Momentum-Current Gravitational Multipoles of Hadrons

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We study multipole expansion of the momentum currents in hadrons with three series $S^{(J)}$, $\tilde{T}^{(J)}$, and $T^{(J)}$ in connection with the gravitational fields generated nearby. The momentum currents are related to their energy-momentum form factors, which in principle can be probed through processes like the deeply virtual Compton scattering currently studied at the Jefferson Lab 12 GeV facility and future Electron-Ion Collider. We define the leading momentum-current multipoles [the "tensor monopole" τ ($T0$) and "scalar quadrupole" $\hat{\sigma}^{ij}$ ($S2$) moments], relating the former to the so-called D term in the literature. We calculate the momentum-current distribution in the hydrogen atom and its monopole moment in the basic unit of $\tau_0 = \hbar^2/4m_e$, showing that the sign of the D term has little to do with mechanical stability. The momentum-current distribution also strongly modifies the static gravitational field inside hadrons.

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I. INTRODUCTION

The energy-momentum tensor (EMT) distribution in a closed system is an important measure of the underlying dynamics; and according to Einstein's general relativity, it is responsible for the space-time geometry nearby. For a stable nuclear system, such as the proton, deeply virtual Compton scattering [1–5] and similar processes discussed in Refs. [6–10] allow one to measure the EMT form factors, which in the static limit (or disregarding the Compton wavelength) can be interpreted in terms of the EMT spatial distributions. The EMT effects from long-range forces on the space-time perturbations have been studied in perturbative quantum field theories [11,12]. Here, we are interested in bound-state systems.

In this paper, we perform the static multipole expansion [13–16] of the mass, momentum, and momentum-current (MC) densities in the hadrons of various spins, trying to understand their physical significance. We relate the hadrons' gravitational form factors to the gravitational multipoles. We find, in particular, that the tensor-monopole moment $T0$ is related to the EMT $C(q^2)$ form factor (the D term in [17]). As a concrete example of MC distribution, we calculate the $C(q^2)$ form factor in the hydrogen atom and find that the monopole moment is $\hbar^2/4m_e$ up to the fractional correction of order $\alpha = 1/137$ (fine structure constant), where m_e is the electron mass. We find its sign different from the so-called "mechanical stability" condition [18,19]. We also remark that the physical significance of "pressure" and "shear pressure" [18,20–22] from the momentum-current density is only limited to the sense of radiation pressure (see, for example, [23]).

Note that the EMT cannot be uniquely derived from the translational symmetry in the flat space-time. Thus, we always consider matter fields (electrons, photons, quarks, gluons, etc.) minimally coupled to a curved space-time with metric $g^{\mu\nu}$, and we derive the energy-momentum tensor through variation [24],

$$T^{\mu\nu} \sim \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}, \tag{1}$$

where S_{matter} is the matter field action in the curved space-time. The flat space-time limit is taken after the functional derivative.

II. STATIC ENERGY-MOMENTUM TENSOR MULTIPOLE EXPANSION

In this section, we consider multipole expansion of the static EMT distribution $T^{\mu\nu}(\vec{r})$ in a finite system: particularly,

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the momentum current $T^{ij}(\vec{r})$. Multipole expansion for electromagnetic systems is well known [25], and we repeat it in the first subsection for the energy current or momentum density T^{0i}, which defines two moment series corresponding to two degrees of freedom of the conserved current. The multipole expansion for gravitational systems has also been worked out in the literature to considerable detail [15,16]. However, most of the studies in gravitational systems focus on generation of gravitational waves. Our interest is in static systems, which only involve time-independent moments. We study the moments of T^{ij} in the second subsection that have three independent series corresponding to three degrees of freedom in the momentum currents. Even though most of the results in this section are not new, they help us to understand the physical significance of the EMT distributions in quantum-mechanical bound-state systems such as hadrons or atoms.

We study the physics of the EMT moments in the context of the linearized Einstein equation in the weak gravitational limit, in which the symmetric metric tensor $g^{\mu\nu}$ can be approximated by the flat space metric $\eta^{\mu\nu} = (1, -1, -1, -1)$ (this choice is opposite to the standard convention in the gravitation literature) plus a small perturbation $h^{\mu\nu}$,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2}$$

The rank-2 tensor $h^{\mu\nu}$ has 10 independent components. Using the coordinates' reparametrization invariance $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu - \eta_{\mu\nu}\partial^\rho\zeta_\rho$, where ζ_μ is an arbitrary vector field, only six components are independent. It is common to define the trace-reversed metric perturbation

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2} h^\rho_\rho, \tag{3}$$

and a convenient gauge choice is then the harmonic or Lorenz gauge defined by four conditions [16],

$$\partial_\mu \bar{h}^{\mu\nu} = 0, \tag{4}$$

which are manifestly consistent with the conservation of the EMT.

An important reason to introduce the harmonic gauge is that the trace-reversed metric perturbation satisfies the linearized Einstein equation:

$$\Box \bar{h}^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}, \tag{5}$$

where $\Box = \partial^\mu\partial_\mu$, G is Newton's constant, c is the speed of light, and $T^{\mu\nu}$ is the EMT of matter fields. Because Eq. (5) is just the standard wave equation, it can be solved as [16]

$$\bar{h}^{\mu\nu}(t, \vec{r}) = \frac{4G}{c^4} \int d^3\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|} T^{\mu\nu}\left(t - \frac{|\vec{r} - \vec{r}'|}{c}, \vec{r}'\right). \tag{6}$$

For a time-dependent source, the above will lead to the generation of gravitational waves ($\sim 1/r$) in regions far away from the source. Among the six independent physical components of the metric, only two correspond to gravitational waves. In the harmonic gauge, they are defined by the transverse-traceless condition [16]. For example, if the wave vector is along the z direction, then the two independent components are $\bar{h}_{xy} = \bar{h}_{yx}$ and $\bar{h}_{xx} = -\bar{h}_{yy}$.

We will not consider the moment series for the energy/mass density T^{00}, which defines

$$M_{i_1 \ldots i_l} = \int d^3\vec{r} r_{(i_1} \ldots r_{i_l)} T^{00}(\vec{r}), \tag{7}$$

where $(\ldots)$ are symmetric and trace-free parts of the tensor. The generation of gravity by mass multipoles is well known [13–16].

A. Multipole expansion for energy current

Consider the conserved energy current or momentum density $T^{0\mu}$ in a static system. Here, for simplicity, we use notation similar to electromagnetism, with j^μ standing for $T^{0\mu}$. The static conservation law becomes

$$\partial_i j^i(\vec{r}) = 0, \tag{8}$$

where $\vec{j}(\vec{r})$ is a static current distribution. Given the vector current, the vector field $\vec{A}$ [standing for $\bar{h}^{i0}c^4/(4G)$], which satisfies the Laplace equation

$$\nabla^2 \vec{A}(\vec{r}) = -4\pi \vec{j}(\vec{r}), \tag{9}$$

can be solved as

$$A_i(\vec{r}) = \int \frac{j_i(\vec{r}') d^3\vec{r}'}{|\vec{r} - \vec{r}'|}. \tag{10}$$

At the large distance of $r \gg r'$, A^i allows the following multipole expansion:

$$A_i(\vec{r}) = \sum_{l=0}^{\infty} \sum_{i_1,\ldots,i_l} \frac{(-1)^l}{l!} j_{i,i_1\ldots i_l} \partial_{i_1} \ldots \partial_{i_l} \frac{1}{r}, \tag{11}$$

where the moments of the vector current

$$j_{i,i_1\ldots i_l} = \int d^3\vec{r} r_{(i_1} \ldots r_{i_l)} j_i(\vec{r}), \tag{12}$$

and the symbol $(\ldots)$ will again be used to denote the symmetric and traceless part between tensor indices i_1 to i_l. Clearly, the subtraction of the trace removes all the moments weighted with r^2, r^4, etc., which do not yield any new tensor structure or contribute to the vector field at large distances.

From a group-theoretic point of view, the moments $j_{i,i_1\ldots i_l}$ form a tensor product of spin-1 and spin-l irreducible representations of the three-dimensional rotation group and can be decomposed into a direct sum of spin-$(l-1)$, l, and $l+1$ representations,

$$[1]\otimes[l]=[l-1]\oplus[l]\oplus[l+1]. \tag{13}$$

In terms of tensor notation, the above decomposition can be written as [15,16]

$$j_{i,i_1\ldots i_l}=U^{(l+1)}_{ii_1\ldots i_l}+\tilde{V}^{(l)}_{ii_1i_2,\ldots,i_l}+\delta_{i(i_1}V^{(l-1)}_{i_2\ldots i_l)}, \tag{14}$$

The spin-$l+1$, l, and $l-1$ parts $U^{(l+1)}$, $\tilde{V}^{(l)}$, and $V^{(l-1)}$ read explicitly as

$$U^{(l+1)}_{ii_1\ldots i_l}\equiv j_{(i,i_1\ldots i_l)}, \tag{15}$$

$$\tilde{V}^{(l)}_{ii_1\ldots i_l}\equiv\frac{l}{l+1}j_{[i,i_1]\ldots i_l}, \tag{16}$$

$$V^{(l-1)}_{i_2\ldots i_l}\equiv\frac{2l-1}{2l+1}j_{i,ii_2,\ldots i_l}. \tag{17}$$

where indices between $[\ldots]$ are antisymmetrized. The above decomposition applies for a generic vector current $\vec{j}$ not necessarily conserved.

For a conserved current, it is easy to show that the totally symmetric $(l+1)$-multipole always vanishes,

$$U^{(l+1)}_{ii_1\ldots i_l}\equiv 0, \tag{18}$$

or

$$\int d^3\vec{r}r_{(i_1}\ldots r_{i_l}j_{i)}(\vec{r})=0, \tag{19}$$

which holds with and without trace subtraction. It generates a large number of identities among the moments after some contractions of indices; for example,

$$2\int d^3\vec{r}\vec{r}\cdot\vec{j}r_{i_1}\ldots r_{i_{l-1}}=-(l-1)\int d^3\vec{r}r^2j_{(i_1}r_{i_2}\ldots r_{i_{l-1})}. \tag{20}$$

For $l=1$, it simply reduces to

$$\int d^3\vec{r}\,\vec{r}\cdot\vec{j}=0, \tag{21}$$

and for $l=2$, the identity reads

$$2\int d^3\vec{r}\,\vec{r}\cdot\vec{j}r_i=-\int d^3\vec{r}r^2j_i. \tag{22}$$

These identities will be useful later.

The contribution of the $V^{(l-1)}_{i_2\ldots i_l}$ multipoles to the vector potential is

$$A_i^{V(l-1)}=\frac{(-1)^l}{l!}V^{(l-1)}_{i_2\ldots i_l}\partial_i\partial_{i_2}\ldots\partial_{i_l}\frac{1}{r} \tag{23}$$

and can be gauged away by the gauge transformation

$$A_i\to A_i-\partial_i\left(\frac{(-1)^l}{l!}V^{(l-1)}_{i_2\ldots i_l}\partial_{i_2}\ldots\partial_{i_l}\frac{1}{r}\right). \tag{24}$$

Therefore, they do not produce a physical effect in static gauge theories. However, the time-varying V multipoles are important for time-dependent effects such as radiation, and they form a useful series for describing the current distribution. The first such a moment is

$$V^{(0)}=\int d^3\vec{r}\,\vec{r}\cdot\vec{j}(r)=0 \tag{25}$$

as a consequence of identity [Eq. (21)]. Thus, the first nonvanishing moment appears at

$$\begin{aligned}V_i^{(1)}&=\frac{3}{5}\int d^3\vec{r}\left(r_i\vec{r}\cdot\vec{j}(r)-\frac{1}{3}r^2j_i\right)\\&=-\frac{1}{2}\int d^3\vec{r}r^2j_i(\vec{r}).\end{aligned} \tag{26}$$

In the second equality, we have used identity [Eq. (22)]. Clearly, this is not independent and relates to the current radius. The independent V-moment series starts from $V^{(2)}_{ij}$, and they can all be related to moments of $\vec{r}\cdot\vec{j}$.

The physically interesting moment series in the static case are

$$\tilde{V}^{(l)}_{ii_1\ldots i_l}\sim\int d^3\vec{r}m_i(\vec{r})r_{(i_1}\ldots r_{i_{l-1})}, \tag{27}$$

where m_i is the well-known "magnetization density" [25] in the case of the electric current or "angular-momentum density" in case of the energy current $T^i=T^{0i}$,

$$\vec{m}(\vec{r})=\vec{r}\times\vec{j}(\vec{r}), \tag{28}$$

$$\vec{J}(\vec{r})=\vec{r}\times\vec{T}(\vec{r}). \tag{29}$$

$\tilde{V}^{(1)}$ is just the magnetic moment in the electromagnetic and total angular-momentum vector $\vec{S}$ for the energy current.

B. Multipole expansion for momentum currents

The momentum current T^{ij} can be decomposed into the three-dimensional trace and traceless parts, calling them tensor and scalar parts, respectively. The scalar multipole expansion defines

$$S^{(l)}_{i_1\ldots i_l} = \int d^3\vec{r} r_{(i_1}\ldots r_{i_l)} T_{kk}(\vec{r}), \tag{30}$$

which is similar to the multipoles of the energy/mass density.

For the tensor part, we can make the following multipole decomposition [15,16,26]:

$$[2]\otimes[l] = [l-2]\oplus[l-1]\oplus[l]\oplus[l+1]\oplus[l+2]. \tag{31}$$

In terms of tensor notation, we first define the moments $T_{ij,i_1\ldots.i_l}$ similar to Eq. (12),

$$T_{ij,i_1\ldots.i_l} = \int d^3\vec{r} T_{ij}(\vec{r}) r_{(i_1\ldots r_l)} \tag{32}$$

where $(i_1\ldots i_l)$ again denotes the traceless and symmetric parts. The tensor decomposition then reads [15,16,26]

$$T_{ij,i_1\ldots.i_l} = U^{(l+2)}_{iji_1\ldots i_l} + \tilde{U}^{(l+1)}_{iji_1\ldots.i_l} + \delta_{ii_1}\bar{S}^{(l)}_{ji_2\ldots i_l} + \delta_{ii_1}\tilde{T}^{(l-1)}_{ji_2\ldots i_l} + \delta_{ii_1}\delta_{ji_2}T^{(l-2)}_{i_3\ldots i_l}, \tag{33}$$

where traceless and symmetric subtractions in ij and in $i_1\ldots.i_l$ are always assumed. The multipole series denote [15,16,26]

$$U^{(l+2)}_{iji_1\ldots i_l} \equiv T_{(ij,i_1\ldots i_l)}, \tag{34}$$

$$\tilde{U}^{(l+1)}_{iji_1\ldots i_l} \equiv \frac{2l}{l+2} T_{i[j,i_1]\ldots i_l}, \tag{35}$$

$$\bar{S}^{(l)}_{i_1\ldots i_l} \equiv \frac{6l(2l-1)}{(l+1)(2l+3)} T_{i(i_1,i_2,\ldots.i_l)i}, \tag{36}$$

$$\tilde{T}^{(l-1)}_{i_1\ldots i_l} \equiv \frac{2(l-1)(2l-1)}{(l+1)(2l+1)} T_{i([i_1,i_2]i_3\ldots i_l)i}, \tag{37}$$

$$T^{(l-2)}_{i_1\ldots.i_{l-2}} \equiv \frac{2l-3}{2l+1} T_{ij,iji_1\ldots i_{l-2}}. \tag{38}$$

where we have included certain coefficients in the definition.

Due to momentum-current conservation $\partial_i T^{ij} = 0$ in a static system, not all the multipoles above are nonvanishing. One can show that the following general identities are true:

$$\frac{1}{k!}\sum_P \int d^3\vec{r} T_{ii_{P(1)}} r_{i_{P(2)}}\ldots. r_{i_{P(k)}} = 0, \tag{39}$$

where P runs over all permutations $P(1),\ldots.P(k)$ of $1,\ldots k$. From the above, one can show that the $l+2$ and $l+1$ moments all vanish;

$$U^{l+2}_{iji_1\ldots i_l} \equiv 0, \qquad \tilde{U}^{l+1}_{iji_1\ldots i_l} \equiv 0. \tag{40}$$

One can form more identities from Eq. (39) by performing contractions or symmetrization/antisymmetrizations. For example, by contracting i with one of the other indices under permutation, one has

$$\int d^3\vec{r} T_{ii} r_{(i_1}\ldots r_{i_k)} = -k \int d^3\vec{r} r_i T_{i(i_1} r_{i_2}\ldots. r_{i_k)}. \tag{41}$$

By contracting two of the indices under permutation, one has

$$2\int d^3\vec{r} T_{ij} r_j r_{i_1}\ldots. r_{i_{k-2}} = -(k-2)\int d^3\vec{r} r^2 T_{i(i_1} r_{i_2}\ldots r_{i_{k-2})}, \tag{42}$$

and so on. For $k=2$ and $k=3$, the above reduces to

$$\int d^3\vec{r} r_i T_{i(i_1} r_{i_2)} = -\frac{1}{2}\int d^3\vec{r} T^{ii} r_{(i_1} r_{i_2)} \tag{43}$$

and

$$\int d^3\vec{x} r^2 T_{(i_1 i_2)} = -2\int d^3\vec{r} r_i T_{i(i_1} r_{i_2)} = \int d^3\vec{r} T^{ii} r_{(i_1} r_{i_2)}. \tag{44}$$

Notice that the preceding holds without trace subtraction as well.

Moreover, by performing antisymmetrization of i with one of the indices under permutation in Eq. (39), one obtains

$$\int d^3\vec{r} T_{i[j} r_{k]} = 0, \tag{45}$$

$$\int d^3\vec{r} (r_{[j} T_{k]i_1} r_{i_2} + r_{[j} T_{k]i_2} r_{i_1}) = 0, \tag{46}$$

and so on.

Given these relations, we can re-express the tensor momentum-current multipole $\bar{S}^{(l)}_{i_1\ldots i_l}$ in terms of that of the scalar momentum-current multipoles. First, let us consider $l=2$; in this case, the above reduces to

$$\bar{S}^{(2)}_{i_1 i_2} = \frac{12}{7}\int d^3\vec{r} \times \left(r_i T_{i(i_1} r_{i_2)} - \frac{1}{3} r^2 T_{(i_1 i_2)} - \frac{1}{3} T_{ii} r_{(i_1} r_{i_2)} \right) \tag{47}$$

that, by using Eqs. (43) and (44), reduces to

$$\bar{S}^{(2)}_{i_1 i_2} = -2\int d^3\vec{r} r_{(i_1} r_{i_2)} T_{ii}(\vec{r}) \equiv -2\sigma_{i_1 i_2}, \tag{48}$$

where the quadrupole of the scalar momentum current $\sigma_{i_1 i_2}$ or scalar quadrupole $S2$ is defined as

$$\sigma_{ij} \equiv S^{(2)}_{ij} = \int d^3\vec{r} T_{kk}(\vec{r}) r_{(i} r_{j)}. \tag{49}$$

In fact, for general l, one can show that the above relation remains valid [15],

$$\bar{S}^{(l)}_{i_1 i_2,\ldots,i_l} = -2S^{(l)}_{i_1\ldots i_l}; \tag{50}$$

therefore, at given order l, there is only one series of linearly independent spin-l multipoles $S^{(l)}_{i_1\ldots i_l}$.

Given the moments, one can study their contribution to $\bar{h}^{ij}$. By standard methods, the contribution of $\bar{S}$ and S reads [15,16]

$$\bar{h}^{Sl}_{ij} = \frac{4G(-1)^l}{l!}\left(\frac{\delta_{ij}}{3}(S^{(l)}_{i_1\ldots i_l} - \bar{S}^{(l)}_{i_1\ldots i_l})\partial_{i_1}\ldots\partial_{i_l}\frac{1}{r} + \bar{S}^{(l)}_{i_1\ldots i_{l-1} i}\partial_j \partial_{i_1}\ldots\partial_{i_{l-1}}\frac{1}{r}\right), \tag{51}$$

where symmetrization between i and j is assumed. Using the relation [Eq. (50)], the above can be written in the form

$$\bar{h}^{Sl}_{ij} = \delta_{ij}\partial_k\zeta_k - \partial_i\zeta_j - \partial_j\zeta_j, \tag{52}$$

where

$$\zeta^{Sl}_i = \frac{4G(-1)^l}{l!} S^{(l)}_{kk,ii_1,\ldots,i_{l-1}}\partial_{i_1}\ldots\partial_{i_{l-1}}\frac{1}{r}; \tag{53}$$

therefore, after a gauge transformation of

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\mu\zeta_\nu - \eta_{\mu\nu}\partial^\alpha\zeta_\alpha, \tag{54}$$

we are left only with the contribution in $\bar{h}^{00}$ [15,16],

$$\bar{h}^{Sl}_{00} = \frac{4G(-1)^l}{l!} S^{(l)}_{i_1\ldots i_l}\partial_{i_1}\ldots\partial_{i_l}\frac{1}{r}, \tag{55}$$

which is just the standard scalar multipole expansion with $T_{kk}(\vec{r})$ as the scalar density, similar to the energy density multipoles. The leading contribution comes from the scalar quadrupole $S2$ moment.

Finally, we come to the other two $\tilde{T}^{(l-1)}$ and $T^{(l-2)}$ multipole series. Their contributions to the trace-reversed metric perturbation $\bar{h}^{ij}$ read [15,16]

$$\bar{h}^{\tilde{T}(l-1)}_{ij} = \frac{2G(-1)^l}{l!}\tilde{T}^{(l-1)}_{i_1 ii_2\ldots i_{l-1}}\partial_j\partial_{i_1}\ldots\partial_{i_{l-1}}\frac{1}{r} + (i \to j) \tag{56}$$

and

$$\bar{h}^{T(l-2)}_{ij} = \frac{4G(-1)^l}{l!} T^{(l-2)}_{i_2\ldots i_l}\partial_i\partial_j\partial_{i_2}\ldots\partial_{i_l}\frac{1}{r}. \tag{57}$$

Similar to the case of the vector current, they can all be gauged away through gauge transformations

$$\zeta^{\tilde{T}(l-1)}_i = \frac{2G(-1)^l}{l!}\tilde{T}^{(l-1)}_{i_1 ii_2\ldots i_{l-1}}\partial_{i_1}\ldots\partial_{i_{l-1}}\frac{1}{r}, \tag{58}$$

and

$$\zeta^{T(l-2)}_i = \frac{2G(-1)^l}{l!} T^{(l-2)}_{i_2\ldots i_l}\partial_i\partial_{i_2}\ldots\partial_{i_l}\frac{1}{r}. \tag{59}$$

Because $\partial^i\zeta^{\tilde{T}(l-1)}_i = \partial^i\zeta^{T(l-2)}_i = 0$ due to antisymmetrization in $T^{(l-2)}$ and $\partial^2\frac{1}{r} = 0$ at a large distance, the gauge transformation will not produce new terms in $\bar{h}^{00}$. Therefore, both series $\tilde{T}^{(l-1)}$ and $T^{(l-2)}$ have no physical effect at a large distance in the static case.

However, moment series $\tilde{T}^{(l-1)}$ and $T^{(l-2)}$ still provide useful characterization of the momentum-current distribution (and do have physical effects in time-varying systems). At $l = 1$, the $\tilde{T}^{(0)}$ vanishes. At $l = 2$, the nonvanishing moment is related to the tensor momentum-current monopole or tensor monopole $T0$ for short:

$$T^{(0)} = \frac{1}{5}\int d^3\vec{r} T_{ij}(\vec{r})\left(r_i r_j - \frac{\delta_{ij}}{3}r^2\right) \equiv \frac{2}{15}\int d^3\vec{r} r^2 s(r), \tag{60}$$

where the second line serves as a definition for $s(r)$ (shear pressure). Using Eq. (43), it is related to the "scalar momentum-current radius,"

$$T^{(0)} = -\frac{1}{6}\int d^3\vec{r} r^2 T_{ii}(\vec{r}) = -\frac{1}{2}\int d^3\vec{r} r^2 p(\vec{r}), \tag{61}$$

where T^{ii} is proportional to the so-called pressure $p(r)$ in the other literature [18,20–22]. We choose to define the tensor-MC monopole moment of a system as

$$\tau = -T^{(0)}/2, \tag{62}$$

which relates to the "D term" $D(0)$ [17] as $\tau = \frac{D(0)}{4M}$, where M is the mass of the whole system.

We next come to the $\tilde{T}1$ dipole $\tilde{T}^{(1)}_{ij}$. It is antisymmetric in ij and can be written as

$$\tilde{T}^{(1)}_{ij} = \frac{2}{5}\int d^3\vec{r} r_k T_{k[i}(\vec{r}) r_{j]}. \tag{63}$$

If one defines the dilatation current at $t = 0$,

$$j^i_D = r_k T^{ki}, \tag{64}$$

then $\tilde{T}^{(1)}$ can be conveniently expressed as the "magnetic moment" of the dilatation current. Due to Eq. (46), $\tilde{T}^{(1)}_{ij} = 0$ identically.

To summarize, there are three series of multipoles for the momentum current: $S^{(l)}$, $\tilde{T}^{(l-1)}$, and $T^{(l-2)}$. The leading-order moments are the tensor monopole $T0$ $\tau = -T^{(0)}/2$ and the scalar quadrupole $S2$ $\sigma_{ij} = S^{(2)}_{ij}$, with the vanishing tensor dipole $\tilde{T}1$. To the next order, one has the scalar octupole $S3$, tensor quadrupoles $\tilde{T}2$ and $T2$, the tensor dipole $T1$, and so on.

III. EMT FORM FACTORS OF HADRONS AND GRAVITATIONAL MULTIPOLES

In this section, we consider examples of the gravitational multipoles in hadrons of different spins. Not all hadrons are capable of generating all types of gravitation multipoles. For the spin-0 particle such as the pion or ^{4}He nucleus, only two multipoles can be generated: one corresponds to the total mass M (mass monopole $M0$), and the other to the momentum-current tensor monopole τ ($T0$). For a spin-1/2 hadron such as the proton and neutron, one can generate in addition the angular-momentum dipole ($\tilde{V}1$). For a spin-1 resonance, such as a ρ meson, one can generate the mass quadrupole $M2$, the scalar quadrupole $S2$, and the tensor quadrupole $T2$. In the following, we will discuss each of them in turn.

We work in the limit in which the hadron masses are large so that their Compton wavelength is negligible [27]. This is true in the large N_c limit for baryons (and certainly not true for a pion). In this case, one can directly Fourier transform the form factors to the position space to obtain the space density distributions. For particles for which the Compton wavelength is not small as compared to its size, an option is to go to the infinite momentum frame [28–30] where one has to be content with a two-dimensional interpretation. In practice, we adopt the standard Breit frame approach as the definition of a spatial density. When studying the gravitational perturbation at the distance r much larger than the Compton wavelength, the formula in terms of the form factors is accurate: independent of the density interpretation.

A. Spin-0 case

Let us first consider a scalar system. The EMT matrix element between the plane wave states $|P^\mu\rangle$ and $|P^{\mu'}\rangle$ defines the gravitational form factors A and C [11,18]:

$$\langle P'|T^{\mu\nu}|P\rangle = 2P^\mu P^\nu A(q^2) + 2(q^\mu q^\nu - g^{\mu\nu} q^2) C(q^2), \quad (65)$$

where $q^\mu = P'^\mu - P^\mu$ is the momentum transfer. The momentum conservation requires $A(0) = 1$. In the Breit frame where $\vec{P} + \vec{P}' = 0$, the Fourier transformation of $MA(q)$ corresponds to a part of the mass density $\rho_m(\vec{r})$.

The form factor $C(q^2)$ is related to the tensor-MC monopole distribution in the system, besides contributing to the mass density $\rho_m(\vec{r})$. In fact, if we Fourier transform T^{ij} to the coordinate space, it has the form

$$T^{ij}(\vec{r}) = (\nabla^2\delta^{ij} - \nabla^i\nabla^j)\frac{C(r)}{M}, \quad (66)$$

where $C(r)$ is the Fourier version of $C(q^2)$; and we have divided a factor $2M$ from the relativistic normalization ($\langle P|P'\rangle = (2\pi)^3 2P^0 \delta^3(\vec{P} - \vec{P}')$). A simple calculation shows that the MC monopole moment $T0$ is just

$$\tau = \frac{C(q=0)}{M}, \quad (67)$$

which relates to the D term [17] as $\tau = \frac{D(0)}{4M}$. For a free spin-0 boson [11],

$$\tau_{\rm boson} = -\frac{\hbar^2}{4M} \quad (68)$$

when the proper SI unit of $\mathrm{kg}\cdot\mathrm{m}^4\cdot\mathrm{s}^{-2}$ is restored. We define a fundamental unit of $\tau_0 = \frac{\hbar^2}{4M}$, we write $\tau_{\rm boson} = g_b\tau_0$, and then $g_b = -1$. For a minimally coupled interacting theory, the monopole moment remains the same [24]. For more complicated examples including nonminimal coupling, QCD pseudo-Goldstone bosons, as well as large nuclei, see Refs. [18,24,31] for extensive discussions. Monopole density distribution is related to $s(r)$ defined in Eq. (60).

For a system with long-range force, such as a charged particle, it can be shown that $C(\vec{q}\to 0)$ is infrared divergent [11,32–36]:

$$\frac{C(\vec{q})}{2M} \to \frac{\alpha\pi}{16|\vec{q}|} + \frac{\alpha}{6\pi M}\ln\frac{\vec{q}^2}{M^2}. \quad (69)$$

The first term is due to the large r asymptotic decay of the Coulomb potential and is classical in nature [11,34], whereas the second term is quantum in nature. It can be shown [11] that a divergent monopole moment $C(0)$ ($\tau_{\rm eff} = +\infty$) will generate $1/r^2$ correction to the space-time metric

$$\bar{h}^{ij}(\vec{r}) = \frac{G\alpha\hat{r}_i\hat{r}_j}{r^2} + \frac{4G\alpha}{3\pi M r^3}(\hat{r}^i\hat{r}^j - \delta^{ij}), \quad (70)$$

where the first term comes form the linearly divergent part $\frac{\pi\alpha}{16|q|}$ of $C(q)$, whereas the second term is due to the logarithmic divergent term $\frac{\alpha}{6\pi M}\ln\frac{\vec{q}^2}{M^2}$.

B. $C(q^2)$ contribution to gravitational potential

According to Sec. II, it appears that the tensor monopole τ does not contribute to the long-distance properties of the gravity: other than it produces a pure gauge contribution.

However, form factor $C(q^2)$ does generate a short-distance static gravitational potential h^{00} through its contribution to the energy density.

It can be shown by solving the linearized Einstein equation that

$$h_C^{00}(\vec{r}) = -\frac{8\pi G}{c^4 M} C(r), \tag{71}$$

$$h_C^{ij}(\vec{r}) = \frac{8\pi G}{c^4 M} C(r)\delta^{ij}, \tag{72}$$

thus $C(r)$ contributes a part of the gravitational potential at a short distance. The total static potential will be added by the form factor $A(q^2)$ contribution

$$h_A^{00}(\vec{r}) = \frac{2}{c^4} V(r), \tag{73}$$

$$h_A^{ij}(\vec{r}) = \frac{2}{c^4} V(r)\delta^{ij}, \tag{74}$$

where

$$V(r) = \int d^3\vec{r}' \frac{GM}{|\vec{r}-\vec{r}'|} A(\vec{r}'). \tag{75}$$

At large r, $h_C^{\mu\nu}$ decays exponentially, whereas $h_A^{\mu\nu}$ reduces to a point-mass Newton potential. For a nonrelativistic probe, only h^{00} matters.

C. Spin-1/2 case

For a spin-$1/2$ system, the matrix element of the EMT is [37–39]

$$\langle P'|T^{\mu\nu}|P\rangle = \bar{u}(P'S')\left[A(q^2)\gamma^{(\mu}\bar{P}^{\nu)} + B(q^2)\frac{\bar{P}^{(\mu}i\sigma^{\nu)\alpha}q_\alpha}{2M} + C(q^2)\frac{q^\mu q^\nu - g^{\mu\nu}q^2}{M}\right]u(PS), \tag{76}$$

where $\bar{P}^\mu = (P+P')^\mu/2$. There are now three dimensionless gravitational form factors :$A(q^2)$, $B(q^2)$, and $C(q^2)$. The physics of $A(q^2)$ is the same as that for the spin-0 case.

The form factor $B(q^2)$ is related to the angular-momentum distribution in the system. Indeed, the momentum density T^{0i} is

$$\vec{T}(\vec{r}) = -\frac{1}{2}\vec{S}\times\nabla(A(r)+B(r)). \tag{77}$$

where $A(r)$ and $B(r)$ are Fourier transformations of A and B form factors, which generate the new $\bar{h}^{i0}$ perturbation through the angular-momentum density $\vec{S}(\vec{r}) = \vec{r}\times\vec{T}(\vec{r})$. The $\tilde{V}1$ moment of the momentum density yields total angular momentum $\vec{S}$. Angular-momentum conservation requires $B(q^2=0)=0$.

The MC monopole moment is zero for a free fermion [40]. In general, a spin-$1/2$ system has

$$\tau = \frac{C(0)}{M}. \tag{78}$$

There have been extensive studies in the literature about $C(0)$ for the nucleon [18,41]. In particular, lattice QCD calculations have been made for the separate contributions from quarks and gluons [22,42]. It appears that τ_N is negative from various nucleon models as well as the preliminary lattice result.

D. Spin-1 case

Unlike the spin-0 and spin-1/2 cases, a spin-1 hadron has six independent gravitational form factors [12,43,44],

$$\begin{aligned}\langle P',\epsilon_f|T^{\mu\nu}(0)|P,\epsilon_i\rangle &= -2\bar{P}^\mu\bar{P}^\nu\left[(\epsilon_f^\star\cdot\epsilon_i)A(q^2) + E^{\alpha\beta}q_\alpha q_\beta\frac{\tilde{A}(q^2)}{M^2}\right] \\ &\quad + J(q^2)\frac{i\bar{P}^{(\mu}S^{\nu)\alpha}q_\alpha}{M} \\ &\quad -2(q_\mu q_\nu - g_{\mu\nu}q^2)\left[(\epsilon_f^\star\cdot\epsilon_i)C(q^2) + E^{\alpha\beta}q_\alpha q_\beta\frac{\tilde{C}(q^2)}{M^2}\right] \\ &\quad -[(E^{\mu\nu}q^2 - E^{\mu\alpha}q^\nu q_\alpha - E^{\alpha\nu}q^\mu q_\alpha + g^{\mu\nu}E^{\alpha\beta}q_\alpha q_\beta]D(q^2).\end{aligned} \tag{79}$$

where ϵ_i and ϵ_f^* are polarization four vectors of the initial and final hadrons, satisfying $\epsilon_i\cdot P = \epsilon_f\cdot P' = 0$. To simplify the expression, we also use symmetric polarization density matrix

$$E^{\mu\nu} = \frac{1}{2}(\epsilon_f^{*\mu}\epsilon_i^\nu + \epsilon_f^{*\nu}\epsilon_i^\mu), \tag{80}$$

and the antisymmetric polarization tensor

$$S^{\mu\nu} = i(\epsilon_f^{*\mu}\epsilon_i^\nu - \epsilon_f^{*\nu}\epsilon_i^\mu). \tag{81}$$

For the convenience of calculating multipoles, the above definition of the dimensionless form factors is slightly different from other conventions. An example of ρ meson EMT form factors can be found in [45] using chiral perturbation theory and in [46] from Nambu–Jona–Lasinio model calculation.

To relate the above form factors to gravitational multipoles, we consider the static limit where initial and final states have small momenta, and $\epsilon^0 = 0$ and $\vec{\epsilon}_i = \vec{\epsilon}_f = \vec{\epsilon}$ are arbitrary. Looking at T^{00}, $A(q^2=0)=1$ gives rise to the mass-monopole contribution, whereas $\tilde{A}(q^2=0)$ contributes to the mass quadrupole $M2$,

$$M_{ij} = \int d^3\vec{r}\left(r_i r_j - \frac{1}{3}\delta_{ij}r^2\right)\rho_m(\vec{r}) = \frac{2\tilde{A}(0)}{M}\hat{E}_{ij} \quad (82)$$

where $\hat{E}^{ij}$ is the traceless part of E^{ij}. The mass quadrupole generates $1/r^3$ perturbation in $\bar{h}^{00}$ in the following form:

$$\bar{h}^{00} = \frac{2GM_{ij}}{r^3}(3\hat{r}^i\hat{r}^j - \delta^{ij}). \quad (83)$$

The momentum density T^{0i} in Fourier space is

$$T^{0i} = \frac{i}{2}S^{ij}q_j J(q^2) = \frac{i}{2}(\vec{S}\times q)^i J(q^2), \quad (84)$$

where the axial vector $\vec{S} = \mathrm{Re}(i\vec{\epsilon}^\star \times \vec{\epsilon})$, from which one identifies $J(\vec{r})$ as the angular-momentum dipole density. Angular-momentum conservation constraints $J(q^2=0)=1$.

From the expression for T^{ij}, one reads off the tensor-MC monopole $T0$ moment

$$\tau = \frac{C(0)}{M}, \quad (85)$$

which is zero for a free photon. The monopole moment of the ρ meson appears close to that of the pion. For other spin-1 systems including deuterons, see Ref. [18].

There is also a new tensor-MC quadrupole $T2$ moment

$$T^{(2)}_{ij} = -\frac{\tilde{C}(0)}{48M^2}\hat{E}_{ij} \quad (86)$$

where the multipole series $T^{(2)}_{ij}$ is defined in Eq. (38). The contribution $\tilde{C}(0)$ to the scalar momentum current is new for spin-1 systems.

Finally, the scalar-MC quadrupole moment $S2$ can be calculated as

$$\sigma_{ij} = \frac{D(q^2=0)}{M}\hat{E}_{ij}, \quad (87)$$

Thus, the tensor quadrupole is proportional to the $D(0)$ form factor defined above. After gauge transformation, it will generate a contribution to $\bar{h}^{00}$ as

$$\bar{h}^{\tau,00} = \frac{2G\sigma_{ij}}{r^3}(3\hat{r}^i\hat{r}^j - \delta^{ij}), \quad (88)$$

which can be combined with the one from the mass quadruple into the form

$$\bar{h}^{00} = \frac{2G(M_{ij}+\sigma_{ij})}{r^3}(3\hat{r}^i\hat{r}^j - \delta^{ij}), \quad (89)$$

in agreement with the general results in Refs. [15,16].

IV. SCALAR MOMENTUM-CURRENT DISTRIBUTION AND $T0$ MOMENT IN HYDROGEN ATOM

In this section, we study the EMT of the hydrogenlike atom. Contrary to the single charged electron, hydrogenlike atoms are charge neutral and are expected to have a finite scalar-MC monopole moment τ. We first show that, in the quantum mechanics, it is possible to construct a conserved EMT using quantum-mechanical wave functions. However, it still has a long-range Coulomb tail due to the interaction between the electron and the proton. We then show that the above conserved EMT can be justified in the field theoretic framework and identified as the leading-order electron kinetic contribution plus the leading-order Coulomb photon exchange contribution. By adding the single electron and single proton contributions, the Coulomb tail gets removed; and the resulting monopole moment τ is equal to the basic unit $\tau_0 = \hbar^2/4m_e$ and positive. We argue that the result is accurate to the leading order in α.

Here, we should emphasize that instead of the so-called D term, it is the tensor-monopole moment with the mass dimension of -1 that is additive for composite systems. For the hydrogen atom, this means that the proton's tensor-monopole moment due to strong interaction can simply be added to the quantum electrodynamics (QED) contribution. The strong interaction part of the proton's tensor-monopole moment, being proportional to the proton's mass inverse $\frac{1}{M}$, is three orders of magnitude smaller as compared with the QED contribution that is of order $\frac{1}{m_e}$. This is similar to the electron magnetic moment $\sim\frac{1}{m_e}$ vs proton magnetic moment $\sim\frac{1}{M_p}$. In particular, in the infinite heavy proton limit, the tensor-monopole moment of the hydrogen atom remains finite and is purely of QED origin, which will be calculated in this section. Because it is well known that the bound state in QED due to the Coulomb force is stable without strong interaction (e.g., the famous "stability of matter"), this example shows that the sign of the EMT has little to do with the mechanical stability.

A. Hydrogen atom: quantum mechanics

The electron wave function ϕ of an hydrogen atom satisfies the Schrödinger equation

$$\left(-\frac{1}{2m}\nabla^2 - eV_p(r)\right)\phi(\vec{r}) = E\phi(\vec{r}), \quad (90)$$

where (e is the proton charge and positive)

$$\nabla^2 V_p(r) = -e\delta^3(\vec{r}) \quad (91)$$

is the potential of the static charge of the proton, and $V_p = e/(4\pi r)$. For convenience, we chose $m_e = 1$. One further defines the static potential V_e for the electron as

$$\nabla^2 V_e(r) = e|\phi(\vec{r})|^2, \tag{92}$$

which can be solved for the ground state as

$$V_e(r) = \frac{e e^{-2\alpha r}(1 + \alpha r - e^{2\alpha r})}{4\pi r}, \tag{93}$$

where $\alpha = e^2/4\pi$.

By nonrelativistic reduction of the Dirac equation, one can construct the following EMT T^{ij}_{QM}, which consists of a kinetic term

$$T^{ij}_K = -\frac{1}{4m_e}(\phi^\dagger \partial^i \partial^j \phi - \partial^i \phi^\dagger \partial^j \phi + \mathrm{c.c.}) \tag{94}$$

plus a potential term made of interacting electric fields of the proton and electron:

$$T^{ij}_V = \delta^{ij} \nabla V_p \cdot \nabla V_e - \partial^i V_e \partial^j V_p - \partial^i V_p \partial^j V_e. \tag{95}$$

The trace of $T^{ij}_{\mathrm{QM}} = T^{ij}_V + T^{ij}_K$ can be calculated as

$$T^{ii}_{\mathrm{QM}} = |\phi|^2(2E + eV_p) + \nabla \cdot (V_p \nabla V_e) + \frac{1}{4m_e}\nabla^2 |\phi|^2. \tag{96}$$

It is easy to show that T^{ij}_{QM} is conserved for the ground state: $\partial_i T^{ij}_{\mathrm{QM}} = 0$.

Therefore, it can be written in a normalized state as

$$T^{ij}_{\mathrm{QM}}(\vec{r}) = (\delta^{ij}\nabla^2 - \nabla^i \nabla^j)\frac{C_{\mathrm{QM}}(r)}{m_e}, \tag{97}$$

$$T^{ii}_{\mathrm{QM}}(r) = 2\nabla^2 \frac{C_{\mathrm{QM}}(r)}{m_e}, \tag{98}$$

from which the C_{QM} can be calculated as

$$\frac{C_{\mathrm{QM}}(r)}{m_e} = \frac{1}{2\nabla^2} T^{ii}_{\mathrm{QM}} = \frac{e^{-2\alpha r}\alpha(2\alpha r + 1)}{16\pi r^2} - \frac{\alpha}{16\pi r^2}. \tag{99}$$

Notice that in the infinite heavy proton limit, it is natural to have the electron mass m_e in the denominator. Clearly, the Coulomb tail of $-\alpha/16\pi r^2$ prevents a finite $C(q = 0)$. Physically, the self-energies of the proton and electron will generate opposite contributions, which cancel the Coulomb tail from the above expression (the EMTs of the electron and proton are separately conserved). However, this piece of physics is outside the usual nonrelativistic quantum mechanics.

For the time being, we can subtract this Coulomb tail and define an effective $C_{\mathrm{eff}}(r)$ in the infrared region where $r \sim \frac{1}{\alpha}$,

$$\frac{C_{\mathrm{eff}}(r)}{m_e} = \frac{e^{-2\alpha r}\alpha(2\alpha r + 1)}{16\pi r^2}, \tag{100}$$

which is of order one when the momentum transfer is of the order of the inverse Bohr radius. In particular, the scalar-MC monopole moment of $\tau = \frac{C(0)}{m_e}$ for the hydrogen atom reads

$$\tau = \frac{C_{\mathrm{eff}}(q = 0)}{m_e} = \tau_0[1 + \mathcal{O}(\alpha \ln \alpha)], \tag{101}$$

where $\tau_0 = \frac{\hbar^2}{4m_e}$ is the basic unit defined before. Below, we show that in quantum field theory, the long-range Coulomb tail is indeed removed and Eq. (101) is the correct MC monopole moment.

B. Hydrogen atom: Field theory

The above calculation can in fact be justified in the field theoretical framework. Let us consider the bound state in quantum electrodynamics between two types of fermions: the standard negative charged electron with mass m_e and the positive charged "proton" with mass M. At an energy scale much smaller than the proton mass M, the proton can be approximated by an infinitely heavy static source N represented by an auxiliary field N. The Lagrangian density of the system reads

$$\mathcal{L} = i\bar{N} v \cdot DN + \mathcal{L}_{\mathrm{QED}}, \tag{102}$$

where N represents the infinitely heavy proton moving along the $v^\mu = (1, 0, 0, 0)$ direction. The Lagrangian preserves Lorentz invariance if v^μ is also treated as an auxiliary field. The EMT of the above system can be shown as

$$T^{\mu\nu} = \frac{i}{4}\bar{N} i v^{(\mu} D^{\nu)} N + T^{\mu\nu}_{\mathrm{QED}}. \tag{103}$$

More precisely, the heavy source only contributes to the T^{0i} part of the EMT. To proceed, we fix the Coulomb gauge as $\nabla \cdot \vec{A} = 0$ [47], in which the static potential A^0 can be solved as

$$A^0(t, \vec{x}) = -\frac{e}{\nabla^2}(\psi^\dagger \psi - \bar{N}N)(t, \vec{x}). \tag{104}$$

By using $\vec{E} = -\partial_t \vec{A}_T - \nabla A_0$ and the explicit solution of A_0, the transverse and longitudinal parts of the electric field decouple from each other; and the Hamiltonian reads

$$H = \frac{1}{2}\int d^3\vec{x}(\vec{E}_T^2 + \vec{B}_T^2) + \int d^3\vec{x}\psi^\dagger(-i\vec{\alpha}\cdot\vec{D}_T + m\gamma^0)\psi + \frac{e^2}{2}\int d^3\vec{x} d^3\vec{y}\frac{(\psi^\dagger\psi - \bar{N}N)(\vec{x})(\psi^\dagger\psi - \bar{N}N)(\vec{y})}{4\pi|\vec{x} - \vec{y}|}, \tag{105}$$

where the last term represents the Coulomb interaction.

We now consider the bound state formed by a pair of electrons and the heavy proton. The leading wave function reads

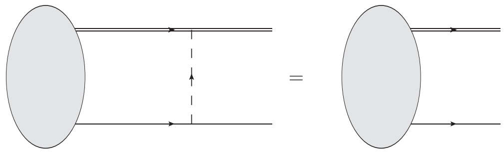

FIG. 1. Bethe–Salpeter equation for wave function ϕ denoted by oval blob. Double line represents propagator of proton field, and single line represents electron propagator. Dashed line represents exchange of a Coulomb photon.

$$|\vec{p}\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\phi(\vec{k})}{2E_k}|\vec{k}\rangle_e|-\vec{k}+\vec{p}\rangle_N, \tag{106}$$

where to the leading order in α, $\phi(\vec{k})$ satisfies the Bethe–Salpeter equation [48] induced by Coulomb-photon exchanges; see Fig. 1 for a depiction of the equation. It can be verified that it is nothing but the standard Schrödinger equation [Eq. (90)] in nonrelativistic limit $|\vec{k}| \ll m_e$, and it is characterized by two scales: the binding energy $\alpha^2 m_e$, and the inverse Bohr radius αm_e.

Given the above wave function, one can calculate the matrix element of $T^{ii}(\vec{r})$. At tree level, there is only one diagram: the electron kinetic contribution shown in Fig. 2. One can show that this contribution is exactly T^{ij}_{K} in Eq. (94).

We then consider radiative corrections starting from $\mathcal{O}(\alpha)$. Due to the fact that the velocities are of order α, only the Coulomb-photon contribution needs to be included. Furthermore, the one-loop contributions can be classified into interference and single electron/single proton diagrams; see Figs. 2 and 3 for a depiction. The interference diagram can be calculated as

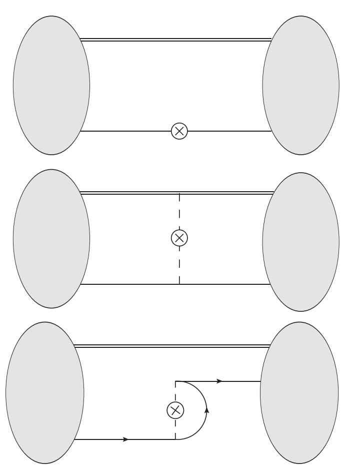

FIG. 2. Order-$\mathcal{O}(1)$ electron kinetic contribution (top), Coulomb photon interference (middle), and single electron (bottom) contributions to T^{ij}. Dashed lines represent Coulomb photons, and crossed circles denote operator insertions. Notice infrared divergences for $C(q)$ at $q=0$ are cancelled between interference and single electron and proton (not shown) contributions.

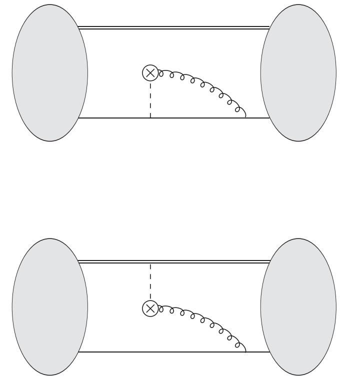

FIG. 3. Mixed contributions between radiative (wavy line) and Coulomb photons. IR divergences regulated by binding energy differences.

$$T^{ii}(\vec{q}) = -e^2\int\frac{d^3k d^3k'}{(2\pi)^6}\phi^\dagger(k')\phi(k-k')\frac{\vec{k}\cdot(\vec{k}-\vec{q})}{\vec{k}^2(\vec{k}-\vec{q})^2}, \tag{107}$$

which is equivalent to

$$T^{ii}_V = \nabla V_e \cdot \nabla V_p. \tag{108}$$

Therefore, we found that the leading-order electron kinetic and interference contributions in Fig. 2 will exactly lead to the conserved T^{ii}_{QM} in Eq. (96) with a Coulomb tail $\frac{\pi\alpha}{8|q|}$.

Clearly, in order to cancel the Coulomb tail, one must add the electron and proton self-energy contributions. The electron contribution is also shown in Fig. 2 and can be calculated in the region $q \sim \mathcal{O}(\alpha m_e)$ as

$$\frac{C_{\mathrm{e}}(q)}{m_e} = \frac{\alpha\pi}{16|q|}\times\frac{16\alpha^4}{(\frac{q^2}{m_e^2}+4\alpha^2)^2}, \tag{109}$$

where the first factor $\frac{\alpha\pi}{16|q|}$ is just the standard Coulomb tail, and the second factor comes from the dressing in the bound-state wave function. Similarly, the proton contribution is of the form in the $M \gg m_e$ limit,

$$\frac{C_{\mathrm{p}}(p)}{m_e} = \frac{\alpha\pi}{16|q|}\times\frac{16\alpha^4}{(\frac{q^2}{M^2}+4\alpha^2)^2}, \tag{110}$$

and can be approximated simply by the Coulomb tail when $q \sim \alpha m_e$. Clearly, the above formulas can be justified only when the momentum transfer is small and are invalid in the ultraviolet region when q is comparable with the particle masses.

In conclusion, in the region $|q| \le \mathcal{O}(\alpha m_e)$, the C form factor of the hydrogen atom reads

$$\frac{C_{\mathrm{H}}(q)}{m_e} = \frac{1}{2m_e(\frac{q^2}{\alpha^2 m_e^2}+4)} - \frac{\alpha}{4|q|}\left(\frac{\pi}{2} - \mathrm{Arctan}\frac{q}{2\alpha m_e}\right) + \frac{\alpha\pi}{|q|}\frac{1}{(\frac{q^2}{\alpha^2 m_e^2}+4)^2} + \frac{\alpha\pi}{|q|}\frac{1}{(\frac{q^2}{\alpha^2 M^2}+4)^2}. \tag{111}$$

From these, the monopole moment for the hydrogen atom is

$$\tau_{\mathrm{H}} = \frac{C_{\mathrm{H}}(0)}{m_e} = \tau_0[1+\mathcal{O}(\alpha\ln\alpha)]. \tag{112}$$

The $g_H = 1$ except for a small correction of order α, which is a result with the opposite sign from a pointlike boson.

Given the form factor $C(q^2)$, we can obtain the scalar momentum current or pressure [18,19],

$$p(r) = \frac{1}{3}T^{ii} = \frac{2}{3m_e}\frac{1}{r^2}\frac{d}{dr}r^2\frac{dC(r)}{dr}, \tag{113}$$

which can be shown to be positive for small r and negative for large r. On the other hand, the momentum-current monopole density distribution

$$\tau(r) = -\frac{2\pi}{5}r^2\left(r^i r^j - \frac{1}{3}r^2\delta^{ij}\right)T_{ij}(r) = -\frac{4\pi}{15}r^4 s(r), \tag{114}$$

where

$$s(r) = -\frac{r}{m_e}\frac{d}{dr}\left(\frac{1}{r}\frac{dC(r)}{dr}\right),$$

has been called shear pressure. Unfortunately, there is no simple expression for it in positional space without using the Meijer G function. A numerical result for $\tau(r)$ is shown in Fig. 4, which is positive definite at all r and finite at $r = 0$. A plot of $C(r)$ is shown in Fig. 5.

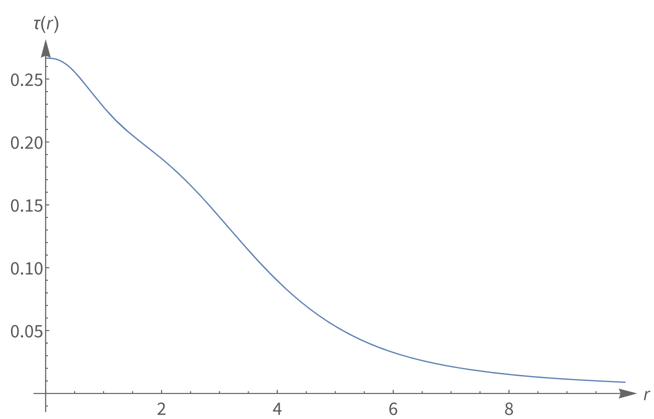

FIG. 4. Tensor momentum-current monopole density $\tau(r)$ in Eq. (114) as function of r in hydrogen atom, where r is in a unit of the Bohr radius $a_0 = \frac{\hbar}{\alpha m_e c}$ and $\tau(r)$ is in a unit of $\frac{\hbar^2}{4m_e}\frac{1}{a_0}$ with the SI dimension of $\frac{\mathrm{kg\cdot m^3}}{\mathrm{s^2}}$. It is positive and finite as $r \to 0$. Proton mass M approximated by ∞.

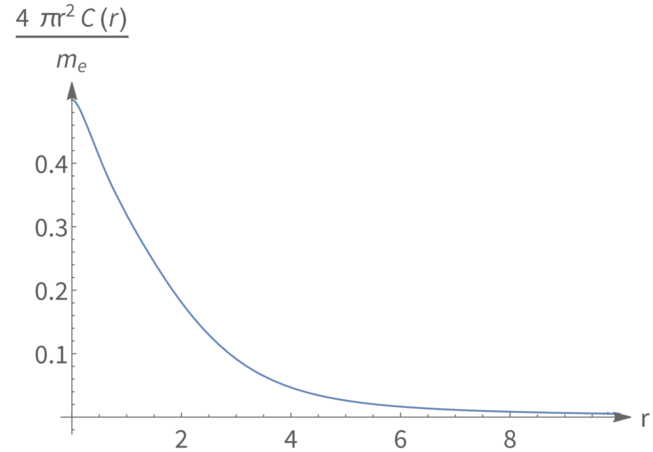

FIG. 5. The $4\pi r^2 \frac{C(r)}{m_e}$ as a function of r in the hydrogen atom, where r is in a unit of the Bohr radius $a_0 = \frac{\hbar}{\alpha m_e c}$ and $4\pi r^2 \frac{C(r)}{m_e}$ is in a unit of $\frac{\hbar^2}{4m_e}\frac{1}{a_0}$. It has the same SI dimension of $\frac{\mathrm{kg\cdot m^3}}{\mathrm{s^2}}$. as that of $\tau(r)$. It is positive and finite as $r \to 0$. According to Eq. (71), it contributes to a part of gravitational potential inside the hydrogen atom.

We also show a plot of the momentum flow T^{ix} in 3-dimensional space in Fig. 6. Any surface integral of the vector field yields the flux of the x-component momentum through the surface or force along the x direction, which would produce a pressure in this direction if the momentum current gets totally absorbed. Any closed surface integral yields zero, indicating momentum conservation or net null force through any size volume. Before ending the section, we add one brief comment on the $\mathcal{O}(\alpha)$ corrections to $C(0)$. By including radiative photons, the degree of infrared divergences is reduced due to the fact that the radiative photon couples to the three-velocity $\vec{v}$ of the electron in the Coulomb gauge. Therefore, the mixing diagram where Coulomb and radiative photons couple to each other can be logarithmically divergent, which is confirmed by the single electron calculation [11,32,33,35,36]. However, the

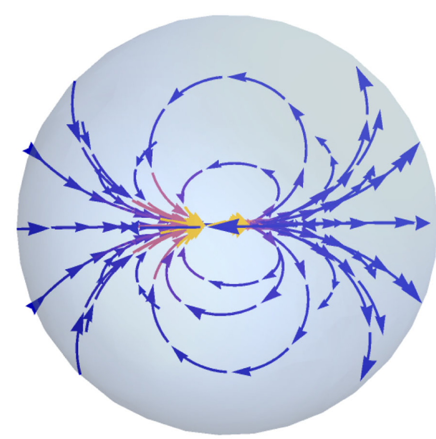

FIG. 6. Conserved x-component-momentum-current distribution $T^{ix}(\vec{r})$ as a vector field in i, with arrows indicating directions of current flow; x direction points to right in horizontal direction.

divergence is expected to be regulated by the binding energy differences in a way similar to the famous Lamb shift, leading to finite $C(0)$ at order $\mathcal{O}(\alpha)$. See Fig. 3 for a depiction. We will leave the calculation of order α corrections in a separate publication [49].

V. COMMENTS ON MECHANICAL STABILITY AND CONCLUSIONS

The QCD EMT of a hadron has often been related to the stress tensor of a continuous medium [50]. If this analogy is of value, one can introduce the concept of pressure, shear pressure, mechanical stability, etc. [18,19,51,52], and try to understand a bound state in quantum theory using the terms of classical mechanics. However, this analogy is of limited use and can even be misleading at times.

The scalar momentum-current $T^{ii}(r)$ has been identified as the pressure $p(r)$, which is true in some models of continuous media. However, the two concepts have different physical significance and are not interchangeable in general. The concept of pressure normally stands for an isotropic force from random microscopic motions in all possible directions and is a positive quantity for stable systems. However, $T^{ii}(r)$ is not positive definite and carries with it the sense of a directional flow, analogous to "acroscopic motion" in a fluid.

The proper analog of the momentum currents T^{ij} may be the radiation pressure [23,25]: Assuming a directional momentum current gets absorbed on a surface, the surface experiences a force or a pressure that can be measured by the momentum passing through the surface per unit time. In this case, the pressure is not a scalar as in thermal systems; rather, it depends on the directions of the momentum flow as well as the surface area. Thus, a negative pressure only means the momentum flow is negative with respect to a reference direction. The Laue condition

$$\int drr^2 p(r) = 0 \tag{115}$$

is trivially related to the conservation of the momentum current [18,53]. Using pressure or "force" to characterize the momentum current may generate confusion because they are not acting on any part of the system itself but on some fictitious surfaces that would absorb the current entirely through some interactions. In our view, the most interesting way to characterize T^{ii} is by its multipoles generating characteristic space-time perturbation. This is similar to using the magnetic moment, etc., to describe a current distribution, which generates a particular type of magnetic field.

Further mechanical stability conditions on T^{ij} derive from comparing it the pressure and shear pressure distributions with a mechanical system. For example, it has been speculated that a negative D term $[C(q=0)]$ is needed for mechanical stability [51]. However, the sign of the scalar-MC monopole moment depends on the flow pattern of the momentum currents. Reversing the direction of the momentum currents at every point in space in classical physics will reverse the D term but should lead to another stable flow pattern. Thus, the sign of the D term cannot be related to mechanical stability. In fact, in the example of the hydrogen atom, the force from the momentum flow is directed toward the center [using the $C_{\rm eff}(r)$ defined in Eq. (100)]; and the D term is positive. On the other hand, we know perfectly well that the hydrogen atom is stable due to quantum physics, which already has a well-defined sense of stability. Thus, using the momentum-current flow to judge the stability of a quantum system appears to not be useful, and is furthermore unnecessary.

To summarize, we have revisited the gravitational fields generated by a static source by performing the multipole expansion [15,16] of the corresponding energy, energy current, and momentum-current densities. For a static and conserved EMT, there are six series of nonvanishing multipoles: one series for T^{00}, two series for T^{0i}, and three series for T^{ij}. They are important characterizations of spatial distribution of the corresponding energy and momentum-current distributions; although, at large distances, only three series will contribute to the physical metric perturbation in the static case.

In particular, the $C(q^2)$ form factor or the D term is related to the tensor-MC monopole moment $T0$, which has a basic unit of $\tau_0 = \hbar^2/4M$. As a concrete example, we have calculated the $C(q^2)$ form factor for the hydrogen atom at the small q region and found that the MC monopole moment is positive, which is opposite to that of a pointlike boson. Moreover, we argue that the notion of mechanical stability or pressure is of limited significance when applied to bound states in quantum field theories.

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