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An a priori error analysis of a type III thermoelastic problem with two porosities

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Abstract

In this work, we study, from the numerical point of view, a type III thermoelastic model with double porosity. The thermomechanical problem is written as a linear system composed of hyperbolic partial differential equations for the displacements and the two porosities, and a parabolic partial differential equation for the thermal displacement. An existence and uniqueness result is recalled. Then, we perform its a priori error numerical analysis approximating the resulting variational problem by using the finite element method and the implicit Euler scheme. The linear convergence of the algorithm is derived under suitable additional regularity conditions. Finally, some numerical simulations are shown to demonstrate the accuracy of the approximations and the dependence of the solution on a coupling coefficient.

KEYWORDS

a priori error estimates, double porosity, finite elements, numerical simulations, type III thermoelasticity

1 | INTRODUCTION

Porous materials are very natural in the daily life. Think, for example, in something as simple and soft as a fairy cake or something more rigid as a piece of limestone travertine. Elastic materials with voids are one of the easiest extensions of the classical theory of elasticity. This theory was proposed forty years ago by Nunziato and Cowin [36]. It describes the behavior of elastic solids with pores. They can be seen as materials with a skeleton or matrix material that is elastic, and the interstices are voids of material. The quantity of the publications analyzing this theory is huge. To cite several examples we can recall [5, 6, 8–11, 27, 29, 30, 32, 37, 38]. Porous elastic solids have been applied to study and to understand geological materials such as rocks or soils, as well as manufactured materials such as ceramics, pressed powders, or biological materials as bones. These materials can also have cracks and/or fissures in their skeleton. Therefore, we can see two kinds of porous structures. It is suitable to consider a more accurate mechanical model to be analyzed from a mathematical point of view. In this case, it is usual to speak about macro-porosity and micro-porosity.

Thermoelastic situations with double porosity received an increasing attention in the last times. They usually were described by the Darcy law and, in such case, it was usual to work with the displacements, pressures associated with the pores and the fissures. In a recent contribution, Ieşan and Quintanilla [19] considered the model of structures with double porosity following the theory proposed in [36]. It is remarkable that, in this case, we have a different set of independent variables, but now the elastic structure and both porosities and strongly coupled. Furthermore, in the work of Ieşan and Quintanilla these structures were also coupled with the usual heat equation based on the Fourier law. After this work, a big interest has been developed to study and to understand this propose. Even more, this approach has been also considered in alternative contexts as viscoelasticity [2, 3, 18, 20–22, 33]. It is worth saying that in several recent contributions the authors provide explicit values for the constitutive parameters (see [1, 23–26, 39]).

However, we should take into account that the Fourier law adjoined to the classical heat equation brings to the parabolic version of the heat conduction, and therefore, it *violates the causality principle*. For this reason, many scientists have been interested proposing alternative laws for the heat conduction. We here recall the theories proposed by Green and Naghdi [13–17], which were established in a rational way. In fact, they proposed three theories and the most general, which is called *type III thermoelasticity*, contains the other two as limit cases. This alternative heat conduction theory is under deep investigation and there is a big quantity of contributions concerning it. In fact, we can recall [12, 28, 31, 34, 35].

In this article, we focus our attention to the type III theory of thermoelasticity with two porosities. As far as we know, the only contribution concerning this theory is proposed in [33]. We believe that it is strictly needed to develop a *body* of studies for this theory. Therefore, some mathematical and physical studies for this theory should be proposed in order to clarify its applicability. The present work is addressed in this direction.

The article is organized as follows. In Section 2, the thermomechanical model is described, its classical and variational formulations presented and a statement of the existence and uniqueness result provided as well as its exponential energy decay. Then, in Section 3 a numerical scheme for the problem is given. Spatially it is approximated by finite elements and the time derivatives are discretized by using the implicit Euler scheme. A discrete stability property and a priori error estimates are proved for the approximate solutions. Finally, the implementation of the numerical scheme is described and some numerical simulations shown in Section 4.

2 | THE THERMOMECANICAL PROBLEM AND ITS VARIATIONAL FORMULATION

Following [33], we describe in this section the type III thermoelastic problem with two porosities, we derive its variational formulation and we recall an existence and uniqueness result (see this recently submitted work for details).

Let us denote by $[0, \ell]$ the one-dimensional rod of length $\ell > 0$ and by $[0, T]$, $T > 0$, the time interval of interest. Moreover, as usual $x \in [0, \ell]$ and $t \in [0, T]$ are the spatial and time variables, and, for every function f, let f_x be the derivative with respect to the spatial variable x. As it has been used in other works, the time derivative is represented as a point (first-order) or two points (second-order) over a variable.

Let u, ϕ_1, ϕ_2, and ψ denote the displacement field, the first porosity (also called the first volume fraction), the second porosity (also called the second volume fraction), and the thermal displacement, respectively.

Therefore, after substitution of the corresponding constitutive equations into the respective evolution equations we obtain the following thermomechanical problem (see [33] for details).

Problem P Find the displacement field $u : [0, \ell] \times [0, T] \to \mathbb{R}$, the first porosity $\phi_1 : [0, \ell] \times [0, T] \to \mathbb{R}$, the second porosity $\phi_2 : [0, \ell] \times [0, T] \to \mathbb{R}$, and the thermal displacement $\psi : [0, \ell] \times [0, T] \to \mathbb{R}$ such that

$$\rho \ddot{u} = \mu u_{xx} + \gamma_1 \phi_{1,x} + \gamma_2 \phi_{2,x} - \beta \dot{\psi}_x \quad \text{in } (0, \ell) \times (0, \mathrm{T}), \tag{2.1}$$

$$J_1 \ddot{\phi}_1 = b_{11} \phi_{1,xx} + b_{12} \phi_{2,xx} + m_1 \psi_{xx} - \xi_{11} \phi_1 - \xi_{12} \phi_2 + d_1 \dot{\psi} - \gamma_1 u_x - \xi^* \dot{\phi}_1 \quad \text{in } (0, \ell) \times (0, \mathrm{T}), \tag{2.2}$$

$$J_2 \ddot{\phi}_2 = b_{12} \phi_{1,xx} + b_{22} \phi_{2,xx} + m_2 \psi_{xx} - \xi_{12} \phi_1 - \xi_{22} \phi_2 + d_2 \dot{\psi} - \gamma_2 u_x \quad \text{in } (0, \ell) \times (0, \mathrm{T}), \tag{2.3}$$

$$a \ddot{\psi} = m_1 \phi_{1,xx} + m_2 \phi_{2,xx} + k \psi_{xx} - d_1 \dot{\phi}_1 - d_2 \dot{\phi}_2 - \beta \dot{u}_x + k^* \dot{\psi}_{xx} \quad \text{in } (0, \ell) \times (0, \mathrm{T}), \tag{2.4}$$

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad \phi_1(x, 0) = \phi_{10}(x) \quad \text{for a.e. } x \in (0, \ell), X_1, \dots, X_n \tag{2.5}$$

$$\dot{\phi}_1(x, 0) = e_{10}(x), \quad \phi_2(x, 0) = \phi_{20}(x), \quad \dot{\phi}_2(x, 0) = e_{20}(x) \quad \text{for a.e. } x \in (0, \ell), \tag{2.6}$$

$$\psi(x, 0) = \psi_0(x), \quad \dot{\psi}(x, 0) = \theta_0(x) \quad \text{for a.e. } x \in (0, \ell), \tag{2.7}$$

$$u(0, t) = \phi_1(0, t) = \phi_2(0, t) = \psi(0, t) = 0 \quad \text{for a.e. } t \in (0, \mathrm{T}), \tag{2.8}$$

$$u(\ell, t) = \phi_1(\ell, t) = \phi_2(\ell, t) = \psi(\ell, t) = 0 \quad \text{for a.e. } t \in (0, \mathrm{T}). \tag{2.9}$$

In the previous mechanical problem we have denoted by ρ the mass density, J_1 and J_2 the product of the mass density by the equilibrated inertias, a the heat capacity, μ the elastic modulus, k the thermal diffusion and $\gamma_1, \gamma_2, \beta, b_{11}, b_{12}, b_{22}, m_1, m_2, \xi_{11}, \xi_{12}, \xi_{22}, d_1, d_2, \xi^*, k^*$ some constitutive parameters. Later, we will impose some assumptions on these coefficients in order to ensure the well-posedness of the problem.

Remark In [33] the authors also considered other thermal problems as the type II thermoelasticity (by assuming that $k^* = 0$ with two viscoporosities) or adding viscous effects in the displacements (by introducing a term in the form $\mu^* \dot{u}_{xx}$ in Equation (2.1)); however, we note that the numerical analysis presented in the next section could be extended to these situations in a straightforward way. Moreover, although we restrict ourselves to the one-dimensional case for the sake of simplicity, the analysis performed in Section 3 could be developed in the multidimensional setting with some minor modifications. Anyway, we point out that the existence and uniqueness, as well as the energy decay are open problems in that case.

Now, we obtain the weak form of this problem. So, let us denote $Y = L^2(0, \ell)$ with scalar product $(\cdot, \cdot)$ and corresponding norm $\| \cdot \|$. Moreover, let $V = H_0^1(0, \ell)$ and denote by $\| \cdot \|_V$ its norm.

Using integration by parts and boundary conditions (2.8)–(2.9) we obtain the following variational formulation of Problem P written in terms of the velocity $v = \dot{u}$, the first porosity speed $e_1 = \dot{\phi}_1$, the second porosity speed $e_2 = \dot{\phi}_2$, and the temperature $\theta = \dot{\psi}$.

Problem VP Find the velocity $v : [0, T] \to V$, the first porosity speed $e_1 : [0, T] \to V$, the second porosity speed $e_2 : [0, T] \to V$, and the temperature $\theta : [0, T] \to V$ such that $v(0) = v_0$, $e_1(0) = e_{10}$, $e_2(0) = e_{20}$, $\theta(0) = \theta_0$ and, for a.e. $t \in (0, T)$ and for all $w, r, z, m \in V$,

$$\rho(\dot{v}(t), w) + \mu (u_x(t), w_x) + \gamma_1 (\phi_1(t), w_x) + \gamma_2 (\phi_2(t), w_x) = -\beta (\theta_x(t), w), \tag{2.10}$$

$$\begin{aligned} J_1 (\dot{e}_1(t), r) &+ b_{11} \left(\phi_{1,x}(t), r_x\right) + b_{12} \left(\phi_{2,x}(t), r_x\right) + m_1 (\psi_x(t), r_x) + \xi_{11} (\phi_1(t), r) \\ &+ \xi_{12} (\phi_2(t), r) + \xi^* (e_1(t), r) + \gamma_1 (u_x(t), r) = d_1(\theta(t), r), \end{aligned} \tag{2.11}$$

$$\begin{aligned} J_2 (\dot{e}_2(t), z) &+ b_{12} \left(\phi_{1,x}(t), z_x\right) + b_{22} \left(\phi_{2,x}(t), z_x\right) + m_2 (\psi_x(t), z_x) + \xi_{12} (\phi_1(t), z) \\ &+ \xi_{22} (\phi_2(t), z) + \gamma_2 (u_x(t), z) = d_2(\theta(t), z), \end{aligned} \tag{2.12}$$

$$\begin{aligned} a(\dot{\theta}(t), m) &+ k (\psi_x(t), m_x) + k^* (\theta_x(t), m_x) + m_1 \left(\phi_{1,x}(t), m_x\right) + m_2 \left(\phi_{2,x}(t), m_x\right) \\ &= -d_1 (e_1(t), m) - d_2 (e_2(t), m) - \beta (v_x(t), m), \end{aligned} \tag{2.13}$$

where the displacements, the first porosity, the second porosity, and the thermal displacements are then recovered from the relations:

$$\begin{aligned} u(t) &= \int_0^t v(s) ds + u_0, \quad \phi_1(t) = \int_0^t e_1(s) ds + \phi_{10}, \\ \phi_2(t) &= \int_0^t e_2(s) ds + \phi_{20}, \quad \psi(t) = \int_0^t \theta(s) ds + \psi_0. \end{aligned} \tag{2.14}$$

The following theorem, which states the existence of a unique solution to Problem VP and an energy decay property, can be proved following the arguments recently used in [33]. We note that the boundary conditions used there are slightly different; however, the proof is similar and therefore, we omit the details.

Theorem 2.1 *Assume that the constitutive coefficients satisfy*

$$\rho > 0, \quad J_1 > 0, \quad J_2 > 0, \quad a > 0, \quad \xi^* > 0, \quad k^* > 0, \tag{2.15}$$

and that the matrices M_1 and M_2 given by

$$M_1 = \begin{pmatrix} b_{11} & b_{12} & m_1 \\ b_{12} & b_{22} & m_2 \\ m_1 & m_2 & k \end{pmatrix}, \quad M_2 = \begin{pmatrix} \mu & \gamma_1 & \gamma_2 \\ \gamma_1 & \xi_{11} & \xi_{12} \\ \gamma_2 & \xi_{12} & \xi_{22} \end{pmatrix},$$

are positive definite. Then, Problem VP has a unique solution with the following regularity:

$$u, \phi_1, \phi_2, \psi \in C^2([0,T];Y) \cap C^1([0,T];V).$$

Moreover, if we also assume that the rank of the matrix:

$$\begin{pmatrix} b_{11} & b_{12} \\ m_1 & m_2 \end{pmatrix}$$

is two and that β, m_2, and γ_2 are different from zero, then this solution is exponentially stable.

3 | FULLY DISCRETE APPROXIMATIONS: AN A PRIORI ERROR ANALYSIS

In this section, we introduce a fully discrete finite element scheme for the numerical approximations of Problem VP. First, in order to obtain the spatial approximation, we assume that the interval $[0, \ell]$ is divided into M subintervals $a_0 = 0 < a_1 < \dots < a_M = \ell$ of length $h = a_{i+1} - a_i = \ell/M$ and so, to approximate the variational space V, we define the finite dimensional space $V^h \subset V$ given by

$$V^h = \Big\{ w^h \in C([0,\ell]) \; ; \; w^h_{|[a_i,a_{i+1}]} \in P_1\left([a_i,a_{i+1}]\right) \; i = 0, \dots, M-1, \\ w^h(0) = w^h(\ell) = 0 \Big\}, \tag{3.1}$$

where $P_1\left([a_i,a_{i+1}]\right)$ represents the space of polynomials of degree less or equal to one in the subinterval $[a_i, a_{i+1}]$; that is, the finite element space V^h is composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter.

Moreover, let us define the discrete initial conditions $u_0^h, v_0^h, \phi_{10}^h, e_{10}^h, \phi_{20}^h, e_{20}^h, \psi_0^h$, and θ_0^h as follows:

$$u_0^h = \mathcal{P}^h u_0, \quad v_0^h = \mathcal{P}^h v_0, \quad \phi_{10}^h = \mathcal{P}^h \phi_{10}, \quad e_{10}^h = \mathcal{P}^h e_{10}, \\ \phi_{20}^h = \mathcal{P}^h \phi_{20}, \quad e_{20}^h = \mathcal{P}^h e_{20}, \quad \psi_0^h = \mathcal{P}^h \psi_0, \quad \theta_0^h = \mathcal{P}^h \theta_0, \tag{3.2}$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over V^h (see [7]).

Second, in order to obtain the time discretization, we consider a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \dots < t_N = T$, with step size $\tau = T/N$ and nodes $t_n = n\tau$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$ let $z_n = z(t_n)$ and, given $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/\tau$ its divided differences.

Therefore, using the implicit Euler scheme, the fully discrete approximations of Problem VP are the following.

Problem VP$^{h\tau}$ Find the discrete velocity $v^{h\tau} = \left\{ v_n^{h\tau} \right\}_{n=0}^N \subset V^h$, the discrete first porosity speed $e_1^{h\tau} = \left\{ e_{1n}^{h\tau} \right\}_{n=0}^N \subset V^h$, the discrete second porosity speed $e_2^{h\tau} = \left\{ e_{2n}^{h\tau} \right\}_{n=0}^N \subset V^h$ and the discrete temperature $\theta^{h\tau} = \left\{ \theta_n^{h\tau} \right\}_{n=0}^N \subset V^h$ such that $v_0^{h\tau} = v_0^h, e_{10}^{h\tau} = e_{10}^h, e_{20}^{h\tau} = e_{20}^h$, $\theta_0^{h\tau} = \theta_0^h$ and, for all $n = 1, \dots, N$ and for all $w^h, r^h, z^h, m^h \in V^h$,

$$\rho\left(\delta v_n^{h\tau}, w^h\right) + \mu\left(u_{n,x}^{h\tau}, w_x^h\right) + \gamma_1\left(\phi_{1n}^{h\tau}, w_x^h\right) + \gamma_2\left(\phi_{2n}^{h\tau}, w_x^h\right) = -\beta\left(\theta_{n,x}^{h\tau}, w^h\right), \tag{3.3}$$

$$J_1\left(\delta e_{1n}^{h\tau}, r^h\right) + b_{11}\left(\phi_{1n,x}^{h\tau}, r_x^h\right) + b_{12}\left(\phi_{2n,x}^{h\tau}, r_x^h\right) + m_1\left(\psi_{n,x}^{h\tau}, r_x^h\right) + \xi_{11}\left(\phi_{1n}^{h\tau}, r^h\right) \\ + \xi_{12}\left(\phi_{2n}^{h\tau}, r^h\right) + \xi^*\left(e_{1n}^{h\tau}, r^h\right) + \gamma_1\left(u_{n,x}^{h\tau}, r^h\right) = d_1\left(\theta_n^{h\tau}, r^h\right), \tag{3.4}$$

$$J_2\left(\delta e_{2n}^{h\tau}, z^h\right) + b_{12}\left(\phi_{1n,x}^{h\tau}, z_x^h\right) + b_{22}\left(\phi_{2n,x}^{h\tau}, z_x^h\right) + m_2\left(\psi_{n,x}^{h\tau}, z_x^h\right) + \xi_{12}\left(\phi_{1n}^{h\tau}, z^h\right)$$
$$+\ \xi_{22}\left(\phi_{2n}^{h\tau}, z^h\right) + \gamma_2\left(u_{n,x}^{h\tau}, z^h\right) = d_2\left(\theta_n^{h\tau}, z^h\right), \tag{3.5}$$

$$a\left(\delta\theta_n^{h\tau}, m^h\right) + k\left(\psi_{n,x}^{h\tau}, m_x^h\right) + k^*\left(\theta_{n,x}^{h\tau}, m_x^h\right) + m_1\left(\phi_{1n,x}^{h\tau}, m_x^h\right) + m_2\left(\phi_{2n,x}^{h\tau}, m_x^h\right)$$
$$= -d_1\left(e_{1n}^{h\tau}, m^h\right) - d_2\left(e_{2n}^{h\tau}, m^h\right) - \beta\left(v_{n,x}^{h\tau}, m^h\right), \tag{3.6}$$

where the discrete displacements, the discrete first porosity, the discrete second porosity and the discrete thermal displacements are then recovered from the relations:

$$u_n^{h\tau} = \tau\sum_{j=1}^{n} v_j^{h\tau} + u_0^h, \quad \phi_{1n}^{h\tau} = \tau\sum_{j=1}^{n} e_{1j}^{h\tau} + \phi_{10}^h,$$
$$\phi_{2n}^{h\tau} = \tau\sum_{j=1}^{n} e_{2j}^{h\tau} + \phi_{20}^h, \quad \psi_n^{h\tau} = \tau\sum_{j=1}^{n} \theta_j^{h\tau} + \psi_0^h. \tag{3.7}$$

Since Problem VP$^{h\tau}$ is linear, using the assumptions of Theorem 2.1 it is straightforward to show that it has a unique discrete solution.

Now, we prove a discrete stability property.

Theorem 3.1 *Under the conditions of Theorem 2.1 we have the following stability estimates*:

$$\|v_n^{h\tau}\|^2 + \|u_{n,x}^{h\tau}\|^2 + \|e_{1n}^{h\tau}\|^2 + \|\phi_{1n,x}^{h\tau}\|^2 + \|\phi_{1n}^{h\tau}\|^2 + \|e_{2n}^{h\tau}\|^2 + \|\phi_{2n,x}^{h\tau}\|^2 + \|\phi_{2n}^{h\tau}\|^2$$
$$+\ \|\theta_n^{h\tau}\|^2 + \|\psi_{n,x}^{h\tau}\|^2 + \tau\sum_{j=1}^{n}\left[\|e_{1j}^{h\tau}\|^2 + \|\theta_{j,x}^{h\tau}\|^2\right] \le C,$$

where $C > 0$ is a positive constant assumed to be independent of the discretization parameters h and τ.

Proof. Taking as a test functions $w^h = v_n^{h\tau}$, $r^h = e_{1n}^{h\tau}$, $z^h = e_{2n}^{h\tau}$ and $m^h = \theta_n^{h\tau}$ in discrete variational Equations (3.3)–(3.6), respectively, it follows that

$$\rho\left(\delta v_n^{h\tau}, v_n^{h\tau}\right) + \mu\left(u_{n,x}^{h\tau}, v_{n,x}^{h\tau}\right) + \gamma_1\left(\phi_{1n}^{h\tau}, v_{n,x}^{h\tau}\right) + \gamma_2\left(\phi_{2n}^{h\tau}, v_{n,x}^{h\tau}\right) = -\beta\left(\theta_{n,x}^{h\tau}, v_n^{h\tau}\right),$$
$$J_1\left(\delta e_{1n}^{h\tau}, e_{1n}^{h\tau}\right) + b_{11}\left(\phi_{1n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + b_{12}\left(\phi_{2n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + m_1\left(\psi_{n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + \xi_{11}\left(\phi_{1n}^{h\tau}, e_{1n}^{h\tau}\right)$$
$$+\ \xi_{12}\left(\phi_{2n}^{h\tau}, e_{1n}^{h\tau}\right) + \xi^*\left(e_{1n}^{h\tau}, e_{1n}^{h\tau}\right) + \gamma_1\left(u_{n,x}^{h\tau}, e_{1n}^{h\tau}\right) = d_1\left(\theta_n^{h\tau}, e_{1n}^{h\tau}\right),$$
$$J_2\left(\delta e_{2n}^{h\tau}, e_{2n}^{h\tau}\right) + b_{12}\left(\phi_{1n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + b_{22}\left(\phi_{2n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + m_2\left(\psi_{n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + \xi_{12}\left(\phi_{1n}^{h\tau}, e_{2n}^{h\tau}\right)$$
$$+\ \xi_{22}\left(\phi_{2n}^{h\tau}, e_{2n}^{h\tau}\right) + \gamma_2\left(u_{n,x}^{h\tau}, e_{2n}^{h\tau}\right) = d_2\left(\theta_n^{h\tau}, e_{2n}^{h\tau}\right),$$
$$a\left(\delta\theta_n^{h\tau}, \theta_n^{h\tau}\right) + k\left(\psi_{n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + k^*\left(\theta_{n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + m_1\left(\phi_{1n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + m_2\left(\phi_{2n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right)$$
$$= -d_1\left(e_{1n}^{h\tau}, \theta_n^{h\tau}\right) - d_2\left(e_{2n}^{h\tau}, \theta_n^{h\tau}\right) + \beta\left(v_n^{h\tau}, \theta_{n,x}^{h\tau}\right).$$

Adding these four discrete variational equations we have

$$\rho\left(\delta v_n^{h\tau}, v_n^{h\tau}\right) + \mu\left(u_{n,x}^{h\tau}, v_{n,x}^{h\tau}\right) + \gamma_1\left(\phi_{1n}^{h\tau}, v_{n,x}^{h\tau}\right) + \gamma_2\left(\phi_{2n}^{h\tau}, v_{n,x}^{h\tau}\right) + J_1\left(\delta e_{1n}^{h\tau}, e_{1n}^{h\tau}\right)$$
$$+\ b_{11}\left(\phi_{1n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + b_{12}\left(\phi_{2n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + m_1\left(\psi_{n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + \xi_{11}\left(\phi_{1n}^{h\tau}, e_{1n}^{h\tau}\right)$$
$$+\ \xi_{12}\left(\phi_{2n}^{h\tau}, e_{1n}^{h\tau}\right) + \xi^*\left(e_{1n}^{h\tau}, e_{1n}^{h\tau}\right) + \gamma_1\left(u_{n,x}^{h\tau}, e_{1n}^{h\tau}\right) + J_2\left(\delta e_{2n}^{h\tau}, e_{2n}^{h\tau}\right)$$
$$+\ b_{12}\left(\phi_{1n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + b_{22}\left(\phi_{2n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + m_2\left(\psi_{n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + \xi_{12}\left(\phi_{1n}^{h\tau}, e_{2n}^{h\tau}\right)$$

$$+\ \xi_{22}\left(\phi_{2n}^{h\tau}, e_{2n}^{h\tau}\right) + \gamma_2\left(u_{n,x}^{h\tau}, e_{2n}^{h\tau}\right) + a\left(\delta\theta_n^{h\tau}, \theta_n^{h\tau}\right) + k\left(\psi_{n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right)$$
$$+\ k^*\left(\theta_{n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + m_1\left(\phi_{1n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + m_2\left(\phi_{2n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) = 0.$$

Now, taking into account that matrices M_1 and M_2 are positive definite and that $v_n^{h\tau} = \delta u_n^{h\tau} = \left(u_n^{h\tau} - u_{n-1}^{h\tau}\right)/\tau$, $e_{1n}^{h\tau} = \delta\phi_{1n}^{h\tau} = \left(\phi_{1n}^{h\tau} - \phi_{1n-1}^{h\tau}\right)/\tau$, $e_{2n}^{h\tau} = \delta\phi_{2n}^{h\tau} = \left(\phi_{2n}^{h\tau} - \phi_{2n-1}^{h\tau}\right)/\tau$ and $\theta_n^{h\tau} = \delta\psi_n^{h\tau} = \left(\psi_n^{h\tau} - \psi_{n-1}^{h\tau}\right)/\tau$ we can easily prove that

$$\begin{aligned}
&\mu\left(u_{n,x}^{h\tau}, v_{n,x}^{h\tau}\right) + \gamma_1\left(\phi_{1n}^{h\tau}, v_{n,x}^{h\tau}\right) + \gamma_2\left(\phi_{2n}^{h\tau}, v_{n,x}^{h\tau}\right) + b_{11}\left(\phi_{1n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + b_{12}\left(\phi_{2n,x}^{h\tau}, e_{1n,x}^{h\tau}\right)\\
&\quad +\ m_1\left(\psi_{n,x}^{h\tau}, e_{1n,x}^{h\tau}\right) + \xi_{11}\left(\phi_{1n}^{h\tau}, e_{1n}^{h\tau}\right) + \xi_{12}\left(\phi_{2n}^{h\tau}, e_{1n}^{h\tau}\right) + \gamma_1\left(u_{n,x}^{h\tau}, e_{1n}^{h\tau}\right) + b_{12}\left(\phi_{1n,x}^{h\tau}, e_{2n,x}^{h\tau}\right)\\
&\quad +\ b_{22}\left(\phi_{2n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + m_2\left(\psi_{n,x}^{h\tau}, e_{2n,x}^{h\tau}\right) + \xi_{12}\left(\phi_{1n}^{h\tau}, e_{2n}^{h\tau}\right) + \xi_{22}\left(\phi_{2n}^{h\tau}, e_{2n}^{h\tau}\right) + \gamma_2\left(u_{n,x}^{h\tau}, e_{2n}^{h\tau}\right)\\
&\quad +\ k\left(\psi_{n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + m_1\left(\phi_{1n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right) + m_2\left(\phi_{2n,x}^{h\tau}, \theta_{n,x}^{h\tau}\right)\\
&\geq \frac{C}{2\tau}\big(\|u_{n,x}^{h\tau}\|^2 + \|\phi_{1n,x}^{h\tau}\|^2 + \|\phi_{2n,x}^{h\tau}\|^2 + \|\phi_{1n}^{h\tau}\|^2 + \|\phi_{2n}^{h\tau}\|^2 + \|\psi_{n,x}^{h\tau}\|^2\\
&\quad -\ \|u_{n-1,x}^{h\tau}\|^2 - \|\phi_{1n-1,x}^{h\tau}\|^2 - \|\phi_{2n-1,x}^{h\tau}\|^2 - \|\phi_{1n-1}^{h\tau}\|^2 - \|\phi_{2n-1}^{h\tau}\|^2 - \|\psi_{n-1,x}^{h\tau}\|^2\big).
\end{aligned}$$

Now, keeping in mind that

$$\begin{aligned}
\rho\left(\delta v_n^{h\tau}, v_n^{h\tau}\right) &\geq \frac{\rho}{2\tau}\left(\|v_n^{h\tau}\|^2 - \|v_{n-1}^{h\tau}\|^2\right),\\
J_1\left(\delta e_{1n}^{h\tau}, e_{1n}^{h\tau}\right) &\geq \frac{J_1}{2\tau}\left(\|e_{1n}^{h\tau}\|^2 - \|e_{1n-1}^{h\tau}\|^2\right),\\
J_2\left(\delta e_{2n}^{h\tau}, e_{2n}^{h\tau}\right) &\geq \frac{J_2}{2\tau}\left(\|e_{2n}^{h\tau}\|^2 - \|e_{2n-1}^{h\tau}\|^2\right),\\
a\left(\delta\theta_n^{h\tau}, \theta_n^{h\tau}\right) &\geq \frac{a}{2\tau}\left(\|\theta_n^{h\tau}\|^2 - \|\theta_{n-1}^{h\tau}\|^2\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{\rho}{2\tau}\left(\|v_n^{h\tau}\|^2 - \|v_{n-1}^{h\tau}\|^2\right) + \frac{J_1}{2\tau}\left(\|e_{1n}^{h\tau}\|^2 - \|e_{1n-1}^{h\tau}\|^2\right) + \frac{J_2}{2\tau}\left(\|e_{2n}^{h\tau}\|^2 - \|e_{2n-1}^{h\tau}\|^2\right)\\
&\quad + \frac{a}{2\tau}\left(\|\theta_n^{h\tau}\|^2 - \|\theta_{n-1}^{h\tau}\|^2\right) +\ \xi^*\|e_{1n}^{h\tau}\|^2 + k^*\|\theta_{n,x}^{h\tau}\|^2\\
&\quad + \frac{C}{2\tau}\big(\|u_{n,x}^{h\tau}\|^2 + \|\phi_{1n,x}^{h\tau}\|^2 + \|\phi_{2n,x}^{h\tau}\|^2 + \|\phi_{1n}^{h\tau}\|^2 + \|\phi_{2n}^{h\tau}\|^2 + \|\psi_{n,x}^{h\tau}\|^2\\
&\quad -\ \|u_{n-1,x}^{h\tau}\|^2 - \|\phi_{1n-1,x}^{h\tau}\|^2 - \|\phi_{2n-1,x}^{h\tau}\|^2 - \|\phi_{1n-1}^{h\tau}\|^2 - \|\phi_{2n-1}^{h\tau}\|^2 - \|\psi_{n-1,x}^{h\tau}\|^2\big) \leq 0,
\end{aligned}$$

and multiplying the above estimates by τ it leads

$$\begin{aligned}
&\|v_n^{h\tau}\|^2 + \|e_{1n}^{h\tau}\|^2 + \|e_{2n}^{h\tau}\|^2 + \|\theta_n^{h\tau}\|^2 + \|u_{n,x}^{h\tau}\|^2 + \|\phi_{1n,x}^{h\tau}\|^2 + \|\phi_{2n,x}^{h\tau}\|^2 + \|\phi_{1n}^{h\tau}\|^2\\
&\quad +\ \|\phi_{2n}^{h\tau}\|^2 + \|\psi_{n,x}^{h\tau}\|^2 + \tau\sum_{j=1}^{n}\left[\|e_{1j}^{h\tau}\|^2 + \|\theta_{j,x}^{h\tau}\|^2\right]\\
&\quad \leq C\left(\|v_0^h\|^2 + \|e_{10}^h\|^2 + \|e_{20}^h\|^2 + \|\theta_0^h\|^2 + \|u_{0,x}^h\|^2 + \|\phi_{10}^h\|_V^2 + \|\phi_{20}^h\|_V^2 + \|\psi_{0,x}^h\|^2\right),
\end{aligned}$$

which implies the discrete stability estimates. ■

In the rest of the section, we prove some a priori error estimates that we summarize in the following.

Theorem 3.2 *If we denote by* (v, e_1, e_2, θ) *and* $\left(v^{h\tau}, e_1^{h\tau}, e_2^{h\tau}, \theta^{h\tau}\right)$ *the respective solutions to Problems* VP *and* $\mathrm{VP}^{h\tau}$ *and we assume the conditions of Theorem 2.1, then we have the following* a priori *error estimates for all* $w^h = \left\{w_n^h\right\}_{n=0}^N$, $r^h = \left\{r_n^h\right\}_{n=0}^N$,

$$z^h = \left\{z_n^h\right\}_{n=0}^N, m^h = \left\{m_n^h\right\}_{n=0}^N \subset V^h:$$

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|v_n - v_n^{h\tau}\|^2 + \|u_n - u_n^{h\tau}\|_V^2 + \|e_{1n} - e_{1n}^{h\tau}\|^2 + \|\phi_{1n} - \phi_{1n}^{h\tau}\|_V^2 + \|e_{2n} - e_{2n}^{h\tau}\|^2 \\
&\quad + \|\phi_{2n} - \phi_{2n}^{h\tau}\|_V^2 + \|\theta_n - \theta_n^{h\tau}\|^2 + \|\psi_n - \psi_n^{h\tau}\|_V^2\Big\} + \tau\sum_{j=1}^N \left[\|e_{1j} - e_{1j}^{h\tau}\|^2 + \|\theta_j - \theta_j^{h\tau}\|_V^2\right] \\
&\le C\tau\sum_{j=1}^N \big(\|\dot v_j - \delta v_j\|^2 + \|\dot u_j - \delta u_j\|_V^2 + \|\dot e_{1j} - \delta e_{1j}\|^2 + \|\dot\phi_{1j} - \delta\phi_{1j}\|_V^2 + \|\dot e_{2j} - \delta e_{2j}\|^2 \\
&\quad + \|\dot\phi_{2j} - \delta\phi_{2j}\|_V^2 + \|\dot\theta_j - \delta\theta_j\|^2 + \|\dot\psi_j - \delta\psi_j\|_V^2 + \|v_j - w_j^h\|_V^2 + \|e_{1j} - r_j^h\|_V^2 \\
&\quad + \|e_{2j} - z_j^h\|_V^2 + \|\theta_j - m_j^h\|_V^2\big) + \frac{C}{\tau}\sum_{j=1}^{N-1}\|v_j - w_j^h - \left(v_{j+1} - w_{j+1}^h\right)\|^2 \\
&\quad + \frac{C}{\tau}\sum_{j=1}^{N-1}\|e_{1j} - r_j^h - \left(e_{1j+1} - r_{j+1}^h\right)\|^2 + \frac{C}{\tau}\sum_{j=1}^{N-1}\|e_{2j} - z_j^h - \left(e_{2j+1} - z_{j+1}^h\right)\|^2 \\
&\quad + \frac{C}{\tau}\sum_{j=1}^{N-1}\|\theta_j - m_j^h - \left(\theta_{j+1} - m_{j+1}^h\right)\|^2 + C\max_{0\le n\le N}\big(\|v_n - w_n^h\|^2 + \|e_{1n} - r_n^h\|^2 \\
&\quad + \|e_{2n} - z_n^h\|^2 + \|\theta_n - m_n^h\|^2\big) + C\big(\|v_0 - v_0^h\|^2 + \|u_0 - u_0^h\|_V^2 + \|e_{10} - e_{10}^h\|^2 \\
&\quad + \|\phi_{10} - \phi_{10}^h\|_V^2 + \|e_{20} - e_{20}^h\|^2 + \|\phi_{20} - \phi_{20}^h\|_V^2 + \|\theta_0 - \theta_0^h\|^2 + \|\psi_0 - \psi_0^h\|_V^2\big),
\end{aligned}$$

where $C > 0$ *is a positive constant assumed to be independent of the discretization parameters h and τ.*

Proof. First, we subtract variational Equation (2.10) at time t_n and for a test function $w = w^h \in V^h \subset V$ and discrete variational Equation (3.3), and we have, for all $w^h \in V^h$,

$$\begin{aligned}
&\rho\left(\dot v_n - \delta v_n^{h\tau}, w^h\right) + \mu\left(u_{n,x} - u_{n,x}^{h\tau}, w_x^h\right) + \gamma_1\left(\phi_{1n} - \phi_{1n}^{h\tau}, w_x^h\right) + \gamma_2\left(\phi_{2n} - \phi_{2n}^{h\tau}, w_x^h\right) \\
&\quad + \beta\left(\theta_{n,x} - \theta_{n,x}^{h\tau}, w^h\right) = 0,
\end{aligned}$$

and therefore, it follows that, for all $w^h \in V^h$,

$$\begin{aligned}
&\rho\left(\dot v_n - \delta v_n^{h\tau}, v_n - v_n^{h\tau}\right) + \mu\left(u_{n,x} - u_{n,x}^{h\tau}, v_{n,x} - v_{n,x}^{h\tau}\right) + \gamma_1\left(\phi_{1n} - \phi_{1n}^{h\tau}, v_{n,x} - v_{n,x}^{h\tau}\right) \\
&\quad + \gamma_2\left(\phi_{2n} - \phi_{2n}^{h\tau}, v_{n,x} - v_{n,x}^{h\tau}\right) + \beta\left(\theta_{n,x} - \theta_{n,x}^{h\tau}, v_n - v_n^{h\tau}\right) = \rho\left(\dot v_n - \delta v_n^{h\tau}, v_n - w^h\right) \\
&\quad + \mu\left(u_{n,x} - u_{n,x}^{h\tau}, v_{n,x} - w_x^h\right) + \gamma_1\left(\phi_{1n} - \phi_{1n}^{h\tau}, v_{n,x} - w_x^h\right) \\
&\quad + \gamma_2\left(\phi_{2n} - \phi_{2n}^{h\tau}, v_{n,x} - w_x^h\right) + \beta\left(\theta_{n,x} - \theta_{n,x}^{h\tau}, v_n - w^h\right). \qquad (3.8)
\end{aligned}$$

Second, subtracting variational Equation (2.11) at time t_n, and for a test function $r = r^h \in V^h \subset V$, and discrete variational Equation (3.4), we obtain, for all $r^h \in V^h$,

$$\begin{aligned}
&J_1\left(\dot e_{1n} - \delta e_{1n}^{h\tau}, r^h\right) + b_{11}\left(\phi_{1n,x} - \phi_{1n,x}^{h\tau}, r_x^h\right) + b_{12}\left(\phi_{2n,x} - \phi_{2n,x}^{h\tau}, r_x^h\right) + m_1\left(\psi_{n,x} - \psi_{n,x}^{h\tau}, r_x^h\right) \\
&\quad + \xi_{11}\left(\phi_{1n} - \phi_{1n}^{h\tau}, r^h\right) + \xi_{12}\left(\phi_{2n} - \phi_{2n}^{h\tau}, r^h\right) + \xi^*\left(e_{1n} - e_{1n}^{h\tau}, r^h\right) + \gamma_1\left(u_{n,x} - u_{n,x}^{h\tau}, r^h\right) \\
&\quad - d_1\left(\theta_n - \theta_n^{h\tau}, r^h\right) = 0,
\end{aligned}$$

and therefore, we have, for all $r^h \in V^h$,

$$\begin{aligned}
&J_1\left(\dot e_{1n} - \delta e_{1n}^{h\tau}, e_{1n} - e_{1n}^{h\tau}\right) + b_{11}\left(\phi_{1n,x} - \phi_{1n,x}^{h\tau}, e_{1n,x} - e_{1n,x}^{h\tau}\right) + b_{12}\left(\phi_{2n,x} - \phi_{2n,x}^{h\tau}, e_{1n,x} - e_{1n,x}^{h\tau}\right) \\
&\quad + m_1\left(\psi_{n,x} - \psi_{n,x}^{h\tau}, e_{1n,x} - e_{1n,x}^{h\tau}\right) + \xi_{11}\left(\phi_{1n} - \phi_{1n}^{h\tau}, e_{1n} - e_{1n}^{h\tau}\right) + \xi_{12}\left(\phi_{2n} - \phi_{2n}^{h\tau}, e_{1n} - e_{1n}^{h\tau}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\xi^*\left(e_{1n}-e_{1n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)+\gamma_1\left(u_{n,x}-u_{n,x}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)-d_1\left(\theta_n-\theta_n^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)\\
&=J_1\left(\dot e_{1n}-\delta e_{1n}^{h\tau},e_{1n}-r^h\right)+b_{11}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{1n,x}-r_x^h\right)+b_{12}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{1n,x}-r_x^h\right)\\
&+m_1\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{1n,x}-r_x^h\right)+\xi_{11}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{1n}-r^h\right)+\xi_{12}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{1n}-r^h\right)\\
&+\xi^*\left(e_{1n}-e_{1n}^{h\tau},e_{1n}-r^h\right)+\gamma_1\left(u_{n,x}-u_{n,x}^{h\tau},e_{1n}-r^h\right)-d_1\left(\theta_n-\theta_n^{h\tau},e_{1n}-r^h\right).
\end{aligned}\tag{3.9}
$$

Proceeding in a similar form with the equations for e_2 and $e_2^{h\tau}$ we find that, for all $z^h\in V^h$,

$$
\begin{aligned}
&J_2\left(\dot e_{2n}-\delta e_{2n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)+b_{12}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)+b_{22}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)\\
&+m_2\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)+\xi_{12}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)+\xi_{22}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)\\
&+\gamma_2\left(u_{n,x}-u_{n,x}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)-d_2\left(\theta_n-\theta_n^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)\\
&=J_2\left(\dot e_{2n}-\delta e_{2n}^{h\tau},e_{2n}-z^h\right)+b_{12}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{2n,x}-z_x^h\right)+b_{22}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{2n,x}-z_x^h\right)\\
&+m_2\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{2n,x}-z_x^h\right)+\xi_{12}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{2n}-z^h\right)+\xi_{22}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{2n}-z^h\right)\\
&+\gamma_2\left(u_{n,x}-u_{n,x}^{h\tau},e_{2n}-z^h\right)-d_2\left(\theta_n-\theta_n^{h\tau},e_{2n}-z^h\right).
\end{aligned}\tag{3.10}
$$

Finally, we subtract variational Equation (2.13) at time t_n, and for a test function $m=m^h\in V^h\subset V$, and discrete variational Equation (3.6), and we obtain, for all $m^h\in V^h$,

$$
\begin{aligned}
&a\left(\dot\theta_n-\delta\theta_n^{h\tau},m^h\right)+k\left(\psi_{n,x}-\psi_{n,x}^{h\tau},m_x^h\right)+k^*\left(\theta_{n,x}-\theta_{n,x}^{h\tau},m_x^h\right)+m_1\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},m_x^h\right)\\
&+\ m_2\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},m_x^h\right)+d_1\left(e_{1n}-e_{1n}^{h\tau},m^h\right)+d_2\left(e_{2n}-e_{2n}^{h\tau},m^h\right)+\beta\left(v_{n,x}-v_{n,x}^{h\tau},m^h\right)=0.
\end{aligned}
$$

So, we find that, for all $m^h\in V^h$,

$$
\begin{aligned}
&a\left(\dot\theta_n-\delta\theta_n^{h\tau},\theta_n-\theta_n^{h\tau}\right)+k\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)+k^*\left(\theta_{n,x}-\theta_{n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)\\
&+\ m_1\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)+m_2\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)+d_1\left(e_{1n}-e_{1n}^{h\tau},\theta_n-\theta_n^{h\tau}\right)\\
&+\ d_2\left(e_{2n}-e_{2n}^{h\tau},\theta_n-\theta_n^{h\tau}\right)+\beta\left(v_{n,x}-v_{n,x}^{h\tau},\theta_n-\theta_n^{h\tau}\right)\\
&=a\left(\dot\theta_n-\delta\theta_n^{h\tau},\theta_n-m^h\right)+k\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\theta_{n,x}-m_x^h\right)+k^*\left(\theta_{n,x}-\theta_{n,x}^{h\tau},\theta_{n,x}-m_x^h\right)\\
&+\ m_1\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\theta_{n,x}-m_x^h\right)+m_2\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\theta_{n,x}-m_x^h\right)+d_1\left(e_{1n}-e_{1n}^{h\tau},\theta_n-m^h\right)\\
&+\ d_2\left(e_{2n}-e_{2n}^{h\tau},\theta_n-m^h\right)+\beta\left(v_{n,x}-v_{n,x}^{h\tau},\theta_n-m^h\right).
\end{aligned}\tag{3.11}
$$

Summing Equations (3.8)–(3.11) we have

$$
\begin{aligned}
&\rho\left(\dot v_n-\delta v_n^{h\tau},v_n-v_n^{h\tau}\right)+\mu\left(u_{n,x}-u_{n,x}^{h\tau},v_{n,x}-v_{n,x}^{h\tau}\right)+\gamma_1\left(\phi_{1n}-\phi_{1n}^{h\tau},v_{n,x}-v_{n,x}^{h\tau}\right)\\
&+\ \gamma_2\left(\phi_{2n}-\phi_{2n}^{h\tau},v_{n,x}-v_{n,x}^{h\tau}\right)+J_1\left(\dot e_{1n}-\delta e_{1n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)+b_{11}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{1n,x}-e_{1n,x}^{h\tau}\right)\\
&+\ b_{12}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{1n,x}-e_{1n,x}^{h\tau}\right)+m_1\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{1n,x}-e_{1n,x}^{h\tau}\right)+\xi_{11}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)\\
&+\ \xi_{12}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)+\xi^*\left(e_{1n}-e_{1n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)+\gamma_1\left(u_{n,x}-u_{n,x}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)\\
&+\ J_2\left(\dot e_{2n}-\delta e_{2n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)+b_{12}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)+b_{22}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)\\
&+\ m_2\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)+\xi_{12}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)+\xi_{22}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)\\
&+\ \gamma_2\left(u_{n,x}-u_{n,x}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)+a\left(\dot\theta_n-\delta\theta_n^{h\tau},\theta_n-\theta_n^{h\tau}\right)+k\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)\\
&+\ k^*\left(\theta_{n,x}-\theta_{n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)+m_1\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)+m_2\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)\\
&=\rho\left(\dot v_n-\delta v_n^{h\tau},v_n-w^h\right)+\mu\left(u_{n,x}-u_{n,x}^{h\tau},v_{n,x}-w_x^h\right)+\gamma_1\left(\phi_{1n}-\phi_{1n}^{h\tau},v_{n,x}-w_x^h\right)\\
&+\ \gamma_2\left(\phi_{2n}-\phi_{2n}^{h\tau},v_{n,x}-w_x^h\right)+\beta\left(\theta_{n,x}-\theta_{n,x}^{h\tau},v_n-w^h\right)+J_1\left(\dot e_{1n}-\delta e_{1n}^{h\tau},e_{1n}-r^h\right)\\
&+\ b_{11}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{1n,x}-r_x^h\right)+b_{12}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{1n,x}-r_x^h\right)+m_1\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{1n,x}-r_x^h\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\ \xi_{11}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{1n}-r^h\right)+\xi_{12}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{1n}-r^h\right)+\xi^*\left(e_{1n}-e_{1n}^{h\tau},e_{1n}-r^h\right)\\
&+\ \gamma_1\left(u_{n,x}-u_{n,x}^{h\tau},e_{1n}-r^h\right)-d_1\left(\theta_n-\theta_n^{h\tau},e_{1n}-r^h\right)+J_2\left(\dot{e}_{2n}-\delta e_{2n}^{h\tau},e_{2n}-z^h\right)\\
&+\ b_{12}\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{2n,x}-z_x^h\right)+b_{22}\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{2n,x}-z_x^h\right)+m_2\left(\psi_{n,x}-\psi_{n,x}^{h\tau},e_{2n,x}-z_x^h\right)\\
&+\ \xi_{12}\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{2n}-z^h\right)+\xi_{22}\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{2n}-z^h\right)+\gamma_2\left(u_{n,x}-u_{n,x}^{h\tau},e_{2n}-z^h\right)\\
&-\ d_2\left(\theta_n-\theta_n^{h\tau},e_{2n}-z^h\right)+a\left(\dot{\theta}_n-\delta\theta_n^{h\tau},\theta_n-m^h\right)+k\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\theta_{n,x}-m_x^h\right)\\
&+\ k^*\left(\theta_{n,x}-\theta_{n,x}^{h\tau},\theta_{n,x}-m_x^h\right)+m_1\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\theta_{n,x}-m_x^h\right)+m_2\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\theta_{n,x}-m_x^h\right)\\
&+\ d_1\left(e_{1n}-e_{1n}^{h\tau},\theta_n-m^h\right)+d_2\left(e_{2n}-e_{2n}^{h\tau},\theta_n-m^h\right)+\beta\left(v_{n,x}-v_{n,x}^{h\tau},\theta_n-m^h\right).
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&\left(\dot{v}_n-\delta v_n^{h\tau},w\right)=(\dot{v}_n-\delta v_n,w)+\left(\delta v_n-\delta v_n^{h\tau},w\right),\\
&\left(\dot{e}_{1n}-\delta e_{1n}^{h\tau},r\right)=(\dot{e}_{1n}-\delta e_{1n},r)+\left(\delta e_{1n}-\delta e_{1n}^{h\tau},r\right),\\
&\left(\dot{e}_{2n}-\delta e_{2n}^{h\tau},z\right)=(\dot{e}_{2n}-\delta e_{2n},z)+\left(\delta e_{2n}-\delta e_{2n}^{h\tau},z\right),\\
&\left(\dot{\theta}_n-\delta\theta_n^{h\tau},m\right)=\left(\dot{\theta}_n-\delta\theta_n,m\right)+\left(\delta\theta_n-\delta\theta_n^{h\tau},m\right),\\
&\left(\delta v_n-\delta v_n^{h\tau},v_n-v_n^{h\tau}\right)\geq\frac{1}{2\tau}\left(\|v_n-v_n^{h\tau}\|^2-\|v_{n-1}-v_{n-1}^{h\tau}\|^2\right),\\
&\left(\delta e_{1n}-\delta e_{1n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)\geq\frac{1}{2\tau}\left(\|e_{1n}-e_{1n}^{h\tau}\|^2-\|e_{1n-1}-e_{1n-1}^{h\tau}\|^2\right),\\
&\left(\delta e_{2n}-\delta e_{2n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)\geq\frac{1}{2\tau}\left(\|e_{2n}-e_{2n}^{h\tau}\|^2-\|e_{2n-1}-e_{2n-1}^{h\tau}\|^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\delta\theta_n-\delta\theta_n^{h\tau},\theta_n-\theta_n^{h\tau}\right)\geq\frac{1}{2\tau}\left(\|\theta_n-\theta_n^{h\tau}\|^2-\|\theta_{n-1}-\theta_{n-1}^{h\tau}\|^2\right),\\
&\left(u_{n,x}-u_{n,x}^{h\tau},v_{n,x}-v_{n,x}^{h\tau}\right)=\left(u_{n,x}-u_{n,x}^{h\tau},\dot{u}_{n,x}-\delta u_{n,x}\right)+\left(u_{n,x}-u_{n,x}^{h\tau},\delta u_{n,x}-\delta u_{n,x}^{h\tau}\right)\\
&\qquad\geq\left(u_{n,x}-u_{n,x}^{h\tau},\dot{u}_{n,x}-\delta u_{n,x}\right)+\frac{1}{2\tau}\left(\|u_{n,x}-u_{n,x}^{h\tau}\|^2-\|u_{n-1,x}-u_{n-1,x}^{h\tau}\|^2\right),\\
&\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{1n,x}-e_{1n,x}^{h\tau}\right)=\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\dot{\phi}_{1n,x}-\delta\phi_{1n,x}\right)+\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\delta\phi_{1n,x}-\delta\phi_{1n,x}^{h\tau}\right)\\
&\qquad\geq\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\dot{\phi}_{1n,x}-\delta\phi_{1n,x}\right)+\frac{1}{2\tau}\left(\|\phi_{1n,x}-\phi_{1n,x}^{h\tau}\|^2-\|\phi_{1n-1,x}-\phi_{1n-1,x}^{h\tau}\|^2\right),\\
&\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)=\left(\phi_{1n}-\phi_{1n}^{h\tau},\dot{\phi}_{1n}-\delta\phi_{1n}\right)+\left(\phi_{1n}-\phi_{1n}^{h\tau},\delta\phi_{1n}-\delta\phi_{1n}^{h\tau}\right)\\
&\qquad\geq\left(\phi_{1n}-\phi_{1n}^{h\tau},\dot{\phi}_{1n}-\delta\phi_{1n}\right)+\frac{1}{2\tau}\left(\|\phi_{1n}-\phi_{1n}^{h\tau}\|^2-\|\phi_{1n-1}-\phi_{1n-1}^{h\tau}\|^2\right),\\
&\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)=\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\dot{\phi}_{2n,x}-\delta\phi_{2n,x}\right)+\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\delta\phi_{2n,x}-\delta\phi_{2n,x}^{h\tau}\right)\\
&\qquad\geq\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\dot{\phi}_{2n,x}-\delta\phi_{2n,x}\right)+\frac{1}{2\tau}\left(\|\phi_{2n,x}-\phi_{2n,x}^{h\tau}\|^2-\|\phi_{2n-1,x}-\phi_{2n-1,x}^{h\tau}\|^2\right),\\
&\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)=\left(\phi_{2n}-\phi_{2n}^{h\tau},\dot{\phi}_{2n}-\delta\phi_{2n}\right)+\left(\phi_{2n}-\phi_{2n}^{h\tau},\delta\phi_{2n}-\delta\phi_{2n}^{h\tau}\right)\\
&\qquad\geq\left(\phi_{2n}-\phi_{2n}^{h\tau},\dot{\phi}_{2n}-\delta\phi_{2n}\right)+\frac{1}{2\tau}\left(\|\phi_{2n}-\phi_{2n}^{h\tau}\|^2-\|\phi_{2n-1}-\phi_{2n-1}^{h\tau}\|^2\right),\\
&\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\theta_{n,x}-\theta_{n,x}^{h\tau}\right)=\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\dot{\psi}_{n,x}-\delta\psi_{n,x}\right)+\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\delta\psi_{n,x}-\delta\psi_{n,x}^{h\tau}\right)\\
&\qquad\geq\left(\psi_{n,x}-\psi_{n,x}^{h\tau},\dot{\psi}_{n,x}-\delta\psi_{n,x}\right)+\frac{1}{2\tau}\left(\|\psi_{n,x}-\psi_{n,x}^{h\tau}\|^2-\|\psi_{n-1,x}-\psi_{n-1,x}^{h\tau}\|^2\right),\\
&\left(\phi_{1n}-\phi_{1n}^{h\tau},v_{n,x}-v_{n,x}^{h\tau}\right)=\left(\phi_{1n}-\phi_{1n}^{h\tau},\dot{u}_{n,x}-\delta u_{n,x}\right)+\left(\phi_{1n}-\phi_{1n}^{h\tau},\delta u_{n,x}-\delta u_{n,x}^{h\tau}\right),\\
&\left(\phi_{2n}-\phi_{2n}^{h\tau},v_{n,x}-v_{n,x}^{h\tau}\right)=\left(\phi_{2n}-\phi_{2n}^{h\tau},\dot{u}_{n,x}-\delta u_{n,x}\right)+\left(\phi_{2n}-\phi_{2n}^{h\tau},\delta u_{n,x}-\delta u_{n,x}^{h\tau}\right),\\
&\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},e_{1n,x}-e_{1n,x}^{h\tau}\right)=\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\dot{\phi}_{1n,x}-\delta\phi_{1n,x}\right)+\left(\phi_{2n,x}-\phi_{2n,x}^{h\tau},\delta\phi_{1n,x}-\delta\phi_{1n,x}^{h\tau}\right),
\end{aligned}
$$

$$\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},e_{2n,x}-e_{2n,x}^{h\tau}\right)=\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\dot{\phi}_{2n,x}-\delta\phi_{2n,x}\right)+\left(\phi_{1n,x}-\phi_{1n,x}^{h\tau},\delta\phi_{2n,x}-\delta\phi_{2n,x}^{h\tau}\right),$$
$$\left(\phi_{2n}-\phi_{2n}^{h\tau},e_{1n}-e_{1n}^{h\tau}\right)=\left(\phi_{2n}-\phi_{2n}^{h\tau},\dot{\phi}_{1n}-\delta\phi_{1n}\right)+\left(\phi_{2n}-\phi_{2n}^{h\tau},\delta\phi_{1n}-\delta\phi_{1n}^{h\tau}\right),$$
$$\left(\phi_{1n}-\phi_{1n}^{h\tau},e_{2n}-e_{2n}^{h\tau}\right)=\left(\phi_{1n}-\phi_{1n}^{h\tau},\dot{\phi}_{2n}-\delta\phi_{2n}\right)+\left(\phi_{1n}-\phi_{1n}^{h\tau},\delta\phi_{2n}-\delta\phi_{2n}^{h\tau}\right),$$
$$\left(v_{n,x}-v_{n,x}^{h\tau},\theta_n-m^h\right)=-\left(v_n-v_n^{h\tau},\theta_{n,x}-m_x^h\right),$$

using several times Cauchy's inequality $ab\leq\epsilon a^2+\frac{1}{4\epsilon}b^2$, for all $a,b,\epsilon\in\mathbb{R}$ with $\epsilon>0$ and classical Cauchy-Schwarz inequality, and the conditions of Theorem 2.1 we have, for all $w^h,z^h,r^h,m^h\in V^h$,

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|v_n-v_n^{h\tau}\|^2-\|v_{n-1}-v_{n-1}^{h\tau}\|^2\right)+\frac{1}{2\tau}\left(\|u_{n,x}-u_{n,x}^{h\tau}\|^2-\|u_{n-1,x}-u_{n-1,x}^{h\tau}\|^2\right)\\
&\quad+\frac{1}{2\tau}\left(\|e_{1n}-e_{1n}^{h\tau}\|^2-\|e_{1n-1}-e_{1n-1}^{h\tau}\|^2\right)+\frac{1}{2\tau}\left(\|\phi_{1n}-\phi_{1n}^{h\tau}\|^2-\|\phi_{1n-1}-\phi_{1n-1}^{h\tau}\|^2\right)\\
&\quad+\frac{1}{2\tau}\left(\|\phi_{1n,x}-\phi_{1n,x}^{h\tau}\|^2-\|\phi_{1n-1,x}-\phi_{1n-1,x}^{h\tau}\|^2\right)+\frac{1}{2\tau}\left(\|e_{2n}-e_{2n}^{h\tau}\|^2-\|e_{2n-1}-e_{2n-1}^{h\tau}\|^2\right)\\
&\quad+\frac{1}{2\tau}\left(\|\phi_{2n}-\phi_{2n}^{h\tau}\|^2-\|\phi_{2n-1}-\phi_{2n-1}^{h\tau}\|^2\right)+\frac{1}{2\tau}\left(\|\phi_{2n,x}-\phi_{2n,x}^{h\tau}\|^2-\|\phi_{2n-1,x}-\phi_{2n-1,x}^{h\tau}\|^2\right)\\
&\quad+\frac{1}{2\tau}\left(\|\theta_n-\theta_n^{h\tau}\|^2-\|\theta_{n-1}-\theta_{n-1}^{h\tau}\|^2\right)+\frac{1}{2\tau}\left(\|\psi_{n,x}-\psi_{n,x}^{h\tau}\|^2-\|\psi_{n-1,x}-\psi_{n-1,x}^{h\tau}\|^2\right)\\
&\quad+\|e_{1n}-e_{1n}^{h\tau}\|^2+\|\theta_{n,x}-\theta_{n,x}^{h\tau}\|^2\\
&\leq C\big(\|\dot{v}_n-\delta v_n\|^2+\|\dot{u}_{n,x}-\delta u_{n,x}\|^2+\|\dot{e}_{1n}-\delta e_{1n}\|^2+\|\dot{\phi}_{1n,x}-\delta\phi_{1n,x}\|^2+\|\dot{\phi}_{1n}-\delta\phi_{1n}\|^2\\
&\quad+\|\dot{e}_{2n}-\delta e_{2n}\|^2+\|\dot{\phi}_{2n,x}-\delta\phi_{2n,x}\|^2+\|\dot{\phi}_{2n}-\delta\phi_{2n}\|^2+\|\dot{\theta}_n-\delta\theta_n\|^2+\|\dot{\psi}_{n,x}-\delta\psi_{n,x}\|^2\\
&\quad+\|v_n-w^h\|_V^2+\|e_{1n}-r^h\|_V^2+\|e_{2n}-z^h\|_V^2+\|\theta_n-m^h\|_V^2+\|v_n-v_n^{h\tau}\|^2+\|e_{1n}-e_{1n}^{h\tau}\|^2\\
&\quad+\|e_{2n}-e_{2n}^{h\tau}\|^2+\|\phi_{1n}-\phi_{1n}^{h\tau}\|^2+\|\phi_{1n,x}-\phi_{1n,x}^{h\tau}\|^2+\|\phi_{2n}-\phi_{2n}^{h\tau}\|^2+\|\phi_{2n,x}-\phi_{2n,x}^{h\tau}\|^2\\
&\quad+\|\theta_n-\theta_n^{h\tau}\|^2+\|\psi_n-\psi_n^{h\tau}\|^2+\left(\delta v_n-\delta v_n^{h\tau},v_n-w^h\right)+\left(\delta e_{1n}-\delta e_{1n}^{h\tau},e_{1n}-r^h\right)\\
&\quad+\left(\delta e_{2n}-\delta e_{2n}^{h\tau},e_{2n}-z^h\right)+\left(\delta\theta_n-\delta\theta_n^{h\tau},\theta_n-m^h\right)\big).
\end{aligned}$$

Multiplying the above estimates by τ and summing up to n, it follows that, for all $w^h,z^h,r^h,m^h\in V^h$,

$$\begin{aligned}
&\|v_n-v_n^{h\tau}\|^2+\|u_{n,x}-u_{n,x}^{h\tau}\|^2+\|e_{1n}-e_{1n}^{h\tau}\|^2+\|\phi_{1n}-\phi_{1n}^{h\tau}\|^2+\|\phi_{1n,x}-\phi_{1n,x}^{h\tau}\|^2+\|e_{2n}-e_{2n}^{h\tau}\|^2\\
&\quad+\|\phi_{2n}-\phi_{2n}^{h\tau}\|^2+\|\phi_{2n,x}-\phi_{2n,x}^{h\tau}\|^2+\|\theta_n-\theta_n^{h\tau}\|^2+\|\psi_{n,x}-\psi_{n,x}^{h\tau}\|^2\\
&\quad+\tau\sum_{j=1}^{n}\left[\|e_{1j}-e_{1j}^{h\tau}\|^2+\|\theta_{j,x}-\theta_{j,x}^{h\tau}\|^2\right]\\
&\leq C\tau\sum_{j=1}^{n}\big(\|\dot{v}_j-\delta v_j\|^2+\|\dot{u}_{j,x}-\delta u_{j,x}\|^2+\|\dot{e}_{1j}-\delta e_{1j}\|^2+\|\dot{\phi}_{1j,x}-\delta\phi_{1j,x}\|^2+\|\dot{\phi}_{1j}-\delta\phi_{1j}\|^2\\
&\quad+\|\dot{e}_{2j}-\delta e_{2j}\|^2+\|\dot{\phi}_{2j,x}-\delta\phi_{2j,x}\|^2+\|\dot{\phi}_{2j}-\delta\phi_{2j}\|^2+\|\dot{\theta}_j-\delta\theta_j\|^2+\|\dot{\psi}_{j,x}-\delta\psi_{j,x}\|^2\\
&\quad+\|v_j-w_j^h\|_V^2+\|e_{1j}-r_j^h\|_V^2+\|e_{2j}-z_j^h\|_V^2+\|\theta_j-m_j^h\|_V^2+\|v_j-v_j^{h\tau}\|^2+\|e_{1j}-e_{1j}^{h\tau}\|^2\\
&\quad+\|e_{2j}-e_{2j}^{h\tau}\|^2+\|\phi_{1j}-\phi_{1j}^{h\tau}\|^2+\|\phi_{1j,x}-\phi_{1j,x}^{h\tau}\|^2+\|\phi_{2j}-\phi_{2j}^{h\tau}\|^2+\|\phi_{2j,x}-\phi_{2j,x}^{h\tau}\|^2\\
&\quad+\|\theta_j-\theta_j^{h\tau}\|^2+\|\psi_j-\psi_j^{h\tau}\|^2+\left(\delta v_j-\delta v_j^{h\tau},v_j-w_j^h\right)+\left(\delta e_{1j}-\delta e_{1j}^{h\tau},e_{1j}-r_j^h\right)\\
&\quad+\left(\delta e_{2j}-\delta e_{2j}^{h\tau},e_{2j}-z_j^h\right)+\left(\delta\theta_j-\delta\theta_j^{h\tau},\theta_j-m_j^h\right)\big)+C\big(\|v_0-v_0^h\|^2+\|u_0-u_0^h\|_V^2\\
&\quad+\|e_{10}-e_{10}^h\|^2+\|\phi_{10}-\phi_{10}^h\|_V^2+\|e_{20}-e_{20}^h\|^2+\|\phi_{20}-\phi_{20}^h\|_V^2+\|\theta_0-\theta_0^h\|^2\\
&\quad+\|\psi_0-\psi_0^h\|_V^2\big).
\end{aligned}$$

Finally, taking into account that

$$\tau\sum_{j=1}^{n}\left(\delta v_j-\delta v_j^{h\tau},v_j-w_j^h\right)=\sum_{j=1}^{n}\left(v_j-v_j^{h\tau}-\left(v_{j-1}-v_{j-1}^{h\tau}\right),v_j-w_j^h\right)$$
$$=\left(v_n-v_n^{h\tau},v_n-w_n^h\right)+\left(v_0^h-v_0,v_1-w_1^h\right)$$
$$+\sum_{j=1}^{n-1}\left(v_j-v_j^{h\tau},v_j-w_j^h-\left(v_{j+1}-w_{j+1}^h\right)\right),$$

where we omit the similar estimates for the terms involving e_{1n}, $e_{2n,}$ and θ_n, applying a discrete version of Gronwall's inequality (see [4]) we conclude the proof. ■

The error estimates provided in Theorem 3.2 can be used to obtain the convergence order of the approximations. Therefore, as an example, we can derive the following results which states the linear convergence of the algorithm.

Theorem 3.3 *Let the assumptions of Theorem 3.2 hold. Therefore, if we assume the following additional regularity*:

$$u,\phi_{1n},\phi_{2n},\psi\in H^3(0,T;Y)\cap W^{1,\infty}\left(0,T;H^2(0,\ell)\right)\cap H^2(0,T;V),$$

it follows that the approximations obtained by Problem VP$^{h\tau}$ *are linearly convergent*; *that is*, *there exists a positive constant C*, *independent of the discretization parameters h and* τ, *such that*

$$\max_{0\le n\le N}\left\{\|v_n-v_n^{h\tau}\|+\|u_n-u_n^{h\tau}\|_V+\|e_{1n}-e_{1n}^{h\tau}\|+\|\phi_{1n}-\phi_{1n}^{h\tau}\|_V+\|e_{2n}-e_{2n}^{h\tau}\|\right.$$
$$\left.+\ \|\phi_{2n}-\phi_{2n}^{h\tau}\|_V+\|\theta_n-\theta_n^{h\tau}\|+\|\psi_n-\psi_n^{h\tau}\|_V\right\}\le C(h+\tau).$$

The proof of the above theorem is done using the classical results on the approximation by finite elements (see [7]), the properties of the finite element interpolation operator $\mathcal{P}^h$ to approximate the initial conditions in (3.2), and the approximation of the terms of the form (see [4]):

$$\frac{C}{\tau}\sum_{j=1}^{N-1}\|v_j-w_j^h-\left(v_{j+1}-w_{j+1}^h\right)\|^2\le Ch^2\|u\|^2_{H^2(0,T;V)}.$$

4 | NUMERICAL RESULTS

In this section, we describe the numerical scheme implemented in MATLAB for solving Problem VP, and we show some numerical examples to demonstrate the accuracy of the approximation and the behavior of the solution.

We solve the following linear problem, for all $w^h,r^h,z^h,m^h\in V^h$,

$$\rho\left(v_n^{h\tau},w^h\right)+\mu\tau^2\left(v_{n,x}^{h\tau},w_x^h\right)=\rho\left(v_{n-1}^{h\tau},w^h\right)-\mu\tau\left(u_{n-1,x}^{h\tau},w_x^h\right)+\gamma_1\tau\left(\phi_{1n,x}^{h\tau},w^h\right)$$
$$+\ \gamma_2\tau\left(\phi_{2n,x}^{h\tau},w^h\right)-\beta\tau\left(\theta_{n,x}^{h\tau},w^h\right),$$

$$J_1\left(e_{1n}^{h\tau},r^h\right)+b_{11}\tau^2\left(e_{1n,x}^{h\tau},r_x^h\right)+\xi_{11}\tau^2\left(e_{1n}^{h\tau},r^h\right)+\xi^*\tau\left(e_{1n}^{h\tau},r^h\right)=J_1\left(e_{1n-1}^{h\tau},r^h\right)$$
$$-\ b_{11}\tau\left(\phi_{1n-1,x}^{h\tau},r_x^h\right)-b_{12}\tau\left(\phi_{2n,x}^{h\tau},r_x^h\right)-m_1\tau\left(\psi_{n,x}^{h\tau},r_x^h\right)-\xi_{11}\tau\left(\phi_{1n-1}^{h\tau},r^h\right)$$
$$-\ \xi_{12}\tau\left(\phi_{2n}^{h\tau},r^h\right)-\gamma_1\tau\left(u_{n,x}^{h\tau},r^h\right)+d_1\tau\left(\theta_n^{h\tau},r^h\right),$$

$$J_2\left(e_{2n}^{h\tau},z^h\right)+b_{22}\tau^2\left(e_{2n,x}^{h\tau},z_x^h\right)+\xi_{22}\tau^2\left(e_{2n}^{h\tau},z^h\right)=J_2\left(e_{2n-1}^{h\tau},z^h\right)$$
$$-\ b_{12}\tau\left(\phi_{1n,x}^{h\tau},z_x^h\right)-b_{22}\tau\left(\phi_{2n-1,x}^{h\tau},z_x^h\right)-m_2\tau\left(\psi_{n,x}^{h\tau},z_x^h\right)-\xi_{12}\tau\left(\phi_{1n}^{h\tau},z^h\right)$$

$$
\begin{aligned}
&- \xi_{22}\tau\left(\phi_{2n-1}^{h\tau}, z^h\right) - \gamma_2\tau\left(u_{n,x}^{h\tau}, z^h\right) + d_2\tau\left(\theta_n^{h\tau}, z^h\right),\\
&a\left(\theta_n^{h\tau}, m^h\right) + k\tau^2\left(\theta_{n,x}^{h\tau}, m_x^h\right) + k^*\tau\left(\theta_{n,x}^{h\tau}, m_x^h\right) = a\left(\theta_{n-1}^{h\tau}, m^h\right) - k\tau\left(\psi_{n-1,x}^{h\tau}, m_x^h\right)\\
&- m_1\tau\left(\phi_{1n,x}^{h\tau}, m_x^h\right) - m_2\tau\left(\phi_{2n,x}^{h\tau}, m_x^h\right) - d_1\tau\left(e_{1n}^{h\tau}, m^h\right) - d_2\tau\left(e_{2n}^{h\tau}, m^h\right) - \beta\tau\left(v_{n,x}^{h\tau}, m^h\right),
\end{aligned}
$$

where the discrete displacements, the discrete first porosity, the discrete second porosity, and the discrete thermal displacements are then recovered from the relations:

$$
\begin{aligned}
u_n^{h\tau} = \tau v_n^{h\tau} + u_{n-1}^h, \quad \phi_{1n}^{h\tau} = \tau e_{1n}^{h\tau} + \phi_{1n-1}^h,\\
\phi_{2n}^{h\tau} = \tau e_{2n}^{h\tau} + \phi_{2n-1}^h, \quad \psi_n^{h\tau} = \tau\theta_n^{h\tau} + \psi_{n-1}^h.
\end{aligned}
$$

This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.01$) took about 1.1 s of CPU time.

4.1 | First example: Numerical convergence

As an academical example, in order to show the accuracy of the approximations the following simpler problem is considered. We solve Problem VP with the following data:

$$
\begin{aligned}
&\ell = 1, \quad T = 1, \quad \rho = 1, \quad \mu = 2, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad \beta = 1, \quad J_1 = 1, \quad J_2 = 1,\\
&b_{11} = 1, \quad b_{12} = 1, \quad b_{22} = 1, \quad m_1 = 1, \quad m_2 = 1, \quad \xi_{11} = 1, \quad \xi_{12} = 1, \quad \xi_{22} = 1,\\
&d_1 = 1, \quad d_2 = 1, \quad \xi^* = 1, \quad a = 1, \quad \kappa = 1, \quad k^* = 1.
\end{aligned}
$$

We assume that the exact solution to Problem P can be easily calculated with the following form, for $(x, t) \in [0, 1] \times [0, 1]$:

$$u(x, t) = \phi_1(x, t) = \phi_2(x, t) = \psi(x, t) = e^t x(x - 1).$$

In this case, we consider homogeneous Dirichlet boundary conditions on the boundaries $x = 0, 1$ and the initial conditions, for $x \in (0, 1)$:

$$u_0(x) = v_0(x) = \phi_{10}(x) = \phi_{20}(x) = e_{10}(x) = e_{20}(x) = \psi_0(x) = \theta_0(x) = x(x - 1).$$

Moreover, we have added the following supply terms, for all $(x, t) \in (0, 1) \times (0, 1)$,

$$
\begin{aligned}
F_1(x, t) = e^t(x(x - 1) - 4 - (2x - 1)), \quad F_2(x, t) = e^t(3x(x - 1) - 6 + (2x - 1)),\\
F_3(x, t) = e^t(2x(x - 1) - 6 + (2x - 1)), \quad F_4(x, t) = e^t(3x(x - 1) - 8 + (2x - 1)).
\end{aligned}
$$

Thus, the approximation errors estimated by

$$
\begin{aligned}
\max_{0 \le n \le N} \big\{ &\|v_n - v_n^{h\tau}\| + \|u_n - u_n^{h\tau}\|_V + \|e_{1n} - e_{1n}^{h\tau}\| + \|\phi_{1n} - \phi_{1n}^{h\tau}\|_V + \|e_{2n} - e_{2n}^{h\tau}\|\\
&+ \|\phi_{2n} - \phi_{2n}^{h\tau}\|_V + \|\theta_n - \theta_n^{h\tau}\| + \|\psi_n - \psi_n^{h\tau}\|_V \big\}
\end{aligned}
$$

are presented in Table 1 for several values of the discretization parameters h and τ. Moreover, the evolution of the error depending on the parameter $h + \tau$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Theorem 3.3 is achieved.

If we assume now that there are not supply terms, and we use the final time $T = 20$, the following data

$$
\begin{aligned}
&\ell = 1, \quad \rho = 1, \quad \mu = 2, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad \beta = 1, \quad J_1 = 1, \quad J_2 = 1,\\
&b_{11} = 5, \quad b_{12} = 1, \quad b_{22} = 3, \quad m_1 = 1, \quad m_2 = 1, \quad \xi_{11} = 7, \quad \xi_{12} = 1, \quad \xi_{22} = 3,\\
&d_1 = 1, \quad d_2 = 1, \quad \xi^* = 1, \quad a = 1, \quad \kappa = 5, \quad k^* = 1,
\end{aligned}
$$

TABLE 1 Example 1: numerical errors for some h and τ

$h \downarrow \tau \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.668060	0.648979	0.644213	0.643371	0.643055	0.642898	0.642851
$1/2^4$	0.359258	0.328250	0.311935	0.308844	0.308182	0.307941	0.307882
$1/2^5$	0.217965	0.180595	0.160309	0.154223	0.151556	0.150696	0.150588
$1/2^6$	0.156602	0.112303	0.087557	0.080215	0.076876	0.075060	0.074573
$1/2^7$	0.132148	0.083145	0.053580	0.044418	0.040297	0.038107	0.037455
$1/2^8$	0.122632	0.071689	0.039025	0.027922	0.022669	0.019882	0.019081
$1/2^9$	0.118769	0.067191	0.033307	0.021026	0.014700	0.011081	0.010027
$1/2^{10}$	0.117174	0.065326	0.031058	0.018342	0.011448	0.007099	0.005700
$1/2^{11}$	0.116572	0.064538	0.030119	0.017272	0.010185	0.005475	0.003789
$1/2^{12}$	0.116381	0.064232	0.029712	0.016816	0.009673	0.004844	0.003026
$1/2^{13}$	0.116330	0.064133	0.029544	0.016616	0.009452	0.004588	0.002728

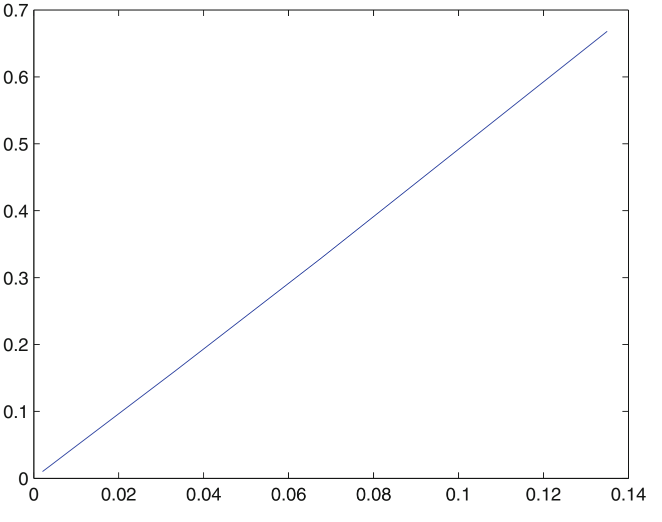

FIGURE 1 Example 1: asymptotic constant error

and the initial conditions

$$u_0 = v_0 = \phi_{20} = e_{10} = e_{20} = \psi_0 = \theta_0 = 0, \quad \phi_{10}(x) = 10x(x-1) \quad \textit{for} \ \ x \in (0,1),$$

taking the discretization parameters $h = 10^{-3}$ and $\tau = 10^{-3}$, the evolution in time of the discrete energy given by

$$\begin{aligned} E_n^{h\tau} = {} & \rho \left\| v_n^{h\tau} \right\|^2 + \mu \left\| u_{n,x}^{h\tau} \right\|^2 + J_1 \left\| e_{1n}^{h\tau} \right\|^2 + J_2 \left\| e_{2n}^{h\tau} \right\|^2 + a \left\| \theta_n^{h\tau} \right\|^2 + b_{11} \left\| \phi_{1n,x}^{h\tau} \right\|^2 \\ & + \ b_{22} \left\| \phi_{2n,x}^{h\tau} \right\|^2 + 2b_{12} \left(\phi_{1n,x}^{h\tau}, \phi_{2n,x}^{h\tau} \right) + \xi_{11} \left\| \phi_{1n}^{h\tau} \right\|^2 + \xi_{22} \left\| \phi_{2n}^{h\tau} \right\|^2 + 2\xi_{12} \left(\phi_{1n}^{h\tau}, \phi_{2n}^{h\tau} \right) \\ & + \ 2\gamma_1 \left(\phi_{1n}^{h\tau}, u_{n,x}^{h\tau} \right) + 2\gamma_2 \left(\phi_{2n}^{h\tau}, u_{n,x}^{h\tau} \right) + \kappa \left\| \psi_{n,x}^{h\tau} \right\|^2 + 2m_1 \left(\phi_{1n,x}^{h\tau}, \psi_{n,x}^{h\tau} \right) + 2m_2 \left(\phi_{2n,x}^{h\tau}, \psi_{n,x}^{h\tau} \right) \end{aligned}$$

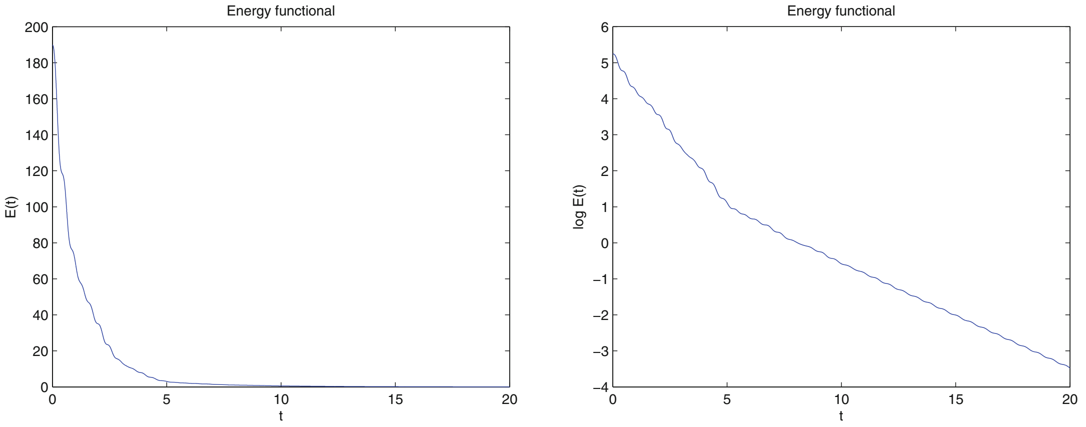

FIGURE 2 Example 1: evolution in time of the discrete energy (natural and semi-log scales)

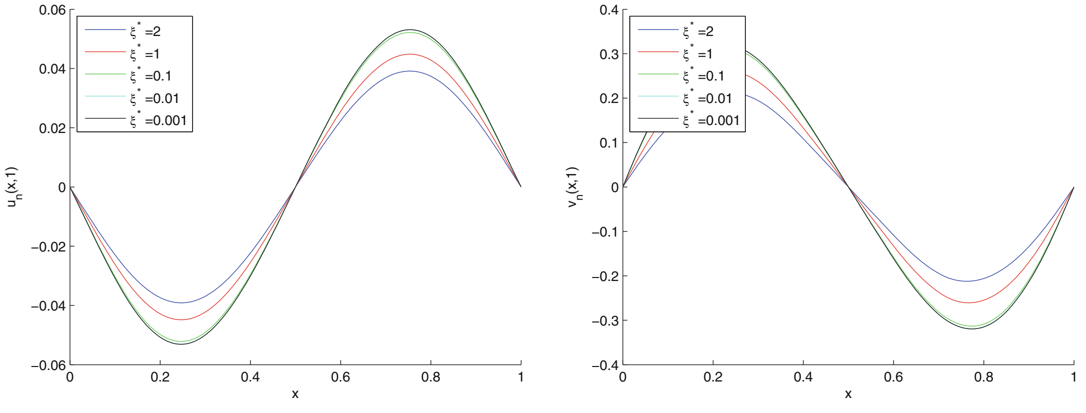

FIGURE 3 Example 2: displacements and velocity fields at final time for some values of ξ^*

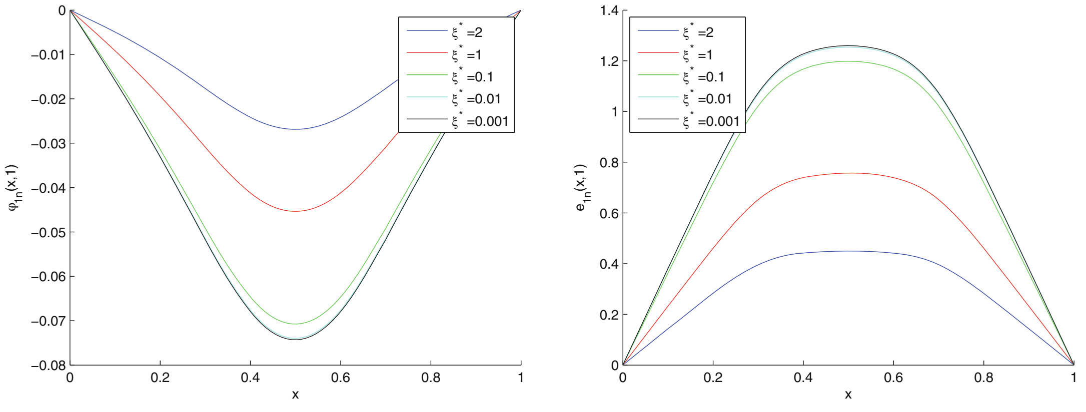

FIGURE 4 Example 2: first porosity and first porosity speed fields at final time for some values of ξ^*

is plotted in Figure 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

4.2 | Second example: Dependence on the parameter ξ^*

As a second example, we analyze the dependence on the solution with respect to parameter ξ^*. Then, we use the following data:

$$\begin{aligned}
&\ell = 1, \quad T = 1, \quad \rho = 1, \quad \mu = 2, \quad \gamma_1 = 1, \quad \gamma_2 = 1, \quad \beta = 1, \quad J_1 = 1, \quad J_2 = 1,\\
&b_{11} = 5, \quad b_{12} = 1, \quad b_{22} = 3, \quad m_1 = 1, \quad m_2 = 1, \quad \xi_{11} = 7, \quad \xi_{12} = 1, \quad \xi_{22} = 3,\\
&d_1 = 1, \quad d_2 = 1, \quad a = 1, \quad \kappa = 5, \quad k^* = 1,
\end{aligned}$$

and the initial conditions

$$u_0 = v_0 = \phi_{20} = e_{20} = \psi_0 = \theta_0 = 0, \quad \phi_{10} = e_{10} = x(x-1) \quad for \ \ x \in (0,1).$$

Taking the discretization parameters $\tau = h = 10^{-3}$, we assume that parameter ξ^* takes values $2, 1, 0.1, 0.01, 0.001$. Therefore, in Figures 3–6 we plot the obtained results at final time. As can be seen, a similar shape is found for all the variables although, as expected, the solutions seem to decrease when the viscosity parameter ξ^* increases.

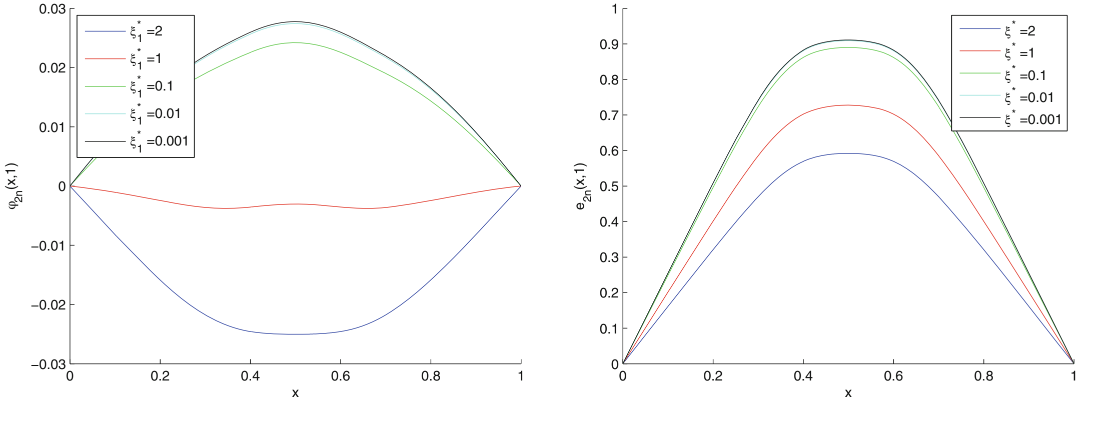

FIGURE 5 Example 2: second porosity and second porosity speed fields at final time for some values of ξ^*

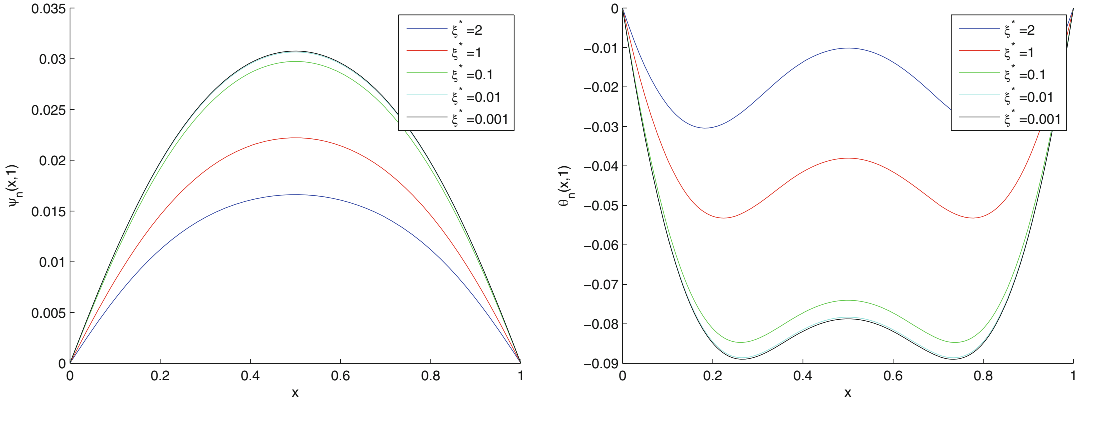

FIGURE 6 Example 2: thermal displacement and temperature at final time for some values of ξ^*

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CONFLICTS OF INTEREST

No conflict of interest exists. We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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