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A new extension of Odd Half-Cauchy Family of Distributions: properties and applications with regression modeling

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ABSTRACT

The paper proposes a new family of continuous distributions called the extended odd half Cauchy-G. It is based on the $T-X$ construction of Alzaatreh et al. (2013) by considering half Cauchy distribution for T and the exponentiated $G(x;\boldsymbol{\xi})$ as the distribution of X. Several particular cases are outlined and a number of important statistical characteristics of this family are investigated. Parameter estimation via several methods, including maximum likelihood, is discussed and followed up with simulation experiments aiming to asses their performances. Real life applications of modeling two data sets are presented to demonstrate the advantage of the proposed family of distributions over selected existing ones. Finally, a new regression model is proposed and its application in modeling data in the presence of covariates is presented.

Key words: $T-X$ method; regression; simulation; estimation

1. Introduction

Following the $T-X$ construction of Alzaatreh et al. (2013), Cordeiro et al. (2017) proposed a new generator of continuous probability distribution by considering Half-Cauchy for T and exponentiated G (Lehmann alternative-I) for X. They called the family generalized odd Half-Cauchy (GOHC-G$(\alpha,\boldsymbol{\xi})$) and investigated its properties and applications.
In the present paper we introduce a new ganerator called extended half Cauchy family of distribution following the same construction by considering exponentiated G (Lehmann alternative-II) for X and T following Half-Cauchy with probability density function (pdf) $q(t)=\frac{2}{\pi(1+t^2)}$, $t>0$, where $G(x;\boldsymbol{\xi})$ is the cumulative distribution function (cdf) of the baseline distribution with parameter vector $\boldsymbol{\xi}$. Now, following Alzaatreh et al. (2013) we define

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the proposed extended odd Half-Cauchy-G with the cdf

$$F(x;\alpha,\boldsymbol{\xi}) = \int_0^{\frac{1-\bar{G}(x;\boldsymbol{\xi})^{\alpha}}{\bar{G}(x;\boldsymbol{\xi})^{\alpha}}} \frac{2}{\pi(1+t^2)}dt = \frac{2}{\pi}\arctan\left[\frac{1-\bar{G}(x;\boldsymbol{\xi})^{\alpha}}{\bar{G}(x;\boldsymbol{\xi})^{\alpha}}\right], \tag{1}$$

where $x \in \boldsymbol{R}$ and $\alpha > 0$ is a parameter. The proposed family is denoted as shortly EOHC-G$(\alpha,\boldsymbol{\xi})$.

The pdf corresponding to (1) is given by

$$f(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha\, g(x;\boldsymbol{\xi})\,\bar{G}(x;\boldsymbol{\xi})^{-\alpha-1}}{\pi\left[1+\left\{1-\bar{G}(x;\boldsymbol{\xi})^{-\alpha}\right\}^2\right]} = \frac{2\alpha\, g(x;\boldsymbol{\xi})\,\bar{G}(x;\boldsymbol{\xi})^{\alpha-1}}{\pi\left[\bar{G}(x;\boldsymbol{\xi})^{2\alpha}+\left\{1-\bar{G}(x;\boldsymbol{\xi})^{\alpha}\right\}^2\right]}, \tag{2}$$

where $g(x;\boldsymbol{\xi}) = \frac{d}{dx}G(x;\boldsymbol{\xi})$ is the baseline pdf. Henceforth, a random variable X with density function (2) is denoted by $X \sim$ EOHC-G$(\alpha,\boldsymbol{\xi})$.

It should be noted that for $\alpha = 1$ both GOHC-G$(\alpha,\boldsymbol{\xi})$ and EOHC-G$(\alpha,\boldsymbol{\xi})$ reduce to the *odd half-Cauchy* (*OHC*) family. Otherwise for $\alpha < 1$, EOHC-G$(\alpha,\boldsymbol{\xi}) >_{st}$ GOHC-G$(\alpha,\boldsymbol{\xi})$ and for $\alpha > 1$ EOHC-G$(\alpha,\boldsymbol{\xi}) <_{st}$ GOHC-G$(\alpha,\boldsymbol{\xi})$. As such the two families give rise to diffrent sets of distributions as special case for $\alpha \neq 1$.

For convenience we shall use $G(x) = G(x;\boldsymbol{\xi})$, $f(x) = f(x;\alpha,\boldsymbol{\xi})$, etc.

The EOHC-G family is related to some distributions as stated below. Let $X \sim$EOHC-G$(\alpha,\boldsymbol{\xi})$. Then, we have the following results.

1. If $Y = \bar{G}(X;\boldsymbol{\xi})^{-\alpha}$, then $F_Y(y) = \frac{2}{\pi}\arctan(y-1)$ and $f_Y(y) = \frac{2}{\pi}\frac{1}{1+(1-y)^2}, \quad y > 1.$

2. If $Y = \bar{G}(X;\boldsymbol{\xi})^{-\alpha} - 1$, then $Y \sim HC(0,1)$ with pdf $f_Y(y) = \frac{2}{\pi}\frac{1}{1+y^2}, \quad y > 0.$

3. If $Y = \bar{G}(X;\boldsymbol{\xi})^{\alpha}$, then $F_Y(y) = \frac{2}{\pi}\arctan(\frac{1-y}{y}), \quad 0 < y < 1.$

The hazard rate function (hrf) of X is

$$h(x;\alpha,\boldsymbol{\xi}) = \frac{2\alpha\, g(x;\boldsymbol{\xi})\,\bar{G}(x;\boldsymbol{\xi})^{\alpha-1}}{\pi\left[\bar{G}(x;\boldsymbol{\xi})^{2\alpha}+\left\{1-\bar{G}(x;\boldsymbol{\xi})^{\alpha}\right\}^2\right]\left[1-\frac{2}{\pi}\arctan\left\{\bar{G}(x;\boldsymbol{\xi})^{-\alpha}-1\right\}\right]}. \tag{3}$$

1.1. Useful relation with the exponentiated class

Based on the following result of Gradshtyn and Ryzhik (2007) page 61, for $x > 0$,

$$\arctan(x) = \frac{\pi}{2} - \sum_{i=0}^{\infty}\frac{(-1)^i}{(2i+1)x^{2i+1}}.$$

We can derive the following mixture representation of the cdf and pdf of EOHC-G:

$$
\begin{aligned}
F(x) &= \frac{2}{\pi}\arctan\left[\frac{1-\bar{G}(x)^{\alpha}}{\bar{G}(x)^{\alpha}}\right] \\
&= 1-\frac{2}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^{i}\,\bar{G}(x)^{\alpha(2i+1)}}{(2i+1)(1-\bar{G}(x)^{\alpha})^{2i+1}} \\
&= 1-\frac{2}{\pi}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{-2i-1}{j}\bar{G}(x)^{\alpha(2i+1)+\alpha j}}{2i+1}. \qquad (4)
\end{aligned}
$$

Hence $F(x)$ can be expressed as an infinite mixture of the exponentiated $G(x)$ (Lehmann alternative-II). Again

$$
\begin{aligned}
F(x) &= 1-\frac{2}{\pi}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}\binom{-2i-1}{j}\binom{\alpha(2i+1)+\alpha j}{k}G(x)^{k}}{2i+1} \\
&= 1-\sum_{k=0}^{\infty}\gamma_k\,G(x)^{k}=\sum_{k=0}^{\infty}\nu_k\,G(x)^{k}, \qquad (5)
\end{aligned}
$$

where

$$
\gamma_k=\frac{2}{\pi}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j+k}\binom{-2i-1}{j}\binom{\alpha(2i+1)+\alpha j}{k}}{2i+1},
$$

$\nu_0 = 1-\gamma_0$ and $\nu_k = -\gamma_k$ for $k \geq 1$

Thus $F(x)$ is seen as an infinite mixture of $G(x)^k$, which is the exponentiated $G(x)$ distribution. Consequently, it is easy to verify that

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty}\nu_k(k+1)g(x)\,G(x)^{k} \\
&= \sum_{k=0}^{\infty}\nu_k h_{k+1}(x), \qquad (6)
\end{aligned}
$$

where $H_{k+1}(x) = G(x)^{k+1}$, $h_{k+1}(x) = \frac{d}{dx}H_{k+1}(x) = (k+1)\,G(x)^k\,g(x)$ and $h_1(x) = g(x)$.

The rest of the paper is organized as follows. A few special cases are presented in Section 2. Important properties like quantile function (qf), moments and moment generating function (mgf) are presented in Section 3. In Section 4 maximum likelihood estimation and its performance assessment via simulation is presented. Some other estimation methods and their performance through simulation is presented in Section 5. In Section 6, a new regression model is presented. In Section 7, data modelling applications with and without covariate are presented. The paper ends with a concluding section.

2. Sub-models of EOHC-G family

2.1. The EOHC-Burr XII (EOHC-BXII) distribution

Considering the BurrXII distribution (Burr, 1942) with pdf and cdf given by $g(x) = \lambda\beta x^{\lambda-1}(1+x^{\lambda})^{-\beta-1}$, $x>0$ and $G(x) = 1-(1+x^{\lambda})^{-\beta}$, $\lambda>0$ and $\beta>0$ the pdf and cdf of EOHC-BXII distribution are given respectively by

$$f^{\text{EOHC\_BXII}}(x;\alpha,\lambda,\beta) = \frac{2\alpha\lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\alpha\beta-1}}{\pi\left[\left(1+x^{\lambda}\right)^{-2\alpha\beta}+\left\{1-\left(1+x^{\lambda}\right)^{-\alpha\beta}\right\}^{2}\right]}, x>0, \quad (7)$$

$$F^{\text{EOHC\_BXII}}(x;\alpha,\lambda,\beta) = \frac{2}{\pi}\arctan\left[\left(1+x^{\lambda}\right)^{\alpha\beta}-1\right], x\geq 0. \quad (8)$$

Figure 1 shows the plots of the pdf and hazard of EOHC-BXII distribution for selected parameter values.

2.2. The EOHC-Fr (EOHC-Fr) distribution

Let $g(x)$ and $G(x)$ be the pdf and cdf of the Frechet distribution, given as $g(x) = \beta\theta^{\beta}x^{-\beta-1}\exp(-(\theta/x)^{\beta})$ and $G(x) = \exp(-(\theta/x)^{\beta})$, $x\geq 0$, $\beta>0, \theta>0$ respectively. Then, the pdf and cdf of the EOHC-Fr distribution are

$$f^{\text{EOHC\_F}}(x;\alpha,\beta,\theta) = \frac{2\alpha\beta\theta^{\beta}x^{-\beta-1}\exp(-(\theta/x)^{\beta})\left[1-\exp(-(\theta/x)^{\beta})\right]^{\alpha-1}}{\pi\left[\left[1-\exp(-(\theta/x)^{\beta})\right]^{2\alpha}+\left[1-\left[1-\exp(-(\theta/x)^{\beta})\right]^{\alpha}\right]\right]}, \quad (9)$$

and

$$F^{\text{EOHC\_F}}(x;\alpha,\beta,\theta) = \frac{2}{\pi}\arctan\left[\left[1-\exp(-(\theta/x)^{\beta})\right]^{-\alpha}-1\right], x\geq 0, \quad (10)$$

respectively.

Figure 2 shows the plots of pdf and hazartd of EOHC-Fr distribution for some selected parameters.

3. Properties of EOHC-G family

3.1. Quantile function and random sample generation

For a $U\sim$ Uniform(0,1) we can generate $X\sim$ EOHC-G by inverting (1) as

$$x = Q_G\left\{\frac{\left[1+\tan(\frac{\pi u}{2})\right]^{\frac{1}{\alpha}}-1}{\left[1+\tan(\frac{\pi u}{2})\right]^{\frac{1}{\alpha}}};\xi\right\}, \quad (11)$$

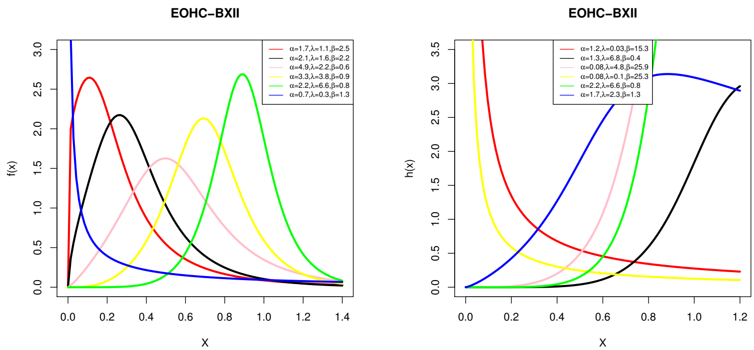

Figure 1: Plots of pdf and hazard for EOHC-BXII.

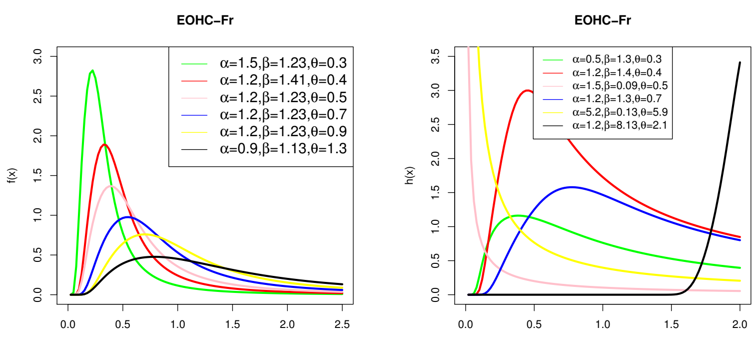

Figure 2: Plots of pdf and hazard for EOHC-Fr.

where $Q_G(\cdot) = G^{-1}(\cdot)$ is the baseline qf. The quantiles of the EOHC-G distributions for any baseline distribution can be obtained by (11). For instance, when $u = 1/2$, we obtain the median of the baseline distribution. Additionally, we can generate random variables from any baseline distribution using the given quantile function, in (11).

3.2. Moments

Let $Y_{k+1} \sim$exp-G$(k+1)$ with pdf $h_{k+1}(x) = (k+1)\, g(x)\, G(x)^k$. An expression for the nth moment of X can be obtained using equation (6) as

$$\mu_n' = E(X^n) = \sum_{k=0}^{\infty} \nu_k\, E(Y_{k+1}^n). \tag{12}$$

Another expression for μ_n' can be derived from equation (6) using the qf $Q_G(u)$ of the baseline distribution G as

$$\mu_n' = \sum_{k=0}^{\infty} (k+1)\, \nu_k\, \tau_{n,k}\ , \tag{13}$$

where $\tau_{n,k} = \int_{-\infty}^{\infty} x^n\, G(x)^k\, g(x)\, dx = \int_0^1 Q_G(u)^n\, u^k du$. $\tau_{n,k}$ is the (n,k)th probability weighted moment (PWM) of G. Thus, the moments of the EOHC-G distribution can be expressed in terms of the PWMs of G.

For integer values of n, let $\mu_n^{'} = E(X^n)$ and $\mu = \mu_1^{'} = E(X)$, then one can also find the n^{th} central moment of the EOHC-BXII distribution as

$$\mu_n = E(X-\mu)^n = \sum_{i=0}^{n} \binom{n}{i} \mu_i^{'} (-\mu)^{n-i}. \tag{14}$$

Using the first four moments of the EOHC-BXII distribution, we obtain the skewness and kurtosis of the EOHC-BXII distribution. Figure 3 shows the behaviour of skewness and kurtosis of the EOHC-BXII distribution.

3.3. Moment generating function

Lemma 1: The condition for $F(x)$ to have a mgf is that $G(x)$ also has a mgf.

Proof: Let $m = \inf\{x|G(x) \geq 0.5\}$, then

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \mathrm{f(x)dx} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}\, \bar{\mathrm{G}}(\mathrm{x})^{\alpha-1}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx} \\
&\leq \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx} \\
&= \int_{-\infty}^{m} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx} + \int_{\mathrm{m}}^{\infty} \mathrm{e}^{\mathrm{tx}}\, \frac{2}{\pi}\, \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha} + [1-\bar{\mathrm{G}}(\mathrm{x})^{\alpha}]^2}\, \mathrm{dx}.
\end{aligned}$$

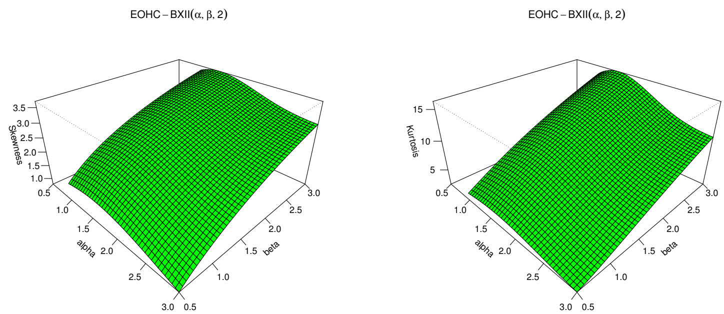

Figure 3: Skewness and Kurtosis for EOHC-BXII.

The second integral above is finite and the first integral is not greater than

$$\int_m^\infty \mathrm{e}^{\mathrm{t\,x}} \frac{2}{\pi} \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha}} \mathrm{dx}.$$

For $x < m$, we have $\bar{G}(x) \geq 0.5$, so that

$$\int_m^\infty \mathrm{e}^{\mathrm{t\,x}} \frac{2}{\pi} \frac{\mathrm{g(x)}}{\bar{\mathrm{G}}(\mathrm{x})^{2\alpha}} \mathrm{dx} < \frac{2^{2\alpha+1}}{\pi} \int_{\mathrm{m}}^\infty \mathrm{e}^{\mathrm{t\,x}} \mathrm{g(x)}\,\mathrm{dx} < \infty.$$

Thus, $M_X(t) < \infty$.

Corollary 1: Using (6), the mgf of $M(t) = E[\exp(tX)]$ of X is

$$M(t) = \sum_{k=0}^{\infty} \nu_k M_{k+1}(t), \tag{15}$$

where $M_{k+1}(t)$ is the mgf of $Y_{k+1} \sim exp - G(k+1)$. Alternatively, using equation (15) we can write

$$M(t) = \sum_{k=0}^{\infty} (k+1)\, \nu_k\, \rho(t,k), \tag{16}$$

where

$$\rho(t,k) = \int_{-\infty}^{\infty} \mathrm{e}^{t\,x}\, G(x)^k\, g(x) dx = \int_0^1 \exp\{t\, Q_G(u)\}\, u^k du.$$

4. Maximum likelihood estimation

Let $x = (x_1, x_2, ..., x_r)$ be a random sample from EOHC-G family with parameter vector $\eta = (\alpha, \xi)$. The log-likelihood function is

$$\ell = \ell(\eta) = r\log\frac{2\alpha}{\pi} + \sum_{i=1}^{r}\log\left[g\left(x_i,\xi\right)\right)] + (\alpha-1)\sum_{i=1}^{r}\log\left[\bar{G}\left(x_i,\xi\right)\right]$$
$$-\sum_{i=1}^{r}\left[\log\bar{G}\left(x_i,\xi\right)^{2\alpha} - \{1-\bar{G}\left(x_i,\xi\right)^{\alpha}\}^2\right].$$

The simultaneous solution of the partial derivatives of the log-likelihood gives the maximum likelihood estimators (MLEs) of the parameter of the EOHC-G family for a given baseline distribution. Unfortunately, it is not possible because of the non-linear structures of these derivatives. In this case, we prefer to maximize the log-likelihood function using the iterative algorithms. It can be done by statistical software such as R, Matlab or Python. Here, we use the R software to do this. The standard errors of the parameters are obtained based on the observed information matrix.

4.1. Performance evaluation of MLE

The MLEs of the parameters of the EOHC-BXII distribution are investigated based on the simulation study. The selected true parameter values are $(\alpha, \lambda, \beta) = (1.5, 2, 1)$. The used sample size is from $n = 20$ to $n = 100$. The simulation is replicated $r = 200$ times. The MLEs are obtained as $(\widehat{\alpha}_i, \widehat{\lambda}_i, \widehat{\beta}_i)$. We compute the biases and mean squared errors for each sample size by using the below equations

$$Bias_r\widehat{\theta} = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\theta}_i - \theta_i)\,\text{and}\,MSE_r\widehat{\theta} = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\theta}_i - \theta_i)^2, \text{for}\,\boldsymbol{\theta} = (\alpha, \lambda, \beta).$$

The simulation results are plotted in Figures 4 and 5, which shows that the biases and mean square errors are near the zero for all parameters. These results confirms that the MLEs of the parameters of the EOHC-BXII distributions are unbiased and consistent.

5. The other estimation methods

Several estimation methods can be used to estimate the unknown model parameters. Here, we focus on four different estimation methods. These are briefly summarized in the rest of this section. See Dey et al. (2018) for detailed information on these estimation methods. Note that, $\{t_{i:n}; i = 1, 2, ..., n\}$ are order statistics and F is the distribution function of EOHC-BXII.

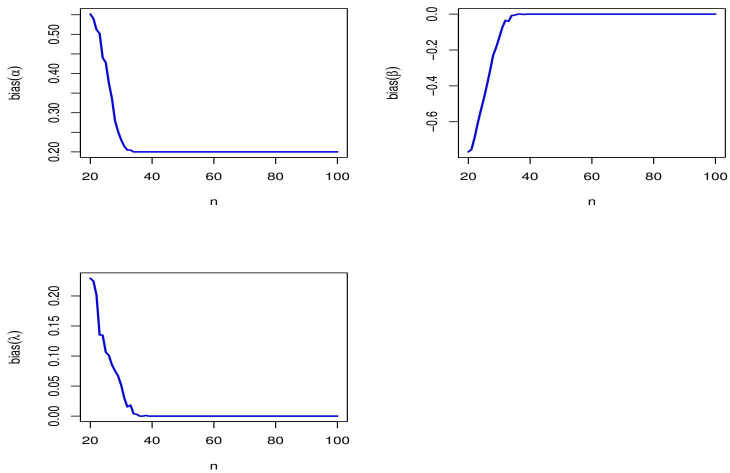

Figure 4: Bias of $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ versus r for EOHC-BXII when $(\alpha, \beta, \lambda) = (1.5, 1, 2)$.

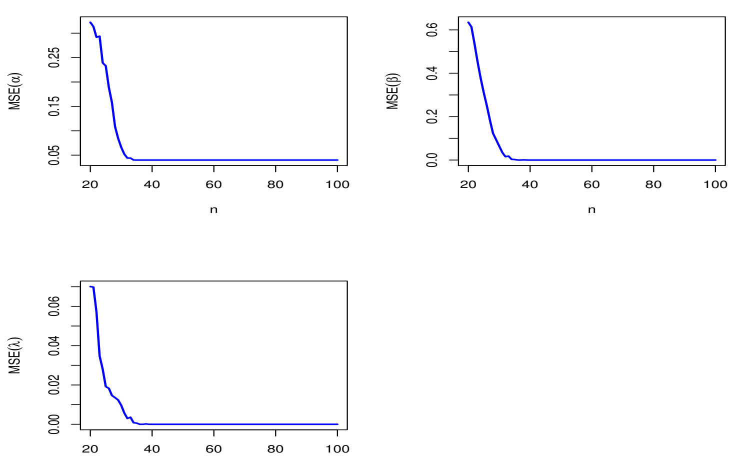

Figure 5: MSE of $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ versus r for EOHC-BXII when $(\alpha, \beta, \lambda) = (1.5, 1, 2)$.

5.1. Least square and weighted least square estimators

Swain et al. (1988) introduced the estimation methods for least square (LSE) and weighted least square estimators (WLSE). These estimators are easily obtained by minimizing the following functions:

$$S_{\mathrm{LSE}}(\alpha,\xi)=\sum_{i=1}^{n}\left(F(t_{i:n};\alpha,\xi)-\frac{i}{n+1}\right)^2$$

and

$$S_{\mathrm{WLSE}}(\alpha,\xi)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(t_{i:n};\alpha,\xi)-\frac{i}{n+1}\right)^2$$

5.2. Cramér–von–Mises estimator

Choi and Bulgren (1968) introduced the method for the Cramér-von-Mises Estimator (CME), which is obtained by minimizing the following function

$$S_{\mathrm{CME}}(\alpha,\xi)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(t_{i:n};\alpha,\xi)-\frac{2i-1}{2n}\right)^2.$$

5.3. Anderson-Darling and right-tailed Anderson-Darling estimators

Anderson-Darling estimators (ADEs) and right-tailed Anderson Darling estimators, shortly denoted as (RTADEs), were introduced by Anderson and Darling (1952) and Macdonald (1971), respectively. The ADEs for the EOHC-BXII distribution can be obtained by minimizing the below function

$$S_{\mathrm{ADE}}(\alpha,\xi)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F(t_{i:n};\alpha,\xi+\log\overline{F}(t_{i:n+1-i};\alpha,\xi\},$$

where $\overline{F}(\cdot)=1-F(\cdot)$.

5.4. Simulation

Again, EOHC-BXII distribution is used to investigate the difference between the estimation methods given in the above section. The true parameter vector is $(\alpha,\lambda,\beta)=(1.5,2,1)$ and the sample size is $n=30,35,\cdots,300$. The simulation is replicated $r=100$ times. The results are plotted in Figure 6.

The following results are obtained.

- For estimating α, AD method has the minimum amount of bias.
- For estimating λ, with small sample size, CVM method and for large sample size, AD has the minimum amount of bias.
- For estimating β, AD method has the minimum amonut of bias.

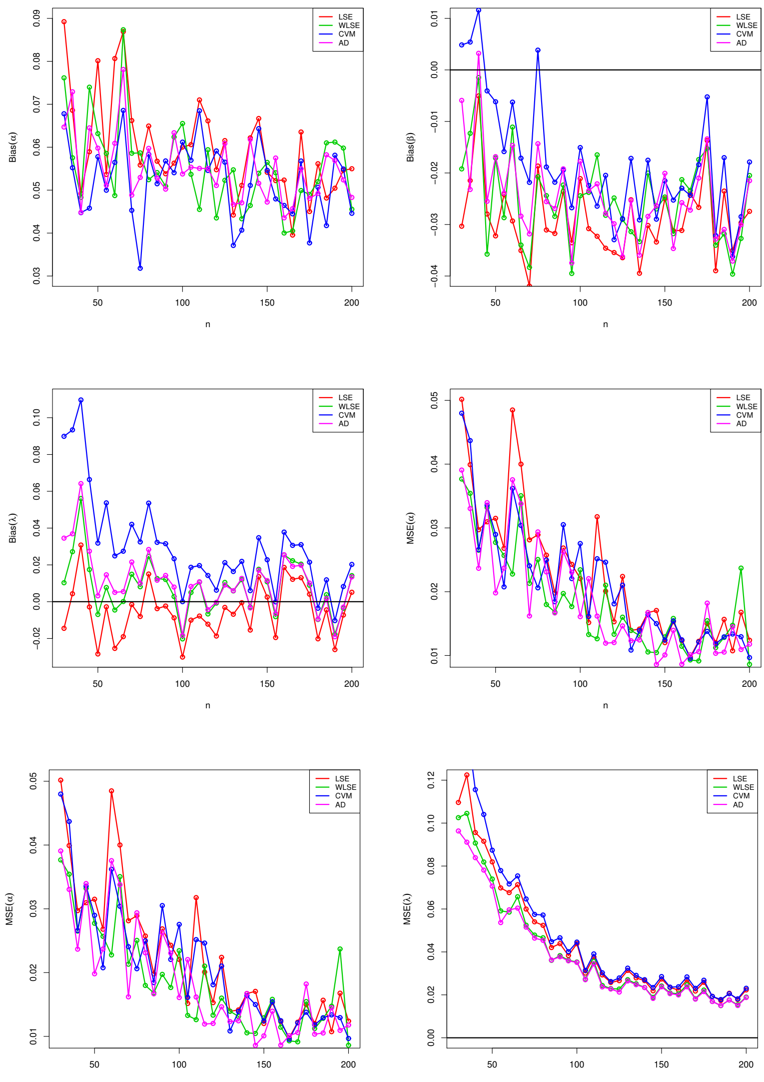

Figure 6: Bias of $\hat{\alpha}$ versus n when $\alpha = 1.5$; Bias of $\hat{\beta}$ versus n when $\beta = 1$; Bias of $\hat{\lambda}$ versus n when $\lambda = 2$; MSE of $\hat{\alpha}$ versus n when $\alpha = 1.5$; MSE of $\hat{\beta}$ versus n when $\beta = 1$; MSE of $\hat{\lambda}$ versus n when $\alpha = 2$

- For estimating α, with small sample size, CVM method and for large sample size, LSE has the minimum amount of MSE.
- For estimating λ, with small sample size, CVM method and for large sample size, AD has the minimum amount of MSE.
- For estimating β, with small sample size, CVM method and for large sample size, AD has the minimum amount of MSE.

6. The log-EOHC-Fr regression model

Consider the EOHC-Fr distribution with three parameters given in (9) and let X be a random variable with EOHC-Fr distribution. Using the transformation $Y = \log(X)$ and the re-parametrizations $\beta = 1/\sigma$ and $\theta = \exp(\mu)$, the pdf of Y is

$$f(y) = \frac{\frac{2\alpha}{\sigma}\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{-\alpha-1}}{\pi\left[1+\left\{1-\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{-\alpha}\right\}^{2}\right]}, \tag{17}$$

where $y \in \Re$. The parameter $\mu \in \Re$ represents the location of Y and the parameter $\sigma > 0$ is treated as a scale parameter and $\alpha > 0$ is the shape parameter. The density in (17) is referred as the Log-EOHC-Fr (LEOHC-Fr) distribution and denoted as $Y \sim$ LEOHC-Fr(α, μ, σ). The survival function of (17) is

$$S(y) = 1 - \frac{2}{\pi}\arctan\left[\frac{1-\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{\alpha}}{\left(1-\left[\exp\left\{-\exp\left\{-\left(\frac{y-\mu}{\sigma}\right)\right\}\right\}\right]\right)^{\alpha}}\right], \tag{18}$$

Now, we introduce a new parametric regression model to analyze the lifetimes of individuals with covariates. To do this, the identity link function is used to link the covariates to location of the response variable. Let y_i be a response variable that follows the density in (17) and $\mathbf{v}_i^{\intercal} = (v_{i1}, \ldots, v_{ip})$ be a explanatory variable vector. We consider the below location-scale regression model

$$y_i = \mathbf{v}_i^{\intercal}\beta + \sigma z_i,\ i = 1, \ldots, n, \tag{19}$$

where y_i has density function (17), $\beta = (\beta_1, \ldots, \beta_p)^{\intercal}$, and $\sigma > 0$, $\alpha > 0$ are unknown parameters.

The unknown parameters of the LEOHC-Fr are obtained by means of MLE method. The response variable is defined as $y_i = \min\{\log(x_i), \log(c_i)\}$. The quantities $\log(x_i)$ and $\log(c_i)$ represent the log-lifetimes and log-censoring times, respectively. We define two sets to represents the log-lifetimes and log-censoring times. These are F and C. The set F contains the log-lifetimes and C contains the log-censoring times. The general equation for the log-likelihood function on the model in (19) is given by

$$l(\Theta) = \sum_{i\in F}\log[f(y_i)] + \sum_{i\in C}\log[S(y_i)]$$

where $\Theta = (\alpha, \sigma, \beta^{\top})$, $l_i(\tau) = \log[f(y_i)]$ and $l_i^{(c)}(\Theta) = \log[S(y_i)]$, $f(y_i)$. Replacing $f(y_i)$ with (17) and $S(y_i)$ with (18) in the above equation, the log-likelihood function of the LEOHC-Fr regression model is

$$\begin{aligned}\ell(\Theta) = & r\log\left(\tfrac{2\alpha}{\sigma}\right) - \sum_{i\in F} z_i - \sum_{i\in F}\exp(z_i) \\ & + (-\alpha-1)\sum_{i\in F}\log\left(1-[\exp\{-\exp\{-z_i\}\}]\right) \\ & - \sum_{i\in F}\log\pi\left[1+\left\{1-(1-[\exp\{-\exp\{-z_i\}\}])^{-\alpha}\right\}^2\right] \\ & + \sum_{i\in C}\log\left(1-\tfrac{2}{\pi}\arctan\left[\tfrac{1-(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}{(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}\right]\right),\end{aligned} \quad (20)$$

where $z_i = (y_i - \mu_i)/\sigma$, and r is the number of uncensored observations. The MLE of the parameter vector, $\ell(\Theta)$ is obtained by direct maximization of (20) using the optim function of R software.

6.1. Residual analysis

Two types of the residuals are considered to study the residual analysis of the LEOHC-Fr regression model.

6.1.1 Martingale residual

The martingale residuals for LEOHC-Fr model is (see Fleming and Harrington, 1994, for details)

$$r_{M_i} = \begin{cases} 1+\log\left(1-\tfrac{2}{\pi}\arctan\left[\tfrac{1-(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}{(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}\right]\right) & \text{if } i\in F, \\ \log\left(1-\tfrac{2}{\pi}\arctan\left[\tfrac{1-(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}{(1-[\exp\{-\exp\{-z_i\}\}])^{\alpha}}\right]\right) & \text{if } i\in C, \end{cases} \quad (21)$$

where $z_i = (y_i - \mu_i)/\sigma$.

6.1.2 Modified deviance residual

The interpretation of the martingale residuals is not easy since it is not symmetrically distributed around zero. Therefore, the modified deviance residual was proposed by Therneau et al. (1990) to remove the lack of the martingale residuals. The modified deviance residual for LEOHC-Fr model is

$$r_{D_i} = \begin{cases} sign(r_{M_i})\{-2[r_{M_i} + \log(1-r_{M_i})]\}^{1/2}, & \text{if } i\in F, \\ sign(r_{M_i})\{-2r_{M_i}\}^{1/2}, & \text{if } i\in C, \end{cases} \quad (22)$$

where r_{M_i} is the martingale residual.

7. Real life applications

7.1. Modelling without covariates

Two different real-life data sets are considered here to study the suitability of the distributions from EOHC-G(α,ξ) family in comparison with some existing distributions taking BurrXII distribution as the baseline distribution. We have used AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), CAIC (Consistent Akaike Information Criterion) and HQIC (Hannan-Quinn Information Criterion) for selecting the best model. The figures of fitted densities and the fitted cdf's presented alongside the corresponding observed histograms and ogives for visual checking.

Here we have considered Burr-XII as the baseline distribution in the EOHC-G family and compared it with the following important extensions of Burr-XII model including GOHC-BXII.

1. BXII distribution:

$$f(x)=\lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta-1},\lambda>0,\beta>0,x>0.$$

2. MOBXII distribution (Arwa Y. Al-Saiari et al., 2014):

$$f(x)=\frac{\lambda\beta\alpha x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta-1}}{\left[1-(1-\alpha)\left(1+x^{\lambda}\right)^{-\beta}\right]^{2}},\ \alpha>0,\ \lambda>0,\ \beta>0,x>0.$$

3. TLBXII distribution (Hesham and Soha, 2017):

$$\begin{aligned}f(x) &= 2\alpha\lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-2\beta-1}\left[1-\left(1+x^{\alpha}\right)^{-2\beta}\right]^{\alpha-1},\\ &\alpha>0,\lambda>0,\beta>0,x>0.\end{aligned}$$

4. KwBXII distribution (Paranaiba et al., 2013):

$$\begin{aligned}f(x) &= \frac{ab\lambda\beta x^{\lambda-1}}{\left(1+x^{\lambda}\right)^{\beta+1}}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a-1}\times\\ &\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a}\right\}^{b-1},\\ &a>0,b>0,\lambda>0,\beta>0,x>0.\end{aligned}$$

5. BBXII distribution (Paranaiba et al., 2011):

$$\begin{aligned}f(x) &= \frac{\lambda\beta}{B(a,b)}x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta(b+1)}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a-1},\\ &a>0,b>0,\lambda>0,\beta>0,x>0.\end{aligned}$$

6. BEBXII distribution (Mead, 2014):

$$\begin{aligned} f(x) &= \frac{\lambda\beta\alpha}{B(a,b)} x^{\alpha-1}\left(1+x^{\lambda}\right)^{-\beta-1}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{a\alpha-1} \times \\ &\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{\alpha}\right\}^{b-1}, \\ & a>0, b>0, \alpha>0, \lambda>0, \beta>0, x>0. \end{aligned}$$

7. FBBXII distribution (Paranaiba et al., 2011):

$$\begin{aligned} f(x) &= \frac{\lambda\beta\alpha^{-\lambda}}{B(a,b)} x^{\lambda-1}\left[1+\left(\frac{x}{\alpha}\right)^{\lambda}\right]^{-\beta b-1}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\lambda}\right]^{-\beta}\right\}^{a-1}, \\ & a>0, b>0, \alpha>0, \lambda>0, \beta>0, x>0. \end{aligned}$$

8. FKwBXII distribution (Paranaiba et al., 2013):

$$\begin{aligned} f(x) &= \frac{ab\lambda\beta x^{\lambda-1}}{\left[1+\left(\frac{x}{\alpha}\right)^{\lambda}\right]^{\beta+1}}\left[1-\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\beta}\right]^{a-1} \times \\ &\left\{1-\left[1-\left(1+\left(\frac{x}{\alpha}\right)^{\lambda}\right)^{-\beta}\right]^{a}\right\}^{b-1}, \\ & a>0, b>0, \alpha>0, \lambda>0, \beta>0, x>0. \end{aligned}$$

9. GOHC-BXII distribution (Cordeiro et al., 2017):

$$\begin{aligned} f(x;\alpha,\lambda,\beta) &= \frac{2\alpha\lambda\beta x^{\lambda-1}\left(1+x^{\lambda}\right)^{-\beta-1}\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{\alpha-1}}{\pi\left[\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{2\alpha\beta}+\left\{1-\left[1-\left(1+x^{\lambda}\right)^{-\beta}\right]^{\alpha}\right\}^{2}\right]}, \\ & \alpha>0, \lambda>0, \beta>0, x>0. \end{aligned}$$

In the first application, we work with the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). It is used also by Shibu and Irshad (2016). The second data set is obtained from Hinkley, (1977). It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. We have presented the descriptive statistics of the data sets I, and II in Table 1. Findings of the data fitting in Tables 2, 3,4, 5. The total time on test (TTT) plot proposed by Aarset (1987) is drawn to get information about the shape of the hazard of a given data set. If the resulting shape of the TTT plot is a straight diagonal line, is of convex shape and concave shape then the corresponding hazard is constant, decreasing and increasing respectively. The TTT plots for the data sets considered here are presented in Figure 7 and

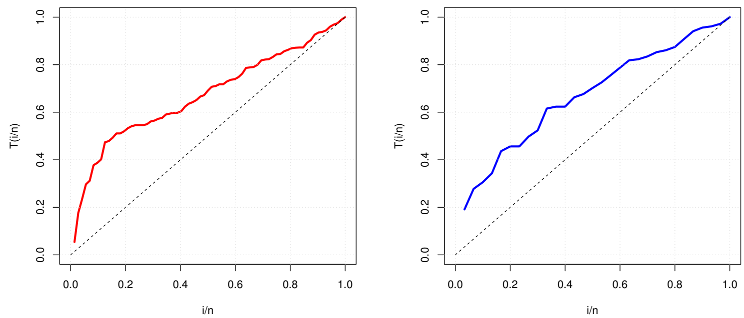

Figure 7: TTT Plots of data I and II from right to left

Table 1: Descriptive Statistics for the data set I, and II

n	Min.	1st Qu.	Third Qu.	Mean	Median	Max.	Variance	Skewness	Kurtosis
72	0.1	1.08	2.30	1.85	1.56	7.00	1.44	1.79	4.16
30	0.32	0.92	2.09	1.67	1.47	4.75	1.00	1.03	0.93

indicate that the all the three data sets are increasing hazard rate.

In Tables 2-5 the MLEs with standard errors of the parameters for all the fitted the models, their AIC, BIC, CAIC and HQIC for the data sets I and II are presented respectively. From these tables it is evident that for both the data sets considered here the EOHC-BXII distribution with lowest AIC, BIC, CAIC, HQIC turned out to be the best model. Moreover, the plots of estimated pdf against the observed histograms and the estimated cdf of EOHC-BXII against empirical cdfs in Figures 8 and 9 reveal that the proposed distribution provides closest fit to both the data sets. It may be mentioned that the proposed three parameter distribution has even outperformed the four and five parameter extensions considered here.

7.2. Modelling with covariates

Yousof et al. (2018) introduced the Log-odd log-logistic-Fréchet (LOLL-Fr) regression model and analysed the Stanford heart transplant data set. The same data set was also analyzed by Korkmaz et al. (2020). Now, we use the same data set to illustrate the importance of the LEOHC-Fr regression model and compare its performance with a regression model of Yousof et al. (2018), LOLL-Fr regression. The data set can be found in an R package, **p3state.msm**. The sample and censoring rate are 103 and 27%, respectively. The aim of the study is to analyze the survival times of individuals, say (y_i) with covariates: v_1- year of acceptance to the program; v_2- age of patient (in years); v_3- previous surgery status $(1 = yes, 0 = no)$; v_4-transplant indicator $(1 = yes, 0 = no)$. The model in (23) is considered and fitted by two models: LEOHC-Fr and LOLL-Fr regression models.

$$y_i = \boldsymbol{\beta_0} + \boldsymbol{\beta_1}\mathrm{v}_{i1} + \boldsymbol{\beta_2}\mathrm{v}_{i2} + \boldsymbol{\beta_3}\mathrm{v}_{i3} + \boldsymbol{\beta_4}\mathrm{v}_{i4} + \sigma z_i \,, \tag{23}$$

Table 2: MLEs, standard errors, confidence interval (in parentheses) for the data set I

Model	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$
BXII (λ,β)	···	···	···	3.102	0.465
				(0.538)	(0.077)
				(2.05,4.16)	(0.31,0.62)
MOBXII (α,λ,β)	8.989	···	···	2.259	1.533
	(4.587)			(0.864)	(0.907)
	(0,17.97)			(0.57,3.95)	(0,3.31)
TLBXII (α,λ,β)	1.796	···	···	2.393	0.488
	(0.915)			(0.907)	(0.244)
	(0.002,3.59)			(0.62,4.17)	(0,0.97)
KwBXII (a,b,λ,β)	···	14.105	7.424	0.525	2.274
		(10.805)	(11.850)	(0.279)	(0.990)
		(0,35.28)	(0,30.65)	(0,1.07)	(0.33,4.21)
BBXII (a,b,λ,β)	···	2.555	6.058	1.800	0.294
		(1.859)	(10.391)	(0.955)	(0.466)
		(0,6.28)	(0,26.42)	(0,3.67)	(0,1.21)
BEBXII $(\alpha,a,b,\lambda,\beta)$	0.572	1.876	2.991	1.780	1.341
	(0.325)	(0.094)	(1.731)	(0.702)	(0.816)
	(0,1.21)	(1.69,2.06)	(0,6.38)	(0.40,3.16)	(0,2.94)
FBBXII $(\alpha,a,b,\lambda,\beta)$	1.655	0.621	0.549	3.398	1.381
	(0.436)	(0.541)	(1.011)	(2.785)	(2.312)
	(0.81,4.48)	(0,1.68)	(0,2.53)	(0,8.86)	(0,5.91)
FKwBXII $(\alpha,a,b,\lambda,\beta)$	1.475	0.588	0.308	3.399	2.131
	(0.361)	(0.442)	(0.314)	(2.082)	(1.833)
	(0.76,2.18)	(0,1.42)	(0,0.92)	(0,7.47)	(0,5.72)
GOHCBXII (α,λ,β)	1.828	···	···	1.981	0.987
	(1.170)			(0.899)	(0.643)
	(0,4.12)			(0.21,3.74)	(0,2.24)
EOHCBXII (α,λ,β)	0.491	···	···	3.126	0.998
	(0.017)			(0.462)	(0.136)
	(0.45,0.52)			(2.22,4.03)	(0.73,1.26)

Table 3: AIC, BIC, CAIC, HQIC values for the data set I

Model	AIC	BIC	CAIC	HQIC
BXII (λ,β)	209.60	214.15	209.77	211.40
MOBXII (α,λ,β)	209.74	216.56	210.09	212.44
TLBXII (α,λ,β)	211.80	218.63	212.15	214.52
KwBXII (a,b,λ,β)	208.76	217.86	209.36	212.38
BBXII (a,b,λ,β)	210.44	219.54	211.03	214.06
BEBXII $(\alpha,a,b,\lambda,\beta)$	212.10	223.50	213.00	216.60
FBBXII $(\alpha,a,b,\lambda,\beta)$	206.80	218.20	207.71	211.30
FKwBXII $(\alpha,a,b,\lambda,\beta)$	206.50	217.90	207.41	211.00
GOHCBXII (α,λ,β)	206.66	213.50	207.01	209.36
EOHCBXII (α,λ,β)	205.96	212.80	206.31	208.66

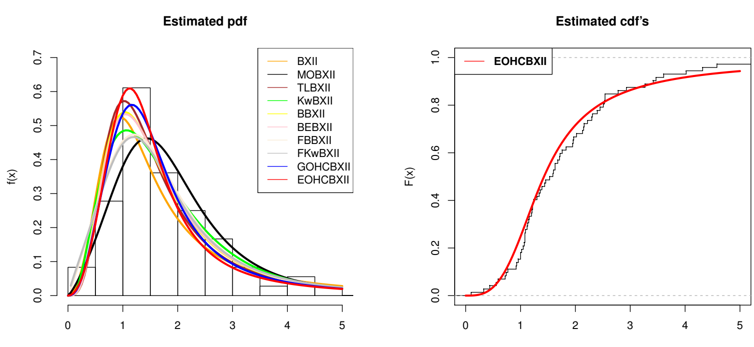

Figure 8: Plots of the observed histogram and estimated pdfs for the BXII, MOBXII, ,TL-BXII, KwBXII, BBXII,BEBXII, FBBXII, FKwBII and EOHCBXII and observed ogive and estimated cdf EOHCBXII for data set I from right to left

Table 4: MLEs, standard errors, confidence interval (in parentheses) for the data set II

Model	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$
BXII (λ,β)	···	···	···	3.255	0.687
				(0.645)	(0.137)
				(1.99,4.52)	(0.41,0.95)
MOBXII (α,λ,β)	5.205	···	···	2.121	1.666
	(8.599)			(1.041)	(1.295)
	(0,22.05)			(0.08,4.16)	(0,4.20)
TLBXII (α,λ,β)	3.949	···	···	1.420	0.977
	(5.685)			(1.019)	(0.904)
	(0,15.09)			(0,3.42)	(0,2.75)
KwBXII (a,b,λ,β)	···	34.377	30.999	0.292	3.006
		(109.591)	(63.785)	(0.368)	(4.298)
		(0,249.17)	(0,156.01)	(0,1.01)	(0,11.43)
BBXII (a,b,λ,β)	···	39.029	15.796	0.389	1.645
		(6.983)	(10.693)	(0.028)	(0.176)
		(0,25.34)	(0,36.75)	(0.33,0.44)	(1.30,1.98)
BEBXII $(\alpha,a,b,\lambda,\beta)$	1.000	15.563	7.818	0.617	1.388
	(1.232)	(22.109)	(13.299)	(0.421)	(0.667)
	(0,3.41)	(0,58.89)	(0,33.88)	(0,1.44)	(0,2.69)
FBBXII $(\alpha,a,b,\lambda,\beta)$	26.693	3.925	58.407	0.889	0.925
	(9.938)	(4.717)	(11.969)	(0.521)	(0.237)
	(7.21,46.17)	(0,13.17)	(34.94,81.86)	(0,1.91)	(0.46,1.39)
FKwBXII $(\alpha,a,b,\lambda,\beta)$	1.929	0.612	0.771	3.344	2.532
	(0.668)	(0.144)	(0.823)	(1.396)	(1.077)
	(0.62,3.24)	(0.32,0.89)	(0,2.38)	(0.61,6.08)	(0.42,4.64)
GOHCBXII (α,λ,β)	4.641	···	···	1.124	2.177
	(8.491)			(0.951)	(2.345)
	(0,21.28)			(0,2.98)	(0,6.77)
EOHCBXII (α,λ,β)	0.432	···	···	2.730	1.347
	(0.174)			(0.570)	(0.098)
	(0.09,0.77)			(1.61,3.84)	(1.15,1.53)

Table 5: AIC, BIC, CAIC, HQIC values for the data set II

Model	AIC	BIC	CAIC	HQIC
BXII (λ,β)	88.50	91.30	88.94	89.38
MOBXII (α,λ,β)	87.28	91.48	88.20	88.60
TLBXII (α,λ,β)	86.62	90.82	87.54	87.94
KwBXII (a,b,λ,β)	86.16	91.76	87.76	87.92
BBXII (a,b,λ,β)	87.14	92.74	88.74	88.90
BEBXII $(\alpha,a,b,\lambda,\beta)$	87.26	94.26	89.76	89.46
FBBXII $(\alpha,a,b,\lambda,\beta)$	87.36	94.36	89.86	89.56
FKwBXII $(\alpha,a,b,\lambda,\beta)$	87.14	94.14	89.64	89.34
GOHCBXII (α,λ,β)	84.78	88.98	85.70	86.12
EOHCBXII (α,λ,β)	84.42	88.62	85.34	85.74

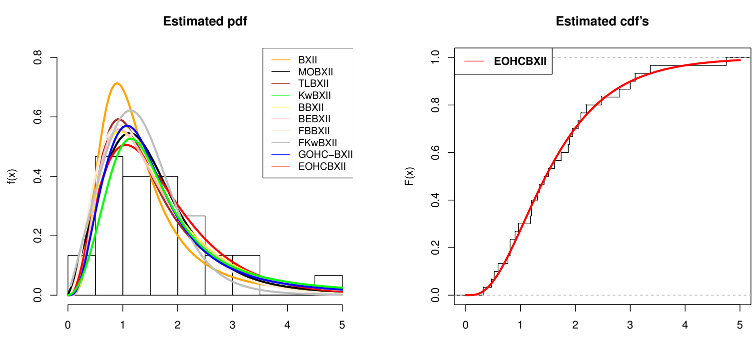

Figure 9: Plots of the observed histogram and estimated pdfs for the BXII, MOBXII, ,TL-BXII, KwBXII, BBXII,BEBXII, FBBXII, FKwBII and EOHCBXII and observed ogive and estimated cdf EOHCBXII for data set II from right to left

Table 6: MLEs of the parameters to Stanford Heart Transplant Data for LOLL-Fr and LEOHC-Fr regression models with corresponding SEs, p-values and $-\ell$, AIC and BIC statistics.

	Models					
	LOLL-Fr			LEOHC-Fr		
Parameters	Estimate	S.E.	p-value	Estimate	S.E.	p-value
α	2.078	0.790	-	24.344	49.796	-
σ	2.886	0.954	-	5.728	2.952	-
β_0	1.252	0.561	0.025	9.661	6.964	0.165
β_1	0.181	0.096	0.061	0.204	0.094	0.031
β_2	-0.047	0.018	0.010	-0.052	0.018	0.004
β_3	-0.151	0.501	0.763	0.206	0.484	0.670
β_4	0.551	0.268	0.039	0.437	0.365	0.230
$-\ell$	160.932			158.965		
AIC	335.865			331.931		
BIC	354.308			350.374		

The results of the fitted regression models including the estimated parameters, standard errors and corresponding p-values as well as model selection criteria such as AIC and BIC values are given in Table 6. As seen from the reported values of AIC and BIC, the LEOHC-Fr regression model has lower values of these statistics than those of the LOLL-Fr regression model. Therefore, we conclude that the LEOHC-Fr regression model is more appropriate than the LOLL-Fr regression model for the data used. Additionally, the regression parameters β_1 and β_2 are statistically significant since the p-values of these parameters are less than 5% significance level.

7.2.1 Residual Analysis of LEOHC-Fr model

Figure 10 displays the residuals analysis results of the LEOHC-Fr model. These figures reveal the applicability and accuracy of the fitted LEOHC-Fr model. Since all residuals are in the plotted envelopes, there is no possible outlier.

8. Conclusion

T-X method is used to generate a new family of continuous distributions. Important statistical properties are investigated. Different estimation methods are discussed to estimate the unknown model parameters via comprehensive simulation studies. Applications of data modelling with distribution fitting and regression modelling have shown favourable results for distributions belonging to the proposed family.

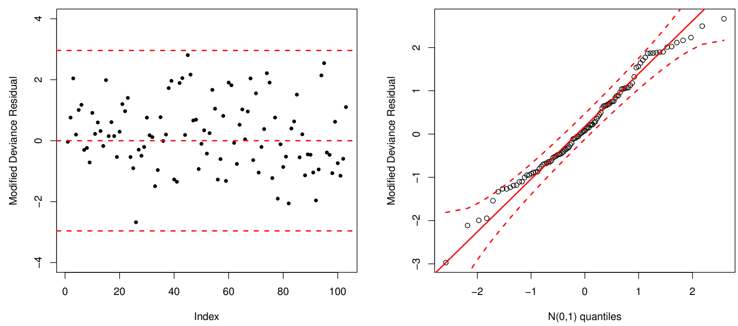

Figure 10: The results of residual analysis: (left) plot of the modified deviance residuals and (right) its quantile-quantile plot

References

Aarset, M. V., (1987). How to identify bathtub hazard rate, *IEEE Transactions on Reliability*, 36, pp. 106–108.

Alzaatreh, A., Lee, C. and Famoye, F., (2013). A new method for generating families of continuous distributions. *Metron*, 71, pp. 63–79.

Anderson, T. W. and Darling, D. A., (1952). Asymptotic theory of certain "goodness-of-fit" criteria based on stochastic processes. *Ann Math Stat*, 23, pp. 93–212.

Arwa, Y., Al- Saiari, Lamya, A., Baharith and Salwa, Mousa A., (2014). Marshall-Olkin extended Burr type XII distribution. *International Journal of Statistics and Probability*, 3, pp. 78–84.

Bjerkedal, T., (1960). Acquisition of resistance in Guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*, 72, pp. 130–148.

Burr, I. W., (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13, pp. 215–232.

Choi, K. and Bulgren, W. G., (1968). An estimation procedure for mixtures of distributions. *Journal of the Royal Statistical Society: Series B (Methodological)*, 30, pp. 444–460.

Cordeiro, G. M., Alizadeh, M., Ramires, T. G. and Ortega, E. M. M., (2017). The generalized odd half-Cauchy family of distributions: Properties and applications. *Communications in Statistics-Theory and Methods*, 46, pp. 5685–5705.

Dey, S., Mazucheli, J. and Nadarajah, S., (2018). Kumaraswamy distribution: different methods of estimation. *Computational and Applied Mathematics*, 37, pp. 2094–2111.

Fleming, T. R. and Harrington, D. P., (1994). *Counting process and survival analysis*, John Wiley, New York.

Gradshteyn, I. S. and Ryzhik, I .M., (2007). *Tables of Integrals, Series, and Products*, Academic Press, New York.

Hesham, M. R. and Soha, A. O., (2017). The Topp-Leone Burr-XII distribution: Properties and applications. British *Journal of Mathematics and Computer Science*, 21, pp. 1–15.

Hinkley, D., (1977). On quick choice of power transformations. *Journal of the Royal Statistical Society, Series (c), Applied Statistics*, 26, pp. 67–69.

Korkmaz, M., Ç., Altun, E., Yousof, H. M. and Hamedani, G. G. (2020). The Hjorth's IDB Generator of Distributions: Properties, Characterizations, Regression Modeling and Applications. *Journal of Statistical Theory and Applications*, 19(1), pp. 59–74.

Macdonald, P. D. M., (1971). Comment on "An estimation procedure for mixtures of distributions" by Choi and Bulgren. *J R Stat Soc B*, 33, pp. 326–329.

Mead, M. E., (2014). A new generalization of Burr XII distribution. *Journal of Statistics: Advances in Theory and Applications*, 12, pp. 53–71.

Paranaíba, P. F., Ortega, E. M. M., Cordeiro, G. M. and Pescim, R. R., (2011). The beta Burr XII distribution with application to lifetime data. *Computational Statistics and Data Analysis*, 55, pp. 1118–1136.

Paranaíba, P. F., Ortega, E. M. M., Cordeiro, G. M. and de Pascoa, M. A. R., (2013). The Kumaraswamy Burr XII distribution: theory and practice. *Journal of Statistical Computation and Simulation*, 83, pp. 2117–2143.

Shibu, D. S. and Irshad, M. R., (2016). Extended new generalized Lindley Distribution . *Statistica*, 76, pp. 41–55.

Swain, J., Venkatraman, S. and Wilson, J., (1988). Least squares estimation of distribution function in Johnson's translation system. *Journal of Statistical Computation and Simulation*, 29, pp. 271–297.

Therneau, T. M., Grambsch, P. M. and Fleming, T. R., (1990). Martingale-based residuals for survival models. *Biometrika*, 77, pp. 147–160.

Yousof, H. M., Altun, E. and Hamedani, G. G., (2018). A new extension of Fréchet distribution with regression models, residual analysis and characterizations. *Journal of Data Science*, 16(4), pp. 743–770.