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Multiple Steady States, Limit Cycles and Chaotic Attractors in Evolutionary Games with Logit Dynamics[*]

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Abstract

This paper investigates, by means of simple, three and four strategy games, the occurrence of periodic and chaotic behaviour in a smooth version of the Best Response Dynamics, the Logit Dynamics. The main finding is that, unlike Replicator Dynamics, *generic* Hopf bifurcation and thus, stable limit cycles, do occur under the Logit Dynamics, even for three strategy games. Moreover, we show that the Logit Dynamics displays another bifurcation which *cannot* to occur under the Replicator Dynamics: the *fold catastrophe*. Finally, we find, in a four strategy game, a period-doubling route to chaotic dynamics under a 'weighted' version of the Logit Dynamics.

JEL classification: C72, C73

Keywords: Evolutionary games, Logit dynamics, Hopf bifurcation, saddle-node bifurcation, chaos

1 Introduction

A large part of the research on evolutionary game dynamics focused on identifying conditions that ensure uniqueness of and/or convergence to point-attractors such as a Nash Equilibrium or an Evolutionary Stable Strategy (ESS). Roughly speaking, within this 'convergence' literature one can further distinguish between literature focusing on classes of games (e.g. Milgrom and Roberts (1991), Nachbar (1990), Hofbauer and Sandholm (2002)) and literature on different classes of evolutionary dynamics (e.g. Cressman (1997), Hofbauer (2000), Hofbauer and Weibull (1996) , Sandholm (2005), Samuelson and Zhang (1992)). Nevertheless, there are some examples of periodic and chaotic behaviour in the literature, mostly under a particular kind of evolutionary dynamics, the Replicator Dynamics. Motivated by the idea of adding an explicit dynamical process to the static concept of Evolutionary Stable Strategy, Taylor and Jonker (1978) introduced the Replicator Dynamics. It soon found applications in biological, genetic or chemical systems, those domains where organisms, genes or molecules evolve over time via replication. The common feature of these systems is that they can be well approximated by an infinite population game with random pairwise matching, giving rise to a replicator-like evolutionary dynamics given by a low-dimensional dynamical system.

In the realm of the non-convergence literature two issues are particularly important: the existence of stable periodic and complicated solutions and their robustness (e.g. to slight perturbations in the payoffs matrix). Hofbauer et al. (1980) and Zeeman (1981) investigate the phase portraits resulting from three-strategy games under the replicator dynamics and conclude that only 'simple' behaviour - sinks, sources, centers, saddles - can occur. In general, an evolutionary dynamics together with a $n-$strategy game defines a proper $n-1$ dynamical system on the $n-1$ simplex. An important result proven by Zeeman (1980), is that under Replicator Dynamics, there are no *generic* Hopf bifurcations on the 2-simplex: *"When n = 3 all Hopf bifurcations are degenerate"*[1]. Bomze (1983, 1995) provides a detailed classification of the planar phase portraits for all 3-strategy games according to the number, location (interior

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[1] Zeeman (1980), pp. 493

or on boundary of the simplex) and stability properties of the Replicator Dynamics fixed points and identify 49 different phase portraits: again, only non-robust cycles are created usually via a degenerate Hopf bifurcation. In Replicator Dynamics only the "hairline" case of a continuum of cycles occurs, which are non-generic and disappear by slightly perturbing the payoff parameters. Another possibility in Replicator Dynamics is a so-called heteroclinic cycle consisting of saddle steady-states on the boundary of the simplex and their connecting saddle paths. Generic limit cycles do not arise in 3-strategy games under Replicator Dynamics.

Hofbauer (1981) proves that in a 4-strategy game stable limit cycles are possible under Replicator Dynamics; the proof consists in finding a suitable Lyapunov function whose time derivative vanishes on the $\omega-$limit set of a periodic orbit. Stable limit cycles are also reported in Akin (1982) in a genetic model where gene 'replicates' via the two allele-two locus selection; this is not surprising as the dynamical system modeling gametic frequencies is three-dimensional, the dynamics is of Replicator type and the Hofbauer (1981) proof applies here, as well. Furthermore, Maynard Smith and Hofbauer (1987) prove the existence of a stable limit cycle for an asymmetric, Battle of Sexes-type genetic model where the allelic frequencies evolution defines again a 3-D system. Their proof hinges on normal form reduction together with averaging and elliptic integrals techniques for computing the phase and angular velocity of the periodic orbit. Stadler and Schuster (1990) perform an impressive systematic search for both generic transitions (fixed points exchanging stability) and degenerate transitions (stable and unstable fixed points colliding into a one or two-dimensional manifold at the critical parameter value) between phase portraits of the replicator equation on 3×3 normal form game.

Chaotic behaviour is found by Schuster et al. (1991) in Replicator Dynamics for a 4-strategy game matrix derived from an autocatalytic reaction network. They report the standard Feigenbaum route to chaos: a cascade of period-doubling bifurcations intermingled with several interior crisis and collapses to a chaotic attractor. Numerical evidence for strange attractors is provided by Skyrms (1992, 2000) for a Replicator Dynamics flow on two examples

of a four-strategy game.

Although periodic and chaotic behaviour is substantially documented in the literature for the Replicator Dynamics, there is much less evidence for such complicated behaviour in classes of evolutionary dynamics that are more appropriate for human interaction (fictitious play, best response dynamics, adaptive dynamics, etc.). While Shapley (1964) constructs an example of a non-zero sum game with a limit cycle under fictitious play and Berger and Hofbauer (2006) find stable periodic behaviour - two limit cycles bounding an asymptotically stable annulus - for a different dynamic - the Brown-von Newmann Nash (BNN) - a systematic characterization of (non) generic bifurcations of phase portraits is still missing for the Best Response dynamics.

The purpose of this paper is to study a smoothed version of the Best Response dynamics - the Logit Dynamics - to study evolutionary dynamics in simple, well-known, three and four strategies games such as Rock-Paper-Scissors and Coordination Game. In particular we focus on generic possibilities of complicated dynamics (i.e. stable limit cycles, multiple steady states, chaos). The qualitative behaviour of the 'evolutionary' games with Logit Dynamics is investigated with respect to changes in the payoff and behavioural parameters. In addition to analytical results on local stability and bifurcations, we use the advanced bifurcation software Matcont (Dhooge et al. (2003)) to numerically detect bifurcation curves in the parameter space.

Most of the earlier discussed evolutionary game examples are inspired from biology and not from social sciences. From the perspective of strategic interaction the main criticism of the 'biological' game-theoretic models is targeted at the intensive use of preprogrammed, simple imitative play with no role for optimization and innovation. Specifically, in the transition from animal contests and biology to humans interactions and economics the Replicator Dynamics seems no longer adequate to model the rationalistic and 'competent' forms of behaviour (Sandholm (2008)). Best Response Dynamics would be more applicable to human interaction as it assumes that agents are able to optimally compute and play a (myopic) 'best

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response' to the current state of the population. But, while the Replicator Dynamics appeared to impose an unnecessarily loose rationality assumption the Best Response dynamics moves to the other extreme: it is too stringent in terms of rationality. Another drawback is that, technically, the best reply is not necessarily unique and this leads to a differential inclusion instead of an ordinary differential equation. One way of solving these problems was to stochastically perturb the matrix payoffs and derive, via the discrete choice theory, a 'noisy' Best Response Dynamics, called the Logit Dynamics. Mathematically it is a 'smoothed', well-behaved dynamics while from the strategic interaction point of view it models a *boundedly rational* player/agent. Moreover, from a nonlinear dynamical systems perspective the Replicator Dynamic is, in fact, non-generic in dimension two and only degenerate Hopf bifurcations can arise on the 2-*simplex*. In sum, apart from its conjectured generic properties, the Logit is recommended by the need for modelling players with different degrees of rationality and for smoothing the Best Response correspondence.

The paper is organized as follows: Section 2 introduces the Logit Dynamics, while Section 3 gives a brief overview of the Hopf bifurcation theory. In Section 4 the Logit Dynamics is implemented on Rock-Scissors-Paper and in Section 5 on a Coordination game. Section 6 discusses an example of chaotic dynamics under a frequency-weighted version of the Logit Dynamics and Section 7 concludes.

2   Replicator vs. Logit Dynamics

Evolutionary game theory deals with games played within a (large) population over a long time horizon (evolution scale). Its main ingredients are the underlying normal form game - with payoff matrix $A[n \times n]$ - and the evolutionary dynamic class which defines a dynamical system on the state of the population. In a symmetric framework, the strategic interaction takes the form of random matching with each of the two players choosing from a finite set of available strategies $E = \{E_1, E_2, ... E_n\}$. For every time t, $\mathbf{x(t)}$ denotes the n-dimensional

vector of frequencies for each strategy/type E_i and belongs to the $n-1$ dimensional simplex $\mathbf{\Delta}^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1\}$. Under the assumption of random interactions strategy E_i fitness would be simply determined by averaging the payoffs from each strategic interaction with weights given by the state of the population $\mathbf{x}$. Denoting with $f(\mathbf{x})$ the payoff vector, its components - individual payoff or fitness of strategy i in biological terms - are:

$$f_i(\mathbf{x}) = (\mathbf{Ax})_i \tag{1}$$

Sandholm (2008) rigorously defines an *evolutionary dynamics* as a map assigning to each population game a differential equation $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ on the simplex $\mathbf{\Delta}^{n-1}$. In order to derive such an 'aggregate' level vector field from *individual choices* he introduces a *revision protocol* $\rho_{ij}(f(\mathbf{x}), \mathbf{x})$ indicating, for each pair (i, j), the rate of switching(ρ_{ij}) from the currently played strategy i to strategy j. The mean vector field is obtained as:

$$
\begin{aligned}
\dot{x}_i &= \mathbf{V}_i(\mathbf{x}) = \text{inflow into strategy } i \text{ - outflow from strategy } i \\
&= \sum_{j=1}^{n} x_j \rho_{ji}(f(\mathbf{x}), \mathbf{x}) - x_i \sum_{j=1}^{n} \rho_{ij}(f(\mathbf{x}), \mathbf{x}).
\end{aligned}
\tag{2}
$$

Based on the computational requirements/quality of the revision protocol ρ the set of evolutionary dynamics splits into two large classes: imitative dynamics and pairwise comparison (*'competent'* play). The first class is represented by the most famous dynamics, the Replicator Dynamic (Taylor and Jonker (1978)) which can be easily derived by substituting into (2) the pairwise proportional revision protocol: $\rho_{ij}(f(\mathbf{x}), \mathbf{x}) = x_j[f_j(\mathbf{x}) - f_i(\mathbf{x})]_+$ (player i switches to strategy j at a rate proportional with the probability of meeting an j-strategist(x_j) and with the excess payoff of opponent j-$[f_j(\mathbf{x}) - f_i(\mathbf{x})]$- iff positive):

$$\dot{x}_i = x_i[f_i(\mathbf{x}) - \bar{f}(\mathbf{x})] = x_i[(Ax)_i - \mathbf{xAx}] \tag{3}$$

where $\bar{f}(\mathbf{x}) = \mathbf{xAx}$ is the average population payoff.

Although widely applicable to biological/chemical models, the Replicator Dynamics lacks the proper individual choice, micro-foundations which would make it attractive for modelling human interactions. The alternative - Best Response dynamics - already introduced by Gilboa and Matsui (1991) requires extra computational abilities from agents, beyond merely sampling randomly a player and observing the difference in payoff: specifically being able to compute a best reply strategy to the current population state:

$$\dot{x}_i = BR(\mathbf{x}) - x_i \tag{4}$$

where,

$$BR(\mathbf{x}) = \arg\max_{\mathbf{y}} \mathbf{y} f(\mathbf{x})$$

Discrete choice models-the Logit evolutionary dynamics

Apart from the highly unrealistic assumptions regarding agents capacity to compute a perfect best reply to a given population state there is also the drawback that (4) defines a differential inclusion, i.e. a set-valued function. The best responses may not be unique and multiple trajectory paths can emerge from the same initial conditions. A 'smoothed' approximation of the Best Reply dynamics - the Logit dynamics - was introduced by Fudenberg and Levine (1998). It was obtained by stochastically perturbing the payoff vector $f(\mathbf{x})$ and deriving the Logit revision protocol:

$$\rho_{ij}(f(\mathbf{x}), \mathbf{x}) = \frac{\exp[\eta^{-1} f_j(\mathbf{x})]}{\sum_k \exp[\eta^{-1} f_k(\mathbf{x})]} = \frac{\exp[\eta^{-1} A\mathbf{x})_i]}{\sum_k \exp[\eta^{-1} A\mathbf{x})_k]} \ , \tag{5}$$

where $\eta > 0$ is the noise level. Here ρ_{ij} represents the probability of player i switching to strategy j when provided with a revision opportunity. For high levels of noise the choice is fully random (no optimization) while for η close to zero the switching probability is almost one.

Another way to obtain (5) is to deterministically perturb the set-valued best reply correspondence (4) with a strictly concave function $V(\mathbf{y})$ (Hofbauer (2000)):

$$BR_\eta(\mathbf{x}) = \arg\max_{\mathbf{y} \in \boldsymbol{\Delta}^{n-1}} [\mathbf{y} \cdot (A\mathbf{x}) + V_\eta(\mathbf{y})]$$

For a particular choice of the perturbation function $V_\eta(\mathbf{y}) = \eta \sum_{i=1,n} y_i \log y_i$, $\mathbf{y} \in \boldsymbol{\Delta}^{n-1}$ the resulting objective function is single-valued and smooth; the first order condition yields the unique logit choice rule:

$$BR_\eta(\mathbf{x})_i = \frac{\exp[\eta^{-1} A\mathbf{x})_i]}{\sum_k \exp[\eta^{-1} A\mathbf{x})_k]}.$$

Plugging the Logit revision protocol (5) back into the general form of the mean field dynamic (2) and making the substitution $\beta = \eta^{-1}$ we obtain a well-behaved system of o.d.e.'s, the Logit dynamics with the *intensity of choice* (Brock and Hommes (1997)) parameter β:

$$\dot{x}_i = \frac{\exp[\beta A\mathbf{x})_i]}{\sum_k \exp[\beta A\mathbf{x})_k]} - x_i \tag{6}$$

The quantal response equilibria of McKelvey and Palfrey (1995), also called 'logit equilibria' for a logistic response function, are fixed points of the Logit Dynamics. When $\beta \to \infty$ the probability of switching to the discrete 'best response' j is close to one while for a very low intensity of choice ($\beta \to 0$) the switching rate is independent of the actual performance of the alternative strategies (equal probability mass is put on each of them). The Logit dynamics displays some properties characteristic to the logistic growth function, namely high growth rates ($\dot{x}_i$) for small values of x_i and growth 'levelling off' when close to the upper bound. This means that a specific frequency x_i grows faster when it is already large in the Replicator Dynamics relative to the Logit dynamics.

3 Hopf and degenerate Hopf bifurcations

Since the possible existence of stable limit cycles is an important theme in this paper, for the convenience of the general reader we briefly review the main bifurcation route towards a stable limit cycle, the Hopf bifurcation. In a one-parameter family of continuous-time systems, the only generic bifurcation through which a limit cycle is created or disappears is the non-degenerate Hopf bifurcation. The planar case will be discussed first and then, briefly, the methods to reduce higher-dimensional systems to the two-dimensional case.

Assume we are given a parameter-dependent, two dimensional system (as in, for example, Kuznetsov (1995)):

$$\dot{\mathbf{x}} = f(\mathbf{x}, \alpha), \mathbf{x} \in \mathbb{R}^2, \alpha \in R, \ f \ \text{smooth}, \tag{7}$$

with a steady state at $\mathbf{x}^* = 0$, i.e. $f(0, \alpha) = 0$ and the Jacobian matrix evaluated at the fixed point $\mathbf{x}^* = 0$ having a pair of purely imaginary, complex conjugate eigenvalues at the bifurcation value $\alpha = 0$, i.e. $\lambda_{1,2} = \mu(\alpha) \pm i\omega(\alpha)$ with $\mu(\alpha) < 0$ for $\alpha < 0, \mu(0) = 0$ and $\mu(\alpha) > 0$ for $\alpha > 0$.

If , in addition, the following *genericity*[2] conditions are satisfied:

(i) $\left[\frac{\partial \mu(\alpha)}{\partial \alpha}\right]_{\alpha=0} \neq 0$ - *transversality* condition

(ii) $l_1(0) \neq 0$, where $l_1(0)$ is the *first* Lyapunov coefficient[3] - *nondegeneracy* condition,

then the system (7) undergoes a Hopf bifurcation at $\alpha = 0$. As α increases the steady state changes stability from a stable focus into an unstable focus.

There are two types of Hopf bifurcation, depending on the sign of the first Lyapunov coefficient $l_1(0)$:

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[2] *Genericity* usually refer to transversality and non-degeneracy conditions. Rougly speaking, the transversality condition means that complex eigenvalues cross the real line at non-zero speed. The nondegeneracy condition implies non-zero higher-order coefficients in equation (10) below. It ensures that the singularity $\mathbf{x}^*$ is typical (i.e. 'nondegenerate') for a class of singularities satisfying certain bifurcation conditions. See Kuznetsov (1995) pp. $89 - 98$ for a complete mathematical description of the Hopf bifurcation.

[3] This is the coefficient of the third order term in the normal form of the Hopf bifurcation (see equation (10) below).

(a) If $l_1(0) < 0$ then the Hopf bifurcation is *supercritical*: the stable focus x becomes *unstable* for $\alpha > 0$ and is surrounded by an *isolated, stable* closed orbit (limit cycle).

(b)If $l_1(0) > 0$ then the Hopf bifurcation is *subcritical*: for $\alpha < 0$ the basin of attraction of the stable focus $\mathbf{x}^*$ is surrounded by an *unstable* cycle which shrinks and disappears as α crosses the critical value $\alpha = 0$ while the system diverges quickly from the neighbourhood of $\mathbf{x}^*$.

In case (a) the stable cycle is created immediately *after* α reaches the critical value and thus the Hopf bifurcation is called *super*critical, while in case (b) the unstable cycle already exists *before* the critical value, i.e. a *sub*critical Hopf bifurcation. The supercritical Hopf is also known as *a soft* or *non-catastrophic* bifurcation because , even when the system becomes unstable, it still lingers within a small neighbourhood of the equilibrium bounded by the limit cycle, while the subcritical case is a *sharp/catastrophic* bifurcation as the system moves quickly far away from the unstable equilibrium.

If the first Lyapunov coefficient $l_1(0) = 0$ then there is a degeneracy in the third order terms of the normal form and , if other, higher order nondegeneracy conditions hold(i.e. non-vanishing *second* Lyapunov coefficient) then the bifurcation is called *Bautin* or *generalized* Hopf bifurcation. This happens when the first Lyapunov coefficient vanishes at the given equilibrium $\mathbf{x}^*$ but the following higher-order *genericity*[4] conditions hold:

(i) $l_2(0) \neq 0$, where $l_2(0)$ is the *second* Lyapunov coefficient - nondegeneracy condition

(ii) the map $\alpha \rightarrow (\mu(\alpha), l_1(\alpha))$ is regular (i.e. the Jacobian matrix is nonsingular) at the critical value $\alpha = 0$ - transversality condition.

Depending on the sign of $l_2(0)$, at the Bautin point the system may display a limit cycle *bifurcating* into two or more cycles , *coexistence* of stable and unstable cycles which collide

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[4]Technically, these 'higher-order' genericity conditions ensure that there are smooth invertible coordinate transformations, depending smoothly on parameters, together with parameter changes and (possibly) time re-parametrizations such that (7) can be reduced to a 'simplest' form, the *normal form.* See Kuznetsov (1995) pp. $309 - 316$ for more details on the Bautin (generalized Hopf) bifurcation and for an expression for the second Lyapunov coefficient $l_2(0)$.

and disappear, together with so-called cycle *blow-up.*

Computation of the first Lyapunov coefficient

For the planar case, the first Lyapunov coefficient $l_1(0)$ can be computed *without* explicitly deriving the normal form, from the Taylor coefficients of a transformed version of the original vector field. The computation of $l_1(0)$ for higher dimensional systems makes use of the Center Manifold Theorem by which the orbit structure of the original system near $(\mathbf{x}^*, \alpha)$, is fully determined by its restriction to the two-dimensional center manifold[5]. On the center manifold (7) takes the form (Wiggins (2003)):

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\lambda(\alpha) & -\operatorname{Im}\lambda(\alpha) \\ \operatorname{Im}\lambda(\alpha) & \operatorname{Re}\lambda(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, \alpha) \\ f^2(x, y, \alpha) \end{pmatrix} \tag{8}$$

where $\lambda(\alpha)$ is an eigenvalue of the linearized vector field around the steady state and the nonlinear functions $f_1(x, y, \alpha)$, $f_2(x, y, \alpha)$ of order $O(|x|^2)$ are derived from the original vector field. Wiggins (2003) also provides a procedure for transforming (7) into (8). Specifically, for any vector field $\dot{\mathbf{x}} = F(\mathbf{x}), \mathbf{x} \in \mathbb{R}^2$ let $DF(\mathbf{x}^*)$ denote the Jacobian evaluated at the fixed point $\mathbf{x}^*$. Then $\dot{\mathbf{x}} = F(\mathbf{x})$ is equivalent to:

$$\dot{\mathbf{x}} = J\mathbf{x} + T^{-1}\bar{F}(T\mathbf{x}) \tag{9}$$

where J stands for the real Jordan canonical form of $DF(\mathbf{x}^*)$, T is the matrix transforming $DF(\mathbf{x}^*)$ into the Jordan form, and $\bar{F}(\mathbf{x}) = F(\mathbf{x}) - DF(\mathbf{x}^*)\mathbf{x}$. At the Hopf bifurcation point α, $\lambda_{1,2} = \pm i\omega$ and the first Lyapunov coefficient is (Wiggins (2003)):

$$l_1(\alpha) = \frac{1}{16}[f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}] + \tag{10}$$
$$+ \frac{1}{16\omega}[f^1_{xy}(f^1_{xx} + f^1_{yy}) - f^2_{xy}(f^2_{xx} + f^2_{yy}) - f^1_{xx}f^2_{xx} + f^1_{yy}f^2_{yy}]$$

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[5]The center manifold is the invariant manifold spanned by the eigenvectors corresponding to the eigenvalues with zero real part (Kuznetsov (1995) pp. 157).

4   Rock-Scissors-Paper Games

The Rock-Paper-Scissors class of games (or games of cyclical dominance) formalize strategic interactions where each strategy E_i is an unique best response to strategy E_{i+1} for $i = 1, 2$ and E_3 is a best response to E_1[6]:

$$
A = \begin{pmatrix}
 & E_1 & E_2 & E_3 \\
E_1 & 0 & \delta_2 & -\varepsilon_3 \\
E_2 & -\varepsilon_1 & 0 & \delta_3 \\
E_3 & \delta_1 & -\varepsilon_2 & 0
\end{pmatrix} ; \delta_i, \varepsilon_i \geq 0 \tag{11}
$$

If matrix (11) is circulant (i.e. $\delta_i = \delta, \varepsilon_i = \varepsilon, i = 1, 2, 3$) then the RSP game is called *circulant*, while for a non-circulant matrix (11) we have a *generalized* RSP game[7]. The behavior of Replicator and Logit Dynamics on the class of circulant RSP games will be investigated, both analytically and numerically, in the first and second part of this subsection, respectively.

Circulant RSP Game and Replicator Dynamics

The circulant RSP game is a first generalization of the classical, *zero-sum* form of RSP game as discussed in, for instance, Hofbauer and Sigmund (2003):

$$
A = \begin{pmatrix}
0 & \delta & -\varepsilon \\
-\varepsilon & 0 & \delta \\
\delta & -\varepsilon & 0
\end{pmatrix} , \delta, \varepsilon > 0 \tag{12}
$$

---

[6]See Weissing (1991) for a thourough charaterization of the *discrete-time* Replicator Dynamics behavior on the most general representation of this class of games, i.e. the matrix (11) before the normalization.

[7]We focus on the simple case of circulant RSP game, which is sufficient for our purpose; see Chapter 2 of Ochea (2009) for the numerical computation of bifurcation curves under continuous-time Logit Dynamics on the generalized RSP games.

Letting $\mathbf{x(t)} = (x(t), y(t), z(t))$ denote the population state at time instance t define a point from the 2-dimensional simplex, the payoff vector $[A\mathbf{x}]$ is obtained via (1):

$$[A\mathbf{x}] = \begin{pmatrix} y\delta - z\varepsilon \\ -x\varepsilon + z\delta \\ x\delta - y\varepsilon \end{pmatrix} \tag{13}$$

Average fitness of the population is:

$$\mathbf{xAx} = x\left(y\delta - z\varepsilon\right) + y\left(-x\varepsilon + z\delta\right) + z\left(x\delta - y\varepsilon\right) \tag{14}$$

The replicator equation (3) with the game matrix (12) induce on the 2-simplex the following vector field:

$$\begin{bmatrix} \dot{x} = x[y\delta - z\varepsilon - (x\left(y\delta - z\varepsilon\right) + y\left(-x\varepsilon + z\delta\right) + z\left(x\delta - y\varepsilon\right))] \\ \dot{y} = y[-x\varepsilon + z\delta - (x\left(y\delta - z\varepsilon\right) + y\left(-x\varepsilon + z\delta\right) + z\left(x\delta - y\varepsilon\right))] \\ \dot{z} = z[x\delta - y\varepsilon - (x\left(y\delta - z\varepsilon\right) + y\left(-x\varepsilon + z\delta\right) + z\left(x\delta - y\varepsilon\right))] \end{bmatrix} \tag{15}$$

While Hofbauer and Sigmund (2003) use the Poincare-Bendixson theorem together with the Dulac criterion to prove that limit cycles cannot occur in games with three strategies under the replicator we will derive this negative result using tools from dynamical systems, in particular the Hopf bifurcation and 'normal form' theory. The same toolkit will be applied next to the Logit Dynamic and a positive result - stable limit cycles do occur - will be derived.

As we are interested in limit cycles within the simplex we only consider interior fixed points of this system (any replicator dynamic has the simplex vertices as steady states, too). For the parameter range $\varepsilon, \delta > 0$ the barycentrum $\mathbf{x}^* = [x = 1/3, y = 1/3, z = 1/3]$ is always an interior fixed point of (15). In order to analyze its stability properties we obtain first, by

substituting $z = 1 - x - y$ into (15), a proper 2 dimensional dynamical system of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x\varepsilon + 2xy\varepsilon - x^2\delta + 2x^2\varepsilon + x^3\delta - x^3\varepsilon + xy^2\delta + x^2y\delta - xy^2\varepsilon - x^2y\varepsilon \\ y\delta - 2xy\delta - 2y^2\delta + y^2\varepsilon + y^3\delta - y^3\varepsilon + xy^2\delta + x^2y\delta - xy^2\varepsilon - x^2y\varepsilon \end{bmatrix}. \quad (16)$$

We can detect the Hopf bifurcation threshold at the point where the trace of the Jacobian matrix of (16) is equal to zero. The Jacobian evaluated at the barycentrical steady state $\mathbf{x}^*$ is $\begin{bmatrix} \varepsilon & \delta + \varepsilon \\ -\delta - \varepsilon & -\delta \end{bmatrix}$, with eigenvalues: $\lambda_{1,2}(\varepsilon, \delta) = \frac{1}{2}(\varepsilon - \delta) \pm i\frac{\sqrt{3}}{2}\sqrt{(\varepsilon + \delta)^2}$ and trace $\varepsilon - \delta$. For $\delta < \varepsilon$, $\mathbf{x}^*$ is an unstable focus, at $\delta = \varepsilon$ a pair of imaginary eigenvalues crosses the imaginary axis($\lambda_{1,2} = \pm i\sqrt{3}\delta$), while for $\delta > \varepsilon$ $\mathbf{x}^*$ becomes a stable focus (see Fig. (1) below). This is consistent with Theorem 6 in Zeeman (1980) which states that the determinant of the matrix A determines the stability properties of the interior fixed point. In the circulant RSP game (12) $Det A = \varepsilon^3 - \delta^3$ vanishes for $\varepsilon = \delta$. By the same Theorem 6 in Zeeman (1980) the vector field (15) has a *center* in the 2-simplex and a *continuum* of cycles if $Det A = 0$.

Alternatively, our local bifurcation analysis shows that a Hopf bifurcation occurs when $\varepsilon = \delta$ and in order to ascertain its features - sub/supercritical or degenerate - we have to further investigate the nonlinear vector field near the $(\mathbf{x}^*, \varepsilon = \delta)$ point. The Hopf bifurcation necessary condition $\varepsilon = \delta$ implies that vector field (16) takes a simpler form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -x\delta + 2xy\delta + x^2\delta \\ y\delta - 2xy\delta - y^2\delta \end{bmatrix} \quad (17)$$

Using equation (8) and the eigenvalues $\lambda_{1,2} = \pm i\sqrt{3}\delta$ at the Hopf bifurcation point ($\varepsilon = \delta$), we can back out the nonlinear functions f^1, f^2 needed for the computation of the first Lyapunov coefficient:

$$\begin{bmatrix} f_1(x, y) = y\sqrt{3}\delta - x\delta + 2xy\delta + x^2\delta \\ f_2(x, y) = -x\sqrt{3}\delta + y\delta - 2xy\delta - y^2\delta \end{bmatrix} \quad (18)$$

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We can now state the following result:

Lemma 1 *All Hopf bifurcations are degenerate in the circulant Rock-Scissor-Paper game under Replicator Dynamics.*

Proof. From the nonlinear functions $f_1(x, y)$, $f_2(x, y)$ derived above we can easily compute the first Lyapunov coefficient (10) as $l_1(\varepsilon^{Hopf} = \delta^{Hopf}) = 0$ which means that there is a first degeneracy in the third order terms from the Taylor expansion of the normal form. The detected bifurcation is a *degenerate Hopf* bifurcation[8]. ∎

Although, in general, the orbital structure at a degenerate Hopf bifurcation may be extremely complicated (see Section 3), for our particular vector field induced by the Replicator Dynamics it can be shown by Lyapunov function techniques (Hofbauer and Sigmund (2003), Zeeman (1980)) that a continuum of cycles is born *exactly* at the critical parameter value[9].

(a) Unstable focus,
$\delta = 0.6, \varepsilon = 1$

(b) Degenerate Hopf,
$\delta = 1, \varepsilon = 1$

(c) Stable focus,
$\delta = 1.1, \varepsilon = 1$

Figure 1: Rock-Scissors-Paper and Replicator Dynamics for fixed $\varepsilon = 1$ and varying δ. Qualitative changes in the phase portraits - unstable focus (Panel (a)), continuum of cycles (Panel (b)) and stable focus (Panel (c)) - occur as we increase δ from below to above ε.

---

[8] Since all 3rd and higher-order terms in (18) are zero, the Hopf bifurcation has, in fact, an "infinite number of degeneracies" with all higher order Lyapunov coefficients $l_i(\varepsilon^{Hopf} = \delta^{Hopf}) = 0, i \geq 2$. This explains why, for the Replicator Dynamics, a continuum of cycles exists after the Hopf bifurcation.

[9] The absence of a generic Hopf bifurcation does not suffice to conclude that the vector field admits no isolated periodic orbits and 'global' results are required (e.g. the Bendixson-Dulac method, positive divergence of the vector field on the simplex).

It is worthwhile pointing out the connection between our local (in)stability results and the static concept of Evolutionary Stable Strategy (ESS). As already noted in the literature (Zeeman (1980), Hofbauer (2000)) ESS implies (global) asymptotic stability under a wide class of evolutionary dynamics - Replicator Dynamics, Best Response and Smooth Best Response Dynamics, Brown-von Newmann-Nash, etc. The reverse implication does not hold in general, i.e. the local stability analysis does not suffice to qualify an attractor as an ESS. However, for $\delta < \varepsilon$ we have shown that $\mathbf{x}^*$ is an unstable focus and we can conclude that, for this class of RSP games, the barycentrum is *not* an ESS. Indeed, the case $\delta < \varepsilon$ is the so called *bad* RSP game (Sandholm (2008)) which is known *not* to have an ESS. For $\delta > \varepsilon$ we are in the *good* RSP game and it does have an interior ESS which coincides with the asymptotically stable rest point $\mathbf{x}^*$ of the Replicator Dynamics. Last, if $\delta = \varepsilon$ (i.e. *standard* RSP) then $\mathbf{x}^*$ is a neutrally stable strategy/state and we proved that in this zero-sum game the Replicator undergoes a degenerate Hopf bifurcation. The Hopf bifurcation scenario is illustrated in Fig. 1 for a fixed value of $\varepsilon = 1$.

Circulant RSP Game and Logit Dynamics

The Logit evolutionary dynamics (6) applied to the circulant normal form game (12) leads to the following vector field:

$$
\begin{bmatrix}
\dot{x} = \frac{\exp(\beta(y\delta-z\varepsilon))}{\exp(\beta(y\delta-z\varepsilon))+\exp(\beta(-x\varepsilon+z\delta))+\exp(\beta(x\delta-y\varepsilon))} - x \\
\dot{y} = \frac{\exp(\beta(-x\varepsilon+z\delta))}{\exp(\beta(y\delta-z\varepsilon))+\exp(\beta(-x\varepsilon+z\delta))+\exp(\beta(x\delta-y\varepsilon))} - y \\
\dot{z} = \frac{\exp(\beta(x\delta-y\varepsilon))}{\exp(\beta(y\delta-z\varepsilon))+\exp(\beta(-x\varepsilon+z\delta))+\exp(\beta(x\delta-y\varepsilon))} - z
\end{bmatrix}
\tag{19}
$$

By substituting $z = 1 - x - y$ into (19) we can reduce to a 2-dimensional system:

$$
\begin{bmatrix}
\frac{\exp(\beta(y\delta-\varepsilon(-x-y+1)))}{\exp(\beta(x\delta-y\varepsilon))+\exp(\beta(-x\varepsilon+\delta(-x-y+1)))+\exp(\beta(y\delta-\varepsilon(-x-y+1)))} - x = 0 \\
\frac{\exp(\beta(-x\varepsilon+\delta(-x-y+1)))}{\exp(\beta(x\delta-y\varepsilon))+\exp(\beta(-x\varepsilon+\delta(-x-y+1)))+\exp(\beta(y\delta-\varepsilon(-x-y+1)))} - y = 0
\end{bmatrix}
\tag{20}
$$

The $2-\text{dim}$ simplex barycentrum $[x = 1/3, y = 1/3, z = 1/3]$ remains a fixed point irrespective of the value of β. The Jacobian of (20) evaluated at this steady state is:

$$\begin{bmatrix} \frac{1}{3}\beta\varepsilon - 1 & \frac{1}{3}\beta\delta + \frac{1}{3}\beta\varepsilon \\ -\frac{1}{3}\beta\delta - \frac{1}{3}\beta\varepsilon & -\frac{1}{3}\beta\delta - 1 \end{bmatrix}, \text{ with eigenvalues: } \lambda_{1,2} = \frac{1}{6}\beta(\varepsilon - \delta) - 1 \pm i\frac{1}{2}\sqrt{\frac{1}{3}}(\beta\delta + \beta\varepsilon).\text{The}$$

Hopf bifurcation (necessary) condition $\text{Re}(\lambda_{1,2}) = 0$ leads to:

$$\beta^{Hopf} = \frac{6}{\varepsilon - \delta}, 0 < \delta < \varepsilon \tag{21}$$

We notice that for the zero-sum RSP game ($\varepsilon = \delta$) - unlike Replicator Dynamics which exhibited a degenerate Hopf at $\varepsilon = \delta$ - the barycentrum is always asymptotically stable ($\text{Re}\,\lambda_{1,2} = -1$) under Logit Dynamics. For $\varepsilon < \delta$, i.e. the "good" RSP game, the interior steady state is always locally stable under Logit Dynamics. We have the following:

Lemma 2 *The Logit Dynamics (19) on the circulant Rock-Scissors-Paper game exhibits a generic Hopf bifurcation and, therefore, has limit cycles. Moreover, all such Hopf bifurcations are supercritical, i.e. the limit cycle is born stable.*

Proof. Condition (21) gives the necessary first-order condition for Hopf bifurcation to occur; in order to show that the Hopf bifurcation is non-degenerate we have to compute the first Lyapunov coefficient $l_1(\beta^{Hopf}, \varepsilon, \delta)$ according to (10) and check whether it is non-zero. The analytical form of this coefficient takes a complicated expression of exponential terms (see Appendix A) which, after some tedious computations, boils down to:

$$l_1(\beta^{Hopf}, \varepsilon, \delta) = -\frac{864(\delta\varepsilon + \delta^2 + \varepsilon^2)}{3(3\varepsilon - 3\delta)^2} < 0, \quad \varepsilon > \delta > 0.$$

∎

Computer simulations of this route to a stable cycle are shown in Fig. 2. We notice that as β increases from 10 to 35 (i.e. the noise level is decreasing) the interior stable steady state looses stability via a supercritical Hopf bifurcation and a small, stable limit cycle emerges

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around the unstable steady state. Unlike Replicator Dynamics, *stable* cyclic behavior does occur under the Logit dynamics even for three-strategy games.

(a) Stable focus, $\beta = 10$ (b) Generic Hopf, $\beta = 30$ (c) Limit cycle, $\beta = 35$

Figure 2: Rock-Scissors-Paper and Logit Dynamics for fixed game $\varepsilon = 1, \delta = 0.8$ and different values of the behavioral parameter β. Qualitative changes in the phase portraits: a stable interior fixed point (Panel (a)) loses stability when the critical threshold $\beta = 30$ is hit, via a generic, supercritical Hopf bifurcation (Panel (b)) and, if β is pushed up even further, a stable limit cycle is born (Panel (c)).

Similar cycles can be detected in the payoff parameter space as in Fig. 3 where the noise level is kept constant and the game parameter δ is allowed to change.

(a) Stable focus, $\delta = 1$ (b) Hopf, $\delta = 0.399$ (c) Limit cycle, $\delta = 0.1$

Figure 3: Rock-Scissors-Paper and Logit Dynamics for fixed behavioral parameter $\beta = 10$ and free game parameter δ [$\varepsilon = 1$]. Qualitative changes in the phase portraits: a stable interior fixed point (Panel (a)) loses stability when the critical threshold $\delta = 0.399$ is hit, via a generic, supercritical Hopf bifurcation (Panel (b)). Panel (c) display a stable limit cycle for $\delta = 0.1$ that attracts trajectories originating both outside and inside the cycle.

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Figure 4 depicts a *curve* of Hopf bifurcations in the $(\beta, \varepsilon - \delta)$ parameter space. As we cross this Hopf curve from below the stable interior fixed point loses stability and a stable periodic attractor surrounds it. The picture summarizes the possible types of dynamical behavior for the "good" and the "bad" Rock-Paper-Scissors game. Thus, for $\varepsilon < \delta$ (i.e. "good" RSP game) the interior, fully mixed steady state is always locally[10] stable under Logit Dynamics, similar to the behavior of Replicator Dynamics on this class of RSP games. For $\varepsilon > \delta$ (i.e. "bad" RSP game) the behavior depends on how sensitive players are to differences in fitness, and, unlike Replicator Dynamics, Logit displays richer dynamics.

Figure 4: Rock-Scissors-Paper and Logit Dynamics: Supercritical Hopf curve in $(\beta, \varepsilon - \delta)$ parameter space. Continuation of a detected co-dimension I singularity (i.e. Hopf bifurcation) with respect to a second parameter gives rise to a curve of supercritical Hopf points in the circulant RSP class of games.

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[10]Numerical simulations suggest that for $\varepsilon < \delta$ the steady state is even globally stable.

5 Coordination Game

Using topological arguments, Zeeman (1980) shows that three-strategies games have at most one interior, isolated fixed point under Replicator Dynamics[11]. This implies that a fold[12] catastrophe in which two *isolated* fixed points collide and disappear when some parameter is varied, cannot occur *in the interior* of the simplex. In this section we show - by means of the classical coordination game - that multiple, isolated, interior steady-states may exist under Logit Dynamics and show that the fold catastrophe occurs when we alter the intensity of choice β. We use advanced numerical tools (Dhooge et al. (2003)) for detecting fold catastrophe bifurcation curves in the parameter space. As our goal is to investigate possible routes to multiplicity of steady states, we consider the simplest version of a symmetric 3×3 pure coordination game, given by the following payoff matrix:

$$A = \begin{pmatrix} 1-\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\varepsilon \end{pmatrix}, 0 < \varepsilon < 1. \tag{22}$$

Coordination Game and Replicator Dynamics

Under Replicator Dynamics, vector field $\dot{\mathbf{x}} = \mathbf{x}[A\mathbf{x} - \mathbf{x}A\mathbf{x}]$ on the Coordination game (22) yields:

$$\begin{bmatrix} \dot{x} = x[x\left(-\varepsilon+1\right) - (y^2 + z^2\left(\varepsilon+1\right) + x^2\left(-\varepsilon+1\right))] \\ \dot{y} = y[y - (y^2 + z^2\left(\varepsilon+1\right) + x^2\left(-\varepsilon+1\right))] \\ \dot{z} = z[z\left(\varepsilon+1\right) - (y^2 + z^2\left(\varepsilon+1\right) + x^2\left(-\varepsilon+1\right))] \end{bmatrix} \tag{23}$$

This systems has the following fixed points in simplex coordinates:

---

[11] See Theorem 3 pp. 478 in Zeeman (1980).

[12] In a parameter-dependent, continous-time dynamical system a fold bifurcation occurs when the Jacobian matrix evaluated at the critical equilibrium has a zero eigenvalue. Technically, other higher-order non-degeneracy conditions must hold, as well. See Kuznetsov (1995) pp. $81-84$ for a complete treatment of the fold bifurcation.

(i) 3 stable nodes at the simplex vertices: $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

(ii) a repelling interior steady state: $O \left(\frac{\varepsilon+1}{3-\varepsilon^2}, \frac{1-\varepsilon^2}{3-\varepsilon^2}, \frac{1-\varepsilon}{3-\varepsilon^2} \right)$

(iii) 3 saddles at the boundaries:

$M \left(\frac{1}{2-\varepsilon}, \frac{1-\varepsilon}{2-\varepsilon}, 0 \right), N \left(\frac{1}{2-\varepsilon}, 0, \frac{1-\varepsilon}{2-\varepsilon} \right), P \left(0, \frac{1}{2-\varepsilon}, \frac{1-\varepsilon}{2-\varepsilon} \right)$

The eigenvalues of the corresponding Jacobian evaluated at each of the above fixed points are:

$A \left[x = 0, y = 1, z = 0 \right]$; eigenvalues: $\lambda_{1,2,3} = -1$

$B \left[x = 1, y = 0, z = 0 \right]$; eigenvalues: $\lambda_{1,2,3} = \varepsilon - 1$

$C \left[x = 0, y = 0, z = 1 \right]$; eigenvalues: $\lambda_{1,2,3} = -\varepsilon - 1$

$O \left[x = \frac{\varepsilon+1}{3-\varepsilon^2}, y = \frac{1-\varepsilon^2}{3-\varepsilon^2}, z = \frac{1-\varepsilon}{3-\varepsilon^2} \right]$; eigenvalues: $\lambda_{1,2} = \frac{\varepsilon^2-1}{\varepsilon^2-3} > 0$, for $\varepsilon \in (0, 1)$

$M \left[x = \frac{1}{2-\varepsilon}, y = \frac{1-\varepsilon}{2-\varepsilon}, z = 0 \right]$; eigenvalues: $\lambda_{1,2} = \pm \frac{1-\varepsilon}{\varepsilon-2}$

$N \left[x = \frac{1}{2-\varepsilon}, y = 0, z = \frac{1-\varepsilon}{2-\varepsilon} \right]$; eigenvalues: $\lambda_{1,2} = \pm \frac{1}{-4\varepsilon+\varepsilon^2+4} \left(2\varepsilon + \varepsilon^2 - \varepsilon^3 - 2 \right)$

$P \left[x = 0, y = \frac{1}{2-\varepsilon}, z = \frac{1-\varepsilon}{2-\varepsilon} \right]$; eigenvalues: $\lambda_{1,2} = \pm \frac{1}{-4\varepsilon+\varepsilon^2+4} \left(\varepsilon + \varepsilon^2 - \varepsilon^3 - 2 \right)$

This fixed points structure is consistent with Zeeman (1980) result that no fold catastrophes (i.e. multiple, isolated and interior steady states) can occur under the Replicator Dynamics. The three saddles together with the interior source define the basins of attractions for the stable nodes at simplex vertices. The boundaries of the simplex are invariant under Replicator Dynamics[13] and it suffices to show that the segment lines $[OM], [ON]$, and $[OP]$, as illustrated in Fig. 5 , are also invariant under this dynamics.

---

[13] see proof in Hofbauer and Sigmund (2003) pp. 67-68.

Figure 5: Coordination Game with $\varepsilon = 0.1$ and Replicator Dynamics - Invariant manifolds

Lemma 3 *The segment lines $[OM], [ON]$, and $[OP]$ are invariant under Replicator Dynamics and they form, along with the simplex edges, the boundaries of the basins of attraction for the three stable steady states A, B and C at the vertices.*

Proof. Using the standard substitution $z = 1 - x - y$ system (23) becomes:

$$
\left[
\begin{array}{c}
\dot{x} = x^2 - x^3 - xy^2 - x^2\varepsilon + x^3\varepsilon - x\left(-x - y + 1\right)^2 - x\varepsilon\left(-x - y + 1\right)^2 \\
\dot{y} = y^2 - y^3 - x^2 y + x^2 y\varepsilon - y\left(-x - y + 1\right)^2 - y\varepsilon\left(-x - y + 1\right)^2
\end{array}
\right]
\tag{24}
$$

Segment $[MO]$ is defined, in simplex coordinates, by $y = x(1 - \varepsilon)$. Along this line we have:

$$
\begin{aligned}
\left[\frac{\dot{y}}{\dot{x}}\right]_{[MO]:y=x(1-\varepsilon)} &= \frac{x\left(-\varepsilon + 1\right)\left(x\left(-\varepsilon + 1\right) - x^2\left(-\varepsilon + 1\right) - x^2\left(-\varepsilon + 1\right)^2 - \left(\varepsilon + 1\right)\left(-x - y + 1\right)^2\right)}{x\left(x\left(-\varepsilon + 1\right) - x^2\left(-\varepsilon + 1\right) - x^2\left(-\varepsilon + 1\right)^2 - \left(\varepsilon + 1\right)\left(-x - y + 1\right)^2\right)} \\
&= 1 - \varepsilon,
\end{aligned}
$$

which is exactly the slope of $[MO]$. Similarly, invariance results are obtained for segments $[ON]$ and $[OP]$ defined by $x(1 - \varepsilon) = z(1 + \varepsilon)$ and $y = z(1 + \varepsilon)$. ∎

Analytically, the size of the basins of attraction of the three different stable nodes are

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$\varepsilon \backslash \mathcal{A}$	$\mathcal{A}(1,0,0)$	$\mathcal{B}(0,1,0)$	$\mathcal{C}(0,0,1)$	Long-Run Average Welfare
0	33.33%	33.33%	33.33%	1
0.1	29.9%	33.2%	36.7%	1.0048
0.5	15.5%	30.3%	54.54%	1.1986
0.6	8.92%	12.18%	78.90%	1.419 9
0.9	1.2%	12%	86.7%	1.512

Table 1: Coordination Game, Replicator Dynamics–relative sizes of basins of attraction of 3 stable steady states and long-run average welfare for different payoff-perturbation parameter ε. As payoff differences ε increases the long-run average welfare increases

determined by the areas of the polygons delineated by the invariant manifolds $[OM], [ON]$, and $[OP]$ and the 2-dim simplex boundaries. These areas are given by:

$$\mathcal{A}(1,0,0) = \mathcal{A}[AMON] = \begin{bmatrix} \left| \frac{1}{8}\frac{\sqrt{3}}{2-\varepsilon} - \frac{1}{4}\sqrt{3}\frac{3-2\varepsilon}{4-2\varepsilon} - \frac{1}{8}\sqrt{3}\frac{\varepsilon+1}{3-\varepsilon^2} + \frac{1}{4}\sqrt{3}\frac{3-\varepsilon}{6-2\varepsilon^2} - \frac{1}{4}\frac{\sqrt{3}}{2-\varepsilon}\frac{3-\varepsilon}{6-2\varepsilon^2} \right| \\ + \left| \frac{1}{4}\sqrt{3}\left(\varepsilon+1\right)\frac{3-2\varepsilon}{(4-2\varepsilon)(3-\varepsilon^2)} + \frac{1}{8}\frac{\sqrt{3}}{2-\varepsilon} - \frac{1}{4}\frac{\sqrt{3}}{4-2\varepsilon} - \frac{1}{8}\sqrt{3}\frac{\varepsilon+1}{3-\varepsilon^2} \right| \\ + \left| \frac{1}{4}\sqrt{3}\frac{3-\varepsilon}{6-2\varepsilon^2} + \frac{1}{4}\sqrt{3}\frac{\varepsilon+1}{(4-2\varepsilon)(3-\varepsilon^2)} - \frac{1}{4}\frac{\sqrt{3}}{2-\varepsilon}\frac{3-\varepsilon}{6-2\varepsilon^2} \right| \end{bmatrix}$$

$$\mathcal{B}(0,1,0) = \mathcal{A}[BMOP] = \left| \frac{1}{4}\sqrt{3}\frac{\varepsilon+1}{(4-2\varepsilon)(3-\varepsilon^2)} - \frac{1}{4}\frac{\sqrt{3}}{2-\varepsilon}\frac{3-\varepsilon}{6-2\varepsilon^2} \right| + \left| \frac{1}{4}\sqrt{3}\left(\varepsilon+1\right)\frac{1-\varepsilon}{(2-\varepsilon)(3-\varepsilon^2)} \right|$$

$$\mathcal{C}(0,0,1) = \mathcal{A}[CNOP] = \left| \frac{\frac{1}{4}\sqrt{3}(\varepsilon+1)(1-\varepsilon)-(1+\varepsilon)}{(2-\varepsilon)(3-\varepsilon^2)} - \frac{1}{4}\sqrt{3}\frac{\varepsilon+1}{3-\varepsilon^2} \right| + \left| \frac{1}{4}\sqrt{3}\left(\varepsilon+1\right)\frac{3-2\varepsilon}{(4-2\varepsilon)(3-\varepsilon^2)} \right|$$

The relative size of the basins vary with the payoff parameter ε as in Table 1. When the payoff asymmetries are small, i.e. ε is small, the simplex of initial conditions is divided equally among the three stable equilibria(Fig. 6a). As the payoff discrepancies increase, i.e. when ε increases the relative size of the welfare-maximizing equilibrium $(0,0,1)$ increases (see Fig. 6b).

(a) $A_\varepsilon[\varepsilon = 0.1]$ (b) $A_\varepsilon[\varepsilon = 0.8]$

Figure 6: Coordination Game and Replicator Dynamics-basins of attraction for different values of the payoff difference ε

Coordination Game and Logit Dynamics

We choose a small payoff perturbation, $\varepsilon = 0.1$, such that system is 'close' to the symmetric basins of attraction $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ in the Replicator Dynamics. With this perturbation size, the Logit Dynamics for the payoff matrix A of the Coordination game (22) defines the following vector field on the simplex of frequencies (x, y, z) of strategies E_1, E_2, E_3, respectively:

$$
\begin{bmatrix}
\dot{x} = \frac{\exp(0.9\beta x)}{\exp(0.9\beta x) + \exp(\beta y) + \exp(1.1\beta z)} - x \\
\dot{y} = \frac{\exp(\beta y)}{\exp(0.9\beta x) + \exp(\beta y) + \exp(1.1\beta z)} - y \\
\dot{z} = \frac{\exp(1.1\beta z)}{\exp(0.9\beta x) + \exp(\beta y) + \exp(1.1\beta z)} - z
\end{bmatrix}
\tag{25}
$$

Multiple Equilibria. In order to ascertain the number of (asymptotically) stable fixed points on the two-dimensional simplex we first run simulations for increasing values of β and for different initial conditions *within* the simplex as illustrated in Fig. 7.

Case I. Low values of β, $\beta \approx 0$ (Fig. 7a). One interior steady state, the barycentrum is

globally attracting, i.e. irrespective of the initial proportions, the population will settle down to a state with almost equal fractions.

Case II. High values of $\beta, \beta \geq 10$ (Fig. 7d-f). Three co-existing stable steady states asymptotically approaching the vertices of the simplex as β increases. The size of their basins of attraction is determined both by the strategy relative payoff advantage and by the value of the intensity of choice. The initial fractions of the strategies are important for which strategy will eventually win the evolutionary competition.

Case III. Intermediate values of $\beta, \beta = 3$ (Fig. 7bc). Co-existence of two steady-states for moderate values of the sensitivity to payoffs differentials parameter, with different initial population mixtures converging to different steady states.

(a) $\beta = 1$, (x_0, y_0, z_0)

(b) $\beta = 3$, $(x_0, y_0, z_0) = (0.2, 0.6, 0.2)$

(c) $\beta = 3$, $(x_0, y_0, z_0) = (0.5, 0.3, 0.2)$

(d) $\beta = 10$, $(x_0, y_0, z_0) = (0.4, 0.3, 0.3)$

(e) $\beta = 10$, $(x_0, y_0, z_0) = (0.25, 0.4, 0.35)$

(f) $\beta = 10$, $(x_0, y_0, z_0) = (0.33, 0.33, 0.34)$

Figure 7: Coordination Game[$\varepsilon = 0.1$] and Logit Dynamics -Unique stable steady state for β low (Panel(a)), two interior, isolated stable steady states for moderate β (Panels (b)-(c)) and three, co-existing stable steady states for high $\beta = 10$ (Panels (d)-(f)).

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Bifurcations. Unlike Replicator, the Logit Dynamics displays multiple, interior isolated steady states created via a fold bifurcation. In a 3-strategy Coordination game, three interior stable steady states emerge through a sequence of two saddle-node bifurcations, as illustrated in Fig. 8.

For small values of β the unique, interior stable steady state is close to the simplex barycentrum $(1/3, 1/3, 1/3)$. As β increases this steady state travels[14] in the direction of the Pareto-superior equilibrium $(0, 0, 1)$. A first fold bifurcation occurs at $\beta = 2.77$ (see Fig. 8a) and two new fixed points are created, one stable and one unstable. If we increase β even further ($\beta \approx 3.26$) a second fold bifurcation takes place and two additional equilibria emerge, one stable and one unstable. Finally, two new fixed points arise at $\beta = 4.31$ via a saddle-source bifurcation[15]. Three stable steady states co-exist for large values of the intensity of choice β. Note that the three stable steady states coincide with the 'logit equilibria' of McKelvey and Palfrey (1995) that converge to the pure strategy Nash equilibria when $\beta \to \infty$. A similar sequence of bifurcations is visible in the payoff parameter space (Fig. 8b) where, for a fixed value of the intensity of choice, a sequence of fold bifurcations occur in which steady states collide and disappear as the payoff difference ε increases. Hence, in the Logit Dynamics for a given "rationality level" β, a unique stable steady state, approaching the Pareto-optimal steady state, will arise when the payoff difference ε is sufficiently large.

---

[14]We point out an interesting similarity with Turocy (2005) "homotopy" method of tracing logit equilibria in normal form games. He shows that, generically, there is unique branch connecting the barrycentrum for $\beta = 0$ and a unique equilibrium - the Pareto-dominant equilbrium $(0, 0, 1)$ in our simple 3×3 pure Coordination game - as $\beta \to \infty$. This homotopy-based branch coincides with the lower branch of our numerically "continued" equilibria (see the bottom (x, β) bifurcation curve in Fig. 8a). Thus, a bifurcation-based method of computing equibria in normal form games may be an alternative to the homotopy method.

[15]A saddle-source bifurcation it is a fold bifurcation where two *unstable* steady states are created: one saddle fixed point (i.e. at least one eigenvalue with positive real part) and one source (i.e. both eigenvalues have positive real part).

(a) Cascade of Fold bifurcations in (x,β) space. $\varepsilon=0.1$

(b) Cascade of Fold bifurcations in (x,ε) space. $\beta=5$.

Figure 8: Coordination Game and Logit Dynamics. Curves of equilibria along with codimension I singularities, in this case fold catastrophe (LP) points. Panel (a): the multiplicity of steady states arises through a sequence of three fold bifurcations when β increases. Panel (b): two fold bifurcations in which steady states collide and disappear as payoff parameter ε increases.

The continuation of the fold curves in the (β, ε) parameter space allows the detection of codimension II[16] bifurcations. Co-dimension II bifurcation points are important, because they act as "organizing centers" of the complete bifurcation diagram with co-dimension I bifurcation curves. Thus, the cusp points (CP) in Fig. 9 organize the entire bifurcating scenario and are endpoints of the saddle-node bifurcation curves along which multiple, interior steady states are created in the Coordination Game under the smoothed best response dynamics. When a fold bifurcation curve is crossed from below two additional equilibria are created: one stable and one unstable after a saddle-node bifurcation and two unstable steady states after a saddle-source bifurcation. If choice is virtually random ($\beta \approx 0$) there is an unique steady state while if β increases, the number of steady states increases from 1 to 3, then from 3 to 5 and, last, from 5 to 7.

---

[16] The codimension of a bifurcation defines the number of parameters that needs to be varied in order for the bifurcation to occur generically (Kuznetsov (1995)).

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(a) Curves of fold points in the (ε, β) space

(b) Cusp points (CP) organize the equilibria structure

Figure 9: Coordination Game and Logit Dynamics. Curves of fold bifurcations along with detected codimension II singularities - in this case, cusp (CP) points - traced in the (ε, β) parameter space. The co-dimension II bifurcation points act as "organizing centers" of the fold bifurcation curves. As the intensity of choice increases, the number of steady states increases from 1 to 3, then from 3 to 5 and, last, from 5 to 7.

Basins of Attraction. The numerical computation of the basins of attraction for different equilibria reveals interesting properties of the Logit dynamics from a social welfare perspective. We construct a measure of long-run aggregate welfare as the payoff at the stable steady state weighted by the size of the corresponding basin of attraction. While for $\beta \to \infty$ the basins of attraction are similar in size as in the Replicator Dynamics (Panels (a), (c), (d) in Fig. 10), for moderate levels of rationality the population manages to coordinate close to the Pareto optimal Nash equilibria (Panel (b) in Fig. 10).

(a) $\beta = 1$, one barrycentrical steady state

(b) $\beta = 10/4$, one steady state 'close' to the Pareto superior equilibrium

(c) $\beta = 10/3$, two stable steady states close to the Pareot optimal and second best equilibrium.

(d) $\beta = 15$, three stable steady states

Figure 10: Coordination Game $[\varepsilon = 0.1]$ and Logit Dynamics: Panels (a)-(d), basins of attraction for increasing values of the intensity of choice β. Fractions converging to each of the steady state are indicated in the box.

Fig. 11 illustrates how the long-run average welfare depends on the parameter β for different levels of the payoff difference ε. Long run average welfare increases with the payoff difference ε but evolves non-monotonically with respect to the behavioural parameter β (Fig. 11ab). Long-run average welfare increases as the fully mixed equilibrium moves towards the Pareto optimal $(0, 0, 1)$ vertex, attains a maximum just before the first fold bifurcation occurs

at $\beta^{LP_1} = 2.77$ and then decreases, approaching the Replicator Dynamics average welfare, in the limit of $\beta \to \infty$. As our measure of average welfare is constructed as payoffs at steady state weighted by the corresponding sizes of basins of attraction, there are two effects driving the welfare peak *before* β^{LP_1} is hit. First, the steady state payoff is higher the closer the steady state is to the Pareto optimal equilibrium. Second, there is a 'basin of attraction' effect: before the first fold bifurcation threshold $\beta = \beta^{LP_1}$ is reached the entire simplex is attracted by the unique steady state lying close to the optimal equilibrium. Intuitively, the noisy choices in the low-beta regime help players escape the path-dependency built into the game and coordinate close to the Pareto-optimal equilibrium.

In the limiting case $\beta \to \infty$, the stable fixed points of the Logit Dynamics (i.e. the logit equilibria) coincide with the pure strategy Nash equilibria of the underlying game which, for this Coordination Game, are exactly the stable nodes of the Replicator Dynamics. Thus the analysis (stable fixed points, basins' of attraction sizes) of the 'unbounded' rationality case is identical to the one pertaining to the Replicator Dynamics in Coordination game (see subsection (5)).

(a) $W - \beta$ plot for different ε (b) surface plot (W, β, ε)

Figure 11: Coordination Game and Logit Dynamics-Long-run average welfare plots as function of payoff perturbation parameter ε and the intensity of choice β. As ε increases the long-run average welfare increases. The long-run average welfare is non-monotonic as a function of the intensity of choice β, with the maximum arising just before the first fold bifurcation.

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6 Weighted Logit Dynamics(wLogit)

As stable limit cycles arise in 3-strategy game under Logit Dynamics, a natural question arises whether more complicated behavior could also occur. One needs a four strategy game to get a proper three dimensional continuous-time evolutionary system which is the minimum dimension that may generate chaotic behavior. In this last section we run computer simulations for a four-strategy Schuster et al. (1991) example , from the perspective of a different type of evolutionary dynamics closely related to the Logit dynamics, namely the frequency-weighted Logit:

$$\dot{x}_i = \frac{x_i \exp[\beta A\mathbf{x})_i]}{\sum_k x_i \exp[\beta A\mathbf{x})_k]} - x_i, \quad \beta = \eta^{-1} \tag{26}$$

This evolutionary dynamic has the appealing property that when β approaches 0 it converges to the Replicator Dynamics (with adjustment speed scaled down by a factor β) and when the intensity of choice is very large it approaches the Best Response dynamic (Hofbauer and Weibull (1996)). Schuster et al. (1991) derive the following payoff matrix from models of biological interaction:

$$A = \begin{pmatrix} 0 & 0.5 & -0.1 & 0.1 \\ 1.1 & 0 & -0.6 & 0 \\ -0.5 & 1 & 0 & 0 \\ 1.7 + \mu & -1 - \mu & -0.2 & 0 \end{pmatrix} \tag{27}$$

The Logit dynamics on this payoff matrix "only" displays stable periodic behavior for intermediate values of the intensity of choice β. The vector field generated by the weighted Logit dynamics (26) along with the underlying game matrix (27) is investigated numerically in the sequel. Schuster et al. (1991) show that, within a specific payoff parameter region [$-0.2 < \mu < -0.105$], a Feigenbaum sequence of period doubling bifurcations unfolds under the Replicator Dynamics and, eventually, chaos sets in. Here we consider the question whether the weighted version of the Logit Dynamics displays similar patterns, for a given payoff perturbation μ, when the intensity of choice β is varied. First we fix μ to -0.2 (the value for

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which the Replicator generates 'only' periodic behaviour in Schuster et al. (1991)) and run computer simulations for different β.Cycles of increasing period multiplicity are reported in Fig. 12ad. For $\beta = 2.5$ the wLogit Dynamics already enters the chaotic regime on a strange attractor (Fig. 12ef).

Figure 12: Schuster et. al. (1991) game and wLogit Dynamics. Period-Doubling route to chaos. Panels (a)-(d): cycles of increasing period are obtained if the intensity of choice β increases. Panels (e)-(f): projections of the strange attractor onto 2 two-dimensional state subspace.

7 Conclusions

The main goal of this paper was to show that, even for 'simple' three-strategy games, periodic *attractors* do occur under a rationalistic way of modelling evolution in games, the Logit dynamics. The resulting dynamical systems were investigated with respect to changes in both the payoff and behavioral parameters. Identifying stable cyclic behaviour in such a system translates into proving that a generic, non-degenerate Hopf bifurcation occurs. By means of normal form computations, we showed first that a non-degenerate Hopf can *not* occur for Replicator Dynamics, when the number of strategies is three for games like Rock-Scissors-Paper. In these games, under the Replicator Dynamics, only a degenerate Hopf bifurcation can occur. However, in Logit dynamics, even for the three strategy case, stable cycles are created, via a generic, non-degenerate, supercritical Hopf bifurcation. Another finding is that the periodic attractors can be generated either by varying the payoff parameters (ε, δ) or the behavioral parameter intensity of choice (β). Moreover, via numerical detection of bifurcation curves in a Coordination game, under the Logit Dynamics we showed that display multiple, isolated, interior steady states exist, created by a fold catastrophe, a bifurcation which cannot occur under the Replicator. Interestingly, a measure of aggregate welfare reaches a maximum only for intermediate values of β, just before the first fold catastrophe, when most of the population manages to coordinate close to the Pareto-superior equilibrium. Last, in a frequency-weighted version of Logit dynamics and for a 4×4 game, period-doubling route to chaos along with strange attractors emerged when the intensity of choice took moderate, 'boundedly rational' values. An interesting topic for future research is to run laboratory experiments with human subjects to find out which evolutionary selection dynamics - either an imitation-based, in the spirit of Replicator or a more involved, belief-based dynamics a la Logit - is more relevant for players actual learning behavior.

Appendix A: Rock-Scissor-Paper game with Logit Dynamics:

Computation of the first Lyapunov coefficient

In order to discriminate between a degenerate and a non-degenerate bifurcations we need to compute the first Lyapunov coefficient. For this, we first use equations (8) to obtain the nonlinear functions:

$$
f_1(x,y) = y\sqrt{3}\frac{\varepsilon+\delta}{\varepsilon-\delta} - x + \frac{\exp\left(\frac{6(y\delta-\varepsilon(-x-y+1))}{-\delta+\varepsilon}\right)}{\exp\left(\frac{6(x\delta-y\varepsilon)}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(-x\varepsilon+\delta(-x-y+1))}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(y\delta-\varepsilon(-x-y+1))}{-\delta+\varepsilon}\right)}
$$

$$
f_2(x,y) = -x\sqrt{3}\frac{\varepsilon+\delta}{\varepsilon-\delta} - y + \frac{\exp\left(\frac{6(-x\varepsilon+\delta(-x-y+1))}{-\delta+\varepsilon}\right)}{\exp\left(\frac{6(x\delta-y\varepsilon)}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(-x\varepsilon+\delta(-x-y+1))}{-\delta+\varepsilon}\right) + \exp\left(\frac{6(y\delta-\varepsilon(-x-y+1))}{-\delta+\varepsilon}\right)}
$$

Next, applying formula (10) for the computation of the first Lyapunov coefficient, we obtain after some further simplifications:

$$
\begin{aligned}
l_1(\varepsilon,\delta) &= \left[\frac{1728\delta\varepsilon - 4320\delta^2 - 4320\varepsilon^2 - 4320\delta\varepsilon + 1728\delta^2 + 1728\varepsilon^2}{19\delta^2 - 38\delta\varepsilon + 19\varepsilon^2 - 16\delta\varepsilon + 8\delta^2 + 8\varepsilon^2}\right] \\
&= \frac{-2592\delta\varepsilon - 2592\delta^2 - 2592\varepsilon^2}{27\delta^2 - 54\delta\varepsilon + 27\varepsilon^2} \\
&= -2592\frac{\delta\varepsilon + \delta^2 + \varepsilon^2}{3(3\varepsilon - 3\delta)^2} < 0.
\end{aligned}
$$

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