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Endomorphisms of Cayley digraphs of rectangular groups

Srichan Arworn, Boyko Gyurov, Nuttawoot Nupo, and Sayan Panma

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Abstract. Let $\mathrm{Cay}(S,A)$ denote the Cayley digraph of the semigroup S with respect to the set A, where A is any subset of S. The function $f : \mathrm{Cay}(S,A) \to \mathrm{Cay}(S,A)$ is called an endomorphism of $\mathrm{Cay}(S,A)$ if for each $(x,y) \in E(\mathrm{Cay}(S,A))$ implies $(f(x), f(y)) \in E(\mathrm{Cay}(S,A))$ as well, where $E(\mathrm{Cay}(S,A))$ is an arc set of $\mathrm{Cay}(S,A)$. We characterize the endomorphisms of Cayley digraphs of rectangular groups $G \times L \times R$, where the connection sets are in the form of $A = K \times P \times T$.

1. Introduction

Hereafter, all sets mentioned in this paper are considered to be finite. For any semigroup S and a subset A of S, the Cayley digraph of S with respect to the set A, denoted by $\mathrm{Cay}(S,A)$, is defined as the digraph with the vertex set S and the arc set $E(\mathrm{Cay}(S,A)) = \{(x,xa) | x \in S, a \in A\}$ (see [6]). The concept of Cayley graphs of groups is introduced by Arthur Cayley in 1878. Many interesting results about Cayley graphs of groups have been obtained and widely studied by various authors (see, for example, [2], [10], [11], and [12]). In addition, Cayley digraphs of semigroups have been considered and many new interesting results are also shown in several journals. The class of rectangular groups is one of the famous classes

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of semigroups and their Cayley digraphs have been studied seriously. Moreover, some properties of the Cayley digraphs of rectangular groups, left groups, and right groups are obtained by many authors (see, for example, [9], [7], [13], [14], [15], [16], and [17]).

The structures of endomorphisms of Cayley digraphs of semigroups are interesting to study. Many authors have studied some results of Cayley digraphs of semigroups by using the properties of homomorphisms (see for examples, [1], [4], [10], [17], and [18]). Here we shall study the structures and give the characterizations of endomorphisms of Cayley digraphs of rectangular groups, right groups, and left groups, respectively.

Note that the Cayley digraph of a rectangular group $G \times L \times R$ with respect to a set $A = K \times P \times T$ is the disjoint union of $|L|$ mutually isomorphic subdigraphs, where each subdigraph is isomorphic to the Cayley digraph of the right group $G \times R$ with respect to the set $\{(a, \alpha) \in G \times R | (a, l, \alpha) \in A$ for some $l \in L\}$. So we are specially interested in the structure of endomorphisms of the Cayley digraph of a right group, where the connection set is in the form of the cartesian product of sets.

As the fact that a left group $G \times L$ is one of the special cases of a rectangular group $G \times L \times R$ when $|R| = 1$, it makes sense to consider their Cayley digraphs. Actually, Cayley digraphs of left groups are the disjoint union of isomorphic copies of Cayley digraphs of groups while Cayley digraphs of right groups are not. Moreover, Cayley digraphs of right groups look more complicated than Cayley digraphs of left groups. Thus we are attentive to characterize endomorphisms of Cayley digraphs of left groups with respect to arbitrary connection sets.

The relevant notations and some terminologies related to our paper will be given in the next section.

2. Preliminaries and notations

In this section, some preliminaries needed in what follows on digrahs and semigroups are given. For more information on digraphs, we refer to [3], and for semigroups see [5]. A semigroup S is called a *left (right) zero semigroup* if $xy = x$ ($xy = y$) for all $x, y \in S$. A semigroup S is said to be a *left (right) group* if it is isomorphic to the direct product $G \times L$ ($G \times R$) of a group G and a left (right) zero semigroup $L(R)$.

A semigroup S is called a *band* if every element in S is idempotent. A *rectangular band* is a band S that satisfies $xyx = x$ for all $x, y \in S$. In fact, there is another classification of rectangular bands. A semigroup S is said to be a rectangular band if it is isomorphic to the direct product

$L \times R$ of a left zero semigroup L and a right zero semigroup R. Moreover, a semigroup S is called a *rectangular group* if it is isomorphic to the direct product $G \times L \times R$ of a group G and a rectangular band $L \times R$. It is obvious that a left (right) zero semigroup, a left (right) group, and a rectangular band are all rectangular groups.

A *digraph (directed graph)* $D = (V, E)$ is a set $V = V(D)$ of vertices together with a binary relation $E = E(D)$ on V. The elements $e = (u, v)$ of E are called the *arcs* of D (see [3]).

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. As in [8], a *digraph homomorphism* $f : D_1 \to D_2$ is a mapping $f : V_1 \to V_2$ such that $(u, v) \in E_1$ implies $(f(u), f(v)) \in E_2$ for all $u, v \in V_1$. In other words, the digraph homomorphism f is also said to be *edge-preserving*. The digraph homomorphism $f : D \to D$ is called an *endomorphism* of D and we denote by $End(D)$ the monoid of all endomorphisms of D.

From now on, $|A|$ denotes the cardinality of A, where A is any finite set and p_i denotes the projection map on the i^{th} coordinate of a triple where $i \in \{1, 2, 3\}$. A subdigraph F of a digraph G is called a strong subdigraph of G if and only if whenever u and v are vertices of F and (u, v) is an arc in G, then (u, v) is an arc in F as well.

3. Main results

In this section, we present the results about endomorphisms of Cayley digraphs of some rectangular groups. We divide this section into three parts. In the first part, we give some results for endomorphisms of Cayley digraphs of rectangular groups and the remaining parts show some characterizations of endomorphisms of Cayley digraphs of right groups and left groups, respectively.

In order to study the structure of endomorphisms of Cayley digraphs of rectangular groups, we need to prescribe some notations used in what follows. For any digraph Ω and $X \subseteq V(\Omega)$, by $[X]$, we mean the strong subdigraph of Ω induced by X. For each function $f : \Omega_1 \longrightarrow \Omega_2$ from a digraph Ω_1 to a digraph Ω_2 and any subdigraph Σ of Ω_1, we mention $f(\Sigma)$ as a strong subdigraph $[f(V(\Sigma))]$ of Ω_2 induced by $f(V(\Sigma))$. We now study the endomorphisms of Cayley digraphs of rectangular groups.

3.1. Endomorphisms of Cayley digraphs of rectangular groups

Throughout this part, we let $S = G \times L \times R$ be a rectangular group and $D = \mathrm{Cay}(S, A)$ the Cayley digraph of the semigroup S with respect to the set A. Before we give some results of endomorphisms of Cayley digraphs of rectangular groups, we first define a useful function using in the sequel.

Let $f : S \to S$ be a function and $l \in L$. For each $\alpha \in R$, we define $\Phi_{l\alpha} : G \to G$ by

$$\Phi_{l\alpha}(a) = b \text{ if there exist } t \in L \text{ and } \beta \in R$$
$$\text{such that } f(a, l, \alpha) = (b, t, \beta) \text{ for all } a \in G.$$

It is easy to verify that $\Phi_{l\alpha}$ is well-defined. We now present some results about endomorphisms of Cayley digraphs of rectangular groups with respect to the given connection sets.

Theorem 3.1. Let $A = K \times P \times T$ be the connection set of D and $f : S \to S$ be a function. Then $f \in \mathrm{End}(D)$ if and only if for each $l \in L$, the following conditions hold:

(i) $f([b\langle K\rangle \times \{l\} \times R])$ is a subdigraph of $[c\langle K\rangle \times \{t\} \times R]$ for some $t \in L$, $c \in G$ and for all $b \in G$;

(ii) $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$ for all $\alpha \in T$;

(iii) for each $g \in K$ and $a \in G$, there exists $g_a \in K$ such that

$$f(ag^{-1}, l, \theta) \in \begin{cases} \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times T & \text{if } \theta \in T, \\ \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times R & \text{if } \theta \in R \setminus T \end{cases}$$

for all $\lambda \in T$ and for some $u \in L$.

Proof. Let $A = K \times P \times T$ be the connection set of D, $l \in L$ and $f : S \to S$ be a function.

($\Rightarrow$) Assume that $f \in \mathrm{End}(D)$. Let $b \in G$. We first show that $f([b\langle K\rangle \times \{l\} \times R])$ is a subdigraph of $[c\langle K\rangle \times \{t\} \times R]$ for some $t \in L$ and$c \in G$. Let $(g, p, r), (h, k, s)$ be two vertices of $f([b\langle K\rangle \times \{l\} \times R])$ such that $(g, p, r) \in c\langle K\rangle \times \{t\} \times R$ and $(h, k, s) \in d\langle K\rangle \times \{u\} \times R$ for some $t, u \in L$ and $c, d \in G$. Thus $p = t$ and $k = u$ and hence

$$(g, t, r) = (g, p, r) = f(g', p', r')$$

for some $(g', p', r') \in b\langle K\rangle \times \{l\} \times R$ and

$$(h, u, s) = (h, k, s) = f(h', k', s')$$

for some $(h', k', s') \in b\langle K\rangle \times \{l\} \times R$. Then $p' = l = k'$. Since

$$((g', l, r'), (g', l, r')(a, i, x)) \in E(D),$$

where $(a, i, x) \in A$ and $f \in \mathrm{End}(D)$, we have

$$(f(g', l, r'), f(g'a, l, x)) \in E(D).$$

Similarly, we conclude that $((h', l, s'), (h', l, s')(a, i, x)) \in E(D)$ implies $(f(h', l, s'), f(h'a, l, x)) \in E(D)$. From $g', h' \in b\langle K\rangle$ and $a \in K$, we get $g'a, h'a \in b\langle K\rangle$. Consider the strong subdigraph $[b\langle K\rangle \times \{l\} \times \{x\}]$ of D. Since $[b\langle K\rangle \times \{l\} \times \{x\}]$ is isomorphic to $\mathrm{Cay}(\langle K\rangle, K)$, and $\mathrm{Cay}(\langle K\rangle, K)$ is connected, we obtain that there exists a dipath connecting between $(g'a, l, x)$ and $(h'a, l, x)$, say the dipath M. We may assume that

$$M := (g'a, l, x), m_1, m_2, \ldots, m_q, (h'a, l, x),$$

where $m_j \in b\langle K\rangle \times \{l\} \times \{x\}$ and $j = 1, 2, \ldots, q$. Since $f \in \mathrm{End}(D)$, we have

$$f(g'a, l, x), f(m_1), f(m_2), \ldots, f(m_d), f(h'a, l, x)$$

is a diwalk in D. Hence there exists a semi-diwalk connecting between $f(g', p', r')$ and $f(h', k', s')$. Since $[c\langle K\rangle \times \{t\} \times R]$ and $[d\langle K\rangle \times \{u\} \times R]$ are maximal semi-connected subdigraphs of D, we conclude that

$$[c\langle K\rangle \times \{t\} \times R] = [d\langle K\rangle \times \{u\} \times R],$$

that is, $t = u$. Therefore, $V(f([b\langle K\rangle \times \{l\} \times R])) \subseteq c\langle K\rangle \times \{t\} \times R$. We now let $((g_1, t_1, r_1), (g_2, t_2, r_2)) \in E(f([b\langle K\rangle \times \{l\} \times R]))$. Then

$$(g_1, t_1, r_1), (g_2, t_2, r_2) \in V(f([b\langle K\rangle \times \{l\} \times R])) \subseteq c\langle K\rangle \times \{t\} \times R.$$

Thus $((g_1, t_1, r_1), (g_2, t_2, r_2)) \in E([c\langle K\rangle \times \{t\} \times R])$ since $[c\langle K\rangle \times \{t\} \times R]$ is a strong subdigraph of D. Consequently, $f([b\langle K\rangle \times \{l\} \times R])$ is a subdigraph of $[c\langle K\rangle \times \{t\} \times R]$.

Next, we will prove that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$ for all $\alpha \in T$. Let $\alpha \in T$ and $(x, y) \in E(\mathrm{Cay}(G, K))$. Thus $y = xa$ for some $a \in K$. Assume that $\Phi_{l\alpha}(x) = u$ and $\Phi_{l\alpha}(y) = v$ for some $u, v \in G$. Then $f(x, l, \alpha) = (u, k, \beta)$ and $f(y, l, \alpha) = (v, q, \gamma)$ for some $k, q \in L$ and $\beta, \gamma \in R$. Since

$$\begin{aligned}((x, l, \alpha), (y, l, \alpha)) &= ((x, l, \alpha), (xa, l, \alpha))\\ &= ((x, l, \alpha), (x, l, \alpha)(a, p, \alpha)) \in E(D),\end{aligned}$$

where $(a, p, \alpha)) \in A$ and $f \in \mathrm{End}(D)$, we have $(f(x, l, \alpha), f(y, l, \alpha)) \in E(D)$, that is, $f(y, l, \alpha) = f(x, l, \alpha)(b, m, \lambda)$ for some $(b, m, \lambda) \in A$. Hence

$$(v, q, \gamma) = f(y, l, \alpha) = f(x, l, \alpha)(b, m, \lambda) = (u, k, \beta)(b, m, \lambda) = (ub, k, \lambda).$$

We obtain that $v = ub$ that means $\Phi_{l\alpha}(y) = v = ub = \Phi_{l\alpha}(x)b$, where $b \in K$. Therefore, $(\Phi_{l\alpha}(x), \Phi_{l\alpha}(y)) \in E(\mathrm{Cay}(G,K))$ and then $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G,K))$.

Now, we will prove (iii). Let $\lambda \in T$ and $\theta \in R$. For each $g \in K$ and $a \in G$, consider $(ag^{-1}, l, \theta) \in S$. Since $(a, l, \lambda) = (ag^{-1}, l, \theta)(g, p, \lambda)$, where $(g, p, \lambda) \in A$, we get $((ag^{-1}, l, \theta), (a, l, \lambda)) \in E(D)$. Because $f \in \mathrm{End}(D)$, we obtain that $(f(ag^{-1}, l, \theta), f(a, l, \lambda)) \in E(D)$. We may assume that $f(ag^{-1}, l, \theta) = (h, u, \delta)$ for some $(h, u, \delta) \in S$. Then there exists $(g_a, i, \mu) \in A$ such that

$$f(a, l, \lambda) = f(ag^{-1}, l, \theta)(g_a, i, \mu) = (h, u, \delta)(g_a, i, \mu) = (hg_a, u, \mu).$$

Hence $\Phi_{l\lambda}(a) = hg_a$. Therefore, $f(ag^{-1}, l, \theta) = (h, u, \delta) = (hg_a g_a^{-1}, u, \delta) = (\Phi_{l\lambda}(a)g_a^{-1}, u, \delta)$.

If $\theta \in T$, then $(g, p, \theta) \in A$. Since $(ag^{-1}, l, \theta) = (ag^{-2}, l, \theta)(g, p, \theta)$, we obtain that $((ag^{-2}, l, \theta), (ag^{-1}, l, \theta)) \in E(D)$. Since $f \in \mathrm{End}(D)$, we have $(f(ag^{-2}, l, \theta), f(ag^{-1}, l, \theta)) \in E(D)$. Suppose that $f(ag^{-2}, l, \theta) = (c, e, \varepsilon)$ for some $(c, e, \varepsilon) \in S$. Hence

$$f(ag^{-1}, l, \theta) = f(ag^{-2}, l, \theta)(m, w, \eta) = (c, e, \varepsilon)(m, w, \eta) = (cm, e, \eta)$$

for some $(m, w, \eta) \in A$. So we can conclude that $(\Phi_{l\lambda}(a)g_a^{-1}, u, \delta) = (cm, e, \eta)$ and hence $\delta = \eta \in T$. Therefore,

$$f(ag^{-1}, l, \theta) \in \begin{cases} \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times T & \text{if } \ \theta \in T, \\ \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times R & \text{if } \ \theta \in R \setminus T. \end{cases}$$

($\Leftarrow$) Suppose that the conditions hold. We will prove that $f \in \mathrm{End}(D)$. Let $((a, l, \rho), (b, j, \lambda)) \in E(D)$. Thus there exists $(k, p, t) \in A$ such that $(b, j, \lambda) = (a, l, \rho)(k, p, t) = (ak, l, t)$. Then $b = ak$, $j = l$ and $\lambda = t \in T$. Since $(a, b) = (a, ak) \in E(\mathrm{Cay}(G,K))$ and $\Phi_{l\lambda} \in \mathrm{End}(\mathrm{Cay}(G,K))$, we get that $(\Phi_{l\lambda}(a), \Phi_{l\lambda}(b)) \in E(\mathrm{Cay}(G,K))$. Hence $\Phi_{l\lambda}(b) = \Phi_{l\lambda}(a)c$ for some $c \in K$. By condition (iii), there exist $u \in L,\ \mu \in R$ and $q \in K$ in which

$$\begin{aligned} f(a, l, \rho) &= f(akk^{-1}, l, \rho) = (\Phi_{l\lambda}(ak)q^{-1}, u, \mu) \\ &= (\Phi_{l\lambda}(b)q^{-1}, u, \mu) = (\Phi_{l\lambda}(a)cq^{-1}, u, \mu). \end{aligned}$$

By the definition of $\Phi_{l\lambda}$, there exist $m \in L$ and $\omega \in R$ such that $f(b, j, \lambda) = f(b, l, \lambda) = (\Phi_{l\lambda}(b), m, \omega)$. Since $\lambda = t \in T$, again by condition (iii), we can conclude that there exists $s \in K$ such that $f(b, j, \lambda) = f(bnn^{-1}, j, \lambda) = (\Phi_{j\lambda}(bn)s^{-1}, v, \xi)$ for some $n \in K$, $v \in L$ and $\xi \in T$. Thus $\omega = \xi \in T$

and hence $f(b,j,\lambda) = (\Phi_{l\lambda}(b), m, \omega) = (\Phi_{l\lambda}(a)c, m, \omega)$. Since $j = l$ and $((a,l,\rho),(b,j,\lambda)) \in E(D)$, we gain that $(a,l,\rho),(b,j,\lambda) \in g\langle K\rangle \times \{l\} \times R$ for some $g \in G$. We get that $f(a,l,\rho), f(b,j,\lambda) \in V(f([g\langle K\rangle \times \{l\} \times R]))$. Since $f([g\langle K\rangle \times \{l\} \times R])$ is a subdigraph of $[h\langle K\rangle \times \{p\} \times R]$ for some $h \in G$ and $p \in L$, both of $f(a,l,\rho), f(b,j,\lambda)$ must belong to the vertex set of the same strong subdigraph of D. From $f(a,l,\rho) = (\Phi_{l\lambda}(a)cq^{-1}, u, \mu)$ and $f(b,j,\lambda) = (\Phi_{l\lambda}(b), m, \omega)$, we can conclude that $f(a,l,\rho), f(b,j,\lambda) \in d\langle K\rangle \times \{u\} \times R$ for some $d \in G$ that means $m = u$. For fixed $y \in P$, we have $(q,y,\omega) \in K \times P \times T = A$ and then

$$\begin{aligned} f(b,j,\lambda) &= (\Phi_{l\lambda}(a)c, m, \omega) = (\Phi_{l\lambda}(a)c, u, \omega) \\ &= (\Phi_{l\lambda}(a)cq^{-1}, u, \mu)(q,y,\omega) \\ &= f(a,l,\rho)(q,y,\omega). \end{aligned}$$

Hence $(f(a,l,\rho), f(b,j,\lambda)) \in E(D)$ and thus $f \in \mathrm{End}(D)$, as required. $\square$

Now, we will illustrate an example of an endomorphism of the Cayley digraph of a rectangular group with respect to the set A as stated in Theorem 3.1 and indicate that the endomorphism satisfies three conditions as shown in Theorem 3.1.

Example 3.2. Let $D = \mathrm{Cay}(\mathbb{Z}_3 \times \{l,k\} \times \{\alpha,\beta\}, A)$, where $A = \{1\} \times \{l\} \times \{\alpha\}$.

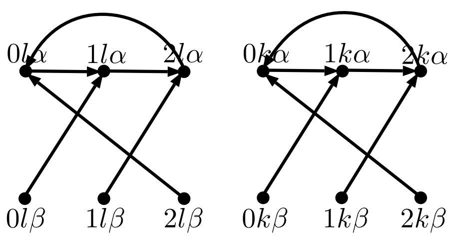

Figure 1. $\mathrm{Cay}(\mathbb{Z}_3 \times \{l,k\} \times \{\alpha,\beta\}, A)$.

We obtain that

$$f = \begin{pmatrix} 0l\alpha & 1l\alpha & 2l\alpha & 0k\alpha & 1k\alpha & 2k\alpha & 0l\beta & 1l\beta & 2l\beta & 0k\beta & 1k\beta & 2k\beta \\ 1k\alpha & 2k\alpha & 0k\alpha & 2l\alpha & 0l\alpha & 1l\alpha & 1k\beta & 2k\alpha & 0k\alpha & 2l\beta & 0l\beta & 1l\beta \end{pmatrix} \in \mathrm{End}(D).$$

From $p_1(A) = \{1\}$, we have $\langle p_1(A)\rangle = \{0,1,2\}$. Consider $f(\langle p_1(A)\rangle \times \{l\} \times \{\alpha,\beta\}) = \{1k\alpha, 2k\alpha, 0k\alpha, 1k\beta\}$, we get that the digraph $f([\langle p_1(A)\rangle \times \{l\} \times$

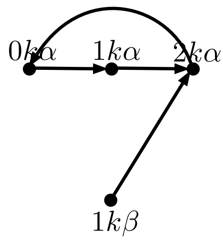

FIGURE 2. Digraph $[\{0k\alpha, 1k\alpha, 2k\alpha, 1k\beta\}]$.

$\{\alpha,\beta\}]) = [\{1k\alpha, 2k\alpha, 0k\alpha, 1k\beta\}]$ shown in *Figure* 2 is the subdigraph of $[\langle p_1(A)\rangle \times \{k\} \times \{\alpha,\beta\}]$.

Similarly, we can observe that $f([\langle p_1(A)\rangle \times \{k\} \times \{\alpha,\beta\}])$ is a subdigraph of $[\langle p_1(A)\rangle \times \{l\} \times \{\alpha,\beta\}]$. Moreover, we have

$$\Phi_{l\alpha} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \Phi_{k\alpha} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

and both of them are endomorphisms of $\mathrm{Cay}(\mathbb{Z}_3, \{1\})$. In addition, f satisfies the condition *(iii)* in *Theorem 3.1* as shown as follows:

$$\begin{aligned}
f(0+1^{-1}, l, \alpha) &= f(2, l, \alpha) = (0, k, \alpha) = (1+2, k, \alpha) = (\Phi_{l\alpha}(0)+1^{-1}, k, \alpha),\\
f(1+1^{-1}, l, \alpha) &= f(0, l, \alpha) = (1, k, \alpha) = (2+2, k, \alpha) = (\Phi_{l\alpha}(1)+1^{-1}, k, \alpha),\\
f(2+1^{-1}, l, \alpha) &= f(1, l, \alpha) = (2, k, \alpha) = (0+2, k, \alpha) = (\Phi_{l\alpha}(2)+1^{-1}, k, \alpha),\\
f(0+1^{-1}, k, \alpha) &= f(2, k, \alpha) = (1, l, \alpha) = (2+2, l, \alpha) = (\Phi_{k\alpha}(0)+1^{-1}, l, \alpha),\\
f(1+1^{-1}, k, \alpha) &= f(0, k, \alpha) = (2, l, \alpha) = (0+2, l, \alpha) = (\Phi_{k\alpha}(1)+1^{-1}, l, \alpha),\\
f(2+1^{-1}, k, \alpha) &= f(1, k, \alpha) = (0, l, \alpha) = (1+2, l, \alpha) = (\Phi_{k\alpha}(2)+1^{-1}, l, \alpha).
\end{aligned}$$

Similarly,for each $t \in \{l, k\}$, we have $f(x+1^{-1}, t, \beta) \in \{\Phi_{t\alpha}(x)+1^{-1}\} \times \{u\} \times R$ for all $x \in \mathbb{Z}_3$ and for some $u \in \{l, k\}$.

The next proposition describes the relation between two useful mappings for studying an endomorphism of the Cayley digraph of a rectangular group with respect to the set mentioned in the above theorem via an edge-preserving property. Before we show the result, we need to define the notation for convenience to use in the proof.

Notation 3.3. Let $f : D \to D$, $l \in L$ and $\alpha \in R$. We denote the restriction function $f_{|_{G\times\{l\}\times\{\alpha\}}} : [G \times \{l\} \times \{\alpha\}] \to D$ by $f_{Gl\alpha}$.

Proposition 3.4. Let $A = K \times P \times T$ be the connection set of D. For each $l \in L$ and $\alpha \in T$, $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$ if and only if $p_1 \circ f_{Gl\alpha} : [G \times \{l\} \times \{\alpha\}] \to \mathrm{Cay}(G, K)$ is a homomorphism.

Proof. Let $l \in L$ and $\alpha \in T$. Suppose that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$. Let $g, h \in G$ be such that $((g, l, \alpha), (h, l, \alpha)) \in E(D)$. Then there exists $(a, q, \lambda) \in A$ such that $(h, l, \alpha) = (g, l, \alpha)(a, q, \lambda) = (ga, l, \lambda)$, that is, $h = ga$, where $a \in K$. This implies that $(g, h) \in E(\mathrm{Cay}(G, K))$. Since $\Phi_{l\alpha}$ is an endomorphism of $\mathrm{Cay}(G, K)$, we have $(\Phi_{l\alpha}(g), \Phi_{l\alpha}(h)) \in E(\mathrm{Cay}(G, K))$. We may assume that $\Phi_{l\alpha}(g) = x$ and $\Phi_{l\alpha}(h) = y$ for some $x, y \in G$. Thus

$$f_{Gl\alpha}(g, l, \alpha) = f(g, l, \alpha) = (x, t, \mu)$$

and

$$f_{Gl\alpha}(h, l, \alpha) = f(h, l, \alpha) = (y, s, \eta)$$

for some $s, t \in L$ and $\mu, \eta \in R$. Hence

$$(p_1 \circ f_{Gl\alpha})(h, l, \alpha) = y = \Phi_{l\alpha}(h) = \Phi_{l\alpha}(g)k = xk = (p_1 \circ f_{Gl\alpha})(g, l, \alpha)k,$$

where $k \in K$. Then $((p_1 \circ f_{Gl\alpha})(g, l, \alpha), (p_1 \circ f_{Gl\alpha})(h, l, \alpha)) \in E(D)$. Therefore, $p_1 \circ f_{Gl\alpha}$ is a homomorphism.

Conversely, assume that $p_1 \circ f_{Gl\alpha}$ is a homomorphism. We will show that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$. Let $x, y \in G$ be such that $(x, y) \in E(\mathrm{Cay}(G, K))$. Thus $y = xa$ for some $a \in K$. Since $\alpha \in T$, there exists $u \in P$ in which $(a, u, \alpha) \in A$ because $A = K \times P \times T$. Hence $(y, l, \alpha) = (xa, l, \alpha) = (x, l, \alpha)(a, u, \alpha)$ and then $((x, l, \alpha), (y, l, \alpha)) \in E(D)$. By our assumption, we obtain that

$$((p_1 \circ f_{Gl\alpha})(x, l, \alpha), (p_1 \circ f_{Gl\alpha})(y, l, \alpha)) \in E(\mathrm{Cay}(G, K)).$$

We will take $f(x, l, \alpha) = (x', l', \alpha')$ and $f(y, l, \alpha) = (y', l', \alpha')$ for some (x', l', α'),
$(y', l', \alpha') \in S$. Hence $\Phi_{l\alpha}(x) = x'$ and $\Phi_{l\alpha}(y) = y'$. We can conclude that

$$\begin{aligned}(\Phi_{l\alpha}(x), \Phi_{l\alpha}(y)) &= (x', y') \\ &= ((p_1 \circ f_{Gl\alpha})(x, l, \alpha), (p_1 \circ f_{Gl\alpha})(y, l, \alpha)) \in E(\mathrm{Cay}(G, K)).\end{aligned}$$

Consequently, $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$. □

3.2. Endomorphisms of Cayley digraphs of right groups

All over this subsection, we let $S = G \times R$ be a right group which is isomorphic to a rectangular group $G \times L \times R$ when $L = \{l\}$. Denote by D the Cayley digraph $\mathrm{Cay}(S, A)$ of the semigroup S with respect to the set A.

We first define the gainful function using in this subsection before we present some results of endomorphisms of Cayley digraphs of right groups. For each $\alpha \in R$ and for all $a \in G$, we define $\varphi_\alpha : G \to G$ by $\varphi_\alpha(a) = \Phi_{l\alpha}(a)$, where $\Phi_{l\alpha}$ is the function defined in Subsection 3.1 with $L = \{l\}$.

In fact, for convenience, we can consider φ_α in another expression as follows.

Let $f : S \to S$ be a function. For each $\alpha \in R$ and for all $a \in G$, we define $\varphi_\alpha : G \to G$ by

$$\varphi_\alpha(a) = b \text{ if there exists } \beta \in R \text{ such that } f(a, \alpha) = (b, \beta).$$

It is not hard to examine that φ_α is well-defined. We now show some results about endomorphisms of Cayley digraphs of right groups with respect to some connection sets.

Theorem 3.5. Let $A = K \times T$ be a connection set of D and $f : S \to S$ be a function. Then $f \in \mathrm{End}(D)$ if and only if the following conditions hold:

(i) $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G, K))$ for all $\alpha \in T$;

(ii) for each $g \in K$ and $a \in G$, there exists $g_a \in K$ such that

$$f(ag^{-1}, \theta) \in \begin{cases} \{\varphi_\lambda(a)g_a^{-1}\} \times T & \text{if } \theta \in T, \\ \{\varphi_\lambda(a)g_a^{-1}\} \times R & \text{if } \theta \in R \setminus T \end{cases}$$

for all $\lambda \in T$.

Proof. ($\Rightarrow$) Actually, $G \times R$ is isomorphic to $G \times L \times R$ when $|L| = 1$. So the result is clear by Theorem 3.1.

($\Leftarrow$) Without loss of generality, suppose that $S = G \times L \times R$, where L is a one-element left zero semigroup. Clearly, condition (i) of Theorem 3.1 holds for S and $K \times T$. On the other hand, since $|L| = 1$, conditions (ii) and (iii) of Theorem 3.1 hold by the assumption. □

We now present an example of an endomorphism of the Cayley digraph of a right group with respect to the set mentioned in Theorem 3.5.

Example 3.6. Let $D = \mathrm{Cay}(\mathbb{Z}_6 \times \{\alpha, \beta\}, A)$, where $A = \{2\} \times \{\alpha\}$.

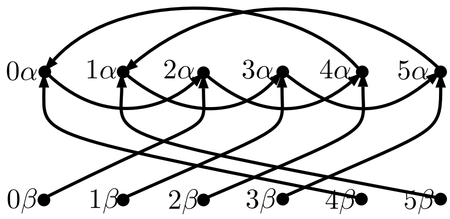

FIGURE 3. $\mathrm{Cay}(\mathbb{Z}_6 \times \{\alpha, \beta\}, A)$.

We obtain that

$$f = \begin{pmatrix} 0\alpha & 1\alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha & 0\beta & 1\beta & 2\beta & 3\beta & 4\beta & 5\beta \\ 5\alpha & 3\alpha & 1\alpha & 5\alpha & 3\alpha & 1\alpha & 5\beta & 3\alpha & 1\alpha & 5\alpha & 3\beta & 1\alpha \end{pmatrix} \in \mathrm{End}(D).$$

Moreover, we have

$$\varphi_\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 5 & 3 & 1 \end{pmatrix} \in \mathrm{End}(\mathrm{Cay}(\mathbb{Z}_6, \{2\})).$$

In addition, f satisfies the condition *(ii)* in *Theorem 3.5* as shown as follows:

$$\begin{aligned}
f(0 + 2^{-1}, \alpha) &= f(4, \alpha) = (3, \alpha) = (5 + 4, \alpha) = (\varphi_\alpha(0) + 2^{-1}, \alpha),\\
f(1 + 2^{-1}, \alpha) &= f(5, \alpha) = (1, \alpha) = (3 + 4, \alpha) = (\varphi_\alpha(1) + 2^{-1}, \alpha),\\
f(2 + 2^{-1}, \alpha) &= f(0, \alpha) = (5, \alpha) = (1 + 4, \alpha) = (\varphi_\alpha(2) + 2^{-1}, \alpha),\\
f(3 + 2^{-1}, \alpha) &= f(1, \alpha) = (3, \alpha) = (5 + 4, \alpha) = (\varphi_\alpha(3) + 2^{-1}, \alpha),\\
f(4 + 2^{-1}, \alpha) &= f(2, \alpha) = (1, \alpha) = (3 + 4, \alpha) = (\varphi_\alpha(4) + 2^{-1}, \alpha),\\
f(5 + 2^{-1}, \alpha) &= f(3, \alpha) = (5, \alpha) = (1 + 4, \alpha) = (\varphi_\alpha(5) + 2^{-1}, \alpha).
\end{aligned}$$

Similarly, we have $f(x + 2^{-1}, \beta) \in \{\varphi_\alpha(x) + 2^{-1}\} \times R$ for all $x \in \mathbb{Z}_6$.

To illustrate the structure of endomorphisms of Cayley digraphs of right groups, we consider the following special connection sets.

Corollary 3.7. Let $A = \{g\} \times R$ be a connection set of D, where $g \in G$, and $f : S \to S$ be a function. Then $f \in \mathrm{End}(D)$ if and only if the following conditions hold:

(i) $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$ for all $\alpha \in R$;

(ii) $\varphi_\beta = \varphi_\gamma$ for all $\beta, \gamma \in R$.

Proof. Suppose that $f \in \mathrm{End}(D)$. By condition (i) of Theorem 3.5, we have $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$ for all $\alpha \in R$. Let $a \in G$. By Theorem 3.5(ii), we obtain that

$$\varphi_\beta(a)g^{-1} = f(ag^{-1}, \theta) = \varphi_\gamma(a)g^{-1} \quad \text{for all } \beta, \gamma \in R.$$

Therefore, $\varphi_\beta = \varphi_\gamma$ for all $\beta, \gamma \in R$.

Conversely, suppose that $\varphi_\beta = \varphi_\gamma$ for all $\beta, \gamma \in R$. Let $a \in G$ and $\theta \in R$. Since $(ag^{-1}, a) \in E(\mathrm{Cay}(G, \{g\}))$ and $\varphi_\theta \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$, we obtain that $(\varphi_\theta(ag^{-1}), \varphi_\theta(a)) \in E(\mathrm{Cay}(G, \{g\}))$. Hence $\varphi_\theta(a) = \varphi_\theta(ag^{-1})g$ and this implies that $\varphi_\theta(ag^{-1}) = \varphi_\theta(a)g^{-1}$. Since $\varphi_\theta = \varphi_\lambda$ for all $\lambda \in R$ as we supposed above, we can conclude that $\varphi_\theta(ag^{-1}) = \varphi_\lambda(a)g^{-1}$. Therefore, there exists $\mu \in R$ such that $f(ag^{-1}, \theta) = (\varphi_\lambda(a)g^{-1}, \mu)$ for all $\lambda \in R$. By the converse of Theorem 3.5, we obtain that $f \in \mathrm{End}(D)$. □

Furthermore, the number of endomorphisms of the Cayley digraph of a right group with respect to the set $\{g\} \times R$ is obtained in the following proposition.

Proposition 3.8. Let G be a group of order n and R be a right zero semigroup of order m. Let $A = \{g\} \times R$ be a connection set of D where $g \in G$.

If $|\mathrm{End}(\mathrm{Cay}(G, \{g\}))| = d$ for some $d \in \mathbb{N}$, then $|\mathrm{End}(D)| = d \cdot m^{mn}$.

Proof. In order to construct an endomorphism f of D, let $\phi \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$ be fixed. Let $\beta \in R$ and define $f : S \to S$ as follows for every $(x, \alpha) \in S$:

$$f(x, \alpha) = (\phi(x), \beta).$$

Now by Corollary 3.7, f is an endomorphism of D. It can be easily seen that β is arbitrary, this means that it does not matter when we choose whatever $\beta \in R$, the function f is always an endomorphism of D. So we can conclude that for each $\phi \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$ and for each $(x, \alpha) \in S$, we have m ways to construct endomorphisms of D. On the other hand, if we pick $f \in \mathrm{End}(D)$, we can obtain by Corollary 3.7 that f must be one of those functions that we defined above. Consequently, $|\mathrm{End}(D)| = |\mathrm{End}(\mathrm{Cay}(G, \{g\}))||R|^{|S|} = d \cdot m^{mn}$, as required. □

3.3. Endomorphisms of Cayley digraphs of left groups

Throughout this subsection, we let $S = G \times L$ be a left group and $D = \mathrm{Cay}(S, A)$ the Cayley digraph of the semigroup S with respect to the set A.

Before we present the characterization of endomorphisms of Cayley digraphs of rectangular groups, we will define the notation for convenience in using.

Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$ where $g_i \in G$ for all $i \in I = \{1, 2, \ldots, k\}$. Let $f : S \to S$ be a function and $l \in L$. By f_{il}, we mean the restriction function $f_{|_{g_i\langle p_1(A)\rangle \times \{l\}}} : [g_i\langle p_1(A)\rangle \times \{l\}] \to D$, where $[g_i\langle p_1(A)\rangle \times \{l\}]$ is the strong subdigraph of D.

Theorem 3.9. Let $f : G \times L \to G \times L$ be a function and A be a subset of $G \times L$. For the Cayley digraph $D = \mathrm{Cay}(G \times L, A)$, following conditions are equivalent:

(i) $f \in \mathrm{End}(D)$;

(ii) f_{il} is edge-preserving for all $l \in L$ and $i \in I$;

(iii) for each $(x, l) \in G \times L$ and $a \in p_1(A)$,

$$f(xa, l) = (p_1(f(x, l))b, p_2(f(x, l)))$$

for some $b \in p_1(A)$.

Proof. Let A be a connection set of D and $f : D \to D$.

(i)$\Rightarrow$(ii) Suppose that $f \in \mathrm{End}(D)$. Let $l \in L$ and $i \in I$. We will prove that f_{il} is edge-preserving. Let $((x, l), (y, l)) \in E([g_i\langle p_1(A)\rangle \times \{l\}])$. Then $((x, l), (y, l)) \in E(D)$. We have $(f_{il}(x, l), f_{il}(y, l)) = (f(x, l), f(y, l)) \in E(D)$ since $f \in \mathrm{End}(D)$. Therefore, f_{il} is edge-preserving, as required.

(ii)$\Rightarrow$(iii) Assume that (ii) is true. Let $(x, l) \in G \times L$ and $a \in p_1(A)$. Then $(x, l) \in g_i\langle p_1(A)\rangle \times \{l\}$ for some $i \in I$. Thus there exists $l' \in p_2(A)$ such that $(a, l') \in A$. Consider $(xa, l) = (x, l)(a, l')$, we obtain that $((x, l), (xa, l)) \in E([g_i\langle p_1(A)\rangle \times \{l\}]) \subseteq E(D)$. Since f_{il} is edge-preserving, we can get that $(f(x, l), f(xa, l)) = (f_{il}(x, l), f_{il}(xa, l)) \in E(D)$. Suppose that $f(x, l) = (y, l_1)$ for some $(y, l_1) \in G \times L$. Then there exists $(b, l_2) \in A$ such that

$$\begin{aligned} f(xa, l) &= f(x, l)(b, l_2) = (y, l_1)(b, l_2) = (yb, l_1) \\ &= (p_1(f(x, l))b, p_2(f(x, l))), \end{aligned}$$

where $b \in p_1(A)$.

(iii)$\Rightarrow$(i) Suppose that the statement (iii) holds. We will show that $f \in \mathrm{End}(D)$. Let $((x, l_1), (y, l_2)) \in E(D)$. Thus $(y, l_2) = (x, l_1)(a, l_3) = (xa, l_1)$ for some $(a, l_3) \in A$. Hence $y = xa$ and $l_1 = l_2$. Assume that $f(x, l_1) = (u, l_4)$ for some $(u, l_4) \in G \times L$. By our supposition, we have

$$f(y, l_2) = f(xa, l_1) = (p_1(f(x, l_1))b, p_2(f(x, l_1))) = (ub, l_4)$$

for some $b_1 \in p_1(A)$. Since $b \in p_1(A)$, there exists $l_5 \in p_2(A)$ such that $(b, l_5) \in A$. We obtain that $f(y, l_2) = (ub, l_4) = (u, l_4)(b, l_5) = f(x, l_1)(b, l_5)$, that is, $(f(x, l_1), f(y, l_2)) \in E(D)$. Therefore, $f \in \mathrm{End}(D)$. $\square$

The above theorem presents characterizations of endomorphisms of Cayley digraphs of left groups. It is more general than Theorem 3.1 in the case of left groups since the connection sets considered in Theorem 3.9 are arbitrary. The last example is presented for guaranteeing the properties of endomorphisms of Cayley digraphs of left groups with respect to arbitrary connection sets.

Example 3.10. Let $D = \mathrm{Cay}(\mathbb{Z}_6 \times \{l, k\}, A)$, where $A = \{(2, l)\}$.

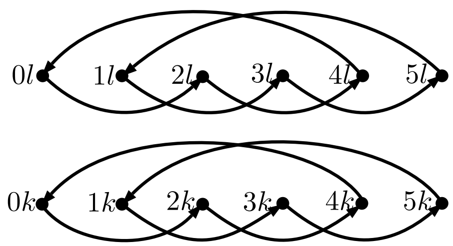

FIGURE 4. $\mathrm{Cay}(\mathbb{Z}_6 \times \{l, k\}, A)$.

We obtain that

$$f = \begin{pmatrix} 0l & 1l & 2l & 3l & 4l & 5l & 0k & 1k & 2k & 3k & 4k & 5k \\ 1k & 2l & 3k & 4l & 5k & 0l & 3l & 5l & 5l & 1l & 1l & 3l \end{pmatrix} \in \mathrm{End}(D).$$

Since $\langle p_1(A) \rangle = \langle \{2\} \rangle = \{0, 2, 4\}$, if we let $g_1 = 0$ and $g_2 = 1$, we obtain that

$$\begin{aligned} (g_1 + \langle p_1(A) \rangle) \times \{l\} &= \{(0, l), (2, l), (4, l)\}; \\ (g_1 + \langle p_1(A) \rangle) \times \{k\} &= \{(0, k), (2, k), (4, k)\}; \\ (g_2 + \langle p_1(A) \rangle) \times \{l\} &= \{(1, l), (3, l), (5, l)\} \end{aligned}$$

and

$$(g_2 + \langle p_1(A) \rangle) \times \{k\} = \{(1, k), (3, k), (5, k)\}.$$

We can conclude that

$$f_{1l} = \begin{pmatrix} 0l & 2l & 4l \\ 1k & 3k & 5k \end{pmatrix} \quad \text{and} \quad f_{1k} = \begin{pmatrix} 0k & 2k & 4k \\ 3l & 5l & 1l \end{pmatrix};$$

$$f_{2l} = \begin{pmatrix} 1l & 3l & 5l \\ 2l & 4l & 0l \end{pmatrix} \quad \text{and} \quad f_{2k} = \begin{pmatrix} 1k & 3k & 5k \\ 5l & 1l & 3l \end{pmatrix}$$

and they are edge-preserving.

The following computation shows that the endomorphism f defined as above satisfies the third condition in *Theorem 3.9*.

$$\begin{aligned}
f(0+2,l) &= f(2,l) = (3,k) = (1+2,k) = (p_1(f(0,l)) + 2, p_2(f(0,l))),\\
f(1+2,l) &= f(3,l) = (4,l) = (2+2,l) = (p_1(f(1,l)) + 2, p_2(f(1,l))),\\
f(2+2,l) &= f(4,l) = (5,k) = (3+2,k) = (p_1(f(2,l)) + 2, p_2(f(2,l))),\\
f(3+2,l) &= f(5,l) = (0,l) = (4+2,l) = (p_1(f(3,l)) + 2, p_2(f(3,l))),\\
f(4+2,l) &= f(0,l) = (1,k) = (5+2,k) = (p_1(f(4,l)) + 2, p_2(f(4,l))),\\
f(5+2,l) &= f(1,l) = (2,l) = (0+2,l) = (p_1(f(5,l)) + 2, p_2(f(5,l))).
\end{aligned}$$

Similarly, we obtain that $f(x+2,k) = (p_1(f(x,k)) + 2, p_2(f(x,k)))$ for all $x \in \mathbb{Z}_6$.

4. Conclusion

In this paper, we have provided related backgrounds of the research and some preliminaries together with notations in section 1 and section 2, respectively. In the third section, some characterizations of endomorphisms of Cayley digraphs of rectangular groups with respect to appropriate connection sets are obtained. In addition, we illustrated examples of endomorphisms of Cayley digraphs of those rectangular groups to guarantee our results.

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CONTACT INFORMATION

Srichan Arworn, Nuttawoot Nupo

Department of Mathematics, Chiang Mai University,
Huay Kaew Road, Chiang Mai, Thailand, 50200
E-Mail(s): srichan28@yahoo.com,
nuttawoot_nupo@cmu.ac.th

Boyko Gyurov

School of Science and Technology Georgia Gwinnett College
1000 University Center Lane Lawrenceville, GA 30043
E-Mail(s): bgyurov@ggc.edu

Sayan Panma
Center of Excellence in Mathematics and Applied Mathematics,
Department of Mathematics, Faculty of Science,
Chiang Mai University, Chiang Mai 50200,
Thailand
E-Mail(s): panmayan@yahoo.com

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