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Principle of Virtual Work as Foundational Framework for Metamaterial Discovery and Rational Design

Le principe des puissances virtuelles comme cadre de base pour la découverte et la conception rationnelle des métamatériaux

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Abstract. Novel theories are needed for the discovery of innovative and exotic metamaterial and for their rational design. The current practice of mechanical analyses based upon moribund classical theories and experimental trial-error campaigns is caught in an inescapable vortex and illusion of inductive reasoning. The needed novel research paradigm is one in which the formulation of theoretical concepts precede their experimental validation. In the absence of theoretical understanding, the design experiments and collection of experimental evidence will remain unavoidably circumscribed. History of science can provide us guidance in the search for the needed powerful tools required for discovery. The principle of virtual work provides the necessary framework for development of theories that can lead to novel metamaterials, as it was the unifying principle which allowed the French-Italian School, headed by D'Alembert, Lagrange and Gabrio Piola, to found modern continuum mechanics. Based upon this framework we have conceived a metamaterial synthesis schema that exploits micro-macro identification traceable to the early days of the formulation of continuum theories for deformable solids. The schema is illustrated with application to metamaterials with pantographic and granular motifs based upon higher-gradient and higher-order theories.

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Résumé. La « Méchanique » Analytique a été une théorie controversée depuis sa formulation par Lagrange en 1788 et la controverse se poursuit jusqu'à nos jours : Truesdell déclare que la Mécanique doit être fondée sur le concept de « force ». Au contraire selon D'Alembert le seul principe unificateur est le Principe des Travaux Virtuels (PTV) : la force étant un concept dérivé, utile dans les applications. Ce débat épistémologique est-il inutile ? La mécanique est-elle aujourd'hui aussi fertile de problèmes théoriques et potentialités dans les applications technologiques ? En effet, la mécanique si basée sur le PTV s'avère être un outil puissant pour favoriser l'invention scientifique et l'avancement technologique : il rends possible, par exemple, la théorie moderne des méta-matériaux ou matériaux architecturés, c'est-à-dire, la théorie qui nous aide à « inventer » des matériaux qui « n'existent pas dans la nature et qui ont des propriétés à l'apparence magique » : donc l'approche épistémologique qui avait inspiré D'Alembert semble n'avoir encore épuisé sa capacité novatrice. En effet, le PTV permet d'homogénéiser le comportement des systèmes complexes et de formuler des théories macroscopiques prédictives du comportement « global » des microstructures qui forment les métamatériaux « exotiques ». La compréhension des résultats du débat épistémologique fondé sur la dichotomie force/travail nous offre un outil puissant pour concevoir et produire des matériaux « réels ». Parmi les infinies possibilités on signale les matériaux i) avec effet Poisson négatif ; ii) qui restent dans le régime élastique même en grandes déformations ; iii) qui se comportent, dans les petites déformations, comme des fluides et, en grandes déformations, comme des solides (pentamode materials) ; iv) avec une structure granulaire, qui ont un comportement chiral au niveaux macroscopique.

Keywords. metamaterials, energy methods, rational design, principle of virtual work, generalized continua, microstructure.

Mots-clés. métamatériaux, méthodes énergétiques, conception rationnelle, principe des travaux virtuels, Milieux continus généralisés, Microstructure.

1. Mechanics, the first Mathematized Theory in History of Science, is still a powerful tool for discovery

A metamaterial may be described as a "material which has been designed to meet a specific purpose governed by a desired specific behavior that is described by a given set of evolution equations" [1, 2]. The impact of the technological applications which are becoming possible because of the use of "exotic" metamaterials cannot be underestimated. Novel metamaterials are changing, just to give some examples, the engineering of bone and blood vessels prostheses [3], airplanes, automobiles, space vehicles and textiles [4, 5].

The discovery of new metamaterials and their design based upon an approach that exemplifies a "trial and error" procedure is expectedly unreliable and inefficient. Indeed, metamaterials have to be discovered through a process of synthesis founded upon sound theoretical reasoning combined with technological methods that can culminate in their fabrication and evaluation. In the theoretical and technological challenges that are to be faced in this synthesis process, an ancient theory, most likely the oldest mathematized theory ever conceived by humankind, plays a crucial role. This theory is Mechanics, albeit, for some readers, this circumstance may appear surprising. The field of Mechanics, to be generative of novel models and for being a powerful tool for discovery, must be founded upon the the principle of virtual work. This overarching principle can serve as the foundational theory that provides the pathway for establishing methodologies and material models that lead to rational design of innovative metamaterials.

We note, in this regard, that alternative postulation paradigms for mechanics are possible. Indeed, history teaches us that paradigm shifts in physical theories are occurring often. So one cannot exclude the possibility of scientific revolution in mechanics as well which could replace the principle of virtual work with another more general and widely encompassing paradigm. Although such a possibility cannot be excluded a priori, it is remarkable that the principle of virtual work has remain unchanged since the 3$^{\text{rd}}$ century BCE as a fundamental pillar of mechanics and that no paradigmatic shift has occurred for thermodynamics which in principle remains unchanged since its foundation by Clausius.

Some of the concepts that we want to describe in this paper were defended in the presentations [6] (in French) and [7] (in English) as well as [8].

1.1. *Evolution in Energy Methods from Ancient Conceptions*

The antecedents of the principle of virtual work may be traced to the apocryphal text Μηχανικά Προβλήματα transmitted as part of the *Corpus Aristotelicum*. In this ancient treatise, whose compilation can be traced to Archytas of Tarentum [9], one can find 35 problems discussed in terms of few basic principles of motion along circular paths, weights and levers. It is argued by modern scholars that the goal of this treatise is to explain technological artefacts in terms of Aristotelian natural philosophy [10].

Although such attribution may appear rather curious given that the treatise predates Aristotle by a few centuries[1], however, they are a tribute to the potency and endurance of the discussed principles [2]. It is interesting that the concepts associated to mechanics based upon motion is also found in the ancient *Vaiśeṣika* sutra of Kanada [12, 13]. A careful and intense investigation about the reciprocal influences and the emergence of certain similar patterns as well as divergences between Indian and Greek science and literature seems one of the most important, and unfortunately neglected, philological challenge to be initiated, ideally with the assistance of scholars steeped in these respective traditions.

A careful reading of pseudo-Aristotelian Μηχανικά Προβλήματα demonstrates how powerful are the rudiments of the principle of virtual work and concepts attributable later to Euclidean geometry, when conjunctly and systematically applied, to explain the functioning of machines.

The reader has to be warned: in the literature, the attribution to Archytas of Tarentum is not generally accepted, further, Diogenes Laertius talks about another Archytas who had written a book about mechanics. However, whoever was the author of Μηχανικά, any mechanician who reads it, cannot avoid to be surprised by its "modern" use of the mathematical modeling techniques in predicting physical phenomena. The style of the presentation and the clear underlying epistemological vision cannot be Aristotelian [14] and the attribution to Archytas of Tarentum may be debated as such vision seems to be too much Hellenistic, that is posterior of some centuries after Archytas. On the other side, as stated in https://plato.stanford.edu/entries/archytas/:

"Diogenes Laertius reports that Archytas was "the first to systematize mechanics by using mathematical first principles" (VIII 83 = A1), and Archytas is accordingly sometimes hailed by modern scholars as the founder of the science of mechanics."

The Μηχανικά Προβλήματα, as such, may be counted amongst the most ancient examples of the unifying capacity of mathematical abstraction to model physical phenomena. In many epochs till modern time periods, the concepts articulated in it have been periodically either revived or abandoned in a cyclic struggle of progression and regression of technology and science, as discussed in [15]. A particularly imaginative mind could even conceive that the textbook which arrived to us was an Attic-dialect Greek (used by Aristotle) text that was an updated version of a more ancient Doric-dialect Greek (used by Archytas) text.

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[1] As in our opinion is definitively proven in [9].

[2] We like to dream that the unknown copyist who included this mathematical masterpiece in the *Corpus* of the works of a famous opposer of mathematics as a tool for understanding Nature (i.e. Aristotle) wanted to preserve it against the destructive intentions of those scholars who wanted to erase it from Greek scientific literature. This circumstance parallels what surely happened more recently: Richard Toupin (Private Communication to the first author) wanted to include a chapter on variational principles in [11] even at the price to have them harshly criticized by Truesdell.

A notable resurrection of these ideas is in the 18th century CE work of D'Alembert *Traité de dynamique* (see for example [16]) that along with the contributions of Euler culminated to the remarkable treatise *Méchanique Analitique* of Lagrange, whose first version dates 1788[3].

In a striking similarity with the ideas expressed in Μηχανικά Προβλήματα, D'Alembert's treatise analyzes the motion of systems of interacting bodies by considering their equilibrium when the total virtual work done by all interactions vanishes for every virtual displacement. Lagrange formulation expands upon these ideas and analyses. Lagrange statement is very general: "If a system consisting of bodies or points, each of which is propelled by any power (for Lagrange a synonym of force), is in equilibrium and if a small movement is imparted to this system, by virtue of which each point will travel an infinitesimally small distance which will express its virtual velocity, then the sum of the powers multiplied by the distance traveled by the points at which it is applied along the line of application of this same power will be equal to zero, if the small distances travelled in the same direction as the power are considered as positive and the distances traveled in the opposite direction as negative."

Although a modern formulation of this principle usually includes the use of concepts from functional analysis, tensor algebra and calculus, the following two points characterize the formulation presented by Lagrange. Firstly, it is given in general terms such that it includes all the versions that appear to have been formulated so far. Secondly, his formulation uses the minimum possible mathematical concepts (only concepts from Euclidean geometry) that are sufficient to rigorously express the principle in its full generality.

1.2. *Force or Work - A Contrarian Viewpoint?*

In both, D'Alembert and Lagrange, the use of the word "force" or equivalent is nominalistic. In these works, the comprehensible quantity is the movement in time that defines the physical basis of the dynamics of bodies, and not its "presumed" cause "force", and therefore motion can only be understood as "indirectly" related to "force". Dynamics is the science developed in order to mathematically predict, from fundamental principles, the laws of motion of considered systems.

The notion of "force" is used in the mathematical theory necessary to produce predictions about physical phenomena from the fundamental principles on which mechanics is based[4]: it is purely an abstract concept which does not correspond to any measurable physical quantities. We further remark that the Newtonian conception of force, as described in his first and second laws of motion, suffers from controversy surrounding Newton's epistemological reasoning, which has been debated widely [17, 18]. Newton's laws use a definition of mass based upon a macro mass density, which ignores the effects of mass distributions at the micro scale (see for example [19]). In this context the difficulties met in the Newtonian postulations has also been underlined, but not solved, by Mach [20]. A more thorough discussion of this point has been developed in a separate paper [21] as the subject requires a lengthy elaboration, which will distract from the focus of the present work. Here we simply want to remark that in the framework of Lagrangian mechanics, forces are mathematical derived concepts defined in terms of the primitive concept related to virtual work.

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[3] It is remarkable that in McLaughlin, Peter. "The Question of the Authenticity of the Mechanical Problems." (2013) one reads: "However, by the end of the 18th century after Euler and Lagrange, the *Mechanical Problems* had ceased to be read as part of science and had become the object of history of science". Against its detractors, it is therefore remarkable that Μηχανικά Προβλήματα has been considered a scientific text until Euler and Lagrange!

[4] Remarkably in McLaughlin one reads that: «the actual text of Diogenes reports [..] that Archytas based mechanics on mechanical principles» and that the editor of Archytas text changed "mechanical principles"into "mathematical principles". Epistemological interpretation of Archytas words may change completely depending on which version is accepted.

Indeed, D'Alembert: states "...forces relative to bodies in motion are obscure and metaphysical entities which can only spread shadows on a science which, otherwise, is crystal clear in itself..." (D'Alembert, *Traité de dynamique* (1743) p. XVI). In D'Alembert's view, this principle has to be the underlying postulate whose consequences can be potentially evaluated through experimental verification of its predictions. Clearly, "force" in D'Alembert understanding is a derived notion used in the development of the theory, but without a direct experimental counterpart. As D'Alembert writes " ...I must warn that, in order to avoid circumlocutions, I have often made use of the obscure term of force [...] but I never pretended to attach to this term any other idea except those which result from the principles that I have established either in this preface or in the first part of this treatise". The fundamental and unifying idea is to be found in the principle of virtual velocities, which was later termed as the principle of virtual work.

In contrast, the Truesdellian criticism of Lagrange presents a contrarian view to the concepts underlying the principle of virtual work. It is claimed by Truesdell that mechanics is to be founded on the concept of "force". In a language replete with baroque and rococo styling (reflecting in many ways his personal lifestyle, see https://judith2you.wordpress.com/2015/01/26/judith-in-alabaster/), Truesdell criticizes the approach articulated in *Méchanique Analitique* in rather sharp terms. For example, Truesdell states (see [22] §14) "Granted his more modest scope, estimates of Lagrange's performance must remain a matter of taste. In music, in painting, in literature, tastes have changed in the past century. Why should they not also change in mechanics? The historians delight in repeating Hamilton's praise of the *Méchanique Analitique* as "a kind of scientific poem", but it is unlikely that many persons today would find Hamilton's recommendations in non-scientific poetry congenial." It is remarkable that Hamilton becomes an incidental target of Truesdell's criticism. In any case, a reading of *Méchanique Analitique* will show that it is written with clarity, with many parts resembling modern textbook such that it can be read with profitability even today. Truesdell further expresses in the essay "V. Whence the Law of Moment of Momentum §3" [23]:

> "Lagrange's best ideas in mechanics derive from his earliest period, when he was studying Euler's papers and had not yet fallen under the personal influence of D'Alembert"

implying that Lagrange was somehow under a nefarious influence of D'Alembert.

The approach elaborated in *Méchanique Analitique* was somehow colored as being of theoretical interest and not for applications although it was always used for solving problems of practical importance [24]. It is remarkable that already in Μηχανικά Προβλήματα the principle of virtual work has been illustrated via practical applications.

With respect to applications of the Lagrangian methods, however, Truesdell in the "Preface to the First Edition" of *A First Course in Rational Continuum Mechanics* [25] states:

> "While the knowledge he thus acquires does not of itself put applications into his hands, it gives him the tools to fashion them efficiently, or at least to classify, describe, and teach the applications already known."

The unifying capacity of the variational principles are diminished by this statement, which seems to imply that variational principles do not allow for the development of innovative theories and for the application of theory to novel applications. This opinion is contradicted by what has been stated by Landau and Feynman, who in their celebrated textbooks claim that variational principles are the most powerful tools for the discovery of novel models.

Truesdell then continues [25]:

> "By consistently leaving applications to the appliers, Lagrange set them on common ground with the theorists who sought to pursue the mathematics further: Both had been trained in the same workshop and spoke the same jargon."

It seems to us that, contrary to what is implied by Truesdell, this is not a negative aspect of the epistemological approach by Lagrange: science advances if and only if theorists and appliers speak the same jargon. The most important feature of the French School of Mechanics is, in our opinion, this unique capacity of his most prominent representatives to conjugate theory with practice.

Truesdell concludes with a typical logical fallacy, mixing a "*Non Sequitur*" with a "Hasty Generalization suppressing evidence", embellished with his baroque style [25]:

> "Even today this comradeship of infancy lingers on, provided discrete systems and rigid bodies exhaust the universe of mechanical discourse."

Moreover, in Truesdell's view Lagrangian methods are of limited value, since, as he states, thus showing that he did not carefully read the last version (1802) of the *Mécanique Analytique* and the works by Piola (albeit he cites them systematically) [25]:

> "In 1788 the mechanics of deformable bodies, which is inherently not only subtler, more beautiful, and grander but also far closer to nature than is the rather arid special case called "analytical mechanics", had been explored only in terms of isolated examples, brilliant but untypical. Unfortunately most of these fitted into Lagrange's scheme; those that did not, he passed over in silence."

Subjected to such severe criticism by Truesdell, arguably one the most influential mechanician of 20th century, it is not difficult to imagine why Lagrange's contributions came to be at times ignored, undervalued and underutilized within the broad field of mechanics.

Notwithstanding Truesdell's negative views regarding principle of virtual work, as expressed in his many writings, its application to modeling of physical phenomena has been unprecedented [26]. The success of the principle of virtual work (and its (only logical or also historical?) antecedent principle of least action) is expressed in the following sentences of a recent article of a widely read weekly magazine "…as Feynman says, there are multiple valid ways of describing so many physical phenomena. But an even stranger fact is that, when there are competing descriptions, one often turns out to be more true than the others, because it extends to a deeper or more general description of reality. Of the three ways of describing objects' motion, for instance, the approach that turns out to be more true is the underdog: the principle of least action.…" (see "A Different Kind of Theory of Everything" by Natalie Wolchover, The New Yorker, February 19, 2019). The principle of least action, first known to have been enunciated by Hellenistic scientists[5], has been rediscovered or restated by many great contributors to the progress of mechanics, including Lagrange, Piola, Landau, Feynman, Toupin, Mindlin, Rivlin,…

The treatise attributed to Archytas is, most likely, the first known mathematical formulations in applied mechanics and, as such, provides a template and direction for the invention of novel theories. The principle of virtual work, in our opinion, provides the most powerful tool for scientific creativity that can drive the technological advancements in mechanics.

2. Metamaterial Synthesis Schema

The variational principles, and in particular principle of virtual work, provide the foundational framework for developing those modern theories, which can lead us to the design of metamaterials or architected materials. Such an assertion can be illustrated through some examples of metamaterials synthesis and design proposed in the recent past, on the basis of a scientific debate started about a century before. Auxetic materials, characterized by negative Poisson's effect,

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[5] "For this would be agreed by all: that Nature does nothing in vain nor labours in vain". Olympiodorus, Commentary on Aristotle's Meteora translated by Ivor Thomas in the Greek Mathematical Works Loeb Classical Library Voll.I-4 1939

represent an exemplary case to this point. Their, sometimes harshly, denied existence has been recently proven on the basis of a procedure bases on two conceptual steps: (i) the characterization of the class of definite positive quadratic energies for isotropic linear elastic materials and (ii) the determination of micro-architectures, which, after homogenization, are described by a Cauchy continuum with any energy in this class.

The synthesis of auxetic metamaterials is a direct consequence of the study of the theory of elasticity based upon the principle of minimum potential energy (a special case of the principle of virtual work). Indeed, it is the analysis performed in the 19th century, by Lamé and Green, based upon the belief that any definite positive isotropic elastic strain energy must be physically realizable, that revealed theoretically the possible existence of auxetic materials. The debate (for a detailed historically precise description of the different viewpoints driving this debate see [27,28]) was confronting the "physical intuition" of the supporters of balance laws who "could not conceive" negative Poisson effect and those who believed in the epistemological principle that "every well-posed model must describe a part of physical reality".

In the 20th century this epistemological principle ultimately led to the construction of auxetic materials and to their technological application, for instance, into blood vessels reconstruction. In a similar vein, more recently Milton and Charkaev [29], considering what elasticity tensors (that are predicated upon the existence of elastic strain energy) are possible, proposed (and such materials have been later realized!) pentamode metamaterials with a tridimensional microstructure, which at the macroscopic level has a fluid-like vanishing shear modulus in linear (small) deformations and behaves as a solid in large deformations.

The challenge in the further discovery and rational design of metamaterials with specific properties must confront the lack of predictive theories that go beyond those of the "classical" continuum theory, in which the strain energy solely depend upon the gradient of placement (that is the so-called first gradient theory or theory of Cauchy continua)[6].

The envisaged novel metamaterials will, by nature, incorporate microscale mechano-morphologies whose energetic contributions and proper description at the macroscale require a shift in the paradigm of how mechanical analysis is widely practiced. In fact, the current practice is caught in the vortex, leading to some unavoidable contradictions, of developing analyses that are pretending to describe the behavior of every kind of possible materials using the existing classical continuum theory.

Instead, for discovering novel metamaterials, the needed paradigm is the one in which the theoretical constructs (equations) mathematically representing the desired behavior is first developed and the microscale mechano-morphologies is subsequently designed in such a way that the macroscale behavior is governed by the chosen equations. The problem of synthesis can thus be stated as the realization of a microstructure, or more generally the micro-mechano-morphology, that, after homogenization, will produce the desired behavior.

One can be tempted to pose the rhetorical question that, mechanics, as being the most ancient mathematized theory, is no longer fertile enough for contributing new theoretical understandings that can lead to further advanced theories and future technological applications. This question, in reality, does not concern mechanics: it concerns the Cauchy version of mechanics, as formulated in modern times by Truesdell. This version is blocked by a kind of conceptual "strait-jacket", which impeded the development of more advanced models. The answer to this question

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[6] The reader should remark that, in a very misleading way, the gradient of placement is called deformation gradient. One is, also, accustomed to call strain or deformation measure the Cauchy–Green tensor built with the placement gradient. As in higher gradient or in micro-structured continua the deformation energy depends also on other tensors, which all together describe some deformation modes, we find all the widely-used-in-literature nomenclature rather inappropriate.

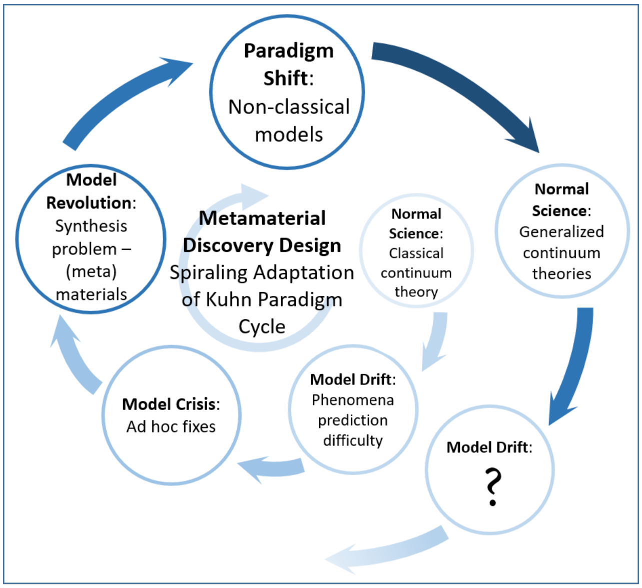

Figure 1. Adaptation of Kuhn paradigm cycle for metamaterial discovery and design. Note that the current practice is based upon classical continuum theory that originated from Cauchy analysis for describing the mechanical behavior of existing materials (recognizable as Kuhn's Normal Science). It is now well known that the theory fails to predict many phenomena such as, nonlinear dispersion in propagation of mechanical waves, finite sized boundary layers formed during deformation, and stress/strain concentration in finite localization zones (representing Model Drift). Attempts have been made to address these shortcomings using *ad hoc* fixes including "artificial" regularizations of the governing equations (Model Crisis). Most significantly, the current theory presents major shortcomings in the problem of synthesis or realization of metamaterials with predefined properties (therefore the need for Model Revolution). Clearly, a paradigm shift is required for developing non-classical models (for which the principle of virtual work is a powerful foundational framework). The ensuing generalized continuum theories, or higher-gradient and higher-order theories, are expected to result to new normal science, which may lead to future model drift in the spiraling cyclic adaptation of Kuhn's paradigm cycle, which we have in mind.

is: if mechanics is based on the principle of virtual work, as it has been since its first Greek formulation, then it has the vitality needed to produce original and extremely innovative theories and powerful technological advancements.

From an epistemic viewpoint, the needed paradigm for metamaterials design is best illustrated through the spiraling adaptation of the widely known Kuhn paradigm cycle [30] in Fig. 1. It could be objected that the epistemological and historical considerations that we are presenting are outside the scope of the practice of mechanicians, and in particular, of those among them who are willing to participate to the challenge involved in the theory of metamaterials.

Instead, a clarification of the origin and evolution of the theoretical understanding of mechanical theory and problems is, in our opinion, a necessary step in the conception and development of every novel theoretical formulations, and in particular, the specific one that can lead to the envisaged paradigm shift. The mathematical conception of mechanics in ancient times, in the era of Archytas on the basis of the virtual work and least action principles, can serve as a faithful guide, even in modern research activity, because deep mathematical ideas, once firmly established, have particular endurance.[7] Moreover, as any "algorithm" or "meta-theory" is not yet available teaching us how to "invent" novel models, the only guidance that we have available is given by the past successful mathematical and modeling inventions.

In this regard, it is notable that the Truesdellian viewpoint that mechanical theories must be founded upon the concept of force and moments of force and their balance postulates, places mechanics on a shaky foundation and poses serious impediments for its future developments. In this viewpoint, the role of micro-scale, the micro-macro identification, the exploitation of symmetry groups, and shape optimization becomes extremely challenging even if one were to adopt certain generalized continuum approaches *à la Eringen* [31] that are based upon balance laws. Symmetry groups and shape optimization in isolation do not provide a principle for founding a physical theory, however their fundamental concepts can be harnessed more efficiently within the framework of variational principles. In this regard, it may be remarked that while kirigami and origami provide interesting internal structures (microstructures) [32,33], they are not basis for predictive theories, as they are specific microstructures among an infinity of others. The problem of synthesis of metamaterials centers around the ability to choose amongst this infinity those that are behaving as desired. Furthermore, in many cases different balance postulates may be needed at the micro- and macro-scales depending upon the different sets of kinematical descriptors necessary for modeling at the two scales. Without recourse to variational approaches, it may not be possible a priori to understand which micro-scale balance postulate relates to a certain macro-scale balance postulate, and further, to establish relationships among the micro- and macro-balance laws. Such force-balance concept, therefore, has to be abandoned in order to achieve a paradigm shift, particularly in light of the existing alternative of the least action principle and its generalization: the principle of virtual work. This viewpoint can be found in many papers that, using the Lagrangian spirit, are able to address problems of complex mechanical systems from several perspectives: 1) theoretical, aimed at designing new materials based on complex sub-structures or involving "multi-physical" aspects (see, e.g., [34–38]); 2) numerical, aimed at solving those complex systems with ad hoc discrete models (see, e.g., [39,40]).

We believe that this principle is the only "firm and immutable" foundation in the art of mathematical modelling of mechanical phenomena since the appearance of Μηχανικά Προβλήματα. Of course, we do not believe that the postulation scheme based on the principle of virtual work is "forever" immutable: we simply claim that a major Kuhn revolution in mechanical sciences will be needed before it can be replaced.

2.1. *Micro-macro Procedure based upon Piola* ansatz

The genesis of micro-macro identification can be traced to the vanguards of continuum mechanics of deformable bodies in the first half of 19th century CE [41–43]. Among these pioneering works, Piola had already conceived a micro-macro identification scheme whose rudiments and conceptual basis can be traced to the systematic use of the principle of virtual work. In Fig. 2 we illustrate the Piola micro-macro identification methodology. Based upon this *ansatz*, a three pronged approach can be outlined that can address the model revolution and lead to a paradigm shift of the Kuhn cycle (1):

- *Macro-scale theory selection*: As a first step, devise a continuum theory, valid at the macro-scale, that seems suitable to describe the desired behavior using a judicious choice of macroscopic descriptors and a suitable formulation of the Principle of Virtual Work.

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[7] In the words of G.H. Hardy "A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words." *A Mathematician's Apology*, Section 10.

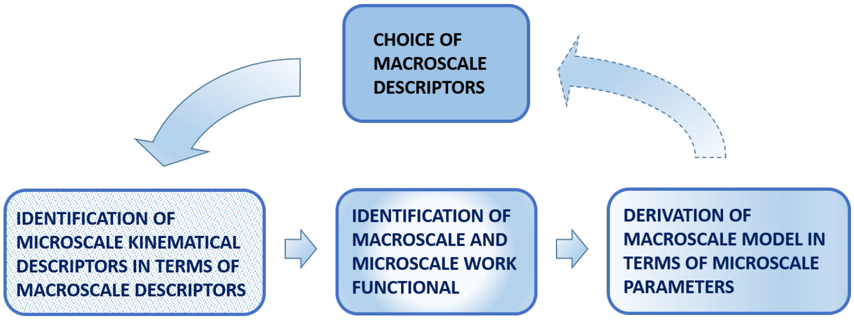

Figure 2. Micro-macro procedure based upon Piola *ansatz*.

- *Micro-mechano-morphology selection*: Next conjecture a subclass –in a specific class of microarchitectures, having a multi-scale mechano-morphology– which represents candidates to be governed by the chosen macro-model, once homogenized.
- *Macro-micro conjectural identification of micro-kinematical descriptors in terms of macro descriptors:* at this stage the modeller must find which micro-kinematical descriptors best represent (energetically) each choice of macro descriptor.
- *Macro-micro constitutive identification*: Finally, identify the macro constitutive equations in terms of suitably parameterized micro properties, in the selected subclass of micro-architectures, by identifying micro-virtual work with macro-virtual work.

Higher gradient theories leading to fabricated pantographic metamaterials (illustrated in Fig. 3) are exemplar of such a micro-macro identification process [44–48]. The three pronged approach outlined above is general enough to allow for variations to be conceived in a larger class of microarchitectures. We consider, as further example, the case of granular microarchitectures, for which the deformation energy can be represented by the aggregation of the deformation energies of interacting grain-pairs. In the sub-class represented by collection of nearly rigid elements (or grains) such that the elastic strain energy is stored in the deformable mechanisms represented through interconnections or interfaces between the grains, the outlined approach could proceed by (i) identification of the chosen continuum kinematic descriptors and deformation measures with the discrete grain-scale kinematics; (ii) consequent identification of the continuum deformation energy density with the volume average of grain-pair interaction energies; and (iii) application of the variational approach for defining stress/force conjugates of the kinematic variables, determining constitutive relations, and the governing macro-scale Euler–Lagrange equations [49, 50].

2.2. *Principle of Virtual Work as a Potent Primogenitor*

The key underlying quantities that drive the micro-macro identification described above are the kinematic descriptors and the associated work functionals. Clearly in this case, the variational methods and, hence, the principle of virtual work provide a general framework to develop the needed equations that govern the behavior at any chosen scale. To illustrate the efficacy of the principle of virtual work in developing non-standard novel theories, we briefly review here the case of metamaterial described by the N^{th} gradient continuum theory. In this case, the only macroscopic descriptor introduced is the placement field, and further, its gradients up to the

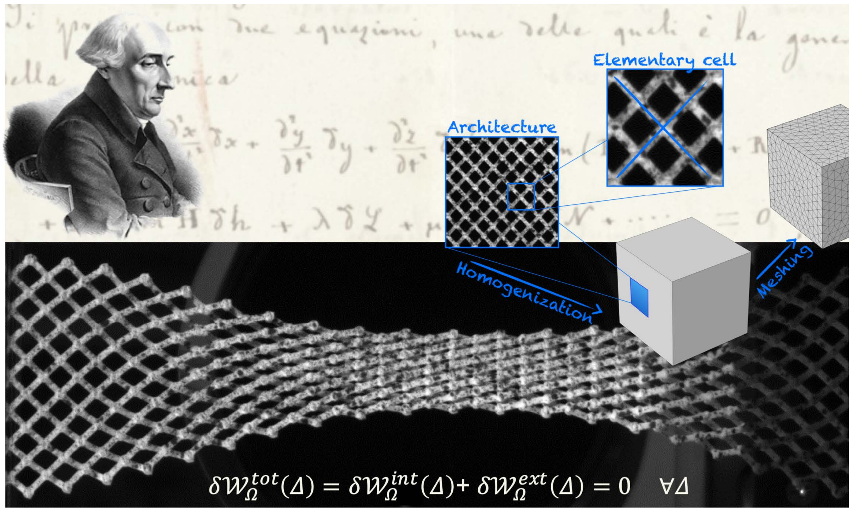

Figure 3. Pantographic metamaterial as an exemplar of the micro-macro identification.

order N are involved in the expression for deformation energy functional. The internal power[8] functional for an N^{th} gradient continua is given as a distribution of order smaller or equal to N, expressed as

$$P^{int}(B,V) = \sum_{\Lambda=0}^{N} \int_B T_\Lambda \cdot \nabla^\Lambda V, \tag{1}$$

in which Λ- order contravariant tensors T_Λ is termed as Λ- order multipolar stress [51]. By performing subsequent integration by parts we arrive at terms related to boundary faces, boundary edges and boundary wedges as illustrated in Fig. 4 expressing the contact loads which are possible in N^{th} gradient continua and therefore the class of external loads which are sustainable by these continua.

Using the principle of virtual power, it is therefore possible to express the class of admissible external loads sustainable by (and applicable to) N^{th} gradient continua as follows (for more details see [51]):

$$\begin{aligned} P^{int}(B,V) = & \int_B F(B,0)V + \int_{\partial B} F(\partial B,0)\cdot V \\ & + \sum_{L=1}^{N-1} \int_{\partial B} F(\partial B,L)\cdot \left(\nabla^\Lambda V\right)_\perp + \int_{\partial\partial B} F(\partial\partial B,0)\cdot V \\ & + \sum_{J=1}^{N-2} \int_{\partial\partial B} F(\partial\partial B,J)\cdot \left(\nabla^J V\right)_\perp + \sum_{J=0}^{N-3} \int_{\partial\partial\partial B} F(\partial\partial\partial B,J)\cdot \nabla^J V_\perp \end{aligned} \tag{2}$$

It is remarkable that externally applicable loads, for N^{th} gradient continua, do NOT reduce simply to forces (i.e. to the quantities dual in work of virtual displacement). One has to introduce k-forces: i.e. the quantities dual in work of the normal gradients of virtual displacements at the body faces, edges and wedges boundaries. This point has been discussed already by [52, 53]

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[8] The reader should not be confused by the introduction, here and in the literature, of different but equivalent nomenclature: one can talk about the principle of virtual work or the principle of virtual power: in the first case one introduces test functions which are virtual displacements, in the second one they are virtual velocities. The conceptual basis is exactly the same and formally one can obtain the power functional from the work functional simply by dividing the virtual displacement by an infinitesimal time.

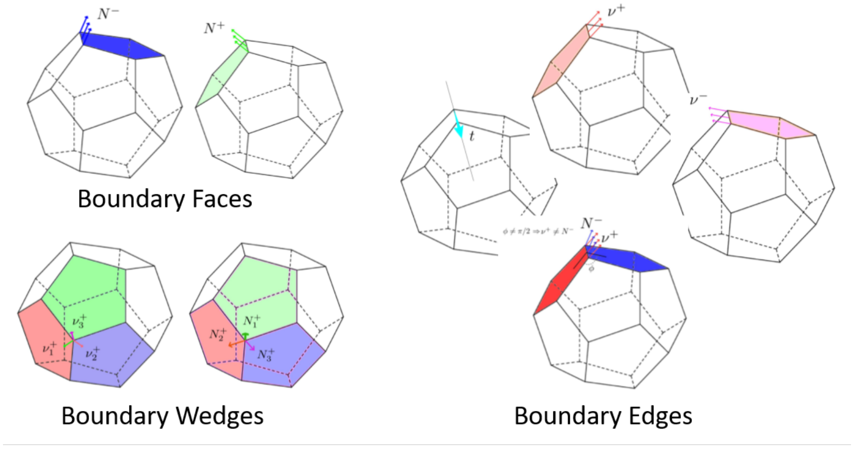

Figure 4. Boundary faces, edges, wedges.

and [54] where the concept of multipolar stresses and forces is introduced. However in [54] only the evolution equations which can be obtained from an invariance principle for total energy are considered.

Instead, using the principle of virtual work one can get not only the appropriate forms of governing equations (which one could call the "balance equations") for every kind of kinematical descriptor that has been introduced, but, more importantly, one can easily *deduce, using an integration by parts argument*, the class of natural and essential boundary conditions leading to well-posed problems. It is remarkable that the determination of boundary conditions in (force, moments of force or any other quantity one may be willing to introduce) balance approaches are often fraught with consistency problems, as they are postulated independently of the bulk balance equations. Aware of this difficulty, many authors recur to the "stratagem" of *inducing* from experiments needed boundary conditions, inductivistic attitude which should be avoided, as we will argue in the following.

Using the principle of virtual work, and beginning with any kind of additional macroscale kinematic descriptors (also some of which are purely Lagrangian, and therefore not at all affected by any kind of Eulerian invariance), one can easily postulate work functionals to introduce any kind higher-order micromorphic theories [49,55–57]. It is, therefore, clear that if we are to achieve the paradigm shift and introduce novel theoretical approaches into the mechanical analyses for metamaterial discovery and design, then the principle of virtual work has a pivotal role.

3. Metamaterial Synthesis based upon the Homogenization Schema

The homogenization procedure outlined in the above section provides a general framework on which the problem of metamaterial synthesis can be based. This problem is not yet solved for the class of N^{th} gradient continua, although it is worthwhile to mention here that it is not completely unprecedented. The mathematical method that are to be developed bears similarity to that used for designing analog circuits whose elementary components are the elements of resistance, inductance, capacitance and transformer [58].

In fact, a general theorem of synthesis has been proven for every quadratic Lagrangian and Hamilton–Rayleigh-dissipation functions in the case of circuit design: it is possible correspondingly to obtain algorithmically a graph, and for every branch of the graph, an elementary linear sub-circuit so that the resulting circuit is governed by the associated Lagrange equations.

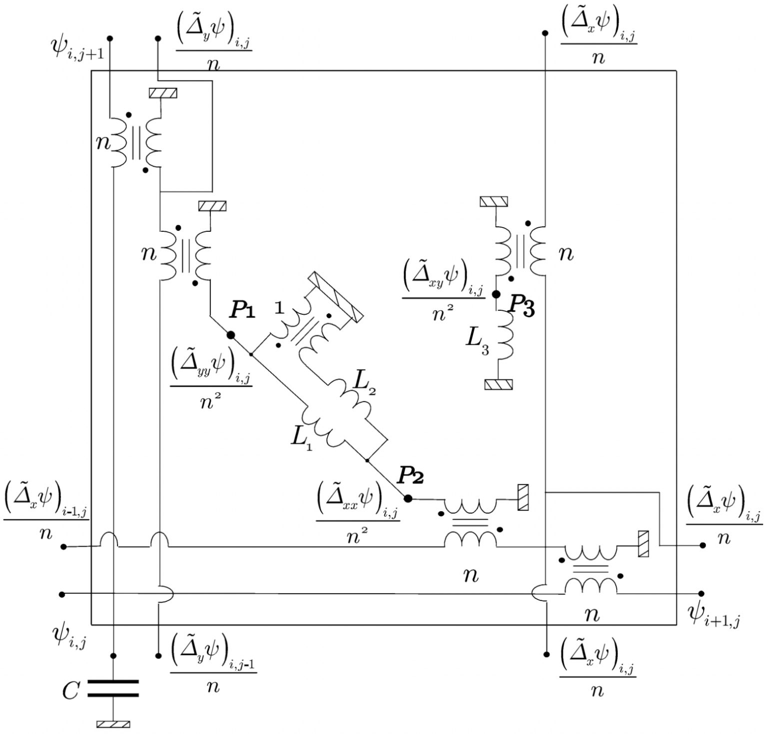

Figure 5. The analog circuit of Kirchhoff plate element.

An example of such an analog circuit for Kirchhoff plate element [59, 60] is illustrated in Fig. 5. The synthesis for a specified class of mechanical metamaterial, therefore, should proceed in a similar manner by finding a minimal set of elementary sub-structures which can be combined using suitable interconnections to represent every internal virtual work functional. To illustrate the paradigm to be developed, we will discuss the case of (1) second, third and n^{th} gradient meta-beam incorporating pantographic motif, and (2) 2D chiral Cosserat metamaterial of granular motif, so giving two examples of the efficacy of the described schema.

3.1. *Second Gradient - Pantograhic Motif*

The most basic substructure in the synthesis of a second gradient material is the pantographic beam shown in Fig. 6. The constituting beams are flexible and are interconnected with perfect pivots. The relevant length scales are three: the beams thickness, the blue pivots distance, and the product of the number of the cells times the blue pivots distance. The linearised pantographic beams are the synthesis of the following macro-scale second gradient quadratic energy [45],

$$E(u) = \int_0^l \left[K\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)^2 + K'\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)^2 \right] ds, \tag{3}$$

It has to be remarked that the pantographic beam incorporates floppy modes, i.e.; non-rigid displacement fields corresponding to vanishing deformation energy. Further, note that the homogenised displacement in Eq. (3) can represent only the position of the blue materials points, which we could call the Piola points. The positions of red points have no macroscopic kinematical

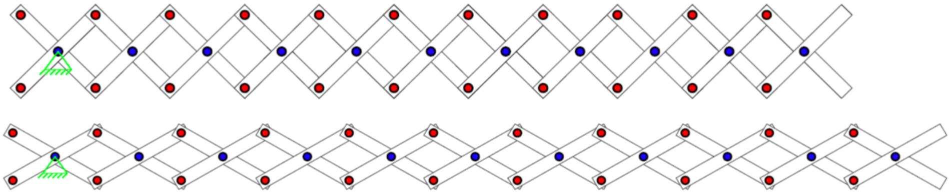

Figure 6. Pantographic beam.

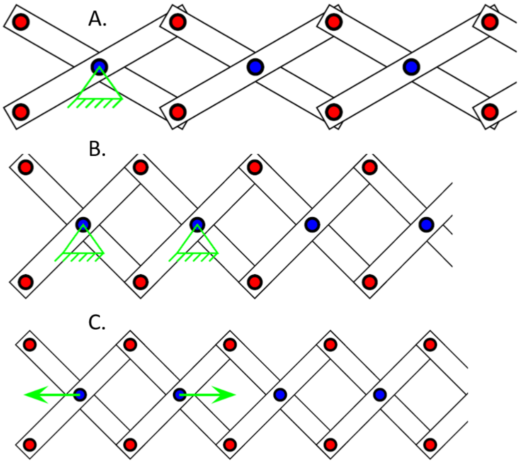

Figure 7. Micro-interpretation of macro-boundary conditions: A. Displacement constraint at the micro-level. B. Normal derivative and rotation constraint at the micro-level. C. Reactive double-force (note: it expends work on elongations and is not a couple).

description and are calculated assuming a local microscopic instantaneous equilibrium condition. The macro boundary conditions can be determined following the approach given in Eq. (2), for a general $N-th$ gradient continuum. These boundary conditions are illustrated in Fig. 7 and includes specification of displacement, rotation and the normal derivative or their energetic duals.

Further, a macro-scale model that can describe large displacements and deformations of pantographic beams can be given by introducing a meso-length scale using Hencky discrete springs that are equivalent to Euler beams, such that macro-deformation energy is expressed in terms of the extensional and rotational kinematic quantities ρ and ϑ as [61],

$$E=\int_0^l K_E K_F\left[\frac{\rho^2-2}{\rho^2\left(K_E-4K_F\right)-2K_E}\vartheta'^2+\frac{\rho^2}{\left(2-\rho^2\right)\left[\rho^2\left(K_E-4K_F\right)+8K_F\right]}\rho'^2\right]ds, \tag{4}$$

whereK_E and K_F are the stiffnesses of the extensional and rotational springs. In this case, the class of architectures, among which we look for our synthesis, is that constituted by several constrained Euler beams as represented by their Hencky discretisation, as illustrated in Fig. 8.

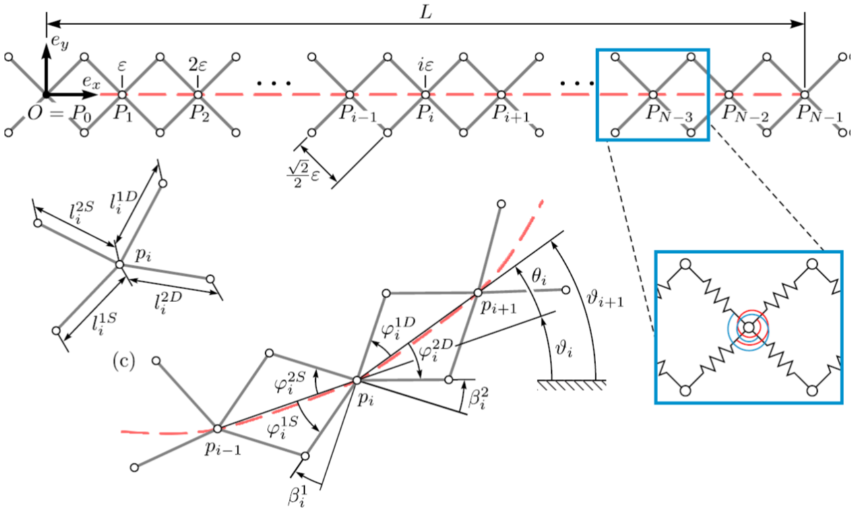

Figure 8. Pantographic beam modeled as collection of Hencky elements representative of Euler beam.

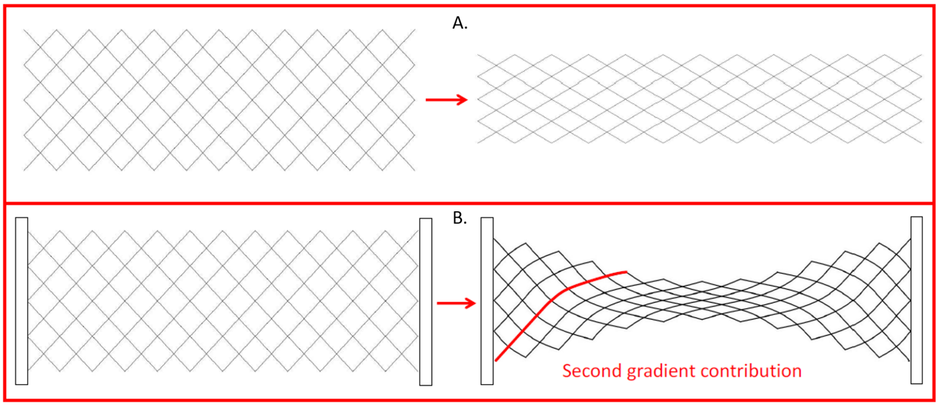

Figure 9. Pantographic 2D-plate: A. Only floppy modes indicating zero first gradient deformation energy. B. Deformation pattern in bias test with fully clamped ends showing the contribution of second gradient deformation energy.

We can now consider the case of 2D plate, in which case the pantographic multi-scale micro-architectures to be synthesized is one whose deformation energy is given as [62],

$$\begin{aligned}\mathscr{U}\left(\chi(\cdot)\right) = &\int_\Omega \sum_\alpha \frac{\mathbb{K}_e}{2}\left(\|\mathbf{FD}_\alpha\| - 1\right)^2 \mathrm{d}A \\ &+ \int_\Omega \sum_\alpha \frac{\mathbb{K}_b}{2}\left[\frac{\nabla\mathbf{F}|\mathbf{D}_\alpha \otimes \mathbf{D}_\alpha \cdot \nabla\mathbf{F}|\, d\mathbf{D}_\alpha \otimes \mathbf{D}_\alpha}{\|\mathbf{FD}_\alpha\|^2} - \left(\frac{\mathbf{FD}_\alpha}{\|\mathbf{FD}_\alpha\|} \cdot \frac{\nabla\mathbf{F}\,|\,\mathbf{D}_\alpha \otimes \mathbf{D}_\alpha}{\|\mathbf{FD}_\alpha\|}\right)\right] \mathrm{d}A \qquad (5) \\ &+ \int_\Omega \frac{\mathbb{K}_p}{2}\left[\cos^{-1}\left(\frac{\mathbf{FD}_1}{\|\mathbf{FD}_1\|} \cdot \frac{\mathbf{FD}_2}{\|\mathbf{FD}_2\|}\right) - \frac{\pi}{2}\right]^\gamma dA\end{aligned}$$

where the exponent γ represents a nonlinear material behavior. It is noted that the deformation energy in Eq. (5) is incomplete since only two second order partial derivatives appear. In this case, the synthesized micro-architecture is illustrated in Fig. 9 (see also [63–66]).

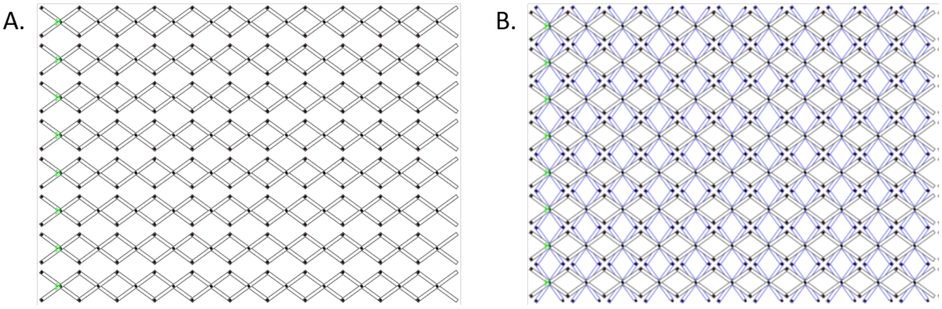

Figure 10. Bi-pantographic sheet micro-architecture: A. Sub-architecture composed of horizontal pantographic beam building blocks. B. Sub-architecture composed of a family of horizontal and vertical pantographic beam building blocks.

Finally we consider the case of bi-pantographic planar sheets, in which Euler beams are replaced by pantographic beams. It is notable that the bi-pantographic sheets do not represent a complete second gradient 2D continuum, even though additional second order derivatives of parallel displacement appear in the deformation energy. In this case, the multi-scale micro-architectures to be synthesized is one whose deformation energy is given as [62],

$$\begin{aligned}\mathfrak{E} = \int_\Omega \sum_\alpha \Bigg\{ K_E K_F \Bigg[& \frac{\rho_\alpha^2 \cos^2\gamma - 1}{\rho_\alpha^2 \cos^2\gamma \left(K_E - 8K_F\cos^2\gamma\right) - K_E} \left(\frac{\partial \vartheta_\alpha}{\partial\alpha}\right)^2 \\ & + \frac{\rho_\alpha^2\cos^2\gamma}{\left(1-\rho_\alpha^2\cos^2\gamma\right)\left[8K_F+\rho_\alpha^2\left(K_E-8K_F\cos^2\gamma\right)\right]}\left(\frac{\partial\rho_\alpha}{\partial\alpha}\right)^2\Bigg] \\ & + K_S\left[\cos^{-1}\left(1-\frac{\rho_\alpha^2}{\frac{1}{2}\cos^2\gamma}\right)-\pi+2\gamma\right]^2\Bigg\} \mathrm{d}A\end{aligned} \tag{6}$$

The bi-pantographic planar sheet is obtained by assembling a sub-architecture constituted by horizontal pantographic beams as illustrated in Fig. 10 combined with a sub-architecture constituted by vertical pantographic beams sharing the same central nodes as the horizontal beams and interconnected using pivots as shown in Fig. 10B.

For the case of perfect pivots, there is zero deformation energy for the homogeneous deformation modes shown in Fig. 11. From a rheological viewpoint, the class of architectures that needs to considered to get the synthesis of the bi-pantographic sheet is, interestingly, again a meso-structure represented by means of extensional and rotational springs as portrayed in Fig. 12. The advent of additive manufacturing (3D printing) has made it possible to fabricate the aforementioned pantographic structures (see, e.g., [67–70]). An example of the 3D printed bi-pantographic sheet is shown in Fig. 12. What makes 3D printing of such structures more significant is the ability of implement experimentally a variety of boundary conditions such that their effect upon the macro-scale behavior can be investigated using traditional mechanical testing devices.

3.2. *Cosserat Continua - Granular motif*

For a Cosserat continua, the most basic structure that can be synthesized is a beam with granular motif. To endow the beam with chirality, such that the axis of chirality is orthogonal to the beam axis, the required synthesis is of the following macro-scale energy [71, 72],

$$E(u) = \frac{1}{2}\int_0^l C^n\left(\frac{\partial u_1}{\partial x_1}\right)^2 + C^s\left(\frac{\partial u_2}{\partial x_1}-\psi\right)^2 + C^\theta\left(\frac{\partial \psi}{\partial x_1}\right)^2 + 2C^{ns}\left(\frac{\partial u_1}{\partial x_1}\right)\left(\frac{\partial u_2}{\partial x_1}-\psi\right), \tag{7}$$

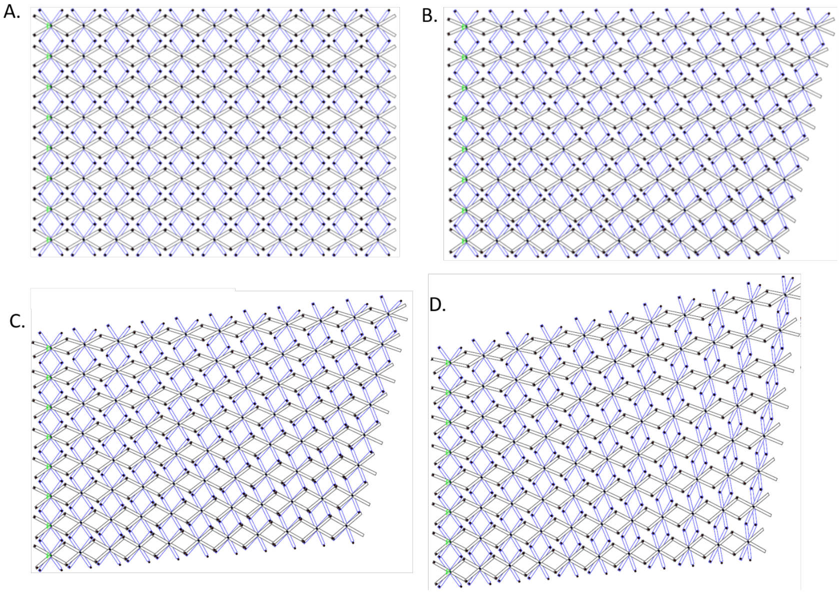

Figure 11. Bi-pantographic zero energy modes under homogeneous deformations: A. Vertical and horizontal elongation. B. Horizontal shear. C. Vertical and horizontal shear. D. Combined elongation and shear.

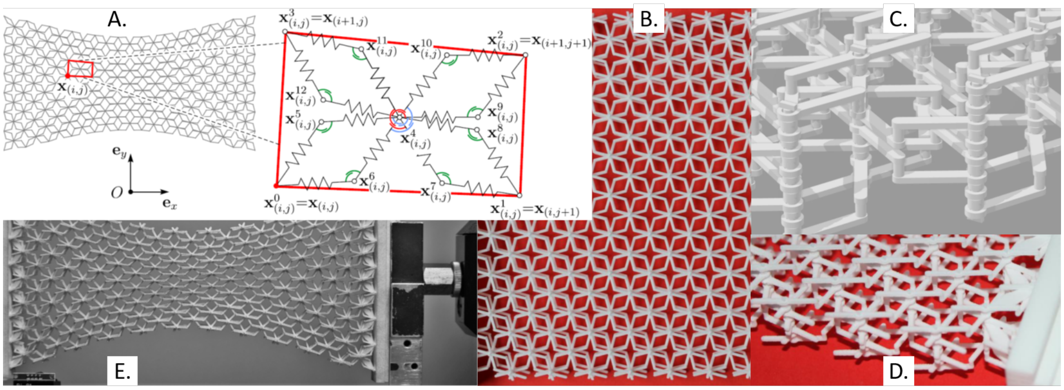

Figure 12. A. Bi-pantographic sheet represented as a collection of Hencky elements representative of Euler beams. B. Physical realization of bi-pantographic sheet using 3D printing. C. CAD rendering of perfect pivot connections. D. The realization of boundary conditions within the 3D printed specimen. E. The fabricated sheet subjected to bias test.

The considered macro-scale kinematic quantities are the axial displacement, u_1, the transversal displacement, u_2, and the rotation, ψ. It is remarked that the macroscale energy in Eq. (7) reduces to that of Timoshenko beam when the 4^{th} term in Eq. (7) is ignored. The synthesis of this macro-scale energy as a granular motif is possible if the interacting grains admit stretch and shear coupling which can be achieved via the grain-grain connection using a duoskelion [35] set of beams as shown in Fig. 13A, which serve as a mechanical analog of grain-pair interactions. The stretch-shear coupling in such a mechanical analog has been shown through finite element (FE)

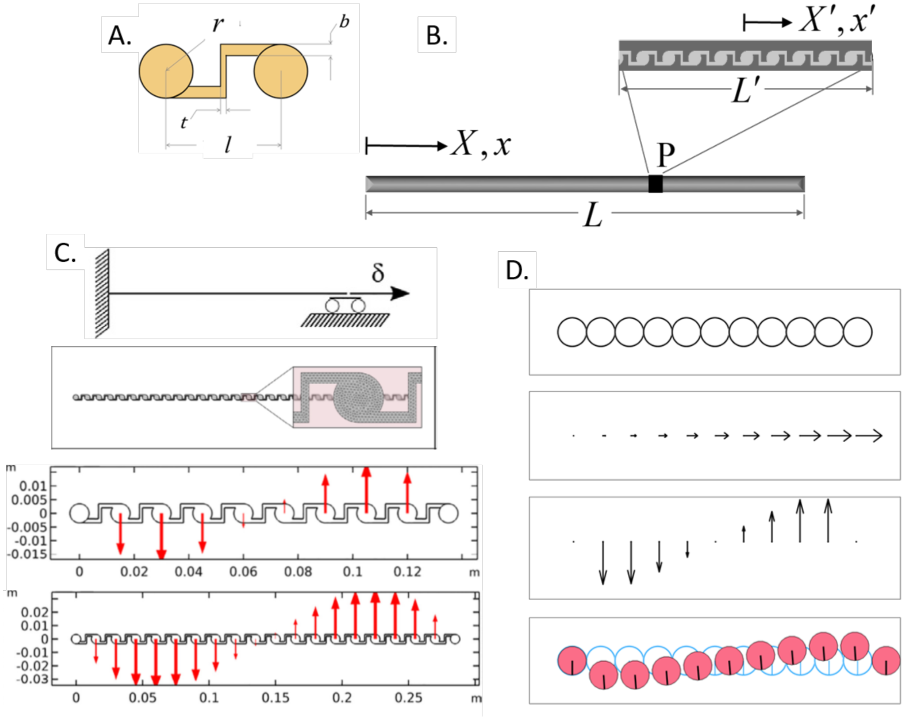

Figure 13. A. Grain-pair with stretch-shear coupling. B. Beam with granular motif emphasized in the inset. C. Results of FE simulation of the beam under extension modeled using standard elastic Cauchy continua showing the non-standard transversal displacements. D. Results of the beam under extension modeled using discrete (rigid) elements interacting via springs representing the grain-pair mechanism in A.

modeling treating the grain-pair and connectors as standard Cauchy continua [71]. The resultant beam is illustrated in Fig. 13B. The chiral nature of the beam is evident from the transversal displacement observed in FE [71] and discrete element simulation of the beam [73] (note that the behavior is independent of the number of grains).

It is noteworthy that the homogenized macro-scale model, Eq. (7), can only represent the positions of the grain barycenter and the grain rotation about its barycenter treating the grains as rigid bodies. Such a treatment is justified, since in these cases the elastic deformation energy is stored in the mechanism that connect a grain-pair. We can further consider the case of a 2D plate, in which case the following deformation energy represents the metamaterial to be synthesized [74]:

$$E(u) = \int_{\Omega} \frac{1}{2}\lambda\left(E_{11}+E_{22}\right)^2 + \mu\left(E_{11}+E_{22}+2E_{12}\right)^2 + \beta\gamma^2 + \alpha\left(\left(\frac{\partial\psi}{\partial x_1}\right)^2 - \left(\frac{\partial\psi}{\partial x_2}\right)^2\right) + \eta\gamma\left(E_{11}+E_{22}\right) \quad (8)$$

The considered macro-scale kinematic quantities are the displacement, u_i, and the micro-rotation, ψ. In this case also, the synthesis of this macro-scale energy as a granular motif is possible if the interacting grains admit stretch and shear coupling (see [74]).

In all of the above examples, one can easily recognize the Piola *ansatz* that proceeds from establishing the equivalence class of micro-deformations corresponding to each macro-deformation and by identifying the micro and macro work functionals on corresponding micro and macro deformations. Then the Piola homogenization is obtained via the application of the principle of virtual work for the two classes of macro and micro virtual work functionals defined on the set of micro or macro admissible virtual displacements (or test functions) and with the identification of corresponding micro and macro virtual works.

From the mathematical viewpoint, it is remarkable that the virtual work functionals are distributions (in the sense of Laurent Schwartz) and virtual displacements are the correspondent test functions ([75]). Therefore the mathematical results obtained in the framework of that theory are of great use in the theory of metamaterials synthesis.

4. Epistemological considerations for experimental validation

In the absence of advanced theories, experimental campaigns for discovering novel phenomena and characterizing unprecedented material behavior will remain inadequate and illusory. While the debate between deductivist and inductivist views of the formulation of scientific ideas may continue, clear advantage is afforded when the formulation of the mathematical concepts precede their validation with experimental evidence. The inductivist view of developing theoretical concepts, typically, reverses this order and seeks to develop mathematical models based upon data obtained from experimentation. Such an approach could be stunting as the domain of possible phenomena is limited to those already observed or to their near interpolations and the developed models serve only as explanations.[9] In fact without having a presumed (or conjectured) theory at hand, it is not possible to design meaningful experiments or interpret the results so as to reveal the novel phenomena.

We illustrate the power of the conceptual epistemological framework given to us by a deductivist viewpoint by discussing its effects when used in the theory of metamaterials. In fact, it has been immediately clear, in the study of pantographic sheets under large deformations, that it was very difficult to recognize where a material sub-body individuated in the reference configuration is placed in the present configuration. The efforts for getting a correct algorithmic elaboration from images, without the guidance of a theory conjecturing where all sub-bodies of considered specimens were presumed to be, were all in vain.

To get an effective algorithmic interpretation and elaboration of the experimental results obtained in the bias test of pantographic sheets using digital image correlation (DIC), it was necessary to use a second gradient theory (valid at macro-scale) guiding the algorithmic analyses designed for interpretation of the deformations measurements as shown in Fig. 14A. In other words: only when the theory suggested where a specific sub-body had to be placed the DIC algorithm could localize it [34, 72, 76, 77]. The second gradient macro-theory was very efficient (as it implied low computational costs) for this aim.

Alternatively, the first gradient theory was applied for a micro scale analyses in which a complete representation of the micro-mechano-morphology of the pantographic sheet is necessary (see Fig. 14B for the result of such an analyses): when this choice was done the needed computational efforts were correspondingly much more time consuming. Only with the availability of both said theoretical frameworks and their predictions of the placement of the sub-bodies it was possible inform the DIC back-tracking procedure to correctly recognize the shape changes in the images the deformed sheet as shown in Fig. 14C and D for the macro- and micro-analyses, respectively.

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[9] "It may be said, therefore, that an explanation is not fully adequate unless its *explanans*, if taken account of in time, could have served as a basis for predicting the phenomenon under consideration. . . . It is this potential predictive force which gives scientific explanation its importance: only to the extent that we are able to explain empirical facts can we attain the major objective of scientific research, namely not merely to record the phenomena of our experience, but to learn from them, by basing upon them theoretical generalizations which enable us to anticipate new occurrences and to control, at least to some extent, the changes in our environment." Hempel & Oppenheim, 1948, "Studies in the Logic of Explanation". *Philosophy of Science*. XV (2): 135–175

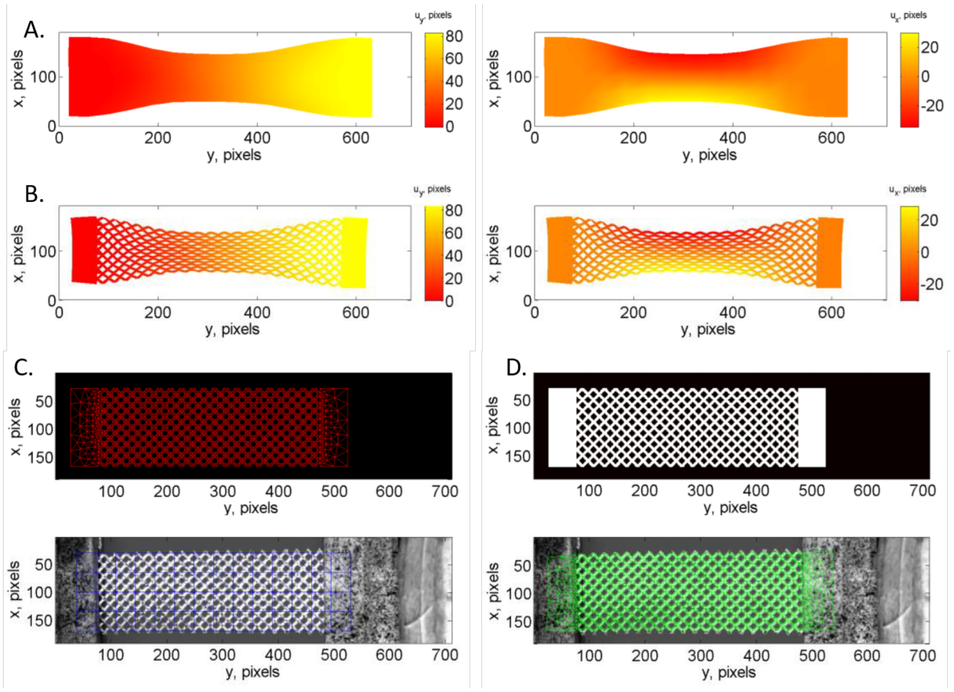

Figure 14. A. Macro-scale analyses using a second gradient theory. B. Micro-scale analyses using a first gradient theory. C. Results of the DIC backtracking for the macro-scale analyses. D. Results of the DIC backtracking for the micro-scale analyses.

Also in the case of the chiral Cosserat beam with granular motif, the accurate determination of the grain motions is only possible by using systematically the introduced presumed conjectural model. In Fig. 15A, we show two possible backtracking scenario, one macro-scale and the other micro-scale, for the DIC determination of deformed shape of the beam. It is clear from Fig. 15A, that the macro-scale analyses provides little useful information (in fact the extracted information could be misleading without the prior knowledge of the precise details of the microstructure). The primary useful information we get from the micro-scale analyses is that the strain energy is localized in the grain connectors and that the bar can be treated as a granular system. Further analyses to extract the rigid motions of the grains leads to kinematic fields of the introduced presumed model as shown in Fig. 15B, where the DIC determined kinematic fields are plotted with those predicted by the introduced model.

5. Conclusion and research perspectives

Through the presented discussion and illustrative examples it is amply clear that the principle of virtual work provides a general framework for developing theories that can be used to scientifically design exotic and innovative metamaterials. The >2500 years old mathematical principle which allowed for the first foundation of mechanics maintains the vitality capable of guiding mechanicians towards innovation, both in their modeling efforts and in their search for technologically relevant applications. The impulse given to mechanics by the Illuministic mechanicians operating in France at the end of 18th century, and their further enhancement by the pupils and scientific descendants of Gabrio Piola in the early 19th century, does not seem to be exhausted yet.

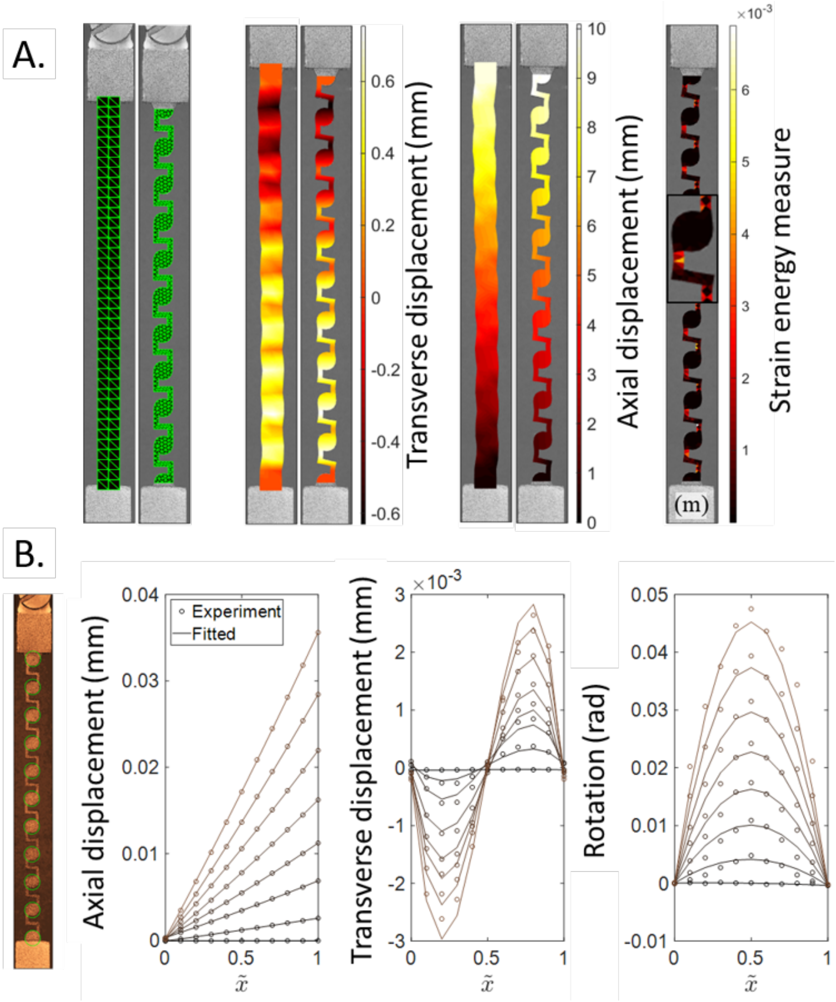

Figure 15. A. Macro-scale and micro-scale determination of beam deformation using DIC. B. Measured grain rigid body motions obtained from DIC and predicted kinematics of the chiral Cosserat beam model.

Indeed, the principle of virtual work provides the most efficient and powerful tool for achieving the paradigm change in mechanical sciences that seems to be urged by the modern demands from applications and by the increased experimental evidence which falsified the first gradient continuum theory in a number of instances. Here, we have discussed selected results taken from more recent literature that reflect the potential progression which can be interpreted in terms of a spiraling adaptation of the classical Kuhn paradigm cycle.

The theoretical analyses based upon the principle of virtual work, as they allow for an effective micro-macro identification process, can be exploited to develop the homogenization schema essential for conceiving a metamaterial synthesis algorithm. In the present paper, to show the impact of the used epistemological concepts, we exemplify the identification-synthesis schema through the discussed synthesis of second gradient and chiral Cosserat media.

Furthermore, from the viewpoint of the further and necessary advances in metamaterials design we want to briefly indicate here how more general synthesis results may be obtained,

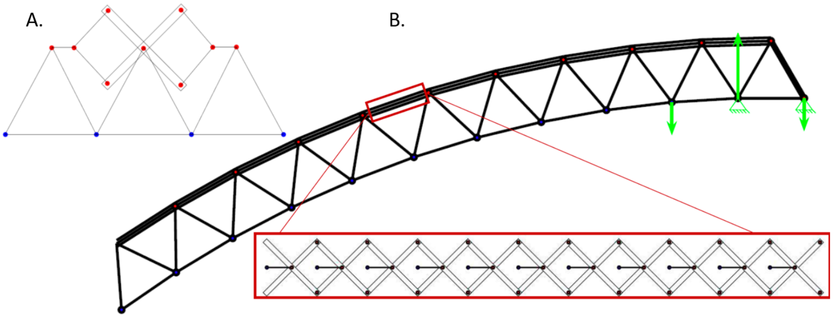

Figure 16. A. Schematic building block of primitive pantograph unit placed within a Warren truss framework for introducing a fourth length-scale. B. Possible realization of a third-gradient beam by incorporating pantographic beam within the Warren bridge architecture.

by discussing the specific synthesis method used for getting linear beams whose deformation energy depends on the third gradient of transverse displacement. Such a third-gradient beam can be synthesized by placing a pantographic beam at the top of a Warren bridge architecture as shown in Fig. 16. This type of beam can be potentially devised by introducing a fourth length scale (the height of the Warren truss triangles) in the pantographic motif as illustrated in Fig. 16A, where a primitive pantograph has been placed within a Warren truss structure [44]. The possible realization of such a system is shown in Fig. 16B, in which a pantographic beam is incorporated within the Warren bridge architecture.

The above approach, most likely, can be extended to N^{th} gradient beams by successively introducing further length scales. Subsequently, by interconnecting such higher order beams, one can expect to get N^{th} gradient 2D and 3D materials.

Similarly, higher-order micromorphic material systems could be synthesized, by introducing additional micro-mechano-morphologies of different length scales or in the case of modulating dynamic behavior, introducing mass distributions that lead to additional micro-inertia as discussed in [19, 78] or using dielectric materials for modulation with electric fields [19].

Conflicts of interest

The authors declare no competing financial interest.

Acknowledgments

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