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**Chapter**

# A Study for Coupled Systems of Nonlinear Boundary Value Problem

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## Abstract

This chapter deals with the existence and uniqueness of solutions for a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions and for the system of two-point boundary value problem when we take the case of integer derivative. The existence results for the fist problem are obtained by using Leray-Shauder nonlinear alternative and Banach contraction principle and for the second problem, we derive explicit eigenvalue intervals of $\lambda$ for the existence of at least one positive solution by using Krasnosel'skii fixed point theorem. An illustrative examples is presented at the end for each problem to illustrate the validity of our results.


**Keywords:** positive solution, uniqueness, Green's function, system of fractional differential equations, system of differential equations, existence, nonlocal boundary value problem, fixed point theorem

## 1. Introduction

In this chapter, we are interested in the existence of solutions for the nonlinear fractional boundary value problem (BVP)

$$\begin{cases} {}^cD^{\alpha}u(t) = f(t, u(t), v(t)), & t \in [0,1], \ \ 2 < \alpha \leq 3, \\ {}^cD^{\beta}v(t) = g(t, u(t), v(t)), & t \in [0,1], \ \ 2 < \beta \leq 3, \\ \lambda u(0) + \gamma u(1) = u(\eta), & \lambda v(0) + \gamma v(1) = v(\eta), \\ u(0) = \int_0^{\eta} u(s)ds, & v(0) = \int_0^{\eta} v(s)ds, \\ \lambda^c D^p u(0) + \gamma^c D^p u(1) = {}^cD^p u(\eta), & 1 < p \leq 2. \end{cases} \tag{1}$$

We also study the integer case of problem

$$\begin{cases} u^{(4)}(t) = \lambda a(t) f(v(t)), & 0 < t < 1, \\ v^{(4)}(t) = \lambda b(t) g(u(t)), & 0 < t < 1, \\ u(0) = 0, u'(0) = 0, u''(1) = 0, & u'''(1) = 0, \\ v(0) = 0, v'(0) = 0, v''(1) = 0, & v'''(1) = 0. \end{cases} \tag{2}$$



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where $D^{\alpha}_{0^+}, D^{\beta}_{0^+}$ are the standard Riemann-Liouville fractional derivative of order $\alpha$ and $\beta$, the functions $f, g \in C\big((0,1) \times \mathbb{R}^2, \mathbb{R}\big)$, the functions $f, g \in C((0,1) \times \mathbb{R}, \mathbb{R})$ in the second problem and $\lambda > 0, a, b \in C([0,1], [0,\infty))$.

The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For more details we refer the reader to [1–6] and the references cited therein.

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. A point of central importance in the study of nonlinear boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions to the boundary value problems. The multi-point boundary conditions are important in various physical problems of applied science when the controllers at the end points of the interval (under consideration) dissipate or add energy according to the sensors located, at intermediate points, see [7, 8] and the references therein. We quote also that realistic problems arising from economics, optimal control, stochastic analysis can be modelled as differential inclusion. The study of fractional differential inclusions was initiated by EL-Sayad and Ibrahim [9]. Also, recently, several qualitative results for fractional differential inclusion were obtained in [10–13] and the references therein.

The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear differential equations (DEs), nonlinear fractional differential equations (FDEs), nonlinear partial differential equations (PDEs), nonlinear fractional partial differential equations (FPDEs), nonlinear stochastic fractional partial differential equations (SFPDEs), plays an essential role in the research of this field, such as establishing the existence, uniqueness and multiplicity of solutions (or positive solutions) and mild solutions for nonlinear of different kinds of FPDEs, FPDEs, SFPDEs, inclusion differential equations and inclusion fractional differential equations with various boundary conditions, by using different techniques (approaches). For more details, see [14–37] and the references therein. For example, iterative method is an important tool for solving linear and nonlinear Boundary Value Problems. It has been used in the research areas of mathematics and several branches of science and other fields. However, Many authors showed the existence of positive solutions for a class of boundary value problem at resonance case. Some recent devolopment for resonant case can be found in [38, 39]. Let us cited few papers. Zhang et al. [40] studied the existence of two positive solutions of following singular fractional boundary value problems:

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, & t \in (0,1) \\ u(0) = 0, \ D^{\beta}_{0+}u(0) = 0, & D^{\beta}_{0+}u(1) = \sum_{j=1}^{\infty} D^{\beta}_{0+}u\left(\eta_j\right), \end{cases} \tag{3}$$

where $D_{0+}^{\alpha}, \ D_{0+}^{\beta}$ are the stantard Riemann-Liouville fractional derivative of order $\alpha \in (2, 3], \beta \in [1, 2], \ f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $a_j, \eta_j \in (0, 1), \alpha - \beta \geq 1$ with $\sum_{\infty}^{i=0} a_j \eta_j^{\alpha-\beta-1} < 1$.

In [41], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = 0, \ D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\eta), \end{cases} \tag{4}$$

where $1 < \alpha \leq 2, \ \ 0 < \eta < 1, \ \ 0 < a, \beta < 1, f \in C\left([0, 1] \times \mathbb{R}^2, \mathbb{R}\right)$ and $D_{0+}^{\alpha}, \ D_{0+}^{\beta}$ are the stantard Riemann-Liouville fractional derivative of order $\alpha$. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

In 2015, Alsulami et al. [42] studied the existence of solutions of the following nonlinear third-order ordinary differential inclusion with multi-strip boundary conditions

$$\begin{cases} u^{(3)}(t) \in F(t, u(t)), \ \ t \in (0, 1), \\ u(0) = 0, \ \ u'(0) = 0, \\ u(1) = \sum_{i=1}^{n-2} \alpha_i {}_{\zeta_i}^{\eta_i} u(s) ds, \\ 0 < \zeta_i < \eta_i < 1, i = 1, 2, \ \dots, n-2, n \geq 3. \end{cases} \tag{5}$$

In 2017, Resapour et al. [43] investigated a Caputo fractional inclusion with integral boundary condition for the following problem

$$\begin{cases} {}^{c}D^{\alpha} u(t) \in F\left(t, u(t), {}^{c}D^{\beta} u(t), u'(t)\right), \\ u(0) + u'(0) + {}^{c}D^{\beta} u(0) = \int_0^{\eta} u(s) ds, \\ u(1) + u'(1) + {}^{c}D^{\beta} u(1) = \int_0^{\nu} u(s) ds, \end{cases} \tag{6}$$

where $1 < \alpha \leq 2, \ \ \eta, \nu, \beta \in (0, 1), \ F : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha$.

Inspired and motivated by the works mentioned above, The goal of this chapter is to establish the existence and uniqueness results for the nonlocal boundary value problem system (1) by using some well-known tools of fixed point theory such as Banach contraction principle and Leray-Shauder nonlinear alternative and the existence of at least one positive solution for the system of two-point boundary value problem (2) by using Krasnosel'skii fixed point theorem. The aim of the last results is to establish some simple criteria for the existence of single positive solutions of the BVPs (2) in explicit intervals for $\lambda$. The chapter is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, for more details; see [44] and we give main results of problem (1). Finally, we give an example to illustrate our result. In Section 3, deals with main results of problem (2) and we give an example to illustrate our results.

## 2. Existence and uniqueness results for problem (1)

### 2.1 Preliminaries

In this section, we introduce some definitions and lemmas, see [2, 4, 44–46].

Definition 2.1. Let $\alpha > 0, \ \ n-1 < \alpha < n, \ \ n = [\alpha] + 1$ and $u \in C([0, \infty), \mathbb{R})$. The Caputo derivative of fractional order $\alpha$ for the function $u$ is defined by

$$ {}^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)ds, \tag{7} $$

where $\Gamma(\cdot)$ is the Eleur Gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$ I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad t > 0, \tag{8} $$

where $\Gamma(\cdot)$ is the Eleur Gamma function, provided that the right side is pointwise defined on $(0, \infty)$.Lemma 2.1. *Let $\alpha > 0, \ \ n-1 < \alpha < n$ and the function $g : [0, T] \to \mathbb{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation ${}^{c}D^{\alpha}g(t) = 0$ is given by*

$$ g(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \tag{9} $$

where $c_0, c_1, \ldots, c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Also, in [19], authors have been proved that for each $T > 0$ and $u \in C([0, T])$ we have

$$ I^{\alpha c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, \tag{10} $$

where $c_0, c_1, \ldots, c_{n-1}$ are real constants and $n = [\alpha] + 1$.

### 2.2 Existence results

Let $X = \{u(t) : u(t) \in C([0,1], \mathbb{R})\}$ endowed with the norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space and the product space $(X \times X, \|(u, v)\|)$ is also a Banach space equipped with the norm $\|(u, v)\| = \|u\| + \|v\|$.

Throughout the first section, we let

$$ \begin{aligned} &M = \frac{\Gamma(3-p)}{|\gamma - \eta^{2-p}|} \neq 0, |\lambda + \gamma - 1| \neq 0, |\gamma - \eta^2| \neq 0, \quad Q = |2(1-\eta)(\gamma - \eta) + \eta^2 |\lambda + \gamma - 1|| \neq 0, \\ &A(t) = |\Lambda_1(t)| = |\lambda + \gamma - 1| \left(\eta^2 + 2(1-\eta)t\right), \\ &B(t) = |\Lambda_2(t)| = \left(\eta^3 |\lambda + \gamma - 1| + 3|\gamma - \eta^2|(1-\eta)\right)\left(\eta^2 + 2(1-\eta)t\right) - Q\left(\eta^3 + 3(1-\eta)t^2\right), \end{aligned} \tag{11} $$

and

$$ Q = 2(1-\eta)(\gamma - \eta) + \eta^2(\lambda + \gamma - 1) \neq 0. \tag{12} $$

Lemma 2.2. *Let* $y \in C([0,1],\mathbb{R})$. *Then the solution of the linear differential system*

$$\begin{cases} {}^cD^\alpha u(t) = y(t),\ {}^cD^\beta v(t) = h(t),\ t\in[0,1],\ 2<\alpha,\beta\le 3 \\ \lambda u(0)+\gamma u(1) = v(\eta),\ \lambda v(0)+\gamma v(1) = u(\eta), \\ u(0) = \int_0^\eta v(s)ds, \quad v(0) = \int_0^\eta u(s)ds, \\ \lambda^c D^p u(0) + \gamma^c D^p u(1) = {}^cD^p v(\eta), \quad 1<p\le 2, \\ \lambda^c D^p v(0) + \gamma^c D^p v(1) = {}^cD^p u(\eta), \quad 1<p\le 2, \end{cases} \tag{13}$$

is equivalent to the system of integral equations

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h(\tau)d\tau\right) ds \\ & - \frac{\Lambda_1(t)}{Q(1-\eta)} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h(\tau)d\tau\right) ds \\ & - \frac{\Lambda_2(t)M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} h(s)ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)ds\right] \\ & + \frac{\Lambda_1(t)}{Q(\lambda+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} h(s)ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds\right], \end{aligned} \tag{14}$$

and

$$\begin{aligned} v(t) = {} & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)d\tau\right) ds \\ & - \frac{\Lambda_1(t)}{Q(1-\eta)} \int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau)d\tau\right) ds \\ & - \frac{\Lambda_2(t)M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} h(s)ds - \gamma\int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\alpha-p)} y(s)ds\right] \\ & + \frac{\Lambda_1(t)}{Q(\lambda+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \gamma\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\alpha)} y(s)ds\right], \end{aligned} \tag{15}$$

where

$$\Lambda_1(t) = (\lambda+\gamma-1)\left(\eta^2 + 2(1-\eta)t\right), \tag{16}$$

and

$$\Lambda_2(t) = \left(\eta^3(\lambda+\gamma-1) + 3\left(\gamma-\eta^2\right)(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right) - Q\left(\eta^3 + 3(1-\eta)t^2\right).$$

Proof. It is well known that the solution of equation ${}^cD^\alpha u(t) = y(t)$ can be written as

$$u(t) = I^\alpha y(t) + c_0 + c_1 t + c_2 t^2, \tag{17}$$

$$v(t) = I^\beta h(t) + d_0 + d_1 t + d_2 t^2, \tag{18}$$

where $c_0, c_{1,} c_2 \in \mathbb{R}$ and and $d_0, d_{1,} d_2 \in \mathbb{R}$ are arbitrary constants.
Then, from (68) we have

$$u'(t) = I^{\alpha-1} y(t) + c_1 + 2c_2 t, \tag{19}$$

and

$${}^cD^p u(t) = I^{\alpha-p} y(t) + c_2 \frac{2t^{2-p}}{\Gamma(3-p)}, \quad 1 < p \leq 2. \tag{20}$$

By using the three-point boundary conditions, we obtain.

$$c_2 = \frac{M}{2}\left(I^{\beta-p}y(\eta) - \gamma I^{\alpha-p}y(1)\right),$$

$$\begin{aligned} c_0 = & -\frac{2\eta^2(\lambda+\gamma-1)}{2(1-\eta)Q}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)d\tau\right)ds + \frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)d\tau\right)ds \\ & - \frac{(\eta^2[\eta^3(\lambda+\gamma-1) + 3(\gamma-\eta^2)(1-\eta)] - \eta^3 Q)M}{2(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}h(s)ds\right. \\ & \left.-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)ds\right] + \frac{\eta^2}{Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\right], \end{aligned} \tag{21}$$

and

$$\begin{aligned} c_1 = & \frac{-2(\lambda+\gamma-1)}{Q}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)d\tau\right)ds \\ & - \frac{(\eta^3(\lambda+\gamma-1) + 3(\gamma-\eta^2)(1-\eta))M}{3Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}h(s)ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)ds\right] \\ & + \frac{2(1-\eta)}{Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}y(s)ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\right]. \end{aligned} \tag{22}$$

Substituting the values of constants $c_0, c_1$ and $c_2$ in (68), we get solution (64). Similarly, we obtain solution (65). The proof is complete.
The following rolations hold:

$$|A(t)| \leq |\beta+\gamma-1|\left(\eta^2 + 2(1-\eta)\right) = A_1, \tag{23}$$

and

$$|B(t)| \leq \left| \left(\eta^3|\beta+\gamma-1| + 3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)\right) - Q\left(\eta^3+3(1-\eta)\right) \right| = B_1, \tag{24}$$

For the sake of brevity, we set

$$\begin{aligned} \Delta_1 &= \frac{\eta^{\beta+1}}{(1-\eta)\Gamma(\beta+2)} + \frac{A_1\eta^{\beta+1}}{Q(1-\eta)\Gamma(\beta+2)} + \frac{MB_1\eta^{\beta-p}}{(1-\eta)Q\Gamma(\lambda-p+1)} \\ &\quad + \frac{A_1\eta^{\beta}}{Q|\beta+\gamma-1|\Gamma(\beta+1)} \\ \Delta_2 &= \frac{MB_1\gamma}{6(1-\eta)Q\Gamma(\alpha-p+1)} + \frac{A_1\gamma}{Q|\lambda+\gamma-1|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \\ \Delta_3 &= \frac{\eta^{\alpha+1}}{(1-\eta)\Gamma(\alpha+2)} + \frac{A_1\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)} + \frac{MB_1\eta^{\alpha-p}}{(1-\eta)Q\Gamma(\alpha-p+1)} \\ &\quad + \frac{A_1\eta^{\alpha}}{Q|\lambda+\gamma-1|\Gamma(\alpha+1)}, \end{aligned} \tag{25}$$

and

$$\Delta_4 = \frac{MB_1\gamma}{6(1-\eta)Q\Gamma(\beta-p+1)} + \frac{A_1\gamma}{Q|\lambda+\gamma-1|\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+1)}. \tag{26}$$

In view of Lemma 2, we define the operator $T : X \times X \to X \times X$ by

$$T(u,v)(t) = \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix}, \tag{27}$$

where

$$\begin{aligned} T_1(u,v)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds + \frac{1}{1-\eta}\int_0^\eta \left( \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau,u(\tau),v(\tau))d\tau \right) ds \\ &\quad - \frac{B(t)M}{6(1-\eta)Q}\left[ \int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} g(s,u(s),v(s))ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f(s,u(s),v(s))ds \right] \\ &\quad + \frac{A(t)}{Q|\beta+\gamma-1|}\left[ \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds. \right] \\ &\quad - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left( \int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau,u(\tau),v(\tau))d\tau \right) ds, \end{aligned} \tag{28}$$

and

$$T_2(u,v)(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds + \frac{1}{1-\eta}\int_0^\eta \left( \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau),v(\tau))d\tau \right) ds$$

$$-\frac{B(t)M}{6(1-\eta)Q}\left[ \int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f(s,u(s),v(s))ds - \gamma \int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\beta-p)} g(s,u(s),v(s))ds \right]$$

$$+\frac{A(t)}{Q|\beta+\gamma-1|}\left[ \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds - \gamma \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} g(s,u(s),v(s))ds. \right]$$

$$-\frac{A(t)}{Q(1-\eta)}\int_0^\eta \left( \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau),v(\tau))d\tau \right) ds. \tag{29}$$

Observe that the boundary value problem (1) has solutions if the operator equation $(u,v) = T(u,v)$ has fixed points.

Now we are in a position to present the first main results of this paper. The existence results is based on Leray-Shauder nonlinear alternative.

Lemma 2.3. [44] *(Leray-Schauder alternative). Let E be a Banach space and* $T : E \to E$ *be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\varepsilon = \{(u,v) \in X \times X : (u,v) = \lambda T(u,v), \textit{for some } 0 < \lambda < 1\}. \tag{30}$$

Then either the $\varepsilon(T)$ is unbounded or $T$ has at least one fixed point.

Theorem 1.1 Assume that $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a continuous function and.

$(H_1)$ there exist a function $k_i \geq 0\, m_i \geq 0, i = 1,2$ and $k_0 > 0, m_0 > 0$ such that $\forall u \in \mathbb{R}, \forall v \in \mathbb{R}, i = 1,2$, we have

$$|f(t,u,v)| \leq k_0 + k_1|u| + k_2|v|, \tag{31}$$

and

$$|g(t,u,v)| \leq m_0 + m_1|u| + m_2|v|. \tag{32}$$

If $(\Delta_2 + \Delta_3)k_1 + (\Delta_1 + \Delta_4)m_1 < 1$ and $(\Delta_2 + \Delta_3)k_2 + (\Delta_1 + \Delta_4), m_3 < 1$, where $\Delta_i, \;\; i = 1,2,3,4$ are given above. Then the boundary value problem (1)–(58) has at least one solution on [0,1].

Proof. It is clear that $T$ is a continuous operator where $T : X \times X \to X \times X$ is defined above. Now, we show that $T$ is completely continuous. Let $\Omega \subset X \times X$ be bounded. Then there exist positive constants $L_1$ and $L_2$ such that

$$|f(t,u(t),v(t))| \leq L_1, \quad |g(t,u(t),v(t))| \leq L_2, \quad \forall (u,v) \in \Omega. \tag{33}$$

Then for any $(u, v) \in \Omega$, we have

$$\begin{aligned}
|T_1(u,v)(t)| \le & \frac{L_2}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds \\
& + \frac{|A(t)|L_2}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds + L_1\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& + \frac{M|B(t)|}{6(1-\eta)Q}\left[L_2\int_0^{\eta}\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \gamma L_1\int_0^{0}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right] \\
& + \frac{|A(t)|}{Q|\lambda+\gamma-1|}\left[L_2\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds + \gamma L_1\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right], \\
\le & L_2\left\{\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds + \frac{A_1}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\beta-1}}{\Gamma(\alpha\beta)}d\tau\right)ds\right. \\
& \left. + \frac{MB_1}{6(1-\eta)Q}\int_0^{\eta}\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \frac{A_1}{6|\lambda+\gamma-1|}\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\right\} \\
& + L_1\left\{\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds + \frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}, \\
\le & L_2\Delta_1 + L_1\Delta_2.
\end{aligned} \tag{34}$$

Hence

$$\|T_1(u,v)\| \le L_2\Delta_1 + L_1\Delta_2. \tag{35}$$

In the same way, we can obtain that

$$\|T_2(u,v)\| \le L_1\Delta_3 + L_2\Delta_4. \tag{36}$$

Thus, it follows from (78) and (95) that the operator $T$ is uniformly bounded, since $\|T(u,v)\| \le L_1(\Delta_1+\Delta_3) + L_2(\Delta_2+\Delta_4)$. Now, we show that $T$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
|T_1(u(t_2),v(t_2)) - T_1(u(t_1),v(t_1))| \le & L_1\int_0^{t_1}\frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& + L_1\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \frac{|A(t_2)-A(t_1)|L_2}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds \\
& + \frac{(B(t_2)-B(t_1))M}{6(1-\eta)Q}\left[L_2\int_0^{\eta}\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \gamma L_1\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right] \\
& + \frac{A(t_2)-A(t_1)}{Q|\lambda+\gamma-1|}\left[L_2\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds - \gamma L_1\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right].
\end{aligned} \tag{37}$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Similarly, we have

$$\begin{aligned}|T_2(u(t_2), v(t_2)) - T_2(u(t_1), v(t_1))| \leq{}& L_2 \int_0^{t_1} \frac{(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}}{\Gamma(\beta)} ds \\ &+ L_2 \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} ds + \frac{|A(t_2) - A(t_1)| L_1}{Q(1-\eta)} \int_0^{\eta} \left( \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \right) ds \\ &+ \frac{(B(t_2) - B(t_1))M}{6(1-\eta)Q} \left[ L_1 \int_0^{\eta} \frac{(\eta - s)^{\alpha-p-1}}{\Gamma(\alpha - p)} ds + \gamma L_2 \int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\beta - p)} ds \right] \\ &+ \frac{A(t_2) - A(t_1)}{Q|\lambda + \gamma - 1|} \left[ L_1 \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} ds - \gamma L_2 \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} ds \right].\end{aligned} \tag{38}$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, the operator $T$ is equicontinuous.

Therefore, the operator $T$ is completely continuous.

Finally, it will be verified that the set $\varepsilon = \{(u, v) \in X \times X : (u, v) = \lambda T(u, v), 0 \leq \lambda \leq 1\}$ is bounded. Let $(u, v) \in \varepsilon$, with $(u, v) = \lambda T(u, v)$ for any $t \in [0, 1]$, we have

$$u(t) = \lambda T_1(u, v)(t), \quad v(t) = \lambda T_2(u, v)(t). \tag{39}$$

Then

$$\begin{aligned}|u(t)| &\leq \Delta_2(k_0 + k_1|u| + k_2|v|) + \Delta_1(m_0 + m_1|u| + m_2|v|), \\ &= \Delta_2 k_0 + \Delta_1 m_0 + (\Delta_2 k_1 + \Delta_1 m_1)|u| + (\Delta_2 k_2 + \Delta_1 m_2)|v|,\end{aligned} \tag{40}$$

and

$$\begin{aligned}|v(t)| &\leq \Delta_3(k_0 + k_1|u| + k_2|v|) + \Delta_4(m_0 + m_1|u| + m_2|v|), \\ &= \Delta_3 k_0 + \Delta_4 m_0 + (\Delta_3 k_1 + \Delta_4 m_1)|u| + (\Delta_3 k_2 + \Delta_4 m_2)|v|.\end{aligned} \tag{41}$$

Hence we have

$$\|u\| = \Delta_2 k_0 + \Delta_1 m_0 + (\Delta_2 k_1 + \Delta_1 m_1)\|u\| + (\Delta_2 k_2 + \Delta_1 m_2)\|v\|, \tag{42}$$

and

$$\|v\| = \Delta_3 k_0 + \Delta_4 m_0 + (\Delta_3 k_1 + \Delta_4 m_1)|u| + (\Delta_3 k_2 + \Delta_4 m_2)|v|, \tag{43}$$

which imply that

$$\begin{aligned}\|u\| + \|v\| = (\Delta_2 + \Delta_3)k_0 + (\Delta_1 + \Delta_4)m_0 + [(\Delta_2 + \Delta_3)k_1 + (\Delta_1 + \Delta_4)m_1]\|u\| \\ + [(\Delta_2 + \Delta_3)k_2 + (\Delta_1 + \Delta_4)m_2]\|v\|.\end{aligned} \tag{44}$$

Consequently,

$$\|(u,v)\| = \frac{(\Delta_2+\Delta_3)k_0 + (\Delta_1+\Delta_4)m_0}{\Delta_0}, \tag{45}$$

where

$$\Delta_0 = min\{1-[(\Delta_2+\Delta_3)k_1 + (\Delta_1+\Delta_4)m_1], 1-[(\Delta_2+\Delta_3)k_2 + (\Delta_1+\Delta_4)m_2]\}, \tag{46}$$

which proves that $\varepsilon$ is bounded. Thus, by Lemma 15, the operator $T$ has at least one fixed point. Hence boundary value problem (1) has at least one solution. The proof is complete.

Now, we are in a position to present the second main results of this paper.

Theorem 1.2 Assume that $f,g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist positive constants $L_1$ and $L_2$ such that for all $t\in[0,1]$ and $u_i, v_i \in \mathbb{R}, i=1,2$, we have.

1. $|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_1(|u_1-v_1| + |u_2-v_2|)$,

2. $|g(t,u_1,u_2) - g(t,v_1,v_2)| \le L_2(|u_1-v_1| + |u_2-v_2|)$.

Then the boundary value problem (1) has a unique solution on [0,1] provided

$$(\Delta_1+\Delta_3)L_1 + (\Delta_2+\Delta_4)L_2 < 1. \tag{47}$$

Proof. Let us set $\sup_{t\in[0,1]} |f(t,0,0)| = N_1 < \infty$ and $\sup_{t\in[0,1]} |g(t,0,0)| = N_2 < \infty$. For $u\in X$, we observe that

$$\begin{aligned} |f(t,u(t),v(t))| &\le |f(t,u(t)) - f(t,0,0)| + |f(t,0,0)|, \\ &\le L_1(|u(t)| + |v(t)|) + N_1, \\ &\le L_1(\|u\| + \|v\|) + N_1, \end{aligned} \tag{48}$$

and

$$|g(t,u(t),v(t))| \le |g(t,u(t)) - g(t,0,0)| + |g(t,0,0)| \le L_2\|u\| + N_2. \tag{49}$$

Then for $u\in X$, we have

$$\begin{aligned} |T_1(u,v)(t)| &\le \frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}[L_2\|(u,v)\| + N_2]d\tau\right)ds \\ &+ \frac{|A(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}[L_2\|(u,v)\| + N_2]d\tau\right)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[L_1\|(u,v)\| + N_1]ds \\ &+ \frac{M|B(t)|}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}[L_2\|(u,v)\| + N_2]ds + \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[L_1\|(u,v)\| + N_1]ds\right] \\ &+ \frac{|A(t)|}{Q|\lambda+\gamma-1|}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}[L_2\|(u,v)\| + N_2]ds + \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[L_1\|(u,v)\| + N_1]ds,\right] \end{aligned}$$

$$\begin{aligned}
&\leq (L_2\|(u,v)\| + N_2)\left\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds + \frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds\right.\\
&\left.+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\right\}\\
&+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg\}\\
&+(L_1\|(u,v)\| + N_1)\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\\
&\left.+\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds + \frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\},\\
&\leq (L_2 r + N_2)\Delta_1 + (L_1 r + N_1)\Delta_2
\end{aligned} \tag{50}$$

Hence

$$\|T_1(u,v)\| \leq (L_2\Delta_1 + L_1\Delta_2)r + N_2\Delta_1 + N_1\Delta_2 \tag{51}$$

In the same way, we can obtain that

$$\|T_2(u,v)\| \leq (L_1\Delta_3 + L_2\Delta_4)r + N_2\Delta_4 + N_1\Delta_3. \tag{52}$$

Consequently,

$$\|T(u,v)\| \leq ((\Delta_2+\Delta_3)L_1 + (\Delta_1+\Delta_4)L_2)r + N_2(\Delta_1+\Delta_4) + N_1(\Delta_2+\Delta_3) \leq r. \tag{53}$$

Now, for $(u_1, v_1), (u_2, v_2) \in X \times X$ and for each $t \in [0,1]$, it follows from assumption $(H_3)$ that

$$\begin{aligned}
|T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)| &\leq L_2(\|u_2-u_1\| + \|v_2-v_1\|)\left\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds\right.\\
&+\frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds\\
&\left.+\frac{MB_1}{6(1-\eta)Q_0}{}^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\right\}\\
&+L_1(\|u_2-u_1\| + \|v_2-v_1\|)\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\\
&\left.+\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds + \frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}\\
&\leq (L_2\Delta_1 + L_1\Delta_2)(\|u_2-u_1\| + \|v_2-v_1\|).
\end{aligned} \tag{54}$$

Thus

$$\|T_1(u_2,v_2) - T_1(u_1,v_1)\| \le (L_2\Delta_1 + L_1\Delta_2)(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (55)$$

Similarly,

$$\|T_2(u_2,v_2) - T_2(u_1,v_1)\| \le (L_2\Delta_3 + L_1\Delta_4)(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (56)$$

It follows from (101) and (110) that

$$\|T(u_2,v_2) - T(u_1,v_1)\| \le (L_2(\Delta_1 + \Delta_3) + L_1(\Delta_2 + \Delta_4))(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (57)$$

Since $L_2(\Delta_1 + \Delta_3) + L_1(\Delta_2 + \Delta_4) < 1$, thus $T$ is a contraction operator. Hence it follows by Banach's contraction principle that the boundary value problem (1) has a unique solution on $[0,1]$.

We construct an example to illustrate the applicability of the results presented.

Example 2.1. Consider the following system fractional differential equation

$$\begin{cases} {}^cD^3u(t) = \frac{t}{8}\left((\cos(t))\sin\left(\frac{|u(t)| + |v(t)|}{2}\right)\right) + \frac{e^{-(u(t)+v(t))^2}}{1+t^2}, & t \in [0,1], \\ {}^cD^3v(t) = \frac{1}{32}\sin(2\pi u(t)) + \frac{|v(t)|}{16(1+|v(t)|)} + \frac{1}{2}, & t \in [0,1], \end{cases} \quad (58)$$

subject to the three-point coupled boundary conditions

$$\begin{cases} \frac{1}{100}u(0) + \frac{1}{10}u(1) = u\left(\frac{1}{2}\right), \\ u(0) = \int_0^{0,5} u(s)ds, \\ \frac{1}{100}{}^cD^{\frac{3}{2}}u(0) + \frac{1}{10}{}^cD^{\frac{3}{2}}u(1) = {}^cD^{\frac{3}{2}}u\left(\frac{1}{2}\right), \end{cases} \quad (59)$$

where $f(t,u,v) = \frac{t}{8}\left((\cos(t))\sin\left(\frac{|u|+|v|}{2}\right)\right) + \frac{e^{-(u+v)^2}}{1+t^2}, t \in [0,1], \eta = 0,5, \lambda = 0,01, \gamma = 0,1, p = 1,5$ *and* $g(t,u,v) = \frac{1}{32\pi}\sin(2\pi u(t)) + \frac{|v(t)|}{16(1+|v(t)|)} + \frac{1}{2}$.

It can be easily found that $M = \frac{20}{3}$ and $Q = \frac{9}{400}$.

Furthermore, by simple computation, for every $u_i, v_i \in \mathbb{R}, \ i = 1,2$, we have

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L(|u_1 - v_1| + |u_2 - v_2|), \quad (60)$$

and

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \le L(|u_1 - v_1| + |u_2 - v_2|), \quad (61)$$

where $L_1 = L_2 = L = \frac{1}{16}$. It can be easily found that $\Delta_1 = \Delta_3 \cong 0,799562, \ \Delta_2 = \Delta_4 \cong 1,182808$.

Finally, since $L_1(\Delta_1 + \Delta_3) + L_2(\Delta_2 + \Delta_4) = 2L(\Delta_1 + \Delta_2) \cong 0,247796 < 1$, thus all assumptions and conditions of Theorem 1.2 are satisfied. Hence, Theorem implies that the three-point boundary value problem (58, 59) has a unique solution.

## 3. Existence result for second problem (2)

We provide another results about the existence of solutions for the problem (2) by using the assumption.

We shall consider the Banach space $B = C([0,1])$ equipped with usual supermum norm and $B^+ = C^+([0,1])$. In arriving our results, we present some notation and preliminary lemmas. The first is well known.

Lemma 3.1. *Let* $y(t) \in C[0,1]$. *If* $u \in C^4[0,1]$, *then the BVP*

$$\begin{cases} u^{(4)}(t) = y(t), & 0 \le t \le 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{62}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{63}$$

where

$$G(t,s) = \begin{cases} \frac{1}{6} t^2 (3s - t), & 0 \le t \le s \le 1, \\ \frac{1}{6} s^2 (3t - s), & 0 \le s \le t \le 1. \end{cases} \tag{64}$$

Lemma 3.2. *For any* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$0 \le G(t,s) \le G(1,s) = \frac{1}{6} s^2 (3 - s) = \psi(s). \tag{65}$$

Proof. The derivatives of the function $G$ with respect to $t$ is

$$\frac{\partial}{\partial t} G(t,s) = \begin{cases} \frac{1}{2} s^2 - \frac{1}{2} (s-t)^2, & 0 \le t \le s \le 1 \\ \frac{1}{2} s^2, & 0 \le s \le t \le 1. \end{cases} \tag{66}$$

Since the derivative of the function $G$ with respect to $t$ is nonnegative for all $t \in [0,1]$, $G$ is nondecreasing function of $t$ that attaints its maximum when $t = 1$. Then

$$\max_{0 \le t \le 1} G(t,s) = G(1,s) = \frac{1}{2} s^2 - \frac{1}{6} s^3. \tag{67}$$

Lemma 3.3. *Let* $0 < \theta < 1$. *Then for* $y(t) \in C^+[0,1]$, *the unique solution* $u(t)$ *of BVP* (14) *is nonnegative and satisfies*

$$\min_{t \in [\theta, 1]} u(t) \ge \frac{2\theta^3}{3} \|u\|. \tag{68}$$

Proof. Let $y(t) \in C^+[0,1]$, then from $G(t,s) \ge 0$ we know $u \in C^+[0,1]$. Set $u(t_0) = \|u\|$, $t_0 \in (0,1]$. We first prove that

$$\frac{G(t,s)}{G(t_0,s)} \geq \frac{2}{3}t^3, \quad t, t_0, s \in (0,1]. \tag{69}$$

In fact, we can consider four cases:

1. if $0 < t, t_0 \leq s \leq 1$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{t^2(3s-t)}{t_0^2(3s-t_0)} \geq \frac{t^2(2s)}{3s-t_0} \geq \frac{t^2(2s)}{3} \geq \frac{t^2(2t)}{3} = \frac{2t^3}{3}, \tag{70}$$

2. if $0 < t \leq s \leq t_0 \leq 1$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{t^2(3s-t)}{t_0^2(3s-t_0)} \geq \frac{t^2(2s)}{3s-t_0} \geq \frac{t^2(2s)}{3} \geq \frac{t^2(2t)}{3} = \frac{2t^3}{3}, \tag{71}$$

3. if $0 < s \leq t, t_0 \leq 1$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{s^2(3t-s)}{s^2(3t_0-s)} = \frac{3t-s}{3t_0-s} \geq \frac{3t-s}{3t_0} \geq \frac{3t-s}{3} \geq \frac{2t+t-s}{3} \geq \frac{2t}{3} \geq \frac{2t^3}{3}, \tag{72}$$

4. if $0 < t_0 \leq s \leq t \leq 1$, then

$$\frac{G(t,s)}{G(t_0,s)} = \frac{s^2(3t-s)}{t_0^2(3s-t_0)} \geq \frac{t_0^2(3t-s)}{t_0^2(3t-t_0)} \geq \frac{3t-s}{3t} \geq \frac{3t-t}{3t} \geq \frac{2t}{3} \geq \frac{2t^3}{3}, \tag{73}$$

Therefore, for $t \in [\theta, 1]$, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds = \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s)y(s)ds \geq \frac{2t^3}{3}u(t_0) \geq \frac{2\theta^3}{3}\|u\|. \tag{74}$$

The proof is complete.
If we let

$$K = \left\{ x \in B : \ x(t) \geq 0 \ \ on \ \ [0,1], \ \ and \ \ \min_{t \in [\theta,1]} x(t) \geq \frac{2\theta^3}{3}\|x\| \right\} \tag{75}$$

then it is easy to see that $K$ a cone in $B$. We not that a pair $(u(t), v(t))$ is a solution of BVPs (2) if, and only if

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds, \quad t \in [0,1], \tag{76}$$

and

$$v(t) = \lambda \int_0^1 G(t,s)b(s)g(u(s))ds, \quad t \in [0,1]. \tag{77}$$

Now, we define an integral operator $T : K \to B$ by

$$(Tu)(t) = \lambda \int_0^1 G(t,s)a(s)f\left(\lambda \int_0^1 G(s,r)b(r)g(u(r))dr\right)ds, \quad u \in K. \tag{78}$$

We adopt the following assumptions:
$(H_1)$ $a, b \in C((0,1),[0,\infty))$ and each does not vanish identically on any subinterval.
$(H_2)$ $f, g \in C([0,\infty),[0,\infty))$ and each to be singular at $t = 0 or t = 1$.
$(H_3)$ All of $f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}$, $g_0 = \lim_{x\to 0^+} \frac{g(x)}{x}$, $f_\infty = \lim_{x\to\infty} \frac{f(x)}{x}$, *and* $g_\infty = \lim_{x\to\infty} \frac{g(x)}{x}$ exist as real numbers.
$(H_4)$ $g(0) = 0$ and $f$ is increasing function.
Lemma 3.4 *Let $\lambda$ be positive number and K be the cone defined above.*

i. If $u \in B^+$ and $v : [0,1] \to [0,\infty)$ is defined by (77), then $v \in K$.

ii. If $T$ is the integral operator defined by (78), then $T(K) \subset K$.

iii. Assume that $(H_1)$, $(H_2)$ hold. Then $T : K \to B$ is completely continuous.

Proof. Let $u \in B^+$ and $v$ be defined by (77).

i. By the nonnegativity of $G$, $b$ and $g$ it follows that $v(t) \geq 0, t \in [0,1]$. In view of $(H_1)$, $(H_2)$, we have

$$\int_0^1 G(t,s)b(s)g(u(s))ds \geq \int_0^1 \min_{t\in[\theta,1]} G(t,s)b(s)g(u(s))ds, \tag{79}$$

from which, we take

$$\min_{t\in[\theta,1]} \int_0^1 G(t,s)b(s)g(u(s))ds \geq \int_0^1 \min_{t\in[\theta,1]} G(t,s)b(s)g(u(s))ds. \tag{80}$$

Consequently, employing (68) and for $\lambda > 0$, we have

$$\begin{aligned}
\lambda \int_0^1 G(t,s)b(s)g(u(s))ds &\geq \lambda \int_0^1 \min_{t\in[\theta,1]} G(t,s)b(s)g(u(s))ds \\
&\geq \frac{2\theta^3}{3} \lambda \int_0^1 G(t_0,s)b(s)g(u(s))ds \\
&\geq \frac{2\theta^3}{3} v(t_0), t_0 \in (0,1] \\
&\geq \frac{2\theta^3}{3} \|v\|.
\end{aligned} \tag{81}$$

Therefore

$$\min_{t\in[\theta,1]} v(t) \geq \frac{2\theta^3}{3}\|v\|. \tag{82}$$

Which give that $v \in K$.

ii. Obviously, for $v\in K,\ \ T(u)\in C^{+}[0,1]$. For $t\in[0,1]$, we have

$$\begin{aligned}\|Tu(t)\| &= \max_{0\leq t\leq 1}\lambda\int_0^1 G(t,s)a(s)f(v(s))ds\\ &\leq \lambda\int_0^1 G(1,s)a(s)f(v(s))ds,\end{aligned} \tag{83}$$

and

$$\begin{aligned}Tu(t) &= \lambda\int_0^1 G(t,s)a(s)f(v(s))ds\\ &= \lambda\int_0^1 \frac{G(t,s)}{G(1,s)}G(1,s)a(s)f(v(s))ds\\ &\geq \frac{2\theta^3}{3}\lambda\int_0^1 G(1,s)a(s)f(v(s))ds\\ &\geq \frac{2}{3}\theta^3\|Tu(t)\|.\end{aligned} \tag{84}$$

Which give that $Tu\in K$. Therefore $T:K\to K$.

iii. By using standard arguments it is not difficult to show that the operator $T:K\to B$ is completely continuous.

The key tool in our approach is the following Krasnoselskii's fixed point theorem of cone expansion-compression type.

Theorem 1.3 (See [47]) Let $B$ be a Banach space and $K\subset B$ be a cone in $B$. Assume $\Omega_1$ and $\Omega_2$ are open subset of $B$ with $0\in\Omega_1$ and $\overline{\Omega_1}\subset\Omega_2$,

$T: K\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)\to K$ be a completely continuous operator such that.

i. $\|Tu\|\leq\|u\|$, $\forall u\in K\cap\partial\Omega_1$ and $\|Tu\|\geq\|u\|$, $\forall u\in K\cap\partial\Omega_2$; or.

ii. $\|Tu\|\geq\|u\|$, $\forall u\in K\cap\partial\Omega_1$ and $\|Tu\|\leq\|u\|$, $\forall u\in K\cap\partial\Omega_2$.

Then, $T$ has a fixed point in $K\cap\left(\overline{\Omega}_2\backslash\Omega_1\right)$. Throughout this section, we shall use the following notations:

$$L_1 = \max\left\{\left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1 \psi(r)a(r)f_\infty dr\right]^{-1}, \left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1 \psi(r)a(r)g_\infty dr\right]^{-1}\right\} \tag{85}$$

and

$$L_2 = \min\left\{\left[\int_0^1 \psi(r)a(r)f_0 dr\right]^{-1}, \left[\int_0^1 \psi(r)b(r)g_0 dr\right]^{-1}\right\}. \tag{86}$$

$$L_3 = \max\left\{\left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1 \psi(r)a(r)f_0 dr\right]^{-1}, \left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1 \psi(r)a(r)g_0 dr\right]^{-1}\right\} \tag{87}$$

and

$$L_4 = \min\left\{\left[\int_\theta^1 \psi(r)a(r)f_\infty dr\right]^{-1}, \left[\int_\theta^1 \psi(r)b(r)g_\infty dr\right]^{-1}\right\}. \tag{88}$$

## 4. Existence results

In this section, we discuss the existence of at least one positive solution for BVPs (2). We obtain the following existence results, by applying the positivity of Green's function $G(t,s)$ and the fixed-point of cone expansion-compression type.

Theorem 1.4 Assume conditions $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied. Then, for each $\lambda$ satisfying $L_1 < \lambda < L_2$ there exists a pair $(u,v)$ satisfying BVPs (2) such that $u(t) > 0$ and $v(t) > 0$ on $(0,1)$.

Proof. Let $L_1 < \lambda < L_2$. And let $\varepsilon > 0$ be chosen such that

$$\max\left\{\left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1 \psi(r)a(r)(f_\infty - \varepsilon)dr\right]^{-1}, \left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1 \psi(r)a(r)(g_\infty - \varepsilon)dr\right]^{-1}\right\} \le \lambda, \tag{89}$$

and

$$\lambda \le \min\left\{\left[\int_\theta^1 \psi(r)a(r)(f_0 + \varepsilon)dr\right]^{-1}, \left[\int_\theta^1 \psi(r)b(r)(g_0 + \varepsilon)dr\right]^{-1}\right\}. \tag{90}$$

From the definitions of $f_0$ and $g_0$ there exists an $R_1 > 0$ such that

$$f(u) \le (f_0 + \varepsilon)u, \quad 0 < u \le R_1, \tag{91}$$

and

$$g(u) \le (g_0 + \varepsilon)u, \quad 0 < u \le R_1, \tag{92}$$

Let $u \in K$ with $\|u\| = R_1$. From (65) and choice of $\varepsilon$, we have

$$\begin{aligned}\lambda\int_0^1 G(t,s)b(r)g(u(r)) &\leq \lambda\int_0^1 \psi(r)b(r)g(u(r))dr\\ &\leq \lambda\int_0^1 \psi(r)b(r)\left(g_0+\varepsilon\right)u(r)dr\\ &\leq \|u\|\lambda\int_0^1 \psi(r)b(r)dr\left(g_0+\varepsilon\right)dr\\ &\leq R_1 = \|u\|.\end{aligned} \tag{93}$$

Consequently, from (65) and choice of $\varepsilon$, we have

$$\begin{aligned}Tu(t) &= \lambda\int_0^1 G(t,s)a(s)f\left(\lambda\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds\\ &\leq \lambda\int_0^1 \psi(s)a(s)f\left(\lambda\int_0^1 G(s,r)b(r)g(u(r))dr\right)ds\\ &\leq \lambda\int_0^1 \psi(s)a(s)\left(f_0+\varepsilon\right)\left[\lambda\int_0^1 G(s,r)b(r)g(u(r))dr\right]ds\\ &\leq \lambda\int_0^1 \psi(s)a(s)\left(f_0+\varepsilon\right)R_1 ds\\ &\leq R_1 = \|u\|.\end{aligned} \tag{94}$$

So, $\|Tu\| \leq \|u\|$. If we set $\Omega_1 = \{u \in B: \ \|u\| < R_1\}$, then

$$\|Tu\| \leq \|u\|, \ \ for \ \ u \in K \cap \partial\Omega_1 \tag{95}$$

Considering the definitions of $f_\infty$ and $g_\infty$ there exists an $\overline{R}_2 > 0$ such that

$$f(u) \geq \left(f_\infty - \varepsilon\right)u, \quad 0 < u \leq \overline{R}_2, \tag{96}$$

and

$$g(u) \geq \left(g_\infty - \varepsilon\right)u, 0 < u \leq \overline{R}_2. \tag{97}$$

Let $u \in K$ and $R_2 = \max\left\{2R_1, \frac{3\overline{R}_2}{2\theta^3}\right\}$ with $\|u\| = R_2$, then

$$\min_{s \in [\theta,1]} u(s) \geq \frac{2}{3}\theta^3\|u\| \geq \overline{R}_2 \tag{98}$$

Thus, from (68) and choice of $\varepsilon$, we have

$$\begin{aligned}\lambda\int_0^1 G(t,s)b(r)g(u(r)) &\geq \frac{2\theta^3}{3}\lambda\int_0^1 G(1,r)b(r)g(u(r))dr\\ &\geq \frac{2\theta^3}{3}\lambda\int_\theta^1 \psi(r)b(r)(g_\infty-\varepsilon)u(r)dr\\ &\geq \|u\|\left(\frac{2\theta^3}{3}\right)^2\lambda\int_\theta^1\psi(r)b(r)dr(g_\infty-\varepsilon)dr\\ &\geq R_2=\|u\|.\end{aligned}\tag{99}$$

Consequently, from (77) and choice of $\varepsilon$, we have

$$\begin{aligned}Tu(t) &\geq \frac{2\theta}{3}\lambda\int_\theta^1\psi(s)a(s)f\left(\lambda\int_\theta^1 G(s,r)b(r)g(u(r))dr\right)ds\\ &\geq \frac{2\theta^3}{3}\lambda\int_\theta^1\psi(s)a(s)(f_\infty-\varepsilon)\left[\lambda\int_\theta^1 G(s,r)b(r)g(u(r))dr\right]ds\\ &\geq \frac{2\theta^3}{3}\lambda\int_\theta^1\psi(s)a(s)(f_\infty-\varepsilon)H_2ds\\ &\geq \left(\frac{2\theta^3}{3}\right)^2\lambda\int_\theta^1\psi(s)a(s)(f_\infty-\varepsilon)H_2ds\\ &\geq R_2=\|u\|.\end{aligned}\tag{100}$$

So, $\|Tu\|\geq\|u\|$. If we set $\Omega_2=\{u\in B:\ \ \|u\|<R_2\}$, then

$$\|Tu\|\geq\|u\|,\ \ for\ \ u\in K\cap\partial\Omega_2.\tag{101}$$

Applying (i) of Theorem 3.1 to (95) and (101), yields that $T$ has a fixed point $u^*\in K\cap(\overline{\Omega}_2/\Omega_1)$. As such and with $v$ defined by

$$v(t)=\lambda\int_0^1 G(t,s)b(s)g(u(s))ds,\tag{102}$$

the pair $(u,v)$ is a desired solution of BVPs (2) for the given $\lambda$. The proof is complete.

Theorem 1.5 Assume conditions $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ are satisfied. Then, for each $\lambda$ satisfying $L_3<\lambda<L_4$ there exists a pair $(u,v)$ satisfying BVPs (2) such that $u(t)>0$ and $v(t)>0$ on $(0,1)$.

Proof. Let $L_3<\lambda<L_4$ and $\varepsilon>0$ be chosen such that

$$max\left\{\left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1\psi(r)a(r)(f_0-\varepsilon)dr\right]^{-1},\left[\left(\frac{2\theta^3}{3}\right)^2\int_\theta^1\psi(r)a(r)(g_0-\varepsilon)dr\right]^{-1}\right\}\leq\lambda,\tag{103}$$

and

$$\lambda \leq min\left\{ \left[ \int_0^1 \psi(r)a(r)(f_\infty + \varepsilon)dr \right]^{-1}, \left[ \int_0^1 \psi(r)b(r)(g_\infty + \varepsilon)dr \right]^{-1} \right\}. \tag{104}$$

From the definitions of $f_0$ and $g_0$ there exists an $R_1 > 0$ such that

$$f(u) \geq (f_0 - \varepsilon)u, \quad 0 < u \leq R_1, \tag{105}$$

and

$$g(u) \geq (g_0 - \varepsilon)u, \quad 0 < u \leq R_1, \tag{106}$$

Now $g(0) = 0$ and so there exists $0 < R_2 \leq R_1$ such that

$$\lambda g(u) \leq \frac{R_1}{\int_0^1 \psi(r)b(r)dr}, \quad 0 \leq u \leq R_2. \tag{107}$$

Let $u \in K$ with $\|u\| = R_2$. Then

$$\begin{aligned} \lambda \int_0^1 G(t,s)b(r)g(u(r)) &\leq \lambda \int_0^1 \psi(r)b(r)g(u(r))dr \\ &\leq \frac{\int_0^1 \psi(r)b(r)R_1 dr}{\int_0^1 \psi(s)b(s)ds} \\ &\leq R_1 = \|u\|. \end{aligned} \tag{108}$$

Therefore, by (68), we have

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f\left( \lambda \int_0^1 G(s,r)b(r)g(u(r))dr \right) ds \\ &\geq \frac{2\theta^3}{3}\lambda \int_\theta^1 \psi(s)a(s)f\left( \frac{2\theta^3}{3}\lambda \int_\theta^1 \psi(r)b(r)g(u(r))dr \right) ds \\ &\geq \frac{2\theta^3}{3}\lambda \int_\theta^1 \psi(s)a(s)(f_0 - \varepsilon)\left[ \left(\frac{2\theta^3}{3}\right)^2 \lambda \int_\theta^1 \psi(r)b(r)(g_0 - \varepsilon)\|u\|dr \right] ds \\ &\geq \frac{2\theta^3}{3}\lambda \int_\theta^1 \psi(r)a(r)(f_0 - \varepsilon)\|u\| \\ &\geq \left(\frac{2\theta^3}{3}\right)^2 \lambda \int_\theta^1 \psi(r)a(r)(f_0 - \varepsilon)\|u\| \\ &\geq \|u\|. \end{aligned} \tag{109}$$

So, $\|Tu\| \geq \|u\|$. If we set $\Omega_1 = \{u \in B : \ \|u\| < R_2\}$, then

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \tag{110}$$

Considering the definitions of $f_\infty$ and $g_\infty$ there exists $\overline{R}_1 > 0$ such that

$$f(u) \leq (f_\infty + \varepsilon)u, \quad u \geq \overline{R}_1, \tag{111}$$

and

$$g(u) \leq (g_\infty + \varepsilon)u, \quad u \geq \overline{R}_1. \tag{112}$$

We consider two cases: $g$ is bounded or $g$ is unbounded.

Case $(i)$. Suppose $g$ is bounded, say $g(u) \leq N, \quad N > 0$ for all $0 < u < \infty$. Then, for $u \in K$

$$\begin{gathered} \lambda \int_0^1 G(t,s)b(r)g(u(r)) \leq \lambda \int_0^1 \psi(r)b(r)g(u(r))dr \\ M = \max\left\{f(u) : \ 0 \leq u \leq N\lambda \int_0^1 \psi(r)b(r)dr\right\} \end{gathered} \tag{113}$$

and let

$$R_3 > \max\left\{2R_2, \ M\lambda \int_0^1 \psi(s)a(s)ds\right\}. \tag{114}$$

Then, for $u \in K$ with $\|u\| = R_3$, we have

$$\begin{aligned} Tu(t) &\leq \lambda \int_0^1 \psi(s)a(s)Mds \\ &\leq R_3 = \|u\|. \end{aligned} \tag{115}$$

So that $\|Tu\| \leq \|u\|$. If we set $\Omega_2 = \{u \in B : \ \|u\| \leq R_3\}$, then, for $u \in K \cap \partial\Omega_2$:

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2 \tag{116}$$

Case$(ii)$. $g$ is unbounded, there exists $R_3 > max\left\{2R_2, \overline{R}_1\right\}$ such that $g(u) \leq g(R_3)$, for $0 < u \leq R_3$.

Similarly, there exists $R_4 > max\left\{2R_3, \ M\lambda\int_0^1\psi(r)b(r)g(R_3)ds\right\}$ such that $f(u) \leq f(R_4)$, for $0 < u \leq R_4$.

Let $u \in K$ with $\|u\| = R_4$, from $(H_4)$, we have

$$\begin{aligned} Tu(t) &\leq \lambda \int_0^1 \psi(s)a(s)f\left(\lambda \int_0^1 \psi(r)b(r)g(R_3)dr\right)ds \\ &\leq \lambda \int_0^1 \psi(r)a(r)f(R_4)ds \\ &\leq \lambda \int_0^1 \psi(r)a(r)(f_\infty + \varepsilon)R_4 ds \\ &\leq R_4 = \|u\|. \end{aligned} \tag{117}$$

So, $\|Tu\| \leq \|u\|$. If we set $\Omega_2 = \{x \in C[0,1] | \|x\| \leq R_4\}$, then

$$\|Tu\| \leq \|u\|, \quad for \quad u \in K \cap \partial\Omega_2. \tag{118}$$

In either of cases, application of part $(ii)$ of Theorem 3.1 yields a fixed point $u^*$ of $T$ belonging to $K \cap (\overline{\Omega}_2/\Omega_1)$, which in turn yields a pair $(u, v)$ satisfying BVPs (2) for the chosen value of $\lambda$. The proof is complete.

We construct an example to illustrate the applicability of the results presented.

Example 4.1. *Consider the two-point boundary value problem*

$$\begin{cases} u^{(4)}(t) = \lambda t v(t)\left(v(t)e^{-v(t)} + \dfrac{v(t)+K}{1+\eta v(t)}\right), & 0 < t < 1, \\ v^{(4)}(t) = \lambda t u(t)\left(u(t)e^{-u(t)} + \dfrac{u(t)+K}{1+\eta u(t)}\right), & 0 < t < 1, \end{cases} \tag{119}$$

and satisfying two-point boundary conditions

$$\begin{cases} u(0) = 0, u'(0) = 0, u''(1) = 0, u'''(1) = 0, \\ v(0) = 0, v'(0) = 0, v''(1) = 0, v'''(1) = 0, \end{cases} \tag{120}$$

where $a(t) = b(t) = t,\ f(v) = v\left(ve^{-v} + \frac{v+K}{1+\eta v}\right),\ g(u) = u\left(1 + \frac{u+K}{1+\eta u}\right)$.

By simple calculations, we find $g(0) = 0, f_\infty = g_\infty = \frac{1}{\eta}, f_0 = g_0 = K$.

Choosing $\theta = \frac{1}{3}, \eta = 100,$ *and* $K = 10^4$, *we obtain* $L_3 \cong 1,1817237, L_4 \cong 9,1666667$.

By Theorem 4, *it follows that for every* $\lambda$ *such that* $1,1817237 < \lambda < 9,1666667$, *there exists a pair* $(u, v)$ *satisfying* BVPs (25–2526).

## 5. Conclusions

This chapter concerns the boundary value problem of a class of fractional differential equations involving the Caputo fractional derivative with nonlocal boundary conditions. By using the Leray-Shauder nonlinear alternative and Banach contraction principle, one shows that the problem has at least one positive solutions and has unique solution. Secondly, we derive explicit eigenvalue intervals of $\lambda$ for the existence of at least one positive solution for the second problem by using Krasnosel'skii fixed point theorem. The results of the present chapter are significantly contribute to the existing literature on the topic.

## Acknowledgements

The authors want to thank the anonymous referee for the thorough reading of the manuscript and several suggestions that help us improve the presentation of the chapter.

## Conflict of interest

The authors declare no conflict of interest.

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