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Bifurcation analysis and optimal control of a network-based SIR model with the impact of medical resources*

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Abstract. A new network-based SIR epidemic model, which incorporates the individual medical resource factor and public medical resource factor is proposed. It is verified that the larger the public medical resource factor, the smaller the control reproduction number, and the larger individual medical resource factor can weaken the spread of diseases. We found that the control reproduction number below unity is not enough to ensure global asymptotic stability of the disease-free equilibrium. When the number of hospital beds or the individual medical resource factor is small enough, the system will undergoes backward bifurcation. Moreover, the existence and uniqueness of the optimal control and two time-varying variables's optimal solutions are obtained. On the scale-free network, the level of optimal control is also proved to be different for different degrees. Finally, the theoretical results are illustrated by numerical simulations. This study suggests that maintaining sufficient both public medical resources and individual medical resources is crucial for the control of infectious diseases.

Keywords: medical resources, network, SIR model, backward bifurcation, optimal control.

1 Introduction

Medical resources have a significant impact on the prevention, control and treatment of infectious diseases. For instance, during the outbreak of COVID-19, in order to treat many

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patients, Chinese government build rapidly Huoshenshan and Leishenshan hospital and many cabin hospitals in Wuhan. When infected people are identified, take the measure of universal nucleic acid testing timely can be effective in inhibiting the large-scale outbreak. Because of the investment of these public medical resources, the COVID-19 has been well controlled.

In addition, individual medical resources also play a vital role in disease controlling. When infectious diseases occur, people will instinctively take some measures, which are related to individual medical resources to protect themselves. During the late stage of the SARS outbreak, some measures like wearing mask, quarantine, etc. have been proved to be useful in reducing the infection rate [12, 27]. In [29], Xiao and Ruan proposed a non-monotonous incidence rate to describe the psychological effect caused by the behavioural changes of susceptible individuals. The problem of psychological effect is also studied in [17] during the SARS outbreak, the protection measures taken by the susceptible individuals can indeed inhibit the spread of disease. Zhang and Sun established a new SIS epidemic model with feedback mechanism (people's instinctive response to infectious diseases) as well as birth and death. It is also proved that the feedback mechanism can weaken the spread of diseases and reduce the level of endemic diseases [31].

Recovery rate is highly dependent on medical conditions, in particular, the amount of medical resources invested. In most models of infectious diseases, the recovery rate is assumed to be proportional to the number of infected individuals, which means that the medical resources are plentiful [2]. In fact, the resources such as the number of health care workers and hospital beds, etc. are very limited, especially, during the stage of diseases outbreak [10,25,26]. Therefore, it is necessary to study the influence of medical resources on the dynamics of epidemic models.

To study the impact of hospital beds, Shan and Zhu [25] proposed a nonlinear recovery rate function $\gamma(b, I) = \gamma_0 + b(\gamma_1 - \gamma_0)/(b + I)$, where γ_0 and γ_1 are the minimum and maximum recovery rates, respectively. b is the number of hospital beds, I is the number of infected individuals. Abdelrazec et al. [1] applied the nonlinear recovery rate to explore the impact of hospital beds on the spread of dengue fever. It is proved that when the number of hospital beds is small enough, backward bifurcation and saddle-node bifurcation may happen [8]. The impact of public medical resources on the spread of infectious diseases is also studied in [19]. In general, these literatures involved theoretical analysis on bifurcation, and the related numerical simulation results are all non-network studies.

To describe the heterogeneity of contact, complex networks are therefore incorporated into the epidemic models. Pastor-Satorras and Vespignani proposed a popular network-based SIS epidemic model [22, 23]. Since then, a large number of related researches on network-based epidemic models have emerged, such as literatures [3, 4, 21]. A saturated treatment function $T(I) = rI/(1 + \alpha I)$ was introduced to describe a limited medical capacity in [33], where r is the cure rate, $\alpha \geqslant 0$ is used to describe the effect of delay due to limited medical resources. Li and Yousef studied a network-based SIR epidemic model with saturated treatment function in [18]. Huang and Li studied the complex dynamical properties of a network-based SIS epidemic model with saturated treatment function [13]. A condition, which can determine the direction of bifurcation at $R_0 = 1$, is derived in [13, 18].

The main problem of optimal control theory is to find the optimal control strategy so that the objective function reaches a maximum or minimum value under some constraints. In the past, optimal control theory was mainly applied to the study of low-dimensional epidemic systems [5, 14–16, 30]. However, there are still relatively few studies on considering the optimal control problems on networks [6, 7, 20, 32].

To our knowledge, there are few results on the bifurcation theory on networks. Besides, few people combined individual and public medical resources to consider the dynamics and optimal control problem of network-based epidemic models. So the research of this paper is still valuable, the highlights are summarized as follows:

- A new network-based SIR epidemic model with the impact of both individual medical resources and public medical resources on diseases transmission is established;
- Due to high dimensionality of the network model, the bifurcation theory of low-dimensional systems cannot be directly generalized to networks. The condition for the existence of the backward bifurcation is given by judging the sign of $\partial\theta/\partial R_0|_{(R_0,\theta)=(1,0)}$. It is proved that when the number of hospital beds in public medical resources or the individual medical resource factor is small enough, the model will exhibit the phenomenon of backward bifurcation;
- An optimal solution which can minimise the sum of infected density and the cost of medical resources is given, and the level of optimal control is also confirmed to be different for different degrees on the scale-free network.

The structure of this paper is as follows. In Section 2, a new network-based SIR epidemic model is proposed. The stability of the disease-free and endemic equilibrium is studied in Section 3. In Section 4, a condition, which decides the direction of bifurcation, is given. In Section 5, the optimal medical resources control strategy is discussed. The numerical simulations are performed in Section 6. Lastly, the conclusions and discussions are presented in Section 7.

2 Model description

A new network-based SIR epidemic model with birth and death will be proposed in this section. On the scale-free network, all the nodes are classified into n groups, and the nodes in the same group have the same degree. Besides, each node is assumed to have one of three states: susceptible, infected, or recovered. Suppose that $S_k(t)$, $I_k(t)$, $R_k(t)$, $N_k(t)$ be the densities of susceptible, infected, recovered and total nodes with degree k at time t, respectively. The dynamical mean-field equations of network-based SIR model are written as

$$\begin{aligned} S_k'(t) &= \Lambda - \beta k\big(1-\alpha\theta(t)\big)S_k(t)\theta(t) - dS_k(t),\\ I_k'(t) &= \beta k\big(1-\alpha\theta(t)\big)S_k(t)\theta(t) - dI_k(t) - \left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta(t)}\right)I_k(t),\\ R_k'(t) &= \left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta(t)}\right)I_k(t) - dR_k(t). \end{aligned} \tag{1}$$

Assume that the network is uncorrelated. Then $\theta(t)=(1/\langle k\rangle)\sum_{k=1}^{n} kp(k)I_k(t)$ is the probability that any given edge is connected to an infected node, and $\langle k\rangle=\sum_{k=1}^{n}kp(k)$ denotes the mean degree of the network, where $p(k)$ is the probability that a node is connected to k other nodes. The parameters in model (1) are described as follows: the birth rate is Λ; β denotes the transmission rate; d stands for the natural death rate; b represents the number of hospital beds in public medical resources; γ_0 is the recovery rate when people rely on their autoimmunity to fight infectious diseases; γ_1 is called public medical resource factor. More specifically, γ_1 denotes the maximum recovery rate when the amount of public medical resources invested increases.

α is called individual medical resource factor and is depended on the degree of self-protection to epidemic diseases. In other words, when epidemic diseases arrive, people will take some preventive measures, which related to individual medical resources (e.g., wear mask, wash hands frequently, get vaccinated and maintain social distance, etc.) to reduce the infection rate. Compared to the general form $\beta kS_k(t)\theta(t)$, we consider the influence of individual medical resource on the spread of infectious diseases. With the increase of infection density, the individual medical resource factor α will become larger, so that $(1-\alpha\theta(t))$ will be smaller, thus the spreading speed will decrease.

It is supposed that newborns are balanced by deaths, hence $\Lambda = d$. We note the solution set of system (1) as the bounded region $\Omega = \{(S_1, I_1, R_1, \ldots, S_n, I_n, R_n)$: $0 \leqslant S_k \leqslant 1,\ 0 \leqslant I_k \leqslant 1,\ 0 \leqslant R_k \leqslant 1,\ S_k + I_k + R_k = 1,\ k = 1, 2, \ldots, n\}$.

Remark 1. Ω is a positive invariant set of system (1). The proof is similar to the proof of Lemma 1 in [18], so it is omitted here.

3 Existence and stability of equilibrium points

3.1 Stability of the disease-free equilibrium

System (1) has a unique disease-free equilibrium given by $E_0 = (1, 0, 0, \ldots, 1, 0, 0) \in R^{3n}$, and the total density $N_k(t)$ are constant, $k = 1, 2, \ldots, n$, so we can only study the following system:

$$\begin{aligned} S_k'(t) &= d - \beta k\big(1-\alpha\theta(t)\big)S_k(t)\theta(t) - dS_k(t),\\ I_k'(t) &= \beta k\big(1-\alpha\theta(t)\big)S_k(t)\theta(t) - dI_k(t) - \left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta(t)}\right)I_k(t). \end{aligned} \tag{2}$$

We can obtain the control reproduction number $R_0 = \rho(FV^{-1})$ by using the next generation matrix method [9]. For model (2), the Jacobian matrices F and V are given by

$$F = \begin{pmatrix} O & O \\ O & F_{22} \end{pmatrix}_{(2n)\times(2n)}, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ O & V_{22} \end{pmatrix}_{(2n)\times(2n)},$$

$$F_{22} = V_{12} = \frac{\beta}{\langle k\rangle}\begin{pmatrix} p(1) & 2p(2) & \cdots & np(n) \\ 2p(1) & 2^2p(2) & \cdots & 2np(n) \\ \vdots & \vdots & \ddots & \vdots \\ np(1) & 2np(2) & \cdots & n^2p(n) \end{pmatrix}_{n\times n},$$

$V_{11} = dE$, $V_{22} = (d+\gamma_1)E$, E represents the $n \times n$ unit matrix, and O represents the $n \times n$ zero matrix. So, the control reproduction number is $R_0 = \beta\langle k^2\rangle/((d+\gamma_1)\langle k\rangle)$, where $\langle k^2\rangle = \sum_{k=1}^{n} k^2 p(k)$.

Remark 2. The control reproduction number is not related to the parameter α, in other words, the individual medical resource factor cannot change the control reproduction number R_0.

Remark 3. The larger the public medical resource factor γ_1, the smaller the control reproduction number R_0.

Furthermore, the Jacobian matrix of system (2) at E_0 is

$$J(E_0) := F - V = \begin{pmatrix} -dE & -F_{22} \\ O & J_{22} \end{pmatrix}_{(2n)\times(2n)},$$

where

$$(J_{22})_{ij} = \begin{cases} \frac{\beta i^2 p(i)}{\langle k\rangle} - d - \gamma_1, & i = j, \\ \frac{\beta ijp(j)}{\langle k\rangle}, & i \neq j. \end{cases}$$

Next, we will prove the local stability of the disease-free equilibrium E_0.

Theorem 1. *If $R_0 < 1$, then the disease-free equilibrium E_0 of system* (2) *is locally asymptotically stable, whereas if $R_0 > 1$ it is unstable.*

Proof. The local stability of E_0 depends on the sign of the real parts of the eigenvalues of the Jacobian matrix $J(E_0)$. Obviously, there is a n-multiple eigenvalue $-d$. The other eigenvalues depend on the block matrix J_{22}. In order to use the results of Lemma 1 in [28], we denote the (i,j)-entries of the matrix $J_{22} - \lambda E$ by

$$-(d+\gamma_1+\lambda)\delta_{ij} + \frac{\beta ijp(j)}{\langle k\rangle}.$$

Further, let $\sigma = -(d+\gamma_1+\lambda)$ and $p_i = \beta i, q_j = jp(j)/\langle k\rangle$. Then we can obtain

$$\det(J_{22} - \lambda E) = (-1)^{n-1}(d+\gamma_1+\lambda)^{n-1}\left(-(d+\gamma_1+\lambda) + \sum_{i=1}^{n} \frac{\beta i^2 p(i)}{\langle k\rangle}\right).$$

So, $\lambda = -(d+\gamma_1)$ is another $(n-1)$-multiple negative eigenvalue. Thus, the stability of E_0 is determined by the sign of the eigenvalue

$$\lambda = \sum_{i=1}^{n} \frac{\beta i^2 p(i)}{\langle k\rangle} - d - \gamma_1 = (d+\gamma_1)(R_0 - 1).$$

Therefore, we can conclude that the eigenvalues of the matrix $J(E_0)$ are all negative when $R_0 < 1$, E_0 is locally asymptotically stable. On the other hand, if $R_0 > 1$, E_0 is unstable. □

The proof of the global stability of E_0 will be given in the following theorem.

Theorem 2. *Denote*

$$\hat{R}_0 = \frac{\beta(1+b)\langle k^2\rangle}{(d+bd+\gamma_0+b\gamma_1)\langle k\rangle}.$$

Then E_0 is globally asymptotically stable if $\hat{R}_0 < 1$.

Proof. Since

$$\begin{aligned}\theta'(t) &= \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kp(k)\left[\beta k\big(1-\alpha\theta(t)\big)S_k(t)\theta(t) - dI_k(t) - \left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta(t)}\right)I_k(t)\right]\\ &\leqslant \frac{1}{\langle k\rangle}\sum_{k=1}^{n} kp(k)\left[\beta k\theta(t) - dI_k(t) - \left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta(t)}\right)I_k(t)\right]\\ &\leqslant \theta(t)\left[\beta\frac{\langle k^2\rangle}{\langle k\rangle} - d - \gamma_0 - \frac{b(\gamma_1-\gamma_0)}{b+1}\right] = \theta(t)\left[d+\gamma_0+\frac{b(\gamma_1-\gamma_0)}{b+1}\right](\hat{R}_0-1),\end{aligned}$$

when $\hat{R}_0 < 1$,

$$\theta'(t) \leqslant \theta(t)\left[d+\gamma_0+\frac{b(\gamma_1-\gamma_0)}{b+1}\right](\hat{R}_0-1) < 0,$$

and hence, $\lim_{t\to\infty}\theta(t) = 0$. Therefore, it can be concluded that $\lim_{t\to\infty} I_k(t) = 0$. Consequently, the disease-free equilibrium E_0 is globally attractive when $\hat{R}_0 < 1$. Combined with the result of Theorem 1, we can conclude that when $\hat{R}_0 < 1$, E_0 is globally asymptotically stable. □

Remark 4. Since $\gamma_0 < \gamma_1$, it can be verified that

$$R_0 = \frac{d+bd+\gamma_0+b\gamma_1}{d+bd+\gamma_1+b\gamma_1}\hat{R}_0 < \hat{R}_0.$$

Remark 5. If $\hat{R}_0 < 1$, that is,

$$R_0 < \frac{d+bd+\gamma_0+b\gamma_1}{d+bd+\gamma_1+b\gamma_1} < 1,$$

then E_0 is globally asymptotically stable.

Remark 6. It seems that $R_0 < 1$ is not enough to guarantee the global asymptotic stability of E_0. So system (1) may exist backward bifurcation, which will be proved in Section 4.

3.2 Existence of the endemic equilibrium

In this subsection, we will prove the existence of the positive equilibrium if $R_0 > 1$.

Theorem 3. *If $R_0 > 1$, system* (1) *admits at least an endemic equilibrium.*

Proof. Assume that $E^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \dots, S_n^*, I_n^*, R_n^*)$ is an endemic equilibrium of system (1), here each $E_k^* = (S_k^*, I_k^*, R_k^*)$, $k = 1, 2, \dots, n$. Then E_k^* satisfy

$$\beta k(1-\alpha\theta^*)S_k^*\theta^* - dI_k^* - \left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta^*}\right)I_k^* = 0,$$

$$\left(\gamma_0 + \frac{b(\gamma_1-\gamma_0)}{b+\theta^*}\right)I_k^* - dR_k^* = 0.$$

Then we can obtain

$$S_k^* = \frac{(d+\gamma_0)\theta^* + b(d+\gamma_1)}{\beta k\theta^*(1-\alpha\theta^*)(b+\theta^*)}I_k^*, \qquad R_k^* = \frac{\gamma_1 b + \gamma_0\theta^*}{bd + d\theta^*}I_k^*.$$

Substituting them into equation $S_k^* + I_k^* + R_k^* = 1$ yields

$$I_k^* = \frac{d\beta k\theta^*(1-\alpha\theta^*)(b+\theta^*)}{M},$$

where

$$M = d(d+\gamma_0)(b+\theta^*) + bd(\gamma_1-\gamma_0) + \beta k\theta^*(1-\alpha\theta^*)\big[(\gamma_0+d)\theta^* + b(\gamma_1+d)\big].$$

Since $\theta^* = (1/\langle k\rangle)\sum_{k=1}^n kp(k)I_k^*$, then a self-consistency equation is obtained as follows:

$$\theta^* = \frac{1}{\langle k\rangle}\sum_{k=1}^n \frac{k^2p(k)d\beta(1-\alpha\theta^*)(b+\theta^*)\theta^*}{M}.$$

We note the function

$$F(\theta) = \frac{1}{\langle k\rangle}\sum_{k=1}^n \frac{k^2p(k)d\beta(1-\alpha\theta)(b+\theta)}{M}.$$

Hence, endemic equilibrium should satisfy the equation $F(\theta) = 1$. Since $F(0) > 1$, $F(1) < 1$. This implies that system (1) admits at least an endemic equilibrium. □

4 Bifurcation analysis

In many of the literatures on network-based epidemic models, the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$. However, from the result of Theorem 2 it seems that $R_0 < 1$ is not enough to ensure the global asymptotic stability of E_0. The dynamical behavior of system (1) is complex. So, in this section, we will analyse whether the system admits a transcritical backward bifurcation at $R_0 = 1$. More specifically, a necessary and sufficient condition, which determines the direction of bifurcation will be derived.

Theorem 4. *System* (1) *exists backward bifurcation at* $R_0 = 1$ *if and only if* $b < \hat{b}$; *and the system undergoes forward bifurcation if and only if* $b > \hat{b}$, *where*

$$\hat{b} = \frac{d(\gamma_1-\gamma_0)\langle k^2\rangle^2}{(d+\gamma_1)[d\alpha\langle k^2\rangle^2 + (d+\gamma_1)\langle k\rangle\langle k^3\rangle]}, \quad \langle k^3\rangle = \sum_{k=1}^n k^3p(k).$$

Proof. The endemic equilibrium can be determined by the equation $F(\theta) = 1$. Replacing β by $(d+\gamma_1)\langle k\rangle R_0/\langle k^2\rangle$, we obtain the equation

$$\sum_{k=1}^{n} k^2 p(k) \frac{d(d+\gamma_1)(1-\alpha\theta)(b+\theta)R_0}{Q_0} = 1, \tag{3}$$

where

$$\begin{aligned} Q_0 = {} & d(d+\gamma_0)(b+\theta)\langle k^2\rangle + bd(\gamma_1-\gamma_0)\langle k^2\rangle \\ & + (\gamma_1+d)k\langle k\rangle\theta(1-\alpha\theta)\big[(\gamma_0+d)\theta + b(\gamma_1+d)\big]R_0. \end{aligned} \tag{4}$$

If we keep in mind that θ is a function of R_0, then the direction of bifurcation is depended on the sign of $\partial\theta/\partial R_0|_{(R_0,\theta)=(1,0)}$. More exactly, if $\partial\theta/\partial R_0|_{(R_0,\theta)=(1,0)} < 0$, then backward bifurcation occurs at $R_0 = 1$. Conversely, the forward bifurcation happens.

Next, taking the deviation of equation (3) associated with R_0, the implicit function theorem yields the following equation:

$$\sum_{k=1}^{n} k^2 p(k) \frac{Q_1 - Q_2}{Q_0^2} = 0, \tag{5}$$

where

$$\begin{aligned} Q_1 = {} & \left[d(d+\gamma_1)(1-\alpha\theta)(b+\theta) + d(d+\gamma_1)(1-b\alpha-2\alpha\theta)R_0\frac{\partial\theta}{\partial R_0}\right]Q_0, \\ Q_2 = {} & d(d+\gamma_1)(1-\alpha\theta)(b+\theta) \\ & \times R_0\bigg\{d(d+\gamma_0)\langle k^2\rangle\frac{\partial\theta}{\partial R_0} + (d+\gamma_1)k\langle k\rangle\theta(1-\alpha\theta)\big[(d+\gamma_0)\theta + b(d+\gamma_1)\big] \\ & + (d+\gamma_1)k\langle k\rangle\big[(d+\gamma_0)\theta + b(d+\gamma_1)\big](1-2\alpha\theta)R_0\frac{\partial\theta}{\partial R_0} \\ & + (d+\gamma_1)(d+\gamma_0)k\langle k\rangle\theta(1-\alpha\theta)R_0\frac{\partial\theta}{\partial R_0}\bigg\}. \end{aligned}$$

Substituting $(R_0,\theta) = (1,0)$ into equation (5), we can obtain the following equation by simple calculation and arrangement:

$$\left(\frac{d+\gamma_0}{b(d+\gamma_1)} + \frac{(d+\gamma_1)\langle k\rangle\langle k^3\rangle}{d\langle k^2\rangle^2} - \frac{1-\alpha b}{b}\right)\frac{\partial\theta}{\partial R_0} = 1.$$

Thus

$$\left.\frac{\partial\theta}{\partial R_0}\right|_{(R_0,\theta)=(1,0)} < 0 \quad\Longleftrightarrow\quad \frac{d+\gamma_0}{b(d+\gamma_1)} + \frac{(d+\gamma_1)\langle k\rangle\langle k^3\rangle}{d\langle k^2\rangle^2} - \frac{1-\alpha b}{b} < 0,$$

that is,

$$b < \frac{d(\gamma_1-\gamma_0)\langle k^2\rangle^2}{(d+\gamma_1)[d\alpha\langle k^2\rangle^2 + (d+\gamma_1)\langle k\rangle\langle k^3\rangle]}.$$

The same holds true also for the other proposition. □

Remark 7. It is a fact that the number of hospital beds in public medical resources does play a key role in determining whether or not backward bifurcation exists. If b is small enough to satisfy the result of Theorem 4, backward bifurcation will occur. Otherwise, when b is big enough, there is no backward bifurcation.

This also indicates the existence of a critical value $\hat{b}$ for the number of hospital beds. Therefore, government departments should provide enough hospital beds such that $b > \hat{b}$, so that the infectious diseases can be eradicated when $R_0 < 1$.

Remark 8. Since

$$\left.\frac{\partial\theta}{\partial R_0}\right|_{(R_0,\theta)=(1,0)} < 0 \quad \Longleftrightarrow \quad \alpha < \frac{d(\gamma_1-\gamma_0)\langle k^2\rangle^2 - b(d+\gamma_1)^2\langle k\rangle\langle k^3\rangle}{bd(d+\gamma_1)\langle k^2\rangle^2},$$

it is suggested that the individual medical resource factor also affects the dynamics of system (1). When α is small enough to satisfy the condition of the inequality, the system will appear backward bifurcation. So, it is also important to provide sufficient individual medical resources for the control of infectious diseases.

5 Optimal medical resources control

Considering the danger of infectious diseases, people will instinctively use their individual medical resources to take some self-protective measures to reduce the probability of being infected. At the same time, the government will invest more resources to increase the public medical resource factor. All of these measures, which can naturally reduce the number of infected, will take a certain cost. The more medical resources invested in health care, the lower the transmission rate, the larger the public medical resource factor. Of course, the corresponding cost will be higher to a certain degree. In order to minimise the density of infected individuals and reduce the medical resources cost at the same time, a feasible approach is to use optimal control theory. Then model (1) is rewritten as the following control system:

$$\begin{aligned}
S_k'(t) &= d - \beta k\big(1-\alpha_k(t)\theta(t)\big)S_k(t)\theta(t) - dS_k(t),\\
I_k'(t) &= \beta k\big(1-\alpha_k(t)\theta(t)\big)S_k(t)\theta(t) - dI_k(t) - \left(\gamma_0 + \frac{b(\gamma_k(t)-\gamma_0)}{b+\theta(t)}\right)I_k(t),\\
R_k'(t) &= \left(\gamma_0 + \frac{b(\gamma_k(t)-\gamma_0)}{b+\theta(t)}\right)I_k(t) - dR_k(t),
\end{aligned} \tag{6}$$

where $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t), \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)$ are the control variables.

Suppose

$$\begin{aligned}
U = \big\{&u(t) = (u_1(t), u_2(t), \ldots, u_n(t), u_{n+1}(t), \ldots, u_{2n}(t))\colon\ u_k(t) \text{ is measurable,}\\
&0 \leqslant u_k(t) = \alpha_k(t) \leqslant \alpha_0,\ \gamma_0 \leqslant u_{n+k}(t) = \gamma_k(t) \leqslant 1,\ t\in[0,T],\ k=1,2,\ldots,n\big\}
\end{aligned}$$

and terminal time $T > 0$.

Since our control goal is to decrease both the prevalence of contagious disease and the medical resources cost, refer to the construction method of the objective function for the optimal control problems on networks [6, 7, 20, 32], we define the objective function as follows:

$$J(u) = \int_0^T \sum_{k=1}^n \left[I_k(t) + \frac{1}{2} B_k \alpha_k^2(t) + \frac{1}{2} C_k \gamma_k^2(t) \right] \mathrm{d}t,$$

with Lagrangian

$$L = \sum_{k=1}^n \left[I_k(t) + \frac{1}{2} B_k \alpha_k^2(t) + \frac{1}{2} C_k \gamma_k^2(t) \right],$$

where $B_k, C_k > 0$, $k = 1, 2, \ldots, n$.

Therefore, the problem of optimal medical resources control can be described as

$$\min J(u) \quad \text{s.t. } u \in U.$$

For obtaining the solution of the optimal control, we will use Corollary 4.1 in [11] to prove the existence of the optimal control firstly.

Theorem 5. *There exists an optimal solution $u^*(t)$ subject to $J(u^*) = \min_{u \in U} J(u)$ and satisfies control system* (6)*.*

Proof. Next, the five conditions [11] in Corollary 4.1 will be proved true.

(i) Obviously, U is closed and convex.

(ii) Because of the definition of U and nonnegativity of $S_k(t)$, $I_k(t)$, $R_k(t)$, so the solution of system (6) exists [24].

(iii) Let $x_k(t) = (S_k(t), I_k(t), R_k(t))^\top$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^\top$, so system (6) can be expressed as $\mathrm{d}x(t)/\mathrm{d}t =: G(x(t))$, $G(x(t))$ is obviously continuous. Besides

$$-\beta k S_k(t) - d S_k(t) \leqslant -\beta k \big(1 - \alpha_k(t)\theta(t)\big) S_k(t)\theta(t) - d S_k(t) \leqslant S_k'(t) \leqslant d,$$

$$(-d-1) I_k(t) \leqslant -d I_k(t) - \left(\gamma_0 + \frac{b(\gamma_k(t) - \gamma_0)}{b + \theta(t)} \right) I_k(t) \leqslant I_k'(t) \leqslant \beta k S_k(t),$$

$$-d R_k(t) \leqslant R_k'(t) \leqslant \left(\gamma_0 + \frac{b(\gamma_k(t) - \gamma_0)}{b + \theta(t)} \right) I_k(t) \leqslant I_k(t),$$

so $G(x(t))$ can be bounded by a linear function of $x(t)$.

(iv) $\partial^2 L/\partial \alpha_k^2 = B_k > 0$, $\partial^2 L/\partial \gamma_k^2 = C_k > 0$, then the function $L(x(t), u_k(t))$ is convex on U.

(v) Since $I_k(t) \geqslant 0$,

$$L\big(x(t), u_k(t)\big) \geqslant \sum_{k=1}^n \left[\frac{1}{2} B_k \alpha_k^2(t) + \frac{1}{2} C_k \gamma_k^2(t) \right] \geqslant \frac{1}{2} \min\{B_k, C_k\} \big\| u(t) \big\|_2^2.$$

There exist $\varphi = 2$, $c_1 = \min\{B_k, C_k\}/2$, $c_2 = 0$ subject to

$$L\big(x(t), u_k(t)\big) \geqslant c_1 \big\|u(t)\big\|_2^{\varphi} + c_2.$$

Therefore, the optimal solution exists. □

The optimal solution will be solved by using the Pontryagin's minimum principle [11]. Define the Hamiltonian H for optimal control problem as follows:

$$\begin{aligned} H = {} & \sum_{k=1}^{n} \left[I_k(t) + \frac{1}{2} B_k \alpha_k^2(t) + \frac{1}{2} C_k \gamma_k^2(t) \right] \\ & + \sum_{k=1}^{n} \big[\lambda_{1k}(t) S_k'(t) + \lambda_{2k}(t) I_k'(t) + \lambda_{3k}(t) R_k'(t) \big]. \end{aligned}$$

$\lambda_{1k}(t)$, $\lambda_{2k}(t)$, $\lambda_{3k}(t)$ are the adjoint variables, which will be determined below.

Theorem 6. *For model* (6)*, suppose* $(S_k^*(t), I_k^*(t), R_k^*(t))$, $k = 1, 2, \ldots, n$, *be the optimal solution associated to the optimal control* $u^*(t) = (u_1^*(t), u_2^*(t), \ldots, u_{2n}^*(t))$ *and* $\theta^*(t) = (1/\langle k \rangle) \sum_{k=1}^{n} kp(k) I_k^*(t)$. *Then the adjoint variables* $\lambda_{1k}(t), \lambda_{2k}(t), \lambda_{3k}(t)$ *satisfy the following equations*:

$$\begin{aligned} \lambda_{1k}'(t) &= \beta k \big(1 - \alpha_k^*(t)\theta^*(t)\big)\theta^*(t)\big(\lambda_{1k}(t) - \lambda_{2k}(t)\big) + d\lambda_{1k}(t), \\ \lambda_{2k}'(t) &= -1 + \frac{\beta k p(k)}{\langle k \rangle} \sum_{i=1}^{n} \big(1 - 2\alpha_i^*(t)\theta^*(t)\big) i S_i^*(t)\big(\lambda_{1i}(t) - \lambda_{2i}(t)\big) + d\lambda_{2k}(t) \\ &\quad + \frac{bkp(k)}{(b + \theta^*(t))^2 \langle k \rangle} \sum_{i=1}^{n} I_i^*(t)\big(\gamma_i^*(t) - \gamma_0\big)\big(\lambda_{3i}(t) - \lambda_{2i}(t)\big) \\ &\quad + \left(\gamma_0 + \frac{b(\gamma_k^*(t) - \gamma_0)}{b + \theta^*(t)}\right)\big(\lambda_{2k}(t) - \lambda_{3k}(t)\big), \\ \lambda_{3k}'(t) &= d\lambda_{3k}(t) \end{aligned} \tag{7}$$

with the transversality condition $\lambda_{1k}(T) = \lambda_{2k}(T) = \lambda_{3k}(T) = 0$, $k = 1, 2, \ldots, n$. *Furthermore, the optimal controls* $\alpha_k^*(t)$, $\gamma_k^*(t)$ *are obtained by*

$$\begin{aligned} \alpha_k^*(t) &= \min\left\{ \max\left(0, \frac{\beta k S_k^*(t) \theta^{*2}(t)(\lambda_{2k}(t) - \lambda_{1k}(t))}{B_k}\right), \alpha_0 \right\}, \\ \gamma_k^*(t) &= \min\left\{ \max\left(\gamma_0, \frac{b I_k^*(t)(\lambda_{2k}(t) - \lambda_{3k}(t))}{C_k(b + \theta^*(t))}\right), 1 \right\}. \end{aligned}$$

Proof. According to the Pontryagin's minimum principle with the Hamiltonian function [11], we can calculate the adjoint variables by the following equations:

$$\begin{aligned} \lambda_{1k}'(t) &= -\frac{\partial H}{\partial S_k}\bigg|_{S_k(t) = S_k^*(t),\, I_k(t) = I_k^*(t),\, R_k(t) = R_k^*(t),\, \alpha_k(t) = \alpha_k^*(t),\, \gamma_k(t) = \gamma_k^*(t)} \\ &= \beta k \big(1 - \alpha_k^*(t)\theta^*(t)\big)\theta^*(t)\big(\lambda_{1k}(t) - \lambda_{2k}(t)\big) + d\lambda_{1k}(t), \end{aligned}$$

$$\begin{aligned}
\lambda'_{2k}(t) &= -\frac{\partial H}{\partial I_k}\bigg|_{S_k(t)=S_k^*(t),\, I_k(t)=I_k^*(t),\, R_k(t)=R_k^*(t),\, \alpha_k(t)=\alpha_k^*(t),\, \gamma_k(t)=\gamma_k^*(t)} \\
&= -1 + \frac{\beta k p(k)}{\langle k\rangle}\sum_{i=1}^{n}\big(1-2\alpha_i^*(t)\theta^*(t)\big)iS_i^*(t)\big(\lambda_{1i}(t)-\lambda_{2i}(t)\big) + d\lambda_{2k}(t) \\
&\quad + \frac{bkp(k)}{\big(b+\theta^*(t)\big)^2\langle k\rangle}\sum_{i=1}^{n} I_i^*(t)\big(\gamma_i^*(t)-\gamma_0\big)\big(\lambda_{3i}(t)-\lambda_{2i}(t)\big) \\
&\quad + \left(\gamma_0 + \frac{b(\gamma_k^*(t)-\gamma_0)}{b+\theta^*(t)}\right)\big(\lambda_{2k}(t)-\lambda_{3k}(t)\big), \\
\lambda'_{3k}(t) &= -\frac{\partial H}{\partial R_k}\bigg|_{S_k(t)=S_k^*(t),\, I_k(t)=I_k^*(t),\, R_k(t)=R_k^*(t),\, \alpha_k(t)=\alpha_k^*(t),\gamma_k(t)=\gamma_k^*(t)} \\
&= d\lambda_{3k}(t)
\end{aligned}$$

with the transversality condition $\lambda_{1k}(T)=\lambda_{2k}(T)=\lambda_{3k}(T)=0$, $k=1,2,\dots,n$.

In addition, the optimal control should satisfy the necessary condition

$$\frac{\partial H}{\partial \alpha_k}\bigg|_{S_k(t)=S_k^*(t),\, I_k(t)=I_k^*(t),\, R_k(t)=R_k^*(t),\, \alpha_k(t)=\alpha_k^*(t),\, \gamma_k(t)=\gamma_k^*(t)} = 0,$$

i.e.,

$$\beta k S_k^*(t)\theta^{*2}(t)\lambda_{1k}(t) - \beta k S_k^*(t)\theta^{*2}(t)\lambda_{2k}(t) + B_k\alpha_k^*(t) = 0, \quad 0 \leqslant \alpha_k^*(t) \leqslant \alpha_0.$$

$$\frac{\partial H}{\partial \gamma_k}\bigg|_{S_k(t)=S_k^*(t),\, I_k(t)=I_k^*(t),\, R_k(t)=R_k^*(t),\, \alpha_k(t)=\alpha_k^*(t),\, \gamma_k(t)=\gamma_k^*(t)} = 0,$$

i.e.,

$$-\frac{bI_k^*(t)}{b+\theta^*(t)}\lambda_{2k}(t) + \frac{bI_k^*(t)}{b+\theta^*(t)}\lambda_{3k}(t) + C_k\gamma_k^*(t) = 0, \quad \gamma_0 \leqslant \gamma_k^*(t) \leqslant 1.$$

So the optimal control

$$\alpha_k^*(t) = \begin{cases} 0, & \frac{\beta k S_k^*(t)\theta^{*2}(t)(\lambda_{2k}(t)-\lambda_{1k}(t))}{B_k} \leqslant 0; \\ \frac{\beta k S_k^*(t)\theta^{*2}(t)(\lambda_{2k}(t)-\lambda_{1k}(t))}{B_k}, & 0 < \frac{\beta k S_k^*(t)\theta^{*2}(t)(\lambda_{2k}(t)-\lambda_{1k}(t))}{B_k} < \alpha_0; \\ \alpha_0, & \frac{\beta k S_k^*(t)\theta^{*2}(t)(\lambda_{2k}(t)-\lambda_{1k}(t))}{B_k} \geqslant \alpha_0. \end{cases}$$

$$\gamma_k^*(t) = \begin{cases} \gamma_0, & \frac{bI_k^*(t)(\lambda_{2k}(t)-\lambda_{3k}(t))}{C_k(b+\theta^*(t))} \leqslant \gamma_0; \\ \frac{bI_k^*(t)(\lambda_{2k}(t)-\lambda_{3k}(t))}{C_k(b+\theta^*(t))}, & \gamma_0 < \frac{bI_k^*(t)(\lambda_{2k}(t)-\lambda_{3k}(t))}{C_k(b+\theta^*(t))} < 1; \\ 1, & \frac{bI_k^*(t)(\lambda_{2k}(t)-\lambda_{3k}(t))}{C_k(b+\theta^*(t))} \geqslant 1. \end{cases}$$

Therefore, substituting the values of $\alpha_k^*(t)$ and $\gamma_k^*(t)$ into system (6) and (7), we can get the optimality system with initial condition $0 < S_k^*(0) < 1$, $0 < I_k^*(0) < 1$, $R_k^*(0) = 0$, $S_k^*(0) + I_k^*(0) = 1$ and the transversality condition $\lambda_{1k}(T) = \lambda_{2k}(T) = \lambda_{3k}(T) = 0$, $k = 1, 2, \dots, n$. □

6 Simulations

Next, we plot all the figures by MATLAB to validate the previous theoretical results. Our simulations are based on the scale-free network with $p(k) = \eta k^{-2.5}$, $k = 1, 2, \ldots, 100$, and the constant η is chosen to satisfy the equation $\sum_{k=1}^{100} p(k) = 1$.

Firstly, we select the following parameters: $d = 0.03$, $\beta = 0.02$, $\alpha = 0.008$, $b = 0.18$, $\gamma_0 = 0.06$, $\gamma_1 = 0.5$. Thus, one can obtain $R_0 = 0.2907$, $\hat{R}_0 = 0.9807$. It is known that E_0 is globally asymptotically stable from the result of Theorem 2. We use 5 different initial conditions to plot the time evolution of the average density $S(t) = \sum_{k=1}^{100} S_k(t)p(k)$, $I(t) = \sum_{k=1}^{100} I_k(t)p(k)$. It can be seen that $\lim_{t\to\infty}(S(t), I(t)) = (1, 0)$ from Fig.1. Hence, this does indeed support the global stability of E_0.

Secondly, we will verify that $R_0 < 1$ is not sufficient to ensure the global asymptotic stability of E_0. The parameters are chosen as follows: $d = 0.01$, $\beta = 0.005$, $\alpha = 0.002$, $\gamma_0 = 0.005$, $\gamma_1 = 0.06$, so $R_0 = 0.5503 < 1$, $\hat{R}_0 = 2.5132 > 1$, so E_0 is not globally asymptotically stable from the result of Theorem 2. From Fig. 2 it can be observed that the trajectories partially converge to zero or move towards to a positive level. The result of the coexistence of two locally asymptotically stable equilibriums is named as the bistable phenomenon.

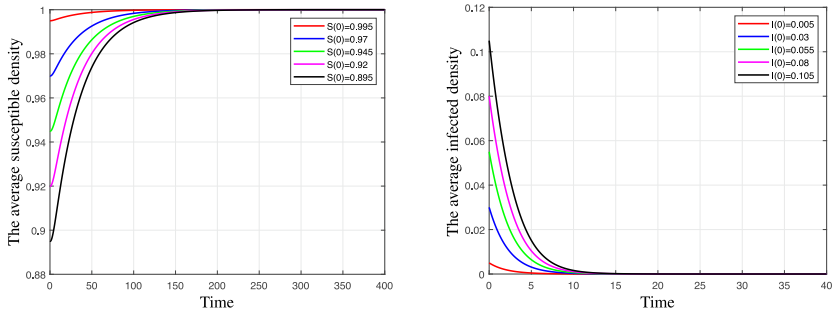


Figure 1. The time evolution of the average density $S(t), I(t)$ with 5 different initial conditions, when $\hat{R}_0 = 0.9807 < 1$.

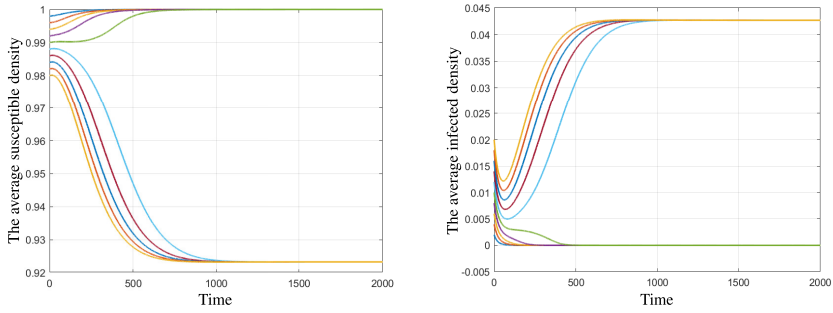
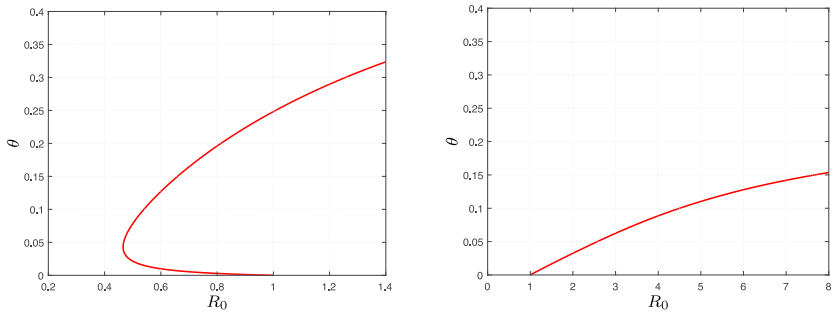


Figure 2. The time evolution of the average density $S(t), I(t)$ with 10 different initial conditions, when $R_0 = 0.5503 < 1$, $\hat{R}_0 = 2.5132 > 1$ and $b = 0.006$.



(a) $b = 0.006$ (b) $b = 0.2$

Figure 3. Bifurcation diagrams in the $R_0\theta$-plane with different values of b.

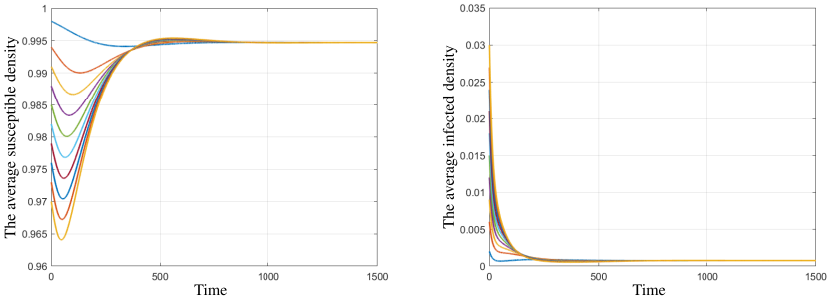


Figure 4. The time evolution of the average density $S(t), I(t)$ with 10 different initial conditions, when $R_0 = 1.1006 > 1$.

Different values of b are taken to show the bifurcation diagrams in Fig. 3. The left column displays the occurrence of backward bifurcation at $R_0 = 1$ when $b = 0.006 < \hat{b} = 0.0239$, and the right column shows forward bifurcation diagram when $b = 0.2 > \hat{b}$. Hence, the numerical simulation results are in agreement with the conclusions of Theorem 4. Besides, for backward bifurcation, one endemic state should be unstable and the other endemic state should be stable, although it is still difficult to prove theoretically.

For the simulation about the existence of the endemic equilibrium, the parameters are set the same values as those in Fig. 2, except that $\beta = 0.01$, $\alpha = 0.008$, $b = 3$. By a simple calculation, $R_0 = 1.1006 > 1$, and hence, model (1) exists endemic equilibrium. Observing the simulation results displayed in Fig. 4, we can see that all trajectories move towards to a positive constant. This phenomenon shows that the endemic equilibrium may be stable, although it is still difficult to prove theoretically.

Thirdly, we will investigate the effect of individual medical resource factor on diseases transmission through numerical simulations. In Fig. 5(a), the parameters are the same as those in Fig. 1 and $\hat{R}_0 = 0.9807 < 1$. Although diseases will eventually become extinct for different α, the larger α can slow down the spread of diseases. The parameters in Fig. 5(b) take the same values as those in Fig. 4, and $R_0 = 1.1006 > 1$. It can be observed that the larger α, the lower the endemic level. This can be explained that the

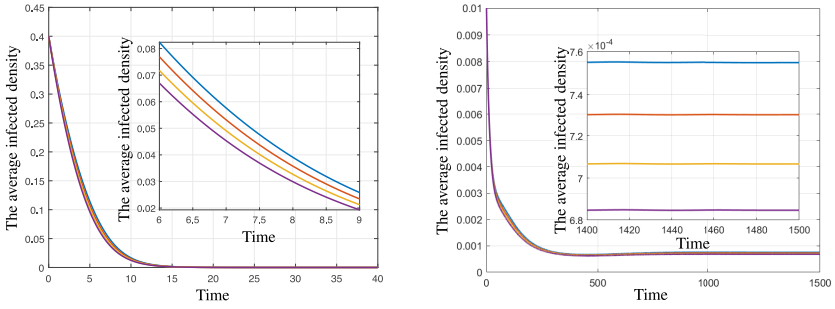


Figure 5. The average density $I(t)$ versus time corresponding to different α, which are $3, 2, 1, 0$ from bottom to top. Subfigures (a) and (b) are achieved with identical initial value $I_k(0) = 0.4$, $I_k(0) = 0.01$, respectively.

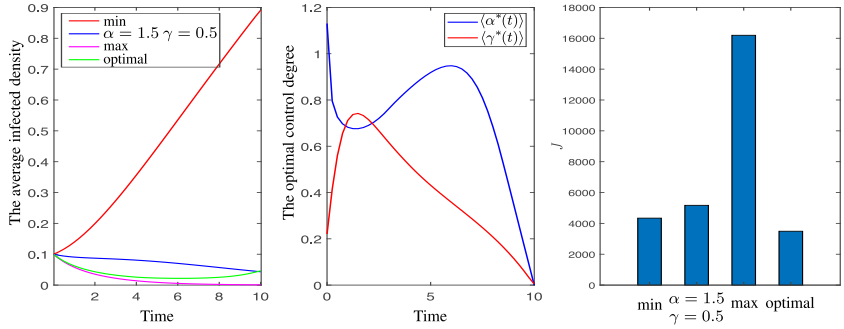


Figure 6. Sensitivity of control strategies with respect to three kinds of constant control ($\alpha_k(t) = \alpha$, $\gamma_k(t) = \gamma$) and optimal control. The minimum control strength corresponds to $\alpha = 0$, $\gamma = \gamma_0 = 0.005$, while $\alpha = \alpha_0 = 3$, $\gamma = 1$ corresponds to the maximum control.

more dangerous infectious diseases, the larger individual medical resource factor, and the density of infected individuals will be smaller, which is in accordance with the laws of real life.

Finally, numerical simulations are presented to illustrate the proposed optimal control solution in Section 5. The optimal control problem is solved with the help of fourth-order Runge–Kutta algorithm. The values of parameters are taken as $d = 0.03$, $\beta = 0.15$, $\alpha_0 = 3$, $\gamma_0 = 0.005$, $b = 0.35$, and we choose the weight parameters $B_k = 0.8$, $C_k = 0.5$, $T = 10$. The initial conditions are $S_k(0) = 0.9$, $I_k(0) = 0.1$, $R_k(0) = 0$ and $0 \leqslant \alpha_k(t) \leqslant \alpha_0, \gamma_0 \leqslant \gamma_k(t) \leqslant 1, k = 1, 2, \ldots, 100$. Through simulation, the mean value of optimal control is obtained $\langle \alpha^*(t) \rangle = \sum_{k=1}^{100} \alpha_k^*(t)/100$, $\langle \gamma^*(t) \rangle = \sum_{k=1}^{100} \gamma_k^*(t)/100$. The average of the control measures are shown in the middle column of Fig. 6. The left column shows the average infected density with different control strategies, and the right column shows the cost. It can be seen from Fig. 6 that in different cases, the optimal control solutions can indeed make the density of infected to relatively smaller, but the cost is the lowest. Therefore, the optimal control does achieve better results than constant control.

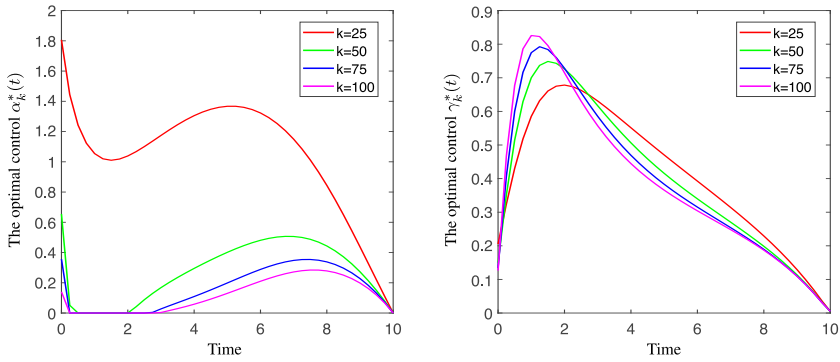


Figure 7. The dynamical change process of different optimal controls $\alpha_k^*(t)$, $\gamma_k^*(t)$, $k = 25, 50, 75, 100$.

The trajectories of the optimal controls $\alpha_k^*(t)$, $\gamma_k^*(t)$ ($k = 25, 50, 75, 100$) can be seen in Fig. 7. For different degrees, the optimal control $\alpha_k^*(t)$ all decrease at first, then increase and decrease again as time evolves. However, the optimal control level is different. The level of the optimal control $\alpha_k^*(t)$ will be lower for the larger degree. The optimal control $\gamma_k^*(t)$ increase at the beginning and then decrease as time goes on. In addition, we can also find that the larger the degree k, the higher the level of optimal control $\gamma_k^*(t)$ within a short time. But as time evolves, the opposite is true.

7 Conclusions and discussions

It is well known that the investment of medical resources is effective in suppressing or reducing outbreaks of epidemics. However, there are few results on considering the epidemic model with the influence of medical resources on networks. Many scholars proved that the number of hospital beds play an important role in the occurrence of backward bifurcation [1, 8, 19, 25], but most of them studied the homogeneously mixed models. In [13, 18], Li and his collaborators given a condition deciding the existence of backward bifurcation on networks, but they did not study quantitatively the impact of medical resources on epidemic spreading process. In this paper, we studied the influence of both individual medical resources and public medical resources on the dynamics of model (1). It has been proved that the system will undergoes backward bifurcation when the number of hospital beds or the individual medical resource factor is small enough, although they can not change the value of R_0. Besides, the larger the public medical resource factor γ_1, the smaller the control reproduction number R_0, and the larger individual medical resource factor α can slow down the spread of diseases and decrease the size of the associated endemic equilibria. Therefore, our work reveals the importance of both the public medical resources and individual medical resources for the suppressing of epidemic spreading.

We also discussed an optimal control problem by adopting two control variables and given their corresponding solutions. The optimal solution is much more effective in minimizing the density of infectious individual as well as the cost of medical resources

(see Fig. 6). In addition, the magnitude of the degree also affects the optimal control strength on the scale-free network.

At present, the bifurcation theory on networks is still less, it will be interesting and important to excavate more abundant theory, such as the study of Hopf bifurcation, saddle-node, transcritical and Bogdanov–Takens bifurcation, etc. The problem of optimal medical resources control has been discussed in this article. In future, we will make efforts to study the distribution problem of medical resources from different angles.

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