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Bayes Estimation of Pareto Distribution Based on Type II Censored Data under the Weighted LINEX Loss Function

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Abstract: The main objective of this article is to develop a linear exponential function risks in Saudi banks (LINEXLF) to estimate the shape parameter, reliability, and hazard rate functions of the Pareto distribution based on Type II Censored Data. By weighting LINEX loss function to produce a modified loss function called weighted linear exponential (WLINEXLF) loss function. We then use WLINEXLF to derive the shape parameter, reliability, and hazard rate functions of the Pareto distribution. Furthermore, to examine the performance of the proposed method WLINEXLF we conduct a Monte Carlo simulation. The comparison is between the proposed method and other methods including maximum likelihood estimation (MLE) and Bayesian estimation under the squared error loss function. The results of the simulation show that the proposed method WLINEXLF in this article has the best performance in estimating shape parameter, reliability, and hazard rate functions, according to the smallest values of mean squared error (MSE). This result means that the proposed method can be applied in real data in banking industrial sectors.
This paper aims to use the modified loss function to estimate the shape parameter, reliability $R(t)$, and hazard rate functions $h(t)$ in Saudi banks of the Pareto distribution based on Type II Censored Data.

Keywords: Bayes, Pareto Distribution, Type II Censored Data, Weighted LINEX.

1 Introduction

The Pareto distribution is one of the important distributions in analyzing the survival times also The Pareto distribution is useful in queuing problems, actuarial modeling, biological science, demographic, and medical fields, it is often used as a model for the distribution of income. Also, the Pareto distribution is useful in queuing problems, actuarial modeling, biological science, demographic, and medical fields. It also plays an important role in economic, Business, and engineering. Many authors estimated the parameters and reliability function of The Pareto distribution using several methods, see [1-8].

Assume that n unit are placed on a lifetime test experiment, and the experiment terminates after the first r-ordered observations are recorded where $r(< n)$ units and$1 \leq r < n$. Say$x_1 < x_2 < \cdots < x_r$. Then, $\underline{x} = x_1, x_2, \dots x_r$ is called a type-II censored sample. The residual $(n-r)$ units are censored and are only known to be greater than x_r. Type-II censoring has been discussed by too many authors, among them [9-13]. The (p.d.f) of the Pareto distribution is [9].

$$f(x;\mu,\lambda) = \frac{\mu\lambda^{\mu}}{(x+\lambda)^{\mu+1}}\,; \qquad x \geq 0;\ \ (\mu > 0\,, \lambda > 0). \tag{1}$$

and the parameters c.d.f are given by

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$$F(x;\mu,\lambda) = 1 - \frac{\lambda^{\mu}}{(x+\lambda)^{\mu}}; \qquad x \geq 0,\ \ \mu > 0\,, \lambda > 0. \tag{2}$$

Therefore, the $R(t)$ and $H(t)$ are given by

$$R(t) = \frac{\lambda^{\mu}}{(t+\lambda)^{\mu}}\ ; \qquad t \geq 0. \tag{3}$$

$$H(t) = \frac{\mu}{t+\lambda}\ ; \qquad t \geq 0. \tag{4}$$

2 Proposed loss function (WINEXLF)

In this section, we will discuss the proposed loss function, which is obtained by weighting the LINEXLF as follows [14,15]:

$$L_w(\hat{\gamma}-\gamma) = w\,(\gamma)\,[exp\ \ [c\Delta] - c\Delta - 1] \qquad ; c \neq 0 \tag{5}$$

where $\Delta = (\hat{\gamma}-\gamma)$, $\hat{\gamma}$ is an estimate of γ. While $w(\gamma)$ represents the proposed weighted function, which given by

$$w(\gamma) = exp[-\omega\gamma]$$

The posterior expectation of the WLINEXLF (5) is

$$E_\gamma\big(L_w(\hat{\gamma}-\gamma)\big) = \exp[a\hat{\gamma}]\ E_\gamma(\exp[-\gamma(\omega+a)|\,x]) - a\,\hat{\gamma}E_\gamma(\exp[-\omega\gamma|x]) \quad + a\,E_\gamma(\gamma\exp[-\omega\gamma|x]) - E_\gamma(\exp[-\omega\gamma|x]) \tag{6}$$

where $E_\gamma(.)$ denoting posterior expectation with respect to the posterior density of γ

It is known that to find the value of $\hat{\gamma}$ that minimize $EL_w(\hat{\gamma},\gamma)$, we have to do the following two steps:

a. $$\frac{\partial E_\gamma L_w(\hat{\gamma},\gamma)}{\partial\hat{\gamma}} = 0$$

$$\frac{\partial E_\gamma L_w(\hat{\gamma},\gamma)}{\partial\hat{\gamma}} = a\ \ \exp[c\hat{\gamma}]\,E_\gamma(\exp[-\gamma(\omega+a)\,|\,X]) - a\,E_\gamma(\exp[-\omega\gamma\ |X]) = 0$$

So, we can find that

$$a\exp[a\hat{\gamma}]\,E_\gamma(\exp[-\gamma(\omega+a)|\,X]) = aE_\gamma\,(\exp[-\omega\gamma\,|\,X\,])$$

Consequently, the Bayes estimator $\hat{\gamma}_{WBL}$ of γ under WLINEXLF is the value $\hat{\gamma}$ which minimizes (6), it is:

$$\hat{\gamma}_{WBL} = \frac{1}{a}\,Ln\left[\frac{E_\gamma(\exp\,[-\omega\gamma]}{E_\gamma(\exp\,[-\gamma(\omega+a)]}\right] \tag{7}$$

provided that $E_\gamma = (e^{-\omega\gamma})$ and $E_\gamma = \left(e^{-(z+a)\,\gamma}\right)$ exists and finite.

b. Show that $\frac{\partial^2 E_\gamma L_w(\hat{\gamma},\gamma)}{\partial\hat{\gamma}^2} > 0$ at the minimum value computed by (a).

$$\frac{\partial^2 E_\gamma L_w(\hat{\gamma},\gamma)}{\partial\hat{\gamma}^2} = \frac{\partial}{\partial\hat{\gamma}}[\frac{\partial E_\gamma L_w(\hat{\gamma},\gamma)}{\partial\hat{\gamma}}] \ = \ a^2\exp[a\hat{\gamma}]\,E_\gamma[\exp[-\gamma(\omega+a)]] > 0$$

Since $\hat{\gamma}_{\,WLINEX}$ satisfies conditions (a) and (b) it follows that $\hat{\gamma}_{\,WLINEX}$ is the minimum value

Not: LINEXLF is a special case of WLINEXLF when $\omega = 0$ in Eq. (7).

3 Maximum Likelihood Estimator (MLE)

Let $x_1, x_2, \ldots x_n$ be a random sample of size n drawn independently from the Pareto distribution defined by (1). in type-II censoring scheme we observe only the first r order statistics. In this case, $L(\underline{x},\mu,\lambda\,)$ takes the form

$$L(\underline{x},\mu,\lambda\,) = Q\prod_{i=1}^{r} f(x_i)[R(x_r)]^{n-r}; \qquad x_1 < x_2 < \cdots < x_r \tag{8}$$

where $Q = \frac{n!}{(n-r)!}$ is a fixed does not depend on the parameters and $\underline{x} = x_1, x_2, \ldots x_r$ is the censored data?. From Eq. (8), the $L(\underline{x},\mu,\lambda\,)$ without the fixed can be written as follows

$$\ell(\underline{x},\mu,\lambda\,) = \log L = r\log\mu + n\mu\log\lambda - \Psi \tag{9}$$

where $\Psi = \mu(n-r)\log(x_r+\lambda) + (\mu+1)\sum_{i=1}^{r}\log(x_i+\lambda)$. and x_r is the time of the rth failure. Assuming that λ is known, The MLE's of μ can be obtained by solving the following equation

$$\frac{\partial\, \ell(\underline{x},\mu,\lambda)}{\partial\, \mu} = \frac{r}{\mu} + \text{n}\ \log\lambda - (n-r)\log(x_r+\lambda) - \sum_{i=1}^{r}\log(x_i+\lambda) = 0\,. \tag{10}$$

$$\hat{\mu}_{MLE} = r/\Psi_1 - \text{n}\ \log\lambda \tag{11}$$

where $\Psi_1 = (n-r)\log(x_r+\lambda) + \sum_{i=1}^{r}\log(x_i+\lambda)$.

and The MLE's of $R(t)$ and $H(t)$ are obtained from (3) and (4) after replacing μ by $\hat{\mu}_{MLE}$ as follows

$$\hat{R}(t)_{MLE} = \frac{\lambda^{\hat{\mu}_{MLE}}}{(t+\lambda)^{\hat{\mu}_{MLE}}}\ ; \qquad t \geq 0. \tag{12}$$

$$\hat{H}(t)_{MLE} = \frac{\hat{\mu}_{MLE}}{t+\lambda} \qquad ; t \geq 0. \tag{13}$$

4 Bayes Estimator

In this section, we derive the Bayes estimates of the shape parameter μ, reliability function R(t), and the hazard rate function H(t) of the Pareto distribution by using WLINEX Loss Function. If λ is known, we assume gamma (υ,δ) to be a conjugate prior distribution for μ as follows

$$\pi(\mu) = \frac{\delta^{\upsilon}}{\Gamma(\upsilon)}\mu^{\upsilon-1}\exp[-\delta\mu]\ ;\quad \delta > 0\,, \mu > 0 \tag{14}$$

Combining the likelihood function in Eq. (8) and the prior density in Eq. (14), we obtain the posterior density of μ in the form

$$\pi^*(\mu|\underline{x}) = \frac{(\mathcal{B}+\delta)^{r+\upsilon}}{\Gamma(r+\upsilon)}\mu^{r+\upsilon-1}\exp[-\mu(\mathcal{B}+\delta)] \tag{15}$$

where $\mathcal{B} = (n-r)\log(x_r+\lambda) + \sum_{i=1}^{r}\log(x_r+\lambda) - n\log\lambda$.

4.1 Bayes estimator using squared error loss function

Under SELF the Bayes estimate $\hat{\mu}_{BSE}$ of μ can be obtained as

$$\hat{\mu}_{BSE} = \int_0^{\infty}\mu\,\pi^*(\mu|\underline{x})\,d\mu$$
$$= \int_0^{\infty}\frac{(\mathcal{B}+\delta)^{r+\upsilon}}{\Gamma(r+\upsilon)}\mu^{r+\upsilon}\exp[-\mu(\mathcal{B}+\delta)]d\mu$$

$$\hat{\mu}_{BS} = \frac{\Gamma(r+\upsilon)}{\mathcal{B}+\delta} \tag{16}$$

The Bayes estimators for $R(t)$ as well as $H(t)$ using squared error loss function have specified by:

$$\hat{R}(t)_{BSE} = \int_0^{\infty}\frac{\lambda^{\mu}}{(t+\lambda)^{\mu}}\pi^*(\mu|\underline{x})\,d\mu$$
$$= \int_0^{\infty}\frac{\lambda^{\mu}}{(t+\lambda)^{\mu}}\frac{(\mathcal{B}+\delta)^{r+\upsilon}}{\Gamma(r+\upsilon)}\mu^{r+\upsilon-1}\exp[-\mu(\mathcal{B}+\delta)]d\mu$$
$$= \left[1+\frac{\log\left(1+\frac{t}{\lambda}\right)}{\mathcal{B}+\delta}\right]^{-(r+\upsilon)} \tag{17}$$

and

$$\hat{H}(t)_{BSE} = \int_0^\infty \frac{\mu}{t+\lambda}\pi^*(\mu|\underline{x})\, d\mu$$
$$= \int_0^\infty \frac{\mu}{t+\lambda}\frac{(\mathcal{B}+\delta)^{r+v}}{\Gamma(r+v)}\mu^{r+v-1}\exp[-\mu(\mathcal{B}+\delta)]d\mu$$
$$= \left[\frac{r+v}{(\mathcal{B}+\delta)(t+\lambda)}\right] \tag{18}$$

4.2 Bayes estimator using weighted LINEX loss function

Under WLINEXLF (5), where $\Delta = (\hat{\mu}-\mu)$, the Bayes estimate $\hat{\mu}_{WBL}$ of μ can be obtained by using Equation (7) as follow:

$$\hat{\mu}_{WBL} = \frac{1}{a}\, Ln\left[\frac{E_\mu(\exp[-\omega\mu]}{E_\mu(\exp[-\mu(\omega+a)]}\right] = \frac{1}{a}\log\left[\frac{\mathcal{A}_1}{\mathcal{A}_2}\right] \tag{19}$$

where :

$$\mathcal{A}_1 = E_\mu(\exp[-\omega\mu]= \int_0^\infty \exp[-\omega\mu]\,\pi^*(\mu|\underline{x})\, d\mu$$
$$= \int_0^\infty \frac{(\mathcal{B}+\delta)^{r+v}}{\Gamma(r+v)}\mu^{r+v-1}\exp[-\mu(\omega+\mathcal{B}+\delta)]d\mu$$
$$= \left(1+\frac{\omega}{\mathcal{B}+\delta}\right)^{-(r+v)} \tag{20}$$

and

$$\mathcal{A}_2 = E_\mu(\exp[-\mu(\omega+a)]= \int_0^\infty \exp[-\mu(\omega+a)]\,\pi^*(\mu|\underline{x})\, d\mu$$
$$= \left(1+\frac{\omega+a}{\mathcal{B}+\delta}\right)^{-(r+v)} \tag{21}$$

Based on WLINEX loss function (5), where $\Delta = \left(\hat{R}(\mathrm{t})-R(t)\right)$, the Bayes estimate $\hat{R}(t)_{WBL}$ of $R(t)$ can be obtained by using Eq. (7) as follow:

$$\hat{R}(\mathrm{t})_{WBL} = \frac{1}{a}\, Ln\left[\frac{E_\mu(\exp[-\omega R(t)]}{E_\mu(\exp[-(\omega+a)R(t)]}\right] = \frac{1}{a}\log\left[\frac{\mathcal{A}_3}{\mathcal{A}_4}\right] \tag{22}$$

where

$$\mathcal{A}_3 = E_\mu(\exp[-\omega R(t)]= \int_0^\infty \exp[-\omega R(t)]\,\pi^*(\mu|\underline{x})\, d\mu$$
$$= \int_0^\infty \exp[-\omega R(t)]\,\frac{(\mathcal{B}+\delta)^{r+v}}{\Gamma(r+v)}\mu^{r+v-1}\exp[-\mu(\mathcal{B}+\delta)]\, d\mu$$
$$= \sum_{i=0}^\infty \frac{(-\omega)^i}{i!}\left(1+\frac{i\,\log\left(1+\frac{t}{\lambda}\right)}{\mathcal{B}+\delta}\right)^{-(r+v)} \tag{23}$$

and

$$\mathcal{A}_4 = E_\mu(\exp[-(\omega+a)R(t)] = \int_0^\infty \exp[-(\omega+a)R(t)]\,\pi^*(\mu|\underline{x})\, d\mu$$
$$= \int_0^\infty \exp[-(\omega+a)R(t)]\,\frac{(\mathcal{B}+\delta)^{r+v}}{\Gamma(r+v)}\mu^{r+v-1}\exp[-\mu(\mathcal{B}+\delta)]\, d\mu$$
$$= \sum_{i=0}^\infty \frac{(-a-\omega)^i}{i!}\left(1+\frac{i\,\log\left(1+\frac{t}{\lambda}\right)}{\mathcal{B}+\delta}\right)^{-(r+v)} \tag{24}$$

where $\Delta = \left(\hat{H}(\mathrm{t})-H(t)\right)$, the Bayes estimate $\hat{H}(t)_{WBL}$ of $H(t)$ can be obtained by using Eq. (7) as follow:

$$\widehat{H}(\mathrm{t})_{WBL} = \frac{1}{a}\, Ln\left[\frac{E_\mu(\exp[-\omega H(t)]}{E_\mu(\exp[-(\omega+a)H(t)]}\right] = \frac{1}{a}\log\left[\frac{\mathcal{A}_5}{\mathcal{A}_6}\right] \tag{25}$$

where

$$\mathcal{A}_5 = E_\mu(\exp[-\omega H(t)] = \int_0^\infty \exp[-\omega H(t)]\,\pi^*(\mu|\underline{x})\,d\mu$$

$$= \int_0^\infty \exp[-\omega H(t)]\,\frac{(\mathcal{B}+\delta)^{r+v}}{\Gamma(r+v)}\mu^{r+v-1}\exp[-\mu(\mathcal{B}+\delta)]\,d\mu$$

$$= \left(1+\frac{\omega}{(\mathcal{B}+\delta)(t+\lambda)}\right)^{-(r+v)} \tag{26}$$

and

$$\mathcal{A}_6 = E_\mu(\exp[-(\omega+a)H(t)] = \int_0^\infty \exp[-(\omega+a)H(t)]\,\pi^*(\mu|\underline{x})\,d\mu$$

$$= \int_0^\infty \exp[-(\omega+a)H(t)]\,\frac{(\mathcal{B}+\delta)^{r+v}}{\Gamma(r+v)}\mu^{r+v-1}\exp[-\mu(\mathcal{B}+\delta)]\,d\mu$$

$$= \left(1+\frac{\omega+a}{(\mathcal{B}+\delta)(t+\lambda)}\right)^{-(r+v)} \tag{27}$$

5 Simulation Study and Comparisons

For the purpose of comparison between the different estimates used in the study the simulation was used according to the following steps:

1. For given values of prior parameters ($\delta = 2$, v =1), we generate μ =1.383 from equation (14).
2. Depending on the used value μ = 1.383 from the step above, taking λ= 2 generate$n$, (n =50, 100, 150, 200) From the pdf specified by equation (1).
3. The different estimates of μ, $R(t)$ and $H(t)$ are computed, from equations (11)- (18),
4. Steps 1 to 3 have been repetitive for 5000 times, while (MSE) for every estimate (say $\hat{\gamma}$) has computed by:
$$MSE(\hat{\gamma}) = \frac{\sum_{i=1}^{L}(\hat{\gamma}-\gamma)^2}{L}$$
where γ can take the μ, $R(t)$ and $H(t)$
5. The results are listed in tables (1- 6).

Table 1: MSEs of the Estimators of μ, with $\omega = 2$.

n	r	$\hat{\mu}_{MLE}$	$\hat{\mu}_{SE}$	$\hat{\mu}_{BL}$			$\hat{\mu}_{WBL}$		
				$a=-1$	$a=0.0001$	$a=1$	$a=-1$	$a=0.0001$	$a=1$
50	15	0.169417	0.088367	0.105879	0.088366	0.086953	0.102531	0.116316	0.077141
	25	0.094239	0.064220	0.069367	0.064219	0.061466	0.067499	0.073098	0.052983
	37	0.057901	0.044405	0.046442	0.044404	0.045232	0.045947	0.049200	0.038886
	50	0.040304	0.033330	0.035652	0.033329	0.033327	0.033992	0.036534	0.077141
100	30	0.074276	0.053095	0.060197	0.053094	0.052308	0.057576	0.059578	0.063694
	50	0.044422	0.035874	0.037217	0.035874	0.035628	0.036366	0.037247	0.037868
	75	0.027528	0.024177	0.024244	0.024176	0.023688	0.023714	0.025377	0.025170
	100	0.019651	0.017708	0.018813	0.017708	0.016996	0.018285	0.018776	0.018091
150	45	0.050031	0.040215	0.041075	0.040215	0.038768	0.038362	0.041165	0.041978
	75	0.028354	0.025075	0.024916	0.025075	0.023640	0.024501	0.024881	0.025934
	112	0.018124	0.016542	0.016786	0.016542	0.016377	0.016133	0.016918	0.017264
	150	0.012996	0.012153	0.012433	0.012153	0.012274	0.012322	0.011880	0.012718
200	60	0.034524	0.029596	0.031103	0.029596	0.029539	0.029325	0.030621	0.032922
	100	0.020749	0.018719	0.018382	0.018719	0.017832	0.018236	0.018534	0.019745

	150	0.013125	0.012284	0.012763	0.012284	0.012238	0.012041	0.012827	0.012634
	200	0.009482	0.009112	0.009554	0.009112	0.009594	0.009245	0.009134	0.009388

Table 2: MSEs of the Estimators of $R(t)$, for with $\omega = 2$

n	r	$\hat{R}(t)_{MLE}$	$\hat{R}(t)_{SE}$	$\hat{R}(t)_{BL}$			$\hat{R}(t)_{WBL}$		
				$a = -1$	$a = 0.0001$	$a = 1$	$a = -1$	$a = 0.0001$	$a = 1$
50	15	0.006241	0.006898	0.00759	0.006897	0.006456	0.005963	0.005650	0.005416
	25	0.004248	0.004461	0.004501	0.004461	0.004099	0.004112	0.003890	0.003673
	37	0.002843	0.002899	0.002977	0.002899	0.002915	0.002758	0.002778	0.002721
	50	0.002074	0.002119	0.002199	0.002119	0.002091	0.002027	0.002075	0.002050
100	30	0.003336	0.003442	0.003731	0.003442	0.003358	0.003349	0.003220	0.003053
	50	0.002129	0.002126	0.002315	0.002126	0.002192	0.002159	0.002072	0.001982
	75	0.001450	0.001464	0.001481	0.001464	0.001442	0.001400	0.001444	0.001380
	100	0.001061	0.001058	0.001133	0.001058	0.001039	0.001090	0.001067	0.001000
150	45	0.002428	0.002482	0.002548	0.002482	0.002406	0.002224	0.002258	0.002130
	75	0.001495	0.001524	0.001494	0.001524	0.001431	0.001426	0.001403	0.001423
	112	0.000981	0.000981	0.000997	0.000981	0.000983	0.000949	0.000976	0.000959
	150	0.000726	0.000726	0.000731	0.000726	0.000728	0.000724	0.000684	0.000715
200	60	0.001791	0.001835	0.001877	0.001835	0.001777	0.001708	0.001715	0.001726
	100	0.001102	0.001101	0.001108	0.001101	0.001067	0.001066	0.001056	0.001090
	150	0.00073	0.000731	0.000763	0.000731	0.000729	0.000705	0.000735	0.000714
	200	0.000537	0.000545	0.000557	0.000545	0.000568	0.000545	0.000529	0.000531

Table 3: MSEs of the Estimators of $H(t)$, with $z = 2$.

n	r	$\hat{H}_{MLE}$	$\hat{H}_{SE}$	$\hat{H}_{BL}$			$\hat{H}_{WBL}$		
				$a = -1$	$a = 0.0001$	$a = 1$	$a = -1$	$a = 0.0001$	$a = 1$
50	15	0.004706	0.002455	0.002634	0.002455	0.002455	0.002486	0.002437	0.001734
	25	0.002618	0.001784	0.001791	0.001784	0.001764	0.001792	0.001779	0.001253
	37	0.001608	0.001233	0.001232	0.001233	0.001285	0.001233	0.001235	0.000961
	50	0.00112	0.000926	0.000954	0.000926	0.000940	0.000923	0.000952	0.001734
100	30	0.002063	0.001475	0.001571	0.001475	0.001485	0.001556	0.001481	0.001470
	50	0.001234	0.000997	0.000999	0.000997	0.001009	0.000999	0.000992	0.000942
	75	0.000765	0.000672	0.000658	0.000672	0.000667	0.000650	0.000691	0.000650
	100	0.000546	0.000492	0.000513	0.000492	0.000475	0.000504	0.000504	0.000477
150	45	0.001390	0.001117	0.001097	0.001117	0.001096	0.001043	0.001064	0.001046
	75	0.000788	0.000697	0.000675	0.000697	0.000662	0.000673	0.000670	0.000682
	112	0.000503	0.000459	0.000458	0.000459	0.000458	0.000447	0.000463	0.000465
	150	0.000361	0.000338	0.000341	0.000338	0.000343	0.000341	0.000323	0.000341
200	60	0.000959	0.000822	0.000839	0.000822	0.000835	0.000816	0.000806	0.000828
	100	0.000576	0.000520	0.000503	0.000520	0.000499	0.000503	0.000502	0.000525
	150	0.000365	0.000341	0.000351	0.000341	0.000342	0.000334	0.000353	0.000343
	200	0.000263	0.000253	0.000263	0.000253	0.000267	0.000257	0.000254	0.000257

Table 4: MSEs of the Estimators of μ, with $\omega = 0.0001$.

n	r	$\hat{\mu}_{MLE}$	$\hat{\mu}_{SE}$	$\hat{\mu}_{BL}$			$\hat{\mu}_{WBL}$		
				$a = -1$	$a = 0.0001$	$a = 1$	$a = -1$	$a = 0.0001$	$a = 1$
50	15	0.169417	0.088367	0.105879	0.088366	0.086953	0.105876	0.088365	0.086953

	25	0.094239	0.064220	0.069367	0.064219	0.061466	0.069365	0.064219	0.061466
	37	0.057901	0.044405	0.046442	0.044404	0.045232	0.046441	0.044404	0.045232
	50	0.040304	0.033330	0.035652	0.033329	0.033327	0.035652	0.033329	0.033327
100	30	0.074276	0.053095	0.060197	0.053094	0.052308	0.060196	0.053094	0.052308
	50	0.044422	0.035874	0.037217	0.035874	0.035628	0.037217	0.035874	0.035628
	75	0.027528	0.024177	0.024244	0.024176	0.023688	0.024244	0.024177	0.023688
	100	0.019651	0.017708	0.018813	0.017708	0.016996	0.018813	0.017708	0.016996
150	45	0.050031	0.040215	0.041075	0.040215	0.038768	0.041074	0.040215	0.038768
	75	0.028354	0.025075	0.024916	0.025075	0.023640	0.024916	0.025075	0.023640
	112	0.018124	0.016542	0.016786	0.016542	0.016377	0.016786	0.016542	0.016377
	150	0.012996	0.012153	0.012433	0.012153	0.012274	0.012433	0.012153	0.012274
200	60	0.034524	0.029596	0.031103	0.029596	0.029539	0.031103	0.029596	0.029539
	100	0.020749	0.018719	0.018382	0.018719	0.017832	0.018382	0.018719	0.017832
	150	0.013125	0.012284	0.012763	0.012284	0.012238	0.012763	0.012284	0.012238
	200	0.009482	0.009112	0.009554	0.009112	0.009594	0.009554	0.009112	0.009594

Table 5: MSEs of the Estimators of $R(t)$, with $\omega = 0.0001$.

n	r	$\hat{R}(t)_{MLE}$	$\hat{R}(t)_{SE}$	$\hat{R}(t)_{BL}$			$\hat{R}(t)_{WBL}$		
				$a=-1$	$a=0.0001$	$a=1$	$a=-1$	$a=0.0001$	$a=1$
50	15	0.006241	0.006898	0.00759	0.006897	0.006456	0.007590	0.006897	0.006456
	25	0.004248	0.004461	0.004501	0.004461	0.004099	0.004501	0.004461	0.004099
	37	0.002843	0.002899	0.002977	0.002899	0.002915	0.002977	0.002899	0.002915
	50	0.002074	0.002119	0.002199	0.002119	0.002091	0.002199	0.002119	0.002091
100	30	0.003336	0.003442	0.003731	0.003442	0.003358	0.003731	0.003442	0.003358
	50	0.002129	0.002126	0.002315	0.002126	0.002192	0.002315	0.002126	0.002192
	75	0.001450	0.001464	0.001481	0.001464	0.001442	0.001481	0.001464	0.001442
	100	0.001061	0.001058	0.001133	0.001058	0.001039	0.001133	0.001058	0.001039
150	45	0.002428	0.002482	0.002548	0.002482	0.002406	0.002548	0.002482	0.002406
	75	0.001495	0.001524	0.001494	0.001524	0.001431	0.001494	0.001524	0.001431
	112	0.000981	0.000981	0.000997	0.000981	0.000983	0.000997	0.000981	0.000983
	150	0.000726	0.000726	0.000731	0.000726	0.000728	0.000731	0.000726	0.000728
200	60	0.001791	0.001835	0.001877	0.001835	0.001777	0.001877	0.001835	0.001777
	100	0.001102	0.001101	0.001108	0.001101	0.001067	0.001108	0.001101	0.001067
	150	0.00073	0.000731	0.000763	0.000731	0.000729	0.000763	0.000731	0.000729
	200	0.000537	0.000545	0.000557	0.000545	0.000568	0.000557	0.000545	0.000568

Table 6: MSEs of the Estimators of $H(t)$, with $\omega = 0.0001$.

n	r	$\hat{H}_{MLE}$	$\hat{H}_{SE}$	$\hat{H}_{BL}$			$\hat{H}_{WBL}$		
				$a=-1$	$a=0.0001$	$a=1$	$a=-1$	$a=0.0001$	$a=1$
50	15	0.004706	0.002455	0.002634	0.002455	0.002455	0.002634	0.002455	0.002455
	25	0.002618	0.001784	0.001791	0.001784	0.001764	0.001791	0.001784	0.001764
	37	0.001608	0.001233	0.001232	0.001233	0.001285	0.001232	0.001233	0.001285
	50	0.00112	0.000926	0.000954	0.000926	0.000940	0.000954	0.000926	0.000940
100	30	0.002063	0.001475	0.001571	0.001475	0.001485	0.001571	0.001475	0.001485
	50	0.001234	0.000997	0.000999	0.000997	0.001009	0.000999	0.000997	0.001009
	75	0.000765	0.000672	0.000658	0.000672	0.000667	0.000658	0.000672	0.000667

	100	0.000546	0.000492	0.000513	0.000492	0.000475	0.000513	0.000492	0.000475
150	45	0.001390	0.001117	0.001097	0.001117	0.001096	0.001097	0.001117	0.001096
	75	0.000788	0.000697	0.000675	0.000697	0.000662	0.000675	0.000697	0.000662
	112	0.000503	0.000459	0.000458	0.000459	0.000458	0.000458	0.000459	0.000458
	150	0.000361	0.000338	0.000341	0.000338	0.000343	0.000341	0.000338	0.000343
200	60	0.000959	0.000822	0.000839	0.000822	0.000835	0.000839	0.000822	0.000835
	100	0.000576	0.000520	0.000503	0.000520	0.000499	0.000503	0.000520	0.000499
	150	0.000365	0.000341	0.000351	0.000341	0.000342	0.000351	0.000341	0.000342
	200	0.000263	0.000253	0.000263	0.000253	0.000267	0.000263	0.000253	0.000267

The results of the simulation are listed in tables 1- 6. From the results we observe that

1. Tables 1, 2 and 3, show that the $\hat{R}(t)_{WBL}$ and $\hat{H}(t)_{WBL}$ have the smallest (MSE's) as compared with $\hat{R}(t)_{SE}$ $\hat{H}(t)_{SE}$,$\hat{R}(t)_{BL}$ $\hat{H}(t)_{BL}$and $\hat{R}(t)_{MLE}$. $\hat{H}(t)_{MLE}$. For all sample sizes and for both complete and censored samples.while the Bayes estimates relative to WLINEXLF have the smallest (MSE's) in estimate the shape parameter as compared with other methods for most sample sizes and for both complete and censored samples.
2. From all tables, the results show that there is an inverse relationship between MSE values and or r.
3. For the purpose of, studying the effect of (ω), we chose different values of ω whenever ($\omega = 0.0001$) it is close to zero, then the Bayes estimates Based on WLINEXLF are almost the same as the Bayes estimates based on LINEXLF. Therefore, it stands for one of the advantageous features of working with proposed loss functions. (See tables 4 - 6.
4. It is clear from the results that the estimates of all methods converge on increasing n, r or or both.

6 Conclusions

In this article, we developed the LINEXLF to estimate the shape parameter, $R(t)$, and $H(t)$ of the Pareto distribution based on Type II Censored Data. the development occurred by weighting LINEXLF to produce a modified loss function called the WLINXLF. After that, we used a modified loss function to estimate the shape parameter, $R(t)$, and $H(t)$ of the Pareto distribution. Finally, a comparison was made between the proposed method and other methods including MLE, BSE, BL, and WBL. The simulation results showed that the performance of the proposed method was best than the other methods used in the research.

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