Chapter

Solving and Algorithm for Least-Norm General Solution to Constrained Sylvester Matrix Equation

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Abstract

Keeping in view that a lot of physical systems with inverse problems can be written by matrix equations, the least-norm of the solution to a general Sylvester matrix equation with restrictions $A_1X_1 = C_1, X_1B_1 = C_2,\ \ A_2X_2 = C_3, X_2B_2 = C_4,\ \ A_3X_1B_3 + A_4X_2B_4 = C_c,$ is researched in this chapter. A novel expression of the general solution to this system is established and necessary and sufficient conditions for its existence are constituted. The novelty of the proposed results is not only obtaining a formal representation of the solution in terms of generalized inverses but the construction of an algorithm to find its explicit expression as well. To conduct an algorithm and numerical example, it is used the determinantal representations of the Moore–Penrose inverse previously obtained by one of the authors.

Keywords: linear matrix equation, generalized Sylvester matrix equation, Moore-Penrose inverse

1. Introduction

Standardly, we state $\mathbb{C}$ and $\mathbb{R}$, respectively, for the complex and real numbers. Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over $\mathbb{C}$, and $\mathbb{C}_r^{m\times n}$ stay for a subset of $m \times n$ complex matrices with rank r. The rank of A is denoted by both symbols $r(A)$ and $rankA$. The (complex) conjugate transpose matrix of $A \in \mathbb{C}^{m\times n}$ is written by A^* and a matrix $A \in \mathbb{C}^{n\times n}$ is said to be Hermitian if $A^* = A$. An identity matrix with feasible shape is denoted by I.

Definition 1.1. The Moore–Penrose (MP-) inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X to the following four Penrose equations

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA. \tag{4}$$

Matrices satisfying the eqs. (1) and (2) are known as reflexive inverses, denoted by A^+.

In addition, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ represent a pair of orthogonal projectors onto the kernels of A and A^*, respectively.

Mathematical models of physical systems with inverse problems especially those has a finite number of model parameters can be written by matrix equations. In particular, the Sylvester-type matrix equations have far-reaching applications in singular system control [1], system design [2], robust control [3], feedback [4], perturbation theory [5], linear descriptor systems [6], neural networks [7] and theory of orbits [8], etc.

Some recent work on generalized Sylvester matrix equations and their systems can be observed in [9–21]. In 2014, Bao [22] examined the least-norm and extremal ranks of the least square solution to the quaternion matrix equations

$$A_1X = C_1, XB_1 = C_2, \;\; A_3XB_3 = C_c. \tag{5}$$

Wang et al. [23] examined the expression of the general solution to the system

$$A_1X_1 = C_1, \quad A_2X_2 = C_3, A_3X_1 \quad B_3 + A_4X_2B_4 = C_c, \tag{6}$$

and as an application, the P-symmetric and P-skew-symmetric solution to

$$A_aX = C_a, A_bXB_b = C_b.$$

has been established. Li et al. [24] established a novel expression of the general solution of the system (6) and they computed the least-norm of general solution to (6). In 2009, Wang et al. [25] constituted the expression of the general solution to

$$\begin{aligned} A_1 \quad X_1 &= C_1, X_1B_1 = C_2, \\ A_2 \quad X_2 &= C_3, X_2B_2 = C_4, \\ A_3 \quad X_1B_3 &+ A_4X_2B_4 = C_c, \end{aligned} \tag{7}$$

and as an application, they explored the (P, Q)-symmetric solution to the system

$$A_aX = C_a, XB_b = C_b, A_cXB_c = C_c.$$

Some latest findings on the least-norm of matrix equations and (P, Q)-symmetric matrices can be consulted in [26–30]. Furthermore, our main system (7) is a special case of the following system

$$\begin{aligned} A_1X_1 &= C_1, X_2B_1 = D_1, \\ A_2X_3 &= C_2, X_3B_2 = D_2, \\ A_3X_4 &= C_3, X_4B_3 = D_3, \\ A_4X_1 + X_2 \quad B_4 + C_4X_3D_4 &+ C_5X_4D_5 = C_c, \end{aligned} \tag{8}$$

which has been investigated by Zhang in 2014.

Motivated by the latest interest of least-norm of matrix equations, we construct a novel expression of the general solution to the system (7) and apply this to investigate the least-norm of the general solution to the system (7) in this chapter. Observing that systems (5) and (6) are particular cases of our system (7), solving system (7) will encourage the least-norm to a wide class of problems.

We commence with the following lemmas which have crucial function in the construction of the chief outcomes of the following sections.

Lemma 1.2. *[31]. Let A, B, and C be given matrices over $\mathbb{C}$ with agreeable dimensions.* Then.

1. $r(A) + r(R_A B) = r(B) + r(R_B A) = r\begin{bmatrix} A & B \end{bmatrix}$.

2. $r(A) + r(CL_A) = r(C) + r(AL_C) = r\begin{bmatrix} A \\ C \end{bmatrix}$.

3. $r(B) + r(C) + r(R_B A L_C) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$.

Lemma 1.3. *[32]. Let A, B, and C be known matrices over $\mathbb{C}$ with right sizes. Then*

1. $A^{\dagger} = (A^* A)^{\dagger} A^* = A^* (AA^*)^{\dagger}$.

2. $L_A = L_A^2 = L_A^*, R_A = R_A^2 = R_A^*$.

3. $L_A (BL_A)^{\dagger} = (BL_A)^{\dagger}, (R_A C)^{\dagger} R_A = (R_A C)^{\dagger}$.

Lemma 1.4. *[33]. Let Φ, Ω be matrices over $\mathbb{C}$ and*

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}, \ \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix}, \ F = \Phi_2 L_{\Phi_1}, \ T = R_{\Omega_1} \Omega_2.$$

Then

$$L_\Phi = L_{\Phi_1} L_F, \ L_\Omega = \begin{bmatrix} L_{\Omega_1} & -\Omega_1^{\dagger} \Omega_2 L_T \\ 0 & L_T \end{bmatrix},$$
$$R_\Omega = R_T R_{\Omega_1}, \ R_\Phi = \begin{bmatrix} R_{\Phi_1} & 0 \\ -R_F \Phi_2 \Phi_1^{\dagger} & R_F \end{bmatrix},$$

where Φ_1^+, Ω_1^+ are any fixed reflexive inverses, L_{Φ_1} and R_{Ω_1} stand for the projectors $L_{\Phi_1} = I - \Phi_1^+ \Phi_1$, $R_{\Omega_1} = I - \Omega_1 \Omega_1^+$ induced by Φ_1, Ω_1, respectively.

Remark 1.5. Since the Moore-Penrose inverse is a reflexive inverse, this lemma can be used for the MP-inverse without any changes. It has taken place in ([32], Lemma 2.4).

Lemma 1.6. *[34]. Suppose that*

$$B_1 X C_1 + B_2 Y C_2 = A \tag{9}$$

is consistent linear matrix equation. Then.

1. *The general solution of the homogeneous equation*

$$B_1 X C_1 + B_2 Y C_2 = 0,$$

can be expressed by

$$X = X_1X_2 + X_3, \quad Y = Y_1Y_2 + Y_3,$$

where $X_1 - X_3$ *and* $Y_1 - Y_3$ *are general solution to the system*

$$B_1X_1 = -B_2Y_1, \;\; X_2C_1 = Y_2C_2, \;\; B_1X_3C_1 = 0, \;\; B_2Y_3C_2 = 0.$$

By computing the value of unknowns in above and using them in X and Y, *we have*

$$\begin{aligned} X &= S_1L_GUR_HT_1 + L_{B_1}V_1 + V_2R_{C_1}, \\ Y &= S_2L_GUR_HT_2 + L_{B_2}W_1 + W_2R_{C_2}, \end{aligned}$$

where $S_1 = \left[I_p, \;\; 0\right], S_2 = [0, \;\; I_s], T_1 = \begin{bmatrix} I_q \\ 0 \end{bmatrix}, \; T_2 = \begin{bmatrix} 0 \\ I_t \end{bmatrix}, G = [B_1, \;\; B_2],$ *and* $H = \begin{bmatrix} C_1 \\ -C_2 \end{bmatrix}$*; the matrices* U, V_1, V_2, W_1 *and* W_2 *are free to vary over* $\mathbb{C}$.

2. *Assume that* Eq. (9) *is solvable, then its general solution can be expressed as*

$$X = X_0 + X_1X_2 + X_3, \quad Y = Y_0 + Y_1Y_2 + Y_3,$$

where X_0 *and* Y_0 *are any pair of particular solutions to (9).*

It can also be written as

$$\begin{aligned} X &= X_0 + S_1L_GUR_HT_1 + L_{B_1}V_1 + V_2R_{C_1}, \\ Y &= Y_0 + S_2L_GUR_HT_2 + L_{B_2}W_1 + W_2R_{C_2}. \end{aligned}$$

Lemma 1.7. *[35]. Let* A_1, B_1, C_1, C_2 *be given matrices over* $\mathbb{C}$ *with agreeable sizes and* X_1 *to be determined. Then the system*

$$A_1X_1 = C_1, X_1B_1 = C_2, \tag{10}$$

is consistent if and only if

$$R_{A_1}C_1 = 0, \;\; C_2L_{B_1} = 0, \;\; A_1C_2 = C_1B_1. \tag{11}$$

Under these conditions, the general solution to (10) *can be established as*

$$X_1 = A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + L_{A_1}U_1R_{B_1},$$

where U_1 *is a free matrix over* $\mathbb{C}$ *with accordant dimension.*

Lemma 1.8. *[36]. Let A, B, and C be known matrices over* $\mathbb{C}$ *with agreeable dimensions, and X be unknown. Then the matrix equation*

$$AXB = C \tag{12}$$

is consistent if and only if $AA^\dagger CB^\dagger B = C$. *In this case, its general solution can be expressed as*

$$X = A^{\dagger}CB^{\dagger} + L_A V + WR_B, \tag{13}$$

where V, W are arbitrary matrices over $\mathbb{C}$ with appropriate dimensions.

In [37], it is proved that (13) is the least squares solution to (12), and its minimum norm least squares solution is $X_{LS} = A^{\dagger}CB^{\dagger}$.

Lemma 1.9. *[25]. Let A_i, B_i, C_i, $(i = 1, \ldots, 4)$, and C_c be given matrices over $\mathbb{C}$ with agreeable dimensions, and X_1, X_2 to be determined. Denote*

$$\begin{aligned}
&A = A_3 L_{A_1}, B = R_{B_1} B_3, C = A_4 L_{A_2}, D = R_{B_2} B_4,\\
&N = DL_B, M = R_A C, S = CL_M,\\
&E = C_c - A_3 A_1^{\dagger} C_1 B_3 - AC_2 B_1^{\dagger} B_3 - A_4 A_2^{\dagger} C_3 B_4 - CC_4 B_2^{\dagger} B_4.
\end{aligned}$$

Then the following conditions are tantamount:

1. *System* (7) *is resolvable.*

2. *The conditions in* (11) *are met and*

$$\begin{aligned}
&R_{A_2} C_3 = 0, \quad C_4 L_{B_2} = 0, \quad A_2 C_4 = C_3 B_2,\\
&R_M R_A E = 0, R_A E L_D = 0, E L_B L_N = 0, R_C E L_B = 0.
\end{aligned} \tag{14}$$

3. *The equalities in* (11) and (14) *are satisfied and*

$$MM^{\dagger} R_A D^{\dagger} D = R_A E, \quad CC^{\dagger} E L_B N^{\dagger} N = E L_B.$$

In these conditions, the general solution to the system (7) *can be written as*

$$\begin{aligned}
X_1 = {} & A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^{\dagger} + L_{A_1} A^{\dagger} E B^{\dagger} R_{B_1} - L_{A_1} A^{\dagger} C M^{\dagger} E B^{\dagger} R_{B_1} -\\
& - L_{A_1} A^{\dagger} S C^{\dagger} E N^{\dagger} D B^{\dagger} R_{B_1} - L_{A_1} A^{\dagger} S V_1 R_N D B^{\dagger} R_{B_1} +\\
& + L_{A_1} (L_A U_1 + Z_1 R_B) R_{B_1},
\end{aligned} \tag{15}$$

$$\begin{aligned}
X_2 = {} & A_2^{\dagger} C_3 + L_{A_2} C_4 B_2^{\dagger} + L_{A_2} M^{\dagger} R_A E D^{\dagger} R_{B_2} + L_{A_2} L_{M_b} S^{\dagger} S C^{\dagger} E N^{\dagger} R_{B_2}\\
& + L_{A_2} L_M \left(V_1 - S^{\dagger} S V_1 N N^{\dagger}\right) R_{B_2} + L_{A_2} W_1 R_D R_{B_2},
\end{aligned} \tag{16}$$

where U_1, V_1, W_1 and Z_1 are free matrices over $\mathbb{C}$ with agreeable dimensions.

Since the general solutions of considered systems are expressed in terms of generalized inverses, another goal of the paper is to give determinantal representations of the least-norm of the general solution to the system (7) based on determinantal representations of generalized inverses.

Due to the important role of generalized inverses in many application fields, considerable effort has been exerted toward the numerical algorithms for fast and accurate calculation of matrix generalized inverse. In general, most existing methods for their obtaining are iterative algorithms for approximating generalized inverses of complex matrices (some recent papers, see, e.g. [38–40]). There are only several direct methods for finding MP-inverse for an arbitrary complex matrix $A \in \mathbb{C}^{m \times n}$. The most famous is method based on singular value decomposition (SVD), i.e. if $A = U\Sigma V^*$, then $A^{\dagger} = V\Sigma^{\dagger} U^*$. The computational cost of this method is dominated by the cost of computing the SVD, which is several times higher than matrix–matrix multiplication. Another approach is constructing determinantal representations of the MP-inverse $A^{\dagger}$. A well-known determinantal representation of an ordinary inverse is the adjugate matrix with the cofactors in entries. It has an important theoretical significance and brings forth Cramer's rule for systems of linear equations. The same is desirable to have for the generalized inverses. Due to looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses (for the MP-inverse, see, e.g. [41, 42]). Because of the complexity of the previously obtained expressions of determinantal representations of the MP-inverse, they have little applicability.

In this chapter, we will use the determinantal representations of the MP-inverse recently obtained in [43].

Lemma 1.10. *[43, Theorem 2.2] If $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank} A = r$, then the Moore-Penrose inverse $A^{\dagger} = \left(a_{ij}^{\dagger}\right) \in \mathbb{C}^{n\times m}$ possess the following determinantal representations*

$$a_{ij}^{\dagger} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \left|(A^*A)_{.i}\left(a_{.j}^*\right)\right|_{\beta}^{\beta}}{\sum_{\beta\in J_{r,n}} |A^*A|_{\beta}^{\beta}} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \left|(AA^*)_{j.}\left(a_{i.}^*\right)\right|_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{r,m}} |AA^*|_{\alpha}^{\alpha}}. \tag{17}$$

Here $|A|_{\alpha}^{\alpha}$ denote a principal minor of A whose rows and columns are indexed by $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$,

$$L_{k,m} := \{\alpha : 1 \le \alpha_1 < \cdots < \alpha_k \le m\}, \text{and } I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}.$$

Also, $a_{.j}^*$ and $a_{i.}^*$ denote the jth column and the ith row of A^*, and $A_{i.}(b)$ and $A_{.j}(c)$ stand for the matrices obtained from A by replacing its ith row with the row vector $b \in \mathbb{C}^{1\times n}$ and its jth column with the column vector $c \in \mathbb{C}^m$, respectively.

The formulas (17) give very simple and elegant determinantal representations of the MP-inverse. So, for any $A \in \mathbb{C}_r^{m\times n}$, we have sum of all principal minors of r order of the matrices A^*A or AA^* in denominators and sum of principal minors of r order of the matrices $(A^*A)_{.i}\left(a_{.j}^*\right)$ or $(AA^*)_{j.}\left(a_{i.}^*\right)$ that contain the ith column or the jth row, respectively, in numerators into (17).

Note that for an arbitrary full-rank matrix A, Lemma 1.10 gives a new way of finding an inverse matrix.

Corollary 1.11. *If $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank} A = \min\{m, n\}$, then the inverse $A^{-1} = \left(a_{ij}^{-1}\right) \in \mathbb{C}^{n\times m}$ possess the following determinantal representations:*

$$a_{ij}^{-1} = \begin{cases} \dfrac{\left|(A^*A)_{.i}\left(a_{.j}^*\right)\right|}{|A^*A|} & if \ \ rank A = n, \\[2ex] \dfrac{\left|(AA^*)_{j.}\left(a_{i.}^*\right)\right|}{|AA^*|} & if \ \ rank A = m. \end{cases}$$

These new determinantal representations of the Moore-Penrose inverse have been obtained by the developed novel limit-rank method in the case of quaternion matrices [44] as well. This method was successfully applied for constructing determinantal representations of other generalized inverses in both cases for complex and quaternion matrices (see e.g. [45–47]). It also yields Cramer's rules of various matrix equations [48–54].

The remainder of our chapter is directed as follows. In Section 2, we provide a new expression of the general solution to our system (7) and discuss its least-norm. The algorithm and numerical example of finding the anti-Hermitian solution to (7) are presented in Section 3. (7). Finally, in Section 4, the conclusions are drawn.

2. A new expression of the general solution to the system

Now we demonstrate the principal theorem of this section (7).

Theorem 2.1. *Assume that* $S_1 = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$, $T_1 = \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} I_{q_2} \\ 0 \end{bmatrix}$, $G = [A \;\; C], H = \begin{bmatrix} B \\ -D \end{bmatrix}$, $H_1 = L_{A_1}L_A, H_2 = L_{A_1}S_1L_G, H_3 = R_H T_1 R_{B_1}, H_4 = L_{A_2}L_C, H_5 = L_{A_2}S_2L_G, H_6 = R_H T_2 R_{B_2}$ *and the system* (7) *is solvable, then the general solution to our system can be formed as*

$$X_1 = A_1^\dagger C_1 + L_{A_1} C_2 B_1^\dagger + L_{A_1} A^\dagger E B^\dagger R_{B_1} - L_{A_1} A^\dagger C M^\dagger E B^\dagger R_{B_1} - \\ - L_{A_1} A^\dagger S C^\dagger E N^\dagger D B^\dagger R_{B_1} + H_1 V_1 R_{B_1} + H_2 U H_3 + L_{A_1} V_2 R_B R_{B_1}, \tag{18}$$

$$X_2 = A_2^\dagger C_3 + L_{A_2} C_4 B_2^\dagger + L_{A_2} M^\dagger R_A E D^\dagger R_{B_2} + L_{A_2} L_M S^\dagger S C^\dagger E N^\dagger R_{B_2} + \\ + H_4 W_1 R_{B_2} + H_5 U H_6 + L_{A_2} W_2 R_D R_{B_2}, \tag{19}$$

where $U, V_1, V_2, W_1,$ *and* W_2 *are free matrices over* $\mathbb{C}$ *with allowable dimensions.*

Proof. Our proof contains three parts. At the first step, we show that the matrices X_1 and X_2 have the forms of

$$X_1 = \phi_0 + H_1 V_1 R_{B_1} + L_{A_1} V_2 R_B R_{B_1} + H_2 U H_3, \tag{20}$$

$$X_2 = \psi_0 + H_4 W_1 R_{B_2} + L_{A_2} W_2 R_D R_{B_2} + H_5 U H_6, \tag{21}$$

where ϕ_0 and ψ_0 are any pair of particular solution to the system (7), V_1, V_2, W_1, $W_{2,}$ and U are free matrices of able shapes over $\mathbb{C}$, are solutions to the system (7). In the second step, we display that any couple of solutions μ_0 and ν_0 to the system (7) can be established as (20) and (21), respectively. In the end, we confirm that

$$\mu = A_1^\dagger C_1 + L_{A_1} C_2 B_1^\dagger + A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger,$$
$$\nu = A_2^\dagger C_3 + L_{A_2} C_4 B_2^\dagger + L_{A_2} M^\dagger R_A E D^\dagger + L_{A_2} L_M S^\dagger S C^\dagger E N^\dagger R_{B_2}$$

are a couple of particular solutions to the system (7).

Now we prove that a couple of matrices X_1 and X_2 having the shape of (20) and (21), respectively, are solutions to the system (7). Observe that

$$A_1^\dagger C_1 B_1 + L_{A_1} C_2 B_1^\dagger B_1 = A_1^\dagger A_1 C_2 + L_{A_1} C_2 = C_2,$$
$$A_2^\dagger C_3 B_2 + L_{A_2} C_4 B_2^\dagger B_2 = A_2^\dagger A_2 C_4 + L_{A_2} C_4 = C_4.$$

It is evident that X_1 having the form (20) is a solution of $A_1X_1 = C_1,$ and $X_1B_1 = C_2$ and X_2 having the form (21) is a solution to $A_2X_2 = C_3, X_2B_2 = C_4$. Now we are left to show that $A_3X_1B_3 + A_4X_2B_4 = C_c$ is satisfied by X_1 and X_2 given in (20) and (21). By Lemma 1.4, we have

$$\begin{aligned} AS_1L_G &= A\begin{bmatrix} I_{p_1} & 0 \end{bmatrix}\begin{bmatrix} L_A & -A^{\dagger}CL_M \\ 0 & L_M \end{bmatrix} = A\begin{bmatrix} L_A & -A^{\dagger}CL_M \end{bmatrix} \\ &= \begin{bmatrix} 0 & -AA^{\dagger}CL_M \end{bmatrix} = \begin{bmatrix} 0 & -(C-M)L_M \end{bmatrix} = \begin{bmatrix} 0 & -CL_M \end{bmatrix} \\ &= -\begin{bmatrix} 0 & S \end{bmatrix} = -CS_2L_G, \end{aligned} \tag{22}$$

and

$$\begin{aligned} R_HT_1B &= \begin{bmatrix} R_B & 0 \\ R_NDB^{\dagger} & R_N \end{bmatrix}\begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix}B = \begin{bmatrix} R_B \\ R_NDB^{\dagger} \end{bmatrix}B \\ &= \begin{bmatrix} 0 \\ R_NDB^{\dagger}B \end{bmatrix} = \begin{bmatrix} 0 \\ R_ND(I-L_B) \end{bmatrix} = \begin{bmatrix} 0 \\ R_ND \end{bmatrix} \\ &= R_HT_2D. \end{aligned} \tag{23}$$

Observe that $AL_A = 0$ and by using (22) and (23), we arrive that

$$A_3X_1B_3 + A_4X_2B_4 = C_c.$$

Conversely, assume that μ_0 and ν_0 are any couple of solutions to our system (7). By Lemma 1.7, we have

$$\begin{aligned} &A_1A_1^{\dagger}C_1 = C_1, C_2B_1^{\dagger}B_1 = C_2, A_2A_2^{\dagger}C_3 = C_3, \\ &C_4B_2^{\dagger}B_2 = C_4, A_1C_2 = C_1B_1, A_2C_4 = C_3B_2. \end{aligned}$$

Observe that

$$\begin{aligned} L_{A_1}\mu_0R_{B_1} &= \left(I - A_1^{\dagger}A_1\right)\mu_0\left(I - B_1B_1^{\dagger}\right) \\ &= \mu_0 - \mu_0B_1B_1^{\dagger} - A_1^{\dagger}A_1\mu_0 + A_1^{\dagger}A_1\mu_0B_1B_1^{\dagger} \\ &= \mu_0 - C_2B_1^{\dagger} - A_1^{\dagger}C_1 + A_1^{\dagger}A_1C_2B_1^{\dagger} \\ &= \mu_0 - L_{A_1}C_2B_1^{\dagger} - A_1^{\dagger}C_1 \end{aligned}$$

produces

$$\mu_0 = L_{A_1}C_2B_1^{\dagger} + A_1^{\dagger}C_1 + L_{A_1}\mu_0R_{B_1}. \tag{24}$$

On the same lines, we can get

$$\nu_0 = L_{A_2}C_4B_2^{\dagger} + A_2^{\dagger}C_3 + L_{A_2}\nu_0R_{B_2}. \tag{25}$$

It is manifest that μ_0 and ν_0 defined in (24)–(25) are also solution pair of

$$AX_1B + CX_2D = E. \tag{26}$$

Since

$$\begin{aligned}AX_1B + CX_2D &= A_3L_{A_1}\mu_0R_{B_1}B_3 + A_4L_{A_2}\nu_0R_{B_2}B_4\\ &= A_3\left(\mu_0 - L_{A_1}C_2B_1^{\dagger} - A_1^{\dagger}C_1\right)B_3 + A_4\left(\nu_0 - L_{A_2}C_4B_2^{\dagger} - A_2^{\dagger}C_3\right)B_4\\ &= A_3\mu_0B_3 - A_3L_{A_1}C_2B_1^{\dagger}B_3 - A_1^{\dagger}C_1B_3 + A_4\nu_0B_4\\ &\quad -A_4L_{A_2}C_4B_2^{\dagger}B_4 - A_4A_2^{\dagger}C_3B_4\\ &= A_3\mu_0B_3 + A_4\nu_0B_4 - AC_2B_1^{\dagger}B_3 - A_1^{\dagger}C_1B_3 - CC_4B_2^{\dagger}B_4 - A_4A_2^{\dagger}C_3B_4\\ &= C_c - AC_2B_1^{\dagger}B_3 - A_1^{\dagger}C_1B_3 - CC_4B_2^{\dagger}B_4 - A_4A_2^{\dagger}C_3B_4\\ &= E.\end{aligned}$$

Hence by Lemma 1.6, μ_0 and ν_0 can be written as

$$\mu_0 = X_{01} + S_1L_GUR_HT_1 + L_AV_1 + V_2R_B, \tag{27}$$

$$\nu_0 = X_{02} + S_2L_GUR_HT_2 + L_CW_1 + W_2R_D, \tag{28}$$

where X_{01} and X_{02} are a couple of special solutions to (26) and U, V_1, V_2, W_1 and W_2 are free matrices with agreeable dimensions. Using (27) and (28) in (24) and (25), respectively, we get

$$\begin{aligned}\mu_0 &= X_{10} + H_2UH_3 + H_1V_1R_{B_1} + L_{A_1}V_2R_BR_{B_1},\\ \nu_0 &= X_{20} + H_5UH_6 + H_4W_1R_{B_2} + L_{A_2}W_2R_DR_{B_2},\end{aligned}$$

where $X_{10} = A_1^{\dagger}C_1 + L_{A_1}C_2B_1^{\dagger} + L_{A_1}X_{01}R_{B_1}$ and $X_{20} = A_2^{\dagger}C_3 + L_{A_2}C_4B_2^{\dagger} + L_{A_2}X_{02}R_{B_2}$. It is evident that X_{10} and X_{20} are a couple of solutions to the system (7). It is clear that μ_0 and ν_0 can be represented by (20) and (21), respectively. Lastly, by putting U_1, V_1, W_1, and Z_1 equal to zero in (15) and (16), we conclude that μ and ν are special solutions to the system (7). Hence the expressions (18) and (19) represent the general solution to the system (7) and the theorem is completed.

Remark 2.2. Due to Lemma 1.3 and taking into account $L_{A_2}L_M = L_ML_{A_2}$, we have the following simplification of the solution pair to the system (7) that is identical for (15)–(16) and (18)–(19) when $U, U_1, V_1, V_2, Z_1, W_1$, and W_2 disappear,

$$\begin{aligned}X_1 &= A_1^{\dagger}C_1 + L_{A_1}C_2B_1^{\dagger} + A^{\dagger}EB^{\dagger} - A^{\dagger}A_4M^{\dagger}EB^{\dagger} - A^{\dagger}SC^{\dagger}EN^{\dagger}B_4B^{\dagger},\\ X_2 &= A_2^{\dagger}C_3 + L_{A_2}C_4B_2^{\dagger} + M^{\dagger}ED^{\dagger} + S^{\dagger}SC^{\dagger}EN^{\dagger}.\end{aligned}$$

Comment 2.3. We have established a novel expression of the general solution to the system (7) in Theorem 2.1 which is different from one created in [25]. With the help of this novel expression, we can explore the least-norm of the general solution which can not be studied with the help of the expression given in [25], which is one of the advantage of our new expression.

Now we discuss some special cases of our system.

If B_1, B_2, C_2 and C_4 disappear in Theorem 2.1, then we gain the following conclusion.

Corollary 2.4. *Denote* $S_1 = \begin{bmatrix} I_{p_1} & 0\end{bmatrix}, S_2 = \begin{bmatrix} 0 & I_{p_2}\end{bmatrix}$, $T_1 = \begin{bmatrix} I_{q_1} \\ 0\end{bmatrix}$, $T_2 = \begin{bmatrix} I_{q_2} \\ 0\end{bmatrix}$, $G = [A \;\; C], H = \begin{bmatrix} B_3 \\ -B_4\end{bmatrix}$, $H_1 = L_{A_1}L_A, H_2 = L_{A_1}S_1L_G, H_3 = R_HT_1, H_4 = L_{A_2}L_C, H_5 =$

$L_{A_2}S_2L_G, H_6 = R_HT_2$ *and the system* (6) *is solvable, then the general solution to* (6) *can be formed as*

$$\begin{aligned} X_1 &= A_1^\dagger C_1 + A^\dagger EB_3^\dagger - A^\dagger A_4 M^\dagger EB_3^\dagger - A^\dagger SC^\dagger EN^\dagger B_4 B_3^\dagger - H_1Y_1 + \\ &\quad + H_2VH_3 + L_{A_1}Y_2R_{B_3}, \\ X_2 &= A_2^\dagger C_3 + M^\dagger EB_4^\dagger + S^\dagger SC^\dagger EN^\dagger + H_4Z_1 + H_5VH_6 + L_{A_2}Z_2R_{B_4}, \end{aligned}$$

where A, C, N, M, S *are the same as in Lemma 1.6,* $E = C_c - A_3A_1^\dagger C_1B_3 - A_4A_2^\dagger C_3B_4$, V, Y_1, Y_2, Z_1, and Z_2 *are free matrices over* $\mathbb{C}$ *obeying agreeable dimensions*.

Comment 2.5. The above consequence is a chief result of [32].

If A_2, B_2, C_3, A_4, B_4 and C_4 vanish in our system (7), then we get the following outcome.

Corollary 2.6. *Suppose that* $A_1, B_1, C_1, C_2, A_3, B_3$ *and* C_c *are given. Then the general solution to system* (5) *is established by*

$$\begin{aligned} X_1 = {} & A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + (A_3L_{A_1})^\dagger \left[C_c - A_3A_1^\dagger C_1B_3 - A_3L_{A_1}C_2B_1^\dagger B_3\right](R_{B_1}B_3)^\dagger + \\ & + L_{A_1}L_{A_3L_{A_1}}W_1R_{B_1} + L_{A_1}W_2R_{R_{B_1}B_3}R_{B_1}, \end{aligned}$$

where W_1 and W_2 *are arbitrary matrices over* $\mathbb{C}$ *with appropriate sizes*.

We experience the least-norm to the system (7) in this section. By the definition and [55], we can get the following result easily.

Lemma 2.7. *Let* $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{n\times m}$. *Then we have.*

(1) $\|A + B\|^2 = \|A\|^2 + \|B\|^2 + 2\,Re\,[\mathrm{tr}(B^*A)]$.

(2) $Re\,[\mathrm{tr}(AB)] = Re\,[\mathrm{tr}(BA)]$.

Theorem 2.8. *Assume that system* (7) *is solvable, then the least-norm of the solution pair* X_1 *and* X_2 *to system* (7) *can be extracted as follows:*

$$\begin{aligned} \|X_1\|_{min} &= A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - \\ &\quad - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger, \end{aligned} \tag{29}$$

$$\|X_2\|_{min} = A_2^\dagger C_3 + L_{A_2}C_4B_2^\dagger + M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger. \tag{30}$$

Proof. By Theorem 2.1 and Remark 2.2, the general solution to (7) can be formed as

$$\begin{aligned} X_1 &= A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger \\ &\quad - H_1V_1R_{B_1} + H_2UH_3 + L_{A_1}V_2R_BR_{B_1}, \\ X_2 &= A_2^\dagger C_3 + L_{A_2}C_4B_2^\dagger + M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger \\ &\quad + H_4W_1R_{B_2} + H_5UH_6 + L_{A_2}W_2R_DR_{B_2}, \end{aligned}$$

where U, V_1, V_2, W_1, and W_2 are free matrices over $\mathbb{C}$ having executable dimensions. By Lemma 2.7, the norm of X_1 can be established as

$$\begin{aligned} \|X_1\|^2 = {} & \|A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - \\ & - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger - H_1V_1R_{B_1} + H_2UH_3 + L_{A_1}V_2R_BR_{B_1}\|^2 \\ = {} & \|A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\|^2 \\ & + \|H_1V_1R_{B_1} + H_2UH_3 + L_{A_1}V_2R_BR_{B_1}\|^2 + J, \end{aligned} \tag{31}$$

where

$$J = 2\,Re\,[\mathrm{tr}((H_1V_1R_{B_1} + H_2UH_3 + L_{A_1}V_2R_BR_{B_1})^* \\ (A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger))]. \tag{32}$$

Now we want to show that $J = 0$. Applying Lemmas 1.3, 1.4 and 2.7, we have

$$\begin{aligned} &Re\left[\mathrm{tr}\left((H_1V_1R_{B_1})^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(R_{B_1}V_1^*H_1^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(R_{B_1}V_1^*L_AL_{A_1}\left(A_1^\dagger C_1\right.\right.\right. \\ &\left.\left.\left. +L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] \\ &= Re\left[\mathrm{tr}\left(R_{B_1}V_1^*L_AL_{A_1}\left(L_{A_1}C_2B_1^\dagger\right)\right)\right] \\ &= Re\left[\mathrm{tr}\left(V_1^*L_AL_{A_1}\left(L_{A_1}C_2B_1^\dagger\right)R_{B_1}\right)\right] = 0, \end{aligned} \tag{33}$$

$$\begin{aligned} &Re\left[\mathrm{tr}\left((L_{A_1}V_2R_BR_{B_1})^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}(R_{B_1}R_BV_2^*L_{A_1}^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger\right.\right. \\ &\left.\left. +A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right))\right] \\ &= Re\left[\mathrm{tr}\left(R_{B_1}R_BV_2^*L_{A_1}\left(L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(V_2^*L_{A_1}\left(L_{A_1}C_2B_1^\dagger\right.\right.\right. \\ &\left.\left.\left. +A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)R_{B_1}R_B\right)\right] \\ &= Re\left[\mathrm{tr}\left(V_2^*L_{A_1}\left(A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)R_B\right)\right] = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} &Re\left[\mathrm{tr}\left((H_2UH_3)^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(H_3^*U^*H_2^*\left(A_1^\dagger C_1 + L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(H_3^*U^*L_GS_1^*L_{A_1}\left(L_{A_1}C_2B_1^\dagger\right.\right.\right. \\ &\left.\left.\left. +A^\dagger EB^\dagger - A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] \\ &= Re\left[\mathrm{tr}\left(H_3^*U^*\begin{bmatrix} L_A & -A^\dagger CL_M \\ 0 & L_M \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix}\left(L_{A_1}C_2B_1^\dagger + A^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(H_3^*U^*L_A\left(A^\dagger EB^\dagger\right.\right.\right. \\ &\left.\left.\left. -A^\dagger A_4M^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger B_4B^\dagger\right)\right)\right] = Re\left[\mathrm{tr}\left(H_3^*U^*L_AL_{A_1}C_2B_1^\dagger\right)\right] \\ &= Re\left[\mathrm{tr}\left(R_{B_1}T_1^*R_HU^*L_AL_{A_1}C_2B_1^\dagger\right)\right] \end{aligned} \tag{35}$$

By using (33)–(35) in (32) produces $J = 0$. Since X_1 is arbitrary, we get (29) from (31). In the same way, we can prove that (30) hold. □

A special case of our system (7) is given below.

If $B_1, B_2, C_2,$ and C_4 become zero matrices in Theorem 2.8, then again we get the principal result of [20].

Corollary 2.9. *Assume that system* (6) *is solvable, then the least-norm of the solution pair X_1 and X_2 to system* (6) *can be furnished as*

$$\|X_1\|_{min} = A_1^\dagger C_1 + A^\dagger E B_3^\dagger - A^\dagger A_4 M^\dagger E B_3^\dagger - A^\dagger S C^\dagger E N^\dagger B_4 B_3^\dagger,$$
$$\|X_2\|_{min} = A_2^\dagger C_3 + M^\dagger E B_4^\dagger + S^\dagger S C^\dagger E N^\dagger.$$

If A_2, B_2, C_3, A_4, B_4 and C_4 vanish in our system, then we get the next consequence.

Corollary 2.10. *Suppose that $A_1, B_1, C_1, C_2, A_3, B_3$ and C_c are given. Then the least-norm of the least square solution to system* (5) *is launched by*

$$\|X_1\|_{min} = A_1^\dagger C_1 + L_{A_1} C_2 B_1^\dagger$$
$$+ (A_3 L_{A_1})^\dagger \left[C_c - A_3 A_1^\dagger C_1 B_3 - A_3 L_{A_1} C_2 B_1^\dagger B_3\right] (R_{B_1} B_3)^\dagger.$$

Comment 2.11. Corollary 2.10 is the key result of [22].

3. Algorithm with example

In this section, we construct the algorithm for finding the least-norm of the solution to (7) that is inducted by Theorem 2.8.

Algorithm 1.

1. By Lemma 1.10 find the matrices $A_i^\dagger$, $B_i^\dagger$ for $i = 1, \ldots, 4$, and $R_{A_i} = I - A_i A_i^\dagger$, $L_{A_i} = I - A_i^\dagger A_i$, $R_{B_i} = I - B_i B_i^\dagger$, and $L_{B_i} = I - B_i^\dagger B_i$ for $i = 1, 2$.

2. By Lemma 1.9 calculate the matrices A, B, C, D, M, S, and E, and by Lemma 1.10 find their MP-inverses and orthogonal projectors when it is needed.

3. Verify the consistence equalities (11) and (14). If these equalities are hold, then we find solutions by the next steps.

4. Finally, by (29) and (30), compute the least-norm of the solution pair X_1 and X_2.

The following example will be considered by using Algorithm 1. Note that our goal is both to confirm correctness of main results from Theorems 2.1 and 2.8, and to demonstrate the technique of applying the determinantal representations of the MP-inverse from Lemma 1.10 by using a not too complicated and understandable example.

Example 1. Given the matrices:

$$A_1 = \begin{bmatrix} 1+i & 1-i & -1+i & -1-i \\ -1+i & 1+i & -1-i & 1-i \\ 2i & 2 & -2 & -2i \end{bmatrix}, B_1 = \begin{bmatrix} 2i & -1 & i+3 \\ -i & 1 & -3-i \\ -1 & i & 1-3i \\ 1 & -i & -1+3i \end{bmatrix}, A_2 = \begin{bmatrix} i & 1 & -1 \\ 1 & -i & i \\ -1 & i & -i \\ -i & -1 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2-i & 2i-1 & i+1 \\ 2i+1 & -i-2 & i-1 \\ -2i+1 & i-2 & -i-1 \\ i+2 & -2i-1 & -i+1 \end{bmatrix}, C_1 = \begin{bmatrix} 8i & -8 & -8i & 8 \\ 4 & 4i & -4 & -4i \\ 2+4i & -4+2i & 2-4i & 4-2i \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 11i & 44i-11 & -44 \\ 22 & 22i+88 & 88i \\ -11i & 44i+11 & 44 \\ -22 & -22i-88 & -88i \end{bmatrix}, A_3 = \begin{bmatrix} 5i+2 & 5-2i & -2+5i & 2i+5 \\ 2i-5 & 5i+2 & -2i-5 & -2+5i \\ 4i & 4 & -4i & -4 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -i & -i+2 & -1 \\ -2 & -2-4i & 2i \\ -2i & 4-2i & -2 \\ 1 & 1+2i & -i \end{bmatrix}, A_4 = \begin{bmatrix} -2i-3 & -3i+2 & 2i+3 \\ -i & 1 & i \\ -3 & -3i & 3 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 3i & 3 & -3 & -3i \\ 3 & -3i & 3i & -3 \\ -3 & 3i & -3i & 3 \\ -3i & -3 & 3 & 3i \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 7i & -i & -2 \\ -7 & -3 & 2i \\ -7i & i & 2 \\ 7 & 3 & -2i \end{bmatrix}, C_4 = \begin{bmatrix} 4-2i & -2+4i & 2+2i \\ 2+4i & -4-2i & -2+2i \\ -2-4i & 4+2i & 2-2i \end{bmatrix},$$

$$C_c = \frac{1}{21}\begin{bmatrix} -1130-502i & -1344+612i & -2798-1250i \\ -1808-688 & -1398+834i & -2942-1538i \\ -1154-946i & -1488+624i & -2654-1394i \end{bmatrix}. \tag{36}$$

Let us find a solution to the system (7) with the given above matrices by Algorithm 1.

1. Thanks to Lemma 1.10, we calculate the Moore-Penrose inverses. So,

$$A_1^\dagger = \frac{1}{32}\begin{bmatrix} 1-i & -1-i & -2i \\ 1+i & 1-i & 2 \\ -1-i & -1+i & -2 \\ -1+i & 1+i & 2i \end{bmatrix}, B_1^\dagger = \frac{1}{44}\begin{bmatrix} -11i & 11i & -11 & 11 \\ 39 & 41 & 20-i & 20+i \\ 7-i & 1+i & 5+3i & 3-3i \end{bmatrix},$$

$$A_2^\dagger = \frac{1}{12}\begin{bmatrix} -i & 1 & -1 & i \\ 1 & i & -i & -1 \\ -1 & -i & i & 1 \end{bmatrix}, B_2^\dagger = \frac{1}{12}\begin{bmatrix} 1 & -i & i & 1 \\ -i & -1 & -1 & i \\ 1-i & -1-i & -1+i & 1+i \end{bmatrix},$$

$$A_3^\dagger = \frac{1}{80}\begin{bmatrix} -2i & -2 & 2-5i \\ 2 & -2i & 5+2i \\ -2i & -2 & 2+5i \\ 2 & -2i & -5+2i \end{bmatrix}, B_3^\dagger = \frac{1}{70}\begin{bmatrix} i & -2 & 2i & 1 \\ 2+i & -2+4i & 4+2i & 1-2i \\ -1 & -2i & -2 & i \end{bmatrix},$$

$$A_4^{\dagger} = \frac{1}{69}\begin{bmatrix} -3+2i & i & -3 \\ 2+3i & 1 & 3i \\ 3-2i & -i & 1 \end{bmatrix}, B_4^{\dagger} = \frac{1}{792}\begin{bmatrix} -35i & -21 & 35i & 21 \\ 47i & -51 & -47i & 51 \\ -52 & -48i & 52 & 48i \end{bmatrix}.$$

Then,

$$A = \frac{1}{2}\begin{bmatrix} 2+5i & 5-2i & 1+8i & 12+9i \\ -5+2i & 2+5i & -8+i & -9+12i \\ 4i & 4 & 4-8i & -8+4i \end{bmatrix},$$

$$B = \frac{1}{22}\begin{bmatrix} -52-31i & 10-135i & -31+52i \\ 8+9i & -10+25i & 9-8i \\ -9+8i & -25-10i & 8+9i \\ 31-52i & 135+10i & -52-31i \end{bmatrix},$$

$$C = \frac{1}{3}\begin{bmatrix} -11-3i & 9-7i & 6+4i \\ -1-3i & 3+i & 2i \\ -9+3 & 3-9i & 6 \end{bmatrix}, D = \begin{bmatrix} 0 & -2i & -2 \\ 0 & -2 & 2i \\ 0 & 2i & 2 \\ 0 & 2 & -2i \end{bmatrix},$$

$$N = \frac{1}{7}\begin{bmatrix} 4+4i & -4-2i & -10-4i \\ 4-4i & -2+4i & -4+10i \\ -4-4i & 4+2i & 10+4i \\ -4+4i & 2-4i & 4-10i \end{bmatrix}, M = \frac{1}{3}\begin{bmatrix} -4-2i & 4-2i & 2+2i \\ -2+4i & -2-4i & 2-2i \\ 0 & 0 & 0 \end{bmatrix}, S = 0$$

$$E = \frac{1}{84}\begin{bmatrix} 19931-108289i & 236509-68427i & -108289-19931i \\ 110417+16211i & 77995+79015i & 16211-110417i \\ 74624+106424i & -138224+255672i & 106424-74624i \end{bmatrix}.$$

2. Confirm that (11) and (14) are true for given matrices.

3. Finally, by (29) and (30), we find that the least-norm of the solution pair X_1 and X_2 to the system (7) is following

$$X_1 = \frac{1}{365760}\begin{bmatrix} -11103239+18670545i & -9851419+14002307i & -5154373+3862099i & -4697553+10234559i \\ 26688873+4258681i & 29888893+5510501i & 12048461+4721147i & 17746081+5177967i \\ 6556168+9656066i & 5321848+2196342i & 4452786+10360112i & -6757414+7845632i \\ -17049264-2930378i & -26304464-11113378i & -10244698-3367816i & -7362609-13720296i \end{bmatrix},$$

$$X_2 = \frac{1}{1344}\begin{bmatrix} 2052-963i & 233-1985i & -2159+3481i & -1465-367i \\ -792+2565i & 1901-205i & 317+445i & -221+317i \\ 171+585i & -146+28i & 868-1714i & 146+2884i \end{bmatrix}.$$

Note that Maple 2021 was used to perform the numerical experiment.

4. Conclusion

We have constructed a novel expression of the general solution to system (7) over $\mathbb{C}$ and used this result to explore the least-norm of the general solution to this system when it is solvable. Some particular cases of our system are also discussed. Our results carry the principal results of [22, 32]. To give an algorithm finding the explicit numerical expression of the least-norm of the general solution, it is used the determinantal representations of the MP-inverse recently obtained by one of the authors. The novelty of the conducted research is obtaining necessary and sufficient conditions to exist a solution, its formal representation of by closed formula in terms of generalized inverses, and the construction of an algorithm to find its explicit expression. A numerical example is also given to interpret the results established in this paper.

Conflict of interest

The authors declare that they have not conflicts of interest.

Data Availability

The data used to support the findings of this study are included within the article titled “Solving and Algorithm for Least-Norm General Solution to Constrained Sylvester Matrix Equation”. The prior studies (and datasets) are cited at relevant places within the text.

Classification

2000 AMS subject classifications: 15A09, 15A15, 15A24.

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References

[1] Shahzad A, Jones BL, Kerrigan EC, Constantinides GA. An efficient algorithm for the solution of a coupled Sylvester equation appearing in descriptor systems. Automatica. 2011;**47**: 244-248. DOI: 10.1016/j.automatica.2010.10.038

[2] Syrmos VL, Lewis FL. Coupled and constrained Sylvester equations in system design. Circuits, Systems, and Signal Processing. 1994;**13**(6):663-694. DOI: 10.1007/BF02523122

[3] Varga A. Robust pole assignment via Sylvester equation based state feedback parametrization: Computer-aided control system design (CACSD). IEEE International Symposium. 2000;**57**: 13-18. DOI: 10.1109/CACSD.2000.900179

[4] Syrmos VL, Lewis FL. Output feedback eigenstructure assignment using two Sylvester equations. IEEE Transaction on Automatic Control. 1993;**38**:495-499. DOI: 10.1109/9.210155

[5] Li RC. A bound on the solution to a structured Sylvester equation with an application to relative perturbation theory. SIAM Journal on Matrix Analysis and Application. 1999;**21**(2): 440-445

[6] Darouach M. Solution to Sylvester equation associated to linear descriptor systems. Systems and Control Letters. 2006;**55**:835-838. DOI: 10.1016/j.sysconle.2006.04.004

[7] Zhang YN, Jiang DC, Wang J. A recurrent neural network for solving Sylvester equation with time-varying coefficients. IEEE Transcation on Neural Networks. 2002;**13**(5):1053-1063. DOI: 10.1109/TNN.2002.1031938

[8] Terán FD, Dopico FM. The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits. Linear Algebra and its Application. 2011; **434**:44-67. DOI: 10.1016/j.laa.2010.08.005

[9] Dehghan M, Hajarian M. An efficient iterative method for solving the second-order Sylvester matrix equation $EVF^2 - AVF - CV = BW$. IET Control Theory and Applications. 2009;**3**: 1401-1408. DOI: 10.1049/iet-cta.2008.0450

[10] Ding F, Chen T. Gradient based iterative algorithms for solving a class of matrix equations. IEEE Transaction on Automatic Control. 2005;**50**(8): 1216-1221. DOI: 10.1109/TAC.2005.852558

[11] Dmytryshyn A, Futorny V, Klymchuk T, Sergeichuk VV. Generalization of Roth's solvability criteria to systems of matrix equations. Linear Algebra and its Application. 2017; **527**:294-302. DOI: 10.1016/j.laa.2017.04.011

[12] He ZH, Wang QWA. Real quaternion matrix equation with applications. Linear and Multilinear Algebra. 2013;**61**(6):725-740. DOI: 10.1080/03081087.2012.703192

[13] He ZH, Wang QW. The η-bihermitian solution to a system of real quaternion matrix equation. Linear and Multilinear Algebra. 2014;**62**(11): 1509-1528. DOI: 10.1080/03081087.2013.839667

[14] Kyrchei II. Explicit representation formulas for the minimum norm least squares solutions of some quaternion

matrix equations. Linear Algebra and its Applications. 2013;**438**(1):136-152. DOI: 10.1016/j.laa.2012.07.049

[15] Kyrchei II. Explicit determinantal representation formulas for the solution of the two-sided restricted quaternionic matrix equation. Journal of Applied Mathematics and Computing. 2018;**58**(1–2):335-365. DOI: 10.1007/s12190-017-1148-6

[16] Rehman A, Wang QW. A system of matrix equations with five variables. Applied Mathematics and Computation. 2015;**271**:805-819. DOI: 10.1016/j.amc.2015.09.066

[17] Rehman A, Wang QW, Ali I, Akram M, Ahmad MO. A constraint system of generalized Sylvester quaternion matrix equations. Adv. Appl. Clifford Algebr. 2017;**27**(4):3183-3196. DOI: 10.1007/s00006-017-0803-1

[18] Rehman A, Wang QW, He ZH. Solution to a system of real quaternion matrix equations encompassing η-Hermicity. Applied Mathematics and Computation. 2015;**265**:945-957. DOI: 10.1016/j.amc.2015.05.104

[19] Rehman A, Akram M. Optimization of a nonlinear hermitian matrix expression with application. Univerzitet u Nišu. 2017;**31**(9):2805-2819. DOI: 10.2298/FIL1709805R

[20] Wang QW, Qin F, Lin CY. The common solution to matrix equations over a regular ring with applications. Indian Journal of Pure and Applied Mathematics. 2005;**36**(12):655-672

[21] Wang QW, Rehman A, He ZH, Zhang Y. Constraint generalized Sylvester matrix equations. Automatica. 2016;**69**:60-64. DOI: 10.1016/j.automatica.2016.02.024

[22] Bao Y. Least-norm and extremal ranks of the Least Square solution to the quaternion matrix equation $AXB = C$ subject to two equations. Algebra Colloq. 2014;**21**(3):449-460. DOI: 10.1142/S100538671400039X

[23] Wang QW, Chang HX, Lin CY. P-(skew)symmetric common solutions to a pair of quaternion matrix equations. Applied Mathematics and Computation. 2008;**195**:721-732. DOI: 10.1016/j.amc.2007.05.021

[24] Li H, Gao Z, Zhao D. Least squares solutions of the matrix equation $AXB + CYD = E$ with the least norm for symmetric arrowhead matrices. Applied Mathematics and Computation. 2014;**226**:719-724. DOI: 10.1016/j.amc.2013.10.065

[25] Wang QW, van der Woude JW, Chang HX. A system of real quaternion matrix equations with applications. Linear Algebra and its Application. 2009;**431**(12):2291-2303. DOI: 10.1016/j.laa.2009.02.010

[26] Peng YG, Wang X. A finite iterative algorithm for solving the least-norm generalized (P, Q) reflexive solution of the matrix equations $A_iXB_i = C_i$. Journal of Computational Analysis and Applications. 2014;**17**(3):547-561

[27] Yuan S, Liao A. Least squares Hermitian solution of the complex matrix equation $AXB + CXD = E$ with the least norm. Journal of Franklin Institute. 2014;**351**(11):4978-4997. DOI: 10.1016/j.jfranklin.2014.08.003

[28] Trench WF. Minimization problems for (R, S)-symmetric and (R, S)-skew symmetric matrices. Linear Algebra and its Applications. 2004;**389**:23-31. DOI: 10.1016/j.laa.2004.03.035

[29] Trench WF. Characterization and properties of matrices with generalized symmetry or skew symmetry. Linear Algebra and its Applications. 2004;**377**: 207-218. DOI: 10.1016/j.laa.2003.07.013

[30] Trench WF. Characterization and properties of (R,S)-symmetric, (R,S)-skew symmetric and (R,S)-conjugate matrices. SIAM Journal on Matrix Analysis and Application. 2005;**26**: 748-757. DOI: 10.1137/S089547980343134X

[31] Marsaglia G, Styan GPH. Equalities and inequalities for ranks of matrices. Linear and Multilinear Algebra. 1974;**2**: 269-292. DOI: 10.1080/03081087408817070

[32] Wang QW, Li CK. Ranks and the least-norm of the general solution to a system of quaternion matrix equations. Linear Algebra and its Application. 2009; **430**:1626-1640. DOI: 10.1016/j.laa.2008.05.031

[33] Wang QW, Chang HX, Ning Q. The common solution to six quaternion matrix equations with applications. Applied Mathematics and Computation. 2008;**198**:209-226. DOI: 10.1016/j.amc.2007.08.091

[34] Tian Y. Solvability of two linear matrix equations. Linear and Multilinear Algebra. 2000;**48**:123-147. DOI: 10.1080/03081080008818664

[35] Wang QW, Wu ZC, Lin CY. Extremal ranks of a quaternion matrix expression subject to consistent systems of quaternion matrix equations with applications. Applied Mathematics and Computation. 2006;**182**:1755-1764. DOI: 10.1016/j.amc.2006.06.012

[36] Wang QW. A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity. Linear Algebra and its Application. 2004;**384**:43-54. DOI: 10.1016/j.laa.2003.12.039

[37] Wensheng C. Solvability of a quaternion matrix equation. Applied Mathematics, Journal of Chinese Universities, Serie B. 2002;**17**(4): 490-498. DOI: 10.1007/s11766-996-0015-2

[38] Artidiello S, Cordero A, Torregrosa JR, Vassileva MP. Generalized inverses estimations by means of iterative methods with memory. Mathematics. 2019;**8**:2. DOI: 10.3390/math8010002

[39] Guo W, Huang T. Method of elementary transformation to compute Moore–Penrose inverse. Applied Mathematics and Computation. 2010; **216**:1614-1617. DOI: 10.1016/j.amc.2010.03.016

[40] Sayevand K, Pourdarvish A, Machado JAT, Erfanifar R. On the calculation of the Moore-Penrose and Drazin inverses: Application to fractional calculus. Mathematics. 2021;**9**:2501. DOI: 10.3390/math9192501

[41] Bapat RB, Bhaskara KPS, Prasad KM. Generalized inverses over integral domains. Linear Algebra and its Applications. 1990;**140**:181-196. DOI: 10.1016/0024-3795(90)90229-6

[42] Stanimirovic PS. General determinantal representation of pseudoinverses of matrices. Matematichki Vesnik. 1996;**48**:1-9

[43] Kyrchei II. Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules. Linear and Multilinear Algebra. 2008;**56**(4): 453-469. DOI: 10.1080/03081080701352856

[44] Kyrchei II. Determinantal representation of the Moore-Penrose inverse matrix over the quaternion skew field. Journal of Mathematical Sciences. 2012;**108**(1):23-33. DOI: 10.1007/s10958-011-0626-x

[45] Kyrchei II. Determinantal representations of the Drazin and W-weighted Drazin inverses over the quaternion skew field with applications. In: Griffin S, editor. Quaternions: Theory and Applications. New York: Nova Sci Publ; 2017. pp. 201-275

[46] Kyrchei II. Determinantal representations of the quaternion weighted Moore-Penrose inverse and its applications. In: Baswell AR, editor. Advances in Mathematics Research 23. New York: Nova Sci Publ; 2017. pp. 35-96

[47] Kyrchei II. Cramer's rule for generalized inverse solutions. In: Kyrchei II, editor. Advances in Linear Algebra Research. New York: Nova Sci Publ; 2015. pp. 79-132

[48] Kyrchei II. Analogs of Cramer's rule for the minimum norm least squares solutions of some matrix equations. Applied Mathematics and Computation. 2012;**218**(11):6375-6384. DOI: 10.1016/j.amc.2011.12.004

[49] Kyrchei II. Determinantal representations of solutions and hermitian solutions to some system of two-sided quaternion matrix equations. Journal of Mathematics. 2018; **2018**:6294672. DOI: 10.1155/2018/6294672

[50] Kyrchei II. Cramer's rules of η-(skew-)Hermitian solutions to the quaternion Sylvester-type matrix equations. Adv. Appl. Clifford Algebr. 2019;**29**(3):56. DOI: 10.1007/s00006-019-0972-1

[51] Kyrchei II. Determinantal representations of solutions to systems of two-sided quaternion matrix equations. Linear and Multilinear Algebra. 2021;**69**(4):648-672. DOI: 10.1080/03081087.2019.1614517

[52] Rehman A, Kyrchei II, Ali I, Akram M, Shakoor A. The general solution of quaternion matrix equation having η-skew-Hermicity and its Cramer's rule. Mathematical Problems in Engineering. 2018;**2018**:7939238. DOI: 10.1155/2019/7939238

[53] Rehman A, Kyrchei II, Ali I, Akram M, Shakoor A. Explicit formulas and determinantal representation for η-skew-Hermitian solution to a system of quaternion matrix equations. Univerzitet u Nišu. 2020;**34**(8):2601-2627. DOI: 10.2298/FIL2008601R

[54] Rehman A, Kyrchei II, Ali I, Akram M, Shakoor A. Constraint solution of a classical system of quaternion matrix equations and its Cramer's rule. Iranian Journal of Science and Technology, Transactions A: Science. 2021;**45**(3):1015-1024. DOI: 10.1007/s40995-021-01083-7

[55] Tian Y. Equalities and inequalities for traces of quaternionic matrices. Algebras Groups Geometry. 2002;**19**(2):181-193