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RESEARCH ARTICLE

Bi-Dbar-Approach for a Coupled Shifted Nonlocal Dispersionless System

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Abstract
We propose a Bi-Dbar approach and apply it to the extended coupled shifted nonlocal dispersionless system. We introduce the nonlocal reduction to solve the coupled shifted nonlocal dispersionless system. Since no enough constraint conditions can be found to curb the norming contants in the Dbar data, the "solutions" obtained by the Dbar dressing method, in general, do not admit the coupled shifted nonlocal dispersionless system. In the Bi-Dbar approach to the extended coupled shifted nonlocal dispersionless system, the norming constants are free. The constraint conditions on the norming constants are determined by the general nonlocal reduction, and the solutions of the coupled shifted nonlocal dispersionless system are derived.

Keywords Bi-Dbar-problem · Dressing method · Shifted nonlocal dispersionless system · General nonlocal reduction

1 Introduction

The coupled integrable dispersionless system can be used to describe the marginally unstable baroclinic wave packets in geophysical fluids and ultra-short pulses in nonlinear optics. The nonlinear coupled dispersionless system is related to the short pulse equation, the Pohlmeyer–Lund–Regge equation and to the Sine–Gordon equation if the potential function q is real [1–3]. For the complex case, the coupled dispersionless system [1, 4] is closely connected to the coupled AB system [5–11]. The coupled dispersionless systems have been studied by many

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methods [4, 10, 12] with interesting explicit solutions, including amplitude growing solitons, decaying solitons, stationary solitons, and loop soliton solutions [3, 13].

$\bar{\partial}$ (Dbar)-problem is a very effective tool to study nonlinear evolution equations and to give their explicit solutions [14–26]. The Dbar approach to the NLS equation with nonzero boundary condition was discussed in [21, 27]. The Dbar-problem to investigate the coupled nonlocal NLS equation is first considered in our recent work [28]. In this paper, we apply the Dbar-approach to study the following extended complex coupled shifted nonlocal dispersionless (ECCSND) system

$$\begin{aligned} &q_{tx}(x,t) + 2w(x,t)q(x,t) = 0, \\ &\hat{q}_{tx}(x,t) + 2\overline{w(x_0-x,t_0-t)}\hat{q}(x,t) = 0, \\ &w_x(x,t) + \sigma\left(q(x,t)\overline{\hat{q}(x_0-x,t_0-t)}\right)_t = 0, \quad \sigma = \mp 1, \end{aligned} \tag{1}$$

where $w(x,t) = 1 - \sigma\partial_x^{-1}\left(q(x,t)\overline{\hat{q}(x_0-x,t_0-t)}\right)_t$. So, the ECCSND system can be regarded as a sub-critical case in [5–8]. It is noted that, for the ECCSND system (1), if $\{q(x,t), \hat{q}(x,t)\}$ is a set of solution, so is $\{\overline{q(x_0-x,t_0-t)}, \overline{\hat{q}(x_0-x,t_0-t)}\}$. In addition, defining $V(x,t) = \sigma q(x,t)\overline{\hat{q}(x_0-x,t_0-t)}$ and $\hat{V}(x,t) = \sigma\hat{q}(x,t)\overline{q(x_0-x,t_0-t)}$ apparently implies $\hat{V}(x,t) = \overline{V(x_0-x,t_0-t)}$.

Equation (1) reduces to the following complex coupled shifted nonlocal dispersionless (CCSND) system

$$\begin{aligned} &q_{tx}(x,t) + 2w(x,t)q(x,t) = 0, \\ &w_x(x,t) + \sigma\left(q(x,t)\overline{q(x_0-x,t_0-t)}\right)_t = 0, \quad \sigma = \mp 1, \end{aligned} \tag{2}$$

if $q(x,t) = \hat{q}(x,t)$ and $\overline{w(x_0-x,t_0-t)} = w(x,t)$. If $x_0 = t_0 = 0$, Eq. (2) can be further reduced to the complex coupled nonlocal dispersionless (CCND) system [29–31]. In fact, the CCND system (or reverse space-time nonlocal version) and the CCSND one are equivalent on certain translation. However, the shifted items play some roles on the norming constants which may let to some special properties which were shown in this paper. We note that the latter is able to be cast to the real reverse space-time nonlocal sine-Gordon ($\sigma = -1$) and sinh-Gordon ($\sigma = 1$) [29], if both q and w are real functions.

In review of the Riemann–Hilbert problem to nonlocal integrable equations, discrete spectrum and norming constants play an important role to construct their solutions. The left and right scattering problems are considered to obtain the potential reconstruction, symmetry conditions are introduced to determine the constraint conditions about the discrete spectrum, and the trace formula are used to find the constraints on the norming constants. Then solutions of the nonlocal integrable equations are obtained [32].

The Dbar approach is mainly discussed in the spectral space, so only the constraint conditions of the discrete spectrum can be found. It is difficult to find the relation between the Dbar problem and the trace formula, and no enough constraint conditions on the norming constrants can be found. As a result, the obtained the formal solutions do not admit the nonlocal integable equations.

Certain remedies are needed to find the constraint conditions on the norming constants from the obtained formal solutions.

We propose a Bi-Dbar problem to consider the extended complex coupled shifted nonlocal dispersionless (ECCSND) system (1). The advantage to solve the extended system is that it only needs the constraint conditions about the discrete spectrum. Hence, solutions of the ECCSND system can be obtained as same as the usual dressing method. The general discrete spectrum and the norming constants imply the solutions, and the special Dbar data give special solution. With the explicit solutions of the ECCSND system (1) in hand, we introduce the general nonlocal reduction to determine the constraint conditions on the norming constants and to construct the solutions of the CCSND system (2).

The (shifted) nonlocal integrable equations reveal the models that the associated field at certain physical point is also determined by other relevant points. For the (shifted) nonlocal equations, it is important to reveal the inner link between these points, which are difficult to find from the results of the classical integrable equations. We know that the field or the solution to a nonlinear equation is recovered from the spectral data which need carefully investigated by the inverse spectral transform. These information can not be caught from the simple transformation between the ECCSND equation and the classical coupled dispersionless system. That is why the NLS equation has been well studied, while carefully discussions for nonlocal NLS was considered. That is also the motivation to carry out our work.

Organization of this paper is as follows. In Sect. 2, we introduce two local Dbar problems (called Bi-Dbar problems), and derive two Lax pairs of the ECCSND equation with different potential matrices. In Sect. 3, we derive the focusing($\sigma = -1$)/defocusing($\sigma = 1$) ECCSND system and its conservation laws. In Sect. 4, we present the explicit solutions for the focusing ECCSND system. In Sect. 5, we discuss the nonlocal reductions to the CCSND system in detail. In last section, some conclusions and discussions are given.

2 Bi-Dbar-Problem and Dressing Method

Consider the first local Dbar-problem

$$\frac{\partial \psi(k)}{\partial \bar{k}} = \psi(k)R(k), \tag{3}$$

with the normalization condition

$$\psi(k) \to I, \quad k \to \infty, \tag{4}$$

where $R(k)$ is the spectral transform matrix. The Dbar-problem (3) and (4) equivalent to the following integral equation

$$\psi(k) = I + \psi(k)R(k)C_k, \tag{5}$$

where the Cauchy–Green operation in complex plane is defined as

$$\psi(k)R(k)C_k = \frac{1}{2\pi i}\iint \psi(k)R(k)\frac{dz\wedge d\bar{z}}{z-k}. \tag{6}$$

The aim of dressing method is construct the relation between the ECCSND potential and the solution of the Bi-Dbar-problem. To this end, a good way is to construct two different Lax pairs of the ECCSND system from each Dbar-problem. It is noted that the Dbar-problem is defined in the spectral space, while the ECCSND system is in the physical space. Thus we need to introduce the physical variables x, t into the function $\psi(k)$, which can be done by extending the spectral transform matrix to be the form $R(k; x, t)$, and letting

$$R_x(k;x,t) = -\frac{ik}{2}[\sigma_3, R(k;x,t)], \tag{7}$$

$$R_t(k;x,t) = [\omega(k)C_k\sigma_3, R(k;x,t)] = \frac{1}{ik}[\sigma_3, R(k;x,t)], \tag{8}$$

where $\omega(k) = -i\pi\delta(k)$. We note that the solution of the system (7) and (8) is not unique.

Under the dressing procedure [20, 23, 24], we find that

$$\begin{aligned} \psi_x(k;x,t) &= -\frac{ik}{2}[\sigma_3, \psi(k;x,t)] + Q(x,t)\psi(k;x,t), \\ Q(x,t) &= -\frac{i}{2}[\sigma_3, \langle\psi(k;x,t)R(k;x,t)\rangle], \end{aligned} \tag{9}$$

and

$$\begin{aligned} \psi_t(k;x,t) &= -\frac{1}{ik}\psi(k;x,t)\sigma_3 + \frac{1}{ik}V(x,t)\psi(k;x,t), \\ V(x,t) &= \psi(0;x,t)\sigma_3\psi^{-1}(0;x,t), \end{aligned} \tag{10}$$

where

$$\langle\psi(k;x,t)R(k;x,t)\rangle = \frac{1}{2\pi i}\int\int \psi(k;x,t)R(k;x,t)dk\wedge d\bar{k}. \tag{11}$$

It is noted that [24]

$$\begin{aligned} U_x(k;x,t) &= -\frac{ik}{2}[\sigma_3, U(k;x,t)] + [Q(x,t), U(k;x,t)], \\ U(k;x,t) &\equiv \psi(k;x,t)\sigma_3\psi^{-1}(k;x,t), \end{aligned} \tag{12}$$

which reduces to

$$V_x(x,t) = [Q(x,t), V(x,t)], \tag{13}$$

in view of $V(x,t) = -i\langle\omega U\rangle = U(k=0;x,t)$. From the definition of Q in (9) and the properties of the Dbar problem, we find [24]

$$Q_t = -\frac{i}{2}[\sigma_3, \langle \omega U \rangle] = \frac{1}{2}[\sigma_3, V(x,t)]. \tag{14}$$

Thus, we obtain

$$V(x,t) = w(x,t)\sigma_3 - Q_t(x,t)\sigma_3, \tag{15}$$

Since $U^2 = I$, then $V^2 = 1$, which implies a normalization condition.

We note that Eqs. (14) and (15) imply the classical coupled dispersionless system [33]. We know that, for the Dbar approach to nonlinear integrable equations, it is important to consider the associated symmetry conditions which determine the properties of the scattering data. For the ECCSND system, we need to consider the second local Dbar problem

$$\begin{aligned} &\frac{\partial \hat{\psi}(k;x,t)}{\partial \bar{k}} = \hat{\psi}(k;x,t)\hat{R}(k;x,t), \\ &\hat{\psi}(k;x,t) \to I, \quad k \to \infty, \end{aligned} \tag{16}$$

where the new spectral transform matrix $\hat{R}(k;x,t)$ is another different solution of the evolution system (7) and (8). It is noted that the local Dbar problem is determined by the spectral transform matrix, and different spectral transform matrix defines different Dbar poblem. Thus two local Dbar problems (3) and (16) may be called the Bi-Dbar problem. Then we have

$$\hat{\psi}(k;x,t) = I + \hat{\psi}(k;x,t)\hat{R}(k;x,t)C_k. \tag{17}$$

A similar procedure gives another potential $\hat{Q}(x,t)$

$$\hat{Q}(x,t) = -\frac{i}{2}[\sigma_3, \langle \hat{\psi}(k;x,t)\hat{R}(k;x,t) \rangle], \tag{18}$$

and the another linear spectral system

$$\hat{\psi}_x(k;x,t) = -\frac{ik}{2}[\sigma_3, \hat{\psi}(k;x,t)] + \hat{Q}(x,t)\hat{\psi}(k;x,t), \tag{19}$$

and

$$\begin{aligned} \hat{\psi}_t(k;x,t) &= -\frac{1}{ik}\hat{\psi}(k;x,t)\sigma_3 + \frac{1}{ik}\hat{V}(x,t)\hat{\psi}(k;x,t), \\ \hat{V}(x,t) &= \hat{\psi}(0;x,t)\sigma_3\hat{\psi}^{-1}(0;x,t). \end{aligned} \tag{20}$$

Similarly, we have

$$\hat{V}_x(x,t) = [\hat{Q}(x,t), \hat{V}(x,t)], \quad \hat{Q}_t = \frac{1}{2}[\sigma_3, \hat{V}(x,t)], \tag{21}$$

as well as

$$\hat{V}(x,t) = \hat{w}(x,t)\sigma_3 - \hat{Q}_t(x,t)\sigma_3. \tag{22}$$

In addition, to get the ECCSND system, one also needs to introduce a symmetry condition about the two potentials

$$\hat{Q}(x,t) = -\Lambda\overline{Q(x_0 - x, t_0 - t)}\Lambda^{-1}, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix}, \tag{23}$$

then the symmetry condition of the eigenfunction takes the following form

$$\hat{\psi}(k;x,t) = \Lambda\overline{\psi(-\bar{k};x_0 - x, t_0 - t)}\Lambda^{-1}. \tag{24}$$

Thus, we have

$$Q = \begin{pmatrix} 0 & q(x,t) \\ \sigma\overline{\hat{q}(x_0 - x, t_0 - t)} & 0 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} 0 & \hat{q}(x,t) \\ \sigma\overline{q(x_0 - x, t_0 - t)} & 0 \end{pmatrix}, \tag{25}$$

and $\hat{w}(x,t) = \overline{w(x_0 - x, t_0 - t)}$ as well as

$$\begin{aligned} &\hat{\psi}_{11}(k;x,t) = \overline{\psi_{22}(-\bar{k};x_0 - x, t_0 - t)}, \quad \hat{\psi}_{12}(k;x,t) = -\sigma\overline{\psi_{21}(-\bar{k};x_0 - x, t_0 - t)}, \\ &\hat{\psi}_{21}(k;x,t) = -\sigma\overline{\psi_{12}(-\bar{k};x_0 - x, t_0 - t)}, \quad \hat{\psi}_{22}(k;x,t) = \overline{\psi_{11}(-\bar{k};x_0 - x, t_0 - t)}. \end{aligned} \tag{26}$$

Under these symmetry conditions, Eqs. (13), (14) and (21) give the ECCSND system (1).

3 The Conservation Laws

From (4) and (9), we know that $\psi(k;x,t)$ has the following asymptotic behaviors

$$\psi(k;x,t) = \sum_{n=0}^{\infty} \frac{a_n(x,t)}{k^n}, \quad k \to \infty, \tag{27}$$

where $a_0(x,t) = I$, and

$$\begin{pmatrix} a_n^{[o]} \\ a_n^{[d]} \end{pmatrix} = (J^{-1}K)^n \begin{pmatrix} 0 \\ I \end{pmatrix}, \tag{28}$$

with $a^{[d]}$ and $a^{[o]}$ are the diagonal and off-diagonal part of the matrix a_n, respectively. Here

$$J = \begin{pmatrix} i\sigma_3\partial_x & 0 \\ 0 & i\sigma_3\partial_x \end{pmatrix}, \quad K = \begin{pmatrix} -\partial_x^2 & \partial_x Q \\ Q\partial_x & -Q^2 \end{pmatrix}, \quad \partial_x \equiv \frac{\partial}{\partial x}.$$

In particular, (28) implies that

$$\begin{aligned} a_1^{[o]} &= iQ\sigma_3, & a_{1,x}^{[d]} &= iQ^2\sigma_3; \\ a_2^{[o]} &= Q_x - Q\partial_x^{-1}Q^2, & a_{2,x}^{[d]} &= QQ_x - Q^2\partial_x^{-1}Q^2. \end{aligned} \tag{29}$$

Let the trace of the spectral transform matrix $R(k;\ x,\ t)$ is zero, then the Dbar problem (3) implies $\bar{\partial}\det\psi(k;x,t)=0$, and further $\det\psi(k;x,t)=1$ in view of the asymptotic behaviors (27). Since $\psi^{-1}=\sigma_2\psi^T\sigma_2$, the temporal linear spectral problem (10) can be rewritten as

$$V\sigma_2 = ik\psi_t(k;x,t)\sigma_2\psi^T(k;x,t) - i\psi\sigma_1\psi^T. \tag{30}$$

Substituting the expansions (27) into (30), and taking $O(k^{-n}), (n=0,1,2,\cdots)$ terms, we obtain

$$V\sigma_2 = ia_{1,t}\sigma_2 - i\sigma_1, \tag{31}$$

and

$$\sum_{m=0}^{n} a_{m+1,t}\sigma_2 a_{n-m}^T = \sum_{m=0}^{n} a_m\sigma_1 a_{n-m}^T, \quad a_0 = I. \tag{32}$$

The diagonal part of (31) implies

$$w(x,t) = 1 - \partial_x^{-1}(Q^2)_t, \tag{33}$$

which gives the third equation in (1). This equation can be regarded as the first conservation law.

The $O(k^{-1})$ terms of the expansion of (30), or $n=1$ in (32), takes the form

$$a_{1,t}\sigma_2 a_1^T + a_{2,t}\sigma_2 = a_1\sigma_1 + \sigma_1 a_1^T, \tag{34}$$

which can be rewritten as the off-diagonal part

$$a_{1,t}^{[d]}\sigma_2 a_1^{[d]} + a_{1,t}^{[o]}\sigma_2 a_1^{[o]T} + a_{2,t}^{[d]}\sigma_2 = 0, \tag{35}$$

and the diagonal part

$$a_{1,t}^{[o]}\sigma_2 a_1^{[d]} + a_{1,t}^{[d]}\sigma_2 a_1^{[o]T} + a_{2,t}^{[o]}\sigma_2 = -2Q\sigma_2. \tag{36}$$

Substituting (29) into (35), we obtain

$$(Q_tQ + \partial_x^{-1}(Q^2)_t \cdot \partial_x^{-1}Q^2)_x = (QQ_x - Q^2\partial_x^{-1}Q^2)_t, \tag{37}$$

which gives the second conservation law. Here and after, we take the integral constants of $a_n^{[d]}$ to be zero. Similarly, the diagonal part (36) reduces to

$$Q_{xt} + 2(1 - \partial_x^{-1}(Q^2)_t)Q = 0, \tag{38}$$

which gives the first two equations in (1) in terms of (33). We note that the diagonal parts of other equations in (36) will also give the ECCSND system in (1), and the off-diagonal parts will reduce to more conservation laws. For example, the off-diagonal parts of (36) for $n=2$, that is

$$
\begin{aligned}
&a_{3,t}^{[d]}\sigma_2 + a_{2,t}^{[o]}\sigma_2 a_1^{[o]^T} + a_{2,t}^{[d]}\sigma_2 a_1^{[d]} + a_{1,t}^{[o]}\sigma_2 a_2^{[o]^T} + a_{1,t}^{[d]}\sigma_2 a_2^{[d]} \\
&\quad = a_2^{[d]}\sigma_1 + a_1^{[o]}\sigma_1 a_1^{[o]^T} + a_1^{[d]}\sigma_1 a_1^{[d]} + \sigma_1 a_2^{[d]^T},
\end{aligned}
\tag{39}
$$

which implies the third conservation law

$$
\begin{aligned}
&\left(Q^2 a_2^{[d]} - Q a_{2,x}^{[o]}\right)_t \\
&\quad = \left(Q^2 - \varpi^2 + a_{2,t}^{[o]} Q - Q_t a_2^{[o]} + \mathrm{tr} a_2^{[d]} - \varpi a_{2,t}^{[d]} + \varpi_t (a_2^{[d]} - \mathrm{tr} a_2^{[d]})\right)_x,
\end{aligned}
\tag{40}
$$

where $\varpi = \partial_x^{-1} Q^2$ and a_2 is given in (29).

4 The Solutions of the ECCND System

In this section, we give the explicit solutions of ECCSND system in the focussing case $\sigma = -1$.

According to the above symmetry conditions, we let the first spectral transform matrix $R(k; x, t)$ has the following form

$$
R(k;x,t) = \pi \begin{pmatrix} 0 & \sum_{j=1}^{\tilde{N}} \bar{d}_j \mathrm{e}^{-2i\theta^\circ(k;x,t)} \delta(k + \bar{\lambda}_j) \\ \sum_{j=1}^{N} c_j \mathrm{e}^{2i\theta(k;x,t)} \delta(k - k_j) & 0 \end{pmatrix},
\tag{41}
$$

and take the second spectral transform matrix be of the form

$$
\hat{R}(k;x,t) = -\pi \begin{pmatrix} 0 & \sum_{j=1}^{N} \bar{c}_j \mathrm{e}^{-2i\theta^\circ(k;x,t)} \delta(k + \bar{k}_j) \\ \sum_{j=1}^{\tilde{N}} d_j \mathrm{e}^{2i\theta(k;x,t)} \delta(k - \lambda_j) & 0 \end{pmatrix},
\tag{42}
$$

where

$$
\theta(k;x,t) = \frac{k}{2}x + \frac{1}{k}t, \quad \theta^\circ(k;x,t) = \theta(k; x - x_0, t - t_0),
\tag{43}
$$

and for any function $f((\cdot);x,t)$, $f^\circ((\cdot),x,t) = f((\cdot);x - x_0, t - t_0)$.

From Eqs. (41) and (9), as well as (42) and (18), we get the reconstruction of the potentials about the Dbar data and the eigenfunctions. Here, the norming constants c_j and d_j are free. In addition, we can show that $r(x,t) = -\hat{q}(x_0 - x, t_0 - t)$ and $\hat{r}(x,t) = -q(x_0 - x, t_0 - t)$, in terms of the symmetry condition (26) and $\theta(-\bar{k}; -x, -t) = \theta(k;x,t)$.

The closed form expressions for the explicit solution for the ECCSND system (1) are

$$q(x,t) = -i\frac{\det \tilde{\Omega}^a}{\det \tilde{\Omega}}, \quad \hat{q}(x,t) = i\frac{\det \Omega^a}{\det \Omega}, \quad \mathbb{E}^2 \backslash (\tilde{\mathcal{P}}_{\mathcal{S}} \cup \mathcal{P}_{\mathcal{S}}), \tag{44}$$

where

$$\tilde{\Omega}^a = \begin{pmatrix} 0 & \tilde{E} \\ \tilde{E}^T & \tilde{\Omega} \end{pmatrix}, \quad \Omega^a = \begin{pmatrix} 0 & E \\ E^T & \Omega \end{pmatrix}. \tag{45}$$

Here the symmetry matrices $\tilde{\Omega}$ and Ω are given by

$$\tilde{\Omega}_{\tilde{N}\times\tilde{N}} = (\tilde{H}^{\circ})^{-1} + \bar{\Lambda} G \bar{\Lambda}^T, \quad \Omega_{N\times N} = (\tilde{G}^{\circ})^{-1} + \Lambda^T H \Lambda, \tag{46}$$

and

$$\begin{aligned} &\tilde{E} = (1,1,\dots,1)_{\tilde{N}}, &&E = (1,1,\dots,1)_N, \\ &\tilde{G} = \mathrm{diag}(\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_N), &&\tilde{H} = \mathrm{diag}(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_{\tilde{N}}), \\ &G = \mathrm{diag}(g_1, g_2, \cdots, g_N), &&H = \mathrm{diag}(h_1, h_2, \cdots, h_{\tilde{N}}), \end{aligned} \tag{47}$$

$$\begin{aligned} g_j(x,t) &= c_j \mathrm{e}^{2i\theta(k_j;x,t)}, \quad &\tilde{g}_j(x,t) &= \bar{c}_j \mathrm{e}^{-2i\theta(-\bar{k}_j;x,t)}, \\ h_l(x,t) &= d_l \mathrm{e}^{2i\theta(\lambda_l;x,t)}, \quad &\tilde{h}_l(x,t) &= \bar{d}_l \mathrm{e}^{-2i\theta(-\bar{\lambda}_l;x,t)}, \end{aligned} \tag{48}$$

with Λ is the $\tilde{N}\times N$ Cauchy type matrix with $\Lambda_{lj} = 1/(\lambda_l + \bar{k}_j)$. We note that $\tilde{g}_j(x,t) = \overline{g_j(-x,-t)}, \tilde{h}_l(x,t) = \overline{h_l(-x,-t)}$. It is noted that, for some scattering data $\mathcal{S} = \{\{c_j, k_j\}_{j=1}^N; \{d_l, \lambda_l\}_{l=1}^{\tilde{N}}\}$, the symmetry matrices $\tilde{\Omega}$ and Ω may degenerate in x-t plane $\mathbb{E}^2$ at certain point sets denoted by $\tilde{\mathcal{P}}_{\mathcal{S}}$ and $\mathcal{P}_{\mathcal{S}}$. In general, $q(x,t)$ and $\hat{q}(x,t)$ have some singularities at the points in $\tilde{\mathcal{P}}_{\mathcal{S}}$ and $\mathcal{P}_{\mathcal{S}}$, respectively. It is important to express the solution of the ECCSND system in the form (44), (45). Because it makes us possible and easier to give the general nonlocal reduction of the ECCSND system, which will be discussed in the next section.

It is remarked that $w(x,t)$ is given by (33), which involves the differential with respect to t and integral about x. So, we may find another way to give the expression of $w(x,t)$. In fact, from (10) and (15), we get $w = (\psi_{12}\psi_{21} + \psi_{11}\psi_{22})|_{k=0}$. By virtue of $\det\psi = 1$ obtained from the Dbar problem with zero trace R, we find the expression of $w(x,t)$ as

$$\begin{aligned} w &= 1 + 2\psi_{12}(0;x,t)\psi(0;x,t) \\ &= 1 - 2\frac{\det \check{\Omega}^b}{\det \check{\Omega}} \frac{\det \tilde{\Omega}^b}{\det \tilde{\Omega}}, \end{aligned} \tag{49}$$

where $\check{\Omega}_{N\times N} = G^{-1} + \bar{\Lambda}^T \hat{H}^{\circ} \bar{\Lambda}$ and

$$\hat{\Omega}^b = \begin{pmatrix} 0 & \alpha \\ E^T & \hat{\Omega} \end{pmatrix}, \quad \tilde{\Omega}^b = \begin{pmatrix} 0 & \tilde{\beta} \\ \tilde{E}^T & \tilde{\Omega} \end{pmatrix},$$
$$\alpha = \left(\frac{1}{k_1}, \cdots, \frac{1}{k_N} \right), \quad \tilde{\beta} = \left(\frac{1}{\bar{\lambda}_1}, \cdots, \frac{1}{\bar{\lambda}_{\tilde{N}}} \right). \tag{50}$$

For $N = \tilde{N} = 1$, we find

$$q(x,t) = \frac{i}{(\tilde{h}_1^\circ)^{-1} + \frac{g_1}{(\bar{\lambda}_1 + k_1)^2}}, \quad \hat{q}(x,t) = \frac{-i}{(\tilde{g}^\circ)_1^{-1} + \frac{h_1}{(\lambda_1 + \bar{k}_1)^2}},$$
$$w = 1 - \frac{2}{k_1 \bar{\lambda}_1} \frac{1}{(\tilde{h}_1^\circ)^{-1} + \frac{g_1}{(\bar{\lambda}_1 + k_1)^2}} \frac{1}{g_1^{-1} + \frac{\tilde{h}_1^\circ}{(\bar{\lambda}_1 + k_1)^2}}, \tag{51}$$

where g_j, h_j and $\hat{g}_j, \hat{h}_j$ are defined in (48). In particular, if $k_1 = ib_1, \lambda_1 = i\eta_1$ and $d_1 = |d_1|\mathrm{e}^{2i\varphi_1}, c_1 = |c_1|\mathrm{e}^{2i\phi_1}$, then (51) reduces to the following forms

$$q = \frac{i}{2} \frac{|d_1|(\mathrm{e}^{2\theta_1^\bullet}\mathrm{e}^{-2i\varphi_1} - \mathrm{e}^{2\tilde{\theta}_1^\circ}\mathrm{e}^{-2i\phi_1})}{\cosh 2(\tilde{\theta}_1^\circ - \theta_1^\bullet) - \cos 2(\varphi_1 - \phi_1)},$$
$$\hat{q} = \frac{i}{2} \frac{|c_1|(\mathrm{e}^{2\theta_1^\circ}\mathrm{e}^{-2i\varphi_1} - \mathrm{e}^{2\tilde{\theta}_1^\bullet}\mathrm{e}^{-2i\phi_1})}{\cosh 2(\tilde{\theta}_1^\bullet - \theta_1^\circ) - \cos 2(\varphi_1 - \phi_1)}, \tag{52}$$

and

$$w = 1 - \frac{(b_1 - \eta_1)^2}{2b_1\eta_1} \frac{\cosh 2(\tilde{\theta}_1^\circ - \theta_1^\bullet)\cos 2(\varphi_1 - \phi_1) - 1}{(\cosh 2(\tilde{\theta}_1^\circ - \theta_1^\bullet) - \cos 2(\varphi_1 - \phi_1))^2}$$
$$- i\frac{(b_1 - \eta_1)^2}{2b_1\eta_1} \frac{\sinh 2(\tilde{\theta}_1^\circ - \theta_1^\bullet)\sin 2(\varphi_1 - \phi_1)}{(\cosh 2(\tilde{\theta}_1^\circ - \theta_1^\bullet) - \cos 2(\varphi_1 - \phi_1))^2}, \tag{53}$$

where

$$\theta_j = \frac{b_j}{2}x - \frac{1}{b_j}t = \theta_j(x,t), \quad \tilde{\theta}_l = \frac{\eta_l}{2}x - \frac{1}{\eta_l}t = \tilde{\theta}_l(x,t),$$
$$\theta_1^\circ = \theta_1(x - x_0, t - t_0), \qquad \tilde{\theta}_1^\circ = \tilde{\theta}_1(x - x_0, t - t_0),$$
$$\theta_1^\bullet = \theta_1 + \log\frac{|b_1 - \eta_1|}{\sqrt{|c_1 d_1|}}, \quad \tilde{\theta}_1^\bullet = \tilde{\theta}_1 + \log\frac{|b_1 - \eta_1|}{\sqrt{|c_1 d_1|}}. \tag{54}$$

If $\varphi_1 - \phi_1 \neq n\pi, n \in \mathbb{Z}$, then $-1 \leq \cos 2(\varphi_1 - \phi_1)) < 1$. Solutions q and $\hat{q}$ in (52) with $b_1\eta_1 < 0$ show amplitude changing solitons, while w in (53) reduces to a one-soliton, (bright/dark one-soliton depends on the value of ϕ_1 and φ_1, if other parameters $b_1, \eta_1, |c_1|, |d_1|$ are fixed). Since

$$\theta_1 + \tilde{\theta}_1 = (\eta_1 + b_1)\left(\frac{x}{2} - \frac{1}{\eta_1 b_1}t\right),$$

then q and $\hat{q}$ are growing solitons for $|\eta_1| > |b_1|$ and decaying soliton for $|\eta_1| < |b_1|$. While if $\eta_1 = -b_1$, then q and $\hat{q}$ are one-solitons, and on different lines.

If $\varphi_1 - \phi_1 = n\pi$, the solutions q, w and $\hat{q}$ in (52) with $b_1\eta_1 < 0$ have singularities on the parallel lines $l_j, j = 1, 2$, respectively, where

$$l_j: \quad \frac{x}{2} + \frac{t}{b_1\eta_1} = \kappa_j,$$

with

$$\kappa_1 = \frac{1}{\eta_1 - b_1}\left(\frac{\eta_1}{2}x_0 - \frac{t_0}{\eta_1} + \log\frac{|b_1 - \eta_1|}{\sqrt{|c_1 d_1|}}\right),$$
$$\kappa_2 = \frac{1}{b_1 - \eta_1}\left(\frac{b_1}{2}x_0 - \frac{t_0}{b_1} + \log\frac{|b_1 - \eta_1|}{\sqrt{|c_1 d_1|}}\right).$$

We note that, if $b_1\eta_1 > 0$, q and $\hat{q}$ will be unstable, but w is still stable and will give a one-soliton or w-shaped wave, which depends on the choice of ϕ_1 and φ_1.

Similarly, for $N = \tilde{N} = 2$, we obtain the solutions q and $\hat{q}$ in (44) with

$$\begin{aligned}
\det\tilde{\Omega}^a &= -\left[(\tilde{h}_1^\circ)^{-1} + (\tilde{h}_2^\circ)^{-1} + \frac{g_1(\bar{\lambda}_2 - \bar{\lambda}_1)^2}{(\bar{\lambda}_1 + k_1)^2(\bar{\lambda}_2 + k_1)^2} + \frac{g_2(\bar{\lambda}_2 - \bar{\lambda}_1)^2}{(\bar{\lambda}_1 + k_2)^2(\bar{\lambda}_2 + k_2)^2}\right],\\
\det\tilde{\Omega} &= (\tilde{h}_1^\circ)^{-1}(\tilde{h}_2^\circ)^{-1} + g_1 g_2\frac{(\bar{\lambda}_2 - \bar{\lambda}_1)^2(k_2 - k_1)^2}{\prod_{j,l=1}^2(\bar{\lambda}_j + k_l)^2}\\
&\quad + \frac{g_1(\tilde{h}_1^\circ)^{-1}}{(\bar{\lambda}_2 + k_1)^2} + \frac{g_2(\tilde{h}_1^\circ)^{-1}}{(\bar{\lambda}_2 + k_2)^2} + \frac{g_1(\tilde{h}_2^\circ)^{-1}}{(\bar{\lambda}_1 + k_1)^2} + \frac{g_2(\tilde{h}_2^\circ)^{-1}}{(\bar{\lambda}_1 + k_2)^2},
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
\det\Omega^a &= -\left[(\tilde{g}_1^\circ)^{-1} + (\tilde{g}_2^\circ)^{-1} + \frac{h_1(\bar{k}_2 - \bar{k}_1)^2}{(\lambda_1 + \bar{k}_1)^2(\lambda_1 + \bar{k}_2)^2} + \frac{h_2(\bar{k}_2 - \bar{k}_1)^2}{(\lambda_2 + \bar{k}_1)^2(\lambda_2 + \bar{k}_2)^2}\right],\\
\det\Omega &= (\tilde{g}_1^\circ)^{-1}(\tilde{g}_2^\circ)^{-1} + h_1 h_2\frac{(\lambda_2 - \lambda_1)^2(\bar{k}_2 - \bar{k}_1)^2}{\prod_{j,l=1}^2(\lambda_j + \bar{k}_l)^2}\\
&\quad + \frac{h_1(\tilde{g}_1^\circ)^{-1}}{(\lambda_1 + \bar{k}_2)^2} + \frac{h_1(\tilde{g}_2^\circ)^{-1}}{(\lambda_1 + \bar{k}_1)^2} + \frac{h_2(\tilde{g}_1^\circ)^{-1}}{(\lambda_2 + \bar{k}_2)^2} + \frac{h_2(\tilde{g}_2^\circ)^{-1}}{(\lambda_2 + \bar{k}_1)^2}.
\end{aligned} \tag{56}$$

In addition, for the solution w in (49), we get

$$
\begin{aligned}
\det \check{\Omega} &= g_1^{-1} g_2^{-1} + \tilde{h}_1^\circ \tilde{h}_2^\circ \frac{(\bar{\lambda}_2 - \bar{\lambda}_1)^2 (k_2 - k_1)^2}{\prod_{j,l=1}^2 (\bar{\lambda}_j + k_l)^2} \\
&\quad + \frac{\tilde{h}_1^\circ g_1^{-1}}{(\bar{\lambda}_1 + k_2)^2} + \frac{\tilde{h}_1^\circ g_2^{-1}}{(\bar{\lambda}_1 + k_1)^2} + \frac{\tilde{h}_2^\circ g_1^{-1}}{(\bar{\lambda}_2 + k_2)^2} + \frac{\tilde{h}_2^\circ g_2^{-1}}{(\bar{\lambda}_2 + k_1)^2}, \\
\det \check{\Omega}^b &= -\frac{1}{k_2} g_1^{-1} - \frac{1}{k_1} g_2^{-1} \\
&\quad + \frac{\tilde{h}_1^\circ \bar{\lambda}_1 (k_2 - k_1)^2}{k_1 k_2 (\bar{\lambda}_1 + k_1)^2 (\bar{\lambda}_1 + k_2)^2} + \frac{\tilde{h}_2^\circ \bar{\lambda}_2 (k_2 - k_1)^2}{k_1 k_2 (\bar{\lambda}_2 + k_1)^2 (\bar{\lambda}_2 + k_2)^2},
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
\det \tilde{\Omega}^b &= -\frac{1}{\bar{\lambda}_2} (\tilde{h}_1^\circ)^{-1} - \frac{1}{\bar{\lambda}_1} (\tilde{h}_2^\circ)^{-1} \\
&\quad + \frac{g_1 k_1 (\bar{\lambda}_2 - \bar{\lambda}_1)^2}{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + k_1)^2 (\bar{\lambda}_2 + k_1)^2} + \frac{g_2 k_2 (\bar{\lambda}_2 - \bar{\lambda}_1)^2}{\bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 + k_2)^2 (\bar{\lambda}_2 + k_2)^2}.
\end{aligned}
\tag{58}
$$

Figure 1 shows the interaction of decaying two-soliton $q(x, t)$ (left),$\hat{q}(x, t)$ (center) and $w(x, t)$ (right) in (55)–(58) and (44).

As an example, we discuss the asymptotic behavior of w. For convenience, we choose $\lambda_j = i\eta_j$ and $k_j = ib_j$ and $b_2 < b_1 < 0 < \eta_1 < \eta_2$ associated with the parameters in Fig. 1. In this case, $M = \det \check{\Omega} \det \tilde{\Omega}$ and $M^b = \det \check{\Omega}^b \det \tilde{\Omega}^b$ take the following forms

$$
M = \left[e^{-X_1} + P e^{X_1} - \frac{e^{-X_2}}{(\eta_1 - b_1)^2} - \frac{e^{X_2}}{(\eta_2 - b_2)^2} - \frac{e^{-X_3}}{(\eta_1 - b_2)^2} - \frac{e^{X_3}}{(\eta_2 - b_1)^2} \right]^2,
\tag{59}
$$

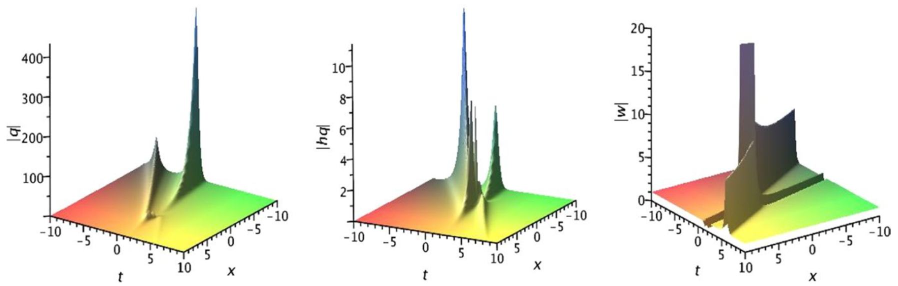

Fig. 1 The interaction of decaying two-soliton $q(x, t)$,$\hat{q}(x, t)$ and two-soliton of $w(x, t)$ in (55) and (44) with $\lambda_1 = 0.5i, \lambda_2 = 1i, k_1 = -i, k_2 = -1.2i, c_1 = i, c_2 = e^{2i\pi/3}, d_1 = e^{-2i\pi/3}, d_2 = e^{i\pi/3}$

$$
\begin{aligned}
M^b = {} & \frac{e^{X_2-X_1}}{b_2\eta_2} + BB_1YY_1e^{X_1-X_2} + \frac{e^{-X_1-X_2}}{b_1\eta_1} + BB_2YY_2e^{X_1+X_2} \\
& + \frac{e^{-X_1-X_3}}{b_2\eta_1} + BB_1YY_2e^{X_1+X_3} + \frac{e^{X_3-X_1}}{b_1\eta_2} + BB_2YY_1e^{X_1-X_3} \\
& - Y\left(\frac{B_2}{b_2}e^{X_2-X_3} + \frac{B_1}{b_1}e^{X_3-X_2}\right) - B\left(\frac{Y_2}{\eta_2}e^{X_2+X_3} + \frac{Y_1}{\eta_1}e^{-X_2-X_3}\right) \\
& - B\left(\frac{B_2}{b_1} + \frac{B_1}{b_2}\right) - B\left(\frac{Y_1}{\eta_2} + \frac{Y_2}{\eta_1}\right),
\end{aligned}
\tag{60}
$$

where $X_l = \alpha_j x - \beta_l t + \alpha_{l0}, (l = 1, 2, 3)$ and

$$
\begin{aligned}
& P = \frac{(b_2-b_1)^2(\eta_2-\eta_1)^2}{\prod_{j,l=1}^{2}(\eta_j-b_l)^2}, \quad B = \frac{(b_2-b_1)^2}{b_1b_2}, \quad Y = \frac{(\eta_2-\eta_1)^2}{\eta_1\eta_2}, \\
& B_j = \frac{b_j}{(\eta_1-b_j)^2(\eta_2-b_j)^2}, \quad Y_j = \frac{\eta_j}{(\eta_j-b_1)^2(\eta_j-b_2)^2}, \quad j = 1, 2, \\
& \alpha_1 = \frac{1}{2}(\eta_1+\eta_2-b_1-b_2), \quad \beta_1 = \frac{1}{\eta_1} + \frac{1}{\eta_2} - \frac{1}{b_1} - \frac{1}{b_2}, \\
& \alpha_2 = \frac{1}{2}(\eta_2-\eta_1+b_1-b_2), \quad \beta_2 = \frac{1}{\eta_2} - \frac{1}{\eta_1} + \frac{1}{b_1} - \frac{1}{b_2}, \\
& \alpha_1 = \frac{1}{2}(\eta_2-\eta_1-b_1+b_2), \quad \beta_3 = \frac{1}{\eta_2} - \frac{1}{\eta_1} - \frac{1}{b_1} + \frac{1}{b_2}.
\end{aligned}
$$

Here $\alpha_1 > \alpha_2 > \alpha_3 > 0$, $\beta_1/\alpha_1 > 0 > \beta_2/\alpha_2 > \beta_3/\alpha_3$ and $\alpha_{l0}, (l = 1, 2, 3)$ are some constants depending on $c_j, \bar{d}_j$ and x_0, t_0.

Now along the direction $x = \frac{\beta_1+\beta_2}{\alpha_1+\alpha_2}t$, where $X_1 + X_2$ is a constant, and

$$
w \sim \begin{cases} 1 + \frac{B}{2}\operatorname{sech}^2\frac{1}{2}(X_2 - X_3 + \tau_1^-),\ t \to -\infty, \\ 1 + \frac{B}{2}\operatorname{sech}^2\frac{1}{2}(X_2 - X_3 + \tau_1^+),\ t \to +\infty, \end{cases}
\tag{61}
$$

where

$$
\tau_1^- = \ln\left(\frac{\eta_2-b_1}{\eta_2-b_2}\right)^2, \quad \tau_1^+ = \ln\left(\frac{\eta_1-b_1}{\eta_1-b_2}\right)^2.
$$

Similarly, along the direction $x = \frac{\beta_1-\beta_3}{\alpha_1-\alpha_3}t$, where $X_1 - X_3$ is a constant, and

$$
w \sim \begin{cases} 1 - \frac{(\eta_1-b_2)^2}{2b_2\eta_1}\operatorname{csch}^2\frac{1}{2}(X_1 - X_3 + \tau_2^-),\ t \to -\infty, \\ 1 - \frac{(\eta_2-b_2)^2}{2b_2\eta_1}\operatorname{csch}^2\frac{1}{2}(X_1 - X_3 + \tau_2^+),\ t \to +\infty, \end{cases}
\tag{62}
$$

where

$$\tau_2^- = \ln\left(\frac{1}{\eta_1 - b_2}\right)^2, \quad \tau_2^+ = \ln(P(\eta_1 - b_2)^2).$$

We note that the wave train along the direction $x = \frac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3} t$ may has singularities at $X_1 - X_3 + \tau_2^{\pm} = 0$, and the amplitude is changed after interaction, see Fig. 1(right).

From the above figures, we find that the two wave trains of $q(x, t)$ and $\hat{q}(x, t)$ locate on different lines which are determined by the Dbar data $\{\lambda_l, d_l\}$ and $\{k_j, c_j\}$. So we can carefully choose certain Dbar data to ensure $q(x, t) = \hat{q}(x, t)$.

5 Nonlocal Reduction to the CCSND System

In general, $q(x, t)$ and $\hat{q}(x, t)$ given in (44) are not equal to each other. From (48), we find that $q(x, t) = \hat{q}(x, t)$ implies the following conditions $k_j = -\bar{k}_j$, $\lambda_l = -\bar{\lambda}_l$ and $N = \tilde{N}$, as well as $|c_j|, |d_l|$ dependenting on $\{k_j, \lambda_l\}$ as well as x_0, t_0. To find the constraint conditions on $|c_j|$ and $|d_l|$ for $q(x, t) = \hat{q}(x, t)$, we take $k_j = ib_j,\ \lambda_l = i\eta_l$, $j, l = 1, 2, \ldots, N$. It is remarked that

$$\begin{aligned} g_j &= c_j \mathrm{e}^{2i\theta(ib_j)}, \quad \tilde{g}_j^{-1} = \frac{1}{\bar{c}_j} \mathrm{e}^{2i\theta(ib_j)}, \\ h_j &= d_j \mathrm{e}^{2i\theta(i\eta_j)}, \quad \tilde{h}_j^{-1} = \frac{1}{\bar{d}_j} \mathrm{e}^{2i\theta(i\eta_j)}. \end{aligned} \tag{63}$$

Since the matrices Ω and $\tilde{\Omega}$ in (45) are symmetry matrix and the matrix $K = i\Lambda = (\frac{1}{\eta_j - b_l})$ is a Cauchy matrix. Using the Cauchy–Binet formula, we give the determinants in (44) as the following form [23, 24]

$$\begin{aligned} \det \tilde{\Omega}^a &= \sum_{\sigma=1}^{N} (-1)^{\sigma} \sum_{\sigma 12} \prod_{l,s,n_\sigma} \frac{(\tilde{h}^{\circ}_{n_\sigma})^{-1} g_s}{(\eta_l - b_s)^2} \prod_{\substack{l<l' \\ s<s'}} (\eta_l - \eta_{l'})^2 (b_{s'} - b_s)^2, \\ \det \tilde{\Omega} &= \prod_{j=1}^{N} (\tilde{h}^{\circ}_j)^{-1} + \sum_{\sigma=1}^{N} (-1)^{\sigma} \sum_{\sigma 11} \prod_{l,m,n_\sigma} \frac{(\tilde{h}^{\circ}_{n_\sigma})^{-1} g_m}{(\eta_l - b_m)^2} \prod_{\substack{l<l' \\ m<m'}} (\eta_l - \eta_{l'})^2 (b_{m'} - b_m)^2, \end{aligned} \tag{64}$$

and

$$\begin{aligned} \det \Omega^a &= \sum_{\sigma=1}^{N} (-1)^{\sigma} \sum_{\sigma 12} \prod_{l,s,n_\sigma} \frac{h_s (\tilde{g}^{\circ}_{n_\sigma})^{-1}}{(\eta_s - b_l)^2} \prod_{\substack{l<l' \\ s<s'}} (\eta_s - \eta_{s'})^2 (b_{l'} - b_l)^2, \\ \det \Omega &= \prod_{j=1}^{N} (\tilde{g}^{\circ}_j)^{-1} + \sum_{\sigma=1}^{N} (-1)^{\sigma} \sum_{\sigma 11} \prod_{l,m,n_\sigma} \frac{h_m (\tilde{g}^{\circ}_{n_\sigma})^{-1}}{(\eta_m - b_l)^2} \prod_{\substack{l<l' \\ m<m'}} (\eta_m - \eta_{m'})^2 (b_{l'} - b_l)^2, \\ & l, l' \in \{j_1, j_2, \cdots, j_\sigma\}, \quad s, s' \in \{r_2, \cdots, r_\sigma\}, \quad m, m' \in \{r_1, r_2, \cdots, r_\sigma\}, \\ & n_\sigma \in \{1, 2, \cdots, N\} \backslash \{j_1, \cdots, j_\sigma\}, \end{aligned} \tag{65}$$

where $\sum\limits_{\sigma 12}$ denotes the summation for indices $1 \leq j_1 \leq j_2 \leq \cdots \leq j_\sigma \leq N, 1 \leq r_2 \leq \cdots \leq r_\sigma \leq N$, and summation $\sum\limits_{\sigma 11}$ for $1 \leq j_1 \leq j_2 \leq \cdots \leq j_\sigma \leq N, 1 \leq r_1 \leq r_2 \leq \cdots \leq r_\sigma \leq N$.

If the determinants in (44) admit

$$\det \tilde{\Omega}^a = (-1)^{N-1} \det \Omega^a, \quad \det \tilde{\Omega} = (-1)^N \det \Omega, \tag{66}$$

which means that $\det \tilde{\Omega}^a \det \Omega + \det \Omega^a \det \tilde{\Omega} = 0$, then $q(x,t) = \hat{q}(x,t)$, and the ECCND system (1) reduces to the CCSND system (2). In addtion, from (64), (65) and (66), the constraint conditions about $|c_j|, |d_j|$ on η_j, b_j and x_0, t_0 can be derived.

For $N = \tilde{N} = 1$, we have, from (64), (65), $\det \tilde{\Omega}^a = \det \Omega^a = -1$ and

$$\det \tilde{\Omega} = (\tilde{h}_1^\circ)^{-1} - \frac{1}{(\eta_1 - b_1)^2} g_1, \quad \det \Omega = (\tilde{g}_1^\circ)^{-1} - \frac{1}{(\eta_1 - b_1)^2} h_1,$$

which can also be obtained from (51). Then $\det \tilde{\Omega} = -\det \Omega$ implies that

$$|c_1| = |\eta_1 - b_1| e^{\theta_1(x_0, t_0)}, \quad |d_1| = |\eta_1 - b_1| e^{\tilde{\theta}_1(x_0, t_0)}, \tag{67}$$

in terms of (54). In this case, we take $c_1 = |c_1| \mathrm{e}^{2i\phi_1}, d_1 = |d_1| \mathrm{e}^{2i\varphi_1}$ and $\varphi_1 - \phi_1 \neq n\pi, n \in \mathbb{N}, (-1 \leq \cos[2(\varphi_1 - \phi_1)] < 1)$, then the solution of CCSND equation (2) takes the form of

$$q(x,t) = \frac{i}{2} \frac{|\eta_1 - b_1| (\mathrm{e}^{2X} \mathrm{e}^{-2i\varphi_1} - \mathrm{e}^{2\tilde{X}} \mathrm{e}^{-2i\phi_1})}{\cosh[2(\tilde{X} - X)] - \cos[2(\varphi_1 - \phi_1)]}, \tag{68}$$

where

$$\tilde{X} = \frac{\eta_1}{2} x - \frac{t}{\eta_1} - \frac{1}{2}\left(\frac{\eta_1}{2} x_0 - \frac{t_0}{\eta_1}\right), \quad X = \frac{b_1}{2} x - \frac{t}{b_1} - \frac{1}{2}\left(\frac{b_1}{2} x_0 - \frac{t_0}{b_1}\right).$$

In addition,

$$w = 1 - 2 \frac{(\eta_1 - b_1)^2}{b_1 \eta_1} \left(\frac{\sinh(\tilde{X} - X) \cos(\varphi_1 - \phi_1) + i \cosh(\tilde{X} - X) \sin(\varphi_1 - \phi_1)}{\cosh[2(\tilde{X} - X)] - \cos[2(\varphi_1 - \phi_1)]} \right)^2, \tag{69}$$

which admits $w(x,t) = \overline{w(x_0 - x, t_0 - t)}$ in terms of

$$X(x_0 - x, t_0 - t) = -X(x,t), \quad \tilde{X}(x_0 - x, t_0 - t) = -\tilde{X}(x,t).$$

It will give a amplitude-changing single-soliton in the case of $\eta_1 b_1 < 0$, see Fig. 2 (left). While if $\eta_1 b_1 > 0$, q will be unstable and not the amplitude-changing single-soliton, but w can still give a soltion, see Fig. 2 (center and right). Furthermore, if $b_1 = -\eta_1$, (68) reduces to a single-soliton with wave train on the line

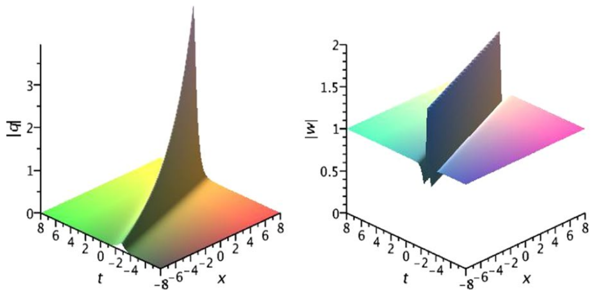

Fig. 2 A growing one-soltion (left) $|q(x,\ t)|$ in (68) and w-shapled one-soliton (center) $w(x,\ t)$ in (69) with $\eta_1 = 1, b_1 = -0.8, x_0 = t_0 = 1, \phi = 0, \varphi_1 = \pi/3$. Single soltion (right) $|w(x,\ t)|$ in (68) with $\eta_1 = 1, b_1 = -1, x_0 = t_0 = 1, \phi = \pi/6, \varphi_1 = \pi/3$

$$\frac{x}{2} + \frac{t}{\eta_1 b_1} = \frac{x_0}{4} + \frac{t_0}{2\eta_1 b_1}.$$

It can be found that the arguments of c_1 and d_1 play an important role on the wave height, for example, the wave height of $|W|$ is

$$1 + \frac{(\eta_1 - b_1)^2}{2\eta_1 b_1 \sin^2(\varphi_1 - \phi_1)}.$$

When $x_0 = t_0 = 0$, the present solutions can reduce to the solutions given in [31]. In particular, if $\varphi_1 - \phi_1 = n\pi, n \in \mathbb{N}$, then we have

$$\begin{aligned} q(x,t) &= \frac{i}{2}|\eta_1 - b_1|e^{-2i\phi_1}\mathrm{e}^{X+\tilde{X}}\mathrm{csch}(X - \tilde{X}), \\ w(x,t) &= 1 - \frac{(\eta_1 - b_1)^2}{4b_1\eta_1}\mathrm{csch}^2(\tilde{X} - X), \end{aligned} \tag{70}$$

which have singularities on the straight line

$$x + \frac{2}{\eta_1 b_1}t = \frac{1}{2}\left(x_0 + \frac{2}{\eta_1 b_1}t_0\right). \tag{71}$$

For $N = \tilde{N} = 2$, using (64), (65) or equivalent (55) and (56), we find, from $\det\tilde{\Omega}^a = -\det\Omega^a$, that

$$\begin{aligned} |c_j| &= \frac{|(\eta_1 - b_j)(\eta_2 - b_j)|}{|\eta_2 - \eta_1|}\mathrm{e}^{\theta_j(x_0,t_0)}, \\ |d_j| &= \frac{|(\eta_j - b_1)(\eta_j - b_2)|}{|b_2 - b_1|}\mathrm{e}^{\tilde{\theta}_j(x_0,t_0)}, \quad j = 1, 2, \end{aligned} \tag{72}$$

and $\det\tilde{\Omega} = \det\Omega$ implies that

$$|c_1 c_2|\mathrm{e}^{\tilde{\theta}_1(x_0,t_0)+\tilde{\theta}_2(x_0,t_0)} = |d_1 d_2|\mathrm{e}^{\theta_1(x_0,t_0)+\theta_2(x_0,t_0)},$$

and

$$\begin{aligned}
&\frac{|c_1|^2}{|d_1|^2}\mathrm{e}^{-2(\theta_1(x_0,t_0)-\tilde{\theta}_1(x_0,t_0))}=\frac{(\eta_2-b_1)^2}{(\eta_1-b_2)^2}=\frac{|d_2|^2}{|c_2|^2}\mathrm{e}^{-2(\tilde{\theta}_2(x_0,t_0)-\theta_2(x_0,t_0))},\\
&\frac{|c_2|^2}{|d_1|^2}\mathrm{e}^{-2(\theta_2(x_0,t_0)-\tilde{\theta}_1(x_0,t_0))}=\frac{(\eta_2-b_2)^2}{(\eta_1-b_1)^2}=\frac{|d_2|^2}{|c_1|^2}\mathrm{e}^{-2(\tilde{\theta}_2(x_0,t_0)-\theta_1(x_0,t_0))}.
\end{aligned}\tag{73}$$

In addition, (72) also gives to $|\eta_2-\eta_1|=|b_2-b_1|$.

For convenience, we let $c_j=|c_j|\mathrm{e}^{2i\phi_j}, d_j=|d_j|\mathrm{e}^{2i\varphi_j}$. Then we find that solution of CCSND equation (2) is

$$q(x,t)=i\frac{M^a}{M},\quad w=1+2\frac{M^b}{M}\frac{\check{M}^b}{\check{M}},\tag{74}$$

where

$$M^a=\tilde{m}_2\mathrm{e}^{-2\tilde{\vartheta}_1}+\tilde{m}_1\mathrm{e}^{-2\tilde{\vartheta}_2}-m_2\mathrm{e}^{-2\vartheta_1}-m_1\mathrm{e}^{-2\vartheta_2},\tag{75}$$

$$\begin{aligned}
M&=\mathrm{e}^{-2(\tilde{\vartheta}_1+\tilde{\vartheta}_2)}+\mathrm{e}^{-2(\vartheta_1+\vartheta_2)}\\
&\quad-p_1\left(\mathrm{e}^{-2(\tilde{\vartheta}_1+\vartheta_2)}+\mathrm{e}^{-2(\tilde{\vartheta}_2+\vartheta_1)}\right)-p_2\left(\mathrm{e}^{-2(\tilde{\vartheta}_1+\vartheta_1)}+\mathrm{e}^{-2(\tilde{\vartheta}_2+\vartheta_2)}\right),\\
\check{M}&=\mathrm{e}^{2(\tilde{\vartheta}_1+\tilde{\vartheta}_2)}+\mathrm{e}^{2(\vartheta_1+\vartheta_2)}\\
&\quad-p_1\left(\mathrm{e}^{2(\tilde{\vartheta}_1+\vartheta_2)}+\mathrm{e}^{2(\tilde{\vartheta}_2+\vartheta_1)}\right)-p_2\left(\mathrm{e}^{2(\tilde{\vartheta}_1+\vartheta_1)}+\mathrm{e}^{2(\tilde{\vartheta}_2+\vartheta_2)}\right),
\end{aligned}\tag{76}$$

and

$$\begin{aligned}
M^b&=\frac{1}{\eta_1\eta_2}(\eta_1\tilde{m}_2\mathrm{e}^{-2\tilde{\vartheta}_1}+\eta_2\tilde{m}_1\mathrm{e}^{-2\tilde{\vartheta}_2}-b_1m_2\mathrm{e}^{-2\vartheta_1}-b_2m_1\mathrm{e}^{2\vartheta_2}),\\
\check{M}^b&=\frac{1}{b_1b_2}(\eta_1\tilde{m}_2\mathrm{e}^{2\tilde{\vartheta}_1}+\eta_2\tilde{m}_1\mathrm{e}^{2\tilde{\vartheta}_2}-b_1m_2\mathrm{e}^{2\vartheta_1}-b_2m_1\mathrm{e}^{2\vartheta_2}).
\end{aligned}\tag{77}$$

Here

$$\begin{aligned}
m_j&=|c_j|\mathrm{e}^{\theta_j(x_0,t_0)}=\frac{|(\eta_1-b_j)(\eta_2-b_j)|}{|\eta_2-\eta_1|},\\
\tilde{m}_j&=|d_j|\mathrm{e}^{\tilde{\theta}_j(x_0,t_0)}=\frac{|(\eta_j-b_1)(\eta_j-b_2)|}{|b_2-b_1|},\\
\vartheta_j&=X_j-i\phi_j,\quad X_j=\frac{b_j}{2}x-\frac{1}{b_j}t-\frac{b_j}{4}x_0+\frac{1}{2b_j}t_0,\\
\tilde{\vartheta}_j&=\tilde{X}_j-i\varphi_j,\quad \tilde{X}_j=\frac{\eta_j}{2}x-\frac{1}{\eta_j}t-\frac{\eta_j}{4}x_0+\frac{1}{2\eta_j}t_0.
\end{aligned}\tag{78}$$

and the constants p_j satisfy the following relations

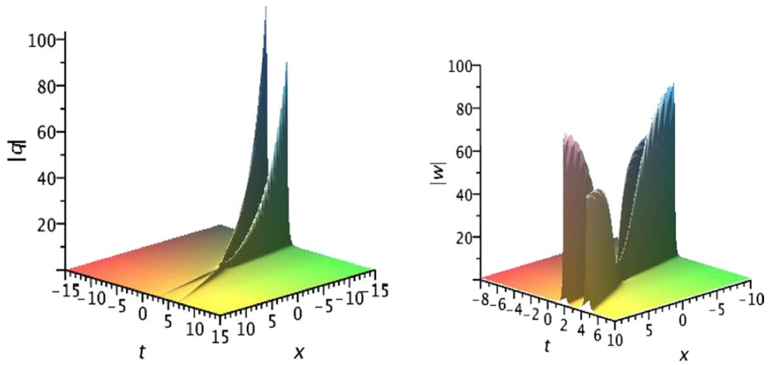

Fig. 3 Decaying two-soliton q and bright two-soliton w (74) with $\eta_1 = 0.5, \eta_2 = 0.8, b_1 = -1, b_2 = -1.3, \phi_1 = -\pi/6, \phi_2 = \pi/4, \varphi_1 = \pi/3, \varphi_2 = -\pi/4$

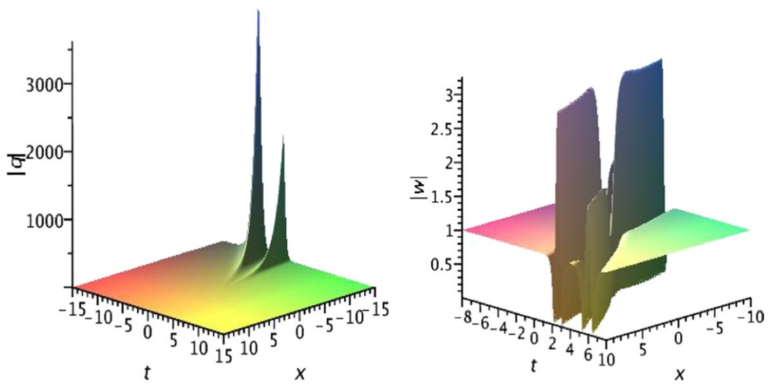

Fig. 4 Decaying two-soliton q and w-shaped two-soliton w in (74) with $\eta_1 = 0.5, \eta_2 = 0.8, b_1 = -1, b_2 = -0.7, \phi_1 = -\pi/6, \phi_2 = \pi/4, \varphi_1 = \pi/3, \varphi_2 = -\pi/4$

$$\begin{aligned} p_1 =& \frac{|d_1 c_1|}{(\eta_1 - b_1)^2} e^{-\theta_1(x_0,t_0) - \tilde{\theta}_1(x_0,t_0)} = \frac{|d_2 c_2|}{(\eta_2 - b_2)^2} e^{-\theta_2(x_0,t_0) - \tilde{\theta}_2(x_0,t_0)}, \\ p_2 =& \frac{|d_1 c_2|}{(\eta_1 - b_2)^2} e^{-\theta_2(x_0,t_0) - \tilde{\theta}_1(x_0,t_0)} = \frac{|d_2 c_1|}{(\eta_2 - b_1)^2} e^{-\theta_1(x_0,t_0) - \tilde{\theta}_2(x_0,t_0)}, \end{aligned} \tag{79}$$

or equivalently

$$p_1 = \frac{|(\eta_1 - b_2)(\eta_2 - b_1)|}{|(\eta_2 - \eta_1)(b_2 - b_1)|}, \quad p_2 = \frac{|(\eta_1 - b_1)(\eta_2 - b_2)|}{|(\eta_2 - \eta_1)(b_2 - b_1)|}.$$

The interaction of solution (74) with $\eta_2 - \eta_1 = b_2 - b_1$ is shown in Fig. 3, and $\eta_2 - \eta_1 = b_1 - b_2$ is shown in Fig. 4. In the two Figures, $q(x, t)$ is decaying two-soliton, while Fig. 3 (right) shows the interaction of bright two-soliton of $w(x, t)$ in the case of

$\eta_2 - \eta_1 = b_1 - b_2$, and Fig. 4 (right) shows the interaction of w-shaped two-soliton of $w(x, t)$ in the case $\eta_2 - \eta_1 = b_2 - b_1$.

In particular, if we take $b_1 = -\eta_2, b_2 = -\eta_1$ and $\phi_1 = \varphi_2 + \pi/2, \phi_2 = \varphi_1 + \pi/2$, then (74) reduces to two-soliton, see Fig. 5 (left). It is interesting that the interaction of the two-soliton q is inelastic.

If we take $\varphi_j = \phi_j$ and $b_j = -\eta_j, (j = 1, 2)$, The solution (74) has some singularities.

We have discussed the reduction conditions for the ECCSND equation (1) to the CCSND equation (2) in the case of $N = \tilde{N} = 1, 2$, and have given the explicit solutions of the CCSND equation (2). It is remarked that the general reduction conditions for the ECCSND equation (1) to the CCSND equation (2) can also be obtained [28]

$$|c_j|^2 = \frac{\prod_{l=1}^{N}(\eta_l - b_j)^2}{\prod_{s=1, s\neq j}^{N}(b_s - b_j)^2} e^{2\theta_j(x_0, t_0)}, \quad |d_j|^2 = \frac{\prod_{l=1}^{N}(\eta_j - b_l)^2}{\prod_{s=1, s\neq j}^{N}(\eta_s - \eta_j)^2} e^{2\tilde{\theta}_j(x_0, t_0)}, \tag{80}$$

and

$$\prod_{1\leq m<m'\leq N}(\eta_m - \eta_{m'})^2 = \prod_{1\leq l<l'\leq N}(b_{l'} - b_l)^2. \tag{81}$$

6 Conclusions and Discussions

A special linear differential system about the spectral transform matrix was introduced to define the Bi-Dbar-problem. The potentials of the ECCSND equation were recovered from the Dbar data, that is, finding two Lax pairs of the ECCSND equation with different potential matrices, by the dressing method. The conservation laws and the ECCSND equation were given. The coupled dispersionless system can be reduced from the AB system

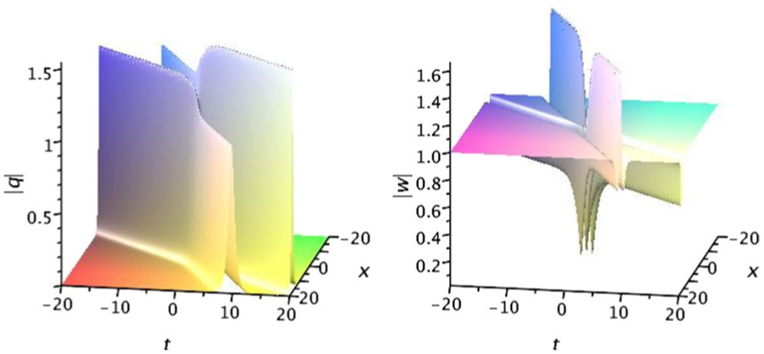

Fig. 5 Bright two-soliton $q(x, t)$ and w-shaped two-soliton $w(x, t)$ in (74) with $\eta_1 = 0.5, \eta_2 = 1.5, \beta_1 = \pi/3, \beta_2 = \pi/4$

$$\left(\frac{\partial}{\partial T}-\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial X}\right)A=\text{sgn}(\epsilon)A-\gamma AB,$$
$$\left(\frac{\partial}{\partial T}-\frac{\partial}{\partial X}\right)B=\left(\frac{\partial}{\partial T}+\frac{\partial}{\partial X}\right)|A|^2, \tag{82}$$

by certain semi-characteristic coordinates transformation. Here ϵ is a parameter measuring the state of the basic flow: the basic flow is super-critical, $\epsilon > 0$; the basic flow is sub-critical, $\epsilon < 0$. γ is a parameter that reflects the interaction of the wave packet and the mean flow. We noted that the presented CCSND equation is obtained by assuming the integral constants of $a_n^{[d]}$ in (29) to be zero. So, the CCSND system here can be regarded as sub-critical. If the integral constants are not zero, one may obtain the super-critical CCSND system, and even the variable-coefficient system [34].

The ECCSND system was derived by the symmetry condition (23), and the CCSND system was given by the nonlocal reduction. If one further assumes that $q(x, t)$ is real, the CCSND system can be reduced to the real reverse space-time shifted nonlocal sine-Gordon :

$$q_{tx}(x,t)+2w(x,t)q(x,t)=0,$$
$$w_x(x,t)-\big(q(x,t)q(x_0-x,t_0-t)\big)_t=0, \tag{83}$$

which can be further reduces to reverse space-time shift nonlocal sine-Gordon [29, 30]. The solution can be obtained by assuming the c_j and d_j in (63) to $c_j = i\tilde{c}_j, d_j = i\tilde{d}_j$, where $\tilde{c}_j, \tilde{d}_j$ are real functions of b_j and η_j as well as x_0, t_0. For example, to equation (83), the amplitude-changing one-soltion [35]

$$q=\frac{(\eta_1-b_1)}{2}\mathrm{e}^{X_1+\tilde{X}_1}\operatorname{sech}(\tilde{X}_1-X_1), \tag{84}$$

and a dark single-soliton

$$w=1+\frac{(b_1-\eta_1)^2}{2b_1\eta_1}\operatorname{sech}^2(\tilde{X}_1-X_1),\quad (b_1\eta_1<0), \tag{85}$$

where X_1 and $\tilde{X}_1$ are defined in (78). In additon, the two-solition (74) of the CCSND system reduces to that of the real reverse space-time shifted nonlocal sine-Gordon (83), which takes the following form

$$q(x,t)=\frac{2m(\eta_1\cosh 2\tilde{X}_2+\eta_2\cosh 2\tilde{X}_1)}{\cosh 2(\tilde{X}_2+\tilde{X}_1)+(p+1)\cosh 2(\tilde{X}_2-\tilde{X}_1)+p},$$
$$w(x,t)=1-\frac{8m^2(\sinh 2\tilde{X}_2+\sinh 2\tilde{X}_1)^2}{[\cosh 2(\tilde{X}_2+\tilde{X}_1)+(p+1)\cosh 2(\tilde{X}_2-\tilde{X}_1)+p]^2}, \tag{86}$$

where $\tilde{X}_j, (j=1,2)$ are defined in (78) and

$$m = \frac{\eta_2 + \eta_1}{|\eta_2 - \eta_1|}, \quad p = \frac{2\eta_1\eta_2}{(\eta_2 - \eta_1)^2} > 0.$$

The other explicit solutions can be obtained similarly from those of the CCSND system.

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Data Availability All data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no competing interests.

Ethics approval Not applicable.

Consent for publication All authors agree for publication.

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