Spring 2023

Ramanujan–Sato series for 1/π

Timothy Huber

Daniel Schultz

Dongxi Ye

ACTA ARITHMETICA
Online First version

Ramanujan–Sato series for $1/\pi$

by

Tim Huber (Edinburg, TX), Daniel Schultz (State College, PA) and Dongxi Ye (Zhuhai)

1. Introduction. In his remarkable paper [19], Ramanujan recorded a total of 17 series for approximating the number $1/\pi$, of which one of the most famous is

$$\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\binom{4n}{2n}\binom{2n}{n}^2\frac{26390n+1103}{396^{4n}}=\frac{1}{\pi}. \tag{1.1}$$

This was used by Gosper in 1985 to compute π up to 17 million digits (see [2, pp. 32, 104, 202, 203, also 229]), then a world record. More incredible is that the formula (1.1) was not proved to be true until two years later, by the Borweins [5], using the theory of elliptic modular functions, through which all of Ramanujan's 17 formulas were proven. In the meantime, Ramanujan's series were studied by the Chudnovskys ([10], see also [4]), who succeeded in extending Ramanujan's list and deriving new series for $1/\pi$ of the same form as Ramanujan's. Interestingly, it was known [5, p. 188] that in both the Borweins' and the Chudnovskys' work, a key ingredient for deriving the relevant series for $1/\pi$ is Clausen's identity for hypergeometric functions. These facts motivated study of series for $1/\pi$, tentatively called Ramanujan-type series, of the form

$$\sum_{n=0}^{\infty}a_n\frac{An+B}{C^n}=\frac{1}{\pi}, \tag{1.2}$$

with Clausen-type transformation formulas as a starting point. However, such a point of view was changed by Sato [20], who discovered a Ramanujan-type series that did not require Clausen's formula in its derivation. This new

2020 *Mathematics Subject Classification*: Primary 11F03; Secondary 11F11.
Key words and phrases: Hauptmodul, modular equations, moonshine groups, pi, singular values.
Received 21 June 2022; revised 12 November 2022.
Published online *.

discovery opened the door to a larger family of series for $1/\pi$ of Ramanujan type not restricted to the Clausen-type transformation framework. Thus, it has now become common practice to give credit to Sato and call a series for $1/\pi$ of the form (1.2) a *Ramanujan–Sato-type* series.

Sato's series motivated Chan, Chan and Liu [6] to derive a general Ramanujan–Sato-type series without using Clausen's identity, and remarkably, they showed that all the existing series for $1/\pi$ are special cases of their general series. Roughly speaking, Chan et al. [6] showed that Ramanujan–Sato-type series can be generally derived from certain complex functions with some prescribed transformation properties which are, in particular, modular functions and modular forms of weight 2. Besides giving a general interpretation to Ramanujan–Sato-type series, the work of Chan et al. also provided a systematic classification of these series according to the level of the modular forms from which they are derived. In the latest terminology, we now call Ramanujan's formula (1.1) a series for $1/\pi$ of level 2.

Based on [6], theories of Ramanujan–Sato-type series corresponding to various levels have been systematically studied and established by many mathematicians. We refer the reader to Cooper's recent book [13] for a nice summary of work regarding levels 1–12, and to [1, 14, 15, 16, 17, 18, 26] for levels ranging from 13 to 35, with some exceptions.

By a careful observation of the previous work, one may note that all of these existing theories are related to some subgroup of $\mathrm{SL}_2(\mathbb{R})$ of genus zero. The connection between their developments and the genus zero property of their associated subgroups has recently been explicitly indicated by the authors of the present work in [18], in which they derive a brand new family of Ramanujan–Sato series corresponding to $\Gamma_0(17)+$, the group obtained from $\Gamma_0(17)$ by adjoining its Fricke involution. By taking a Hauptmodul for $\Gamma_0(17)+$ as a starting point, whose existence is due to the genus zero property of $\Gamma_0(17)+$, they construct a family of series from this Hauptmodul and its values at imaginary quadratic points. The prototype case in [18] motivates the present work, in which the aim is to establish a general theory of Ramanujan–Sato-type series and show how they naturally arise from the general Hauptmodul for the so-called moonshine groups and their modular equations. The study is limited to these groups because the corresponding differential equations for the modular forms in terms of the Hauptmodul have rational coefficients (cf. Theorem 2.1 below).

Much of the theory needed to derive Ramanujan–Sato series at each level is well known and beautifully presented in a number of works. However, no guide exists that incorporates comprehensive theoretical and algorithmic details allowing one to formulate complete classes of series at each level. The following provides such a primer. For each genus zero subgroup and a corresponding Hauptmodul, a uniquely determined modular form of weight 2 will be constructed in Section 2 that satisfies a third order differential equation whose polynomial coefficients are explicitly determined. In Section 3, fundamental properties of the modular equation satisfied by each Hauptmodul are derived. The other important contribution in the present work is a recipe for a complete list of singular values of the Hauptmodul with fixed degree over $\mathbb{Q}$ in a fundamental domain. Restricted singular values are established from modular equations satisfied by the Hauptmodul. Formulas for coefficients of Ramanujan–Sato series of the form (1.2) are given in terms of the singular values. This allows one to formulate all Ramanujan–Sato series in which the coefficients have fixed degree over $\mathbb{Q}$. We conclude the paper with a set of tables collecting the q-expansions of the Hauptmoduln for the groups from [12]; singular values for the Hauptmoduln of select levels; and explicit formulations of the coefficients in the differential equations that give rise to the series expansion defining the Ramanujan–Sato series.

2. Differential equations. In this section, we determine a procedure for constructing a weight 2 modular form z from a Hauptmodul x for moonshine groups. This parameter z satisfies a linear third order differential equation with respect to x that can be described explicitly from the construction. The form of the equation may be anticipated from a general theorem [22, 25].

THEOREM 2.1. *Let Γ be subgroup of $\mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$. If $t(\tau)$ is a non-constant meromorphic modular function and $F(t(\tau))$ is a meromorphic modular form of weight k with respect to Γ, then $F, \tau F, \ldots, \tau^k F$ are linearly independent solutions to a $(k+1)$st order differential linear equation with coefficients that are algebraic functions of t. The coefficients are rational functions when $\Gamma \setminus \mathfrak{H}$ has genus zero and t generates the field of modular functions on Γ. The differential equation takes the form*

$$(2.1)\quad 0 = \frac{W(y, F, \tau F, \ldots, \tau^k F)}{W(F, \tau F, \ldots, \tau^k F)}, \qquad W(f_0, \ldots, f_m) = \det\left(\frac{d^i f_j}{dt^i}\right)_{i,j=0,\ldots,m}.$$

We first recall that for a genus zero congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ commensurable with $\mathrm{SL}_2(\mathbb{Z})$, the function field on $X(\Gamma)$ can be generated by a single modular function $t_\Gamma(\tau)$, and such a function is called a *Hauptmodul* for Γ if it has a unique simple pole of residue 1 at the cusp $i\infty$, i.e., it has Fourier expansion of the form $q^{-1/h} + c(0) + c(1)q^{1/h} + \cdots$ with $q = \exp(2\pi i\tau)$ at the cusp $i\infty$ where h is the width of the cusp $i\infty$. In their *Monstrous Moonshine* paper [12], Conway and Norton proposed an interesting family of congruence subgroups which are defined as follows. For any natural number N and $e \,\|\, N$, i.e., $e \mid N$ and $\gcd(e, N/e) = 1$, consider the set of so-called Atkin–Lehner involutions

$$W_e = \left\{ \begin{pmatrix} ea & b \\ Nc & ed \end{pmatrix} \,\middle|\, \begin{matrix} (a,b,c,d) \in \mathbb{Z}^4 \\ ead - \frac{N}{e}bc = 1 \end{matrix} \right\}.$$

Each W_e is a coset of $\Gamma_0(N)$ with the multiplication rule

$$W_e W_f \equiv W_{ef/\gcd(e,f)^2} \bmod \Gamma_0(N).$$

For any set of indices e closed under this rule, the group $\Gamma = \bigcup_e W_e$ is a subgroup of the normalizer of $\Gamma_0(N)$. Such a group is denoted as $\Gamma_0(N) + W_{e_1}, W_{e_2}, \ldots, W_{e_r}$, or more succinctly as $N+e_1, e_2, \ldots, e_r$. This is shortened to $N+$ when all of the indices are present, and shortened to $N-$ when no indices (except 1) are present. It is known [12] that such a group of genus zero is of particular interest due to its connection with the famous moonshine conjecture, and we now call it a *moonshine group*. In [11], Conway, McKay and Sebbar determined a full list of genus zero $\Gamma_0(N)+W_{e_1}, \ldots, W_{e_r}$, whose Hauptmoduln have been constructed by Conway and Norton [12]. In Section 4, we explicitly list all of these groups and Hauptmoduln as a reference for the reader.

Although polynomial coefficients in the differential equation from Theorem 2.1 may be obtained by clearing denominators, we show next that for each choice for $x(\tau)$ as a Möbius transformation of the Hauptmodul t_Γ, there exists a polynomial choice for $w(x)$ resulting in an a priori polynomial expansion in x for other parameters in the differential equation arising from Theorem 2.1. For consistency, from this point on we assume the Hauptmodul t_Γ to be normalized with a pole at the cusp $[i\infty]$, i.e., $t_\Gamma(\tau) = \frac{1}{q} + O(q)$, which is uniquely determined. We will later establish a class of Möbius transformations that, in some cases, decrease the degree of the polynomials appearing in the differential equation. Here and throughout the paper, let

$$f_x = x\frac{df}{dx}.$$

THEOREM 2.2. *For any choice of $x(\tau)$ as a Möbius transformation of the Hauptmodul t_Γ, there exists a polynomial $w(x)$ and a weight 2 modular form $z = (\log x)_q/\sqrt{w(x)}$ such that*

$$R = \frac{2zz_{qq} - 3z_q^2}{z^4}$$

is a polynomial in x.

Proof. Around every point $\tau_0 \in \mathbb{H}$ where $x(\tau)$ does not have a pole, it has an expansion of the form

$$x(\tau) = a + (\tau-\tau_0)^r(b + c(\tau-\tau_0)^1 + d(\tau-\tau_0)^2 + O(\tau-\tau_0)^3), \tag{2.2}$$

where $r \geq 1$ is an integer and $a \neq 0$. Points τ_0 where $r \geq 2$ will be called *ramification points* and are finite in number modulo $\mathit{\Gamma}$, as these points correspond to the zeros of the form $x'(\tau)$ of weight 2. Assume first that $x(\tau)$ has its only pole at ∞ and that its zero is not at any cusp or ramification point. Around every cusp $\tau = -D/C$ that is not $\mathit{\Gamma}$-equivalent to $i\infty$, $x(\tau)$ has an expansion of the form

$$x(\tau) = a + bQ + cQ^2 + dQ^3 + O(Q^4), \tag{2.3}$$

where $Q = \exp\bigl(2\pi i \frac{A\tau+B}{C\tau+D}\bigr)$.

Set

$$w(x) = \prod_{\substack{\tau_0\in\mathit{\Gamma}\backslash\mathbb{H}\\ r=2}} \left(1-\frac{x}{a}\right) \prod_{\substack{\tau_0\in\mathit{\Gamma}\backslash\mathbb{H}\\ r\geq 3}} \left(1-\frac{x}{a}\right)^2 \prod_{\substack{\tau_0\in\mathit{\Gamma}\backslash\mathbb{Q}\\ x(\tau_0)\neq\infty}} \left(1-\frac{x}{a}\right)^2, \tag{2.4}$$

where the definitions of r and a in each case are as in (2.2) and (2.3), so that $w(0) = 1$, and the finiteness of the first two products is guaranteed by the valence formula for $\mathit{\Gamma}$.

Define the auxilary function

$$\mathrm{P} = \frac{2\zeta\zeta_{qq} - 3\zeta_q^2}{\zeta^4}, \quad \text{where} \quad \zeta = (\log x)_q, \tag{2.5}$$

and note that

$$R = w\mathrm{P} + \frac{3w_x^2}{4w} - w_{xx}. \tag{2.6}$$

As the function P is invariant under $\mathit{\Gamma}$, it is a rational function of x. Hence R is also a rational function of x. By considering each of the following cases separately, we may deduce that R has poles only at ∞, which means that it is in fact a polynomial in x.

- Around the point $\tau_0 \in \mathbb{H}$ where $r = 1$, we have $w = (\alpha + \beta(x-a) + \gamma(x-a)^2 + O(x-a)^3)$ and the series expansion
$$\frac{R}{3} = \frac{a^2\beta^2}{4\alpha} - \frac{4\alpha a^2c^2}{b^4} + \frac{4\alpha a^2 d}{b^3} - \frac{2a^2\gamma}{3} - \frac{a\beta}{3} + \frac{\alpha}{3} + O(\tau-\tau_0).$$
- Around a point $\tau_0 \in \mathbb{H}$ where $r = 2$, we have $w = (x-a)(\alpha + \beta(x-a) + \gamma(x-a)^2 + O(x-a)^3)$ and the series expansion
$$\frac{R}{3} = -\frac{3\alpha a^2c^2}{8b^3} + \frac{\alpha a^2 d}{2b^2} - \frac{a^2\beta}{6} - \frac{\alpha a}{3} + O(\tau-\tau_0).$$
- Around a point $\tau_0 \in \mathbb{H}$ where $r \geq 3$, we have $w = (x-a)^2(\alpha + \beta(x-a) + \gamma(x-a)^2 + O(x-a)^3)$ and the series expansion
$$\frac{R}{3} = \frac{a^2\alpha}{3r^2} + O(\tau-\tau_0).$$

- Around a point $\tau_0 \in \mathbb{Q}$, we have $w = (x-a)^2(\alpha + \beta(x-a) + \gamma(x-a)^2 + O(x-a)^3)$ and the series expansion
$$\frac{R}{3} = \left(-\frac{a^2 b\beta}{3} + \frac{2\alpha a^2 c}{3b} - \frac{2\alpha ab}{3}\right)Q + O(Q^2).$$

Finally, we show that the existence of w is unaffected by changing the function x by a Möbius transformation. Suppose that x is such that there is a polynomial choice for $w(x)$ such that $R(x)$ is a polynomial in x. If x is replaced by any new x' with $x = \frac{\mathcal{A}'x'+\mathcal{B}'}{\mathcal{C}'x'+\mathcal{D}'}$, then the new choice
$$w'(x') = (\mathcal{C}'x' + \mathcal{D}')^{2+\deg(w)} w\left(\frac{\mathcal{A}'x'+\mathcal{B}'}{\mathcal{C}'x'+\mathcal{D}'}\right)$$
is a choice for $w'(x')$ such that the new $R'(x')$ is also a polynomial in x'. Justification for this claim may be given as above. The extra 2 in the exponent can be omitted if $\mathcal{D}' = 0$. ∎

THEOREM 2.3. *If z, w, R are as in the last theorem, then the differential equation for z with respect to x takes the form*
$$2wz_{xxx} + 3w_x z_{xx} + (w_{xx} - 2R)z_x - R_x z = 0, \tag{2.7}$$
with polynomial coefficients.

Proof. From Theorem 2.1, the third order linear equation satisfied by $f = z$ can be written
$$\det\begin{pmatrix} f & f_x & f_{xx} & f_{xxx} \\ (z) & (z)_x & (z)_{xx} & (z)_{xxx} \\ (z\log q) & (z\log q)_x & (z\log q)_{xx} & (z\log q)_{xxx} \\ (z\log^2 q) & (z\log^2 q)_x & (z\log^2 q)_{xx} & (z\log^2 q)_{xxx} \end{pmatrix} = 0.$$
Using the conditions and definitions given,
$$x\frac{\partial}{\partial x} = \frac{1}{z\sqrt{w}} q\frac{\partial}{\partial q}, \qquad x_q = \sqrt{w}\, zx, \qquad z_{qq} = \frac{3z_q^2}{2z} + \frac{Rz^3}{2},$$
the vanishing of the determinant may be written as in the theorem. ∎

EXAMPLE 2.4. We first consider the level 1 case where Γ is the full modular group. Define
$$x(\tau) = \frac{1}{j(\tau)},$$
where the modular j-invariant $j(\tau)$ is given by
$$j(\tau) = 1728\frac{E_4^3}{E_4^3 - E_6^2} = t_{\mathrm{SL}_2(\mathbb{Z})} + 744 = \frac{1}{q} + 744 + 19688q + O(q^2),$$

with

$$E_4 = E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n} \quad \text{and} \quad E_6 = E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n},$$

the normalized Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ of weight 4 and 6, respectively. Now

$$j'(\tau) = -\frac{j(\tau)E_6}{E_4},$$

from which it follows that the only zeros of $j'(\tau)$ occur at $\tau = i$. Since

$$j(i) = 1728, \qquad x(i) = \frac{1}{j(i)} = \frac{1}{1728},$$

we see that

$$x(\tau) = \frac{1}{1728} + O((\tau - i)^2).$$

Therefore (2.4) implies

$$w(x) = 1 - 1728x.$$

Since P is a modular function, it is a rational function of x. From (2.5), we obtain

$$\mathrm{P} = \frac{2x^2x_{qqq}x_q - 3x^2x_{qq}^2 + x_q^4}{x_q^4}, \tag{2.8}$$

Since x_q has only a simple pole at $i\infty$, the valence formula implies that the number of zeros counting multiplicity of x_q is bounded by $\lfloor 1/6 + 1\rfloor$ in x. Thus, as a polynomial in x, the denominator of (2.8) is of degree at most $\lfloor 2/3 + 4\rfloor$. Similarly, one can check that the numerator of (2.8) has the only pole at $i\infty$ of order bounded by 2, so the number of zeros of P is bounded by $\lfloor 1/6 + 2\rfloor$. Therefore, for some coefficients a_i, b_i,

$$\frac{a_0 + a_1x + a_2x^2}{b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4} - \mathrm{P} = 0. \tag{2.9}$$

The vanishing of the coefficients of the q-expansion of the left side of (2.9) implies $a_0 = b_3 = b_4 = 0$, $a_1 \neq 0$ and

(2.10)

$$a_2 = -\frac{1}{31}(6912a_1), \qquad b_0 = -\frac{a_1}{1488}, \qquad b_1 = \frac{72a_1}{31}, \qquad b_2 = -\frac{1}{31}(62208a_1).$$

Hence, from (2.9)–(2.10),

$$\mathrm{P} = \frac{48x(6912x - 31)}{(1728x - 1)^2}, \qquad R = 240x, \tag{2.11}$$

so that the weight 2 modular form

$$z(\tau) = \frac{1}{\sqrt{1 - 1728x}}\,\frac{x'(\tau)}{2\pi i x(\tau)} = 1 + 120q - 6120q^2 + \cdots = \sqrt{E_4}$$

satisfies

$$(1728x-1)z_{xxx}+2592xz_{xx}+1104xz_x+120xz=0.$$

One can verify that a holomorphic local solution at $x=0$ to this equation is given by

$$z={}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};1,1;x\right),$$

which coincides with Ramanujan's result of signature 6 (see, e.g., [13, Chapter 4]).

The previous example demonstrates a general method for computing P as a rational function of x. The recipe is formalized in the next lemma.

LEMMA 2.5. *For a Hauptmodul x of $\Gamma \le \Gamma_0(N)$, we have* $\mathrm{P}=f(x)/g(x)$ *for polynomials f,g with*

$$\deg f(x) \le \left\lfloor \frac{1}{6}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]+2\right\rfloor,$$

$$\deg g(x) \le \left\lfloor \frac{2}{3}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]+4\right\rfloor.$$

Proof. Since P is invariant under Γ, it must be a rational function in x. From (2.8), the poles of P occur only at the zeros of x_q. Since x_q has only a simple pole at $i\infty$, the valence formula implies that the number of zeros counting multiplicity of x_q is bounded by $\frac{1}{6}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]+1$, since x_q may be considered a modular form of weight 2 on some $\Gamma\subset\Gamma_0(N)$. Thus the number of poles of P is bounded by $\frac{2}{3}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]+4$. Similarly, since the numerator of P has the only pole at $i\infty$ of order bounded by 2, the number of zeros of P is bounded by $\frac{1}{6}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]+2$. Therefore, $\mathrm{P}=f(x)/g(x)$ for some polynomials f and g in x of degrees up to the given bounds. From these bounds and from

$$[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]=N\prod_{p\mid N}\left(1+\frac{1}{p}\right),$$

the coefficients of f and g may be determined from the q-expansion of P. ∎

The construction in the last proof implies that knowledge of ramification points of $x(\tau)$ is not needed to compute $w(x)$. As it is only necessary to compute the rational function P in (2.5), the calculations can stay in $\mathbb{Q}$. The polynomial $w(x)$ can then be built up from factors of the denominator of P. The resulting polynomial may not agree with that from Theorem 2.2 defined in terms of ramification points of x. In some cases, a different choice of $w(x)$ can be made so that $R(x)$ is a polynomial in x. The resulting degree of both $w(x)$ and $R(x)$ may be less than the degree of the corresponding polynomials in Theorem 2.2. We illustrate the optimization of the choice of $w(x)$ in the next example. A full list of such choices for $w(x)$ for each level is given in Table 5 (see Section 4).

EXAMPLE 2.6. For the group $\Gamma = 7+$, we have the normalized Hauptmodul

$$t(\tau) = \frac{\eta(\tau)^4}{\eta(7\tau)^4} + 49\frac{\eta(7\tau)^4}{\eta(\tau)^4} + 4 = \frac{1}{q} + 51q + 204q^2 + \cdots.$$

The choice $x(\tau) = \frac{1}{t(\tau)} = q - 51q^3 - 204q^4 + \cdots$ and an application of Lemma 2.5 result in

$$\mathrm{P}(x) = \frac{24x^2(109350x^4 + 34020x^3 + 414x^2 - 527x - 34)}{(9x+1)^2(10x+1)^2(18x-1)^2}.$$

The square root of the denominator of $\mathrm{P}(x)$ suggests that $w(x)$ is at least a cubic in x. However, if we make the Möbius transformation $X = (at+b)/(ct+d)$, then the denominator of $\mathrm{P}(x) = \mathcal{P}(X)$ as a function of X is given by

$$(18a+b-18cX-dX)^2(10a-b-10cX+dX)^2(9a-b-9cX+dX)^2. \tag{2.12}$$

It is impossible to choose c, d such that the expression (2.12) is the square of a linear factor. The degree of the polynomial in (2.12) will be minimized and non-constant when either

$$d = -18c, \quad d = 10c, \quad \text{or} \quad d = 9c. \tag{2.13}$$

Any of these choices will make (2.12) the product of two squared linear factors. Choosing $d = 9c$ leads to

$$(9a-b)^2(18a+b-27cX)^2(10a-b-cX)^2. \tag{2.14}$$

Later we will require that $w(0) = 1$. This means

$$9a - b = \pm 1, \quad 18a + b = \pm 1, \quad 10a - b = \pm 1. \tag{2.15}$$

Only two of these eight sets of equations have a solution, namely $a = 0$ and $b = \pm 1$. Therefore, with $a = 0$ and $b = 1$, (2.12) becomes

$$(1+cX)^2(1-27cX)^2. \tag{2.16}$$

Hence, with $c = 1$, we have

$$x(\tau) = \frac{1}{t(\tau)+9} = q - 9q^2 + 30q^2 + \cdots.$$

This leads via (2.5) to

$$\mathrm{P} = \frac{3x(27x^3 + 188x^2 - 41x - 6)}{(x+1)^2(27x-1)^2}.$$

With $w(x) = (1+x)(1-27x)$, equation (2.6) gives the polynomial $R = 8x(1+3x)$. Finally, Theorem 2.3 gives the differential equation

$$(x+1)(27x-1)z_{xxx} + 3x(27x+13)z_{xx} + 3x(26x+7)z_x + 4x(6x+1)z = 0$$

for the function

$$\begin{aligned} z(\tau) &= \frac{1}{\sqrt{(1+x(\tau))(1-27x(\tau))}} \frac{x'(\tau)}{2\pi i x(\tau)} = 1 + 4q + 12q^2 + \cdots \\ &= \frac{7E_2(7\tau) - E_2(\tau)}{6}, \end{aligned}$$

where $E_2(\tau)$ denotes the Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

The last equality coincides with Cooper's result on the case of level 7 [13, Theorem 7.23].

Since we have established a mechanism for deriving the polynomial $w(x)$ for which the differential equation for z with respect to x has polynomial coefficients, we can attempt to write z as a power series in x and solve for the coefficients A_n in the expansion

$$z = \sum_{n=0}^{\infty} A_n x^n. \tag{2.17}$$

We will require $A_0 = 1$, and the rest of the A_n to be integers when possible. As the q-series coefficients of the normalized Hauptmodul $t(\tau)$ are integers, this will require that $1/x(\tau) = t(\tau) + C$ for some integer C such that $x(\tau) = q + O(q^2)$ with all integer coefficients. This will also require that the polynomial $w(x)$ has integer coefficients and constant term 1. The fact that this last property can be satisfied will be clear in Section 3, where it will be shown that the values of $t(\tau)$ at cusps and ramification points are algebraic integers. The differential equation (2.7) implies a recurrence relation for the series coefficients A_n in (2.17) of order

$$\max\big(\deg(w(x)), \deg(R(x))\big).$$

These sequences have appeared at many places in the literature and many formulas for A_n are known (see [7] and the references therein). When the order of the recurrence relation is 1, the solution for A_n follows immediately in Table 1. When the order of the recurrence relation is higher, the formula may be verified using other means such as the WZ-algorithm [23]. A label from *The Online Encyclopedia of Integer Sequences* (OEIS) is given for each indexed sequence.

Table 1

Γ	$1/x$	$w(x)$	$R(x)$
Formula for A_n			OEIS ref.
1+	$t+744$	$1-1728x$	$240x$
$A_n=\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}$			A001421
2+	$t+104$	$1-256x$	$48x$
$A_n=\binom{2n}{n}^2\binom{4n}{2n}$			A008977
3+	$t+42$	$1-108x$	$24x$
$A_n=\binom{2n}{n}^2\binom{3n}{n}$			A184423
4+	$t+42$	$1-64x$	$16x$
$A_n=\binom{2n}{n}^3$			A002897
5+	$t+16$	$1-44x-16x^2$	$12x(1+x)$
$A_n=\sum_{k=0}^n\binom{2n}{n}\binom{n}{k}^2\binom{k+n}{k}$			A274786
6+	$t+10$	$(1+4x)(1-32x)$	$8x(1+12x)$
$A_n=\sum_{k=0}^n\binom{2n}{n}\binom{n}{k}^3$			A181418
7+	$t+9$	$(1+x)(1-27x)$	$8x(1+3x)$
$A_n=\sum_{k=0}^n\binom{2k}{n}\binom{n}{k}^2\binom{k+n}{k}=\sum_{k=0}^n\binom{n}{k}^2\binom{k+n}{k}\binom{2n-k}{n}$			A183204
8−	t	$(4x-1)^2(4x+1)^2$	$64x^2(1-4x)(1+4x)$
$A_n=\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2$			A036917
9+	$t+6$	$1-18x-27x^2$	$3x(2+9x)$
$A_n=\sum_{k_1=0}^n\sum_{k_2=0}^{k_1}\binom{k_1}{k_2}\binom{n}{k_1}^2\binom{n}{k_2}\binom{k_1+k_2}{n}$			A290576
10+	$t+4$	$(1+4x)(1-16x)$	$4x(1+15x)$
$A_n=\sum_{k=0}^n\binom{n}{k}^4$			A005260
12+	$t+6$	$(4x-1)(16x-1)$	$8x(1-8x)$
$A_n=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}$			A002895

3. Modular equations and series for $1/\pi$. We begin with the set of matrices

$$\Delta_n^*(N)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\,\middle|\,\begin{array}{l}\gcd(\alpha,\beta,\gamma,\delta)=1\\ \alpha\delta-\beta\gamma=n\\ N\mid\gamma,\ \gcd(\alpha,N)=1\end{array}\right\}\subset\mathbb{Z}^{2\times 2}.$$

The following lemma collects some facts proven in [8].

Lemma 3.1. *If* $\gcd(n,N)=1$, *then*

(1) $\Delta_n^*(N)$ *has the decomposition*

$$\Delta_n^*(N)=\bigsqcup_{\substack{\gcd(\alpha,\beta,\delta)=1\\ 0\le\beta<\delta\\ \alpha\delta=n}}\Gamma_0(N)\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix},$$

(2) *and the double coset representation*

$$\Delta_n^*(N)=\Gamma_0(N)\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\Gamma_0(N).$$

The following proposition is associated to a Hauptmodul $x(\tau)$ with the Fourier expansion $q^{-1}+O(1)$ at the cusp $i\infty$. Note that in most instances in the rest of the paper, including the derivation of the differential equation, we take a Hauptmodul with the Fourier expansion $q+O(q^2)$ at the cusp $i\infty$.

Proposition 3.2. *Set* $x(\tau)=t_\Gamma(\tau)$. *For any integer* $n\ge 2$ *satisfying* $\gcd(n,N)=1$, *there is a polynomial* $\Psi_n(X,Y)$ *of degree*

$$\psi(n)=n\prod_{\substack{q\mid n\\ q\ \text{prime}}}\left(1+\frac{1}{q}\right)$$

in X *and* Y *such that:*

(1) $\Psi_n(X,Y)$ *is irreducible and has degree* $\psi(n)$ *in* X *and degree* $\psi(n)$ *in* Y.
(2) $\Psi_n(X,Y)$ *is symmetric in* X *and* Y.
(3) $\Psi_n(X,X)$ *has leading coefficient* $\pm e^{\Lambda(\sqrt{n})}$, *where* Λ *is the Mangoldt function.*
(4) *The roots of* $\Psi_n(x(\tau),Y)=0$ *are precisely the numbers* $Y=x((\alpha\tau+\beta)/\delta)$ *for integers* α, β *and* δ *such that* $\alpha\delta=n$, $0\le\beta<\delta$, *and* $\gcd(\alpha,\beta,\delta)=1$.

Proof. This modular equation, without the computation of the leading coefficient, is proven in [8, Section 2]. The polynomial Ψ_n can be written as

$$\Psi_n(X,Y)=\prod_{\substack{(\alpha,\beta,\delta)=1\\ 0\le\beta<\delta\\ \alpha\delta=n}}\left(Y-x\left(\frac{\alpha\tau+\beta}{\delta}\right)\right), \tag{3.1}$$

where the coefficients of Y^k on the right hand side should be expressed as polynomials in X for $X=x(\tau)$. This can be done since the coefficients of Y^k are symmetric polynomials in $x\left(\frac{\alpha\tau+\beta}{\delta}\right)$, and the product runs over the equivalence class of the set of matrices of discriminant n modulo the action of $\Gamma_0(N)$. Therefore, one can check that these polynomials are actually modular functions for $\Gamma_0(N)$, and thus are polynomials in $x(\tau)$. The coefficient of $X^{\psi(n)}Y^{\psi(n)}$ in $\Psi_n(X,Y)$ is the constant term of the product on the right hand side of (3.1), which is clearly non-zero because the function $x(\tau)$ does not have poles at the cusps of $\mathbb{H}/\Gamma_0(N)$. Therefore, $\Psi_n(X,Y)$ has the claimed degree $\psi(n)$ in X and Y. The symmetry can be proven by noting that $\tau \mapsto -1/(Nn\tau)$ interchanges $x(\tau)$ and $x(n\tau)$.

For the third assertion, we first assume that $n = p^{2t}$ for some prime p and positive integer t. Note that

$$\prod_{\substack{(\alpha,\beta,\delta)=1\\ 0\le\beta<\delta\\ \alpha\delta=n}} \left(Y - x\left(\frac{\alpha\tau+\beta}{\delta}\right)\right)$$
$$= Y^{\psi(n)} + \sum_{k=1}^{\psi(n)} (-1)^k S_k\left(x(n\tau),\ldots,x\left(\frac{\tau+n-1}{n}\right)\right) Y^{\psi(n)-k},$$

where S_k is a homogeneous symmetric polynomial of degree k in the variables

$$F = \left\{x(n\tau),\ldots,x\left(\frac{\tau+n-1}{n}\right)\right\}.$$

We know that S_k is a modular function for $\Gamma_0(N)$ with poles supported at $i\infty$ only, and thus it must be a polynomial in $x(\tau)$, which we will denote by $s_k(X)$ for $X = x(\tau)$. The degree of $s_k(X)$ is determined by the order of vanishing of S_k at $\tau = i\infty$. Since the Fourier expansion of $x(\tau)$ is of the form $q^{-1} + O(1)$, this order, in turn, is determined by terms of S_k with k factors in which the sum of the order of vanishing of the factors at $i\infty$ is maximized under the constraints in the product (3.1). Define

$$E_k(\tau) := \sum_{f\in F} f_1\cdots f_k.$$

From conditions in the product (3.1), non-constant factors of E_k consist of $x(p^{2t}\tau)$ and

$$x\left(\frac{p^{2t-m}\tau+j}{p^m}\right), \quad 1\le m\le 2t,\ j\in(\mathbb{Z}/p^m\mathbb{Z})^\times. \tag{3.2}$$

The order of vanishing at $i\infty$ of the expression in (3.2) is decreasing in m. For $m \ge t$, this order is less than or equal to 1. For $m < t$, the order of vanishing is at least 2. In particular, for $k = p^{t-1}$, the product of factors comprising the unique term of S_k with the greatest order of vanishing is

$$E_{p^{t-1}}(\tau) = x(p^{2t}\tau) \prod_{\substack{1\le m\le t-1\\ j\in(\mathbb{Z}/p^m\mathbb{Z})^\times}} x\left(\frac{p^{2t-m}\tau+j}{p^m}\right),$$

with each factor having order of vanishing at least p^2. The order of vanishing of $E_{p^{t-1}}(\tau)$ at $i\infty$ is

$$p^{2t} + \sum_{m=1}^{t-1} (p^m - p^{m-1}) p^{2t-2m} = p^{2t} + p^{2t-1} - p^t = \psi(n) - p^t.$$

For $k = p^{t-1}+1$, the terms of S_k with maximal order of vanishing $\psi(n)-p^t+1$ are

$$E_{p^{t-1}+1}(\tau) = x(p^{2t}\tau) \sum_{j\in(\mathbb{Z}/p^t\mathbb{Z})^\times} E_{p^{t-1}}(\tau) x\left(\frac{p^{2t-t}\tau + j}{p^t}\right).$$

Similarly, for $k = p^{t-1}+2$, the terms of S_k with maximal order of vanishing $\psi(n) - p^t + 2$ are

$$\begin{aligned} &E_{p^{t-1}+2}(\tau) \\ &\quad = x(p^{2t}\tau) \sum_{j_1<j_2\in(\mathbb{Z}/p^t\mathbb{Z})^\times} E_{p^{t-1}}(\tau) x\left(\frac{p^{2t-t}\tau + j_1}{p^t}\right) x\left(\frac{p^{2t-t}\tau + j_2}{p^t}\right). \end{aligned}$$

As k increases, each additional factor in the inductive formulation has order of vanishing 1. Therefore, for $p^{t-1} \le k \le p^t$, the order of vanishing of $E_k(\tau)$ is $\psi(n) - p^t + k - p^{t-1}$. From the above calculations, and the fact that the order of vanishing of $X = x(\tau)$ at $i\infty$ is 1, we get

$$\deg s_k(X) X^{\psi(n)-k} = 2\psi(n) - p^t - p^{t-1}, \qquad p^{t-1} \le k \le p^t.$$

Since one fewer factor of order at least p^2 appears in the product comprising terms of $E_{p^{t-1}-1}$ than in that for $E_{p^{t-1}}$, the order of vanishing of $E_{p^{t-1}-1}(\tau)$ is at most $\psi(n) - p^t - p^2$. Therefore, the contribution to the degree of $\Psi_n(X, X)$ is

$$\begin{aligned} \deg s_{p^{t-1}-1}(X) X^{\psi(n)-(p^{t-1}-1)} &\le 2\psi(n) - p^t - p^{t-1} + (1 - p^2) \\ &< 2\psi(n) - p^t - p^{t-1}. \end{aligned}$$

Similarly, for $k \le p^{t-1} - 1$, we have $\deg s_k(x) Y^{\psi(n)-k} < 2\psi(n) - p^t - p^{t-1}$. For $k > p^t$, we observe that products contributing to $E_{p^t+1}(\tau)$ have one more factor of order at most p^{-2} than those for $E_{p^t}(\tau)$. Thus,

$$\begin{aligned} \deg s_{p^t+1}(X) X^{\psi(n)-(p^t+1)} &\le 2\psi(n) - p^t - p^{t-1} + (p^{-2} - 1) \\ &\le 2\psi(n) - p^t - p^{t-1}. \end{aligned}$$

By similar reasoning, we see that $\deg s_k(X) X^{\psi(n)-k} < 2\psi(n) - p^t - p^{t-1}$ for all $k > p^t$. This shows that terms of maximal degree for $\Psi_n(X, X)$ come solely from terms of degree $2\psi(n) - p^t - p^{t-1}$ in the polynomial

$$\sum_{k=p^{t-1}}^{p^t} (-1)^k s_k(X) X^{\psi(n)-k}. \tag{3.3}$$

The corresponding terms of $\Psi_n(X,Y)$ of maximal degree are, up to sign,

$$\begin{aligned}
&\sum_{k=p^{t-1}}^{p^t} (-1)^k E_k(\tau) Y^{\psi(n)-k} \\
&\quad = (-1)^{p^{t-1}} Y^{\psi(n)-p^{t-1}} x(p^{2t}\tau) \prod_{\substack{1\le m\le t-1\\ j\in(\mathbb{Z}/p^m\mathbb{Z})^\times}} x\left(\frac{p^{2t-m}\tau+j}{p^m}\right) \\
&\qquad + (-1)^{p^{t-1}+1} Y^{\psi(n)-p^{t-1}-1} x(p^{2t}\tau) \prod_{\substack{1\le m\le t-1\\ j\in(\mathbb{Z}/p^m\mathbb{Z})^\times}} x\left(\frac{p^{2t-m}\tau+j}{p^m}\right) \\
&\qquad\qquad \times \sum_{j\in(\mathbb{Z}/p^t\mathbb{Z})^\times} x\left(\frac{p^{2t-t}\tau+j}{p^t}\right) \\
&\qquad + (-1)^{p^{t-1}+2} Y^{\psi(n)-p^{t-1}-2} x(p^{2t}\tau) \prod_{\substack{1\le m\le t-1\\ j\in(\mathbb{Z}/p^m\mathbb{Z})^\times}} x\left(\frac{p^{2t-m}\tau+j}{p^m}\right) \\
&\qquad\qquad \times \sum_{j_1<j_2\in(\mathbb{Z}/p^t\mathbb{Z})^\times} x\left(\frac{p^{2t-t}\tau+j_1}{p^t}\right) x\left(\frac{p^{2t-t}\tau+j_2}{p^t}\right) \\
&\qquad + \cdots + (-1)^{p^t} Y^{\psi(n)-p^t} x(p^{2t}\tau) \prod_{\substack{1\le m\le t\\ j\in(\mathbb{Z}/p^m\mathbb{Z})^\times}} x\left(\frac{p^{2t-m}\tau+j}{p^m}\right) \\
&\quad = (-1)^{p^{t-1}} Y^{\psi(n)-p^{t-1}} X^{\psi(n)-p^t} \\
&\qquad + (-1)^{p^{t-1}+1} Y^{\psi(n)-p^{t-1}-1} X^{\psi(n)-p^t+1} \sum_{j\in(\mathbb{Z}/p^t\mathbb{Z})^\times} \zeta_{p^t}^j \\
&\qquad + (-1)^{p^{t-1}+2} Y^{\psi(n)-p^{t-1}-2} X^{\psi(n)-p^t+2} \sum_{j_1<j_2\in(\mathbb{Z}/p^t\mathbb{Z})^\times} \zeta_{p^t}^{j_1}\zeta_{p^t}^{j_2} \\
&\qquad + \cdots + (-1)^{p^t} Y^{\psi(n)-p^t} X^{\psi(n)-p^{t-1}}.
\end{aligned}$$

Thus the leading term of $\Psi_n(X,X)$ is

$$(-1)^{p^{t-1}}\Big(1 - \sum_{j\in(\mathbb{Z}/p^t\mathbb{Z})^\times} \zeta_{p^t}^j + \sum_{\substack{j_1<j_2\\ j_1,j_2\in(\mathbb{Z}/p^t\mathbb{Z})^\times}} \zeta_{p^t}^{j_1}\zeta_{p^t}^{j_2} + \cdots + (-1)^{p^t-p^{t-1}}\Big),$$

where ζ_{p^t} is the primitive p^tth root of unity, which is

$$(-1)^{p^{t-1}} \prod_{j\in(\mathbb{Z}/p^t\mathbb{Z})^\times} (1-\zeta_{p^t}^j) = (-1)^{p^{t-1}} p = (-1)^{p^{t-1}} e^{\Lambda(\sqrt{p^{2t}})}.$$

Now for n a composite square with at least two distinct prime factors, a similar argument can be applied, and one can check that the leading coefficient of $\Psi_n(X,X)$ turns out to be, up to sign,

$$\prod_{j\in(\mathbb{Z}/\sqrt{n}\mathbb{Z})^\times} (1-\zeta_{\sqrt{n}}^j) = 1 = e^0 = e^{\Lambda(\sqrt{n})}.$$

For n not a square, the contribution of terms with maximal degree in the expansion of $s_k(x)x^{\psi(n)-k}$ over multiple values of k is due to factors of order 1

$$x\left(\frac{\frac{n}{m}\tau + j}{m}\right),$$

which offset the decrease of the degree of $x^{\psi(n)-k}$ when k increases by 1. However, this will not occur for n not a square since, for $m \mid n$, we have $\frac{n}{m^2} \neq 1$, so the order of vanishing of each factor

$$x\left(\frac{\frac{n}{m}\tau + j}{m}\right)$$

of $E_k(\tau)$ cannot be 1. By reasoning similarly to the base case and grouping factors with contributions at least 1, we observe that the leading coefficient of $\Psi_n(x,x)$ has a coefficient, up to sign,

$$\prod_{d\mid n} \prod_{j\in(\mathbb{Z}/d\mathbb{Z})^\times} \zeta_d^j = 1 = e^0 = e^{\Lambda(\sqrt{n})}.$$

This establishes the claimed properties of the modular equation $\Psi_n(X,Y)$. ■

PROPOSITION 3.3. *Let* $X(\tau) = t_{11+}$, *where* t_{11+} *is as given in Table* 3 (*see Section* 4). *Then*

$$\begin{aligned}
\Phi_2(X,Y) &= 44 - 56X + 20X^2 - X^3 - 56Y + 53XY \\
&\quad - 12X^2Y + 20Y^2 - 12XY^2 + X^2Y^2 - Y^3, \\
\Phi_3(X,Y) &= 144X^2 + 24X^3 + X^4 \\
&\quad - 112XY - 48X^2Y - 57X^3Y + 144Y^2 - 48XY^2 \\
&\quad + 24X^2Y^2 + 18X^3Y^2 + 24Y^3 - 57XY^3 + 18X^2Y^3 - X^3Y^3 + Y^4, \\
\Phi_4(X,X) &= 2(-1+X)^2(11 - X - X^2)^2(242 - 176X + 54X^2 - X^3).
\end{aligned}$$

From these modular equations we can deduce that $x(\tau)$ is an algebraic integer when τ is quadratic irrational. A complete list of the τ where $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$ in the case of $\Gamma = 17+$ and $\Gamma = 11+$ is given in Tables 6 and 7, respectively.

Proposition 3.4. *There is a polynomial $\Upsilon(X,J)$ of the form*

$$A_0(X)J^t + A_1(X)J^{t-1} + \cdots + A_t(X),$$

where $t = [\Gamma : \Gamma_0(N)]$ for some Γ of level N with Hauptmodul $x(\tau)$, such that $\Upsilon(x(\tau), j(\tau)) = 0$.

Now suppose that $\tau \in \mathbb{H}$ satisfies $a\tau^2 + b\tau + c = 0$ with $a > 0$ and $\gcd(a,b,c) = 1$, and $\mathbb{Q}(\tau)$ has no non-trivial roots of unity. If $M \in \Delta_n^*(N)$ and $x(M\tau) = x(\tau)$, then $w_e M\tau = \tau$ for some $w_e = \left(\begin{smallmatrix} \alpha e & \beta \\ \gamma N & \delta e \end{smallmatrix}\right) \in W$. Since the non-trivial matrices fixing τ have the form $\left(\begin{smallmatrix} x & -cy \\ ay & x+by \end{smallmatrix}\right)$ for $x, y \in \mathbb{R}$ and $y \neq 0$, we see, after replacing x by ex, that

$$\begin{pmatrix} e\alpha & \beta \\ N\gamma & e\delta \end{pmatrix} M = \begin{pmatrix} ex & -cy \\ ay & ex + by \end{pmatrix}.$$

From this equality it follows that $ex, y \in \mathbb{Z}$ since $\gcd(a,b,c) = 1$. Multiplying this equality on the left by $\left(\begin{smallmatrix} 0 & -1 \\ e & 0 \end{smallmatrix}\right)^{-1}$ gives

$$\begin{pmatrix} \frac{N\gamma}{e} & \delta \\ -e\alpha & -\beta \end{pmatrix} M = \begin{pmatrix} \frac{a}{e}y & x + \frac{b}{e}y \\ -ex & cy \end{pmatrix}. \tag{3.4}$$

Since the first matrix on the left hand side is in $\Gamma(1)$ and M is primitive, the matrix on the right must also be primitive. It follows that $x, y \in \mathbb{Z}$, $\gcd(ex, y) = 1$, and $e \mid \gcd(a,b)$. Set

$$M_e(x,y) = \begin{pmatrix} \frac{a}{e}y & x + \frac{b}{e}y \\ -ex & cy \end{pmatrix}. \tag{3.5}$$

The determinant of this matrix is the quadratic form

$$|M_e|(x,y) = ex^2 + bxy + \frac{ac}{e}y^2, \tag{3.6}$$

which equals n by (3.4). Now set $Y(\tau) = x(\left(\begin{smallmatrix} 0 & -1 \\ e & 0 \end{smallmatrix}\right) M_e \tau)$ and $X = x(\tau)$. Notice that

$$\frac{1 + (\partial_X Y)^{1/r}}{1 - (\partial_X Y)^{1/r}} = \frac{b + 2ex/y}{\sqrt{b^2 - 4ac}}, \tag{3.7}$$

where r is the ramification index of the function x at τ. This implies that the slopes of the tangents to $\Phi_n(X,Y) = 0$ at $(X(\tau), X(\tau))$ are never 0 or 1 and are distinct for different choices of $M_e(x,y)$ and fixed $n = |M_e|(x,y)$. Therefore the modular equation $\Psi_n(X,Y) = 0$ has only ordinary singularities along the diagonal $X = Y$. Hence knowledge of Ψ_n and the value of the tangent in (3.7) is sufficient to expand Y as a power series in X in the neighborhood of the singular value.

Theorem 3.5 (Series for $1/\pi$). *Suppose $\tau_0 \in \mathbb{H}$ satisfies $a\tau_0^2 + b\tau_0 + c = 0$ with $a > 0$ and $\gcd(a,b,c) = 1$. Choose a matrix $M_e(x,y)$ with $w_e \in \Gamma$*

so that $w_e^{-1}\left(\begin{smallmatrix}0 & -1\\ e & 0\end{smallmatrix}\right)M_e(x,y) \in \varDelta_n^*(N)$ *with* $n = |M_e|(x,y)$. *Let* X_0 *be the appropriate root of* $\varPsi_n(X,X) = 0$, *and determine the series expansion of* $Y(\tau) = X(M\tau)$,

$$Y = X_0 + \partial_X Y(X_0)(X - X_0) + \tfrac{1}{2}\partial_X^2 Y(X_0)(X - X_0)^2 + \cdots,$$

from the modular equation $\varPsi_n(X,Y) = 0$ *and the value of* $\partial_X Y(X_0)$ *given in* (3.7). *If* Z *denotes the function* $z(\tau)$ *as a function of* X *and* W *denotes* $\sqrt{w(\tau)}$, *then*

$$\frac{1}{2\pi\,\mathrm{Im}(\tau_0)} = W\left(X\partial_X Z + \left(1 + \frac{X\partial_X W}{W} + \frac{X\partial_X^2 Y}{\partial_X Y(1-\partial_X Y)}\right)Z\right)\bigg|_{X=X_0}. \tag{3.8}$$

Proof. Let τ_0 be fixed by the matrix $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. There is an expansion

$$x(Mt) = X_0 + \partial_X Y(X_0)(x(t) - X_0) + \tfrac{1}{2}\partial_X Y(X_0)(x(t) - X_0)^2 + \cdots. \tag{3.9}$$

Applying $\frac{1}{2\pi i}\frac{d}{d\tau}$ to (3.9) gives

$$\begin{aligned}&\frac{\alpha\delta - \beta\gamma}{(\gamma\tau+\delta)^2} w(M\tau)x(M\tau)z(M\tau)\\ &\qquad = (\partial_X Y(X_0) + \partial_X^2 Y(X_0)(x(\tau) - X_0) + \cdots)w(\tau)x(\tau)z(\tau).\end{aligned} \tag{3.10}$$

Setting $\tau = \tau_0$ gives

$$\partial_X Y(X_0) = \frac{\alpha\delta - \beta\gamma}{(\gamma\tau_0 + \delta)^2}, \tag{3.11}$$

which is in agreement with (3.7). Applying now $\frac{1}{2\pi i}\frac{d}{d\tau}$ to (3.10) gives

$$\begin{aligned}&\frac{i\gamma}{\pi}\frac{\alpha\delta - \beta\gamma}{(\gamma\tau+\delta)^3} w(M\tau)x(M\tau)z(M\tau)\\ &\quad + \frac{(\alpha\delta-\beta\gamma)^2}{(\gamma t+\delta)^4}\{\partial_x w(M\tau)x(M\tau)z(M\tau)\\ &\quad + w(M\tau)z(M\tau) + w(M\tau)x(M\tau)\partial_x z(M\tau)\}w(M\tau)x(M\tau)z(M\tau)\\ &= (\partial_X^2 Y(X_0) + \cdots)w(\tau)^2x(\tau)^2z(\tau)^2\\ &\quad + (\partial_X Y(X_0) + \partial_X^2 Y(X_0)(x(\tau) - X_0) + \cdots)\{\partial_x w(\tau)x(\tau)z(\tau)\\ &\quad + w(\tau)z(\tau) + w(\tau)x(\tau)\partial_x z(\tau)\}w(\tau)x(\tau)z(\tau).\end{aligned} \tag{3.12}$$

Setting $\tau = \tau_0$ and suppressing τ_0 from the arguments, we get

(3.13)

$$\begin{aligned}&\frac{i\gamma}{\pi}\frac{\alpha\delta - \beta\gamma}{(\gamma\tau_0+\delta)^3}WXZ + \frac{(\alpha\delta-\beta\gamma)^2}{(\gamma\tau_0+\delta)^4}\{XZ\partial_X W + WZ + WX\partial_X Z\}WXZ\\ &\qquad = W^2X^2Z^2\partial_X^2 Y + \partial_X Y\{XZ\partial_X W + WZ + WX\partial_X Z\}WXZ.\end{aligned}$$

Transposing the second term on the left and dividing by $\partial_X Y(1-\partial_X Y)W^2XZ$ while keeping in mind (3.11) gives

$$\frac{i\gamma}{\pi W(1-\partial_X Y)(\gamma\tau_0+\delta)} = X\partial_X Z + \left(1 + \frac{X\partial_X W}{W} + \frac{X\partial_X^2 Y}{\partial_X Y(1-\partial_X Y)}\right)Z.$$

Simplifying the left hand side while keeping in mind that the minimal polynomial of τ_0 is $a\tau_0^2 + b\tau_0 + c = 0$ gives the desired series. ∎

As $Z = \sum_{n=0}^{\infty} A_n X^n$, the series for $\frac{1}{\pi}$ is of the form

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} A_n(Bn+C)X_0^n. \tag{3.14}$$

The most tedious part of evaluating such a series for π is the calculation of the coefficient of Z, which is the number C in (3.14). This requires the second order term from the modular equation. Since an application of Theorem 3.5 implies a computation of $\Phi_n(X,Y)$, which can be very large in practice, it is necessary to give a more effective procedure.

Example 3.6. It should be remarked at this point that in their series

$$\frac{426880\sqrt{10005}}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}\binom{3n}{n}\binom{6n}{3n} \frac{13591409 + 545140134n}{(-640320)^{3n}},$$

D. V. Chudnovsky and G. V. Chudnovsky do not specify how the number 13591409 was obtained. In similar series, J. M. Borwein and P. B. Borwein indicated that this term was calculated using known approximations of π. By rearranging Theorem 3.5 slightly, we see that what we are in fact calculating is the value of

$$\frac{1}{z(\tau)}\left(\frac{1}{2\pi i\,\mathrm{Im}(\tau)\sqrt{w(\tau)}} - x(\tau)z_x(\tau)\right) \tag{3.15}$$

for imaginary quadratic points τ.

Ramanujan's series converge rapidly and have been used to calculate record numbers of digits of π (cf. [3]). The extensions to formulas derived by the Chudnovskys [10] are still the basis of most record-breaking calculations. Less is known about the efficiency of subsequently derived series. A comprehensive study of the rate of convergence of Ramanujan–Sato series should be undertaken. In the next remark, we provide evidence that Ramanujan's original series expansions converge more rapidly than series corresponding to groups from [12] of higher level.

Remark 3.7. For a Hauptmodul $t(\tau)$ of level n, it is known [15] that the Galois conjugates over $\mathbb{Q}$ of $t(\tau_0)$, where $\tau_0 = \frac{d+\sqrt{d}}{2}$, are $t(\tau_Q)$ as Q ranges over $\mathcal{Q}_d(n)/\Gamma_0(n)$, where d is a negative fundamental discriminant, $\mathcal{Q}_d(n)$ denotes the set of positive definite quadratic forms $aX^2 + bXY + cY^2$

of discriminant d with $(a,n)=1$, and τ_Q is the imaginary quadratic point in $\mathbb{H}$ induced by the quadratic form Q. Then for two Hauptmoduln $t_1(\tau)$ and $t_2(\tau)$ of level n_1 and n_2, respectively, with $n_1 \mid n_2$ and simple poles at $i\infty$, the rational norm of $t_1(\tau_0)$ is roughly larger than that of $t_2(\tau_0)$. This is apparent since, by the assumption $n_1 \mid n_2$, the algebraic integrality of $t_2(\tau_0)$, and the fact that $t(\tau_0) \gg 1$, we have

$$t_2(\tau_0) \approx t_1(\tau_0)^{1/k}$$

for some

$$k \geq [\Gamma_0(n_1) : \Gamma_0(n_2)] = \left(\frac{n_2}{n_1}\right)^2 \prod_{\substack{p \mid n_2 \\ p \nmid n_1}} (1+p^{-1}).$$

By the surjection

$$\mathcal{Q}_d(n_2)/\Gamma_0(n_2) \twoheadrightarrow \mathcal{Q}_d(n_1)/\Gamma_0(n_1),$$

one can see that roughly

$$\mathrm{N}_{\mathbb{Q}}(t_2(\tau_0)) \approx \mathrm{N}_{\mathbb{Q}}(t_1(\tau_0))^{r/k}$$

where

$$r = [\mathcal{Q}_d(n_2)/\Gamma_0(n_2) : \mathcal{Q}_d(n_1)/\Gamma_0(n_1)] = \frac{n_2}{n_1} \prod_{\substack{p \mid n_2 \\ p \nmid n_1}} (1-\chi_d(p)p^{-1}) < k.$$

Therefore, in the order of divisibility, a Ramanujan–Sato series of prime level has a smaller dominant exponential decay term than a series of composite level. Since Ramanujan's series correspond to small levels, these tend to converge more rapidly than most of the recently discovered series.

Theorem 3.5 shows that (3.15) is algebraic, and gives a rigorous procedure for evaluating it. The procedure implied by Theorem 3.8 is much more effective, as it essentially bounds the denominator of (3.15).

THEOREM 3.8. *Set $t(\tau) = t_\Gamma(\tau)$ and suppose that $t(\tau_0) = t(M\tau_0)$ for some $M \in \Delta_n^*(N)$. Suppose further that either $t(\tau)$ does not ramify at $\tau = \tau_0$ or $\mathbb{Q}(\tau_0)$ contains no roots of unity. Let $\Psi_n(T,S)$ be the modular equation from Proposition* 3.2.

(1) *$T_0 = t(\tau_0)$ is a root of the monic polynomial $\Psi(T,T) \in \mathbb{Z}[T]$.*
(2) *Set $S = s(\tau) = t(M\tau)$, let k denote the multiplicity of T_0 as a root of $\Psi(T,T)$, and let $\eta_1, \ldots, \eta_k$ be the set of tangents $\partial_T S$ to the modular curve $\Psi(T,S) = 0$ at $(T,S) = (T_0,T_0)$ as found, for example, in* (3.7)*. Then*

$$k\left(\frac{\partial_T^k \Psi(T,T)}{k!}\bigg|_{T=T_0}\right)^3 \left(\prod_{i=1}^{k} \frac{\eta_i}{1-\eta_i}\right)^2 \left(\frac{\partial_T^2 S}{\partial_T S(1-\partial_T S)}\bigg|_{T=T_0}\right) \in \mathbb{Z}[T_0].$$

(3) *If* $x(\tau) = \frac{\alpha t(\tau)+\beta}{\gamma t(\tau)+\delta}$ *is another choice of the Hauptmodul and* $y(\tau) = x(M\tau)$*, then*

$$\frac{\partial_X^2 Y}{\partial_X Y(1-\partial_X Y)}\bigg|_{X=X_0} = \frac{\gamma(\delta+\gamma T_0)}{\alpha\delta-\beta\gamma} - \frac{(\delta+\gamma T_0)^2}{\alpha\delta-\beta\gamma}\frac{\partial_T^2 S}{\partial_T S(1-\partial_T S)}\bigg|_{T=T_0}.$$

Proof. Item (1) is a restatement of Proposition 3.2, and (3) is a straightforward calculation. For (2), expand the modular equation in the form

$$\Psi(T_0+u, T_0+\eta u) = f(\eta)u^k + g(\eta)u^{k+1} + O(u^{k+2}),$$

where $f(\eta), g(\eta) \in \mathbb{Z}[T_0][\eta]$. By the symmetry of Ψ, we know that f and g are self-reciprocal polynomials and $\deg(f) \le k$ and $\deg(g) \le k+1$. In fact, f has degree exactly k and its k distinct roots are the tangents to the modular curve at (T_0, T_0), none of which is 0 or 1. By setting $\eta = \eta_i$ to be one of the actual tangents, which fixes the branch on which S lies, we arrive at the equality

$$\xi := \frac{\partial_T^2 S}{\partial_T S(1-\partial_T S)} = \frac{2g(\eta_i)}{\eta_i(1-\eta_i)f'(\eta_i)}. \tag{3.16}$$

However, the presence of ξ in, for example, the main formula (3.8) of Theorem 3.5 implies that its value is independent of the particular choice of branch of the function S. Therefore, we have the *identity*

$$\begin{aligned}(3.17)\quad & \xi f(0)f(1)\eta(1-\eta)f'(\eta) \\ &= 2f(0)f(1)g(\eta) + 2g(0)f(1)(1-\eta)f(\eta) + 2g(1)f(0)\eta f(\eta)\end{aligned}$$

of polynomials in $\mathbb{Z}[T_0][\eta]$. Since the leading coefficient of $f(\eta)$ is also $f(0)$, extracting the coefficient of η^k in (3.17) shows that

$$-k\xi f(0)f(1)f(0) \in \mathbb{Z}[T_0]. \tag{3.18}$$

The proof of (2) is complete once the following formula for $f(\eta)$ is substituted into (3.18):

$$f(\eta) = \left(\frac{\partial_T^k \Psi(T,T)}{k!}\bigg|_{T=T_0}\right)\prod_{i=1}^k \frac{\eta-\eta_i}{1-\eta_i}. \ \blacksquare$$

Example 3.9. Let $\Gamma = 7+$, $n = 11$, and $\tau = \tau(7,7,11)$. Table 2 lists all roots of $\Psi_{11}(T,T) = 0$ and their multiplicities. In the table, $M = T^4 - 2564T^3 - 184242T^2 - 2945940T - 24113575$.

Accordingly,

$$\begin{aligned}\Psi_{11}(T,T) &= (T-116)^2(T-18)^2(T+1)^2(T+10)^2(T+17)^2 \\ &\quad\times (T^2+10T+225)^2(T^2+1378T+10593)^2 \\ &\quad\times (T^4-2564T^3-184242T^2-2945940T-24113575),\end{aligned}$$

Table 2

$\tau(a,b,c)$	Branch(es) and tangent value(s)	Minimal polynomial for T_0
$(14,-14,9)$	$s=t\left(\frac{\tau+5}{11}\right)$, $\partial_T S=-1$	M
$(21,-14,6)$	$s=t\left(\frac{\tau+7}{11}\right)$, $\partial_T S=-1$	M
$(14,-14,9)$	$s=t\left(\frac{\tau+5}{11}\right)$, $\partial_T S=-1$	M
$(7,0,11)$	$s=t\left(\frac{\tau+0}{11}\right)$, $\partial_T S=-1$	M
$(7,7,11)$	$\begin{pmatrix} s=t\left(\frac{\tau+0}{11}\right),\ \partial_T S=-\frac{15}{22}+\frac{\sqrt{-259}}{22} \\ s=t\left(\frac{\tau+1}{11}\right),\ \partial_T S=-\frac{15}{22}-\frac{\sqrt{-259}}{22} \end{pmatrix}$	$T^2+1378T+10593$
$(35,-21,5)$	$\begin{pmatrix} s=t\left(\frac{\tau+4}{11}\right),\ \partial_T S=-\frac{15}{22}+\frac{\sqrt{-259}}{22} \\ s=t\left(\frac{\tau+2}{11}\right),\ \partial_T S=-\frac{15}{22}-\frac{\sqrt{-259}}{22} \end{pmatrix}$	$T^2+1378T+10593$
$(7,-14,11)$	$\begin{pmatrix} s=t\left(\frac{\tau+0}{11}\right),\ \partial_T S=\frac{3}{11}-\frac{4\sqrt{-7}}{11} \\ s=t\left(\frac{\tau+9}{11}\right),\ \partial_T S=\frac{3}{11}+\frac{4\sqrt{-7}}{11} \end{pmatrix}$	$T-116$
$(7,-4,2)$	$\begin{pmatrix} s=t\left(\frac{\tau+3}{11}\right),\ \partial_T S=-\frac{9}{11}-\frac{2\sqrt{-10}}{11} \\ s=t\left(\frac{\tau+9}{11}\right),\ \partial_T S=-\frac{9}{11}+\frac{2\sqrt{-10}}{11} \end{pmatrix}$	$T^2+10T+225$
$(7,-10,5)$	$\begin{pmatrix} s=t\left(\frac{\tau+6}{11}\right),\ \partial_T S=-\frac{9}{11}-\frac{2\sqrt{-10}}{11} \\ s=t\left(\frac{\tau+7}{11}\right),\ \partial_T S=-\frac{9}{11}+\frac{2\sqrt{-10}}{11} \end{pmatrix}$	$T^2+10T+225$
$(7,-7,3)$	$\begin{pmatrix} s=t\left(\frac{\tau+4}{11}\right),\ \partial_T S=-\frac{13}{22}+\frac{3\sqrt{-35}}{22} \\ s=t\left(\frac{\tau+6}{11}\right),\ \partial_T S=-\frac{13}{22}-\frac{3\sqrt{-35}}{22} \end{pmatrix}$	$T+17$
$(7,-14,8)$	$\begin{pmatrix} s=t\left(\frac{\tau+4}{11}\right),\ \partial_T S=-\frac{103}{121}+\frac{24\sqrt{-7}}{121} \\ s=t\left(\frac{\tau+5}{11}\right),\ \partial_T S=-\frac{103}{121}-\frac{24\sqrt{-7}}{121} \end{pmatrix}$	$T-18$
$(7,-3,1)$	$\begin{pmatrix} s=t\left(\frac{\tau+1}{11}\right),\ \partial_T S=\frac{3}{22}+\frac{5\sqrt{-19}}{22} \\ s=t\left(\frac{\tau+8}{11}\right),\ \partial_T S=\frac{3}{22}-\frac{5\sqrt{-19}}{22} \end{pmatrix}$	$T+1$
$(7,-7,2)$	$\begin{pmatrix} s=t\left(\frac{\tau+2}{11}\right),\ \partial_T S=-\frac{103}{121}+\frac{24\sqrt{-7}}{121} \\ s=t\left(\frac{\tau+8}{11}\right),\ \partial_T S=-\frac{103}{121}-\frac{24\sqrt{-7}}{121} \end{pmatrix}$	$T+10$

and

$$\begin{aligned} f(1)(T) &:= \frac{\partial_T^2 \Psi(T,T)}{2!} \bmod (T^2+1378T+10593) \\ &= -2^{35}3^{18}7^{14}261257(531055149531498561216467T \\ &\qquad + 410537313688533697958885\,1). \end{aligned}$$

Therefore, when $\tau_0=(7,7,11)$ or $(35,-21,5)$, the following equation should hold for some integers a_0 and a_1:

$$2(f(1)(T_0))^3\left(\frac{11}{37}\right)^2\left(\frac{\partial_T^2 S}{\partial_T S(1-\partial_T S)}\bigg|_{T=T_0}\right)=a_0+a_1T_0.$$

Evaluating the third term on the left hand side numerically for $\tau_0 = (7, 7, 11)$ and $(35, -21, 5)$ gives integer solutions for a_0 and a_1 of size about 180 decimal digits, which is the minimum precision needed when applying Theorem 3.8 to this example.

THEOREM 3.10. *For each Hauptmodul x of $\Gamma_0(p)+$, the following algorithm results in a complete list of algebraic $(\tau, x(\tau))$ with $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$:*

Let $P(x, j)$ be the polynomial relation between $x(\tau)$ and $j(\tau)$. For each discriminant $-1555 \leq d \leq -1$:

(1) *List all primitive reduced $\tau = \tau(a, b, c)$ of discriminant d in a fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$. Translate these values via a set of coset representatives for $\Gamma_0(p)+$ to a fundamental domain for $\Gamma_0(p)+$.*
(2) *Factor the resultant of $P(X, Y)$ and the class polynomial*

$$H_d(Y) = \prod_{\substack{(a,b,c) \text{ reduced, primitive} \\ d=b^2-4ac}} \left(Y - j\left(\frac{-b+\sqrt{d}}{2a}\right)\right).$$

The linear and quadratic factors of the resultant correspond to a complete list of $x = x(\tau)$, for τ of discriminant d, such that $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$. Associate candidate values τ from Step (1) *to x by numerically approximating $x(\tau)$. For each tentative pair, (τ, x), prove $x = x(\tau)$ by deriving a modular equation for which $x(\tau)$ is a root.*

Proof. We use well-known facts about the j-invariant [21]. For algebraic τ, the only algebraic values of $j(\tau)$ occur at $\mathrm{Im}(\tau) > 0$ satisfying $a\tau^2 + b\tau + c = 0$ for $a, b, c \in \mathbb{Z}$, with $d = b^2 - 4ac < 0$ not necessarily a fundamental discriminant. Moreover, $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(d)$, where $h(d)$ is the class number. Since there is a polynomial relation $P(x, j)$ between x and j of degree $W = [\Gamma : \Gamma_0(N)]$ (cf. [8, Remark 1.5.3]), we have

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \leq [\mathbb{Q}(j(\tau), x(\tau)) : \mathbb{Q}] \leq W[\mathbb{Q}(x(\tau)) : \mathbb{Q}],$$

and so values τ with $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$ satisfy $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(d) \leq 2W$, which is a necessary condition for $x(\tau)$ being quadratic. Therefore, an appropriate bound (from [24]) on $|d|$ for which $h(d) \leq 2W$ allows us to apply the above steps to find a complete list of algebraic $(\tau, x(\tau))$ satisfying $[\mathbb{Q}(x(\tau)) : \mathbb{Q}] \leq 2$. ∎

REMARK 3.11. Suppose that τ_0 is an imaginary quadratic point arising from a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with $(a, N) = 1$ and $d = b^2 - 4ac$, a fundamental discriminant. For a Hauptmodul t_Γ of level N, it is known [9] that $t_\Gamma(\tau_0)$ generates the ring class field of conductor N over $\mathbb{Q}(\sqrt{d})$. Then one has

$$[\mathbb{Q}(t_\Gamma(\tau_0) : \mathbb{Q}] = h(N^2 d) = \frac{h(d)N}{[\mathcal{O}_k^\times : \mathcal{O}_k(N)^\times]} \prod_{p|N} \left(1 - \left(\frac{d}{p}\right)\frac{1}{p}\right),$$

where $\mathcal{O}_k$ is the ring of integers of $k = \mathbb{Q}(\sqrt{d})$ and $\mathcal{O}_k(N)$ is an order of conductor N of k, i.e., a full rank $\mathbb{Z}$-submodule of index N of $\mathcal{O}_k$.

Remark 3.12. The differential equations resulting from Table 3 correspond to results in the literature or improve upon those results by decreasing the degree of the coefficients.

(1) For the cases $\Gamma = 14+, 15+$, $\Gamma = 20+$ and $\Gamma = 21+, 22+, 33+, 35+$, the results given in Table 3 agree with those given in [16], [18] and [1]. The explicit differential equations for the cases of $\Gamma = 22+$ and $33+$ are omitted in the original references, so we present them for the reader's convenience. The differential equation associated with $\Gamma = 22+$ is given by

$$x^2(1-8x)(4x^3-4x^2+1)\frac{d^3z}{dx^3} - 3x(96x^4-90x^3+8x^2+12x-1)\frac{d^2z}{dx^2} - (600x^4-448x^3+28x^2+28x-1)\frac{dz}{dx} - 2(120x^3-66x^2+2x+1)z = 0,$$

and the differential equation associated with $\Gamma = 33+$ is given by

$$\begin{aligned} x^2(1-2x-11x^2)(4x^3+8x^2+4x+1)\frac{d^3z}{dx^3} \\ - 3x(154x^5+288x^4+140x^3+22x^2-3x-1)\frac{d^2z}{dx^2} \\ - (1188x^5+1844x^4+712x^3+81x^2-8x-1)\frac{dz}{dx} \\ - (660x^4+808x^3+228x^2+15x-1)z = 0. \end{aligned}$$

(2) We improve the result for the case of $\Gamma = 13+$ given in [15]. The authors of [15] obtained a third order differential equation of degree 7. Our choice of Möbius transformation of t_{13+} results in a third order differential equation of degree 5 as follows:

$$8x^2(1+x)(1-10x-27x^2)\frac{d^3z}{dx^3} - 12x(135x^3+148x^2+27x-2)\frac{d^2z}{dx^2} - 2(3x+2)(437x^2+63x-2)\frac{dz}{dx} - (693x^2+350x+12)z = 0.$$

The differential equation for the case $\Gamma = 17+$ from Table 3 has lower degree than the one given in our previous work [17], which is of degree 9. The differential equation in the present work is of degree 6 and reads

$$\begin{aligned} x^2(1-6x-27x^2-28x^3-16x^4)\frac{d^3z}{dx^3} \\ - 3x(3x+1)(16x^3+18x^2+12x-1)\frac{d^2z}{dx^2} \\ - (300x^4+348x^3+197x^2+20x-1)\frac{dz}{dx} - (120x^3+102x^2+35x+1)z = 0. \end{aligned}$$

(3) The respective results in [13, Table 14.1, levels 8 and 9], [26] and [13, pp. 420–421] do not correspond to the cases $\Gamma = 8-, 9+$, $\Gamma = 16+$ and $\Gamma = 18+$ given here. Rather, the results from [13, 26] correspond to other subgroups of $\mathrm{SL}_2(\mathbb{R})$ in which $\Gamma = 8-$ is of index 4, and $\Gamma = 9+$, $\Gamma = 16+$ and $\Gamma = 18+$ are of index 2. This explains why the formulas for A_n's for $\Gamma = 8-, 9+$ given in Table 1 are different from those given in [13, Table 14.1, levels 8 and 9]. For reference, we give relations between the x's used in [26, 13] (called X in both references), and the Hauptmoduln t_{8-}, t_{9+}, t_{16+} and t_{18+} defined in Section 4. The values of the parameters x used in [13, Table 14.1, levels 8 and 9] are respectively

$$x = \frac{t_{8-}(t_{8-}-4)(t_{8-}+4)}{(t_{8-}^2 - 8t_{8-} - 16)^2} \quad \text{and} \quad x = \frac{t_{9+}+6}{t_{9+}^2 - 6t_{9+} - 99},$$

the value used in [26] is

$$x = \frac{t_{16+}+4}{t_{16+}^2 - 4t_{16+} - 28},$$

and the corresponding value used in [13, pp. 420–421] is

$$x = \frac{t_{18+}+2}{(t_{18+}+5)^2}.$$

4. Tables. The canonical Hauptmodul for each group from [12] is listed in Table 3. Each Hauptmodul is written in normalized form except for those given implicitly. The table uses the following notation for theta and eta functions:

$$(4.1)\qquad \begin{aligned} \eta_a &= q^{a/24}\prod_{n=1}^{\infty}(1-q^{an}), & \theta_{a,b,c} &= \sum_{x,y=-\infty}^{\infty} q^{(ax^2+bxy+cy^2)/2}, \\ \theta\mathrm{x}_{a,b,c} &= \sum_{\substack{x,y=-\infty \\ x \text{ odd}}}^{\infty} q^{(ax^2+bxy+cy^2)/2}, & \theta\mathrm{y}_{a,b,c} &= \sum_{\substack{x,y=-\infty \\ y \text{ odd}}}^{\infty} q^{(ax^2+bxy+cy^2)/2}. \end{aligned}$$

Moreover, we write t_{N+} for $\Gamma = N+$ and $S(d,N) = t_{N+}(\tau) + t_{N+}(d\tau)$. Those not given by formulas from [12] are determined from the q-expansions of the Hauptmodul.

Table 3

Γ	t_Γ	Γ	t_Γ
2−	$\frac{\eta_1^{24}}{\eta_2^{24}} + 24$	4−	$\frac{\eta_1^{8}}{\eta_4^{8}} + 8$
2+	$\frac{\eta_1^{24}}{\eta_2^{24}} + 2^{12}\frac{\eta_2^{24}}{\eta_1^{24}} + 24$	4+	$\frac{\eta_2^{48}}{\eta_1^{24}\eta_4^{24}} - 24$
3−	$\frac{\eta_1^{12}}{\eta_3^{12}} + 12$	5−	$\frac{\eta_1^{6}}{\eta_5^{6}} + 6$
3+	$\frac{\eta_1^{12}}{\eta_3^{12}} + 3^6\frac{\eta_3^{12}}{\eta_1^{12}} + 12$	5+	$\frac{\eta_1^{6}}{\eta_5^{6}} + 5^3\frac{\eta_5^{6}}{\eta_1^{6}} + 6$

Table 3 [cont.]

Γ	t_Γ	Γ	t_Γ
6+6	$\frac{\eta_2^{12}\eta_3^{12}}{\eta_1^{12}\eta_6^{12}} - 12$	15+	$\frac{\eta_1^2\eta_5^2}{\eta_3^2\eta_{15}^2} + 9\frac{\eta_3^2\eta_{15}^2}{\eta_1^2\eta_5^2} + 2$
6+3	$\frac{\eta_1^6\eta_3^6}{\eta_2^6\eta_6^6} + 6$	16−	$\frac{\eta_8\eta_1^2}{\eta_2\eta_{16}^2} + 2$
6+2	$\frac{\eta_1^4\eta_2^4}{\eta_3^4\eta_6^4} + 4$	16+	$\frac{\eta_2^6\eta_8^6}{\eta_1^4\eta_4^4\eta_{16}^4} - 4$
6+	$\frac{\eta_2^{12}\eta_3^{12}}{\eta_1^{12}\eta_6^{12}} + \frac{\eta_1^{12}\eta_6^{12}}{\eta_2^{12}\eta_3^{12}} - 12$	17+	$\frac{(\theta x_{1/2,0,17/2} - \theta y_{1/2,0,17/2})^2}{4\eta_1^2\eta_{17}^2} - 2$
7−	$\frac{\eta_1^4}{\eta_7^4} + 4$	18−	$\frac{\eta_6\eta_9^3}{\eta_3\eta_{18}^3}$
7+	$\frac{\eta_1^4}{\eta_7^4} + 7^2\frac{\eta_7^4}{\eta_1^4} + 4$	18+2	$\frac{\eta_1\eta_2}{\eta_9\eta_{18}} + 1$
8−	$\frac{\eta_1^4\eta_4^2}{\eta_2^2\eta_8^4} + 4$	18+9	$\frac{\eta_1^3\eta_6^2\eta_9^3}{\eta_2^3\eta_3^2\eta_{18}^3} + 3$
8+	$\frac{\eta_2^8\eta_4^8}{\eta_1^8\eta_8^8} - 8$	18+18	$\frac{\eta_2^3\eta_9^3}{\eta_1^3\eta_{18}^3} - 3$
9−	$\frac{\eta_1^3}{\eta_9^3} + 3$	18+	$\frac{\eta_3^4\eta_6^4}{\eta_1^2\eta_2^2\eta_9^2\eta_{18}^2} - 2$
9+	$\frac{\eta_3^{12}}{\eta_1^6\eta_9^6} - 6$	19+	$\frac{4\theta_{2,2,10}^2}{(\theta_{1,2,20} - \theta_{4,2,5})^2} - 4$
10−	$\frac{\eta_2\eta_5^5}{\eta_1\eta_{10}^5} - 1$	20+20	$\frac{\eta_4^2\eta_5^2}{\eta_1^2\eta_{20}^2} - 2$
10+2	$\frac{\eta_1^2\eta_2^2}{\eta_5^2\eta_{10}^2} + 2$	20+4	$\frac{\eta_1^2\eta_4^2\eta_{10}^2}{\eta_2^2\eta_5^2\eta_{20}^2} + 2$
10+5	$\frac{\eta_1^4\eta_5^4}{\eta_2^4\eta_{10}^4} + 4$	20+	$\frac{\eta_2^8\eta_{10}^8}{\eta_1^4\eta_4^4\eta_5^4\eta_{20}^4} - 2$
10+10	$\frac{\eta_2^6\eta_5^6}{\eta_1^6\eta_{10}^6} - 6$	21+21	$\frac{\eta_3^2\eta_7^2}{\eta_1^2\eta_{21}^2} - 2$
10+	$\frac{\eta_1^2\eta_2^2}{\eta_5^2\eta_{10}^2} + 25\frac{\eta_5^2\eta_{10}^2}{\eta_1^2\eta_2^2} + 2$	21+3	$\frac{\eta_1\eta_3}{\eta_7\eta_{21}} + 1$
11+	$\frac{\theta_{2,2,6}^2}{\eta_1^2\eta_{11}^2} - 6$	21+	$\frac{\eta_1\eta_3}{\eta_7\eta_{21}} + \frac{7\eta_7\eta_{21}}{\eta_1\eta_3} + 1$
12−	$\frac{\eta_3^3\eta_4}{\eta_1\eta_{12}^3} - 1$	22+11	$\frac{\eta_1^2\eta_{11}^2}{\eta_2^2\eta_{22}^2} + 2$
12+12	$\frac{\eta_3^4\eta_4^4}{\eta_1^4\eta_{12}^4} - 4$	22+	$\frac{\eta_1^2\eta_{11}^2}{\eta_2^2\eta_{22}^2} + \frac{4\eta_2^2\eta_{22}^2}{\eta_1^2\eta_{11}^2} + 2$
12+4	$\frac{\eta_1^4\eta_4^4\eta_6^4}{\eta_2^4\eta_3^4\eta_{12}^4} + 4$	23+	$\frac{\theta_{2,2,12}}{\eta_1\eta_{23}} - 3$
12+3	$\frac{\eta_1^2\eta_3^2}{\eta_4^2\eta_{12}^2} + 2$	24+8	$\frac{\eta_1^2\eta_6\eta_8^2\eta_{12}}{\eta_2\eta_3^2\eta_4\eta_{24}^2} + 2$
12+	$\frac{\eta_2^{12}\eta_6^{12}}{\eta_1^6\eta_3^6\eta_4^6\eta_{12}^6} - 6$	24+24	$\frac{\eta_2\eta_3^2\eta_8^2\eta_{12}}{\eta_1^2\eta_4\eta_6\eta_{24}^2} - 2$
13−	$\frac{\eta_1^2}{\eta_{13}^2} + 2$	24+	$\frac{\eta_2^2\eta_4^2\eta_6^2\eta_{12}^2}{\eta_1^2\eta_3^2\eta_8^2\eta_{24}^2} - 2$
13+	$\frac{\eta_1^2}{\eta_{13}^2} + 13\frac{\eta_{13}^2}{\eta_1^2} + 2$	25−	$\frac{\eta_1}{\eta_{25}} + 1$
14+7	$\frac{\eta_1^3\eta_7^3}{\eta_2^3\eta_{14}^3} + 3$	25+	$\frac{\eta_1}{\eta_{25}} + 5\frac{\eta_{25}}{\eta_1} + 1$
14+14	$\frac{\eta_2^4\eta_7^4}{\eta_1^4\eta_{14}^4} - 4$	26+	$\frac{\eta_2^2\eta_{13}^2}{\eta_1^2\eta_{26}^2} + \frac{\eta_1^2\eta_{26}^2}{\eta_2^2\eta_{13}^2} - 2$
14+	$\frac{\eta_1^3\eta_7^3}{\eta_2^3\eta_{14}^3} + \frac{8\eta_2^3\eta_{14}^3}{\eta_1^3\eta_7^3} + 3$	26+26	$\frac{\eta_2^2\eta_{13}^2}{\eta_1^2\eta_{26}^2} - 2$
15+5	$\frac{\eta_1^2\eta_5^2}{\eta_3^2\eta_{15}^2} + 2$	27+	$t_{27+}^2 + 3t_{27+} + 3 = \frac{\eta_3^3\eta_9^3}{\eta_1^3\eta_{27}^2}$
15+15	$\frac{\eta_3^3\eta_5^3}{\eta_1^3\eta_{15}^3} - 3$	28+7	$\frac{\eta_1\eta_7}{\eta_4\eta_{28}} + 1$

Table 3 [cont.]

Γ	t_Γ
28+	$\frac{\eta_2^6\eta_{14}^6}{\eta_1^3\eta_4^3\eta_7^3\eta_{28}^3} - 3$
29+	$\frac{\theta x_{1/2,0,29/2} - \theta y_{1/2,0,29/2}}{2\eta_1\eta_{29}} - 1$
30+15	$\frac{\eta_3\eta_5}{\eta_2\eta_{30}}$
30+6,10,15	$\frac{\eta_1^3\eta_6^3\eta_{10}^3\eta_{15}^3}{\eta_2^3\eta_3^3\eta_5^3\eta_{30}^3} + 3$
30+3,5,15	$\frac{\eta_1\eta_3\eta_5\eta_{15}}{\eta_2\eta_6\eta_{10}\eta_{30}} + 1$
30+2,15,30	$\frac{\eta_3\eta_5\eta_6\eta_{10}}{\eta_1\eta_2\eta_{15}\eta_{30}} - 1$
30+5,6,30	$\frac{\eta_2^2\eta_3^2\eta_{10}^2\eta_{15}^2}{\eta_1^2\eta_5^2\eta_6^2\eta_{30}^2} - 2$
30+	$\frac{\eta_3\eta_5\eta_6\eta_{10}}{\eta_1\eta_2\eta_{15}\eta_{30}} + \frac{\eta_1\eta_2\eta_{15}\eta_{30}}{\eta_3\eta_5\eta_6\eta_{10}} - 1$
31+	$\frac{(\theta_{2,2,16} - \theta_{4,2,8})^3}{8\eta_1^3\eta_{31}^3}$
32+	$\frac{\eta_2^3\eta_{16}^3}{\eta_1^2\eta_4\eta_8\eta_{32}^2} - 2$
33+11	$\frac{\eta_1\eta_{11}}{\eta_3\eta_{33}} + 1$
33+	$\frac{\eta_1\eta_{11}}{\eta_3\eta_{33}} + 3\frac{\eta_3\eta_{33}}{\eta_1\eta_{11}} + 1$
34+	$\left(\frac{\theta x_{1/2,1,9} - \theta y_{9/2,1,1}}{2\eta_1\eta_{17}}\right)^2$
35+35	$\frac{\eta_5\eta_7}{\eta_1\eta_{35}} - 1$
36+4	$\frac{\eta_1\eta_4\eta_{18}}{\eta_2\eta_9\eta_{36}} + 1$
36+36	$\frac{\eta_4\eta_9}{\eta_1\eta_{36}} - 1$
36+	$\frac{\eta_1\eta_4\eta_6^{16}\eta_9\eta_{36}}{\eta_2^4\eta_3^6\eta_{12}^6\eta_{18}^4} + 1$
38+	$\frac{1}{2}\left(\sqrt{1 - 4(-t_{19+}(\tau) - t_{19+}(2\tau) - 4)} - 1\right)$
39+39	$\frac{\eta_3\eta_{13}}{\eta_1\eta_{39}} - 1$
39+	$\frac{\eta_3\eta_{13}}{\eta_1\eta_{39}} + \frac{\eta_1\eta_{39}}{\eta_3\eta_{13}} - 1$
41+	$\frac{\theta x_{3/2,2,15/2} - \theta y_{3/2,2,15/2}}{2\eta_1\eta_{41}} - 1$
42+6,14,21	$\frac{\eta_1^2\eta_6^2\eta_{14}^2\eta_{21}^2}{\eta_2^2\eta_3^2\eta_7^2\eta_{42}^2} + 2$
42+3,14,42	$\frac{\eta_2\eta_6\eta_7\eta_{21}}{\eta_1\eta_3\eta_{14}\eta_{42}} - 1$
42+	$\frac{\eta_2\eta_6\eta_7\eta_{21}}{\eta_1\eta_3\eta_{14}\eta_{42}} + \frac{\eta_1\eta_3\eta_{14}\eta_{42}}{\eta_2\eta_6\eta_7\eta_{21}} - 1$
44+	$\frac{\eta_2^4\eta_{22}^4}{\eta_1^2\eta_4^2\eta_{11}^2\eta_{44}^2} - 2$
45+	$\frac{\eta_3^2\eta_{15}^2}{\eta_1\eta_5\eta_9\eta_{45}} - 1$
46+23	$\frac{\eta_1\eta_{23}}{\eta_2\eta_{46}} + 1$
46+	$\frac{\eta_1\eta_{23}}{\eta_2\eta_{46}} + \frac{2\eta_2\eta_{46}}{\eta_1\eta_{23}} + 1$
47+	$\frac{\theta_{2,2,24} - \theta_{4,2,12}}{2\eta_1\eta_{47}}$

Table 3 [cont.]

Γ	t_Γ
49+	$t_{49+}^3+2t_{49+}^2-t_{49+}-1=\frac{\eta_7^4}{\eta_1^2\eta_{49}^2}$
50+50	$\frac{\eta_2\eta_{25}}{\eta_1\eta_{50}}-1$
50+	$t_{50+}^2+t_{50+}-4=S(2,25)$
51+	$S(2,19)=t^2+t-4$
54+	$t^2+t+1=\frac{\eta_3\eta_6\eta_9\eta_{18}}{\eta_1\eta_2\eta_{27}\eta_{54}}$
55+	$S(5,11)=t^5-10t^3-5t^2+16t$
56+	$\frac{\eta_2\eta_4\eta_{14}\eta_{28}}{\eta_1\eta_7\eta_8\eta_{56}}-1$
59+	$\frac{2\theta_{6,2,10}}{\theta_{2,2,30}-\theta_{6,2,10}}$
60+12,15,20	$\frac{\eta_1\eta_{12}\eta_{15}\eta_{20}}{\eta_3\eta_4\eta_5\eta_{60}}+1$
60+4,15,60	$\frac{\eta_2\eta_3\eta_5\eta_{12}\eta_{20}\eta_{30}}{\eta_1\eta_4\eta_6\eta_{10}\eta_{60}}-1$
60+	$\frac{\eta_2^2\eta_6^2\eta_{10}^2\eta_{30}^2}{\eta_1\eta_3\eta_4\eta_5\eta_{12}\eta_{15}\eta_{20}\eta_{60}}-1$
62+	$t_{62+}^2+t_{62+}-2=S(2,31)$
66+6,11,66	$\frac{\eta_2\eta_3\eta_{22}\eta_{33}}{\eta_1\eta_6\eta_{11}\eta_{66}}-1$
66+	$t_{66+}^2+t_{66}+-4=S(2,33)$
69+	$t_{69+}^3-2t_{69+}-3=S(3,23)$
70+10,14,35	$\frac{\eta_1\eta_{10}\eta_{14}\eta_{35}}{\eta_2\eta_5\eta_7\eta_{70}}+1$
71+	$\frac{\theta_{4,2,18}-\theta_{6,2,12}}{2\eta_1\eta_{71}}$
78+6,26,39	$\frac{\eta_1\eta_6\eta_{26}\eta_{39}}{\eta_2\eta_3\eta_{13}\eta_{78}}+1$
87+	$t_{87+}^3+t_{87+}-3=S(3,29)$
92+	$\frac{\eta_2^2\eta_{46}^2}{\eta_1\eta_4\eta_{23}\eta_{92}}-1$
94+	$t_{94+}^2+t_{94+}-2=S(2,47)$
95+	$t_{95+}^5-5t_{95+}^3+t_{95+}-5=S(5,19)$
105+	$t_{105+}^3-2t_{105+}-3=S(3,35)$
110+	$\frac{1}{2}\left(\sqrt{1-4(-t_{55+}(\tau)-t_{55+}(2\tau))}-1\right)$
119+	$t_{119+}^7-7t_{119+}^3-7t_{119+}^2-6t_{119+}-7=S(7,17)$

Table 4. Transformations $x=1/(t_\Gamma+c)$ of the normalized Hauptmoduln that decrease the degree D of the coefficients in the corresponding differential equation compared to $x=1/t_\Gamma$

Γ	c	D	Γ	c	D	Γ	c	D
1+	744	1	2−	−24	1	2+	104	1
3−	−12	2	3+	42	1	4−	8	2
4+	24	1	5−	−6	2	5+	16	2
6−	3	4	6+6	12	2	6+3	−6	2
6+2	−4	2	6+	10	2	7−	−4	4
7+	9	2	8−	4	4	8+	8	2

Table 4 [cont.]

Γ	c	D	Γ	c	D	Γ	c	D
9−	−3	4	9+	6	2	10−	1	6
10+2	−2	4	10+5	−4	3	10+10	6	3
10+	4	2	11+	6	3	12−	0	8
12+12	4	4	12+4	5	4	12+3	2	4
12+	6	2	13−	−2	6	13+	3	3
14+7	−3	4	14+14	4	4	14+	9	3
15+5	−2	4	15+15	3	4	15+	4	3
16−	0	8	16+	4	4	17+	2	4
18−	0	12	18+2	−1	6	18+9	1	6
18+18	3	6	18+	1	4	19+	3	4
20+	4	3	20+4	−2	6	20+20	2	6
21+21	2	6	21+3	−1	6	21+	0	3
22+	2	4	22+11	−2	6	23+	2	6
24+8	−1	8	24+24	2	8	24+	2	4
25+	1	6	25−	−1	10	26+26	2	7
26+	0	4	27+	3	7	28+7	−1	8
29+	2	6	30+15	−1	12	30+6,10,15	−3	6
30+3,5,15	−1	6	30+2,15,30	1	6	30+5,6,30	2	6
30+	−1	4	31+	0	6	32+	2	8
33+11	−1	8	33+	0	5	34+	2	5
35+35	1	8	35+	2	5	36+4	0	12
36+36	1	12	36+	2	6	38+	0	6
39+	3	5	39+39	1	10	41+	0	8
42+6,14,21	−2	8	42+3,14,42	1	8	42+	−1	5
44+	2	6	45+	1	6	46+	0	8
46+23	−1	12	47+	0	10	49+	2	10
50+50	−1	8	50+	−1	8	51+	2	7
54+	1	10	55+	1	7	56+	1	8
59+	0	12	60+12,15,20	−1	12	60+4,15,60	1	12
60+	1	6	62+	0	10	66+6,11,66	1	12
66+	0	7	69+	0	10	70+10,14,35	−1	12
71+	0	14	78+6,26,39	−1	14	87+	0	12
92+	1	12	94+	0	14	95+	−1	11
105+	−1	9	110+	−1	10	119+	0	14

With the Hauptmodul $x(\tau)$ as given in Table 4, we list the parameters $w(x)$ and $R(x)$ defining the differential equation (2.7) from Theorem 2.3.

Table 5. Parameters $w(x)$ and $R(x)$ defining the differential equation (2.7) for the Hauptmodul $x(\tau)$

Γ	$w(x)$	$R(x)$
1+	$1-1728x$	$240x$
2−	$1+64x$	$-16x$
2+	$1-256x$	$48x$
3−	$(1+27x)^2$	$-3x(10+243x)$,
3+	$1-108x$	$24x$
4−	$(1-16x)^2$	$16x(1-16x)$
4+	$1-64x$	$16x$
5−	$1+22x+125x^2$	$-5x(2+25x)$
5+	$1-44x-16x^2$	$12x(1+x)$
6−	$(1+x)^2(1-8x)^2$	$-8x(1+x)(1+4x)(-1+8x)$
6+6	$1-34x+x^2$	$x(10-x)$
6+3	$(1+4x)(1+16x)$	$-8x(1+8x)$
6+2	$1+14x+81x^2$	$-3x(2+27x)$
6+	$(1+4x)(1-32x)$	$8x(1+12x)$
7−	$(1+13x+49x^2)^2$	$-x(18+433x+3430x^2+9604x^3)$
7+	$(1+x)(1-27x)$	$8x(1+3x)$
8−	$(1-4x)^2(1-8x)^2$	$16x(1-4x)(1-8x)^2$
8+	$1-24x+16x^2$	$8x(1-2x)$
9−	$(1+9x+27x^2)^2$	$-12x(1+9x)(1+9x+27x^2)$
9+	$1-18x-27x^2$	$3x(2+9x)$
10−	$(1+x)^2(1-4x)^2(1+4x^2)$	$4x(1+x)(1-4x)$ $\times(1+3x+13x^2+36x^3)$
10+2	$(1+6x+25x^2)^2$	$-4x(2+38x+200x^2+625x^3)$
10+5	$(1+4x)(1+12x+16x^2)$	$-8x(1+10x+18x^2)$
10+10	$(1-x)(1-18x+x^2)$	$x(28-85x+9x^2)/4$
10+	$(1+4x)(1-16x)$	$4x(1+15x)$
11+	$1-20x+56x^2-44x^3$	$8x(1-8x+11x^2)$
12−	$(1-x)^2(1+x)^2$ $\times(-1+3x)^2(1+3x)^2$	$48(1-x)x^2(1+x)(1-3x)$ $\times(1+3x)(1-3x^2)$
12+12	$(1+x)^2(1-14x+x^2)$	$4x(1+x)(1+8x-x^2)$
12+4	$(1-8x)^2(1-9x)^2$	$24x(1-8x)(1-9x)(1-12x)$
12+3	$(1-4x)^2(1-4x+16x^2)$	$8x(1-4x)(1-8x+32x^2)$
12+	$(1-4x)(1-16x)$	$8x(1-8x)$
13−	$(1+5x+13x^2)^2$ $\times(1+6x+13x^2)$	$-12x-236x^2-1858x^3$ $-7904x^4-18252x^5-19773x^6$
13+	$(1+x)(1-10x-27x^2)$	$x(12+175x+231x^2)/4$
14+7	$(1+x)(1+8x)(1+5x+8x^2)$	$-8x(1+4x)(1+7x+8x^2)$
14+14	$1-14x+19x^2-14x^3+x^4$	$x(6-25x+34x^2-4x^3)$

Table 5 [cont.]

Γ	$w(x)$	$R(x)$
14+	$(1-4x)(1-18x+49x^2)$	$x(10-141x+392x^2)$
15+5	$1+10x+47x^2+90x^3+81x^4$	$-3x(2+23x+78x^2+108x^3)$
15+15	$(-1-x+x^2)(-1+11x+x^2)$	$4x(1+4x-6x^2-x^3)$
15+	$(1-12x)(1-2x+5x^2)$	$3x(2-11x+40x^2)$
16−	$(1-2x)^2(1+2x)^2(1+4x^2)^2$	$-256x^4(-1+2x)$
16+	$(1-2x)^2(1-12x+4x^2)$	$8x(1-2x)(1-8x+4x^2)$
17+	$1-6x-27x^2-28x^3-16x^4$	$x(2+35x+68x^2+60x^3)$
18−	$(1+x)^2(-1+2x)^2$ $\times(1-x+x^2)^2(1+2x+4x^2)^2$	$72x^3(1+x)(1-2x)$ $\times(1-x+x^2)$ $\times(1+2x+4x^2)(1+4x^3)$
18+2	$(1+2x+9x^2)(1+3x+9x^2)^2$	$-3x(1+3x+9x^2)$ $\times(2+27x+81x^2+243x^3)$
18+9	$(1-3x)^2(1-4x)^2(-1+12x^2)$	$12x(1-3x)(1-4x)$ $\times(1-3x-33x^2+108x^3)$
18+18	$(1-10x+x^2)(1-x+x^2)^2$	$3x(1-x+x^2)$ $\times(2-11x+23x^2-3x^3)$
18+	$(1+x)^2(1+4x)(1-8x)$	$24x^2(1+x)(2+5x)$
19+	$(1+x)(1-13x+35x^2-27x^3)$	$x(6-31x-24x^2+105x^3)$
20+	$(1-4x)(1-12x+16x^2)$	$8x(1-10x+18x^2)$
20+4	$(1+x)^2(1+5x)^2(1+2x+5x^2)$	$-5x(1+x)(1+5x)$ $\times(2+13x+40x^2+45x^3)$
20+20	$(1+x)^2$ $\times(1-8x-2x^2-8x^3+x^4)$	$x(1+x)(2+25x+31x^2$ $+47x^3-9x^4)$
21+21	$(1-x)^2$ $\times(1-6x-17x^2-6x^3+x^4)$	$4x+4x^2-70x^3$ $+16x^4+52x^5-9x^6$
21+3	$(1+x+7x^2)^2(1+5x+7x^2)$	$-(x/4)(20+257x+1453x^2$ $+4487x^3+9212x^4+12348x^5)$
21+	$(1+4x)(1-2x-27x^2)$	$-x(2-47x-240x^2)$
22+	$(1-8x)(1-4x^2+4x^3)$	$4x(1-3x)(1+4x-10x^2)$
22+11	$(1+4x+8x^2+4x^3)$ $\times(1+8x+16x^2+16x^3)$	$-8x(1+12x+57x^2$ $+132x^3+160x^4+72x^5)$
23+	$(1-x^2+x^3)$ $\times(1-8x+3x^2-7x^3)$	$4x(1-x-x^2$ $+12x^3-15x^4+14x^5)$
24+8	$(1-x)^2(1+2x)^2$ $\times(1+4x+8x^2-8x^3+4x^4)$	$4x(1-x)(1+2x)(-1-3x$ $+13x^2+46x^3-60x^4+32x^5)$
24+24	$(1-x)^2(1+x)^2$ $\times(1-8x+2x^2+8x^3+x^4)$	$-4(-1+x)x(1+x)(1+x$ $-22x^2+3x^3+25x^4+4x^5)$
24+	$(1+4x^2)(1-8x+4x^2)$	$-4x(-1+3x-20x^2+16x^3)$
25+	$(1+x-x^2)^2(1-4x-16x^2)$	$20x^2(1+x-x^2)(2+2x-7x^2)$

Table 5 [cont.]

Γ	$w(x)$	$R(x)$
25−	$(1+2x+5x^2)$ $\times(1+5x+15x^2+25x^3+25x^4)^2$	$-5x(1+5x+15x^2+25x^3+25x^4)$ $\times(2+25x+125x^2$ $+375x^3+625x^4+625x^5)$
26+26	$(1-x)(1-8x+8x^2-18x^3$	$(x/4)(20-109x+339x^2-521x^3$
26+	$(1+4x)(1-2x-15x^2-16x^3)$	$x(1+2x)(-2+35x+126x^2)$
27+	$(1-3x+3x^2)^2$ $(1-12x+36x^2-36x^3)$	$12x(-1+3x)(1-3x+3x^2)$ $\times(-1+12x-36x^2+36x^3)$
28+7	$(1+2x)^2(1+x+2x^2)$ $\times(1+3x+4x^2)(1+2x+8x^2)$	$-8x(1+2x)(1+4x+8x^2)$ $\times(1+7x+22x^2+32x^3+32x^4)$
29+	$1-10x+23x^2-10x^3$ $-15x^4+20x^5-16x^6$	$6-37x+26x^2+68x^3$ $-132x^4+140x^5$
30+15	$(1-x^2)^2(1+4x-x^2)$ $\times(1+x+2x^2-x^3+x^4)$ $\times(1+x-x^2)$	$4x(1-x^2)(1+2x-9x^2-31x^3$ $-31x^4+43x^5+9x^6$ $-53x^7+46x^8-9x^9$
30+6,10,15	$(1+3x+x^2)(1+6x+x^2)$ $\times(1+7x+x^2)$	$-x(10+123x+420x^2$ $+375x^3+106x^4+9x^5)$
30+3,5,15	$(1+x)(1+4x)$ $\times(1+4x^2)(1+x+4x^2)$	$-4x(1+8x+40x^2$ $+83x^3+160x^4+144x^5)$
30+2,15,30	$(1-6x+x^2)$ $\times(1-x+x^2)(1+3x+x^2)$	$-x(-2-17x+28x^2$ $-53x^3-26x^4+9x^5)$
30+5,6,30	$(1-7x+x^2)$ $\times(1-3x+x^2)(1+x+x^2)$	$-(x/4)(-20+77x+25x^2$ $+245x^3-236x^4+36x^5)$
30+	$(1+x)(1-16x^2)(1+5x)$	$4x(-1+4x+58x^2+75x^3)$
31+	$(1+4x+3x^2+x^3)$ $\times(1-17x^2-27x^3)$	$4x(-1+5x+69x^2$ $+180x^3+161x^4+60x^5)$
32+	$(1-2x+2x^2)^2$ $(1-8x+12x^2-16x^3+4x^4)$	$-8x(1-2x+2x^2)(-1+10x$ $-36x^2+60x^3-60x^4+16x^5)$
33+11	$(1+x+3x^2)$ $\times(1+7x+28x^2+59x^3$ $+84x^4+63x^5+27x^6)$	$-3x(2+25x+126x^2$ $+407x^3+810x^4$ $+1137x^5+918x^6+432x^7)$
33+	$(1-2x-11x^2)$ $\times(1+4x+8x^2+4x^3)$	$x(-2+15x+152x^2$ $+404x^3+264x^4)$
34+	$(1-x)(1-x-4x^2)$ $\times(1-9x+16x^2)$	$-(x/4)(-28+197x$ $+15x^2-1484x^3+1584x^4)$
35+35	$(1+x-x^2)$ $\times(1-5x-9x^3-5x^5-x^6)$	$-x(-2-9x-14x^2-47x^3$ $+30x^4-57x^5+50x^6+16x^7)$
35+	$(1-2x+5x^2)$ $\times(1-8x+16x^2-28x^3)$	$x(6-61x+296x^2-580x^3+840x^4)$
36+4	$(-1+x)^2(1+2x)^2$ $\times(1+x+x^2)^2(1-2x+4x^2)^2$	$72(-1+x)x^3(1+2x)(1+x+x^2)$ $\times(1-2x+4x^2)(1-4x^3)$

Table 5 [cont.]

Γ	$w(x)$	$R(x)$
36+36	$(1-x^2)^2(1+x+x^2)^2$ $(1-4x-6x^2-4x^3+x^4)$	$12(1-x^2)x^2(1+x+x^2)$ $\times(2+5x-x^2-17x^3$ $-20x^4-8x^5+3x^6)$
36+	$(1+x)^2(-1+3x)^2$ $\times(1-6x-3x^2)$	$3x(1+x)(1-3x)$ $\times(2-3x-48x^2-27x^3)$
38+	$(1+4x+4x^2+4x^3)$ $\times(1-2x-7x^2-8x^3)$	$x(-2+15x+116x^2$ $+316x^3+392x^4+280x^5)$
39+	$(-1+4x)(-1+3x+x^2)$ $\times(1-11x+27x^2)$	$4x(3-46x+210x^2)$ $-253x^3-168x^4$
39+39	$(1+x)^2$ $\times(1-7x+11x^2-7x^3+x^4)$ $\times(1+x-x^2+x^3+x^4)$	$x(2+17x-48x^2-25x^3$ $+194x^4-45x^5-168x^6$ $+137x^7+82x^8-25x^9)$
41+	$1+4x-8x^2-66x^3$ $-120x^4-56x^5+53x^6$	$x(1+2x)(-4+20x$ $+174x^2+240x^3$
42+6,14,21	$(1+x+x^2)(1+5x+x^2)$ $\times(1+7x+16x^2+7x^3+x^4)$	$-(x/4)(36+437x+1915x^2$ $+3852x^3+4443x^4$ $+2517x^5+660x^6+64x^7)$
42+3,14,42	$(1-5x+x^2)(1-x+x^2)$ $\times(1+x+4x^2+x^3+x^4)$	$-(x/4)(-12+41x-287x^2$ $+360x^3-639x^4$ $+201x^5-252x^6+64x^7)$
42+	$(1+x)(1-3x)(1+4x)$ $\times(1+5x+8x^2)$	$-(x/4)(20+49x$ $-567x^2-2508x^3-2304x^4)$
44+	$(-1+4x-8x^2+4x^3)$ $\times(-1+8x-16x^2+16x^3)$	$-8x(-1+12x-57x^2$ $+132x^3-160x^4+72x^5)$
45+	$(1+x-x^2)(1-3x+3x^2)$ $\times(1-3x-9x^2)$	$-(3x/4)(-4-13x+123x^2$ $-87x^3-396x^4+324x^5)$
46+	$(1-2x-7x^2)$ $\times(1+2x-3x^2+x^3)$ $\times(1+2x+x^2+x^3)$	$x(-2+23x+104x^2+28x^3$ $-66x^4+13x^5-144x^6+105x^7)$
46+23	$(1+x+2x^2+x^3)$ $\times(1+4x+4x^2+8x^3)$ $\times(1+5x+14x^2+25x^3$ $+28x^4+20x^5+8x^6)$	$-8x(1+13x+81x^2+316x^3$ $+880x^4+1851x^5+2996x^6$ $+3772x^7+3636x^8+2560x^9$ $+1232x^{10}+288x^{11})$
47+	$(1+4x+7x^2+8x^3+4x^4+x^5)$ $\times(1-5x^2-20x^3-24x^4-19x^5)$	$4x(-1-3x+23x^2+177x^3$ $+560x^4+1087x^5+1347x^6$ $+1098x^7+500x^8+114x^9)$
49+	$(1-4x+3x^2+x^3)^2$ $\times(1-10x+27x^2-10x^3-27x^4)$	$7x(1-4x+3x^2+x^3)$ $\times(2-27x+108x^2-105x^3$ $-171x^4+216x^5+96x^6)$

Table 5 [cont.]

Γ	$w(x)$	$R(x)$
50+	$(1+4x)(1+3x+x^2)^2$ $\times(1+2x-7x^2-16x^3)$	$5x(1+3x+x^2)(-2-13x+17x^2$ $+247x^3+472x^4+204x^5)$
50+50	$(1+4x)(1+3x+x^2)^2$ $\times(1+2x-7x^2-16x^3)$	$5x(1+3x+x^2)(-2-13x+17x^2$ $+247x^3+472x^4+204x^5)$
51+	$(1-8x+16x^2-12x^3)$ $\times(1-6x+15x^2-22x^3+17x^4)$	$x(10-141x+798x^2-2488x^3$ $+4656x^4-4996x^5+2448x^6)$
54+	$(1-x+x^2)^2(1+4x^3)$ $\times(1-6x+9x^2-8x^3)$	$3x(1-x+x^2)(2-15x+39x^2$ $+x^3-156x^4+408x^5$ $-424x^6+264x^7)$
55+	$(1+x-x^2)(1-7x+11x^2)$ $\times(1-4x^2-4x^3)$	$-4x(-1-2x+34x^2+7x^3$ $-148x^4-64x^5+132x^6)$
56+	$(-1+x)(-1+2x)(1+x+2x^2)$ $\times(1-4x-8x^3+4x^4)$	$-4x(-1+4x-16x^2+59x^3$ $-64x^4+88x^5-152x^6+64x^7)$
59+	$(1+2x-4x^2-21x^3-44x^4$ $-60x^5-61x^6-46x^7$ $-24x^8-11x^9)(1+2x+x^3)$	$x(-4-4x+86x^2+448x^3$ $+1216x^4+2217x^5+3024x^6$ $+3128x^7+2600x^8+1748x^9$ $+720x^{10}+385x^{11})$
60+12,15,20	$(1+3x+8x^2+3x^3+x^4)$ $\times(1+4x+10x^2+4x^3+x^4)$ $\times(1+x)^2(1+x+x^2)$	$-4x(1+x)(2+26x+145x^2$ $+483x^3+1013x^4+1379x^5$ $+1257x^6+736x^7+290x^8$ $+68x^9+9x^{10})$
60+4,15,60	$(1-x)^2(1+x)^2$ $\times(1-x-x^2)(1-4x-x^2)$ $\times(1-x+2x^2+x^3+x^4)$	$-4(-1+x)x(1+x)(1-2x$ $-9x^2+31x^3-31x^4-43x^5$ $+9x^6+53x^7+46x^8+9x^9)$
60+	$(-1+x)(-1+4x)(1+4x^2)$ $\times(1-x+4x^2)$	$-4x(-1+8x-40x^2+83x^3$ $-160x^4+144x^5)$
62+	$(1+x^2-x^3)(1+4x+5x^2+3x^3)$ $\times(1-2x-3x^2-4x^3+4x^4)$	$x(-2+7x+76x^2+224x^3$ $+214x^4-7x^5-460x^6$ $-351x^7-80x^8+288x^9)$
66+	$(-1+x)(1+3x)(-1+x+8x^2)$ $\times(1-4x^2+4x^3)$	$-(1/4)x(4-147x-133x^2$ $+1756x^3-264x^4$ $-5468x^5+4608x^6)$
66+6,11,66	$(1-x+x^2)(1-3x-4x^2$ $-3x^3+x^4)(1-3x+2x^2$ $+x^3+2x^4-3x^5+x^6)$	$-(1/4)x(-20+97x-109x^2$ $+133x^3-770x^4+1286x^5$ $-1226x^6+325x^7-757x^8$ $+1441x^9-860x^{10}+144x^{11})$
69+	$(1-x^2+x^3)(1+4x+7x^2+5x^3)$ $\times(1-2x-5x^2+6x^3-3x^4)$	$x(-2+15x+82x^2+68x^3$ $-186x^4-77x^5+618x^6$ $+51x^7-480x^8+360x^9)$

Table 5 [cont.]

Γ	$w(x)$	$R(x)$
70+10,14,35	$(1+3x+x^2)(1+x+4x^2 + x^3+x^4)(1+5x+10x^2 + 17x^3+10x^4+5x^5+x^6)$	$-(1/4)x(28+305x+1651x^2 + 5373x^3+11702x^4+18902x^5 + 20462x^6+16701x^7+9643x^8 + 3953x^9+1108x^{10}+144x^{11})$
71+	$(1+4x+5x^2+x^3 - 3x^4-2x^5+x^7) \times (1-7x^2-11x^3+5x^4 + 18x^5+4x^6-11x^7)$	$4x(-1+x+38x^2+112x^3 + 29x^4-362x^5-563x^6+25x^7 + 717x^8+475x^9-248x^{10} - 372x^{11}-42x^{12}+132x^{13})$
78+6,26,39	$(1+3x+x^2+3x^3+x^4) \times (1+3x+5x^2+3x^3+x^4) \times (1+4x+8x^2+6x^3+8x^4 + 4x^5+x^6)$	$-x(8+96x+514x^2+1678x^3 + 3916x^4+6973x^5+9970x^6 + 11173x^7+10328x^8+7663x^9 + 4378x^{10}+1741x^{11}+428x^{12} + 49x^{13})$
87+	$(1-2x-x^2-x^3)(1+2x + 3x^2+3x^3)(1+2x+7x^2 + 6x^3+13x^4+4x^5+8x^6)$	$x(-2-17x+6x^2+124x^3 + 646x^4+1851x^5-2996x^6 + 3772x^7-3636x^8+2560x^9 - 1232x^{10}+288x^{11})$
92+	$(-1+x-2x^2+x^3) \times (-1+4x-4x^2+8x^3) \times (1-5x+14x^2-25x^3 + 28x^4-20x^5+8x^6)$	$-8x(-1+13x-81x^2+316x^3 - 880x^4+1851x^5-2996x^6 + 3772x^7-3636x^8+2560x^9 - 1232x^{10}+288x^{11})$
94+	$(1-8x-48x^2+256x^3-1024x^4) \times (1-20x+144x^2-576x^3 + 1024x^4-1024x^5) \times (1-36x+464x^2-2624x^3 + 7168x^4-13312x^5)$	$2x(-1-15x+23x^2+417x^3 - 736x^4+1851x^5-2996x^6 + 3772x^7-3636x^8+2560x^9 - 1232x^{10}+288x^{11})$
95+	$(1+4x+4x^2+4x^3) \times (1+5x+7x^2+5x^3+x^4) \times (1+5x+3x^2-15x^3-19x^4)$	$4x(-3-40x-199x^2-403x^3 + 127x^4+2516x^5+6309x^6 + 8606x^7+7036x^8+3264x^9 + 570x^{10})$
105+	$(1+x-x^2)(-1-x+5x^2) \times (1+5x+7x^2) \times (-1-4x-4x^2+4x^3)$	$x(-10-143x-832x^2-2475x^3 - 3146x^4+3007x^5+17636x^6 + 27496x^7+18000x^8+3960x^9)$
110+	$(1+3x+x^2)(1+3x+5x^2) \times (1+4x+8x^2+4x^3) \times (1+2x+x^2-8x^3)$	$x(-10-143x-832x^2-2475x^3 - 3146x^4+3007x^5+17636x^6 + 27496x^7+18000x^8+3960x^9)$
119+	$(1+2x+3x^2+6x^3+5x^4) \times (1+2x+3x^2+6x^3+4x^4+x^5) \times (1-2x+3x^2-6x^3-7x^5)$	$x(-2-17x-66x^2-26x^3 + 190x^4+1077x^5+3578x^6 + 7492x^7+12836x^8+17746x^9 + 18692x^{10}+15617x^{11} + 7644x^{12}+1680x^{13})$

Table 6. Complete list of values of $t_{17+}(\tau)$ with $[\mathbb{Q}(t_{17+}(\tau)) : \mathbb{Q}] \leq 2$ for the group $\Gamma = 17+$. The value of τ is listed by the coefficients (a, b, c) of its minimal polynomial. Starred values correspond to the reciprocals of the singular values within the radius of convergence of the corresponding Ramanujan–Sato series.

$b^2 - 4ac$	$\tau(a,b,c)$	$t_{17+}(\tau)$
-1411	$(85, -17, 5)$	$-515 + 126\sqrt{17}$
-1411	$(17, 17, 25)^*$	$-515 - 126\sqrt{17}$
-1003	$(187, -85, 11)$	$7(-25 + 6\sqrt{17})$
-1003	$(17, 17, 19)^*$	$7(-25 - 6\sqrt{17})$
-595	$(85, -85, 23)$	$\frac{1}{2}(-95 + 21\sqrt{17})$
-595	$(17, -17, 13)^*$	$\frac{1}{2}(-95 - 21\sqrt{17})$
-427	$(17, -27, 17)^*$	$\frac{1}{2}(25 + 33i\sqrt{7})$
-427	$(17, -41, 31)^*$	$\frac{1}{2}(25 - 33i\sqrt{7})$
-408	$(17, -34, 23)^*$	$25 + 12\sqrt{2}$
-408	$(34, -68, 37)^*$	$25 - 12\sqrt{2}$
-340	$(17, -34, 22)^*$	$2(6 + \sqrt{85})$
-340	$(34, -34, 11)$	$2(6 - \sqrt{85})$
-323	$(51, -17, 3)$	$\frac{1}{2}(-27 + 7\sqrt{17})$
-323	$(17, -17, 9)^*$	$\frac{1}{2}(-27 - 7\sqrt{17})$
-187	$(17, -17, 7)^*$	-13
-136	$(17, -34, 19)^*$	$\frac{1}{2}(7 + 3\sqrt{17})$
-136	$(85, -102, 31)$	$\frac{1}{2}(7 - 3\sqrt{17})$
-123	$(17, -25, 11)$	$-1 + 4i\sqrt{3}$
-123	$(17, -9, 3)$	$-1 - 4i\sqrt{3}$
-115	$(17, -19, 7)$	$\frac{1}{2}(-15 + i\sqrt{23})$
-115	$(17, -15, 5)$	$\frac{1}{2}(-15 - i\sqrt{23})$
-100	$(17, -28, 13)$	$3 + 2i\sqrt{5}$
-100	$(17, -6, 2)$	$3 - 2i\sqrt{5}$
-72	$(34, -8, 1)$	$2i\sqrt{3}$
-72	$(17, -8, 2)$	$-2i\sqrt{3}$
-67	$(17, -33, 17)$	7
-64	$(17, -2, 1)$	$2 + 3\sqrt{2}$
-64	$(85, -66, 13)$	$2 - 3\sqrt{2}$
-60	$(17, -22, 8)$	$\frac{1}{2}(-7 + 3i\sqrt{3})$
-60	$(17, -12, 3)$	$\frac{1}{2}(-7 - 3i\sqrt{3})$
-52	$(17, -30, 14)$	$\sqrt{13}$
-52	$(34, -38, 11)$	$-\sqrt{13}$
-51	$(51, -51, 13)$	-5
-43	$(17, -29, 13)$	2
-36	$(17, -24, 9)$	$-2 + i\sqrt{3}$
-36	$(17, -10, 2)$	$-2 - i\sqrt{3}$

Table 6 [cont.]

b^2-4ac	$\tau(a,b,c)$	$t_{17+}(\tau)$
-35	$(17,-21,7)$	$\frac{1}{2}(-7+i\sqrt{7})$
-35	$(17,-13,3)$	$\frac{1}{2}(-7-i\sqrt{7})$
-32	$(17,-28,12)$	$-1+\sqrt{2}$
-32	$(51,-40,8)$	$-1-\sqrt{2}$
-19	$(17,-7,1)$	-1
-16	$(17,-18,5)$	-4
-15	$(17,-23,8)$	$\frac{1}{2}(-5+i\sqrt{3})$
-15	$(17,-11,2)$	$\frac{1}{2}(-5-i\sqrt{3})$
-8	$(17,-20,6)$	-3
-4	$(17,-8,1)$	-2

Table 7. Complete list of values of $t_{11+}(\tau)$ with $[\mathbb{Q}(t_{11+}(\tau)) : \mathbb{Q}] \le 2$ for the group $\Gamma = 11+$

b^2-4ac	$\tau(a,b,c)$	$t_{11+}(\tau)$
-1507	$(11,11,37)$	$10(-3266-279\sqrt{137})$
-1507	$(143,-77,13)$	$10(-3266+279\sqrt{137})$
-1243	$(11,11,31)$	$2(-5902-555\sqrt{113})$
-1243	$(187,-231,73)$	$2(-5902+555\sqrt{113})$
-1012	$(11,0,23)$	$2(2183+465\sqrt{23})$
-1012	$(22,-22,17)$	$2(2183-465\sqrt{23})$
-715	$(55,-55,17)$	$2(-520+231\sqrt{5})$
-715	$(11,11,19)$	$2(-520-231\sqrt{5})$
-627	$(33,-33,13)$	$2(-322+55\sqrt{33})$
-627	$(11,11,17)$	$2(-322-55\sqrt{33})$
-403	$(11,9,11)$	$10(-26+3i\sqrt{31})$
-403	$(11,-31,31)$	$10(-26-3i\sqrt{31})$
-352	$(11,-22,19)$	$103+33\sqrt{11}$
-352	$(44,-44,13)$	$103-33\sqrt{11}$
-275	$(33,-11,3)$	$2(-28+13\sqrt{5})$
-275	$(11,-11,9)$	$2(-28-13\sqrt{5})$
-220	$(11,-22,16)$	$\frac{1}{2}(65+33\sqrt{5})$
-220	$(77,-44,7)$	$\frac{1}{2}(65-33\sqrt{5})$
-187	$(11,-11,7)$	-50
-132	$(11,-22,14)$	$10(1+\sqrt{3})$
-132	$(22,-22,7)$	$10(1-\sqrt{3})$
-123	$(11,-19,11)$	$2(8+5i\sqrt{3})$
-123	$(11,-25,17)$	$2(8-5i\sqrt{3})$
-112	$(11,-14,7)$	$-14+15i$

Table 7 [cont.]

b^2-4ac	$\tau(a,b,c)$	$t_{11+}(\tau)$
-112	$(11,-8,4)$	$-14-15i$
-99	$(11,-11,5)$	-18
-88	$(11,-22,13)$	16
-72	$(22,-4,1)$	$5(1+i\sqrt{3})$
-72	$(22,-40,19)$	$5(1-i\sqrt{3})$
-55	$(22,-11,2)$	$\frac{1}{2}(-13+3\sqrt{5})$
-55	$(11,-11,4)$	$\frac{1}{2}(-13-3\sqrt{5})$
-52	$(11,-16,7)$	$-2+6i$
-52	$(11,-6,2)$	$-2-6i$
-51	$(11,-13,5)$	$2(-4+i\sqrt{3})$
-51	$(11,-9,3)$	$2(-4-i\sqrt{3})$
-43	$(11,-23,13)$	10
-40	$(11,-2,1)$	$1+3\sqrt{5}$
-40	$(22,-20,5)$	$1-3\sqrt{5}$
-35	$(11,-19,9)$	$2\sqrt{5}$
-35	$(33,-47,17)$	$-2\sqrt{5}$
-32	$(11,-12,4)$	$-7+i$
-32	$(11,-10,3)$	$-7-i$
-28	$(11,-18,8)$	1
-24	$(11,-14,5)$	$-5+i\sqrt{3}$
-24	$(11,-8,2)$	$-5-i\sqrt{3}$
-19	$(11,-5,1)$	-2
-11	$(11,-11,3)$	-6
-8	$(11,-6,1)$	-4
-7	$(22,-31,11)$	-5

Acknowledgements. The authors thank the anonymous referee for his/her useful comments, suggestions and corrections.

Dongxi Ye is supported by the Natural Science Foundation of China (Grant No. 11901586), the Natural Science Foundation of Guangdong Province (Grant No. 2019A1515011323), the Fundamental Research Project of Guangzhou (Grant No. 202102080195) and the Fundamental Research Funds for the Central Universities, Sun Yat-sen University (Grant No. 22qntd2901).

References

[1] T. Anusha, E. N. Bhuvan, S. Cooper and K. R. Vasuki, *Elliptic integrals and Ramanujan-type series for* $1/\pi$ *associated with* $\Gamma_0(N)$*, where* N *is a product of two small primes*, J. Math. Anal. Appl. 472 (2019), 1551–1570.

[2] J. Arndt and C. Haenel, *Pi – Unleashed*, Springer, Berlin, 2001.

[3] N. D. Baruah, B. C. Berndt and H. H. Chan, *Ramanujan's series for* $1/\pi$*: a survey*, Amer. Math. Monthly 116 (2009), 567–587.

[4] B. C. Berndt and H. H. Chan, *Eisenstein series and approximations to* π, Illinois J. Math. 45 (2001), 75–90.

[5] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.

[6] H. H. Chan, S. H. Chan and Z. Liu, *Domb's numbers and Ramanujan–Sato type series for* $1/\pi$, Adv. Math. 186 (2004), 369–410.

[7] H. H. Chan and S. Cooper, *Rational analogues of Ramanujan's series for* $1/\pi$, Math. Proc. Cambridge Phillos. Soc. 153 (2012), 361–383.

[8] I. Chen and N. Yui, *Singular values of Thompson series*, in: Groups, Difference Sets, and the Monster (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ. 4, de Gruyter, Berlin, 1996, 255–326.

[9] S. Y. Choi and J. K. Koo, *Class fields from the fundamental Thompson series of level* $N = o(g)$, J. Korean Math. Soc. 42 (2005), 203–222.

[10] D. V. Chudnovsky and G. V. Chudnovsky, *Approximations and complex multiplication according to Ramanujan*, in: Ramanujan Revisited (Urbana-Champaign, IL, 1987), Academic Press, Boston, MA, 1988, 375–472.

[11] J. Conway, J. McKay and A. Sebbar, *On the discrete groups of Moonshine*, Proc. Amer. Math. Soc. 132 (2004), 2233–2240.

[12] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. 11 (1979), 308–339.

[13] S. Cooper, *Ramanujan's Theta Functions*, Springer, Cham, 2017.

[14] S. Cooper, J. Ge and D. Ye, *Hypergeometric transformation formulas of degrees 3, 7, 11 and 23*, J. Math. Anal. Appl. 421 (2015), 1358–1376.

[15] S. Cooper and D. Ye, *The Rogers–Ramanujan continued fraction and its level 13 analogue*, J. Approx. Theory 193 (2015), 99–127.

[16] S. Cooper and D. Ye, *Level 14 and 15 analogues of Ramanujan's elliptic functions to alternative bases*, Trans. Amer. Math. Soc. 368 (2016), 7883–7910.

[17] T. Huber, D. Schultz and D. Ye, *Series for* $1/\pi$ *of level 20*, J. Number Theory 188 (2018), 121–136.

[18] T. Huber, D. Schultz and D. Ye, *Level 17 Ramanujan–Sato series*, Ramanujan J. 52 (2020), 303–322.

[19] S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Math. 45 (1914), 350–372.

[20] T. Sato, *Apéry numbers and Ramanujan's series for* $1/\pi$, abstract of a talk presented at the annual meeting of the Mathematical Society of Japan, 28–31 March, 2002.

[21] Th. Schneider, *Arithmetische Untersuchungen elliptischer Integrale*, Math. Ann. 113 (1937), 1–13.

[22] P. Stiller, *Special values of Dirichlet series, monodromy, and the periods of automorphic forms*, Mem. Amer. Math. Soc. 49 (1984), no. 299, iv+116 pp.

[23] A. Tefera, *What is ... a Wilf–Zeilberger pair?*, Notices Amer. Math. Soc. 57 (2010), 508–509.

[24] M. Watkins, *Class numbers of imaginary quadratic fields*, Math. Comp. 73 (2004), 907–938.

[25] Y. Yang, *On differential equations satisfied by modular forms*, Math. Z. 246 (2004), 1–19.

[26] D. Ye, *Level 16 analogue of Ramanujan's theories of elliptic functions to alternative bases*, J. Number Theory 164 (2016), 191–207.

Tim Huber
School of Mathematical and Statistical Sciences
University of Texas Rio Grande Valley
Edinburg, TX 78539, USA
E-mail: timothy.huber@utrgv.edu

Daniel Schultz
Department of Mathematics
Pennsylvania State University
State College, PA 16802, USA
E-mail: tthsqe12@gmail.com

Dongxi Ye
School of Mathematics (Zhuhai)
Sun Yat-sen University
Zhuhai 519082, Guangdong, People's Republic of China
E-mail: yedx3@mail.sysu.edu.cn

Abstract (will appear on the journal's web site only)

We compute Ramanujan–Sato series systematically in terms of Thompson series and their modular equations. A complete list of rational and quadratic series corresponding to singular values of the parameters is derived.