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On unicyclic graphs of metric dimension 2 with vertices of degree 4

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Abstract. We show that if G is a unicyclic graph with metric dimension 2 and $\{a, b\}$ is a metric basis of G then the degree of any vertex v of G is at most 4 and degrees of both a and b are at most 2. The constructions of unispider and semiunispider graphs and their knittings are introduced. Using these constructions all unicyclic graphs of metric dimension 2 with vertices of degree 4 are characterized.

Introduction

The concept of metric dimension of a connected graph was introduced by Harary and Melter [1]. Slater [2] described the usefulness of this concept in connection to the long range aids to navigation.

A metric dimension as a graph parameter has numerous applications. In particular, to the representation of chemical compounds [3] in chemistry and to the problems of pattern recognition and image processing. Some of them involve hierarchical data structures [4]. Also, it is used in sonar [2] and combinatorial optimization [5].

The metric dimension decision problem is among the classical NP-hard problems [6], but for some families of graphs, for example, for trees [7], the polynomial algorithm can be built. It is easy to see, that if G has n vertices

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then $1 \leqslant \dim G \leqslant n-1$. There are results describing graphs having a given value of metric dimension. In particular, in [1] it is proved that a graph has metric dimension 1 if and only if it is a path. Moreover, metric dimensions of the cycle C_n, the complete graph K_n and the complete bipartite graph $K_{s,t}$ are 2, $n-1$ and $n-2$, respectively [1], [8].

S. Khuller, B. Raghavachari and A. Rosenfeld [7] considered some properties of graphs with metric dimension 2. They proved that if $\{a, b\}$ is a metric basis of G with metric dimension 2 then degrees of both a and b are at most 3 and the maximum degree of any vertex u in the shortest path between a and b is 5. Later G. Sudhakara and A. R. Hemanth Kumar [9] evolved these properties and showed that the maximum degree of any vertex in a graph G with metric dimension 2 is 8. For unicyclic graphs, i.e. graphs containing exactly one cycle, these estimates can be determined more precisely. We prove that if G is a unicyclic graph with metric dimension 2 and $\{a, b\}$ is a metric basis of G then for any vertex v from G the following condition holds: $\deg(v) \leqslant 4$ and degrees of both a and b are at most 2. Moreover, a vertex v such that $\deg(v) = 4$ exists only for odd graphs.

In [10] and [11] all unicyclic graphs that have metric dimension 2 and vertices of degree at most 3 were widely studied.

In this paper we characterize all unicyclic graphs of metric dimension 2 with vertices of degree 4. For this purpose we introduce the constructions of unispider and semiunispider graphs and their knittings. We show that if a unicyclic graph G with $\dim G = 2$ has a vertex of degree 4 then G is unispider or a semiunispider graph or a knitting of one of graphs of such type.

1. Preliminaries

Throughout this article, we consider only simple, finite, undirected, connected and nontrivial graphs.

For a graph $G = (V, E)$ we define a *distance* $d_G(u, v)$ between two vertices u and v as the length of the shortest path between u and v; we put $d_G(u, v) = 0$ when $u = v$.

A vertex u of G *resolves* vertices v_1 and v_2 of G, if $d_G(u, v_1) \neq d_G(u, v_2)$.

A *resolving* set is an ordered vertex subset S of V such that every two distinct vertices of G are resolved by some vertex of S. A resolving set also is called a *metric generator*. A resolving set of minimum cardinality is said to be a *metric basis* of G. A *metric dimension*, $\dim G$, of G is the cardinality of its metric basis.

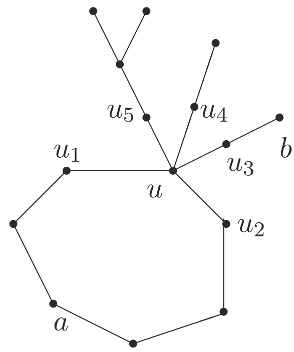

FIGURE 1.

We use the standard notation: by the symbols C_n and L_n we denote a cycle and a path on n vertices respectively. For unspecified notions in graph theory we refer to [12].

Denote by $\hat{G} = (\hat{V}, \hat{E})$ a subgraph of a unicyclic graph $G = (V, E)$, which is isomorphic to a cycle C_m for some positive integer m. The subgraph $\hat{G}$ is called the *cycle* of a graph G.

In [7] S. Khuller, B. Raghavachari and A. Rosenfeld considered some properties of graphs with metric dimension 2. In particular, they proved, that if $G = (V, E)$ is a graph with metric dimension 2 and $\{a, b\}$ is a metric basis in G, then the degrees of a and b are at most 3 and every other node u has the degree at most 5.

For unicyclic graphs there are some more accurate estimates.

Proposition 1 ([10]). *Let $G = (V, E)$ be a unicyclic graph. If metric dimension of G is 2, then $\deg_G(v) \leqslant 3$ for any $v \in V \setminus \hat{V}$.*

For the following proposition we need some definitions.

Following [10], a vertex $u \in V \setminus \hat{V}$ of a unicyclic graph G is said to be *projected* to a vertex $w \in \hat{V}$ if

$$d_G(u, w) < d_G(u, q)$$

for any $q \in \hat{V}$, $q \neq w$.

Proposition 2. *Let $G = (V, E)$ be a unicyclic graph. If the metric dimension of G is 2, then $\deg_G(v) \leqslant 4$ for any $v \in \hat{V}$.*

Proof. Assume that a unicyclic graph G with $\dim G = 2$ has a vertex u such that $\deg(u) \geqslant 5$. Let u_1, u_2, u_3, u_4 and u_5 be the vertices adjacent to u, and $u_1, u_2 \in \hat{V}$, $u_3, u_4, u_5 \in V \setminus \hat{V}$ (see Figure 1). Suppose that a metric basis of G consists of vertices a, b. As $\{a, b\}$ is a basis, vertices u_i, $1 \leqslant i \leqslant 5$, are resolved by a, b. But $u_1, u_2 \in \hat{V}$, so one of vertices a, b is projected to some vertex $w \in \hat{V}$, $w \neq u$ (may be $a = w$ or $b = w$). We may assert that $a \in \hat{V}$ or a is projected to w and resolves u_1 and u_2. Hence,

$$d_G(a, u_3) = d_G(a, u_4) = d_G(a, u_5) = d_G(a, u) + 1.$$

In other words, a does not resolve u_3, u_4, u_5. Then b is projected to u. Without loss of generality we may assert that the shortest path, which connects vertices b and u, contains u_3. But in this case, we have

$$d_G(b, u_4) = d_G(b, u_5) = d_G(b, u) + 1.$$

So, b does not resolve u_4, u_5. Therefore, our assumption is incorrect. □

Note, that from Proposition 1 and Proposition 2 it follows that if a unicyclic graph has metric dimension 2, then the degree of any vertex from a cycle of G is less than 5 and the degree of any vertex out of the cycle of G is less than 4. In the next section, we describe in detail the structure of such graphs.

Recall that the inner vertices are the vertices of degree at least 3. An inner vertex $v \in \hat{V}$ is called a *main vertex* if there is an inner vertex $w \in V \setminus \hat{V}$ that is projected to v.

An inner vertex v *is close to a leaf* a (see [2]) if there is no other inner vertex w in the unique path between v and a in G, i.e. for every other inner vertex w of G the inequality

$$d_G(a, v) < d_G(a, w)$$

holds. If an inner vertex v is close to two different leaves, then we say that v is a *two-leaf* vertex.

We need the lemma from [13].

Lemma 1 ([13])**.** *Let $G = (V, E)$ be a unicyclic graph and* $\dim G = 2$*. Then there exist at most two main vertices in G.*

2. Odd and even unicyclic graphs with vertices of degree 4

Following [10], a unicyclic graph G is *even*, if $|\hat{V}| = 2k$ for some positive integer k. If there is some positive integer k, such that $|\hat{V}| = 2k + 1$, then the unicyclic graph G is *odd*.

Theorem 1. *Let $G = (V, E)$ be a unicyclic graph and* $\dim G = 2$*. If G is even, then the degree of any vertex of G is less than 4. If G is odd, then the maximum number of its vertices of degree 4 is 2.*

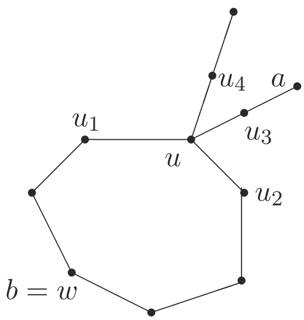

FIGURE 2.

Proof. Assume that a vertex u of G has the degree 4. Let u_1, u_2, u_3 and u_4 be the vertices adjacent to u, and $u_1, u_2 \in \hat{V}$, $u_3, u_4 \in V \setminus \hat{V}$ (see Figure 2). Denote by symbols $\{a, b\}$ the metric basis of G. As $\{a, b\}$ is a basis, the vertices a, b resolve u_i, $1 \leqslant i \leqslant 4$. But u_3 and u_4 are projected to the single vertex u. Then one of the basis vertices is also projected to u. Without loss of generality we may assume that it is a. We may assert that the shortest path, which connects a and u, contains u_3. Then

$$d_G(a, u_1) = d_G(a, u_2) = d_G(a, u_4) = d_G(a, u) + 1.$$

Hence, a does not resolve u_1, u_2, u_4. So, u_1, u_2, u_4 are resolved by b.

Let b is projected to w, $w \in \hat{V}$ (may be $b = w$). As

$$d_G(b, u_i) = d_G(b, w) + d_G(w, u_i), \qquad 1 \leqslant i \leqslant 4,$$

vertices u_1, u_2, u_4 are resolved by b if and only if they are resolved by w. Hence, w resolves vertices u_1, u_2, u_4, and $d_G(w, u_1) \neq d_G(w, u_2)$. Assume that $d_G(w, u_1) < d_G(w, u_2)$. If the shortest path, which connects w and u_2, contains u_1, then its length equals the length of the shortest path between w and u_4. In other words, in this case w does not resolve u_2 and u_4. Therefore, the shortest path between w and u_2 does not contain u_1. So, if m is a number of vertices in the cycle of G, we obtain

$$d_G(w, u_2) + d_G(w, u_1) + 2 = m.$$

Moreover, $d_G(w,u_2) < \frac{m}{2}$, $d_G(w,u_1) < \frac{m}{2}$. But, $d_G(w,u_1) < d_G(w,u_2)$, then from direct calculations we obtain that

$$\frac{m-2}{2} < d_G(w,u_2) < \frac{m}{2}.$$

If $m = 2k$, for some positive integer k, then

$$\frac{2k-2}{2} = k-1 < d_G(w,u_2) < k.$$

So, these inequalities never hold. Hence, if G is even, G does not have vertices of degree 4.

If $m = 2k+1$, for some positive integer k, then

$$\frac{2k+1-2}{2} = k - \frac{1}{2} < d_G(w,u_2) < k + \frac{1}{2}.$$

Hence,

$$d_G(w,u_2) = k, \quad d_G(w,u_1) = k-1, \quad d_G(w,u_4) = k+1.$$

Similarly, we can show that in this case w can also be a vertex of degree 4. Note in addition, if G has a vertex of degree 4, then one of the vertices from the metric basis of G is projected to this vertex. Therefore, if G is odd, then the maximum number of its vertices of degree 4 is 2. $\square$

3. Unispider and semiunispider graphs

Definition 1. An odd unicyclic graph G with $|\hat{V}| = 2k+1$ is said to be a *unispider graph* if the following conditions hold (see Figures 3–7):

1) for any vertex v from $V \setminus \hat{V}$, $\deg_G(v) \leqslant 3$;
2) for any main vertex w of G there exists exactly one two-leaf vertex, that is projected to w;
3) in the cycle $\hat{G}$ of G there are exactly two vertices w and u with the degree greater than 2, moreover, at least one of w and u has the degree 4, each of w and u is the main vertex or (and) a vertex of degree 4 and

$$d_G(w,u) = k.$$

Note, that from the definition of unispider graphs it follows that for any unispider graph G one of the following cases holds:

1) the vertices w and u are the main vertices of degree 4 (see Figure 3);

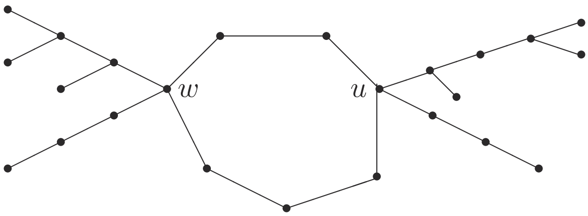

FIGURE 3. Two main vertices that are vertices of degree 4

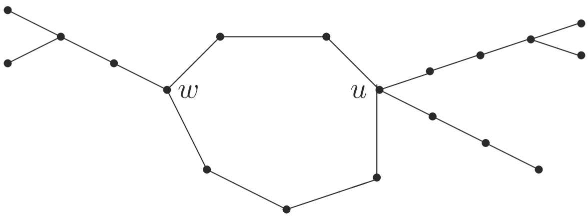

FIGURE 4. Two main vertices, one of them is vertex of degree 4

2) the vertices w and u are main vertices, but one of them is a vertex of degree 4 (see Figure 4);
3) the vertices w and u are vertices of degree 4, but only one of them is a main vertex (see Figure 5);
4) one of vertices w and u is a vertex of degree 4 and the other one is a main vertex (see Figure 6).
5) the vertices w and u are vertices of degree 4, but G does not have main vertices (see Figure 7);

Proposition 3. *Let $G = (V, E)$ be a unispider graph. Then*

$$\dim G = 2.$$

Proof. Let G be unispider graph with two main vertices u and w, that are vertices of degree 4 (see Figure 8). For the other types of unispider graphs proof is very similar. Denote the leaves that are projected to u and w by a and b, respectively. Moreover, if there are inner vertices that are projected to u, then one of them is close to a. Similar, if there are inner vertices that are projected to w, then one of them is close to b (see Figure 8).

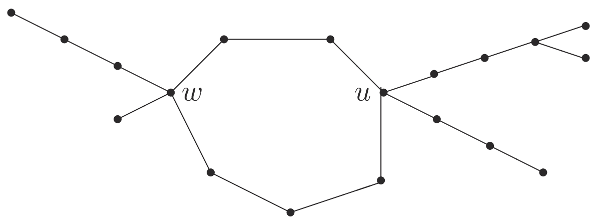

Figure 5. Two vertices of degree 4, one of them is the main vertex

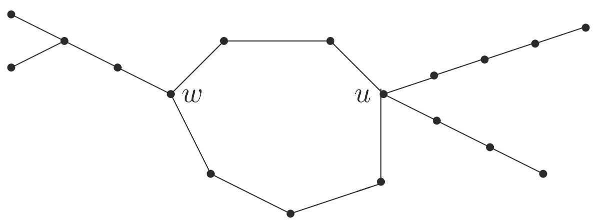

Figure 6. One main vertex and one vertex of degree 4

We need to show, that $\{a, b\}$ is a metric basis of G.

Note, that a resolves all vertices that are adjacent to b, and vice versa.

In addition, if some pair of vertices from a subgraph $\hat{G}$ is not resolved by the vertex b, say w_3 and w_4, then this pair is resolved by a, because G is odd, $|\hat{V}| = 2k+1$, a and b are projected to u and w, respectively, and $d_G(w, u) = k$ by the definition of unispider graphs. Similar, if some pair of vertices from $\hat{G}$ is not resolved by a, say u_3 and u_4, then this pair is resolved by b. Moreover, from the proof of Theorem 1 it follows that b resolves any pair of vertices from the set $\{u_1, u_2, u_3, u_4\}$ and a resolves any pair of vertices from the set $\{w_1, w_2, w_3, w_4\}$. So, any pair of vertices from G is resolved by a or b. Therefore, $\{a, b\}$ is a metric basis of G. $\square$

Definition 2. An odd unicyclic graph G with $|\hat{V}| = 2k+1$ is said to be a *semiunispider graph* if the following conditions hold (see Figures 9 and 10):

1) $\deg_G(v) \leqslant 3$ for any v from $V \setminus \hat{V}$;

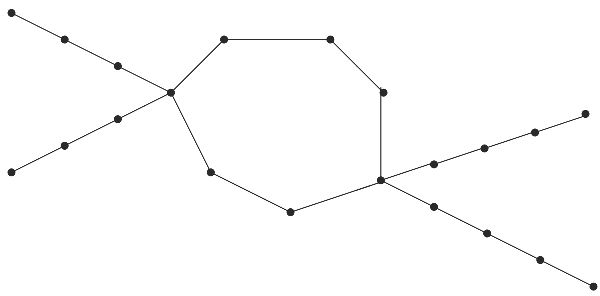

FIGURE 7. There are two vertices of degree 4, but the graph without main vertices

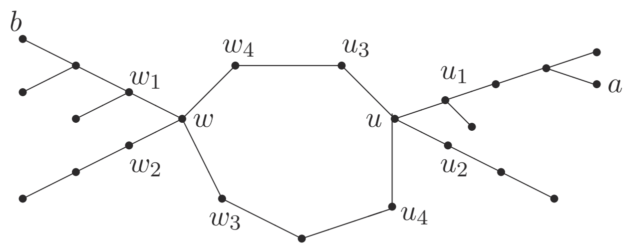

FIGURE 8.

2) in the cycle $\hat{G}$ of G there is exactly one vertex w with the degree greater than two, moreover, $\deg(w) = 4$;
3) the vertex w may be the main vertex, in this case there exists exactly one two-leaf vertex, that is projected to w.

Proposition 4. *Let $G = (V, E)$ be a semiunispider graph. Then*

$$\dim G = 2.$$

Proof. Let G be a semiunispider graph with one main vertex w of degree 4. For the other type of semiunispider graphs the proof is similar. Denote the leaf that is projected to w by a (see Figure 9). Some inner vertex is close to a. Let b be a vertex of cycle $\hat{V}$ such that $d_G(w, b) = k$.

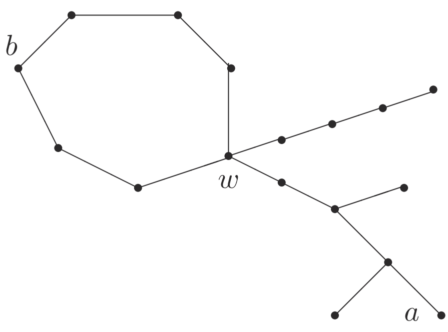

Figure 9. Semiunispider graphs with the main vertex

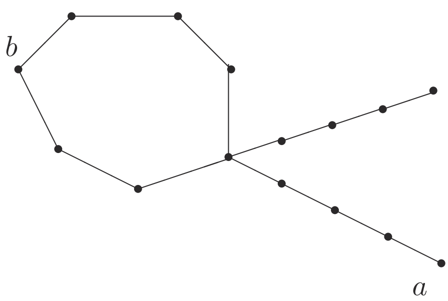

Figure 10. Semiunispider graphs without the main vertex

Similarly to the proof of Proposition 3 the set of vertices $\{a, b\}$ is a metric basis of G. $\square$

4. The main result

The main result is based on the notion of a *knitting* of a graph ([10]). To introduce it we need the following definition.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be simple graphs. Assume that $v_1 \in V_1$ and $v_2 \in V_2$. Define the equivalence relation $\sim$ on the set $V_1 \cup V_2$ by the following way: the vertex u is equivalent to the vertex w if and only if $v_1 = u$ and $v_2 = w$ for all $u \in V_1$ and $w \in V_2$.

A graph $G = (V_1 \cup V_2 \backslash \sim, E_1 \cup E_2)$ is built from G_1 and G_2 by *gluing* along vertices v_1 and v_2. Roughly speaking, the vertex v_2 is replaced by the vertex v_1 for all edges of G_2.

Definition 3. Let G_1 be a unispider graph. Denote by u and w the vertices of the degree greater than 2 of G_1. A unicyclic graph G is called a *knitting* of G_1 by chains $L_1, \dots, L_r$ if G is obtained from G_1 by gluing vertices of degree 2 from the cycle of G_1 and the leaves of chains $L_1, \dots, L_r$ such that any vertex of degree 2 from the cycle G_1 may be glued with one leaf of one chain L_j, $1 \leqslant j \leqslant r$ (see Figure 11).

Note that, if G_1 is a unispider graph with one vertex u of degree 4, then we can glue the vertex w of degree 3 with the leaf of the chain L_0. In this case we have a knitting of the unispider graph G_2 with two vertices of degree 4 (G_2 is from Figures 3 or 5) by chains $L_1, \dots, L_r$.

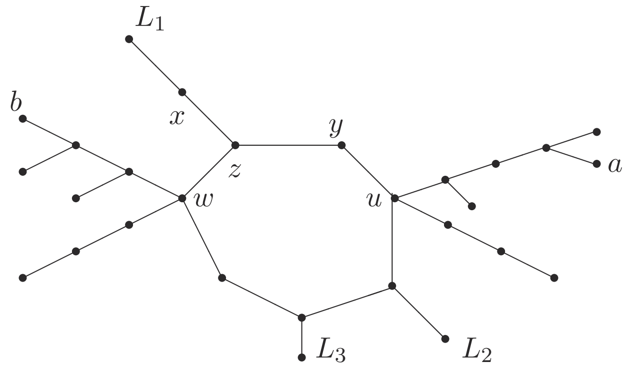

FIGURE 11. Knitting of a unispider graph

Let now G_1 be a semiunispider graph with the vertex w of degree 4. Assume that b is a vertex from a cycle of G_1 such that $\deg(b) = 2$ and $d_{G_1}(w, b) = k$, where $2k + 1$ is the number of vertices in the cycle of G_1.

Definition 4. A unicyclic graph G is called a *knitting* of G_1 by chains $L_1, \dots, L_r$ if G is obtained from G_1 by gluing vertices with degree 2 from the cycle of G_1 and leaves of the chains $L_1, \dots, L_r$ such that any vertex with degree 2 of the cycle of G_1 may be glued with the leaf of exactly one chain.

Note that, if the vertex b is glued with the leaves of two chains, then graph G is a knitting of the unispider graph G_1 (see Figure 5 or Figure 7) by some chains.

The main result of this paper is the following.

Theorem 2. *Let* $G = (V, E)$ *be a unicyclic graph with vertices of degree* 4. *Then* $\dim G = 2$ *if and only if one of the following conditions holds:*

(1) *G is a unispider graph;*
(2) *G is a knitting of some unispider graph;*
(3) *G is a semiunispider graph;*
(4) *G is a knitting of some semiunispider graph;*

Proof. Let $G = (V, E)$ be a unicyclic graph with vertices of degree 4. We need to show that if one of the conditions (1)–(4) holds then $\dim G = 2$. From Propositions 3 and 4 it follows that unispider and semiunispider graphs have metric dimension 2. So, if G satisfies (1) or (3) then $\dim G = 2$.

Let G be a knitting of some unispider graph. Denote the vertices of G with degree 4 by u and w and let a and b be the leaves that are projected to u and w, respectively (see Figure 11).

As $d_G(w, u) = k$ and $2k+1$ is the number of all vertices in the cycle $\hat{G}$ of G, a and b resolve all pairs of vertices from $\hat{G}$. Moreover, from the proofs of Propositions 3 and 4 it follows that a resolves all the pairs vertices that are adjacent to b, and vice versa.

Let a chain L_j, $1 \leqslant j \leqslant n$ be glued with unicyclic graph G in point z. Without loss of generality, we may claim that this is L_1 (see Figure 11). Assume that vertices x, y and w are adjacent to z. Moreover, x belongs to L_1, and y is a vertex of $\hat{G}$. Then x and y are not resolved by the basis vertex b. But they are resolved by another basis vertex a. Hence, any pair of vertices from G is resolved by a or b. Therefore, $\{a, b\}$ is a metric basis of G.

Let G be a knitting of some semiunispider graph, i. e. G satisfies condition (4). Assume that w is a vertex of G with $\deg(w) = 4$. Denote the leaf that is projected to w by a. Let b be a vertex of the cycle $\hat{V}$ such that $d_G(w, b) = k$. Similarly to the case of a unispider graph, $\{a, b\}$ is a metric basis of G.

Therefore, if G is a knitting of a unispider or semiunispider graph, then $\dim G = 2$.

Now let $G = (V, E)$ be a unicyclic graph with vertices of degree 4 and $\dim G = 2$. From Theorem 1 it follows that G is odd and the maximum number of vertices of degree 4 is two. From Lemma 1 (see [13]) we have, that G can contain at most two main vertices. Moreover, from the proofs of Lemma 1 and Theorem 1 it follows that a and b are projected to main vertices and to vertices of degree 4, if they exist. Note that G has no less than one vertex of degree 4, then if basis vertices are projected to u and w, then u or w (may be u and w) has degree 4 and

$$d_G(w, u) = k.$$

Hence, G is unispider, or semiunispider, or a knitting of some unispider graph, or a knitting of some semiunispider graph.

For example, if G has exactly two vertices u and w of degree 4 and two main vertices, then any vertex of degree 4 is the main vertex. Hence, G is a unispider graph or its knitting (see Figure 3, Figure 11).

For the other number of vertices of degree 4 and main vertices the proof is very similar. $\square$

Corollary 1. *Let $G = (V, E)$ be a unicyclic graph with vertices of degree 4, and let $\{a, b\}$ be a metric basis of G. Then the degrees of a and b are no greater than 2.*

The proof of Corollary 1 directly follows from the proof of Theorem 2.

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