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# A NOTE FOR THE DUNKL-CLASSICAL POLYNOMIALS

**Abstract.** In this paper, we give a new characterization for the Dunkl-classical orthogonal polynomials. The previous characterization has been illustrated by some examples.

**Key words:** *orthogonal polynomials, Dunkl operator, Dunkl-classical polynomials*

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**1. Introduction and preliminary results.** Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$. An orthogonal polynomial set (OPS for short) $\{P_n\}_{n\geqslant 0}$ in $\mathcal{P}$ is called classical (resp. $\Delta$-classical, $H_q$-classical) if $\{DP_n\}_{n\geqslant 1}$ (resp. $\{\Delta P_n\}_{n\geqslant 1}$, $\{H_q P_n\}_{n\geqslant 1}$) is also an OPS, where $D$ (resp. $\Delta$, $H_q$) denotes the derivative operator $D = \frac{d}{dx}$ (resp. $\Delta$ the difference operator, $H_q$ the Hahn operator given, respectively, by $\Delta f(x) = f(x+1) - f(x)$ and $H_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}, q \neq 1, f \in \mathcal{P}$).

In [10], the authors characterized the so-called classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) by a new characterization. In particular, they showed that a MOPS $\{P_n\}_{n\geqslant 0}$ is classical if and only if there exists a polynomial $\alpha_n$ of degree $n \geqslant 0$, and a polynomial $\Phi$ (monic) of degree less or equal to 2, such that $P_{n+1}u = D(\alpha_n \Phi u)$, $n \geqslant 0$, where $u$ is the corresponding form to $\{P_n\}_{n\geqslant 0}$. Later on, this characterization has been extended for the classical discrete and $q$-classical (discrete) polynomials (see [2]).

A natural question arises: Is there a similar characterization for Dunkl-classical orthogonal polynomials?

The aim of this paper is to answer this question. Namely, we prove the Theorem 2 (see section 2).

We begin by reviewing some preliminary results needed in the sequel. Let $\mathcal{P}'$ be the dual of $\mathcal{P}$. We denote by $\langle u,f \rangle$ the action of $u \in \mathcal{P}'$ on

$f \in \mathcal{P}$. In particular, we denote by $(u)_n = \langle u, x^n \rangle$, $n \geqslant 0$, the moments of the form $u$ (linear functional).

Let us introduce some useful operations in $\mathcal{P}'$. For any form $u$, any polynomial $p$ and any $(a, c) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}$, let $fu$, $h_a u$, $\delta_c$ and $(x-c)^{-1}u$ be the forms defined by duality:

$$\langle fu, p \rangle = \langle u, fp \rangle \, ; \quad \langle h_a u, p \rangle = \langle u, h_a p \rangle \, ,$$

$$\langle \delta_c, p \rangle = p(c); \quad \left\langle (x-c)^{-1}u, p \right\rangle = \langle u, \theta_c p \rangle \, ,$$

where $h_a p(x) = p(ax)$ and $(\theta_c p)(x) = \dfrac{p(x) - p(c)}{x - c}$.

Then, it is straightforward to prove that for $c \in \mathbb{C}$ and $u \in \mathcal{P}'$

$$(x-c)^{-1}((x-c)u) = u - (u)_0 \delta_c.$$

Let $\{P_n\}_{n \geqslant 0}$ be a sequence of monic polynomials (MPS for short) with $\deg P_n = n$, $n \geqslant 0$. The dual sequence for $\{P_n\}_{n \geqslant 0}$ is the sequence $\{u_n\}_{n \geqslant 0}$, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m \rangle = \delta_{n,m}$, $n, m \geqslant 0$, where $\delta_{n,m}$ is the kronecker symbol.

The linear form $u$ is called regular if there exists a MPS $\{P_n\}_{n \geqslant 0}$, such that [8]:

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m}, \; n, m \geqslant 0, \;\; r_n \neq 0, \, n \geqslant 0.$$

The sequence $\{P_n\}_{n \geqslant 0}$ is then said to be orthogonal with respect to $u$. In this case, we have

$$u_n = (\langle u_0, P_n^2 \rangle)^{-1} P_n u_0, \; n \geqslant 0. \tag{1}$$

Moreover, $u = \lambda u_0$, where $(u)_0 = \lambda \neq 0$ [13].

In what follows, all regular linear functionals $u$ are assumed to be normalized, i.e, $(u)_0 = 1$.

A polynomial set $\{P_n\}_{n \geqslant 0}$ is called symmetric if and only if $P_n(-x) = (-1)^n P_n(x)$, $n \geqslant 0$.

According to Favard's theorem, a monic orthogonal polynomial sequence (MOPS) is characterized by the following three-term recurrence relation [8]:

$$\begin{cases} P_0(x) = 1, P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \; n \geqslant 0. \end{cases} \tag{2}$$

with $(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C} \setminus \{0\}, n \geqslant 0$. The first associated with $\{P_n\}_{n\geqslant 0}$ is the MOPS $\{P_n^{(1)}\}_{n\geqslant 0}$, defined by

$$\begin{cases} P_0^{(1)}(x) = 1,\ P_1^{(1)}(x) = x - \beta_1, \\ P_{n+2}^{(1)}(x) = (x - \beta_{n+2})P_{n+1}^{(1)}(x) - \gamma_{n+2}P_n^{(1)}(x),\ n \geqslant 0. \end{cases} \tag{3}$$

Let us introduce the Dunkl operator [9]:

$$T_\mu(f) = f' + 2\mu H_{-1}f,\ (H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x},\ f \in \mathcal{P}, \mu \in \mathbb{C}.$$

By transposition, we define the operator $T_\mu$ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows:

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle,\ f \in \mathcal{P},\ u \in \mathcal{P}'.$$

In particular, this yields

$$(T_\mu u)_n = -\mu_n (u)_{n-1},\ n \geqslant 0,$$

with the convention $(u)_{-1} = 0$ where

$$\mu_n = n + 2\mu\, \xi_n,\ \xi_n = \frac{1 - (-1)^n}{2},\ n \geqslant 0. \tag{4}$$

Note that $T_0$ is reduced to the derivative operator $D$.

Using the previous definitions, we get the following formula [5]:

$$T_\mu(fu) = fT_\mu u + (T_\mu f)u + 2\mu(H_{-1}f)(h_{-1}u - u),\ f \in \mathcal{P},\ u \in \mathcal{P}'. \tag{5}$$

Now, consider a MOPS $\{P_n\}_{n\geqslant 0}$ and let

$$P_n^{[1]}(x, \mu) = \frac{1}{\mu_{n+1}}(T_\mu P_{n+1})(x),\ \ \mu \neq -n - \frac{1}{2},\ n \geqslant 0.$$

Denoting by $\{u_n^{[1]}(\mu)\}_{n\geqslant 0}$ the dual sequence of $\{P_n^{[1]}(\cdot, \mu)\}_{n\geqslant 0}$, we have [14]

$$T_\mu u_n^{[1]}(\mu) = -\mu_{n+1} u_{n+1},\ n \geqslant 0. \tag{6}$$

**Definition 1**. *[4,7,14] A monic orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is said to be $T_\mu$-classical (or Dunkl-classical) polynomial sequence if $\{T_\mu P_n\}_{n\geqslant 1}$ is an orthogonal polynomial sequence. In this case, the form $u$ corresponding to $\{P_n\}_{n\geqslant 0}$ is called $T_\mu$-classical form.*

B. Bouras proved in [5] the following theorem:

**Theorem 1**. *Let $\{P_n\}_{n\geqslant 0}$ be a MPS orthogonal with respect to a linear form $u_0$. For $\mu \neq \frac{1}{2}$ and $\mu \neq 0$, the following statements are equivalent:*

*(a) The sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical.*

*(b) There exist a non-zero complex number $K$ and three polynomials $\Phi$ (monic), $\widetilde{\Phi}$ and $\Psi$ with $\deg\Phi \leqslant 2$, $\deg\widetilde{\Phi} \leqslant 3$ and $\deg\Psi = 1$, such that*

$$\Psi'(0) + \frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2\xi_n - n) + \frac{K\widetilde{\Phi}'''(0)}{3(1-4\mu^2)}\mu(\xi_n - n) \neq 0, \qquad (7)$$

*and*

$$T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big) + \frac{1-4\mu^2}{K}\Psi u_0 = 0, \qquad (8)$$

*with*

$$x\Phi(x)u_0 = h_{-1}(\widetilde{\Phi}(x)u_0). \qquad (9)$$

**Remark 1**. *Symmetric Dunkl-classical forms are well described in [4]. In particular, two canonical forms appear: the generalized Hermite and the generalized Gegenbauer forms; however, for the non-symmetric case one canonical case appears: it is the regular perturbed generalized Gegenbauer form [6]*

$$\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}) = \lambda(x-1)^{-1}\mathcal{G}(\alpha,\mu-\frac{1}{2}) + \delta_1, \qquad (10)$$

*where*

$$\lambda = -\frac{2\alpha}{2\alpha+2\mu+1}, \qquad (11)$$

*and $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is the generalized Gegenbauer form [1], [3].*

*The MOPS corresponding to $\mathcal{G}(\alpha,\mu-\frac{1}{2})$, which we denote $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$, satisfies the three-term recurrence relation (2) with [8]*

$$\beta_n = 0 \quad \text{and} \quad \gamma_{n+1} = \frac{\mu_{n+1}(\mu_{n+1}+2\alpha)}{(2n+2\alpha+2\mu+1)(2n+2\alpha+2\mu+3)}, n \geqslant 0, \qquad (12)$$

*where $\mu_{n+1}$ is given in (4).*

**Lemma 1**. *[5], [7]. If $\{P_n\}_{n\geqslant 0}$ is a Dunkl-classical MOPS, then $u_0^{[1]}(\mu)$ satisfies*

$$\langle u_0^{[1]}(\mu), (P_n^{[1]}(\cdot,\mu))^2\rangle = \Big(\Psi'(0) + \frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2\xi_n - n) +$$

$$+\frac{K\widetilde{\Phi}'''(0)}{3(1-4\mu^2)}\mu(\xi_n-n)\Big)\frac{\langle u_0,P_{n+1}^2\rangle}{\mu_{n+1}}. \quad (13)$$

**2. Main Result.** The main result of this section is as follows:

**Theorem 2.** *Let $\{P_n\}_{n\geqslant 0}$ be a MPS orthogonal with respect to a linear form $u_0$. For $\mu\neq 0, \frac{1}{2}$, the following statements are equivalent.*

*(a) The sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical.*

*(b) There exist a non-zero complex number $K$ and three polynomials $\Phi$ (monic), $\deg\Phi\leqslant 2$, $\widetilde{\Phi}$, $\deg\widetilde{\Phi}\leqslant 3$ and $\Psi$, $\deg\Psi=1$ and a polynomial $Q_n$, $\deg(Q_n)=n, n\geqslant 0$, such that*

$$P_{n+1}u_0=\frac{K}{1-4\mu^2}T_\mu\Big(Q_n(\Phi u_0-2\mu h_{-1}(\Phi u_0))\Big),\ n\geqslant 0, \quad (14)$$

$$\Psi'(0)+\frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2\xi_n-n)+\frac{K\widetilde{\Phi}'''(0)}{3(1-4\mu^2)}\mu(\xi_n-n)\neq 0, \quad (15)$$

*with*

$$x\Phi(x)u_0=h_{-1}(\widetilde{\Phi}(x)u_0). \quad (16)$$

**Proof.** $(a)\Rightarrow(b)$ From the assumption, we have

$$u_n=(\langle u_0,P_n^2\rangle)^{-1}P_nu_0,\ n\geqslant 0 \quad (17)$$

and

$$u_n^{[1]}(\mu)=(\langle u_0^{[1]}(\mu),(P_n^{[1]}(\cdot,\mu))^2\rangle)^{-1}P_n^{[1]}(\cdot,\mu)u_0^{[1]}(\mu),n\geqslant 0. \quad (18)$$

Substitution of (17) and (18) in (6) gives

$$T_\mu(P_n^{[1]}(\cdot,\mu)u_0^{[1]}(\mu))=-\mu_{n+1}\frac{r_n^{[1]}}{r_{n+1}}P_{n+1}u_0,\ n\geqslant 0, \quad (19)$$

where $r_n^{[1]}=\langle u_0^{[1]}(\mu),(P_n^{[1]}(\cdot,\mu))^2\rangle$ and $r_{n+1}=\langle u_0,P_{n+1}^2\rangle$.

For $n=0$, equation (19) becomes

$$T_\mu u_0^{[1]}(\mu)=-\frac{1+2\mu}{\gamma_1}P_1u_0. \quad (20)$$

Using formula (5), equation (19) is transformed to

$$P_n^{[1]}(\cdot,\mu)T_\mu u_0^{[1]}(\mu)+(T_\mu P_n^{[1]}(\cdot,\mu))u_0^{[1]}(\mu)+2\mu(H_{-1}P_n^{[1]}(\cdot,\mu))\times$$

$$\times\Big(h_{-1}u_0^{[1]}(\mu)-u_0^{[1]}(\mu)\Big)=-\mu_{n+1}\frac{r_n^{[1]}}{r_{n+1}}P_{n+1}u_0,\ n\geqslant 0. \quad (21)$$

For $n=1$, equation (21) becomes

$$P_1^{[1]}(\cdot,\mu)T_\mu u_0^{[1]}(\mu)+u_0^{[1]}(\mu)+2\mu h_{-1}u_0^{[1]}(\mu)=-2\frac{r_1^{[1]}}{r_2}P_2u_0. \quad (22)$$

Substitution of (20) in (22) gives

$$u_0^{[1]}(\mu)+2\mu h_{-1}u_0^{[1]}(\mu)=K\Phi u_0, \quad (23)$$

where

$$K\Phi=\frac{1+2\mu}{\gamma_1}P_1P_1^{[1]}(\cdot,\mu)-2\frac{r_1^{[1]}}{r_2}P_2, \quad (24)$$

and the non-zero constant $K$ is chosen to make $\Phi$ monic.

Applying the operator $h_{-1}$ to (23), we get

$$h_{-1}u_0^{[1]}(\mu)+2\mu u_0^{[1]}(\mu)=Kh_{-1}(\Phi u_0). \quad (25)$$

Multiplying (25) by $2\mu$ and subtracting the result from (23), we get

$$u_0^{[1]}(\mu)=\frac{K}{1-4\mu^2}(\Phi u_0-2\mu h_{-1}(\Phi u_0)). \quad (26)$$

Substitution of (13) and (26) in (19) gives

$$\frac{K}{1-4\mu^2}T_\mu\Big(P_n^{[1]}(\cdot,\mu)(\Phi u_0-2\mu h_{-1}(\Phi u_0))\Big)=$$
$$=-\Big(\Psi'(0)+\frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2\xi_n-n)+\frac{K\widetilde{\Phi}'''(0)}{3(1-4\mu^2)}\mu(\xi_n-n)\Big)P_{n+1}u_0,\ n\geqslant 0.$$

Thus, (14) follows, where

$$Q_n=-\frac{T_\mu P_{n+1}}{\Big(\Psi'(0)+\frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2\xi_n-n)+\frac{K\widetilde{\Phi}'''(0)}{3(1-4\mu^2)}\mu(\xi_n-n)\Big)\mu_{n+1}},\ n\geqslant 0. \quad (27)$$

Now, putting $n=2$ in (21), we obtain

$$P_2^{[1]}(\cdot,\mu)T_\mu u_0^{[1]}(\mu)+(T_\mu P_2^{[1]}(\cdot,\mu))u_0^{[1]}(\mu)+ \\ +2\mu H_{-1}P_2^{[1]}(\cdot,\mu)\Big(h_{-1}u_0^{[1]}(\mu)-u_0^{[1]}(\mu)\Big)=-\chi_2 P_3 u_0. \quad (28)$$

Taking into account (20) and (26), we get

$$\frac{-2\mu K}{1-4\mu^2}\Big(T_\mu P_2^{[1]}(\cdot,\mu)-(1+2\mu)H_{-1}P_2^{[1]}(\cdot,\mu)\Big)h_{-1}(\Phi u_0)= \\ =\Big(\frac{1+2\mu}{\gamma_1}P_1P_2^{[1]}(\cdot,\mu)-\frac{K}{1-4\mu^2}\Phi T_\mu P_2^{[1]}(\cdot,\mu)+ \\ +\frac{2\mu K}{1-2\mu}\Phi H_{-1}P_2^{[1]}(\cdot,\mu)-\chi_2P_3\Big)u_0. \quad (29)$$

Applying the operator $h_{-1}$ to the last equation and taking into account the fact that

$$(T_\mu P_2^{[1]}(\cdot,\mu))(x)-(1+2\mu)(H_{-1}P_2^{[1]}(\cdot,\mu))(x)= \\ =(P_2^{[1]})'(x,\mu)-(H_{-1}P_2^{[1]})(x,\mu)=2x$$

and the formulas

$$h_{-1}(xv)=-xh_{-1}v \quad\text{and}\quad h_{-1}(h_{-1}v)=v, v\in\mathcal{P}',$$

we obtain (15), where

$$\widetilde{\Phi}(x)=\frac{1-4\mu^2}{4\mu K}\Big(\frac{1+2\mu}{\gamma_1}P_1(x)P_2^{[1]}(x,\mu)-\frac{K}{1-4\mu^2}\Phi(x)(T_\mu P_2^{[1]})(x,\mu)+ \\ +\frac{2\mu K}{1-2\mu}\Phi(x)(H_{-1}P_2^{[1]})(x,\mu)-\chi_2P_3(x)\Big). \quad (30)$$

$(b)\Rightarrow(a)$ Putting $n=0$ in (14), we get

$$P_1u_0=\frac{K}{1-4\mu^2}Q_0T_\mu\Big(\Phi u_0-2\mu h_{-1}(\Phi u_0)\Big). \quad (31)$$

Then, according to Theorem 1, the sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical with $\Psi=-\dfrac{P_1}{Q_0}$. □

**3. Examples.** In this section, we will illustrate Theorem 2 by giving some examples. For this, we need the following results.

Let $\{\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ be the sequence of orthogonal polynomials with respect to the form $\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})$ (see (10)).

The sequence $\{\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ satisfies the recurrence relation

$$\begin{aligned} &\widetilde{S}_0^{(\alpha,\mu-\frac{1}{2})}(x)=1,\ \widetilde{S}_1^{(\alpha,\mu-\frac{1}{2})}(x)=x-\widetilde{\beta}_0,\\ &\widetilde{S}_{n+2}^{(\alpha,\mu-\frac{1}{2})}(x)=(x-\widetilde{\beta}_{n+1})\widetilde{S}_{n+1}^{(\alpha,\mu-\frac{1}{2})}(x)-\widetilde{\gamma}_{n+1}\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}(x),\ n\geqslant 0, \end{aligned} \tag{32}$$

with [12]

$$\widetilde{\beta}_0=-a_0^{(\alpha)}=1+\lambda,\ \widetilde{\beta}_{n+1}=a_n^{(\alpha)}-a_{n+1}^{(\alpha)},\ \widetilde{\gamma}_{n+1}=-a_n^{(\alpha)}(1+a_n^{(\alpha)}),\geqslant 0,$$

where $a_n^{(\alpha)}$ is given by Maroni [12]

$$a_n^{(\alpha)}=-\frac{S_{n+1}^{(\alpha,\mu-\frac{1}{2})}(1)+\lambda(S_n^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}{S_n^{(\alpha,\mu-\frac{1}{2})}(1)+\lambda(S_{n-1}^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)},n\geqslant 0. \tag{33}$$

The relationship between $\{\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ and $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ is (see [12])

$$\widetilde{S}_{n+1}^{(\alpha,\mu-\frac{1}{2})}=S_{n+1}^{(\alpha,\mu-\frac{1}{2})}+a_n^{(\alpha)}S_n^{(\alpha,\mu-\frac{1}{2})},\ n\geqslant 0. \tag{34}$$

**Lemma 2**. *The coefficient $a_n^{(\alpha)}$ is given by*

$$a_n^{(\alpha)}=-\frac{\mu_{n+1}}{2n+2\alpha+2\mu+1},\ n\geqslant 0. \tag{35}$$

**Proof.** We will prove (35) by induction on $n$. Using (3), (11), (12), and (33), we get

$$-\frac{S_1^{(\alpha,\mu-\frac{1}{2})}(1)+\lambda(S_0^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}{S_0^{(\alpha,\mu-\frac{1}{2})}(1)+\lambda(S_{-1}^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}=-(1+\lambda)=-\frac{\mu_1}{2\alpha+2\mu+1}. \tag{36}$$

Hence, (35) is true for $n=0$.

Assume that (35) is true until $n$ and let us prove it for $n+1$. From (33), the recurrence hypothesis, and the three-term recurrence relation fulfilled by $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$, we have

$$a_{n+1}^{(\alpha)}=-\frac{S_{n+2}^{(\alpha,\mu-\frac{1}{2})}(1)+\lambda(S_{n+1}^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}{S_{n+1}^{(\alpha,\mu-\frac{1}{2})}(1)+\lambda(S_n^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}=$$

$$= -\Big(1 - \gamma_{n+1}\frac{S_n^{(\alpha,\mu-\frac{1}{2})}(1) + \lambda(S_{n-1}^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}{S_{n+1}^{(\alpha,\mu-\frac{1}{2})}(1) + \lambda(S_n^{(\alpha,\mu-\frac{1}{2})})^{(1)}(1)}\Big) = -\Big(1 + \frac{\gamma_{n+1}}{a_n^{(\alpha)}}\Big) =$$

$$= -\Big(1 - \frac{\mu_{n+1} + 2\alpha}{2n + 2\alpha + 2\mu + 3}\Big) \quad (\text{by } (12)) =$$

$$= -\frac{n + 2 + \mu - \mu(-1)^n}{2n + 2\alpha + 2\mu + 3} = -\frac{\mu_{n+2}}{2n + 2\alpha + 2\mu + 3}.$$

This completes the proof. □

**Remark 2.** *From (35), it is easy to see that $a_n^{(\alpha)}$ satisfies the following relation:*

$$\mu_{n+1} \times a_{n+1}^{(\alpha)} = \mu_{n+2} \times a_n^{(\alpha+1)}, n \geqslant 0. \tag{37}$$

**Lemma 3.** *We have the following results:*

1) *The generalized Hermite polynomials $\mathcal{H}_n^{(\mu)}$ satisfy [11]*

$$T_\mu \mathcal{H}_{n+1}^{(\mu)}(x) = \mu_{n+1}\mathcal{H}_n^{(\mu)}(x), n \geqslant 0. \tag{38}$$

2) *The generalized Gegenbauer polynomials $S_n^{(\alpha,\mu-\frac{1}{2})}$ satisfy [4]*

$$T_\mu S_{n+1}^{(\alpha,\mu-\frac{1}{2})}(x) = \mu_{n+1} S_n^{(\alpha+1,\mu-\frac{1}{2})}(x), n \geqslant 0. \tag{39}$$

3) *The sequence of orthogonal polynomials $\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}$ satisfy*

$$T_\mu \widetilde{S}_{n+1}^{(\alpha,\mu-\frac{1}{2})}(x) = \mu_{n+1} \widetilde{S}_n^{(\alpha+1,\mu-\frac{1}{2})}(x), n \geqslant 0. \tag{40}$$

**Proof.** We aim at proving (40); from (34) and (39), we have:

$$T_\mu \widetilde{S}_{n+2}^{(\alpha,\mu-\frac{1}{2})}(x) = T_\mu S_{n+2}^{(\alpha,\mu-\frac{1}{2})}(x) + a_{n+1}^{(\alpha)} T_\mu S_{n+1}^{(\alpha,\mu-\frac{1}{2})}(x) =$$

$$= \mu_{n+2} S_{n+1}^{(\alpha+1,\mu-\frac{1}{2})}(x) + a_{n+1}^{(\alpha)}\mu_{n+1} S_n^{(\alpha+1,\mu-\frac{1}{2})}(x) =$$

$$= \mu_{n+2}\Big(S_{n+1}^{(\alpha+1,\mu-\frac{1}{2})}(x) + a_{n+1}^{(\alpha)}\frac{\mu_{n+1}}{\mu_{n+2}} S_n^{(\alpha+1,\mu-\frac{1}{2})}(x)\Big) =$$

$$= \mu_{n+2}\Big(S_{n+1}^{(\alpha+1,\mu-\frac{1}{2})}(x) + a_n^{(\alpha+1)} S_n^{(\alpha+1,\mu-\frac{1}{2})}(x)\Big) \text{ by } (37) =$$

$$= \mu_{n+2}\widetilde{S}_{n+1}^{(\alpha+1,\mu-\frac{1}{2})}(x),\ n \geqslant 0.$$

Moreover, it is clear that $T_\mu \widetilde{S}_1^{(\alpha,\mu-\frac{1}{2})}(x) = 1 + 2\mu = \mu_1 \widetilde{S}_0^{(\alpha+1,\mu-\frac{1}{2})}(x)$.

Hence the desired result. □

**Example 1. Generalized Hermite polynomials.** The sequence of generalized Hermite polynomials $\{H_n^{(\mu)}\}_{n\geqslant 0}$ is symmetric Dunkl-classical and its associated form $\mathcal{H}(\mu)$ satisfies (7)–(9) with [5]

$$\Phi(x) = 1,\, \widetilde{\Phi}(x) = -x,\, \Psi(x) = 2x,\, K = 1 + 2\mu. \tag{41}$$

Using (27), (38), and (41), we can easily prove that

$$Q_n(x) = -\frac{1}{2}\, H_n^{(\mu)}(x),\, n \geqslant 0.$$

The form $\mathcal{H}(\mu)$ is symmetric, i.e, $\mathcal{H}(\mu) = h_{-1}\mathcal{H}(\mu)$. Then

$$\mathcal{H}(\mu) - 2\mu h_{-1}\mathcal{H}(\mu) = (1-2\mu)\mathcal{H}(\mu).$$

According to Theorem 2, the sequence $\{H_n^{(\mu)}\}_{n\geqslant 0}$ satisfies the following relation:

$$H_{n+1}^{(\mu)}\mathcal{H}(\mu) = -\frac{1}{2}\, T_\mu(H_n^{(\mu)}\mathcal{H}(\mu)),\, n \geqslant 0.$$

**Example 2. Generalized Gegenbauer polynomials.** The sequence of generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ is symmetric Dunkl-classical and its associated form $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ satisfies (7)–(9) with [5]

$$\begin{aligned}\Phi(x) = x^2-1,\, \widetilde{\Phi}(x) = -x(x^2-1),& \\ \Psi(x) = (2\alpha+2\mu+3)x, K = -\frac{(1+2\mu)(\alpha+\mu+\frac{3}{2})}{\alpha+1}.&\end{aligned} \tag{42}$$

From (27), (39) and (42), we can easily prove that

$$Q_n(x) = -\frac{\alpha+1}{(\alpha+\mu+\frac{3}{2})(n+2+2\alpha+\mu(1-(-1)^n))}\, S_n^{(\alpha+1,\mu-\frac{1}{2})}(x), n \geqslant 0.$$

Moreover, since $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is symmetric, then we have

$$\mathcal{G}(\alpha,\mu-\frac{1}{2}) = h_{-1}(\mathcal{G}(\alpha,\mu-\frac{1}{2})),$$

Multiplying the last equation by $x^2-1$, we obtain

$$(x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2}) = h_{-1}((x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2})),$$

Therefore,

$$(x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2})-2\mu h_{-1}((x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2})) = (1-2\mu))(x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2}).$$

Thus, according to Theorem 2, the sequence $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ satisfies the following relation:

$$S_{n+1}^{(\alpha,\mu-\frac{1}{2})}\mathcal{G}(\alpha,\mu-\frac{1}{2}) = \frac{1}{n+2+2\alpha+\mu(1-(-1)^n)}\times \\ \times T_\mu(S_n^{(\alpha+1,\mu-\frac{1}{2})}(x^2-1)\mathcal{G}(\alpha,\mu-\frac{1}{2})),\ n\geqslant 0.$$

**Example 3. Non-symmetric Dunkl-classical orthogonal polynomials.** The sequence $\{\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ is non-symmetric Dunkl-classical and its associated form $\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})$ satisfies (7)–(9) with [5]

$$\Phi(x) = (x-1)(x+\frac{1+2\mu}{1-2\mu}),\ \widetilde{\Phi}(x) = x(x-1)(x-\frac{1+2\mu}{1-2\mu}),$$
$$\Psi(x) = \frac{(1+2\mu+2\alpha)^2}{2\alpha}\Big(x-\frac{1+2\mu}{1+2\mu+2\alpha}\Big),\ K = \frac{(2\mu-1)(1+2\mu+2\alpha)}{2\alpha} \tag{43}$$

for $\alpha\neq 0$.

On the one hand, we use (27), (40), and (43) and we get

$$Q_n(x) = -\frac{2\alpha}{(2\alpha+2\mu+1)(n+1+2\alpha+\mu(1+(-1)^n))}\widetilde{S}_n^{(\alpha+1,\mu-\frac{1}{2})}(x),\ n\geqslant 0.$$

On the other hand, from (10) we have

$$(x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}) = \lambda\mathcal{G}(\alpha,\mu-\frac{1}{2}). \tag{44}$$

Since $\mathcal{G}(\alpha,\mu-\frac{1}{2})$ is symmetric, we have $\mathcal{G}(\alpha,\mu-\frac{1}{2}) = h_{-1}(\mathcal{G}(\alpha,\mu-\frac{1}{2}))$, or, equivalently, in (44):

$$(x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}) = h_{-1}((x-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})).$$

Multiplying the last equation by $x-\frac{1+2\mu}{1-2\mu}$, we obtain

$$(x-1)\Big(x-\frac{1+2\mu}{1-2\mu}\Big)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}) = -h_{-1}((x-1)\Big(x+\frac{1+2\mu}{1-2\mu}\Big)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})). \tag{45}$$

Now, from the first equality in (43) and (45), we have

$$\Phi(x)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})-2\mu h_{-1}(\Phi(x)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}))=(1+2\mu)(x^2-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2}). \tag{46}$$

Consequently, according to Theorem 2, the sequence $\{\widetilde{S}_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ satisfies the following relation:

$$\widetilde{S}_{n+1}^{(\alpha,\mu-\frac{1}{2})}\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})=\frac{1}{n+1+2\alpha+\mu(1+(-1)^n)}\times$$
$$\times T_\mu(\widetilde{S}_n^{(\alpha+1,\mu-\frac{1}{2})}(x^2-1)\widetilde{\mathcal{G}}(\alpha,\mu-\frac{1}{2})), n\geqslant 0.$$

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