differential form  $\omega = dy \wedge dx$ ; that is, on a pair  $v = (v_1, v_2)$ ,  $w = (w_1, w_2) \in \mathbb{R}^2$ , the form  $\omega$  takes the value

$$\omega(v, w) = \langle v, Ew \rangle = v_2 w_1 - v_1 w_2.$$

Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be of the form  $\psi = (\psi_1(x, y), \psi_2(x, y))$ . Let  $D\psi$  denote the  $2 \times 2$  Jacobi matrix

$$D\psi = \begin{pmatrix} \frac{\partial \psi_1}{\partial x} & \frac{\partial \psi_1}{\partial y} \\ \frac{\partial \psi_2}{\partial x} & \frac{\partial \psi_2}{\partial y} \end{pmatrix}.$$

The pull-back  $\psi^* \omega$  of  $\omega$  under  $\psi$  is defined as

$$\psi^* \omega(v, w) = \omega(D\psi v, D\psi w). \quad (7)$$

The map  $\psi$  is called *symplectic*, if

$$\psi^*\omega = \omega.$$

It is called *conformally symplectic*, if there is a function  $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\psi^*\omega = \lambda\omega.$$

**Proposition 1.1.** *The phase map  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , given by (6) is conformally symplectic. More precisely, it satisfies*

$$\varphi^*\omega = e^\rho\omega.$$

*This implies directly that*

$$\det D\varphi = e^\rho. \tag{8}$$

The proposition is a corollary of the more general propositions A.5 and B.2, which are stated and proved in the appendix.

Note that  $\det D\varphi = e^\rho$  implies that the map  $\varphi$  multiplies phase volume by a factor  $e^\rho$ , as

$$\text{area}(\varphi(A)) = \iint_{\varphi(A)} dx dy = \iint_A \det D\varphi dx dy = e^\rho \text{area}(A).$$

It follows that there are no bounded regions that are invariant under  $\varphi$ ; this implies for instance that  $\varphi$  has no invariant circles.

**1.3 Comparison with continuous time case.** Note that the properties of the phase map have well-known analogues in continuous time. We sketch this briefly.

The continuous time problem asks to maximise a functional

$$J = \int_0^\infty g(x, u) e^{-\rho t}$$

under the side condition that

$$\dot{x} = f(x, u).$$

The *continuous time present-value Pontryagin function* takes the form

$$P(x, y, u) = g(x, u) + yf(x, u).$$

An interior optimising orbit satisfies necessarily (cf. equation (3))

$$P_u = 0, \quad \dot{x} = P_y, \quad \rho y - \dot{y} = -\frac{d}{dt}(e^{-\rho t}y) \Big|_{t=0} = P_x.$$

If  $P_{uu} < 0$  everywhere, we can solve  $u = U(x, y)$  from  $P_u = 0$ , and obtain the *continuous time present-value Hamilton function*

$$H(x, y) = P(x, y, U(x, y)).$$

In terms of  $H$ , the necessary conditions read as (cf. (5))

$$\dot{x} = H_y, \quad \rho y - \dot{y} = H_x.$$

Letting  $X(x, y) = (H_y, \rho y - H_x)$  denote the vector field defined by these equations, we have

$$\operatorname{div} X = \rho.$$

Now if  $\Phi_t = e^{tX}$  is the phase map defined by the vector field  $X$ , then

$$\Phi_t^* \omega = e^{\rho t} \omega$$

for all  $t$ . This implies

$$\det D\Phi_t = e^{t \operatorname{div} X} = e^{t\rho}.$$

The first equality can be verified easily by differentiation with respect to  $t$ . In particular (cf. (8))

$$\det D\Phi_1 = e^\rho.$$

**1.4 Description of the context.** We proceed to describe the class of problems which we are interested in. Let the phase map  $\varphi = \varphi_\mu : \mathcal{M} \rightarrow \mathcal{M}$  depend on a parameter  $\mu \in \mathbb{R}$ . We make the following assumptions on this dependence.

**Assumption 2.** For all values of the parameter  $\mu$ ,

1. the map  $\varphi = \varphi_\mu$  has two saddle fixed points  $z_- = (x_-, y_-)$  and  $z_+ = (x_+, y_+)$ , and  $x_- < x_+$ ;
2. for  $i \in \{+, -\}$ , the eigenvalues  $\lambda_i^u$  and  $\lambda_i^s$  of  $D\varphi_\mu(z_i)$  satisfy

$$0 < \lambda_i^s < 1 < \lambda_i^u.$$

Recall that the *stable manifold*  $W^s$  of a fixed point  $\bar{z}$  is the set of all points  $z \in \mathcal{M}$  whose forward iterates converge to  $\bar{z}$ :

$$W^s = \{z \in \mathcal{M} : \lim_{t \rightarrow \infty} \varphi^t(z) = \bar{z}\}.$$

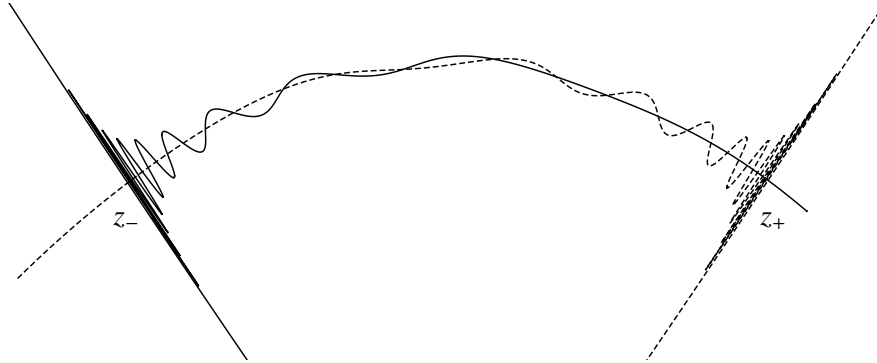
Analogously the *unstable manifold* of  $\bar{z}$  consists of the points backward asymptotic to  $\bar{z}$ . By the invariant manifold theorem, these sets are in fact differentiable manifolds. We shall denote the stable and unstable manifold of  $z_\pm$  by  $W_\pm^s$  and  $W_\pm^u$  respectively.

As in the continuous time case, orbits on the stable manifolds are candidates for optimal trajectories, as they satisfy the transversality condition automatically. We postulate that in our class the optimisation problem has a solution, and that orbits on the stable manifolds of  $z_-$  and  $z_+$  are the only candidates for the optimal orbits.



**Assumption 3.** For every  $x_0 \in \mathcal{X}$ , the problem to optimise  $J(\mathbf{x}, \mathbf{u})$  subject to equation (2) has a solution. Moreover, the state-costate trajectory  $\mathbf{z}$  of such a solution is either on  $W_-^s$  or  $W_+^s$ .

As mentioned above, the genesis of indifference points and indifference thresholds is intimately connected with the occurrence of heteroclinic orbits in the system. A point  $z$  is called *heteroclinic*, or a *heteroclinic intersection* of  $W_-^u$  and  $W_+^s$ , if  $z \in W_-^u \cap W_+^s$ . Note that if  $z$  is heteroclinic, so is  $\varphi(z)$ , and in fact every iterate  $\varphi^k(z)$ . The orbit  $O(z) = \{\varphi^k(z) \mid k \in \mathbb{Z}\}$  through a heteroclinic point  $z$  is therefore called a heteroclinic orbit. Note that as a consequence of assumption 1, the inverse  $\varphi^{-1}(z)$  of a point is uniquely defined, if it exists.



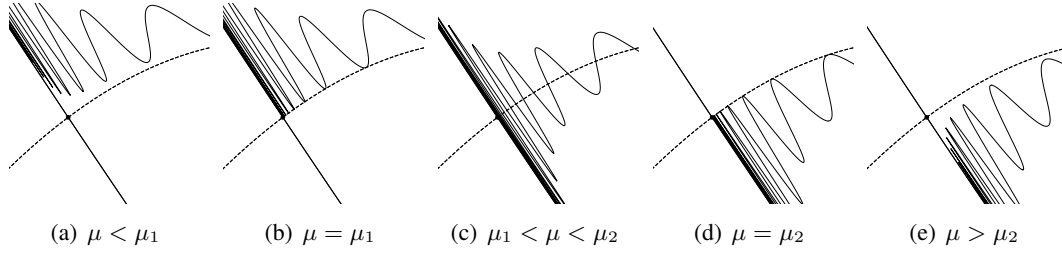
**Figure 1:** Stable manifold  $W_+^s$  of  $z_+$  (solid) and an unstable manifold  $W_-^u$  of  $z_-$  (dashed). The stable manifold  $W_+^s$  is composed of all points that are forward asymptotic to  $z_+$ ; likewise,  $W_-^u$  is composed of all points backward asymptotic to  $z_-$ . A heteroclinic point is an intersection of  $W_+^s$  and  $W_-^u$ , hence a point that is forward asymptotic to  $z_+$  and backward asymptotic to  $z_-$ . As both manifolds contain infinitely many orbits, they do not necessarily coincide (unlike the continuous time case).

A heteroclinic intersection  $z$  is *transversal*, if at  $z$  the tangent vectors to  $W_-^u$  and  $W_+^s$  are linearly independent, see for instance figure 1 or figure 2(c). As invariant manifolds and their tangent spaces depend continuously on parameters, we see that if for a given parameter value  $\mu_0$  there is a transversal heteroclinic intersection, then this is the case for all values of  $\mu$  sufficiently close to  $\mu_0$ . A non-transversal heteroclinic intersection is called a heteroclinic *tangency* (as in figures 2(b) and 2(d)).

The family  $\varphi_\mu$  is said to go through a heteroclinic bifurcation scenario, involving for instance  $W_-^u$  and  $W_+^s$ , if there is a parameter interval  $[\mu_1, \mu_2]$  such that for  $\mu < \mu_1$  and  $\mu > \mu_2$ , the manifolds  $W_-^u$  and  $W_+^s$  have no points in common, and such that for  $\mu \in [\mu_1, \mu_2]$  there is at least one heteroclinic orbit. Necessarily for  $\mu = \mu_1$  and  $\mu = \mu_2$ , all heteroclinic orbits are tangencies. Figure 2 illustrates the basic scenario. In general, the scenario may be more complex, featuring also tangencies for intermediate values of  $\mu$ .

The family  $\varphi_\mu$  of phase maps is assumed to go through a heteroclinic bifurcation scenario:

**Assumption 4.** If  $\mu < \mu_1$  or  $\mu > \mu_2$ , then  $W_+^s$  and  $W_-^u$  have no points in common. On the other hand, if  $\mu_1 \leq \mu \leq \mu_2$ , then there are heteroclinic intersections of  $W_-^u$  and  $W_+^s$ . Moreover,



**Figure 2:** Relative position of  $W_-^u$  (dashed) and  $W_+^s$  at  $z_-$ , depending on the parameter  $\mu$ . At  $\mu = \mu_1$  and  $\mu = \mu_2$ ,  $W_-^u$  and  $W_+^s$  exhibit heteroclinic tangencies; for  $\mu_1 < \mu < \mu_2$ , the manifolds intersect transversally.

if  $\mu > \mu_2$ , then  $W_-^s$  does not intersect the line  $x = x_+$ , nor does  $W_+^s$  intersect  $x = x_-$ . If  $\mu < \mu_1$ , then  $W_+^s$  intersects the line  $x = x_-$ ,

A direct implication of assumption 3 together with the second half of assumption 4 is that if  $\mu > \mu_2$ , then both  $x_-$  and  $x_+$  are locally optimal fixed points.

For a given heteroclinic intersection  $z \in W_-^u \cap W_+^s$ , let  $C$  be the curve obtained by taking the part of  $W_-^u$  that connects  $z_-$  to  $z$  and the part of  $W_+^s$  that connects  $z$  to  $z_+$ . If  $C$  is a curve without self-intersections, then  $z$  is called a *primary* heteroclinic intersection.

The next assumption postulates that the map  $\varphi$  has some generic properties.

**Assumption 5.**

1. For  $\mu = \mu_1$  and  $\mu = \mu_2$ , there is a single orbit of heteroclinic tangencies of  $W_-^u$  and  $W_+^s$ .
2. There is a finite set  $F \subset [\mu_1, \mu_2]$  such that for each  $\mu \in [\mu_1, \mu_2] \setminus F$ , the manifolds  $W_-^u$  and  $W_+^s$  have only transversal primary intersection points. If  $\mu \in F$ , there is one orbit of primary quadratic heteroclinic tangencies of  $W_-^u$  and  $W_+^s$ , as well as at least two orbits of primary heteroclinic transversal intersections.

Remark that the conditions of the assumptions determine an open set of phase maps  $\varphi$ . Whether this set is also dense, in some suitable function topology, is not immediately clear, due to the indirect definition of  $\varphi$ . We leave this question to a future investigation and instead only conjecture that the conditions of assumption 5 determine an open and dense set, with respect to the  $C^\infty$  topology, of optimisation problems that satisfy assumptions 2–4.

Note however that without the restriction to primary intersection points, the conjecture might well be false, as there may be generically infinitely many values of  $\mu$  for which there is a heteroclinic tangency (cf. Palis and Takens, 1993, chapter 6).

The next assumption is necessary since the inverse of  $\varphi$  is not necessarily defined in every part of the phase space  $\mathcal{M}$ ; consequently, the stable manifold need not be connected.

**Assumption 6.** For each  $\mu \in (\mu_1, \mu_2)$  and every orbit  $O$  of heteroclinic intersections of  $W_+^s$  and  $W_-^u$ , there exists  $z \in O$  and two smooth curves in  $\mathcal{M}$  that connect  $z$  to  $\varphi(z)$  along  $W_-^u$  and  $\varphi(z)$  to  $z$  along  $W_+^s$  respectively.

**1.5 The main result and its interpretation.** We call a state  $\bar{x}$  an (*optimal*) *steady state*, if the optimal trajectory starting at  $\bar{x}$  is given by  $x_t = \bar{x}$  for all  $t$ . An optimal steady state  $\bar{x}$  is *globally optimal*, if every optimal trajectory  $\{x_t\}$  converges to  $\bar{x}$ ; the steady state  $\bar{x}$  is *locally optimal*, if for all initial states  $x_0$  in a neighbourhood of  $\bar{x}$ , the optimal trajectory starting at  $x_0$  converges to  $\bar{x}$ .

Now we can state our main result.

**Main Theorem.** *Let the assumptions 1–6 be satisfied. There is a value  $\mu_1 < \mu_c < \mu_2$  such that*

1. *If  $\mu < \mu_c$ , the steady state  $x_+$  is globally optimal;*
2. *if  $\mu > \mu_c$ , both steady states  $x_-$  and  $x_+$  are locally optimal, and there is a state  $x_s < x_s < x_+$  such that  $x_s$  is initial state to two optimal solutions, one converging to  $x_-$  and the other converging to  $x_+$ ;*
3. *if  $\mu = \mu_c$ , the steady state  $x_-$  is semi-stable: optimal solutions starting at  $x_0 \leq x_-$  tend to  $x_-$ , whereas optimal solutions starting at  $x_0 > x_-$  tend to  $x_+$ ;*
4. *moreover, if  $\mu = \mu_c$ , there is an infinite sequence*

$$x_i^{(1)} > x_i^{(2)} > \dots > x_-,$$

*such that*

$$\lim_{k \rightarrow \infty} x_i^{(k)} = x_-$$

*and such that each  $x_i^{(k)}$  is initial point to two optimal sequences, both converging to  $x_+$ .*

The proof is a direct corollary of theorems 1–3 below.

We can interpret this theorem most easily, if we relate it to the *optimal dynamics* in state space. By this we mean the following. Any optimal state control trajectory  $(\mathbf{x}, \mathbf{u})$  corresponds one-to-one with a state-costate trajectory  $\mathbf{z}$ , which in turn is determined by its initial state  $z_0 = (x_0, y_0)$ .

It is a consequence of the structure of  $J$  that by the principle of optimality, if  $\mathbf{z} = \{z_t\}_{t=0}^{\infty}$  is an optimal state-costate trajectory with initial point  $z_0$ , and if  $n$  is a positive integer, then  $\sigma_n \mathbf{z} \stackrel{\text{def}}{=} \{z_{t+n}\}_{t=0}^{\infty}$  is also an optimal trajectory, but with the initial state  $z_n$ . For we have that

$$J(\mathbf{z}) = \sum_{t=1}^n e^{-\rho t} g(x_t, U(x_t, y_t)) + e^{-n\rho} J(\sigma_n \mathbf{z}),$$

and it is clear that if  $\sigma_n \mathbf{z}$  did not maximise  $J$  over the set of admissible trajectories starting at  $z_n$ , then  $\mathbf{z}$  would not maximise  $J$  over the set of admissible trajectories starting at  $z_0$ .

The set of optimal state-costate trajectories can therefore be described by a set-valued map

$$Y^o = Y^o(x) \subset \mathcal{Y}.$$

If  $z_0 = (x_0, y_0)$ , with  $y_0 \in Y^o(x_0)$ , then  $z_0$  is the initial point of an optimal trajectory. We shall call  $Y^o$  the *optimal costate map*. In the present context, it follows from assumption 3 that  $Y^o(x)$  is a set of either one or two elements. Analogously, we define the *optimal state map*

$$\Psi^o(x) = \varphi_1(x, Y^o(x)).$$

Note that this is also a set-valued map. Finally, the map

$$U^o(x) = U(x, Y^o(x))$$

is the policy function. Note that for all practical purposes, all points  $x$  where  $U(x)$  consists of two elements are jump points of the policy function.

The complexity of the optimal state dynamics is not as great as would appear at first sight. Indeed, if a state  $\xi$  is such that

$$\xi \in \Psi^o(x_0) = \varphi_1(x_0, y_0)$$

for some  $x_0$  and some  $y_0 \in Y^o(x_0)$ , then  $\psi^o(\xi)$  contains exactly one element; otherwise, there would be two optimal state-costate orbits with initial point  $(x_0, y_0)$ , which contradicts the fact that the phase map  $\varphi$  is well-defined (and hence single-valued).

If  $\Psi^o(x)$  contains only one element, we define  $\psi^o(x)$  by setting

$$\Psi^o(x) = \{\psi^o(x)\}.$$

Note that an optimal steady state as defined above is just a fixed point of the map  $\psi^o$ . If  $\Psi^o(x)$  contains two elements, the state  $x$  is an *indifference state*, as there are two optimal state trajectories starting at  $x$ . If these two optimal trajectories have different  $\omega$ -limit sets, then  $x$  is called an *indifference threshold*.

We can now rephrase the main theorem in terms of the (parameter-dependent) optimal state dynamics  $\Psi_\mu^o$ : if the assumptions are satisfied, and if  $\mu < \mu_c$ , then all orbits of the optimal dynamics tend to  $x_+$ , and  $x_+$  is a global attractor for the optimal dynamics. If  $\mu > \mu_c$ , there is one indifference threshold  $x_s$ , and all orbits starting at  $x_0 < x_s$  tend to  $x_-$ , whereas all orbits starting at a point  $x_0 > x_s$  tend to  $x_+$ ; both  $x_-$  and  $x_+$  are local attractors of the optimal dynamics. If  $\mu = \mu_c$ , then the orbit  $x = x_-$  is semi-stable: all orbits starting to the left of it converge to  $x_-$ , while all orbits starting to the right converge to  $x_+$ . We summarise these facts by saying that at  $\mu = \mu_c$ , a locally stable attractor and an indifference threshold of the optimal dynamics are generated through an *indifference-attractor bifurcation*. The last statement of the main theorem is that for  $\mu = \mu_c$  the optimal dynamics has an infinity of indifference points that are not indifference thresholds.

The bifurcation value  $\mu_c$  is determined by a geometric criterion, which is contained in the statement of theorem 3, and which can be used to compute the indifference-attractor bifurcation curve numerically; we plan to present the numerical details in a future paper.

## 2 Application to the discrete time lake problem

This section illustrates our results for a variant of the lake problem introduced by Mäler et al. (2003). In this problem, a social planner tries to optimally manage a phosphorus pollution stream  $\mathbf{u} = \{u_t\}_{t=1}^{\infty}$  that originates from the use of artificial fertilisers in agriculture. By rainfall, these fertilisers are washed into a lake; the concentration  $x_t$  of phosphorus in the lake is assumed to follow the dynamics

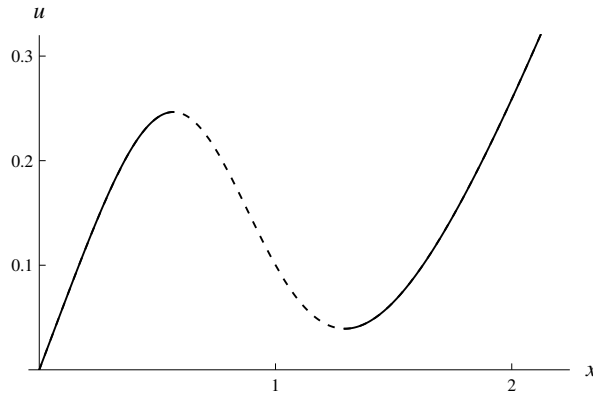
$$x_t = u_t + (1 - b)x_{t-1} + \frac{x_t^q}{1 + x_t^q}.$$

Here  $b$  is the *sedimentation rate*, and  $q$  is the responsiveness of the lake. Typically, for lakes, a value of  $q = 2$  is taken; to illustrate our results more clearly, in this section we take  $q = 4$ , which would correspond to a deeper lake.

For a constant pollution loading  $u_t = u$  for all  $t$ , the fixed points of the lake are solutions  $x$  of the equation

$$u = bx - \frac{x^q}{1 + x^q},$$

which is illustrated in figure 3. There is a range of  $u$ -values such that there are multiple steady



**Figure 3:** Location of fixed points for constant pollution streams  $u_t = u$  for all  $t$ , for  $b = 0.6$  and  $q = 4$ . Indicated are stable (solid) and unstable fixed points (dashed).

states; also, if the system starts in a low pollution steady state, and if then  $u$  is raised past the tipping value (in the figure approximately  $u \approx 0.25$ ), then it switches to a high pollution steady state. A small subsequent decrement of  $u$  will not move the system back to the clean branch of steady states again. For this, the pollution flow has to be lowered significantly, below  $u \approx 0.04$  in the figure.

In the lake pollution problem, the social manager has to weigh the interest of the farmers, deriving income from the use of artificial fertilisers, against that of the lake users, suffering from pollution damage to the lake. The resulting social utility functional is modelled as

$$J = \sum_{t=1}^{\infty} (\log u_t - cx_{t-1}^2) e^{-\rho t}.$$

Here  $c$  is the social preference parameter, and  $\rho > 0$  the discount rate.

The state space and control space are given as  $\mathcal{X} = \mathcal{U} = (0, \infty)$ . The discrete Pontryagin function is

$$P = \log u - cx^2 + y \left( u + (1-b)x + \frac{x^q}{1+x^q} \right).$$

Note that  $P_{uu} < 0$  for all  $u > 0$ . The necessary condition  $P_u = 0$  takes the form

$$0 = P_u = \frac{1}{u} + y.$$

Solving for  $u$  yields that  $u = U = -1/y$ . The costate space is given as  $\mathcal{Y} = (-\infty, 0)$ , and the phase space  $\mathcal{M} = \mathcal{X} \times \mathcal{Y} = (0, \infty) \times (-\infty, 0)$ . Substituting out  $u$ , we find the discrete Hamilton function

$$H = -\log(-y) - cx^2 - 1 + y \left( (1-b)x + \frac{x^q}{1+x^q} \right).$$

Since  $H_{yy} = y^{-2} > 0$  and

$$H_{xy} = 1 - b + q \frac{x^{q-1}}{(1+x^q)^2} > 0,$$

assumption 1 is satisfied.

We obtain the necessary conditions

$$\begin{aligned} x_t = H_y &= -\frac{1}{y_t} + (1-b)x_{t-1} + \frac{x_{t-1}^q}{1+x_{t-1}^q}, \\ e^\rho y_{t-1} = H_x &= -2cx_{t-1} + y_t \left( (1-b) + q \frac{x_{t-1}^{q-1}}{(1+x_{t-1}^q)^2} \right). \end{aligned}$$

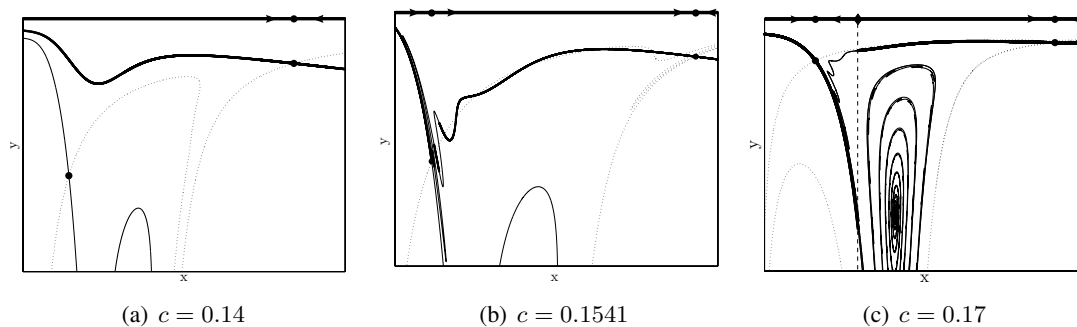
Solving the second equation for  $y_t$  and substituting into the first yields the phase map  $\varphi$  as

$$\varphi(x, y) = \left( -\frac{(1-b) + q \frac{x^{q-1}}{(1+x^q)^2}}{e^\rho y + 2cx} + (1-b)x + \frac{x^q}{1+x^q}, \frac{e^\rho y + 2cx}{(1-b) + q \frac{x^{q-1}}{(1+x^q)^2}} \right).$$

By introducing  $g(x) = (1-b)x + x^q/(1+x^q)$ , this expression simplifies somewhat to

$$\varphi(x, y) = \left( -\frac{g'(x)}{e^\rho y + 2cx} + g(x), \frac{e^\rho y + 2cx}{g'(x)} \right).$$

Fixing the parameters at  $b = 0.6$ ,  $\rho = 0.03$  and  $q = 4$ , in figure 4 we plot fixed points and their stable and unstable manifolds for a range of values of  $c$ ; for all values, the phase map has two saddle fixed points  $z_-$  and  $z_+$ . It can be shown, though we shall not do this here, that assumption 3 is satisfied. Taking  $\mu = c$  and accepting the geometric evidence from the

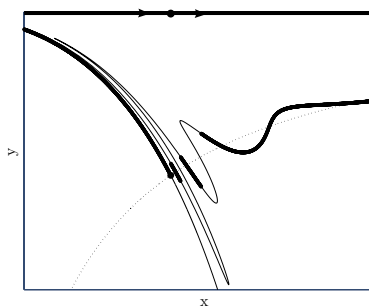


**Figure 4:** Solid lines indicate stable manifolds, dotted lines unstable manifolds; optimal solutions are marked by thick lines; the vertical line through the indifference threshold is dashed. Note that  $y < 0$  throughout, so that the  $x$ -axis is at the top of the figure. On the  $x$ -axis, the optimal dynamics are indicated; attractors are marked by a circle, the indifference threshold by a diamond.

plots in figure 4, the intermediate value theorem implies that assumption 4 is satisfied. At least at  $c = 0.1541$ , geometric evidence also supports assumptions 5 and 6. Granting the assumptions, the main theorem applies.

We note that for  $c = 0.14$ , that is, in a situation where the returns from agriculture weight relatively heavily, it is for every initial state  $x_0 \in \mathcal{X}$  optimal to steer the lake to the high pollution state  $x_+$ . The value  $c = c_{IA} \approx 0.1541$  corresponds to the case that  $\Omega(A) = 0$  of section 3; it follows from the main theorem that then for  $x_0 \leq x_-$  the optimal policy steers the lake to the low pollution state  $x_-$ , while if  $x_0 > x_-$ , it is optimal to end at the high pollution state  $x_+$ .

Moreover, the main theorem implies that for  $c = c_{IA}$  there is a countable infinity of indifference states. Recall that indifference states are initial states to two distinct optimal policies. At these points the policy function jumps; two of these jumps can be seen in figure 5.



**Figure 5:** For  $c = 0.1541$  we have  $\Omega(A) = 0$ , and consequently there is an infinity of indifference points.

Finally, for  $c = 0.17$ , both  $x_-$  and  $x_+$  are locally optimal, and their basins of attraction are separated by an indifference threshold.

### 3 About regions and orientations

In this section, we formulate three theorems that together give precise conditions to determine whether the states  $x_-$  and  $x_+$  are both locally optimal, or whether  $x_+$  is globally optimal. These conditions are formulated in terms of the oriented area of a certain region in phase space. We briefly introduce the geometric concepts needed: regions, intersection numbers, cochains. These will allow us to state our theorems, and to provide a brief sketch of the proof, of which the details will be given in section 4.

In appendix B.2, the precise definition of a region is given; here we paraphrase it by stating that a region is a collection of oriented open and bounded sets that are simply connected and that have well-behaved boundaries. We construct a certain region  $A$  as follows. First, we assume that the parts of  $W_-^u$  and  $W_+^s$  that interact in the heteroclinic bifurcation are parametrised by arclength, starting from the respective fixed points. That is, the parametrisations  $\gamma_u(s)$  and  $\gamma_s(s)$  satisfy  $\gamma_u(0) = z_-$  and  $\gamma_s(0) = z_+$ , as well as  $\|\gamma_u'(s)\| = \|\gamma_s'(s)\| = 1$ . Note that this determines an orientation of  $W_-^u$  and  $W_+^s$ .

Using these parametrisations, we say that a transversal heteroclinic intersection  $z = \gamma_u(s_1) = \gamma_s(s_2)$  has intersection number  $+1$  (cf. Hirsch, 1976), if

$$\det \begin{pmatrix} \gamma_u'(s_1) & \gamma_s'(s_2) \end{pmatrix} > 0.$$

Intersections of intersection number  $-1$  are defined analogously; see figure 6(a). The intersection number of a quadratic heteroclinic tangency is set to be 0.

Since  $\det D\varphi = e^\rho > 0$ , if  $p_t$  is a transversal heteroclinic intersection with intersection number  $+1$ , then so is  $\varphi(p_t)$ . Therefore, the intersection number of a heteroclinic orbit  $\mathbf{p}$  is well-defined as the intersection number of any of its elements. Let  $\mathbf{p} = \{p_k\}_{k=-\infty}^{\infty}$  be a transversal heteroclinic intersection of  $W_-^u$  and  $W_+^s$  with intersection number  $+1$ . Heteroclinic orbits of this type will be called *upward* orbits.

Differential forms can be integrated over oriented regions (see e.g. Spivak, 1965): for example, if  $A$  is an open connected set that has the standard orientation of  $\mathbb{R}^2$ , then

$$\int_A \omega = -\text{area}(A).$$

We use this equality to define a function  $\Omega$  taking regions as arguments, a *cochain*, by setting

$$\Omega(A) = \int_A \omega.$$

The cochain  $\Omega$  will allow us to formulate both the results and the proofs succinctly.

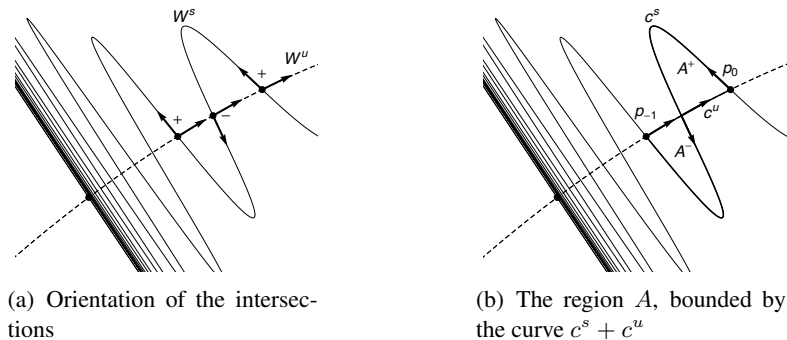
If  $\mathbf{p}$  is an upward orbit, assume that  $p_0$  is such that smooth curves  $c^u$ ,  $c^s$  as postulated in assumption 6 exist, connecting  $p_0$  to  $p_{-1}$ . Let  $c$  be the closed curve obtained by first following  $c^s$  from  $p_0$  to  $p_{-1}$  and then  $c^u$  from  $p_{-1}$  to  $p_0$ . Then  $c$  is the boundary of a region  $A$ , with positively and negatively oriented components  $A^+$  and  $A^-$  respectively. Recall that

$$\Omega(A^-) = \int_{A^-} \omega = \text{area}(A^-) > 0, \quad \Omega(A^+) = \int_{A^+} \omega = -\text{area}(A^+) < 0,$$



and

$$\Omega(A) = \Omega(A^+) + \Omega(A^-) = \text{area}(A^-) - \text{area}(A^+).$$



**Figure 6:** Definition of the region  $A$ .

Theorems 1–3 give precise conditions which enable us to determine whether there are two locally attracting or one globally attracting steady state. Together, these theorems imply the main theorem.

**Theorem 1.** *If  $\mu > \mu_2$  or if  $\Omega(A) \geq 0$  for each upward orbit  $\mathbf{p}$ , then both  $x_-$  and  $x_+$  are locally optimal fixed points.*

**Theorem 2.** *If  $\mu < \mu_1$  or if  $\Omega(A) < 0$  for some upward orbit  $\mathbf{p}$ , then  $x_+$  is a globally optimal fixed point.*

Moreover, we can characterise the codimension one situation separating the two generic cases.

**Theorem 3.** *Let  $\mu_2 \in [\mu_1, \mu_2]$  be such that  $\Omega(A) = 0$  for some upward orbit  $\mathbf{p}$ . Moreover, let  $\tilde{A}$  be the corresponding region for another upward orbit  $\mathbf{q}$ . We assume that  $\Omega(\tilde{A}) > 0$  for any such orbit. Then  $x_t = x_-$  for all  $t$  is an optimal trajectory. For each  $x_0 > x_-$ , the optimal trajectories beginning at  $x_0$  converge to  $x_+$ . Moreover, there are infinitely many points  $x > x_-$  which are initial point to two distinct optimal trajectories.*

Remark that though the magnitude of  $\Omega(A)$  depends on the choice of  $p_0$  of the heteroclinic orbit, the sign of  $\Omega(A)$  is independent of that choice, since  $\Omega(\varphi(A)) = e^\rho \Omega(A)$ .

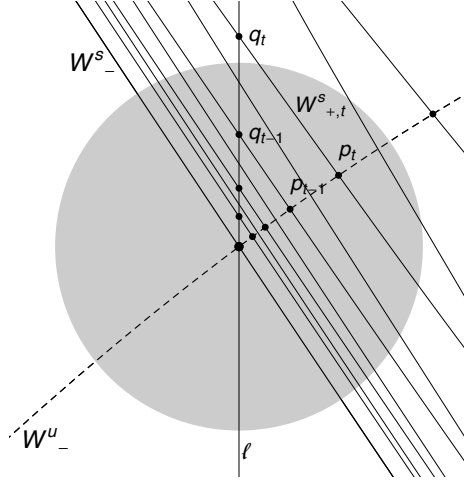
We sketch the idea of the proof. Consider an upward heteroclinic orbit  $\mathbf{p} = \{p_t\}_{t=-\infty}^{\infty}$ , and let  $U$  be a small convex open neighbourhood of the fixed point  $z_-$ . If  $p_t \in U$ , introduce the set  $W_{+,t}^s$  as the largest connected component of  $W^s \cap U$  that contains  $p_t$ ; otherwise, let  $W_{+,t}^s = \emptyset$ .

Assume first that  $p_t$  is a transversal heteroclinic intersection of  $W_+^s$  and  $W_-^u$ . The inclination lemma from the theory of dynamical systems, which is quoted in section 4, implies that for  $t <$

$-T$ , where  $T > 0$  is a sufficiently large constant, the set  $W_{+,t}^s$  is a curve segment that is  $C^1$ -close through the line through  $p_t$  that is parallel to the linear stable eigenspace  $E_-^s$  of the fixed point  $z_-$ . As this line intersects the vertical line  $\ell = \{(x, y) : x = x_+\}$  through  $z_-$ , so does  $W_{+,t}^s$ . Introduce

$$q_t = \ell \cap W_{+,t}^s.$$

The situation is illustrated in figure 7. Such an intersection arising from an upward orbit shall be



**Figure 7:** Definition of the segment  $W_{+,t}^s$ , as well as the points  $q_t$ .

called an upward intersection.

To every point on the stable manifold  $W_+^s$  we can associate a value by evaluating the objective functional for the phase trajectory starting at the point. It follows from proposition B.4 that we obtain the same result by integrating  $y dx$  along the stable manifold; since the manifold is Lagrangian, the result of the integration is independent of the integration path (see subsection B.1).

Let  $\gamma^s(s) = (x(s), y(s))$  be the parametrisation of  $W_+^s$  by arc length, such that  $z(0) = z_+$ , and such that heteroclinic points correspond to positive values of the parameters  $s$ . To a point  $\alpha \in W_+^s$  we associate the value  $v(\alpha)$  given by the phase trajectory  $\mathbf{z}$  starting at  $\alpha$

$$v(\alpha) = J(\mathbf{x}, \mathbf{u}),$$

where  $u_t = U(x_{t-1}, y_t)$ . This implies for the fixed point  $z_+$  that

$$v(z_+) = \frac{g(x_+, U(x_+, y_+))}{e^\rho - 1}.$$

Given a point  $\alpha \in W_+^s$ , let  $s_\alpha$  be such that  $\alpha = \gamma^s(s_\alpha)$ . Then, by proposition B.4,

$$v(\alpha) = v(z_+) + \int_0^{s_\alpha} y(s)x'(s) ds. \quad (9)$$

Note that  $v$  is only defined for points on  $W_+^s$ .

We shall establish that for every  $t < 0$  such that  $W_{+,t}^s$  intersects the line  $\ell$ , there is a region  $C_t$  such that

$$v(q_{t-1}) - v(q_t) = e^{-\rho t} \Omega(A) + e^{-\rho t} \text{area}(C_t). \quad (10)$$

If  $\Omega(A) > 0$ , this implies that  $v(q_t)$  is an increasing sequence as  $t$  decreases towards minus infinity. It follows from proposition B.9 that  $v(q_t) \uparrow V_-(x_-)$  as  $t \rightarrow -\infty$ . Theorem 1 insures that in this way every upward orbit gives rise to an increasing sequence of values. An intersection of  $W_{+,t}^s$  and  $\ell$  that is not upward can be shown to have a smaller value than the upward intersection immediately preceding it.

If equation (10) holds for every upward orbit  $\mathbf{p}$ , it follows that  $V_-(x_-)$  is larger than any value  $v(z)$  for  $z \in W_+^s \cap \ell$ , and consequently, that it is optimal to remain in  $z_-$ . A similar argument then holds if  $\ell$  is replaced by any vertical line through a point sufficiently close to  $z_-$ , demonstrating local optimality of  $x_-$ .

Equation (10) is also helpful for analysing the case that  $\Omega(A) < 0$  for some upward orbit  $\mathbf{p}$ , for we can show that

$$\frac{\text{area}(C_t)}{\Omega(A)} \rightarrow 0$$

as  $t \rightarrow -\infty$ . This implies that the sequence  $v(q_t), v(q_{t-1}), \dots$  is eventually decreasing. Note that the limit of the sequence is still  $V_-(x_-)$ ; therefore, there is some  $T$  such that

$$v(q_T) > V_-(x_-),$$

and the steady state  $x_-$  cannot be optimal in this case.

## 4 Proofs of the theorems

In this section, the proofs of theorems 1, 2 and 3 are given. General background results like the area rule and the iterated area rule are given in the appendices.

### 4.1 Local preliminaries.

**Proposition 4.1.** *If  $v = (v_1, v_2)$  is a nonzero eigenvector of  $D\varphi$ , then  $v_1 \neq 0$ .*

*Proof.* Assume that  $v_1 = 0$ . Then

$$\lambda v = \begin{pmatrix} 0 \\ \lambda v_2 \end{pmatrix} = \begin{pmatrix} e^\rho H_{yy} H_{xy}^{-1} v_2 \\ e^\rho H_{xy}^{-1} v_2 \end{pmatrix}.$$

If  $\lambda = 0$ , then  $H_{xy}^{-1} v_2 = 0$  and consequently  $v_2 = 0$ ; but then  $v$  would be trivial. If  $\lambda \neq 0$ , then  $v_2 = (e^\rho / \lambda) H_{xy}^{-1} v_2$ . Substituting into the first equation yields that

$$0 = H_{yy} v_2.$$

As  $H_{yy}$  is positive definite, it follows that  $v_2 = 0$ , again implying that  $v$  is trivial. ■

It follows from this proposition that there is a neighbourhood  $U = (x_- - \delta, x_- + \delta) \times (y_- - \delta, y_- + \delta)$  of  $z_-$  such that  $W_-^s$  and  $W_-^u$  restricted to  $U$  can be represented as the graphs of functions  $w^s$  and  $w^u$  respectively.

**Proposition 4.2.** *Assume that  $H_{xy} > 0$ . If  $v^u = (1, v_2^u)$  and  $v^s = (1, v_2^s)$  are the stable and unstable eigenvectors of  $D\varphi$ , then  $v_2^u > v_2^s$ .*

*Proof.* The derivative  $D\varphi$  takes the form

$$D\varphi = \begin{pmatrix} H_{xy} - \frac{H_{yy}H_{xx}}{H_{xy}} & e^\rho \frac{H_{yy}}{H_{xy}} \\ -\frac{H_{xx}}{H_{xy}} & \frac{e^\rho}{H_{xy}} \end{pmatrix}. \quad (11)$$

Moreover, if  $v = (1, v_2)$  is an eigenvector with eigenvalue  $\lambda$ , we have

$$(H_{xy}^2 - H_{xx}H_{yy}) - \lambda H_{xy} + e^\rho H_{yy}v_2 = 0.$$

This can be written as

$$v_2 = \frac{H_{xx}H_{yy} - H_{xy}^2}{e^\rho H_{yy}} + \frac{\lambda}{e^\rho} \frac{H_{xy}}{H_{yy}}.$$

The result now follows from the fact that  $H_{xy} > 0$  and  $H_{yy} > 0$ . ■

**Corollary 4.1.** *We have identically that*

$$\det D\varphi = e^\rho.$$

*Proof.* The computation of the determinant from equation (11) is straightforward. ■

From proposition 4.2, we obtain immediately the following corollary.

**Proposition 4.3.** *Let  $\Delta$  be the triangle bounded by the line connecting  $(0, 0)$  to  $v^u$ , followed by the line connecting  $v^u$  to  $v^u + v^s = (0, v_2^u + v_2^s)$  and the line connecting  $v^u + v^s$  to  $0$ . Then  $\Delta$  is positively oriented.*

Recall that a map  $\Phi$  is symplectic if  $\Phi^*\omega = \omega$ .

**Proposition 4.4.** *There is an open neighbourhood  $U \subset \mathcal{M}$  of  $z_-$ , an open neighbourhood  $\tilde{U} \subset \mathbb{R}^2$  of  $(0, 0)$  and a symplectic coordinate transformation  $\Phi : U \rightarrow \tilde{U}$  of the form*

$$\zeta = (\xi, \eta) = \Phi(x, y) = \Phi(z),$$

*such that in the new coordinates the map  $\varphi$  has the form*

$$\varphi(\zeta) = \begin{pmatrix} \lambda^s & 0 \\ 0 & \lambda^u \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} \xi\psi_1 \\ \eta\psi_2 \end{pmatrix},$$

*where restricted to  $\tilde{U}$  we have  $|\psi_i(\zeta)| \leq K|\zeta|$ ,  $i = 1, 2$  for some  $K > 0$ .*

*Proof.* We perform two successive symplectic coordinate transformations, such that in the new coordinates, the stable and the unstable manifolds coincide with the coordinate axes. As the first transformation, we take

$$\tilde{x} = x - x_-, \quad \tilde{y} = y - w^s(x).$$

This transformation is symplectic, since

$$\varphi^* \omega = d\tilde{y} \wedge d\tilde{x} = (dy - (w^s)'(x) dx) \wedge dx = dy \wedge dx = \omega.$$

Note that in  $(\tilde{x}, \tilde{y})$  coordinates, the fixed point  $z_-$  is given by  $(\tilde{x}, \tilde{y}) = (0, 0)$ , the stable manifold  $W_-^s$  by the equation  $\tilde{y} = 0$ , and the unstable manifold  $W_-^u$  by

$$\tilde{y} = w_1^u(\tilde{x}) = w^u(x_- + \tilde{x}) - w^s(x_- + \tilde{x}).$$

Note that the function  $w_1^u$  defined by this equation satisfies  $(w_1^u)'(0) = (w^u)'(\tilde{x}) - (w^s)'(\tilde{x}) \neq 0$ . As the second transformation, take

$$\xi = \tilde{x} - (w_1^u)^{-1}(\tilde{y}), \quad \eta = \tilde{y}.$$

Note that this transformation is well-defined on  $U$  — possibly  $\delta > 0$  has to be taken smaller to ensure the invertibility of  $w_1^u$  — that it is symplectic, that it preserves the location  $\eta = 0$  of the stable manifold, and that it maps the unstable manifold to  $\xi = 0$ .

The map  $\varphi$  has then in the new coordinates necessarily the form given in the proposition. ■

Finally, we recall the inclination lemma or  $\lambda$ -lemma (see Palis and Takens, 1993, p. 155).

**Inclination lemma.** Let  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  be a  $C^k$  diffeomorphism,  $k \geq 1$ , with a hyperbolic fixed point  $z$ . Let  $W \subset M$  be a  $C^k$  submanifold such that  $\dim(W) = \dim(W^s(z))$ , and such that  $W$  has a point  $p$  of transversal intersection with  $W^u(z)$ .

Then for each  $n$ , one can choose a disk  $D_t \subset \varphi^{-t}(W)$ , which is a neighbourhood of  $\varphi^{-t}(p)$  in  $\varphi^{-t}(W)$ , such that

$$\lim_{t \rightarrow \infty} D_t = D,$$

where  $D$  is a disk-neighbourhood of  $p$  in  $W^s(z)$ . Convergence means here that for  $t$  sufficiently large  $D_t$  and  $D$  are  $C^k$ -near embedded disks.

**4.2 Estimating value differences using the area rule.** In this subsection, equation (10) is stated precisely, derived, and an estimate of the term  $\text{area}(C_t)$  is given. Moreover a variant of equation (10), needed to prove theorem 1, is derived as well.

Let  $\mathbf{p}$  be an upward heteroclinic orbit. Let moreover  $T$  be such that if  $t < -T$ , then the part of  $W_+^s$  connecting  $p_t$  to  $p_{t-1}$  intersects the line  $\ell = \ell_\xi$  given by  $x = \xi$ . Let  $q_t$  be the first intersection of  $W_+^s$  with  $\ell$  following  $p_0$ , that is, let  $q_t$  be such that the segment of  $W_+^s$  connecting  $p_t$

to  $q_t$  has no other points in common with  $\ell$ . Define  $\ell_t$  as the segment of  $\ell$  connecting  $q_{t-1}$  to  $q_t$ . Then  $\varphi^{-t}\ell_t$  is a curve, connecting  $\varphi^{-t}(q_{t-1})$  to  $\varphi^{-t}(q_t)$ , which are both located on  $W_+^s$ .

Consider the curve  $c$  given by the part of  $W_+^s$  connecting  $\varphi^{-t}(q_t)$  to  $\varphi^{-t}(q_{t-1})$ , followed by the curve  $\varphi^{-t}\ell_t$  connecting  $\varphi^{-t}(q_{t-1})$  to  $\varphi^{-t}(q_t)$ . Then  $c$  is the boundary of a region  $B_t$  with positively and negatively oriented components  $B_t^+$  and  $B_t^-$  respectively.

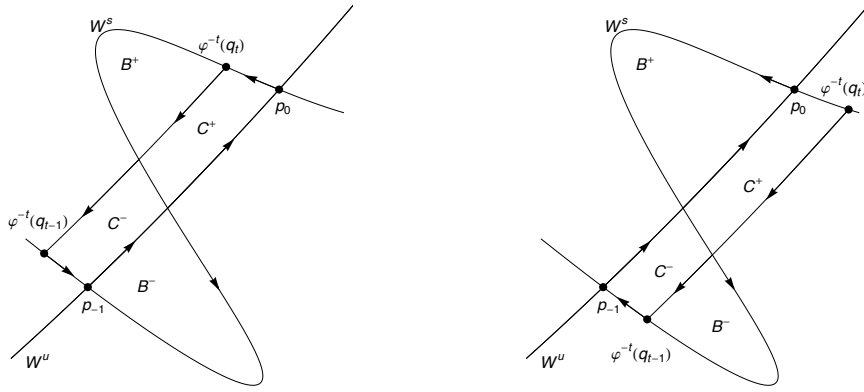
Define

$$C_t^+ = A^+ - B_t^+, \quad C_t^- = A^- - B_t^-,$$

and

$$C = A - B = C_t^+ + C_t^-.$$

See figure 8.



**Figure 8:** The regions  $B_t^\pm$  and  $C_t^\pm$ . The regions  $B^+$  and  $B^-$  are respectively positively and negatively oriented by definition. In the situation depicted in the left subfigure,  $C^+$  and  $C^-$  are both positively oriented, whereas in the right subfigure, both are negatively oriented.

**Proposition 4.5.** Let  $v$  be as in equation (9). For  $t < -T$ , we have

$$v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(A) - e^{\rho t} \Omega(C_t).$$

In particular, if all simple components of  $C_t$  are positively oriented, then

$$v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(A) + e^{\rho t} \text{area}(C_t),$$

whereas if all simple components are negatively oriented, then

$$v(q_{t-1}) - v(q_t) = e^{\rho t} \Omega(A) - e^{\rho t} \text{area}(C_t).$$

*Proof.* Recall that  $B_t = A - C_t$ . By the iterated area rule

$$e^{-\rho t} (v(q_{t-1}) - v(q_t)) = \Omega(B_t) = \Omega(A) - \Omega(C_t).$$

The result follows. ■

We now consider the case that  $\ell$  is the line  $x = x_-$ . The following proposition states that in that case, all simple components of  $C_t$  are positively oriented (see figure 8), and it gives an estimate of  $\Omega(C_t)$ .

**Proposition 4.6.** *If  $\ell$  is the line  $x = x_-$ , then all simple components of  $C_t^+$  and  $C_t^-$  are positively oriented. Moreover, there are constants  $T_0 > 0$  and  $K > 0$  such that for all  $t < -T_0$ , the inequality*

$$-K\lambda_u^{2t} \leq e^{\rho t}\Omega(C_t) \leq 0$$

holds.

*Proof.* Since  $\det D\varphi = e^\rho > 0$ , the phase map  $\varphi$  preserves orientation. Note that if  $T_0 > 0$  is sufficiently large, if  $t < -T_0$ , then the regions  $\varphi^t C_t^\pm$  are contained in the curvilinear triangle  $\tilde{\Delta}$  formed by the part  $W_t^s$  of  $W_+^s$  connecting  $p_t$  to  $q_t$  — the first intersection of  $W_+^s$  and  $\ell$  that follows  $p_t$  — the part of  $\ell$  connecting  $q_t$  to  $z_-$  and the part of  $W_+^s$  connecting  $z_-$  to  $p_t$ . By the  $\lambda$ -lemma, for large values of  $t$  the curve segment  $W_t^s$  is  $C^1$ -close to  $W_-^s$ . Therefore the curvilinear triangle  $\tilde{\Delta}$  has the same orientation as the triangle  $\Delta$  introduced in proposition 4.3. But that proposition states that  $\Delta$  is positively oriented.

Let  $m_0$  and  $m_{-1}$  be lines through  $p_0$  and  $p_{-1}$  respectively that intersect  $W_-^u$  transversally, and which are such that the region bounded by  $m_0$ ,  $\varphi^{-t}\ell$ ,  $m_{-1}$  and  $W_-^u$  contains  $C_t$ .

Moreover, in local coordinates, let

$$p_{-t} = (\xi_{-t}, 0).$$

By the  $\lambda$ -lemma, the iterates of the  $m_i$  have the property that the intersections  $\varphi^{-t}m_i \cap U$  tend to  $W_-^s \cap U$  in the  $C^1$ -norm. That is, given  $\varepsilon > 0$ , there is a  $T > 0$  such that for  $t > T$  in local  $(\xi, \eta)$ -coordinates, the intersections takes the form

$$\varphi^t m_i \cap U : \xi = \chi_i(\eta),$$

with  $\chi_0(0) = \xi_t$ ,  $\chi_{-1}(0) = \xi_{t-1}$  and  $\max_{|\eta| < \delta} |\chi_i'(\eta)| < \varepsilon$ .

In local coordinates, the curve  $\ell$  takes the form

$$\eta = w_1^u(\xi) = w^u(x_- + \xi) - w^s(x_- + \xi).$$

Note that  $w_1^u(0) = 0$  and  $|(w_1^u)'(\xi)| < C$  for all  $\xi$  such that  $(\xi, w_1^u(\xi)) \in U$ . The area  $R$  bounded by  $W_-^u$ ,  $\varphi^t m_0$  and  $\ell$  contains  $\varphi^t C_t$ . Consequently

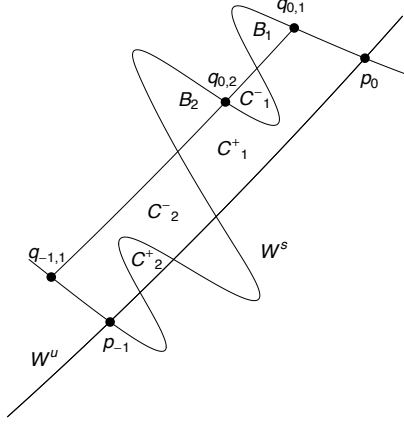
$$e^{\rho t} \text{area}(C_t) = \text{area}(\varphi^t C_t) \leq \text{area}(R).$$

The region  $R$  itself is contained in the triangle formed by the lines  $\eta = C\xi$ ,  $\eta = \xi_t + \frac{1}{\varepsilon}\xi$  and  $\eta = 0$ ; it follows that

$$\text{area}(R) \leq \frac{C}{2\varepsilon(1 - C\varepsilon)} \xi_t^2 = C' \xi_t^2.$$

The fact that  $\xi_t = \bar{\xi}\lambda_u^t + O(\lambda_u^{2t})$ , uniformly in  $t$ , proves the proposition. ■

As noted in the sketch of the proof, proposition 4.5 shows consequently that if  $\Omega(A) > 0$ , then  $v(q_t)$  increases towards  $V_-(x_-)$  as  $t \rightarrow -\infty$ . However, not all intersections of  $W_+^s$  with  $\varphi^{-t}\ell_t$  follow directly on an upward intersection of  $W_+^s$  with  $W_-^u$ ; we may have a configuration as the one depicted in figure 9.



**Figure 9:** Several intersections of  $W_+^s$  and  $\varphi^{-t}\ell_t$  following an upward intersection.

Define  $q_{0,1} = q_0, q_{0,2}, \dots, q_{0,K+1} = q_{-1,0}$  as the consecutive positive intersections of  $W_+^s$  with  $\varphi^{-t}\ell_t$  that follow  $p_t$ . Set  $q_{t,i} = \varphi^t(q_{0,i})$ .

Denote moreover by  $B_{t,1}, B_{t,2}$  the components of the region  $B_t$  that are such that for  $1 \leq i \leq K$ , the point  $q_{0,i}$  is contained in the boundary of  $B_{t,i}$ . Likewise, denote by  $C_{t,1}^-, C_{t,2}^-$  the components of  $C_t^-$  that are such that for  $1 \leq i \leq K$  the point  $q_{0,i+1}$  is contained in  $C_{t,i}^-$ .

**Proposition 4.7.** *Let  $1 \leq k \leq K$  be such that  $q_{0,k}$  follows  $p_0$ , but precedes any other upward intersection of  $W_+^s$  with  $W_-^u$ . Then*

$$v(q_{t-1}) - v(q_{t,k}) \geq e^{\rho t} \Omega(A).$$

*Proof.* The condition exactly implies that

$$e^{-\rho t} (v(q_{t,k}) - v(q_{t,1})) = \sum_{i=1}^{k-1} \text{area}(C_i^-) - \text{area}(B_i^-).$$

Using proposition 4.5 and the equality

$$\begin{aligned} v(q_{t-1}) - v(q_{t,i}) &= v(q_{t-1}) - v(q_t) + v(q_t) - v(q_{t,i}) \\ &= v(q_{t-1}) - v(q_t) + e^{\rho t} \left( \text{area}(B_{t,i}) - \text{area}(C_{t,i}^-) \right) \end{aligned}$$

then yields that

$$v(q_{t-1}) - v(q_{t,k}) = e^{\rho t} \Omega(A) + e^{\rho t} \left( \text{area}(C_t) - \sum_{i=1}^{k-1} \text{area}(C_{t,i}^-) + \sum_{i=1}^{k-1} \text{area}(B_{t,i}) \right).$$



As  $\bigcup_{i=1}^{k-1} C_{t,i}^- \subset C_t$ , the result follows. ■

### 4.3 Proof of theorem 1.

*Proof.* The first part of the proposition is immediate: if  $\mu > \mu_2$ , then by assumption 4, there are open neighbourhoods  $N_-$ ,  $N_+$  of  $x_-$  and  $x_+$  respectively, such that  $W_+^s \cap N_- \times \mathbb{R} = \emptyset$  and  $W_-^s \cap N_+ \times \mathbb{R} = \emptyset$ . But by assumption 3, optimal solutions correspond to trajectories on either  $W_-^s$  or  $W_+^s$ . It follows that all optimal state trajectories starting in  $N_-$  tend to  $x_-$ , and those starting in  $N_+$  tend to  $x_+$ .

To prove the second part of the proposition, let as before  $\mathbf{p}$  be an upward heteroclinic intersection of  $W_+^s$  and  $W_-^u$  such that  $p_0$  satisfies ..., let  $A$  be ... . Let moreover  $\ell$  be the line  $x = x_-$  and let  $q_{t,i}$ ,  $i = 1, \dots, k_t$  be the positive intersections of  $W_+^s$  with  $\ell$  that follow  $p_t$  and that precede the next upward intersection of  $W_+^s$  and  $W_-^u$ . Set  $q_t = q_{t,1}$ .

Using  $\Omega(A) \geq 0$  together with proposition 4.5, we obtain that

$$v(q_{t-1}) - v(q_t) > 0.$$

and therefore  $v(q_t)$  is an increasing sequence. Since  $q_t \rightarrow z_-$ , it follows from proposition B.9 that  $v(q_t) \rightarrow V_-(x_-)$ . We conclude that

$$\dots < v(q_t) < v(q_{t-1}) < v(q_{t-2}) < \dots < V_-(x_-).$$

Moreover, from proposition 4.7, it follows that for  $1 \leq i \leq k_t$ , we have

$$v(q_{t,i}) < v(q_{t-1}).$$

It is immediate that the remaining intersections of  $W_+^s$  and  $\ell_t$  yield even smaller values. But then no orbit on  $W_+^s$  yields a value that is as high as  $V_-(x_-)$ , and the proposition follows. ■

### 4.4 Proof of theorem 2.

*Proof.* For the first part of theorem 2, we make use of proposition B.8. Let  $N_-$  be an open neighbourhood of  $z_-$  that is such that  $W_-^s$  restricted to  $N_-$  can be represented as the graph of a function. Take  $\alpha \in W_-^s \cap N_-$  and  $\beta \in W_+^s$  such that  $x_\alpha = x_\beta$ . Then  $\Omega(A) > 0$  and

$$V_-(x_\alpha) = v_-(\alpha) < v_+(\beta) \leq V_+(x_\beta),$$

and  $\alpha$  cannot be optimal. But this implies that no solution tending towards  $z_-$  can be optimal.

The second part of theorem 2 follows from propositions B.9 and 4.5, as we find a sequence of points  $\{q_t\}$  which is such that  $q_t \rightarrow z_-$  as  $t \rightarrow -\infty$ , implying  $v(q_t) \rightarrow V_-(x_-)$ , and which satisfies for all  $t < -T_0$  the inequalities

$$a_t = e^{\rho t} \Omega(A) - C'' \lambda_u^{2t} \leq v(q_{t-1}) - v(q_t) \leq e^{\rho t} \Omega(A) + C'' \lambda_u^{2t} = b_t.$$

Using  $\Omega(A) < 0$  and the fact that  $e^\rho = \det D\varphi(z_-) = \lambda_u \lambda_s$ , we have

$$e^{-\rho t} b_t = \Omega(A) + C'' \left( \frac{\lambda_u}{\lambda_s} \right)^t,$$

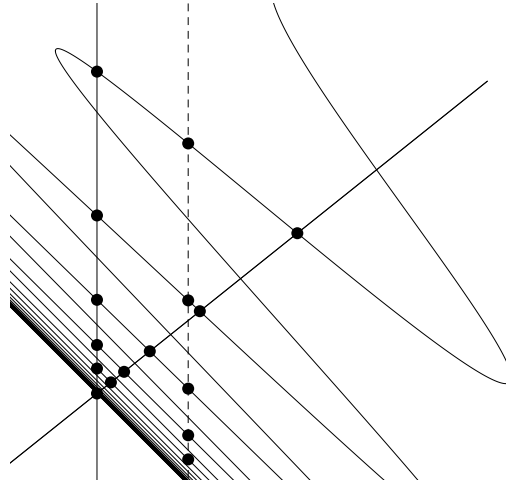
and there is some  $T' > 0$  such that for all  $t < -T'$  we have that  $b_t < 0$ . But then the sequence  $v(q_t)$  is eventually decreasing as  $t \rightarrow -\infty$ . Therefore there is some  $t_0$  such that  $v(q_{t_0}) > V_-(x_-)$ , and the state trajectory remaining at  $x = x_-$  cannot be optimal.

This implies that the optimal solution starting at  $x = x_-$ , converges to  $x = x_+$ . Consequently, no solution on  $W_-^s$  can be optimal, and therefore, by assumption 3, every optimal solution converges to  $x = x_+$ . ■

#### 4.5 Proof of theorem 3.

*Proof.* The optimality of the trajectory  $x_t = x_-$  for all  $t$  follows from theorem 1.

Fix a small neighbourhood  $N$  of  $x_-$ , and take  $x_0 \in N$  such that  $x_0 > x_-$ . Let  $\ell$  be the vertical line  $x = x_0$ , and denote, as before, by  $q_t$  the first intersection of  $W_+^s$  with  $\ell$ , starting from a point  $p_t$  of an upward orbit  $\mathbf{p}$  (see figure 10). Moreover, let  $t_0$  be such that for  $t \geq t_0$ , the curve segment from  $p_t$  to  $q_t$  is oriented in the same way as  $W_+^s$ , while for  $t < t_0$  that curve segment is oriented the opposite direction.



**Figure 10:** Intersections of  $W_+^s$  with the line  $x = x_-$  (solid) and the line  $x = x_0$  (dashed).

Then we have that the orientation of  $C_t$  is positive for  $t > t_0$ , while it is negative for  $t < t_0$ . Since by assumption  $\Omega(A) = 0$ , proposition 4.5 implies for all  $t > t_0$  that

$$v(q_{t-1}) > v(q_t),$$

while for  $t < t_0$ , we have

$$v(q_{t-1}) < v(q_t).$$

As  $v(q_t) \rightarrow V_-(x_0)$  as  $t \rightarrow -\infty$ , it follows that

$$v(q_{t_0}) > V_-(x_0),$$

and consequently that the optimal state trajectory starting at  $x_0$  will tend to  $x_+$ .

For the final claim of the theorem, note that

$$V_+(x_0) = \max\{v(q_{t_0}), v(q_{t_0-1})\},$$

and that  $t_0 = t_0(x_0)$  as function of  $x_0$  decreases towards  $-\infty$  as  $x_0 \rightarrow x_-$ .

■

## A First variation and Hamiltonian formalism.

In this section, discrete optimal control theory for systems with  $n$ -dimensional state space are formulated in a way that is suitable for the main purposes in this thesis. In particular, the ideas in section 1 are developed more formally.

To keep notation minimal the following notations for derivatives are used. For a function  $f : \mathbb{R}^{\times} \rightarrow \mathbb{R}$  with  $f = f(x)$  of the vector variable  $x$ , the following notation is employed

$$\frac{df}{dx} = \left( \frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right);$$

Likewise, for a function  $g = g(x, y)$ ,  $g : \mathbb{R}^{\times} \times \mathbb{R}^{\triangleright} \rightarrow \mathbb{R}$  the following is used

$$g_x = \left( \frac{\partial g}{\partial x_1} \quad \cdots \quad \frac{\partial g}{\partial x_n} \right)$$

etc.

**A.1 Definitions.** This section is begun by recalling some general results; at the same time, this will serve as an opportunity to introduce notation.

Time  $t$  is discrete, and takes values  $0, 1, 2, \dots$ . Let the state space  $\mathcal{X}$  and the control set  $\mathcal{U}$  be open and convex subsets of  $\mathbb{R}^n$ . On the state space, let the state dynamics

$$x_t = f(x_{t-1}, u_t) \tag{12}$$

be given, where  $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$  is a *smooth* function. A function is smooth if it has as many derivatives as necessary; ordinarily, we shall think of  $C^\infty$  functions, but the reader can substitute  $C^k$  with  $k > 0$  sufficiently large.

For technical convenience the special assumption is made that for all  $(x, u) \in \mathcal{X} \times \mathcal{U}$  we have

$$\det f_u(x, u) \neq 0.$$

Note that this encompasses a large class of practical optimal control problems, as well as all discrete calculus of variations problems, where  $f(x, u) = u$ .

If  $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$  and  $\mathbf{u} = \{u_t\}_{t=1}^{\infty}$  are sequences in  $\mathcal{X}$  and  $\mathcal{U}$  respectively, the pair  $(\mathbf{x}, \mathbf{u})$  is called *weakly admissible*, if we have that equation (12) holds for all  $t \geq 1$ . Let  $\mathcal{W}$  denote the set of weakly admissible pairs of sequences  $(\mathbf{x}, \mathbf{u})$ .

Let  $\rho > 0$  be a positive real number, and let  $g : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$  be another smooth function. For each integer  $T \geq 1$ , define a functional  $J_T : \mathcal{W} \rightarrow \mathbb{R}$  by setting

$$J_T(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^T g(x_{t-1}, u_t) e^{-\rho t}.$$

A sequence  $\mathbf{a} = \{a_t\}$  of positive real numbers is called *summable* if  $\sum_t a_t < \infty$ . A pair of weakly admissible sequences  $(\mathbf{x}, \mathbf{u}) \in \mathcal{W}$  is called *admissible*, if there is a positive summable sequence  $\mathbf{a}$  such that for all  $t \geq 1$

$$|g(x_{t-1}, u_t) e^{-\rho t}| \leq a_t.$$

The set of admissible pairs  $(\mathbf{x}, \mathbf{u})$  is denoted  $\mathcal{A}$ . Define the functional  $J : \mathcal{A} \rightarrow \mathbb{R}$  by

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t}.$$

Note that  $J$  is well-defined on  $\mathcal{A}$ .

The problem is to maximise an objective

$$J = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t}, \tag{13}$$

where  $\rho > 0$ , under the side condition that for all  $t \geq 1$  we have

$$x_t = f(x_{t-1}, u_t). \tag{14}$$

The state  $x_{t-1}$  and the control  $u_t$  take values in  $\mathcal{X}$  and  $\mathcal{U}$  respectively. Moreover, the initial state  $x_0$  is assumed to be given. Furthermore, we shall assume that  $f$  takes its values in  $\mathcal{X}$ ; this implies that there are no binding state constraints. Given this optimisation problem, in the rest of this chapter the followings are discussed: derivation of the necessary optimality conditions and the transversality condition for the optimisation problem; construction of the local and associated value functions corresponding to the optimisation problem.

**A.2 Variations.** Given an admissible pair  $(\mathbf{x}, \mathbf{u})$ , we consider *variations*

$$(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)) = (\mathbf{x} + \varepsilon \boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon \mathbf{v}(\varepsilon))$$

for  $0 \leq \varepsilon \leq 1$ . Throughout, it will be assumed that the variations are weakly admissible for all  $\varepsilon \in [0, 1]$ , and that for all  $t$  the functions  $\xi_t(\varepsilon)$  and  $v_t(\varepsilon)$  are smooth. We write

$$\xi_t^0 = \xi_t(0) \quad \text{and} \quad v_t^0 = v_t(0).$$

Then we can define a function  $j_T : [0, 1] \rightarrow \mathbb{R}$  by setting

$$j_T(\varepsilon) = J_T(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)).$$

To compute the derivative of  $j_T$  at  $\varepsilon = 0$ , note that if the pair  $(\mathbf{x} + \varepsilon \boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon \mathbf{v}(\varepsilon))$  is weakly admissible for every  $\varepsilon \in [0, 1]$ , then

$$x_t + \varepsilon \xi_t(\varepsilon) = f(x_{t-1} + \varepsilon \xi_{t-1}(\varepsilon), u_t + \varepsilon v_t(\varepsilon)).$$

Expanding and solving for  $v_t$  with the implicit function theorem then yields

$$v_t(\varepsilon) = f_u^{-1}(\xi_t(\varepsilon) - f_x \xi_{t-1}(\varepsilon)) + \varepsilon r, \tag{15}$$

where  $|r(x, u, \varepsilon, \xi_1, \xi_2)| \leq C|\xi|^2$ , uniformly in  $(x, u, \varepsilon)$ . Note that in equation (15) the arguments  $(x_{t-1}, u_t)$  has been omitted; we shall do this whenever there is no chance for confusion. Taking  $\varepsilon \rightarrow 0$  yields

$$v_t^0 = f_u^{-1}(\xi_t^0 - f_x \xi_{t-1}^0). \tag{16}$$

Moreover

$$\begin{aligned} \frac{j_T(\varepsilon) - j_T(0)}{\varepsilon} &= \frac{J_T(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)) - J_T(\mathbf{x}, \mathbf{u})}{\varepsilon} \\ &= \sum_{t=1}^T [g_x \xi_{t-1}^0 + g_u v_t^0] e^{-\rho t} + \mathcal{O}(\varepsilon) \\ &= \sum_{t=1}^T [(g_x - g_u f_u^{-1} f_x) \xi_{t-1}^0 + g_u f_u^{-1} \xi_t^0] e^{-\rho t} + \mathcal{O}(\varepsilon) \end{aligned} \tag{17}$$

Introduce the sequence of costates  $\mathbf{y} = \{y_t\}_{t=0}^\infty$  by setting

$$\begin{cases} y_t = -g_u f_u^{-1} & \text{for } t \geq 1, \text{ and} \\ y_0 = e^{-\rho} (g_x(x_0, u_1) + y_1 f_x(x_0, u_1)). \end{cases} \tag{18}$$

Note that the  $y_t$  are row vectors. Taking in (17) the limit  $\varepsilon \rightarrow 0$ , we obtain the following result.

**Proposition A.1.** *The right derivative  $D_+j_T(0)$  at  $\varepsilon = 0$  exists and equals*

$$D_+j_T(0) = y_0\xi_0^0 - e^{-\rho T}y_T\xi_T^0 + \sum_{t=2}^T [(g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}^0] e^{-\rho t}. \quad (19)$$

From this, we obtain the following easy corollary.

**Proposition A.2.** *If  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{W}$  is such that  $J_T(\mathbf{x}^*, \mathbf{u}^*) \geq J_T(\mathbf{x}, \mathbf{u})$  for all  $(\mathbf{x}, \mathbf{u}) \in \mathcal{W}$  such that  $x_0 = \alpha$ , then  $y_T = 0$  and*

$$e^\rho y_{t-1} = g_x(x_{t-1}, u_t) + y_t f_x(x_{t-1}, u_t) \quad (20)$$

for all  $1 \leq t \leq T$ .

Note that (20) holds for  $t = 1$  by definition of  $y_0$ . Any admissible pair  $(\mathbf{x}, \mathbf{u})$  that satisfies (20) for all  $t \geq 1$  is called *extremal*.

As  $J_T$  depends only on finitely many variables, there is no real problem in finding the first variation formula (19). To find the analogous expression for the infinite horizon functional  $J$ , we have to be able to interchange differentiation and infinite summation. This is permitted if the variations are strongly admissible.

**Definition 1 (Strong admissibility)** *An admissible variation  $(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$  is called strongly admissible, if there is a fixed positive summable sequence  $\{\mathbf{a}_t\}$  such that for all  $t \geq 1$  and for all  $\varepsilon \in (0, 1)$*

$$\varepsilon^{-1} \left| g(x_{t-1} + \varepsilon \xi_{t-1}(\varepsilon), u_t + \varepsilon v_t(\varepsilon)) - g(x_{t-1}, u_{t-1}) \right| e^{-\rho t} \leq a_t. \quad (21)$$

**Proposition A.3.** *Let the variation  $(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon))$  be strongly admissible, and let*

$$j(\varepsilon) = J(\mathbf{x}(\varepsilon), \mathbf{u}(\varepsilon)).$$

*Then the right-hand derivative  $D_+j(0)$  exists. Moreover, there is a positive summable sequence  $\{\mathbf{a}_t\}$  and a sequence  $\{R_t\}$  such that*

$$D_+j(0) = y_0\xi_0^0 - e^{-\rho T}y_T\xi_T^0 + \sum_{t=2}^T [(g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}^0] e^{-\rho t} + R_T$$

and

$$|R_T| \leq \sum_{t=T+1}^{\infty} a_t.$$

for every  $T \geq 1$ .

*Proof.* The conditions of strong admissibility precisely guarantee that the series is uniformly convergent and that we may pass to the limit  $\varepsilon \rightarrow 0$  under the summation sign; see for instance Knopp (1996).  $\blacksquare$

The formulation of proposition A.3 using a remainder term  $R_T$  allows to derive the transversality condition  $\lim_{t \rightarrow \infty} e^{-\rho t} y_t = 0$ .

**Proposition A.4.** *Let  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{A}$  be such that  $J(\mathbf{x}^*, \mathbf{u}^*) \geq J(\mathbf{x}, \mathbf{u})$  for all  $(\mathbf{x}, \mathbf{u}) \in \mathcal{A}$  with  $x_0 = \alpha$ , and let  $\mathbf{y}^*$  be the associated sequence of costates, given by (18). Assume that there is an  $\delta > 0$  such that for every variation  $\boldsymbol{\xi}$  which is such that  $\xi_0 = 0$  and  $|\xi_t| \leq \delta$  for all  $t \geq 0$  there is a sequence  $v(\varepsilon)$  such that  $(\mathbf{x}^* + \varepsilon \boldsymbol{\xi}, \mathbf{u}^* + \varepsilon v(\varepsilon))$  is strongly admissible for  $\varepsilon \in [0, 1]$ . Then*

$$e^\rho y_{t-1}^* = g_x(x_{t-1}^*, u_t^*) + y_t^* f_x(x_{t-1}^*, u_t^*) \quad (22)$$

for all  $t \geq 1$  and

$$\lim_{t \rightarrow \infty} e^{-\rho t} y_t^* = 0. \quad (23)$$

*Proof.* Since  $(\mathbf{x}^*, \mathbf{u}^*)$  maximises  $J$ , necessarily  $D_+ j(0) \leq 0$ . Using  $\xi_0 = 0$ , we see that

$$0 \geq D_+ j(0) = \sum_{t=2}^T [(g_x + y_t f_x - e^\rho y_{t-1}) \xi_{t-1}] e^{-\rho t} - e^{-\rho T} y_T \xi_T + R_T.$$

Let  $\text{sign}(x)$  denote the sign function

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Setting

$$\begin{cases} \xi_{t-1} &= \delta \text{sign}(g_x + y_t f_x - e^\rho y_{t-1}) & \text{for } 2 \leq t \leq T, \text{ and} \\ \xi_T &= -\delta \text{sign}(y_T) \end{cases}$$

yields that

$$\delta \sum_{t=2}^T |g_x + y_t f_x - e^\rho y_{t-1}| e^{-\rho t} + \delta e^{-\rho T} |y_T| \leq |R_T| \leq \sum_{t=T+1}^{\infty} a_t.$$

Since this inequality has to hold for all  $T \geq 2$ , and since  $\sum_{t=T+1}^{\infty} a_t \rightarrow 0$  as  $T \rightarrow \infty$ , the result follows.  $\blacksquare$

The equation 23 is a so called *transversality condition*. The optimality conditions for an infinite horizon problem are identical to those of a finite horizon problem with the exception of the transversality condition. Hence, in solving the problem the most important change is how we deal with the need for the transversality conditions. Although economically intuitive, as shown here, 23 can only be directly derived by variational approach that considers specific perturbations. However, it has to be noticed that there is no need to assume that the present value of the stock at the infinity should be zero.

**A.3 The discrete Hamiltonian.** The results of the previous subsection can be formulated very elegantly if we introduce the discrete Pontryagin and Hamilton functions. The former is given as

$$P(x, y, u) = g(x, u) + yf(x, u).$$

In terms of  $P$ , equations (12), (18) and (22) can be formulated as

$$0 = P_u, \quad x_t = P_y, \quad e^\rho y_{t-1} = P_x;$$

here the argument of the derivatives of  $P$  is always  $(x_{t-1}, y_t, u_t)$ .<sup>1</sup>

We shall make the assumption that  $P_{uu}$  is always negative definite. Then equation  $P_u = 0$  can be solved for  $u = U(x, y)$ , allowing to introduce the discrete Hamilton function by

$$\begin{aligned} H(x, y) &= P(x, y, U(x, y)) \\ &= g(x, U(x, y)) + yf(x, U(x, y)). \end{aligned} \quad (24)$$

Note, for later reference, that since  $P_u(x, y, U(x, y)) = 0$  identically in  $(x, y)$ , we have that

$$g(x, U(x, y)) = H(x, y) - yH_y(x, y). \quad (25)$$

In order to find state-costate dynamics in generation function form, the necessary equations are written in the (present-value) Hamiltonian form

$$x_t = H_y(x_{t-1}, y_t), \quad e^\rho y_{t-1} = H_x(x_{t-1}, y_t). \quad (26)$$

By extension, the pair  $(\mathbf{x}, \mathbf{y})$  is called extremal, if equation (26) is satisfied for every  $t \geq 1$ . Note that if  $(\mathbf{x}, \mathbf{y})$  is extremal, and if a control sequence  $\mathbf{u}$  is obtained by setting  $u_t = U(x_{t-1}, y_t)$  for  $t \geq 1$ , then the pair  $(\mathbf{x}, \mathbf{u})$  is extremal in the former sense.

**A.4 The phase map.** The next step is to solve the present-value Hamiltonian equations 26 for a phase map  $\varphi$  that satisfies

$$(x_t, y_t) = \varphi(x_{t-1}, y_{t-1})$$

for every  $t \geq 1$ .

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<sup>1</sup>This is weaker than the maximum principle, however, as it will be assumed that  $P$  is concave in  $u$  this is fine.



Introduce first the costate space

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^n \mid y = -g_u(x, u) f_u^{-1}(x, u), \quad (x, u) \in \mathcal{X} \times \mathcal{U} \right\}.$$

Since  $\mathcal{X} \times \mathcal{U}$  is connected and

$$Y(x, u) = -g_u(x, u) / f_u(x, u)$$

is continuous, it follows that  $\mathcal{Y} \subset \mathbb{R}^n$  is connected as well. We shall call  $\mathcal{M} = \mathcal{X} \times \mathcal{Y}$ , the domain of definition of  $\varphi$ , the *phase space*, to distinguish it from the *state space*  $\mathcal{X}$ .

Let  $F : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  be given as

$$F(z, \varphi) = \begin{pmatrix} \varphi_1 - H_y(x, \varphi_2) \\ e^\rho y - H_x(x, \varphi_2) \end{pmatrix},$$

where  $z = (x, y) \in \mathcal{M}$ . Let the map  $\varphi$  be implicitly defined by the equation

$$F(z, \varphi) = 0$$

**Proposition A.5.** *If  $H_{xy}$  is invertible, then the equation  $F = 0$  can be solved for  $\varphi = \varphi(z)$ . Moreover,*

$$D\varphi = \begin{pmatrix} H_{xy} - H_{yy}H_{xy}^{-1}H_{xx} & e^\rho H_{yy}H_{xy}^{-1} \\ -H_{xy}^{-1}H_{xx} & e^\rho H_{xy}^{-1} \end{pmatrix}$$

and  $\det D\varphi = e^{n\rho}$ .

*Proof.* Compute

$$D_\varphi F = \begin{pmatrix} I & -H_{yy} \\ 0 & -H_{xy} \end{pmatrix}.$$

Under the assumption of the lemma, this matrix is invertible at  $(z_0, \varphi_0)$ , and

$$(D_\varphi F)^{-1} = \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix}.$$

By the implicit function theorem, the solution  $\varphi = \varphi(z)$  of  $F(z, \varphi) = 0$  satisfies

$$\begin{aligned} D\varphi &= -(D_\varphi F)^{-1} D_z F \\ &= \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix} \begin{pmatrix} H_{xy} & 0 \\ H_{xx} & -e^\rho I \end{pmatrix} = \begin{pmatrix} H_{xy} - H_{yy}H_{xy}^{-1}H_{xx} & e^\rho H_{yy}H_{xy}^{-1} \\ -H_{xy}^{-1}H_{xx} & e^\rho H_{xy}^{-1} \end{pmatrix}. \end{aligned}$$

Moreover

$$\det D\varphi = \det \begin{pmatrix} I & -H_{yy}H_{xy}^{-1} \\ 0 & -H_{xy}^{-1} \end{pmatrix} \det \begin{pmatrix} H_{xy} & 0 \\ H_{xx} & -e^\rho I \end{pmatrix} = e^{n\rho}. \quad \blacksquare$$

Summarising, we have found a phase map  $\varphi$  such that the orbits  $\mathbf{z} = \{z_t\} = \{(x_t, y_t)\}$  of  $\varphi$  are extremal and such that  $\varphi$  satisfies

$$\varphi_1(x, y) = H_y(x, \varphi_2(x, y)), \quad e^\rho y = H_x(x, \varphi_2(x, y)). \quad (27)$$

**A.5 Example.** As a simple example, consider the standard growth problem to maximise

$$J = \sum_{t=1}^{\infty} e^{-\rho t} \log c_t$$

subject to the capital dynamics

$$k_t = f(k_{t-1}) - c_t, \quad k_0 = \alpha.$$

The discrete Pontryagin function reads  $P(k, p, c) = \log c + p(f(k) - c)$ . Consumption  $c$  is determined by  $P_c = \frac{1}{c} - p = 0$ , leading to  $C(p) = 1/p$ . The discrete Hamilton function takes the form

$$H(k, p) = -\log p - 1 + pf(k).$$

The necessary conditions read as

$$k_t = H_p = f(k_{t-1}) - \frac{1}{p_t}, \quad e^\rho p_{t-1} = H_k = p_t f'(k_{t-1}).$$

If  $f'(k_{t-1}) \neq 0$ , these equations can be solved for the phase map

$$(k_t, p_t) = \varphi(k_{t-1}, p_{t-1}) = \left( e^\rho \frac{f(k_{t-1})}{f'(k_{t-1})}, \frac{e^\rho}{f'(k_{t-1})} p_{t-1} \right).$$

This is consistent with the condition of proposition A.5, which reads here

$$H_{kp} = f'(k) \neq 0.$$

## B Local and associated value functions

In this section, “local value functions” are constructed from the stable manifolds of the saddle points of the phase map  $\varphi$ . As in section A, the ideas are developed for systems with an  $n$ -dimensional state space.

From now on, the investigation will be restricted to the case that the optimal state trajectories  $\mathbf{x}$  all converge to a long term steady state. These states correspond to fixed points of the phase map  $\varphi$ ; since  $\det D\varphi = e^{n\rho} > 1$ , such fixed points are necessarily saddles. In proper coordinates around a saddle  $\bar{z}$ , the linear map  $D\varphi$  takes the form

$$D\varphi(\bar{z}) = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix},$$

where all eigenvalues of  $\Lambda_s$  are inside the unit circle, while all eigenvalues of  $\Lambda_u$  are outside. The case that the dimensions of the stable and the unstable (generalised) linear eigenspaces are equal is of special interest. Orbits on the stable manifold of these saddles, i.e. the manifolds of orbits tending to these saddle fixed points, are natural candidates for maximisers, as for these orbits the transversality condition (23) is automatically satisfied.

Moreover, if the stable manifold  $W^s$  can be represented as the graph of a function  $y = \psi(x)$ , then the value function  $V$  of the trajectories in  $W^s$  should satisfy  $\partial V / \partial x_i = \psi_i$  for all  $i$ . To recover the value function from  $\psi$  requires an integration, and an argument that the integration is indeed possible: the components of  $\psi$  have to satisfy an integrability condition. To formulate this condition, and to demonstrate that it is satisfied, the language of differential forms is introduced. A couple of proposition dealing with the comparison of values at different points will be demonstrated; in the proofs, differential forms will be an indispensable tool. For more information, the reader is referred to Spivak (1965) or Arnol'd (1989).

**B.1 Invariant manifolds.** Let  $\bar{z} = (\bar{x}, \bar{y})$  be a fixed saddle point of  $\varphi$ . The linear stable and unstable manifolds are the eigenspaces  $E^s$  and  $E^u$  of the eigenvalues that are lesser and greater than one respectively. The *stable manifold*  $W^s$  and the *unstable manifold*  $W^u$  of  $\bar{z}$  are defined as the set of all points  $z \in \mathcal{M}$  such that the forward orbit respectively the backward orbit of  $\varphi$  through  $z$  tends to  $\bar{z}$ :

$$\begin{aligned} W^s &= \{z \in \mathcal{X} \times \mathcal{Y} \mid \varphi^t(z) \rightarrow \bar{z}, t \rightarrow \infty\} \\ W^u &= \{z \in \mathcal{X} \times \mathcal{Y} \mid \varphi^t(z) \rightarrow \bar{z}, t \rightarrow -\infty\} \end{aligned}$$

The basic result about the sets  $W^s$  and  $W^u$  is the invariant manifold theorem (e.g. Hirsch et al., 1977), which states that  $W^s$  and  $W^u$  are smooth manifolds, thus justifying the names.

**Invariant Manifold Theorem.** *Let  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  be a  $C^k$  invertible map,  $k \geq 1$ , and let  $\bar{z}$  be a saddle fixed point. Then the sets  $W^s$  and  $W^u$  are both  $C^k$ -smooth manifolds, tangent to the corresponding eigenspaces.*

A value function can be associated to the stable manifold  $W^s$ ; in order to show this, some concepts have to be introduced. First, note that  $\mathcal{M} = \mathcal{X} \times \mathcal{Y} \subset T^*(\mathcal{X})$ , where  $T^*(\mathcal{X}) = \mathcal{X} \times \mathbb{R}^n$  is the *cotangent bundle* of  $\mathcal{X}$ . On  $T^*(\mathcal{X})$ , a canonical 1-form  $\eta = y dx = \sum y_i dx_i$  is defined as well as its derivative  $\omega = d\eta = dy \wedge dx$ , the *symplectic 2-form*.

The symplectic form  $\omega$  is said to vanish on a submanifold  $\mathcal{N}$  of  $\mathcal{M}$ , if for any point  $z \in \mathcal{N}$  and any tangent vectors  $v, w$  to  $\mathcal{N}$  at  $z$  equality  $\omega_z(v, w) = 0$  holds. A  $n$ -dimensional submanifold  $\mathcal{N}$  of  $\mathcal{M}$  is called *Lagrangian*, if  $\omega$  vanishes on  $\mathcal{N}$ . Being Lagrangian is an integrability condition: to see this assume that  $\mathcal{N}$  can be represented as the graph  $y = \psi(x)$  of a function  $\psi : \mathcal{X} \rightarrow \mathcal{Y}$ . If  $\mathcal{N}$  is Lagrangian, then there exists a function  $V : \mathcal{X} \rightarrow \mathbb{R}$  such that  $dV = \psi dx = \sum_i \psi_i dx_i$ .

Recall that since  $\mathcal{X}$  is convex, it is topologically trivial, and hence there is a function  $V : \mathcal{X} \rightarrow \mathbb{R}$  that satisfies

$$\frac{\partial V}{\partial x_i} = \psi_i$$

for  $i = 1, 2, \dots, n$  if and only if the integrability conditions

$$\frac{\partial \psi_j}{\partial x_i} - \frac{\partial \psi_i}{\partial x_j} = 0$$

are satisfied for all  $i, j$ .

Being Lagrangian expresses the same thing. To see this, let  $\Psi : \mathcal{X} \rightarrow \mathcal{M}$  be given by  $\Psi(x) = (x, \psi(x))$ . The manifold  $\mathcal{N}$  is Lagrangian if and only if  $\Psi^*\omega = 0$ . Compute

$$\begin{aligned} 0 &= \Psi^*\omega \\ &= \sum_i d\psi_i(x) \wedge dx_i &= \sum_i \sum_j \frac{\partial \psi_i}{\partial x_j} dx_j \wedge dx_i \\ &= \sum_i \sum_{j < i} \left( \frac{\partial \psi_j}{\partial x_i} - \frac{\partial \psi_i}{\partial x_j} \right) dx_i \wedge dx_j; \end{aligned}$$

and the classical integrability conditions have been recovered.

The phase map preserves the symplectic form up to a constant factor  $e^\rho$ . Of course, the presence of this factor is an echo of the fact that we formulate the optimisation problem in current value variables.

**Proposition B.1.** *We have that  $\varphi^*\omega = e^\rho\omega$ . Moreover, if  $\psi$  satisfies  $\varphi = e^{\rho/2}\psi$ , then  $\psi^*\omega = \omega$ .*

*Proof.* Using that  $\varphi_1 = H_y(x, \varphi_2)$  (equation (27)), we compute

$$\varphi^*\omega = d\varphi_2 \wedge d\varphi_1 = d\varphi_2 \wedge (H_{xy} dx + H_{yy} dy) = H_{xy} d\varphi_2 \wedge dx.$$

Analogously, using  $e^\rho y = H_x(x, \varphi_2)$ , we find

$$e^\rho dy \wedge dx = dH_x(x, \varphi_2) \wedge dx = (H_{xx} dx + H_{xy} d\varphi_2) \wedge dx = H_{xy} d\varphi_2 \wedge dx.$$

The proof for  $\psi$  runs similarly, using equations (27) in the form

$$\psi_1 = e^{-\rho/2} H_y(x, e^{\rho/2} \psi_2) \quad \text{and} \quad e^\rho y = H_x(x, e^{\rho/2} \psi_2).$$

This proves the proposition. ■

**Definition 2 (Symplectic transformation)** *A differential map  $\psi$  that preserves the 2-form  $\omega$ , that is, which is such that  $\psi^*\omega = \omega$ , is called symplectic.*

The fact that  $\psi$  is symplectic has implications for the spectrum of the Jacobian matrix  $D\psi$ .

**Proposition B.2.** *If  $\psi = e^{-\rho/2}\varphi$  is symplectic, then if  $\lambda$  is an eigenvalue of  $D\psi$ , so is  $1/\lambda$ . Consequently, if  $\lambda$  is an eigenvalue of the phase map  $D\varphi$ , then so is*

$$\frac{e^\rho}{\lambda}.$$

*Proof.* See Abraham and Marsden (1978), proposition 3.1.12, p. 168. ■

The next thing to show is that the invariant manifolds of a saddle point  $\bar{z}$  satisfy the integrability condition.

**Proposition B.3.** *Let  $\bar{z}$  be a saddle fixed point of the phase map  $\varphi$ , and let  $W^s$  and  $W^u$  be the associated stable and unstable manifolds. Assume that both  $W^s$  and  $W^u$  are  $n$ -dimensional. Then the symplectic form  $\omega$  vanishes on  $W^s$  and  $W^u$ .*

*Proof.* Assume that  $W^s$  is not Lagrangian; that is, assume that there are vectors  $v, w$ , tangent to  $W^s$  at some point  $z \in W^s$  such that  $|z - \bar{z}| \leq \varepsilon$ , for which  $\omega(v, w) \neq 0$ ; we may assume that  $\omega(v, w) = 1$ . Denote, as above, the restriction of  $D\varphi$  to the stable eigenspace  $E^s$  by  $\Lambda_s$ , and let  $|\Lambda_s| = \lambda_s < 1$ , where  $|\Lambda_s|$  is the matrix norm associated to the Euclidean vector norm  $|\cdot|$ . Note that for  $v$  tangent to  $W^s$  at  $z$ , we have

$$|D\varphi^t(z)v| \leq (\lambda_s + C\varepsilon)^t |v| < c^t |v|$$

for some  $0 < c < 1$ . Consequently

$$1 \leq e^{\rho t} = e^{\rho t} \omega(v, w) = (\varphi^t)^* \omega(v, w) = \omega(D\varphi^t v, D\varphi^t w) \leq c^{2t} |v| |w|.$$

But for  $t > 0$  sufficiently large, this entails a contradiction.

If  $W^s$  is  $n$ -dimensional, there are  $n$  eigenvalues  $\lambda_i$  of  $D\varphi(\bar{z})$  such that  $|\lambda_i| < 1$ ,  $i = 1, \dots, n$ . Proposition B.2 implies that the other  $n$  eigenvalues then have to satisfy  $|\lambda_{n+i}| > e^\rho$ ,  $i = 1, \dots, n$ . It follows that  $|\Lambda_u| = \lambda_u > e^\rho$ , and that for  $v$  tangent to  $W^u$  at  $z$ , we have

$$|D\varphi^{-t}(z)v| \leq (\lambda_u^{-1} + C\varepsilon)^t |v| < c^t |v|$$

for some  $0 < c < e^{-\rho}$ . It follows then that

$$1 = \omega(v, w) = e^{\rho t} (\varphi^{-t})^* \omega(v, w) = e^{\rho t} \omega(D\varphi^{-t} v, D\varphi^{-t} w) \leq e^{\rho t} c^{2t} |v| |w| \leq c^t |v| |w|.$$

This also leads to a contradiction. ■

We have shown that if  $W^s$  (and  $W^u$ ) can be parametrised as the graph of a function  $y : \mathcal{X} \rightarrow \mathcal{Y}$ , then there is a function  $W : \mathcal{X} \rightarrow \mathbb{R}$  such that  $dW = y dx$ . The next thing to demonstrate is that up to a constant  $W(x)$  is actually the value function for orbits of  $\varphi$  starting at  $(x, y(x))$ . We do this by showing that the value function  $\tilde{V}$  for orbits on  $W^s$  is differentiable and satisfies  $d\tilde{V} = y dx$ .

To formulate this more precisely, choose a smooth parametrisation  $z : \mathbb{R}^n \rightarrow \mathcal{X} \times \mathcal{Y}$  of the stable manifold  $W^s$  of  $\bar{z}$ . Write

$$z(\sigma) = (x(\sigma), y(\sigma)).$$

and assume that  $z(0) = \bar{z}$ . Introduce

$$\sigma_{t+1} = \psi(\sigma_t)$$

be the smooth map induced by  $\varphi$  on  $\mathbb{R}^n$ . That is, if  $z_t = (x_t, y_t) = (x(\sigma_t), y(\sigma_t))$  is an orbit of  $\varphi$  on  $W^s$ , then

$$\varphi(z(\sigma_t)) = \varphi(x_t, y_t) = (x_{t+1}, y_{t+1}) = z(\sigma_{t+1}) = z(\psi(\sigma_t)).$$

Let  $z(\sigma_0) = (x(\sigma_0), y(\sigma_0)) \in W^s$ . If  $dx/d\sigma(\sigma_0) \neq 0$ , then on a neighbourhood of  $x(\sigma)$  we can find a function  $y = y(x)$  such that  $y(\sigma) = y(x(\sigma))$  for  $\sigma$  close to  $\sigma_0$ . The value of the orbit  $(x_t, y_t)$  starting at  $z = (x, y(x))$  is then given as

$$\tilde{V}(x) = \sum_{t=1}^{\infty} g(x_{t-1}, U(x_{t-1}, y_t)) e^{-\rho t}.$$

**Proposition B.4.** *If  $dx/d\sigma(\sigma_0) \neq 0$ , then  $d\tilde{V}/dx(x(\sigma_0)) = y(\sigma_0)$ .*

*Proof.* Let  $z_0 + \varepsilon\zeta_0(\varepsilon)$  be an arbitrary curve of initial points in  $W^s$ ; let  $\mathbf{z} + \varepsilon\boldsymbol{\zeta}(\varepsilon)$  be the trajectories in  $W^s$  defined by these initial points, and let  $(\mathbf{x} + \varepsilon\boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon\boldsymbol{\nu}(\varepsilon))$  be the corresponding state-control trajectories.

Let

$$j(\varepsilon) = J(\mathbf{x} + \varepsilon\boldsymbol{\xi}(\varepsilon), \mathbf{u} + \varepsilon\boldsymbol{\nu}(\varepsilon));$$

then we have that

$$j(\varepsilon) = \tilde{V}(x_0 + \varepsilon\xi_0(\varepsilon)).$$

If we assume that  $j$  is differentiable for all curves of initial points through  $z_0$ , then

$$j'(0) = \frac{d\tilde{V}}{dx}(x_0)\xi_0(0).$$

But it follows from proposition A.3 that

$$j'(0) = y_0\xi_0(0).$$

Since  $\xi_0(0)$  is arbitrary, the theorem follows.

It remains therefore to show that  $j$  is differentiable at  $\varepsilon = 0$ ; this will follow from proposition A.3. We turn to verifying the hypotheses of that proposition.

If  $\Lambda_s$  denotes the stable part of  $D\varphi(\bar{z})$ , then it is possible to choose the parametrising coordinate  $\sigma$  of the stable manifold  $W^s$  such that

$$\psi(\sigma) = \Lambda^s \sigma + O(|\sigma|^2).$$

Define

$$u(\sigma) = U(x(\psi^{-1}(\sigma)), y(\sigma))$$

and note that with this definition  $u(\sigma_t) = U(x_{t-1}, y_t) = u_t$ .

Let  $\mathbf{z} = \{z_t\}$  be the orbit in  $W^s$  starting at  $z_0$ , and let  $\boldsymbol{\sigma} = \{\sigma_t\}$  be its associated orbit of parameters  $z_t = z(\sigma_t)$ . Let moreover  $c_0 : (-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}^n$  be a smooth curve of the form

$$c_0(\varepsilon) = \sigma_0 + \varepsilon\tau_0,$$

with  $\tau_0 \in \mathbb{R}^n$ . Consider the forward iterates  $c_t = \psi^t(c_0)$ , parametrised as

$$c_t(\varepsilon) = \sigma_t + \varepsilon\tau_t(\varepsilon),$$

where  $\sigma_t = \psi^t(\sigma_0)$ . Note that for all  $t \geq 1$

$$\psi(\sigma_{t-1}) + \varepsilon\tau_t(\varepsilon) = \psi(\sigma_{t-1} + \varepsilon\tau_{t-1}(\varepsilon))$$

and hence

$$\begin{aligned} \tau_t(\varepsilon) &= D\psi(\sigma_{t-1})\tau_{t-1}(\varepsilon) + \varepsilon\Psi_1(\varepsilon, \sigma_{t-1}, \tau_{t-1}) \\ &= \Lambda^s \tau_{t-1}(\varepsilon) + \varepsilon\Psi_1(\varepsilon, \sigma_{t-1}, \tau_{t-1}) + \Psi_2(\sigma_{t-1}, \tau_{t-1}), \end{aligned}$$

where  $|\Psi_1(\varepsilon, \sigma, \tau)| \leq C|\tau|^2$  and  $|\Psi_2(\sigma, \tau)| \leq C|\sigma||\tau|$ . Choosing  $T > 0$  sufficiently large and  $\varepsilon_0 > 0$  sufficiently small, this implies that for  $t > T$

$$|\tau_t(\varepsilon)| \leq (\lambda^s + \delta)|\tau_{t-1}(\varepsilon)|,$$

where  $0 < \lambda^s + \delta < 1$ . As a consequence

$$|\tau_t(\varepsilon)| \rightarrow 0,$$

uniformly in  $\varepsilon$ , as  $t \rightarrow \infty$ .

It follows that the curves  $z(c_t)$  in  $W^s$  take the form

$$z(c_t) = z(\sigma_t + \varepsilon\tau_t(\varepsilon)) = z_t + \varepsilon\zeta_t(\varepsilon) = (x_t + \varepsilon\xi_t(\varepsilon), y_t + \varepsilon\eta_t(\varepsilon)),$$

and  $|\zeta_t(\varepsilon)| \rightarrow 0$  uniformly in  $\varepsilon$ . Using the control map  $u$ , we find that the associated control sequence is also of the form  $u_t + \varepsilon v_t(\varepsilon)$  with  $v_t(\varepsilon) \rightarrow 0$  uniformly in  $\varepsilon$ . As a consequence, the family  $(\mathbf{x} + \varepsilon\xi(\varepsilon), \mathbf{u} + \varepsilon v(\varepsilon))$  is extremal and strongly admissible. We conclude that proposition A.3 can be applied.  $\blacksquare$

We can now define a value function  $\bar{V}$  associated to the stable manifold  $W^s$  of  $\bar{z}$  by setting

$$\bar{V}(x) = \sup\{j(\sigma) \mid x(\sigma) = x\}. \quad (28)$$

**B.2 Regions and area.** Here, we define simple regions, oriented simple regions, regions, as well as their area. Regions will be the domains of the integral  $\Omega(A) = \int_A \omega$ , which is used extensively in the following.

We call a *simple region* in  $\mathbb{R}^2$  any simply connected bounded submanifold  $A$  of  $\mathbb{R}^2$  which is such that its boundary  $\partial A$  is a closed piecewise smooth curve. An *oriented simple region* is a pair  $(A, \mu)$ , where  $A$  is a simple region and  $\mu = \pm 1$  is the orientation; we shall often fail to give the orientation explicitly. We define the oriented simple region  $-A$  as follows:

$$(-A, \mu) = (A, -\mu).$$

A region  $A$  in  $\mathbb{R}^2$  is a formal sum of oriented simple regions  $A_i$ :

$$A = A_1 + A_2 + \cdots + A_k.$$

Moreover, we set

$$A - A = 0;$$

that is, equal regions with opposite orientations cancel. It is evident how these concepts generalise to subsets of 2-dimensional oriented manifolds that are diffeomorphic to  $\mathbb{R}^2$ ; these will be called surface regions, if we want to stress the difference.

If  $(A, \mu)$  is an oriented simple region, and  $z$  the boundary curve of  $A$ , then  $z$  is oriented *consistently* with  $A$ , if the winding number  $n_p(z)$  of  $z$  relative to any point  $p$  in the interior of  $A$  is  $\mu$ ; recall that if  $\vartheta_p$  is the angle of  $z - p$  with the positive horizontal axis, the winding number is defined as

$$n_p(z) = \frac{1}{2\pi} \int_z d\vartheta_p.$$

We always choose the boundary curve  $\partial A$  of an oriented region consistently with the orientation of  $A$ . Inversely, to any closed curve  $z$  without self-intersection, we associate an oriented simple region  $A$  such that  $\partial A = z$ . More generally, if  $z$  is a piecewise smooth curve with a finite number of self-intersections then it divides the plane in a finite number of bounded regions  $A_i$ , which are such that the boundary of a  $A_i$  is made up of segments of the curve  $z$ . Again, we choose the orientation of  $A_i$  consistently with that of its boundary arcs. Let now  $I$  be the index set of positively oriented simple regions  $A_i$ ; that is, if  $i \in I$ , then the orientation of  $A_i$  is positive, otherwise it is negative. Define then the positively and negatively oriented parts of  $A$  by respectively

$$A^+ = \sum_{i \in I} A_i, \quad \text{and} \quad A^- = \sum_{i \notin I} A_i.$$

It is clear how these definitions extend to surface regions.

If  $(A, \mu)$  is an oriented simple region, the area of  $A$  is given as

$$\text{area}(A) = \int_A dx dy = \mu \int_A dx \wedge dy = -\mu \int_A \omega.$$

Introduce for simple regions the map  $\Omega$  as

$$\Omega(A) = \int_A \omega = -\mu \text{area}(A).$$

If the  $A_i, i = 1, 2, \dots$  are simple regions and  $A = \sum A_i$ , then we define

$$\Omega(A) = \sum_i \Omega(A_i),$$



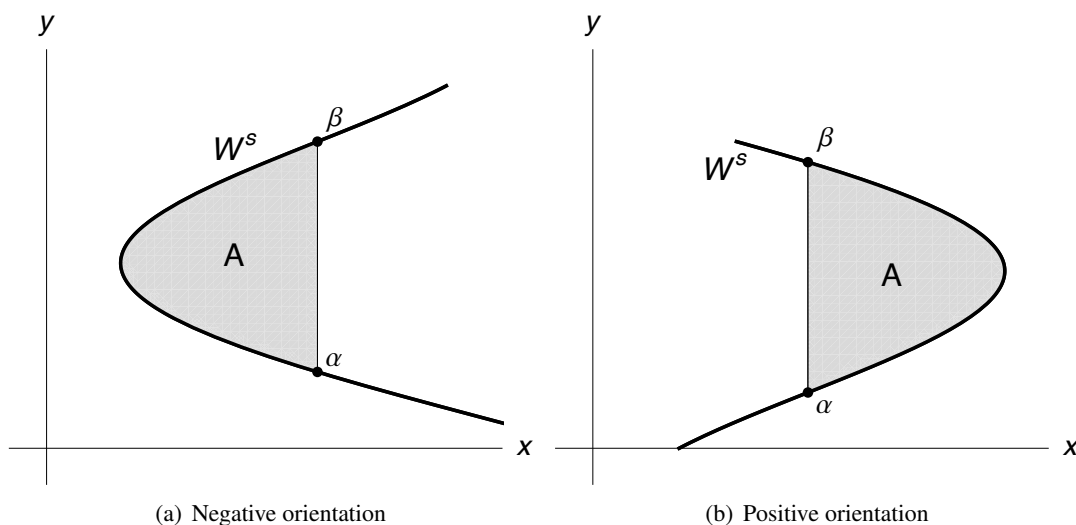
and we call the simple regions  $A_i$  the *simple components* of the region  $A$ . In particular we have

$$\Omega(A) = \sum_i -\mu_i \text{area}(A_i).$$

Moreover, since for  $\eta = y dx$  we have that  $d\eta = \omega$ , for a simple region  $A$  we can also write

$$\text{area}(A) = -\mu \int_A \omega = -\mu \int_{\partial A} \eta.$$

**B.3 The area rule.** In this subsection a result is derived that links the location of discontinuities of the derivative of  $\bar{V}$  to the geometry of the manifold  $W^s$ . First we recall some more geometrical facts.



**Figure 11:** *The area rule. Let  $z$  be the curve from  $\alpha$  to  $\beta$  along  $W^s$  and from  $\beta$  to  $\alpha$  along the straight connecting line. If  $z$  surrounds  $A$  negatively, then  $v(\beta) - v(\alpha) = \text{area}(A)$ ; if positively, then  $v(\beta) - v(\alpha) = -\text{area}(A)$ .*

Assume that we have a saddle fixed point  $\bar{z}$  and a curve  $z = (x, y) : [0, 1] \rightarrow \mathcal{M}$  on the stable manifold  $W^s$  of  $\bar{z}$  such that  $z(0) = \bar{z}$ . Writing  $\alpha = z(1)$ , equation (9) implies that the value  $v(\alpha)$  of the orbit starting at  $\alpha$  is given by the following integral:

$$v(\alpha) = v(\bar{z}) + \int_z \eta = v(\bar{z}) + \int_0^1 y(\sigma)x'(\sigma) d\sigma;$$

here  $v(\bar{z})$  is the value of the constant orbit  $\bar{z}$ . Note that  $v(\alpha)$  does not depend on the curve  $z$ ; if  $z_1$  and  $z_2$  are two curves on  $W^s$  that connect  $\bar{z}$  to  $\alpha$ , then  $z_1 - z_2$  is a closed curve that encloses an oriented surface region  $A$ . By Stokes' theorem,

$$\int_{z_1} y dx - \int_{z_2} y dx = \int_A dy \wedge dx = \int_A \omega = 0,$$

since  $W^s$  is Lagrangian.

Consider now a curve  $z_1 : [0, 1] \rightarrow \mathcal{M}$  lying on  $W^s$  and connecting two phase points  $\alpha = z_1(0)$  and  $\beta = z_1(1)$  which have equal state coordinates:

$$x_1(0) = x_1(1),$$

whereas

$$x_1(\sigma) \neq x_1(0) = x_1(1) \quad \text{for all } 0 < \sigma < 1.$$

Let moreover  $z_2$  be the straight line joining  $\beta$  to  $\alpha$ , and let  $z = z_1 + z_2$ . Then  $z$  is a closed curve without self-intersections; let  $A$  be an oriented surface region that is bounded by  $z$ .

Using Stokes' theorem and  $\int_{z_2} y dx = 0$  yields

$$v(\beta) - v(\alpha) = \int_{z_1} y dx = \int_z y dx = \int_A \omega.$$

In the special case that  $\mathcal{M}$  is 2-dimensional, if the orientation of  $A$  is positive, then

$$v(\beta) - v(\alpha) = \int_A \omega = -\text{area}(A).$$

If however  $A$  is negatively oriented, then

$$v(\beta) - v(\alpha) = \text{area}(A).$$

Both relations are illustrated in figure 11.

**B.4 Results about differential forms.** Proposition B.1 states that the phase map  $\varphi$  leaves the symplectic form  $\omega = dy \wedge dx$  invariant up to a factor. A directly related result can be derived for the canonical 1-form  $\eta = y dx$ .

Introduce the function  $G : \mathcal{M} \rightarrow \mathbb{R}$  by setting

$$G(x, y) = g\left(x, U(x, \varphi_2(x, y))\right);$$

note that with this definition

$$G(x_t, y_t) = g(x_t, U(x_t, y_{t+1})) = g(x_t, u_{t+1}).$$

If  $\mathbf{y}$  is the sequence of costates associated to an extremal pair  $(\mathbf{x}, \mathbf{u})$ , and if  $z_t = (x_t, y_t)$  for all  $t$ , introduce

$$v(z_0) = J(\mathbf{x}, \mathbf{u}) = \sum_{t=1}^{\infty} g(x_{t-1}, u_t) e^{-\rho t} = \sum_{t=1}^{\infty} G(z_{t-1}) e^{-\rho t}.$$

Using equations (25) and (27) yields the relation

$$G(z) = G(x, y) = H(x, \varphi_2) - \varphi_1 \varphi_2. \quad (29)$$

**Proposition B.5.** Let  $z : [0, 1] \rightarrow \mathcal{M}$  be a  $C^1$  curve in  $\mathcal{M}$ , joining  $\alpha = z(0)$  to  $\beta = z(1)$ , and let  $\varphi_* z = \varphi \circ z$  be its image under  $\varphi$ . Then

$$\int_z e^\rho y \, dx = \int_{\varphi_* z} y \, dx + (G(\beta) - G(\alpha)).$$

In the language of differential forms, putting  $\eta = y \, dx$ , this can be formulated equivalently as

$$e^\rho \eta - \varphi^* \eta = dG. \quad (30)$$

*Proof.* This is a simple computation. Deriving equation (29) and using (27) yields

$$\begin{aligned} dG &= dH - \varphi_1 d\varphi_2 - \varphi_2 d\varphi_1 \\ &= H_x dx + H_y d\varphi_2 - \varphi_1 d\varphi_2 - \varphi_2 d\varphi_1 \\ &= e^\rho y dx - \varphi_2 d\varphi_1. \end{aligned}$$

Since  $\eta = y \, dx$  and  $\varphi^* \eta = \varphi_2 d\varphi_1$ , this shows the result. ■

If the curve  $z$  is vertical, that is, if  $dx = 0$  everywhere along the curve, then the form  $\eta = y \, dx$  vanishes on  $z$ . For such curves proposition B.5 yields

**Proposition B.6.** Let  $z : [0, 1] \rightarrow \mathcal{M}$  join  $\alpha = z(0)$  to  $\beta = z(1)$ , and let  $dx = 0$  along  $z$ . Then

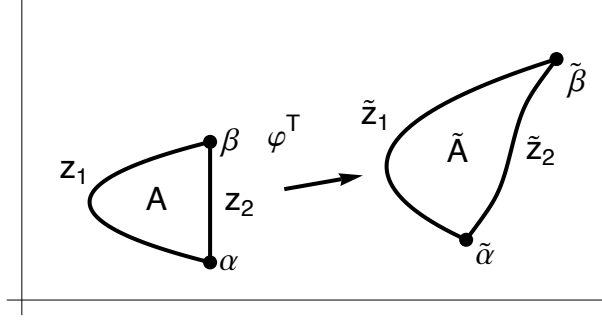
$$-\int_{\varphi_* z} y \, dx = G(\beta) - G(\alpha).$$

**B.5 The iterated area rule.** In actual optimisation problems, the phase map  $\varphi$  may not be a diffeomorphism; in particular, it may not be surjective everywhere. This has consequences for the stable manifold: there may be “holes” in it. If we want to apply an area rule to compare values of orbits, we have to make sure that the surfaces featuring in the rule are actually defined. In the next section, we shall want to determine the values of backward iterates of  $\varphi$  that have very high order, whose existence shall be ensured by the fact that they are close to some fixed point. However, when comparing two such points, there may be no way to connect them with a continuous curve in  $W^s$ .

The iterated area rule, which is stated and proved next, is used to formulate a value comparison result with respect to a single fixed surface, whose existence is not a very strong assumption.

Assume then the following situation:  $\alpha$  and  $\beta$  are both points on the stable manifold of  $W^s$  with the same  $x$ -coordinate and with associated values  $v(\alpha)$  and  $v(\beta)$ , but there is no curve in  $W^s$  joining them. There is, however, a curve  $\tilde{z}_1 : [a, b] \rightarrow \mathcal{M}$  in  $W^s$  that joins  $\tilde{\alpha} = \varphi^T(\alpha) = \tilde{z}_1(a)$  to  $\tilde{\beta} = \varphi^T(\beta) = \tilde{z}_1(b)$ ; for this curve

$$\int_{\tilde{z}_1} \eta = v(\varphi^T(\beta)) - v(\varphi^T(\alpha)) \quad (31)$$



**Figure 12:** The iterated area rule: as  $z_1 + z_2$  is negatively oriented with respect to  $A$ , we have  $v(\beta) - v(\alpha) = \text{area}(A)$ . Even if  $z_1$  and hence  $A$  were not defined, we still have  $v(\beta) - v(\alpha) = e^{-\rho T} \text{area}(\tilde{A})$ .

From the representations

$$v(\alpha) = \sum_{t=1}^{\infty} G(\varphi^{t-1}(\alpha)) e^{-\rho t}, \quad v(\beta) = \sum_{t=1}^{\infty} G(\varphi^{t-1}(\beta)) e^{-\rho t},$$

we obtain easily

$$v(\beta) - v(\alpha) = \sum_{t=1}^T [G(\varphi^{t-1}(\beta)) - G(\varphi^{t-1}(\alpha))] e^{-\rho t} + e^{-\rho T} (v(\varphi^T(\beta)) - v(\varphi^T(\alpha))). \quad (32)$$

Let now  $z_2 : [a, b] \rightarrow \mathcal{M}$  be the vertical curve joining  $\beta$  to  $\alpha$ , and let  $\tilde{z}_2 = \varphi^T \circ z_2$ . Applying proposition B.5 repeatedly, we obtain

$$\begin{aligned} v(\beta) - v(\alpha) &= G(\beta) - G(\alpha) + e^{-\rho} (G(\varphi(\beta)) - G(\varphi(\alpha))) + \dots \\ &\quad + e^{-\rho T} (v(\varphi^T(\beta)) - v(\varphi^T(\alpha))) \\ &= \left( \int_z e^\rho \eta - \int_{\varphi_* z} \eta \right) + e^{-\rho} \left( \int_{\varphi_* z} e^\rho \eta - \int_{\varphi_*^2 z} \eta \right) + \dots \\ &\quad + e^{-\rho T} (v(\varphi^T(\beta)) - v(\varphi^T(\alpha))) \\ &= \int_{z_2} \eta - e^{-\rho T} \int_{\varphi_*^T z_2} \eta + e^{-\rho T} (v(\varphi^T(\beta)) - v(\varphi^T(\alpha))). \end{aligned}$$

Using (31), as well as  $\int_{z_2} \eta = 0$  and  $\tilde{z}_2 = \varphi_*^T z_2$  leads to

$$v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{z}_1 + \tilde{z}_2} \eta.$$

The curve  $\tilde{z} = \tilde{z}_1 + \tilde{z}_2$  is closed; let  $\tilde{A}$  be a surface region that is bounded by this closed curve. Then

$$v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{A}} \omega.$$

We summarise this discussion in the following proposition.

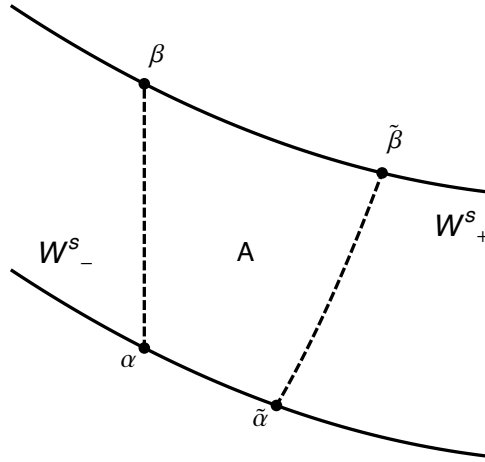
**Proposition B.7.** Let  $\alpha, \beta \in W^s$  be points with the same  $x$ -coordinate,  $\alpha, \beta$  the associated orbits of  $\varphi$ , and set  $v(\alpha) = J(\alpha)$  and  $v(\beta) = J(\beta)$ . Assume that there is a curve  $\tilde{z}_1$  on  $W^s$  joining  $\varphi^T(\alpha)$  and  $\varphi^T(\beta)$ . Let  $z_2$  be a vertical curve connecting  $\beta$  to  $\alpha$ , and set  $\tilde{z}_2 = \varphi^T \circ z_2$ . Let finally  $\tilde{A}$  be a surface region such that  $\partial \tilde{A} = \tilde{z} = \tilde{z}_1 + \tilde{z}_2$ . Then

$$v(\beta) - v(\alpha) = e^{-\rho T} \int_{\tilde{A}} \omega$$

The proposition is illustrated in figure 12.

**B.6 Value differences.** Consider now the situation illustrated in figure 13. There are two stable manifolds  $W_-^s$  and  $W_+^s$ , associated to the fixed points  $z_-$  and  $z_+$ , and two points  $\alpha \in W_-^s$  and  $\beta \in W_+^s$ , such that their  $x$ -coordinates are equal (see figure 13). Let  $z$  be the line segment joining  $\alpha$  to  $\beta$  and let  $A$  be the oriented surface region bounded by the concatenation of  $z$ , the part  $w_+$  of  $W_+^s$  joining  $\beta$  to  $\tilde{\beta} = \varphi(\beta)$ , the negative of  $\varphi_* z$  joining  $\tilde{\beta}$  to  $\tilde{\alpha} = \varphi(\alpha)$ , and the negative  $-w_-$  of the part of  $W_-^s$  joining  $\alpha$  to  $\tilde{\alpha}$ . Define finally

$$\Omega(A) = \int_A \omega.$$



**Figure 13:** Relation between values and area:  $v(\beta) - v(\alpha) = \text{area}(A)/(e^\rho - 1)$ . The boundary of  $A$  is the curve  $\alpha \rightarrow \beta \rightarrow \tilde{\beta} \rightarrow \tilde{\alpha} \rightarrow \alpha$ ; it is negatively oriented, consequently the orientation of  $A$  is negative as well and  $\Omega(A) = \text{area}(A)$

**Proposition B.8.** In the situation sketched above, we have

$$v(\beta) - v(\alpha) = \frac{\Omega(A)}{e^\rho - 1}.$$

*Proof.* Again by Stokes' theorem:

$$\begin{aligned}\Omega(A) &= \int_A \omega = \int_{\partial A} \eta = \int_z \eta + \int_{w_+} \eta - \int_{\varphi_* z} \eta - \int_{w_-} \eta \\ &= v(\tilde{\beta}) - v(\beta) - \int_{\tilde{z}} \eta - v(\tilde{\alpha}) + v(\alpha).\end{aligned}$$

Using proposition B.5 yields that

$$- \int_{\tilde{z}} \eta = G(\beta) - G(\alpha).$$

Moreover, using  $v(\alpha) = \sum_1^\infty G(z_t) e^{-\rho t}$  yields

$$v(\tilde{\alpha}) = e^\rho v(\alpha) - G(\alpha).$$

Eliminating with these relations the quantities  $\int_{\tilde{z}} y dx$  as well as  $v(\tilde{\alpha})$  and  $v(\tilde{\beta})$ , we arrive at

$$\Omega(A) = (e^\rho - 1) \left( v(\beta) - v(\alpha) \right),$$

as claimed in the proposition. ■

**B.7 An approximation result.** Let  $\mathbf{z} = \{z_t\}_{t=0}^\infty$  be an orbit of the phase map  $\varphi$ . We write

$$J(\mathbf{z}) = \sum_{t=1}^\infty G(z_{t-1}) e^{-\rho t}.$$

Then the following proposition holds true.

**Proposition B.9.** *Let  $\mathbf{a}$  be a summable sequence of positive real numbers. Assume that there is a sequence  $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \dots$  of orbits of the phase map  $\varphi$ , such that*

$$\left| G\left(z_t^{(k)}\right) e^{-\rho t} \right| \leq a_t$$

for all  $k \geq 1$  and all  $t \geq 1$ . Assume moreover that

$$z_0^{(k)} \rightarrow z_0^{(\infty)} \quad \text{as } k \rightarrow \infty.$$

Then

$$J(\mathbf{z}^{(k)}) \rightarrow J(\mathbf{z}^{(\infty)}).$$

*Proof.* Choose  $\varepsilon > 0$ . Then there is an  $T > 0$  such that for any  $k$  we have

$$\left| J(\mathbf{z}^{(k)}) - J_T(\mathbf{z}^{(k)}) \right| = \left| \sum_{t=T+1}^\infty G(z_{t-1}) e^{-\rho t} \right| \leq \sum_{t=T+1}^\infty a_t < \frac{\varepsilon}{3}.$$

Moreover, if  $\mathbf{z}$  is an orbit of  $\varphi$ , note that  $J_T(\mathbf{z})$  only depends on the initial segment

$$\mathbf{z}_T = (z_0, \dots, z_T) = (z_0, \varphi(z_0), \varphi^2(z_0), \dots, \varphi^T(z_0)),$$

which is a continuous function of  $z_0$ . Therefore, there is a constant  $\delta > 0$  such that

$$|z_0^{(k)} - z_0^{(\infty)}| < \delta \quad \Rightarrow \quad |J_T(\mathbf{z}^{(k)}) - J_T(\mathbf{z}^{(\infty)})| < \frac{\varepsilon}{3}.$$

Take now  $N > 0$  such that  $|z_0^{(k)} - z_0^{(\infty)}| < \delta$  for all  $n \geq N$ . Then

$$|J(\mathbf{z}^{(k)}) - J(\mathbf{z}^{(\infty)})| \leq |J(\mathbf{z}^{(k)}) - J_T(\mathbf{z}^{(k)})| + |J_T(\mathbf{z}^{(k)}) - J_T(\mathbf{z}^{(\infty)})| + |J_T(\mathbf{z}^{(\infty)}) - J(\mathbf{z}^{(\infty)})| \leq \varepsilon.$$

This proves the claim of the lemma. ■

Note that the lemma can be applied if the orbits  $\mathbf{z}^{(k)}$  lie all in some set  $S \subset \mathcal{M}$  on which  $G$  is bounded.

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