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Abstract

Flood and Garber (1983), Smith (1991), and Froot and Obstfeld (1991a,b) examined the return of the United Kingdom to the gold standard in 1925 as an example of state-contingent process switching. They calculated the exchange rate via the density function of the first-passage time through the announced parity (Flood and Garber, 1983; Smith, 1991) or via solving a differential equation under suitable boundary conditions (Froot and Obstfeld, 1991a,b). We alternatively employ the underlying transition probabilities and confirm the solution obtained in the literature. In addition, our approach allows us to critically evaluate intuitive arguments in the literature that actually relied on transition probabilities without the latter actually having been derived. The transition probabilities also have obvious appeal for econometric analyses, derivative pricing, and decision making under the potential of "extinction".

Keywords: Absorption, asset price equation, Brownian motion, stochastic process switching, transition probability.

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1 Introduction

Flood and Garber (1983) modelled the return of the United Kingdom to the gold standard in 1925 as an example of state-contingent process switching. This choice was based on the announcement by monetary authorities already in 1918 of their wish to fix the pound to the dollar at the pre-war parity of $4.86 conditional on attaining again the pre-war purchasing power parity. Or, the exchange rate would be fixed, i.e. absorbed, upon first hitting the pre-war parity. Flood and Garber (1983) approached the determination of the exchange rate via a probabilistic reasoning. More in particular, they focussed on the probability density function for the first-passage time through the announced parity, but failed to solve the resulting integrals. This first-passage time problem was ultimately solved in Smith (1991). Froot and Obstfeld (1991a,b) alternatively employed a mathematically elegant and simpler approach in which the problem was rephrased in terms of a differential equation for which the relevant integration constant was obtained from the no-jump condition upon switching.

This article solves the above problem again via a probabilistic approach, but now by relying on transition probabilities, i.e. the conditional likelihood that some state can be attained at a certain future point of time. This method, that Froot and Obstfeld (1991a, p. 241) correctly anticipated as "likely to be burdensome", is complementary to the aforementioned approaches as it provides us with interesting additional information such as the probability of absorption. In addition, it allows us to support but also correct some intuitive arguments in the literature that were actually phrased in terms of transition probabilities without the latter actually having been derived. The transition probabili-

ties also have obvious appeal for econometric exercises based on, for instance, maximum likelihood,

derivative pricing, (cash) management under the potential of "extinction", and other decision making

problems.

The remainder of this article is structured as follows. In Section 2, we briefly specify the economic

model and introduce the transition probabilities required for solving it. Section 3 applies our approach

under the assumption of Brownian motion with drift.

2   The economic model and the required transition probabilities

Flood and Garber (1983) started from the so-called asset price equation:

$$x\left(t\right) = k\left(t\right) + \alpha \frac{E\left[dx\left(t\right) | I\left(t\right)\right]}{dt},$$

where $x\left(t\right)$ denotes the (log) spot exchange rate, $k\left(t\right)$ is the fundamental and E represents the expec-

tations operator. The time-t information set is given by $I\left(t\right)$. The fundamental can be interpreted as

an indicator of the relative supply to demand conditions of the home currency and can be given more

specific content via the flexible-price monetary model as in Flood and Garber (1983). Finally, the

parameter α measures the sensitivity of the exchange rate to its own expected future course and in

the monetary model can be seen as the semi-elasticity of money demand with respect to the interest

rate.

Ruling out speculative bubbles, the unique saddle-path solution for any sequence of future funda-

mentals is given by (see Froot and Obstfeld, 1991a,b):

$$x\left(t\right) = \frac{1}{\alpha} \int_{t}^{+\infty} E\left[k\left(s\right)|k\left(t\right)\right] \exp\left\{-\frac{s-t}{\alpha}\right\} ds, \tag{1}$$

where $E\left[k\left(s\right)|k\left(t\right)\right]$ is the conditional expectation for the fundamental at time s given the present

state $k\left(t\right)$.

Flood and Garber (1983) showed that the stochastic process of the fundamental switches when

it hits the pre-announced switching value, $\overline{k}$. At reaching that point, it will be absorbed, i.e.

will permanently remain at $\overline{k}$. Hence, the expectations term in Equation (1) can be rewritten as

$E\left[k\left(s\right) \leqslant \overline{k}|k\left(t\right)\right]$ in order to highlight that $\overline{k}$ is the absorbing maximum value of the fundamental:

$$x\left(t\right) = \frac{1}{\alpha} \int_{t}^{+\infty} E\left[k\left(s\right) \leqslant |k\left(t\right)\right] \exp\left\{-\frac{s-t}{\alpha}\right\} ds. \tag{2}$$

We now proceed toward specifying the conditional expectation in Equation (2). The transition prob-

ability density function, in short the transition density, conditional on absorption not taking place

within the prediction interval $(s-t)$ is denoted by $p\left[k(s) < \overline{k}|k\left(t\right)\right]$. It specifies the probability of

attaining $k\left(s\right) < \overline{k}$ at time s given that the process currently is at the source point $k\left(t\right)$. The transi-

tion probability distribution function, in short the transition distribution, conditional on absorption

not occurring in $(s-t)$ is $P\left[k(s) < \overline{k}|k\left(t\right)\right]$. The transition distribution of absorption is defined

as $P\left[k(s) = \overline{k}|k\left(t\right)\right]$. Together both transition distributions specify $P\left[k(s) \leqslant \overline{k}|k\left(t\right)\right]$ which is the

cumulative transition density over the entire domain of the fundamental, namely $\left(-\infty, \overline{k}\right]$, and it

obviously equals 1:

$$P\left[k(s) \leqslant \overline{k}|k\left(t\right)\right] = P\left[k(s) < \overline{k}|k\left(t\right)\right] + P\left[k(s) = \overline{k}|k\left(t\right)\right],$$

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with

$$P\left[k(s) < \overline{k}\,|k\left(t\right)\right] \;\; = \;\; \int_{-\infty}^{\overline{k}} p\left[k(s) < \overline{k}\,|k\left(t\right)\right]\;dk\left(s\right), \tag{3}$$

$$P\left[k(s) = \overline{k}\,|k\left(t\right)\right] \;\; = \;\; 1 - P\left[k(s) < \overline{k}\,|k\left(t\right)\right]. \tag{4}$$

A similar distinction arises for the conditional expectation $E\left[k\left(s\right) \leqslant \overline{k}\,|k\left(t\right)\right]$ in Equation (2). It is

the sum of the conditional expectation provided absorption does not come about within the predic-

tion interval, $E\left[k\left(s\right) < \overline{k}\,|k\left(t\right)\right]$, and the conditional expectation located on the absorbing boundary,

$E\left[k\left(s\right) = \overline{k}\,|k\left(t\right)\right]$:

$$E\left[k(s) \leqslant \overline{k}\,|k\left(t\right)\right] = E\left[k(s) < \overline{k}\,|k\left(t\right)\right] + E\left[k(s) = \overline{k}\,|k\left(t\right)\right],$$

with

$$E\left[k(s) < \overline{k}\,|k\left(t\right)\right] \;\; = \;\; \int_{-\infty}^{\overline{k}} k\left(s\right)\;p\left[k(s) < \overline{k}\,|k\left(t\right)\right]\;dk\left(s\right), \tag{5}$$

$$E\left[k(s) = \overline{k}\,|k\left(t\right)\right] \;\; = \;\; \overline{k}\;P\left[k(s) = \overline{k}\,|k\left(t\right)\right]. \tag{6}$$

3   Conditional expectations and the exchange rate for Brownian motion with drift

In order to solve the above economic model, Flood and Garber (1983) assumed k to be a Brownian

motion with drift prior to the regime switch:

$$dk\left(t\right) = \eta dt + \sigma dz\left(t\right), \tag{7}$$

where η and σ^2 denote the drift and diffusion coefficients, respectively and $dz(t)$ is the increment of a Wiener process.

The transition density $p\left[k(s) < \overline{k} \,|k(t)\right]$ can be obtained via the Fokker-Planck or the (Kolmogorov) forward equation:[1]

$$\frac{1}{2}\sigma^2\frac{\partial^2 p\left[k(s) < \overline{k}\,|k(t)\right]}{\partial k(s)^2} - \eta\frac{\partial p\left[k(s) < \overline{k}\,|k(t)\right]}{\partial k(s)} = \frac{\partial p\left[k(s) < \overline{k}\,|k(t)\right]}{\partial s} \qquad (8)$$

for $-\infty < k(s), k(t) < \overline{k}$ and $s > t$.

Equation (8) is to solved subject to one boundary condition and one initial condition. Imposing absorption at $\overline{k}$ is tantamount to requiring that the boundary $\overline{k}$ can gather no probability mass since $p\left[k(s) < \overline{k}\,|k(t)\right]$ explicitly conditions on absorption not taking place within the prediction horizon:

$$\lim_{k(s)\uparrow\overline{k}}\left[p\left[k(s) < \overline{k}\,|k(t)\right]\right] = 0.$$

The relevant initial condition is:

$$\lim_{s\downarrow t}\left[p\left[k(s) < \overline{k}\,|k(t)\right]\right] = \delta\left(k(s) - k(t)\right)\delta\left(s - t\right),$$

with $\delta(\cdot)$ being the Dirac delta function. This condition requires all initial probability mass to be located at the initial value and point of time, which clearly is the relevant initial condition for Brownian motion processes.

The above initial-boundary value problem can be solved via, for instance, the method of images.[2]

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[1] See, for instance, Risken (1989).
[2] See Sommerfeld (1949) for more detail.

This yields:

$$
p\left[k(s) < \overline{k}\,|k(t)\right] \;=\; \frac{1}{\sigma\sqrt{2\pi(s-t)}}\exp\left[-\frac{(k(s)-k(t)-\eta(s-t))^2}{2\sigma^2(s-t)}\right] -
$$

(9)

$$
\frac{1}{\sigma\sqrt{2\pi(s-t)}}\exp\left[-\frac{2\eta\left(k(t)-\overline{k}\right)}{\sigma^2}\right]\exp\left[-\frac{\left(k(s)+k(t)-2\overline{k}-\eta(s-t)\right)^2}{2\sigma^2(s-t)}\right].
$$

The transition distribution conditional on no absorption taking place, i.e. Equation (3), emerges as:[3]

$$
P\left[k(s) < \overline{k}\,|k(t)\right] = \Phi\left[\frac{\overline{k}-k(t)-\eta(s-t)}{\sigma\sqrt{s-t}}\right] - \exp\left[-\frac{2\eta\left(k(t)-\overline{k}\right)}{\sigma^2}\right]\Phi\left[\frac{k(t)-\overline{k}-\eta(s-t)}{\sigma\sqrt{s-t}}\right], \quad (10)
$$

with $\Phi(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{q}\exp\left(-\frac{1}{2}w^2\right)dw$ being the cumulative standard normal distribution function. As required, increasing $\overline{k}$ to $+\infty$, and thus effectively precluding absorption, yields 1. Under driftless Brownian motion, the transition distribution of no absorption simplifies to:

$$
\lim_{\eta\to0}\left[P\left[k(s) < \overline{k}\,|k(t)\right]\right] = 1 - 2\Phi\left[\frac{k(t)-\overline{k}}{\sigma\sqrt{s-t}}\right].
$$

The transition distribution of absorption within the prediction horizon, see Equation (4), then is:

$$
P\left[k(s) = \overline{k}\,|k(t)\right] = 1 - \Phi\left[\frac{\overline{k}-k(t)-\eta(s-t)}{\sigma\sqrt{s-t}}\right] + \exp\left[-\frac{2\eta\left(k(t)-\overline{k}\right)}{\sigma^2}\right]\Phi\left[\frac{k(t)-\overline{k}-\eta(s-t)}{\sigma\sqrt{s-t}}\right].
$$

(11)

Equation (11) directly specifies the conditional probability that absorption, i.e. the switch in the stochastic process, manifests itself within the time horizon in question. Unsurprisingly, higher drift in the fundamental steps up the likelihood of absorption as the derivative to η is strictly positive.

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[3]Analytic expressions for the various integrals in this article can be obtained via software such as Mathematica®. However, this was not possible for a number of integrals in which case we had to use and extend solution 4.12.(12) in Erdélyi et al. (1954). All derivations in this article can be obtained from the author upon simple demand.

Or, monetary authorities can design appropriate policies concerning the drift parameter in order

to influence the likelihood of the regime switch. Under driftless Brownian motion, the transition

distribution in Equation (11) collapses into:

$$\lim_{\eta \to 0} \left[P \left[k(s) = \overline{k} \, | k(t) \right] \right] = 2\Phi \left[\frac{k(t) - \overline{k}}{\sigma \sqrt{s - t}} \right].$$

The latter result shows that the likelihood of the regime switch exceeds 0 even for $\eta = 0$, i.e. when

policy refrains from bringing k to $\overline{k}$ via a positive value for η.[4]

Plugging Equation (9) into Equation (5) yields the conditional expectation provided that absorp-

tion does not materialize within the prediction interval. The conditional expectation located on the

absorbing state is obtained by inserting Equation (11) into Equation (6). Adding these two expecta-

tions then yields the conditional expectation for $k(s) \leqslant \overline{k}$:

$$
\begin{aligned}
E \left[k(s) \leqslant \overline{k} \, | k(t) \right] \quad = \quad & \overline{k} + \left(k(t) - \overline{k} + \eta(s - t) \right) \Phi \left[\frac{\overline{k} - k(t) - \eta(s - t)}{\sigma \sqrt{s - t}} \right] + \\
& \left(k(t) - \overline{k} - \eta(s - t) \right) \exp \left[-\frac{2\eta \left(k(t) - \overline{k} \right)}{\sigma^2} \right] \Phi \left[\frac{k(t) - \overline{k} - \eta(s - t)}{\sigma \sqrt{s - t}} \right].
\end{aligned}
\tag{12}
$$

Under driftless Brownian motion, the conditional expectation simplifies into:

$$\lim_{\eta \to 0} \left[E \left[k(s) \leqslant \overline{k} \, | k(t) \right] \right] = k(t).
\tag{13}$$

Note that the result in Equation (13) does not depend on $\overline{k}$ such that $k(t)$ is also the conditional

expectation for the unrestricted driftless process, i.e. when additionally $\overline{k} \to +\infty$.

The final, but rather lengthy, step consists in plugging Equation (12) into Equation (2). After

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[4]We will come back to this property later in this article.

rearranging and simplifying, the following expression for the exchange rate emerges:

$$x\left(t\right) = k\left(t\right) + \alpha\eta\left(1 - \exp\left[\lambda_1\left(k\left(t\right) - \overline{k}\right)\right]\right), \tag{14}$$

with

$$\lambda_1 = \frac{-\alpha\eta + \sqrt{\alpha^2\eta^2 + 2\alpha\sigma^2}}{\alpha\sigma^2} > 0,$$

which is identical to the solution obtained in Froot and Obstfeld (1991a,b) and Smith (1991). Note that the free-float exchange rate x^{FF} is given by:

$$x^{FF}\left(t\right) = k\left(t\right) + \alpha\eta. \tag{15}$$

Indeed, the exponential term in Equation (14) vanishes for increasing $\overline{k}$, i.e. when moving toward the unrestricted case for both the fundamental and the exchange rate given the one-to-one relation between the two in Equation (1).

We now can use the above relations to discuss a number of intuitive statements in the literature that actually referred to transition probabilities. For instance, Froot and Obstfeld (1991a,b) argued that the conditional expectation of the fundamental is its present state if drift is absent. This is confirmed by Equation (13) or in terms of conditional expectations the possibility of a regime switch will generate no effect when drift is absent. Froot and Obstfeld (1991b, p. 217) and Miller and Sutherland (1994, p. 807)[5] then extended the absence of an effect also to the transition probabilities, still under $\eta = 0$, when arguing that the distribution of possible movements in k is symmetric upward and downward or

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[5] Miller and Sutherland (1994) actually studied a mean-reverting stochastic process for which, however, driftless Brownian motion emerges as a limiting case.

equivalently that the expectation of absorption removes the same amount of upward and downward potential for the fundamental. However, such symmetry in transition probabilities is actually not present even when $\eta = 0$. This lack of symmetry can most easily be illustrated via the transition distribution conditional on the future fundamental not exceeding its current state, $P\left[k(s) < k(t) < \overline{k}\,|k(t)\right]$, which emerges as the integral of the density in Equation (9) over $(-\infty, k(t))$:

$$
P\left[k(s) < k(t) < \overline{k}\,|k(t)\right] = \Phi\left[\frac{-\eta\sqrt{s-t}}{\sigma}\right] - \exp\left[-\frac{2\eta\left(k(t) - \overline{k}\right)}{\sigma^2}\right]\Phi\left[\frac{2\left(k(t) - \overline{k}\right) - \eta(s-t)}{\sigma\sqrt{s-t}}\right].
$$
(16)

Symmetry around $k(t)$, and thus $P\left[k(s) < k(t) < \overline{k}\,|k(t)\right] = 0.5$, requires both $\eta = 0$ and $\overline{k} \to +\infty$. This means evaluating the limit for an unrestricted driftless Brownian motion for which symmetry is a logical consequence of the symmetric nature of the diffusion component in the no-drift specialization of Equation (7). Generally, $P\left[k(s) < k(t) < \overline{k}\,|k(t)\right]$ will be (well) below 0.5 as the second term in Equation (16) is negative. This is due to the fact that $\overline{k}$ absorbs the fundamental and thus prevents $k(s)$ from straying below $k(t)$ again after first hitting. The absorbing boundary thus removes conditional likelihood below the present level of the fundamental even when $\eta = 0$.

Thus, the symmetry in transition probabilities is lost under absorption but the conditional expectation in the driftless case will still equal its value under the unrestricted and thus symmetric process. This may at first sight be surprising and requires some additional elaboration. The loss of symmetry in probabilities induces also an asymmetry in conditional expectations even for $\eta = 0$. Indeed, if one splits up the conditional expectation into the contribution of the domain below $k(t)$ and the one

above $k\left(t\right)$, it is immediately clear that the contribution from the domain above $k\left(t\right)$ is more than 50%, i.e. more than $\frac{1}{2}k\left(t\right)$ for $\eta=0$ (see Equation 13). Indeed, this is inevitable when coupling the above asymmetry in transition probabilities with the fact that the fundamental is higher above $k\left(t\right)$ than below that value. Nevertheless, the addition of these two unequal contributions in the driftless case still gives $k\left(t\right)$. This actually is in line with the definition of the Brownian motion in Equation (7) for which the expected change in k, for $\eta=0$, is 0.

We now briefly discuss the relation between the transition probabilities under absorption and under the unrestricted case in order to examine the implications for the relation between the regime-switch and the free-float exchange rates. For instance, Smith and Smith (1990, p. 170) argued that the prospect of absorption removes probability above the actual fundamental due to the truncation of the upper support of the probability density function which then would negatively impact on the value of the exchange rate. Miller and Sutherland (1994) added that absorption also removes probability mass at the lower end of the domain when compared to the unrestricted case and this would have an upward effect on the exchange rate. We discuss these claims using $\Delta P[k\left(s\right)<k\left(t\right)|k\left(t\right)]$ that is defined as the transition distribution under the unrestricted process minus its homologue under absorption for the domain below the present fundamental $k\left(t\right)$:

$$\Delta P[k\left(s\right)<k\left(t\right)|k\left(t\right)]=\exp\left[-\frac{2\eta\left(k\left(t\right)-\overline{k}\right)}{\sigma^2}\right]\Phi\left[\frac{2\left(k\left(t\right)-\overline{k}\right)-\eta\left(s-t\right)}{\sigma\sqrt{s-t}}\right]>0. \qquad (17)$$

Thus, the support below $k\left(t\right)$ gathers more conditional probability mass under the unrestricted process and absorption thus indeed removes probability mass below $k\left(t\right)$. This is due to the fact that, as

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mentioned earlier, the absorbing nature of $\overline{k}$ prevents the fundamental from straying again (deep) into the lower part of the domain after hitting the switching state. However, Equation (17) also implies that the region outside $(-\infty, k\,(t))$ will have more, rather than less, probability mass under absorption than under the unrestricted process. The aforementioned removal of probability due to truncation of the upper support then must be explicitly expressed with respect to the region above $\overline{k}$ for which absorption per definition precludes the presence of probability mass.

Announcing absorption thus affects transition probabilities for sure and subsequently can alter conditional expectations and the exchange rate. Under absence of drift, transition probabilities under the unrestricted process and the absorption case will differ. However, conditional expectations and the exchange rates are the same as can be seen from Equations (12)-(15) for $\eta = 0$. Positive drift shifts probability mass upward in both regimes but the effect on the conditional expectations and thus the exchange rate will obviously be more pronounced under the unrestricted process. Indeed, the region above $\overline{k}$ then attracts more probability mass and the fundamentals there are obviously higher than $\overline{k}$, which is the maximum value under absorption. The free-float exchange rate $x^{FF}(t)$ then will exceed the regulated exchange rate $x\,(t)$. Negative drift strongly restricts the likelihood of finding fundamentals above $\overline{k}$ under the unrestricted case and it increases the probability mass under $\overline{k}$ where the free float anyway gathers more probability.[6] This then brings the free-float exchange rate below the exchange rate under absorption or $x^{FF}(t) < x\,(t)$.

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[6]It can be shown that also $P[k\,(s) < \overline{k}\,|k\,(t)] > 0$.

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