## BENEMÉRITA UNIVERSIDAD AUTÓNOMA DE PUEBLA

FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS

## INFORMACIÓN CUÁNTICA CON VARIABLES CONTINUAS

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## INFORMACIÓN CUÁNTICA CON VARIABLES CONTINUAS

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## BENEMÉRITA UNIVERSIDAD AUTÓNOMA DE PUEBLA

 FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS
## PHD THESIS

## QUANTUM INFORMATION WITH CONTINUOUS VARIABLES

A thesis submitted by Julio Abraham Mendoza Fierro as partial requirement for the degree of Doctor en Ciencias.

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## ABSTRACT

This thesis identifies the model of simultaneous measurement of the position and momentum observables seminally posed by Arthurs and Kelly as a system described entirely by observables with continuous eigenspectrum; in particular, we treat the model in the regime of Gaussian states by assuming a minimum uncertainty state as the system under measurement. Under this consideration, the mathematical framework used to describe these states in quantum information processing tasks finds applicability. First, we consider the free energies of each quantum system defining the measurement setting in the measurement dynamics; then, we study how this consideration affects the retrodictive and predictive aspects of accuracy for the simultaneous measurement of the position and momentum observables of the system under examination. We find that the accuracy of the simultaneous measurement is affected by the degree of coupling between the detectors of the measurement apparatus and the system under observation.

On the other hand, we identify the measurement process as an entanglement generator in the particular regime of the Gaussian states. We study this aspect in both the completely dynamical and the strong coupling regime raised originally by Arthurs and Kelly. In particular, we research the qualitative entanglement properties of the system, categorizing it as a function of its separability properties; then, we establish the presence of genuine tripartite entanglement in the measurement setup by proving the non-separability of each mode bipartition of the system. Besides, we investigate the quantitative entanglement properties developed in the system; then, we study the amount and structure of the tripartite entanglement by two measures defined for three-mode Gaussian states; that is, the minimum residual Gaussian contangle and the residual tripartite Rényi-2 entanglement.

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## LIST OF PUBLICATIONS

This thesis collect the results of three publications

1. Mendoza-Fierro, J. A., Arévalo Aguilar, L. M., and Velázquez Aguilar, V. M. (2021). Effects of the free evolution in the Arthurs and Kelly model of simultaneous measurement and in the retrodictive predictions of the Heisenberg uncertainty relations. The European Physical Journal Plus, 136(9), 1-25.

In this work, we analyze the effect of the full dynamics of the simultaneous measurement process of position and momentum observables in the retrodictive and predictive accuracy aspects in the measurement. We find that the degree of coupling between the measuring device and system under inspection, regulate the degree of accuracy in the measurement.
2. Mendoza-Fierro J. A. and Arévalo Aguilar L.M. (2022). Gaussian tripartite entanglement in the simultaneous measurement of position and momentum. To be published.

In this article, we demonstrate the generation of genuine tripartite entanglement in the simultaneous measurement process raised seminally by Arthurs and Kelly; hence, we treat the model in the strong coupling regime. In particular, we appeal to the system under measurement as the most general single-mode Gaussian state. Besides, we analyze the quantitative structure of the generated entanglement; then, we study the amount of entanglement in all global and reduced mode bipartitions of the system.
3. Mendoza-Fierro J. A. and Arévalo Aguilar L.M. (2022). Gaussian entanglement properties in a completely dynamical ArthursKelly measurement process of the position and momentum observables.Quantum Information Processing. To be published.

## LIST OF PUBLICATIONS

In this paper, we demonstrate the generation of genuine tripartite entanglement in the completely dynamical Arthurs-Kelly measurement process; therefore, we include the free energy operator of each measurement component in the dynamics of the measurement process. In particular, we consider a squeezed vacuum state as the system under measurement. We find that the entanglement properties are not affected by the degree of coupling between the measurement apparatus and the Gaussian system under observation. Moreover, we establish the quantitative entanglement properties of the system, finding that the degree of coupling between the measurement device and the Gaussian system has a role in the amount of entanglement in the system.

## I

## CONTINUOUS VARIABLE SYSTEMS

> "[...] I don't feel frightened not knowing things, by being lost in a mysterious universe without any purpose, which is the way it really is as far as I can tell."

Richard Feynman

### 1.1 CONTINUOUS VARIABLE NON-INTERACTING BOSONIC SYSTEMS

We refer to a continuous-variable (CV) quantum system when it is defined on an infinite-dimensional Hilbert space. Besides, the proper observables of the system have continuous eigenspectra [1]. A particular example is an electromagnetic field composed of $N$ non-interacting quantum harmonic oscillators, each one with a frequency $\omega_{k}$; then, every single oscillator is termed as a mode. This system has associated a Hilbert space given by a tensor product of the form: $\mathcal{H}^{\otimes N}=\otimes_{k=1}^{N} \mathcal{H}_{k}$, where to each mode corresponds a Hilbert space $\mathcal{H}_{k}$. This system is associated with $N$ pairs of annihilation and creation operators $\left\{\hat{a}_{k}, \hat{a}_{k}^{\dagger}\right\}_{k=1}^{N}$, which satisfy the following bosonic commutation relations

$$
\begin{equation*}
\left[\hat{a}_{k}, \hat{a}_{l}^{\dagger}\right]=\delta_{k l}, \quad\left[\hat{a}_{k}, \hat{a}_{l}\right]=\left[\hat{a}_{k}^{\dagger}, \hat{a}_{l}^{\dagger}\right]=0 . \tag{1.1.1}
\end{equation*}
$$

Adopting units of $h=1$, we define the so-called quadrature phase operators for each $k$-mode as

$$
\begin{equation*}
\hat{q}_{k}=2^{-\frac{1}{2}}\left(\hat{a}_{k}+\hat{a}_{k}^{\dagger}\right), \quad \hat{p}_{k}=i 2^{-\frac{1}{2}}\left(\hat{a}_{k}^{\dagger}-\hat{a}_{k}\right) . \tag{1.1.2}
\end{equation*}
$$

The quadrature field operators represent canonical observables for each mode and act similarly to the position and momentum operators of a quantum harmonic oscillator; besides, they are observables with continuous eigenspectra; therefore, they have improper eigenstates ${ }^{1}|q\rangle$ and $|p\rangle$ such that they satisfy the following eigenvalue equations

$$
\begin{equation*}
\hat{q}_{k}|q\rangle_{k}=q_{k}|q\rangle_{k}, \quad \hat{p}_{k}|p\rangle_{k}=p_{k}|p\rangle_{k}, \tag{1.1.3}
\end{equation*}
$$

with the continuous-variable eigenvalues $q_{k}, p_{k} \in \mathbb{R}$. The two sets $\left\{|q\rangle_{k}\right\}$ and $\left\{|p\rangle_{k}\right\}$ constitute two bases for $\mathcal{H}_{k}$ and are connected by a Fourier transform

$$
\begin{align*}
|q\rangle_{k} & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} d p e^{-i q p}|p\rangle  \tag{1.1.4}\\
|p\rangle_{k} & =\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} d q e^{i q p}|q\rangle_{k} \tag{1.1.5}
\end{align*}
$$

On the other hand, we can array the quadrature operators of the $N$-mode system in a vectorial operator just as

$$
\begin{equation*}
\hat{\mathbf{R}}=\left(\hat{q}_{1}, \hat{p}_{1}, \ldots \hat{q}_{N}, \hat{p}_{N}\right)^{T}, \tag{1.1.6}
\end{equation*}
$$

whose entries satisfy the commutation relations

$$
\begin{equation*}
\left[\hat{R}_{j}, \hat{R}_{k}\right]=i \Omega_{j k}, \quad(i, j=1, \ldots, 2 N) \tag{1.1.7}
\end{equation*}
$$

where $\hat{R}_{j}$ is the $j$-component of vector $\hat{\mathbf{R}}$ and $\Omega_{i j}$ is the generic element of the $2 N \times 2 N$ matrix $\Omega$, which is given by

$$
\boldsymbol{\Omega}=\bigoplus_{k=1}^{N} \boldsymbol{w}=\left[\begin{array}{lll}
\boldsymbol{w} & &  \tag{1.1.8}\\
& \ddots & \\
& & \boldsymbol{w}
\end{array}\right], \quad \text { with } \quad \boldsymbol{w}=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

known as the symplectic form. The Hilbert space of this system is separable and infinite dimensional since a single Hilbert space $\mathcal{H}_{k}$ is spanned by a countable basis $\{|n\rangle\}_{n=0}^{\infty}$, called number state basis, composed of the eigenstates of the number operator $\hat{n}=\hat{a}^{\dagger} \hat{a}$, that is

$$
\begin{equation*}
\hat{a}|n\rangle=\sqrt{n}|n-1\rangle \quad(\text { for } \quad n \geq 1) \tag{1.1.9}
\end{equation*}
$$

[^0]\[

$$
\begin{equation*}
\hat{a}^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle \quad(\text { for } \quad n \geq 0) \tag{1.1.10}
\end{equation*}
$$

\]

The Hamiltonian of the whole system is

$$
\begin{equation*}
\hat{H}=\sum_{k=1}^{N} \hat{H}_{k}, \quad \hat{H}_{k}=\hbar w_{k}\left(\hat{a}_{k}^{\dagger} \hat{a}_{k}+1 / 2\right), \tag{1.1.11}
\end{equation*}
$$

where each $\hat{H}_{k}$ refers to the Hamiltonian of the $k^{t h}$ mode of the complete set. The Hamiltonian of each mode is bounded from bellow ensuring the stability of the system [3]; then, for each mode $k$ there exists a vacuum state $|0\rangle_{k}$ such that

$$
\begin{equation*}
\hat{a}_{k}|0\rangle_{k}=0 . \tag{1.1.12}
\end{equation*}
$$

The vacuum state of the whole $N$-mode system is denoted by $|0\rangle=$ $\otimes_{k}|0\rangle_{k}$, that is, a tensor product of the vacuum states of each mode. An alternative and overcomplete basis for each $\hat{H}_{k}$ is provided by the coherent states, which are right-eigenstates of the annihilation operator $\hat{a}_{k}$. Tensor products of coherent states for the $N$-mode system are obtained by applying the N -mode Weyl operator:

$$
\begin{equation*}
\hat{D}(\boldsymbol{\xi})=\exp \left(i \hat{\mathbf{R}}^{T} \hat{\Omega} \boldsymbol{\xi}\right) \tag{1.1.13}
\end{equation*}
$$

to the global vacuum state, that is $|\boldsymbol{\xi}\rangle=\hat{D}(\boldsymbol{\xi})|0\rangle$; where the vector

$$
\begin{equation*}
\boldsymbol{\xi}=\left(\xi_{1}, \xi_{1}^{\prime}, \cdots \xi_{k}, \xi_{k}^{\prime}\right)^{T} \tag{1.1.14}
\end{equation*}
$$

belongs to the real $2 N$-dimensional space $\Gamma=\left(\mathbb{R}^{2 N}, \hat{\Omega}\right)$ called phase space, which is equipped with a symplectic form $\hat{\Omega}$.

## Example.

For $N=1$ :
$\hat{\Omega}=\left[\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right], \quad \boldsymbol{\xi}=\left(\xi_{1}, \xi_{1}^{\prime}\right)^{T}=\left(\xi, \xi^{\prime}\right)^{T}, \quad \hat{\mathbf{x}}^{T}=\left(\hat{q_{1}}, \hat{p_{1}}\right)=(\hat{q}, \hat{p})$,
where, we have suppressed the subscript in order to simplify the notation (this is possible because we are dealing with only one mode); then

$$
\begin{equation*}
\exp \left(i \hat{\mathbf{x}}^{T} \hat{\Omega} \boldsymbol{\xi}\right)=\exp \left(i\left[\xi^{\prime} \hat{q}-\xi \hat{p}\right]\right) \tag{1.1.16}
\end{equation*}
$$

by using Eqs. (1.1.2), we have

$$
\begin{align*}
\exp \left(i\left[\xi^{\prime} \hat{q}-\xi \hat{p}\right]\right) & =\exp \left(a_{k}^{\dagger}\left[\frac{\xi+i \xi^{\prime}}{\sqrt{2}}\right]-a_{k}\left[\frac{\xi-i \xi^{\prime}}{\sqrt{2}}\right]\right) \\
& =\exp \left(a_{k}^{\dagger} \alpha-a_{k} \alpha^{*}\right) \\
& =\hat{D}_{k}(\alpha) \tag{1.1.17}
\end{align*}
$$

where we have take the coherent amplitude $\alpha=\left(\xi+i \xi^{\prime}\right) / \sqrt{2}$, so, we recover the usual expression for a single-mode displacement operator. In this manner, $D_{k}(\alpha)|0\rangle_{k}=|\alpha\rangle_{k}$

From the last example we can obtain the explicit definition for a single-mode coherent state by disentangling the displacement operator through the theorem ([4], pag. 49):

$$
\begin{align*}
e^{\hat{A}+\hat{B}} & =e^{-\frac{1}{2}[\hat{A}, \hat{B}]} e^{\hat{A}} e^{\hat{B}} \\
& =e^{\frac{1}{2}[\hat{A}, \hat{B}]} e^{\hat{B}} e^{\hat{A}} \tag{1.1.18}
\end{align*}
$$

valid for the conditions

$$
\begin{equation*}
[\hat{A}, \hat{B}] \neq 0, \quad[\hat{A},[\hat{A}, \hat{B}]]=[\hat{B},[\hat{A}, \hat{B}]]=0 \tag{1.1.19}
\end{equation*}
$$

then, identifying $\hat{A}=\alpha \hat{a}_{k}^{\dagger}$ and $\hat{B}=-\alpha^{*} \hat{a}_{k}$ and using $[\hat{A}, \hat{B}]=|\alpha|^{2}$, then

$$
\begin{equation*}
\hat{D}_{k}(\alpha)=e^{-\frac{1}{2}|\alpha|^{2}} e^{\alpha \hat{a}_{k}^{\dagger}} e^{-\alpha^{*} \hat{a}_{k}} \tag{1.1.20}
\end{equation*}
$$

Now, applying the first exponential operator to the vacuum

$$
\begin{equation*}
e^{-\alpha^{*} \hat{a}_{k}}|0\rangle_{k}=\sum_{n=0}^{\infty} \frac{\left(-\alpha^{*}\right)^{n}}{n!} \hat{a}_{k}^{n}|0\rangle_{k}=|0\rangle_{k}=|0\rangle_{k}, \tag{1.1.21}
\end{equation*}
$$

where we have used Eq. (1.1.9) and the fact that $\hat{a}_{k}^{n}|0\rangle_{k}=|0\rangle_{k}$ except for $n=0$, besides, we have developed the exponential operator in McLaurin series. Applying the second and third operator exponential operators of Eq. (1.1.20), then

$$
\begin{equation*}
e^{-\frac{1}{2}|\alpha|^{2}} e^{\alpha \hat{a}_{k}^{\dagger}}|0\rangle_{k}=e^{-\frac{1}{2}|\alpha|^{2}} \sum_{n=0}^{\infty} \frac{(\alpha)^{n}}{n!}\left(\hat{a}_{k}^{\dagger}\right)^{n}|0\rangle_{k}=e^{-\frac{1}{2}|\alpha|^{2}} \sum_{n=0}^{\infty} \frac{(\alpha)^{n}}{\sqrt{n!}}|n\rangle_{k}, \tag{1.1.22}
\end{equation*}
$$

where we have used Eq. (1.1.10) recursively to verify that $\left(\hat{a}_{k}^{\dagger}\right)^{n}|0\rangle_{k}=$ $\sqrt{n!}|n\rangle$; hence

$$
\begin{equation*}
\hat{D}(\alpha)|0\rangle_{k}=e^{-\frac{1}{2}|\alpha|^{2}} \sum_{n=0}^{\infty} \frac{(\alpha)^{n}}{\sqrt{n!}}|n\rangle_{k} . \tag{1.1.23}
\end{equation*}
$$

The coherent states belong to the particular class of CV quantum states called Gaussian states; we will review their main characteristics in Sec. 1.3. The coherent states are usually referred to as the most classicalquantum states since the mean value of the quantum electromagnetic field in these states has a classical expression. Besides, the fluctuations of the field (quantified by an adequate measure of dispersion) equals those of the vacuum; hence, they saturate the Heisenberg uncertainty relation; therefore, they are minimum uncertainty states ([4], pag. 45).

### 1.2 Phase-space Representation

The fundamental postulate of quantum representation establishes that the entire physical information about a quantum system is contained in its quantum representation [5]; the most general form is achieved through a density operator $\hat{\rho}$, which is trace-one positive operator acting on the corresponding Hilbert space. Any density operator has an equivalent representation in terms of a $s$-ordered characteristic function of the form [3]

$$
\begin{equation*}
\chi_{\hat{\rho}}^{s}(\boldsymbol{\xi})=\operatorname{Tr}[\hat{\rho} \hat{D}(\boldsymbol{\xi})] e^{s\|\boldsymbol{\xi}\|^{2} / 2} \tag{1.2.1}
\end{equation*}
$$

where $\|\cdot\|$ stands for the euclidean norm on $\mathbb{R}^{2 N}$, and $\boldsymbol{\xi} \in \Gamma=\left(\mathbb{R}^{2 N}, \boldsymbol{\Omega}\right)$, as defined in previous section. In this representation, the tensor product
structure of the Hilbert space is replaced by a direct sum structure so that the $N$-mode phase space decomposes as $\Gamma=\oplus_{k} \Gamma_{k}$, where $\Gamma_{k}=\left(\mathbb{R}^{2}, w\right)$ is the local phase space associated with the mode $k$. The family of characteristic functions is related, via complex a Fourier transform, to real quasi-probability distributions $W_{\hat{\rho}}^{s}(\boldsymbol{\xi})$

$$
\begin{equation*}
W_{\hat{\rho}}^{s}(\boldsymbol{\xi})=\int_{\mathbb{R}^{2 N}} \frac{d^{2 N} \boldsymbol{\xi}}{(\pi)^{2 N}} \exp \left(i \hat{\boldsymbol{R}}^{T} \hat{\Omega} \boldsymbol{\xi}\right) \chi_{\hat{\rho}}^{S}(\boldsymbol{\xi}) . \tag{1.2.2}
\end{equation*}
$$

These distributions are usually called "quasi-probability" functions because they do not behave like a classical probability distribution, that is, there exist quantum states for which $W_{\hat{\rho}}^{s}(\boldsymbol{\xi})$ is not a regular probability distribution in the sense that can, for example, assume negative values or be singular at determined points of the phase space. The case for $s=-1$ corresponds to the Husimi "Q-function", which constitutes the only case for which a regular behaviour arises; then, the Husimi function becomes non-negative and well behaved in the phase space. The situation for $s=1$ corresponds to the P-representation, which represent the expansion of the density operator in the over-complete basis of coherent states. On the other hand, we only consider the case for which $s=0$, thus, $\chi_{\hat{\rho}}^{s=0}(\boldsymbol{\xi})$ and $W_{\hat{\rho}}^{s=0}$ corresponds to the so-called "Wigner representation" representation [3], [1], which we simply denote by $W_{\hat{\rho}}$ and $\chi_{\hat{\rho}}(\boldsymbol{\xi})$. For an operator $\hat{A}=f\left(\hat{a}_{k}, \hat{a}_{k}^{\dagger}\right)$ given by a symmetrically ordered function of the $k^{\text {th }}$ bosonic field operator (with $k=1, \cdots, 2 N$ ), we have for its expected value

$$
\begin{equation*}
\operatorname{Tr}[\hat{\rho} \hat{A}]=\int_{\mathbb{R}^{2 N}} W_{\hat{\rho}}^{0}(\boldsymbol{\xi}) \tilde{f}(\boldsymbol{\xi}) d^{2 N} \boldsymbol{\xi} \tag{1.2.3}
\end{equation*}
$$

where $f\left(\boldsymbol{\xi}_{k}+i \boldsymbol{\xi}_{k+1}, \boldsymbol{\xi}_{k}-i \boldsymbol{\xi}_{k+1}\right)$; hence Eq. (1.2.3) entails the following properties for the trace of the density operator

$$
\begin{equation*}
\operatorname{Tr}[\hat{\rho}]=\int_{\mathbb{R}^{2 N}} W_{\hat{\rho}}^{0}(\boldsymbol{\xi}) d^{2 N} \boldsymbol{\xi}=\chi_{\hat{\rho}}^{0}(0), \tag{1.2.4}
\end{equation*}
$$

and the purity

$$
\begin{equation*}
\mu_{\hat{\rho}}=\operatorname{Tr}\left[\hat{\rho}^{2}\right]=(2 \pi)^{N} \int_{\mathbb{R}^{2 N}}\left[W_{\hat{\rho}}^{0}(\boldsymbol{\xi})\right]^{2} d^{2 N} \boldsymbol{\xi}=\int_{\mathbb{R}^{2 N}}\left|\chi_{\hat{\rho}}^{0}(\boldsymbol{\xi})\right|^{2} d^{2 N} \boldsymbol{\xi} . \tag{1.2.5}
\end{equation*}
$$

Hence the Wigner function can be written in terms of the non-normalized eigenvectors $|\boldsymbol{q}\rangle=\otimes_{k=1}^{N}|q\rangle_{k}$ of the quadrature operators (such that $\left.\hat{q}_{j}|\boldsymbol{q}\rangle=q_{j}|\boldsymbol{q}\rangle, j=1, \cdots, N\right)$ as

$$
\begin{equation*}
W_{\hat{\rho}}(\boldsymbol{q}, \boldsymbol{p})=\frac{1}{\pi^{N}} \int_{\mathbb{R}^{2 N}}\langle\boldsymbol{q}+\boldsymbol{x}| \hat{\rho}|\boldsymbol{q}-\boldsymbol{x}\rangle e^{2 i \boldsymbol{x} \cdot \boldsymbol{p}} d^{N} \boldsymbol{x}, \quad\{\boldsymbol{q}, \boldsymbol{p}\} \in \mathbb{R}^{N} . \tag{1.2.6}
\end{equation*}
$$

The marginal integral of the Wigner function over the set $\left\{p_{1}, \cdots, p_{N}, q_{1}, \cdots, q_{N-1}\right\}$, that is,

$$
\begin{equation*}
\int_{\mathbb{R}^{2 N-1}} W_{\hat{\rho}}(\boldsymbol{q}, \boldsymbol{p}) d p_{1} \cdots d p_{N} d q_{1} \cdots d q_{N-1}=\left\langle q_{N}\right| \hat{\rho}\left|q_{N}\right\rangle \tag{1.2.7}
\end{equation*}
$$

gives the probability distribution for the quadrature $q_{N}$. The most relevant quantities that characterize the quasi-probability distributions are the statistical moments [1]. The first moment is given by the displacement vector $\boldsymbol{R}$ (it must be noted the absence of the hat ${ }^{\wedge}$ ), whose components $R_{j}$ are the mean values of the quadrature field operator, Eq. (1.1.6), of the $N$-mode system, that is,

$$
\begin{equation*}
R_{j}=\langle\hat{\boldsymbol{R}}\rangle_{j} \tag{1.2.8}
\end{equation*}
$$

The first moments for arbitrary CV N-mode systems can be adjusted by local unitary operations, as instances, rotations, or displacements in phase space; therefore, leaving properties as correlations, entropy or entanglement measures invariant [3].

The second moment is built up by the covariance matrix $\mathbf{V}$, whose arbitrary element $V_{i j}$ is defined by

$$
\begin{equation*}
V_{i j}=\operatorname{Tr}\left[\hat{\rho}\left\{\Delta \hat{R}_{i}, \Delta \hat{R}_{j}\right\}\right] \tag{1.2.9}
\end{equation*}
$$

where $\Delta \hat{R}_{i}=\hat{R}_{i}-\left\langle\hat{R}_{i}\right\rangle$, and $\{\cdot\}$ denotes the anticommutator. It is important to note that the expression Eq. (1.2.9) is defined for $h=1$, other definitions can vary by the unit convention for $\hbar$. The diagonal elements of expression, Eq. (1.2.9), represent the double variance of the quadrature operators $\hat{R}_{i}$, that is,

$$
\begin{equation*}
V_{i i}=2\left(\left\langle\hat{R}_{i}^{2}\right\rangle-\left\langle\hat{R}_{i}\right\rangle^{2}\right) \tag{1.2.10}
\end{equation*}
$$

The covariance matrix is a $2 N \times 2 N$ real, symmetric, and positive definite matrix which must satisfy the Robertson-Schrödinger uncertainty relation:

$$
\begin{equation*}
\mathbf{V}+i \boldsymbol{\Omega} \geq 0 \tag{1.2.11}
\end{equation*}
$$

so that the continuous variable system represent a physical state. The Ineq. (1.2.11) comes from the semi-positive definiteness of the density operator $\hat{\rho}$ and the canonical commutation relations, Eq. (1.1.7). In section 1.3, we will review that these two moments are sufficient to describe all properties of the archetypical system of continuous variables, that is, the Gaussian states.

### 1.3 GAUSSIAN STATES

The paradigm of a continuous-variable system is entirely played by the so-called Gaussian states. They are the most familiar system appearing in many theoretical models for physical systems; the ground state of a quantized harmonic oscillator, thermal radiation coming from a black body source, the light emitted by a laser, or that emerging from non-linear optical processes as the parametric down-conversion, vibrational modes of solids, ions confined in tramps, etc. The inclination for Gaussian states in the continuous-variable quantum information scenario lies in its kindly properties in both theoretical and experimental sense; namely, they constitute the set quickly accessible and manipulable through current technology mainly from quantum optics laboratories; therefore, supplying a rapidly way for creating entanglement $[6,7,8]$, which is an elemental ingredient to do non-classical tasks, for communication protocols as quantum teleportation or quantum cryptography. On the other hand, the mathematical description of Gaussian states becomes relatively simple since it is relegated only with the first and second statistical moments, being the last aspect that only intervenes in the characterization of quantum entanglement [9]. Therefore, despite living in an infinitely dimensional Hilbert space, Gaussian states can be described by only two finite parameters. Then, for that reasons, they have been played a prominent role in several informational protocols such as quantum communication, and quantum teleportation $[7,10$, 11, 12, 13, 14], quantum key distribution and quantum cryptography $[6,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33]$. Further, they have constituted the test field to study fundamental aspects as non-separability and entanglement sharing in the most general case of the multipartite scenario, giving rise to theoretical advances to test and quantify the presence of quantum entanglement [9, 34, 35, 36, 37].

### 1.3.1 General description and definition

A Gaussian system is defined as that whose characteristic function, Eq. (1.2.1), and quasi-probability distributions, Eq. (1.2.2), have a Gaussian mathematical structure [3, 9, 23]; hence when the state has a Gaussian
shape in the phase space [38]. In this manner, a generic multivariate Gaussian function has the form

$$
\begin{equation*}
f(\boldsymbol{r})=N e^{-\frac{1}{2} \boldsymbol{r}^{T} M \boldsymbol{r}+C^{T} r} \tag{1.3.1}
\end{equation*}
$$

where $N$ stands as an adequate normalization constant, $C=\left(c_{1}, c_{2}, \ldots c_{n}\right)^{T}$ is a constant vector with real or complex components as necessary, and $\boldsymbol{M}$ is a $n \times n$ dimensional matrix which must be positive definite. As said before, a Gaussian state is completely defined through its first and second moments, this means that its characteristic and quasi-probabilistic distributions, Eqs. (1.2.1) and (1.2.2), can be expressed in terms of such moments, hence we have

$$
\begin{gather*}
\chi_{\hat{\rho}}(\boldsymbol{\xi})=\exp \left[-\frac{1}{4} \boldsymbol{\xi}^{\mathrm{T}} \boldsymbol{\Omega} \boldsymbol{V} \boldsymbol{\Omega}^{\mathrm{T}} \boldsymbol{\xi}-i(\boldsymbol{\Omega} \boldsymbol{R})^{\mathrm{T}} \boldsymbol{\xi}\right],  \tag{1.3.2}\\
W_{\hat{\rho}}(\boldsymbol{X})=\left(\pi^{N} \sqrt{\operatorname{det}(\boldsymbol{V})}\right)^{-1} \exp \left[-(\boldsymbol{X}-\boldsymbol{R})^{\mathrm{T}} \boldsymbol{V}^{-1}(\boldsymbol{X}-\boldsymbol{R})\right] \tag{1.3.3}
\end{gather*}
$$

where $\boldsymbol{\xi}, \boldsymbol{X} \in \mathbb{R}^{2 N}$, and $\boldsymbol{V}$ is the positive definite covariance matrix of the Gaussian state which must obey the uncertainty relation, Eq. (1.2.11). It is instructive to say that for an arbitrary Gaussian state with density operator $\hat{\rho}$ and first moments equal to zero, the trace of the covariance matrix is related with the average energy of each mode, therefore with the mean energy of the non-interacting Hamiltonian, Eq (1.1.11); hence

$$
\begin{equation*}
\bar{n}_{k}=\left\langle\hat{a}_{k}^{\dagger} \hat{a}_{k}\right\rangle=\frac{1}{4}[\operatorname{Tr}[\boldsymbol{V}]-2] . \tag{1.3.4}
\end{equation*}
$$

On the other hand, we recall that pure Gaussian states are the only one continuous variable systems with positive Wigner distribution in phase space [39].

### 1.3.2 Gaussian unitaries and symplectic group

Unitary transformations in Hilbert space that preserves the trace of the density operator are equivalent to symplectic transformations of the first and second moments of a Gaussian state in phase space, that is,

$$
\hat{U} \hat{\rho} \hat{U} \longrightarrow\left\{\begin{array}{l}
\boldsymbol{R}^{\prime}=\boldsymbol{S} \boldsymbol{d}  \tag{1.3.5}\\
\boldsymbol{V}^{\prime}=\boldsymbol{S} \boldsymbol{V} \boldsymbol{S}^{\mathrm{T}}
\end{array}\right.
$$

where $\boldsymbol{S}$ is a symplectic matrix belonging to the symplectic group $S p(2 N, \mathbb{R})$. The set of real symplectic matrices are defined according to

$$
\begin{equation*}
\boldsymbol{S} \boldsymbol{\Omega} \boldsymbol{S}^{\mathrm{T}}=\boldsymbol{\Omega} \tag{1.3.6}
\end{equation*}
$$

where $\boldsymbol{\Omega}$ is the symplectic form defined in Eq. (1.1.8). Hence Eq. (1.3.6) implies that symplectic operations preserve the commutation relations given by the matrix $\boldsymbol{\Omega}$; consequently, they do not modify the kinematics of the system. We recall that all symplectic matrices are always $(2 \times 2)$-dimensional invertible matrices that satisfy $\operatorname{det}[\boldsymbol{S}]=$ 1. Symplectic transformations that preserve the Gaussian nature of Gaussian states are known as Gaussian unitaries [23]; they are generated via $\hat{U}=e^{\alpha \hat{H}}$ from Hamiltonians $\hat{H}$ of second-order polynomials in the bosonic field operators. Quantum information with Gaussian states is built on symplectic transformations; however, for this work, we do not delve into such matters; the interested reader can see the reference [40].

### 1.3.3 Symplectic eigenvalues and the Williamson theorem

Williamson showed [41] that any positive-definite matrix can be expressed in diagonal form through a symplectic transformation; then, we introduce the so-called Williamson theorem:

Theorem 1. Given a $2 N \times 2 N$ positive definite matrix $\boldsymbol{V}$, there exists a symplectic matrix $\boldsymbol{S} \in S p(2 N, \mathbb{R})$ which diagonalize $\boldsymbol{V}$ according to:

$$
\boldsymbol{V}=\boldsymbol{S} \bigoplus_{k=1}^{N}\left[\begin{array}{cc}
\nu_{k} & 0  \tag{1.3.7}\\
0 & \nu_{k}
\end{array}\right] \boldsymbol{S}^{\mathrm{T}} .
$$

a proof for this theorem can be found in [42]. In Eq. (1.3.7) the $\nu_{k}$ are called the symplectic eigenvalues of the matrix $\boldsymbol{V}$; hence we define a diagonal matrix $\boldsymbol{\nu}=\operatorname{diag}\left(\nu_{1}, \cdots, \nu_{n}\right)$ which is called the symplectic spectrum of $\boldsymbol{V}$; for a physical state it must be satisfied $\nu_{k} \geq 1 \forall k$ [35], which, is in principle equivalent to the uncertainty relation given by Eq. (1.2.11). A much more convenient manner to find the spectrum $\boldsymbol{\nu}$ is from [3]

$$
\begin{equation*}
\boldsymbol{\nu}=\operatorname{Eig}_{+}(i \boldsymbol{\Omega} \boldsymbol{V}) \tag{1.3.8}
\end{equation*}
$$

where $\operatorname{Eig}_{+}(\boldsymbol{A})$ represent the diagonal matrix of positive eigenvalues of the matrix $\boldsymbol{A}$; therefore, the Eq. (1.3.8) implies that the $N$ symplectic eigenvalues can be obtained by solving the characteristic polynomial of the matrix $|i \boldsymbol{\Omega} \boldsymbol{V}|[43]$. Knowing the symplectic eigenvalues is a powerful tool to compute several informational measures as the purity which we will review in the following section.

### 1.3.4 Purity of Gaussian states

Let us assume a physical system which is prepared through a known procedure; if we want to learn a physical property about it, the laws of quantum mechanics dictate that before a measurement, we cannot predict with complete certitude the result of the test; instead, all that is at our disposal is the set of probabilities $p_{k}$ associated with the possible outputs (see [2], pag. 253). The quantity of information we gain on average about the observable of the system after the measurement is carried out is in a complementary relationship with the certainty that we have about it before the test [44], page. 500; besides, such uncertainty goes hand in hand with our knowledge about the composition of the system. In general, the quantity of information available from a quantum state (inside any interpretation) will depend on its preparation procedure and the set of probabilities $p_{k}$ related to the observable we want to know. As an instance, take the case of the Stern-Gerlach experiment. When the particle interacts with the inhomogeneous magnetic field, its associated quantum state splits into a couple of entangled components according to the two projection numbers for $s=1 / 2[45,46]$. If we block one of them, we prepare the representation of the particle either the up or dawn component; hence we have the maximal knowledge about the preparation of the system in the eigenbasis of the measured observable, any subsequent measurement of the same observable gives no more new information about the physical property in that system. In opposition, there is the case for a statistical ensemble or mixed state, where the exact composition for the spin components is unknown; then, each possible spin polarization has a probability of preparation of $p=1 / 2$; then, for any unbiased measurement of spin, we have a fifty-fifty likelihood to get both components and the gained information about the physical property of the system is maximal for each measurement. For pure quantum states,
the correspondent density matrix is represented by a projector such a way $\hat{\rho}^{2}=\hat{\rho}$, while for mixed states, the density matrix is proportional to the identity $\rho=\mathbb{I}_{N} / N$. Then, the purity (or degree of mixedness) $\mu$ is a measure of information that can quantify the degree of knowledge about the composition of a quantum state or the quantity of information available for the inspection of an observable before the measurement or the certainty about her after the test; where each interpretation of the concept will depend on the nature of the problem [47].

There exists several ways to quantify the purity of a quantum state; the first approximation is given by the trace of the correspondent density operator of the system according to

$$
\begin{gathered}
\operatorname{Tr}\left[\hat{\rho}^{2}\right]=1, \text { for pure states, } \\
\operatorname{Tr}\left[\hat{\rho}^{2}\right]<1, \text { for mixed states. }
\end{gathered}
$$

The purity of quantum states living in Hilbert spaces with $\operatorname{dim}[\mathcal{H}]=N$ varies in the range

$$
\begin{equation*}
\frac{1}{N} \leq \mu \leq 1 \tag{1.3.9}
\end{equation*}
$$

taking its minimum value for totally mixed states. Effectively, the absolute minimum is reached for continuous variable systems where $N \longrightarrow \infty$.

Concerning Gaussian states, there are several measures to characterize the purity. One of them is given by the symplectic rank $\aleph$ of the covariance matrix $\boldsymbol{V}$ of the system. The symplectic rank corresponds to the number of symplectic eigenvalues of $\boldsymbol{V}$ different from 1 ; this corresponds to the number of normal modes which are not in the vacuum state [48].

Also, we can define pure and mixed Gaussian states according to the determinant of the covariance matrix

$$
\begin{gather*}
\operatorname{det}(\boldsymbol{V})=1, \text { for pure states, }  \tag{1.3.10}\\
\operatorname{det}(\boldsymbol{V})>1, \text { for mixed states. } \tag{1.3.11}
\end{gather*}
$$

In general, the purity of a $N$-mode Gaussian state is related with the trace of its density operator by the formula

$$
\begin{equation*}
\mu=\operatorname{Tr}\left[\hat{\rho}^{2}\right]=\frac{1}{\sqrt{\operatorname{det} \boldsymbol{V}}}=\prod_{k=1}^{n}\left(\frac{1}{\nu_{k}}\right), \tag{1.3.12}
\end{equation*}
$$

where $\left\{\nu_{k}\right\}$ is the set of symplectic eigenvalues of the covariance matrix $\boldsymbol{V}$. Here we remark that the saturation of the uncertainty principle, Eq. (1.2.11), is achieved by $N$-mode Gaussian states for which $\nu_{k}=1, \forall k \in N$. The conjugate quantity $S_{L}=1-\operatorname{Tr}\left[\hat{\rho}^{2}\right]$ is the so-called linear entropy, ranging from 0 for pure states to 1 for maximally mixed states.

Another proper way to quantify the purity of a Gaussian state is the Von Neumann entropy $S_{V}$, which is defined in terms of the set $\left\{\nu_{k}\right\}$ of symplectic eigenvalues according to

$$
\begin{equation*}
S_{V}=\sum_{k=1}^{N} f\left(\nu_{k}\right) \tag{1.3.13}
\end{equation*}
$$

where

$$
\begin{equation*}
f(x)=\frac{x+1}{2} \ln \left(\frac{x+1}{2}\right)-\frac{x-1}{2} \ln \left(\frac{x-1}{2}\right) . \tag{1.3.14}
\end{equation*}
$$

Since the symplectic eigenvalues are invariants under symplectic transformations, the Von Neumann entropy of Gaussian states also is invariant under these operations. To get a step-by-step derivation of Eq. (1.3.14), see the reference [38].

On the other hand, the purity, or the lack of information about the constitution of a quantum state also can be measured by generalized entropic measures, as the Bastiaans-Tsallis entropies [49, 50, 51] $S_{p}=$ $\left(1-\operatorname{Tr}\left[\hat{\rho}^{p}\right]\right) /(p-1)$ which reduce to the linear entropy $S_{L}$ for the power $p=2$ and the Rényi entropies $[52] S_{p}^{R}=\left(\log \operatorname{Tr}\left[\hat{\rho}^{p}\right]\right) /(1-p)$. Besides, both entropies are related through [53]

$$
\begin{equation*}
\lim _{p \longrightarrow 1+} S_{p}=\lim _{p \longrightarrow 1+} S_{p}^{R}=S_{V} \tag{1.3.15}
\end{equation*}
$$

hence, the Shannon-Von Neumann entropy can be defined in terms of generalized entropies.

### 1.3.5 Entanglement of pure Gaussian states

Quantum entanglement represents one of the most characteristic features of quantum theory; in a general conception, it constitutes the lack of
individuality for each of the subsystems which make up a composite system, to give way to a holistic description for the whole system. Since its first conceptions recognized by Einstein, Podolsky, Rosen (EPR), and Schrödinger [54, 55], the entanglement concept has evolved from a striking spooky feature of quantum formalism to a helpful resource to do tangible tasks, giving rise to the advent of quantum information technologies and the relatively new research area of quantum information [44, 56]. Then, because of its resource character, finding new ways to generate and study the quantum entanglement of Gaussian states offers the direct possibility to exploit fundamental and practical applications in quantum information processing. In the following, we review tools to determine the quantitative and qualitative entanglement properties of two and three-mode Gaussian systems; this will be helpful to study the characteristics of the entanglement in the model of simultaneous measurement of position and momentum observables for a Gaussian state as the system under measurement. Also, we will review a general mathematical mechanism to detect quantum entanglement in general $(1+N)$-mode Gaussian states. This helpful tool allows to directly verify genuine tripartite entanglement in Gaussian states composed of three modes.

### 1.3.5.1 Qualification

First, let us consider the simpler case of entanglement, that is, the bipartite case. A quantum pure state composed by two systems is called separable, factorizable or non-entangled if it can not be written as tensor product of the individual states; for example, for two states $|a\rangle$ and $|b\rangle$ we mean $|a\rangle \otimes|b\rangle$. In the regime of mixed states, we have a little different situation. A bipartite mixed state is separable if it can be represented as a convex sum of pure products, that is,

$$
\begin{equation*}
\hat{\rho}=\sum_{i} p_{i}|a\rangle\langle a| \otimes|b\rangle\langle b|, \tag{1.3.16}
\end{equation*}
$$

where $\sum_{i} p_{i}=1$. It must be noted that the Eq. (1.3.16) is straightforward reduced to the pure condition with $p_{i}=1$. The condition, Eq. (1.3.16), can be arbitrarily generalized for $n$ subsystems, that is, to the Werner condition [57]:

Let $\hat{\rho}$ the density operator describing a composite CV system consisting of $n$ subsystems. The state is called non-entangled or separable through its subsystems if the density operator can be written as the convex sum:

$$
\begin{equation*}
\hat{\rho}=\sum_{i} p_{i} \hat{\rho}_{1}^{i} \otimes \cdots \otimes \hat{\rho}_{n}^{i} . \tag{1.3.17}
\end{equation*}
$$

The above condition defined for mixed states is again reduced for the pure case by taking $p_{i}=1$, resulting in the separability of the system if it is described by a tensor product of the individual states [58].

Notably, in multipartite systems, it is possible to obtain other classes of separability by considering the different $k$-splits for the subsystems composing the global state [59]; in this case, each of the parts resulting from the split is considered as a system in its own right. Therefore, we can have a non-separable state for one determined $k$-split but entangled for another $k$-split. Therefore, we need to consider all possible splits for all possible $k$ to obtain a complete classification. According to this scheme, a $N$-partite state is genuinely $N$-partite entangled if it is not separable respect to any split [59]. However, in general, there are states which can be written as a convex sum of $k$-separable states which are not separable for any split.

In particular, three parties labelled by the indexes 1, 2, and 3 are genuinely tripartite entangled if the corresponding density operator for such system cannot be written as

$$
\begin{equation*}
\hat{\rho}=P_{1} \sum_{i} p_{i} \hat{\rho}_{1}^{i} \hat{\rho}_{23}^{i}+P_{2} \sum_{j} p_{j} \hat{\rho}_{2}^{j} \hat{\rho}_{13}^{j}+P_{3} \sum_{k} p_{k} \hat{\rho}_{3}^{k} \hat{\rho}_{12}^{k} ; \tag{1.3.18}
\end{equation*}
$$

this is the so-called biseparability condition [59, 60, 61], which we will use in Section 3.5.1. In Eq. (1.3.18) we have $\sum_{m=1}^{3} P_{m}=1$ and $\sum_{n} p_{n}=1$. Besides, $\hat{\rho}_{j}^{i}$ represents the quantum density operator for the single-mode system $j$, while $\hat{\rho}_{k l}^{i}$ is the quantum density operator which encompasses the modes $k$ and $l$ (with $j, k, l \in\{1,2,3\}, j \neq k \neq l$ ) as a system in its own right. Each single product $\hat{\rho}_{j}^{i} \hat{\rho}_{k l}^{i}$ in Eq. (1.3.18) implies that the modes $k$ and $l$ are entangled, but there no exists any entanglement relation between $j$ with $k$, or $j$ with $l$; therefore, this individual pure state implies its separability respect to a mode split $j \mid k l$ which groups together the modes $k$ and $l$ in single subsystem but consider apart the mode $j$.

Then, for pure Gaussian states, we can give a qualification, i.e., a classification of the entanglement properties of the system according to the separability of all its bipartitions. For example, using this line of thinking, in reference [9] is given a complete classification for three-mode Gaussian states in five distinct categories according to the separability of each of its ( 1 vs $n$ )-mode bipartitions. Notably, there is a helpful resource for testing the separability of all (1 vs $n$ )-mode bipartitions in generic pure $1+n$-mode Gaussian states; this tool is called the positivity of partial transposition criterion. We review this statute in the following subsection.

### 1.3.5.2 The PPT criterion

Let us consider a general Gaussian state consisting of $N$ continuousvariable modes; then, this system is divided into two parties: a subsystem $A$ consisting of $m$ modes, and a subsystem $B$ composed by $n$ modes. This system have a $2(m+n) \times 2(m+n)$-dimensional covariance matrix $\boldsymbol{V}$. On the other hand, let us recall for any bipartite quantum state $\hat{\rho}$, the positivity of the partially transposed density matrix $\hat{\tilde{\rho}}$ is a necessary condition for the separability of the bipartition of the state; this is the so-called PPT, or Peres-Horodecki criterion [62]. The partial transpose of a density matrix is equivalent to the transposition of the variables of only one of the two subsystems on some basis; this action amounts to a flip in the sign of the moments belonging to the system with respect is transposing [63]; then, the transposed covariance matrix $\tilde{\boldsymbol{V}}$ differs from $\boldsymbol{V}$ by a sign flip in det $\boldsymbol{V}$. For our $(m+n)$-mode system, the transposition with respect to the partie $A$ amounts to the following covariance matrix

$$
\tilde{\boldsymbol{V}}=\boldsymbol{T} \boldsymbol{V} \boldsymbol{T}, \quad \text { where } \boldsymbol{T}=\bigoplus_{1}^{m}\left(\begin{array}{cc}
1 & 0  \tag{1.3.19}\\
0 & -1
\end{array}\right) \oplus \mathbb{I}_{2 n},
$$

where $\mathbb{I}_{2 n}$ represents the $(2 n \times 2 n)$-dimensional identity matrix. Thus, in this general bipartition, the separability of the state is only guaranteed if the partially transposed covariance matrix $\widetilde{\boldsymbol{V}}$ satisfies the uncertainty relation given by $E q$. (1.2.11), that is $\widetilde{\boldsymbol{V}}+i \boldsymbol{\Omega} \geq 0$. Remarkably, the PPT is a necessary and sufficient condition for the separability of $(1+n)$-mode Gaussian states $[63,64]$ and, notably, a special class of $(m+n)$-mode

Gaussian states known as bisymmetric. For the aims of this work, the PPT criterion will be applied for a bipartition "1 vs various" modes, therefore supplying a powerful tool for detecting quantum entanglement en these class of particular bipartitions. Notably, the PPT criterion imposes the following condition on the symplectic eigenvalues $\left\{\tilde{\nu}_{k}\right\}$ of the partially transposed covariance matrix $\tilde{\boldsymbol{V}}$ [35]

$$
\begin{equation*}
\tilde{\nu}_{k} \geq 1, \quad \forall k=1, \cdots, n . \tag{1.3.20}
\end{equation*}
$$

Alternatively to the PPT criterion, there exists an operational criterion which is independent of the PPT criterion; it constitutes a necessary and sufficient condition of all $(m+n)$-mode Gaussian states of any $m \times n$ bipartitions [9].

### 1.3.5.3 Quantification of entanglement in two-mode Gaussian states

In this subsection, we board the task for quantifying the entanglement of the simplest continuous variable Gaussian states, that is, those formed by only two modes. The covariance matrix of this kind of states is written in terms of the $2 \times 2$ matrices referred as $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, that is,

$$
\boldsymbol{\sigma}=\left(\begin{array}{cc}
\boldsymbol{\alpha} & \boldsymbol{\gamma}  \tag{1.3.21}\\
\boldsymbol{\gamma}^{T} & \boldsymbol{\beta}
\end{array}\right)
$$

By partially transposing the last covariance matrix through the definition given by Eq. (1.3.19), we can obtain the corresponding partially transposed symplectic eigenvalues according to the following formula

$$
\begin{equation*}
\tilde{\nu}_{ \pm}=\sqrt{\frac{\tilde{\Delta}(\boldsymbol{\sigma}) \mp \sqrt{\tilde{\Delta}^{2}(\boldsymbol{\sigma})-4 \operatorname{Det} \boldsymbol{\sigma}}}{2}}, \tag{1.3.22}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta(\boldsymbol{\sigma})=\operatorname{Det} \boldsymbol{\alpha}+\operatorname{Det} \boldsymbol{\beta}+2 \operatorname{Det} \boldsymbol{\gamma}, \tag{1.3.23}
\end{equation*}
$$

(see (1.4.4) for a generalization for $N$-modes) and Det $\boldsymbol{\sigma}$ are invariant under symplectic operations on the covariance matrix. Hence, since the act of partially transpose a determined covariance matrix changes the sign of the moments of the system with respect is transposing (see Section 1.3.5.2), the symplectic invariant given by Eq. (1.3.23) is transformed to

$$
\begin{equation*}
\tilde{\Delta}(\boldsymbol{\sigma})=\Delta(\tilde{\boldsymbol{\sigma}})=\operatorname{Det} \boldsymbol{\alpha}+\operatorname{Det} \boldsymbol{\beta}-2 \operatorname{Det} \boldsymbol{\gamma}, \tag{1.3.24}
\end{equation*}
$$

Thus, the PPT criterion establishes that the symplectic eigenvalue $\tilde{\nu}_{-}$ must satisfy the Eq. (1.3.20) to establish separability between the modes [65]. In the following we will review some entanglement measures for two-mode Gaussian states.

## Negativity

The negativities, first proposed in [66] and later adapted for the continuousvariable context [67, 68] is a computable entanglement measure for general Gaussian states which is defined according to

$$
\begin{equation*}
\aleph(\hat{\rho})=\frac{\|\tilde{\hat{\rho}}\|_{1}-1}{2}, \tag{1.3.25}
\end{equation*}
$$

where $\tilde{\hat{\rho}}$ represent the partially transposed density matrix and $\|\hat{\rho}\|=$ $\operatorname{Tr}|\hat{o}|=\operatorname{Tr} \sqrt{\hat{o} \hat{o}^{\dagger}}$ is the trace norm of the hermitian operator $\hat{o}$; this operation represent the sum of the singular values of the operator $\hat{o}$. The negativity $\aleph(\hat{\rho})$ is equal to the modulus of the sum of the negative eigenvalues of the $\tilde{\hat{\rho}}$, that is

$$
\begin{equation*}
\aleph(\hat{\rho})=\left|\sum_{i} \lambda_{i}\right| . \tag{1.3.26}
\end{equation*}
$$

Finally, we emphasize that negativity quantifies the degree to which the partially transposed density matrix $\tilde{\hat{\rho}}$ fails to be positive.

## Logarithmic negativity

Related to $\aleph(\hat{\rho})$, there is the so-called logarithmic negativity, defined as

$$
\begin{equation*}
E_{\aleph}(\hat{\rho})=\log \|\tilde{\rho}\|_{1} ; \tag{1.3.27}
\end{equation*}
$$

this quantity constitutes an upper bound to the distillable entanglement of the density operator $\hat{\rho}$ [69]. Both negativities are monotone under local operations and classical communications (LOCC) ${ }^{2}$, one of the principal

[^1]characteristics to be a good entanglement measure. Also it is important to recall, that for any two-mode Gaussian state the lowest symplectic eigenvalue $\tilde{\nu_{-}}$of the partially transposed covariance matrix $\boldsymbol{V}$, completely qualifies and quantifies in terms of negativities the entanglement of any two mode Gaussian state [35]; besides, for $\tilde{\nu}_{-} \geq 1$ the state is separable (non-entangled); on the other hand, as $\tilde{\nu}_{-} \longrightarrow 0$, the negativities, and therefore, the entanglement, diverge.

For a two-mode Gaussian state with associated density matrix both the negativity and the logarithmic negativity are decreasing functions of the symplectic eigenvalue $\tilde{\nu}_{-}[67,72]$, that is,

$$
\begin{gather*}
\|\tilde{\hat{\rho}}\|_{1}=\frac{1}{\tilde{\nu}_{-}} \Rightarrow \aleph(\hat{\rho})=\max \left[0, \frac{1-\tilde{\nu}_{-}}{2 \tilde{\nu}_{-}}\right],  \tag{1.3.28}\\
E_{\aleph}(\hat{\rho})=\max \left[0,-\ln \tilde{\nu}_{-}\right] . \tag{1.3.29}
\end{gather*}
$$

where $\max [a, b]=\left\{\begin{array}{l}a, \text { if } a \geq b \\ b \text { if } a<b\end{array}\right.$. The expressions given by Eqs. (1.3.28) and (1.3.29) quantify the amount by which the PPT condition (expresed by Eq. (1.3.20) is violated). Therefore the symplectic eigenvalue $\tilde{\nu}_{-}$ qualifies and quantifies in terms of negativities the amount of entanglement in two-mode Gaussian states. Besides, in the limit of $\tilde{\nu}_{-} \rightarrow 0$, both negativities grow to infinity.

## Entanglement of formation

Let us recall that any pure state is symmetric; therefore $\mu=1$ (see Eq. (1.3.12)). Hence in the case of symmetric two-mode Gaussian states, the entanglement of formation (EoF) constitute another valid entanglement measure. Let us recover the definition of EoF of an arbitrary state with associated density operator $\hat{\rho}$, this is,

$$
\begin{equation*}
\operatorname{EoF}(\hat{\rho})=\min _{p_{i},\left|\psi_{i}\right\rangle} \sum_{i} p_{i} E(|\psi\rangle), \tag{1.3.30}
\end{equation*}
$$

where the minimum is taken over all the pure states realizations of $\hat{\rho}$ :

$$
\begin{equation*}
\hat{\rho}=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| . \tag{1.3.31}
\end{equation*}
$$

the absolute minimum in the Eq. (1.3.30) is given within the set of pure two-mode Gaussian states according to

$$
\begin{equation*}
E_{F}=\max \left[0, f\left(\tilde{\nu}_{-}\right)\right], \tag{1.3.32}
\end{equation*}
$$

where

$$
\begin{equation*}
f(x)=\frac{(1+x)^{2}}{4 x} \log \left[\frac{(1+x)^{2}}{4 x}\right]-\frac{(1-x)^{2}}{4 x} \log \left[\frac{(1-x)^{2}}{4 x}\right] . \tag{1.3.33}
\end{equation*}
$$

Therefore, this quantity is a monotonically decreasing function of $\nu_{-}$. The EoF is an entanglement measure equivalent to the negativities defined in the last two subsections.

## Gaussian convex-roof extended versions

It is possible to define entanglement measures for exclusively Gaussian states by using the formalism of Gaussian entanglement measures proposed in reference [73]. The general formalism presented in that reference allows defining generic Gaussian entanglement measures through the convex roof formalism applied over pure Gaussian decompositions to any valid measure of bipartite entanglement for Gaussian states. Then, this formalism has been used in reference [65] to compute Gaussian entanglement measures for other relevant classes of two-mode Gaussian states which are non-symmetric; these states present extremal (maximal and minimal) negativity as characteristic. We do not delve into this signature; the interested reader can review the cited references. Hence, in the following section, we walk to the first step for a pure Gaussian multipartite system, that is, those formed by three subsystems; these relevant classes of states can be associated with the Arthurs-Kelly measurement process when the system under observation is also a pure Gaussian state.

### 1.4 PURE THREE-MODE GAUSSIAN STATES

In this section, we review the main properties of the simplest multipartite Gaussian continuous variable system: the three-mode (pure) Gaussian states. The properties here derived are adequate for the study of the entanglement properties in the CV Gaussian measurement setting of Arthurs and Kelly. In the subsequent, we will refer to the local symplectic
quantities and the local covariance matrices of the local modes according to the label of the single-mode $i$; in the same manner, for the reduced two-mode Gaussian states, we will refer to their covariance matrix by the double subscript labeling the pertinent modes $j k$.

### 1.4.1 General properties

Let us define the generic form of the CM of a three-mode pure Gaussian state, expressed in block form

$$
\boldsymbol{V}_{N=3}^{(p)}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{1} & \boldsymbol{\varepsilon}_{1,2} & \boldsymbol{\varepsilon}_{1,3}  \tag{1.4.1}\\
\boldsymbol{\varepsilon}_{1,2}^{T} & \boldsymbol{\sigma}_{2} & \boldsymbol{\varepsilon}_{2,3} \\
\boldsymbol{\varepsilon}_{1,3}^{T} & \boldsymbol{\varepsilon}_{2,3}^{T} & \boldsymbol{\sigma}_{3}
\end{array}\right)
$$

where all elements defining this matrix are $(2 \times 2)$-dimensional matrices; the diagonal components describe the CMs of the reduced one-mode systems, and the off-diagonal contains the correlations (quantum and classical) between them. The symplectic eigenvalues $\nu_{\mp}$ of a reduced two-mode covariance matrix $\boldsymbol{\sigma}_{i j}$ are recovered according to the formula given by Eq. (1.3.22). The general purity condition, Eq. (1.3.12) requires

$$
\begin{equation*}
\text { Det } \boldsymbol{V}_{N=3}^{(p)}=1 \text {. } \tag{1.4.2}
\end{equation*}
$$

Besides, we have

$$
\begin{equation*}
\Delta_{1,2,3}=3, \tag{1.4.3}
\end{equation*}
$$

where $\Delta_{1, \ldots, N}$ is an invariant under symplectic transformations for any number $N$ of modes defined as [43]:

$$
\begin{equation*}
\Delta_{1, \cdots, N} \equiv \sum_{j=1}^{N} \operatorname{Det} \boldsymbol{V}_{j}+2 \sum_{j<k} \operatorname{Det} \boldsymbol{\epsilon}_{j k} ; \tag{1.4.4}
\end{equation*}
$$

besides, the conditions, Eqs. (1.4.2) and (1.4.3), imply that the local purities of any ( 1 vs 2 )-mode bipartitions are equal [35]

$$
\begin{equation*}
\text { Det } \boldsymbol{V}_{i j}=\operatorname{Det} \boldsymbol{V}_{k}, \quad i \neq j \neq k \text {; } \tag{1.4.5}
\end{equation*}
$$

then, combining Eqs. (1.4.2), (1.4.3) and (1.4.5), it can be proved that (see Appendix 5.3)

$$
\begin{equation*}
\left(\Delta_{12}-\operatorname{Det} \boldsymbol{V}_{12}\right)+\left(\Delta_{13}-\operatorname{Det} \boldsymbol{V}_{13}\right)+\left(\Delta_{23}-\operatorname{Det} \boldsymbol{V}_{23}\right)=3, \tag{1.4.6}
\end{equation*}
$$

where we recall that the $\boldsymbol{V}_{i j}$ and $\Delta_{i j}$ stands as the reduced CMs and the local symplectic invariants given by Eq. (1.4.4), for the reduced two-mode states comprising the modes $i$ and $j$, the Eq. (1.4.6) implies that

$$
\begin{equation*}
\Delta_{i j}=\operatorname{Det} \boldsymbol{V}_{i j}+1, \quad \forall i, j: i \neq j \tag{1.4.7}
\end{equation*}
$$

Invoking the saturation of the uncertainty relation for two-mode Gaussian states [35]:

$$
\begin{equation*}
\Delta_{i j}-\text { Det } \boldsymbol{V}_{i j} \leq 1, \tag{1.4.8}
\end{equation*}
$$

we conclude that any reduced two-mode state of a pure three-mode Gaussian state saturates the partial uncertainty relation. The states which saturate the partial uncertainty relation are states with minimal negativity for given global and local purities [72, 74].

Another important fact emerging from Eq. (1.4.5) is that the entanglement between any pair of modes belonging to a three-mode pure Gaussian state is fully determined by the local symplectic invariants: Det $\boldsymbol{V}_{i}, i=1,2,3$, for any adequate entanglement measure. Besides, the entanglement of any $i \mid j k$-mode bipartition of a pure three-mode state can be established by the entropy of any of the reduced states, hence again by Det $\boldsymbol{V}_{i}$. Thus, for a pure Gaussian three-mode state, all entanglement properties are determined by the local symplectic invariants Det $\boldsymbol{V}_{i}[35]$, hence (by virtue of Eq. (1.3.12)) by the local single-mode purities of the system. Then, defining

$$
\begin{equation*}
a_{i} \equiv \sqrt{\operatorname{Det} \boldsymbol{\sigma}_{i}}, \quad i=1,2,3 \tag{1.4.9}
\end{equation*}
$$

the local single-mode purities are

$$
\begin{equation*}
\mu_{i}=\frac{1}{a_{i}} \tag{1.4.10}
\end{equation*}
$$

It has been proved that the $a_{i}$ are restricted to vary according to the triangular inequality [35]

$$
\begin{equation*}
\left|a_{j}-a_{k}\right|+1 \leq a_{i} \leq a_{j}+a_{k}-1 \tag{1.4.11}
\end{equation*}
$$

for any permutation of the indexes $\{i, j, k\}=\{1,2,3\}$, in order to the CM $\boldsymbol{V}_{N=3}^{(p)}$ represent a physical state. Then, the inequality, Eq. (1.4.11) fully characterizes the local single-mode symplectic eigenvalues $a_{i}$, hence
providing a full characterization of entanglement in three-mode pure Gaussian states [35]. Besides, it is important to say that the quantities given by Eq. (1.4.9) represent the local symplectic eigenvalues of the reduced single-mode covariance matrices.

In general, the CM of any Gaussian state can be simplified through local unitary symplectic operations on the individual modes; such operations do not change the informational properties of the system. These simpler forms are called standard forms. For future convenience, let us write the standard form of the CM for a pure three-mode Gaussian state

$$
\boldsymbol{V}_{s f}^{(p)}=\left(\begin{array}{cccccc}
a_{1} & 0 & b_{12}^{+} & 0 & b_{13}^{+} & 0  \tag{1.4.12}\\
0 & a_{1} & 0 & b_{12}^{-} & 0 & b_{13}^{-} \\
b_{12}^{+} & 0 & a_{2} & 0 & b_{23}^{+} & 0 \\
0 & b_{12}^{-} & 0 & a_{2} & 0 & b_{23}^{-} \\
b_{13}^{+} & 0 & b_{23}^{+} & 0 & a_{3} & 0 \\
0 & b_{13}^{-} & 0 & b_{23}^{-} & 0 & a_{3}
\end{array}\right),
$$

with

$$
\begin{align*}
b_{i j}^{ \pm} \equiv \frac{1}{4 \sqrt{a_{i} a_{j}}} & \left\{\sqrt{\left[\left(a_{i}-a_{j}\right)^{2}-\left(a_{k}-1\right)^{2}\right]\left[\left(a_{i}-a_{j}\right)^{2}-\left(a_{k}+1\right)^{2}\right]}\right. \\
& \left. \pm \sqrt{\left[\left(a_{i}+a_{j}\right)^{2}-\left(a_{k}-1\right)^{2}\right]\left[\left(a_{i}+a_{j}\right)^{2}-\left(a_{k}+1\right)^{2}\right]}\right\} . \tag{1.4.13}
\end{align*}
$$

We recall that for the compute of all entanglement properties in a three-mode Gaussian state, we only need the symplectic quantities $a_{i}$ composing the main diagonal of the matrix given by Eq. (1.4.12).

### 1.4.2 Classification

When we consider a $N$-partite system, we can classify the qualitative entanglement properties of the system according to the different groupings for the subsystems composing the entangled state. Based on this fact, in [75] it has been considered a pure 3 -qubit state $(A, B, C$ for each subsystem) and its entanglement properties based on its separability for different qubits bipartitions. Hence such scheme considers all possibilities to group the $N$ parties in $m \leq N$ subsets: that is, the groupings in which $A B, A C, B C$ are together (entangled) and, $C, B$ and $A$ being separate
(non-entangled), respectively. It must be noted that these cases are described by the three $i \mid j k$-mode bipartitions of the tripartite system. By taking the CV Gaussian analogous, the entanglement properties also can be determined according to all $i \mid j k$-mode bipartitions of the system, since the PPT criterion is a necessary and sufficient condition for the separability of all ( 1 vs $N-1$ )-mode Gaussian states as we established in subsection 1.3.5.1. According to these facts, in [9], is given a classification of five categories for pure three-mode Gaussian states according to the separability of their three $i \mid j k$-mode bipartitions:
(C1) Fully inseparable states, which are inseparable through any bipartition.
(C2) One-mode biseparable states, being separable under only one bipartition.
(C3) Two-mode biseparable states, being separable through two bipartitions.
(C4) Three-mode biseparable states, also called bound entangled states, resulting separable in all three bipartitions, but impossible to write in the form of a convex sum of pure states.
(C5) Fully separable states, which are separable in the three bipartitions and can be written as the convex sum given by Eq. (1.3.17).

Examples belonging to class (C1) are the well-known continuousvariable Greenberger-Horne-Zeilinger (GHZ) states [10, 76] (see section 1.4.3.1), and the tripartie version of the continuous-variable Einstein-Podolsky-Rosen (EPR) states (see Section 1.4.3.2), generated in [77, 78]. For class (C2), we have two-mode squeezed states and a single vacuum in the third mode. The above classification will be helpful to define the qualitative entanglement properties for the Arthurs-Kelly simultaneous measurement process for position and momentum in a Gaussian configuration.

### 1.4.3 Examples of three-mode Gaussian states

### 1.4.3.1 Continuous variable Greenberger-Horne-Zeilinger (GHZ) states

We start with the generating configuration of the GHZ states [60], this is a simple combination of three squeezed vacuum states and two optical beam splitters; see figure 1 and reference [60].

First, we have two orthogonal squeezed vacuum states combined on a first beam splitter BS1. This process generates a couple of entangled modes at the outputs; the entanglement for this two-mode system is the continuous-variable version of the Einstein-Podolsky-Rosen argument for the completion of quantum mechanics [54]. Then, one of the arms is the mode $A$ with associated quadratures vector $\hat{\boldsymbol{R}}_{a}$. Now, the second arm of the beam splitter BS1 is interfered with another vacuum squeezed state into a second beam splitter BS2. The squeeze parameters $r_{j}, j=1,2,3$ of each vacuum state are equal; then, we have $\left|r_{1}\right|=\left|r_{2}\right|=\left|r_{3}\right|$. Besides, it is important to note that, in all the above, we are considering the ideal case of only pure states.

Now, let us going more quantitatively across the diagram presented in figure 1. First, each squeezed vacuum state is represented according to [79]

$$
\begin{equation*}
\left|0, V_{j}\right\rangle_{p}=\left(\pi V_{j}\right)^{-\frac{1}{4}} \int d x_{j} e^{-x_{j}^{2} / 2 V_{j}}\left|x_{j}\right\rangle, \quad j=1,2,3 \tag{1.4.14}
\end{equation*}
$$

where in the following, the labels $j=1,2,3$ denote the first, the two, and the third inputs states. The associated variance of the state given by Eq. (1.4.14) is $\sigma_{q_{s}}^{2}=V_{s} / 2$. Through a Fourier transform we obtain the state in momentum representation; then, it is trivial to verify the variance $\sigma_{p_{s}}^{2}=2 / V_{s}$. For the states given by Eq. (1.4.14) we have the saturation of the Heisenberg uncertainty relation $\sigma_{q_{s}}^{2} \sigma_{p_{s}}^{2}=1$ (units free $\hbar=2$ ). Therefore, if $\sigma_{q_{s}}^{2}<1$, we have squeezing in the variance of the position quadrature; however, if $\sigma_{p_{s}}^{2}<1$, we have squeezing in the variance of the momentum quadrature. Then, for the
first input of figure 1 we have $\sigma_{x_{1}}^{2}<1$, while for the second input we have $\sigma_{p_{2}}^{2}<1$.

The production of squeezed vacuum states can be carried out by the spontaneous parametric down conversion method in the degenerated regime [80]. In this process a powerful laser is irradiated on a second-


Figure 1: Scheme sketching the generation of the (finite squeezing) continuous variable GHZ states. In the diagram, BS1 and BS2 represent two (phase-free) beam splitters with reflectivities $\mathcal{R}_{1}=2 / 3\left(\mathcal{T}_{1}=1 / 3\right)$ and $\mathcal{R}_{2}=1 / 2\left(\mathcal{T}_{2}=1 / 2\right)$. We have orthogonal quadrature-phase amplitudes $\left(\hat{x}_{j}, \hat{p}_{j}\right)$ for each optical mode, which are given by Eqs. (1.4.19) and (1.4.20). The $x-p$ axis and the ellipses depict the orientation of squeezing.
order nonlinear optical medium; then, single photon of frequency $2 \omega$ may split into pairs of photons with frequency $\omega$, which are indistinguishable in frequency, direction and polarization. The quantum state representing the photon pairs exhibits squeezing. The corresponding Gaussian unitary to this process is the one-mode squeezing operator, which is

$$
\begin{equation*}
S(r)=\exp \left\{\left[\hat{a}^{2}-\left(\hat{a}^{\dagger}\right)^{2}\right] / 2\right\}, \tag{1.4.15}
\end{equation*}
$$

being $r \in \mathbb{R}$ the squeezing factor, governing the degree of squeezing of the state, in the Heisenberg picture the annihilation and creation operators, $\hat{a}$ and $\hat{a}^{\dagger}$ of a single-mode, are transformed according to

$$
\begin{gather*}
\hat{a}^{\prime} \longrightarrow(\cosh r) \hat{a}-(\sinh r) \hat{a}^{\dagger}  \tag{1.4.16}\\
\hat{a}^{\dagger} \longrightarrow(\cosh r) \hat{a}^{\dagger}-(\sinh r) \hat{a} . \tag{1.4.17}
\end{gather*}
$$

Besides, arranging the quadrature operators $\hat{x}$ and $\hat{p}$ of an arbitrary quantum state in the quadratures vector $\hat{\boldsymbol{R}}=(\hat{x}, \hat{p})^{T}$, the transformation carried out by the unitary operator, Eq. (1.4.15), on $\hat{\boldsymbol{R}}$ is given by $\boldsymbol{S}(r) \hat{\boldsymbol{R}}$, where

$$
\boldsymbol{S}(r)=\left(\begin{array}{cc}
e^{-r} & 0  \tag{1.4.18}\\
0 & e^{r}
\end{array}\right)
$$

Therefore, we have the quadratures

$$
\begin{align*}
& \hat{x}_{j}=e^{-r_{j}} \hat{x}_{0}^{j},  \tag{1.4.19}\\
& \hat{p}_{j}=e^{+r_{j}} \hat{p}_{0}^{j}, \tag{1.4.20}
\end{align*}
$$

with $j=1,2,3$, for the squeezed vacuum states given in Eq. (1.4.14). Besides, $\hat{x}_{0}^{j}$ and $\hat{p}_{0}^{j}$ represent the initial position and momentum quadratures of the vacuum state $|0\rangle$. Taking the variance $\sigma_{\hat{A}}^{2}=\left\langle\hat{A}^{2}\right\rangle-\langle\hat{A}\rangle^{2}$ of the observable $\hat{A}$, we verify the variances

$$
\begin{align*}
& \sigma_{\hat{x}_{j}}^{2}=e^{-2 r_{j}} \sigma_{\hat{x}_{0}^{j}}^{2}  \tag{1.4.21}\\
& \sigma_{\hat{p}_{j}}^{2}=e^{+2 r_{j}} \sigma_{\hat{p}_{0}^{j}}^{2} . \tag{1.4.22}
\end{align*}
$$

On the other hand, we recall the Gaussian unitary associated with the beam splitter transformation [23]; the operator associated with this simple interferometer is

$$
\begin{equation*}
B(\theta)=\exp \left[\theta\left(\hat{a}^{\dagger} \hat{b}-\hat{a} \hat{b}^{\dagger}\right)\right] \tag{1.4.23}
\end{equation*}
$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators of the two input modes. Besides, $\theta$ is related with the transmittance of the beam splitter: $\mathcal{T}=$ $\cos ^{2} \theta \in[0,1]$, and with the reflectance according to $\mathcal{R}=\sin ^{2} \theta \in[0,1]$. Then, in the Heisenberg picture, the annihilation operators are transformed via the following unitary transformation

$$
\binom{\hat{a}^{\prime}}{\hat{b}^{\prime}} \rightarrow\left(\begin{array}{cc}
\sqrt{1-\mathcal{T}} & \sqrt{\mathcal{T}}  \tag{1.4.24}\\
-\sqrt{\mathcal{T}} & \sqrt{1-\mathcal{T}}
\end{array}\right)\binom{\hat{a}}{\hat{b}},
$$

besides, the quadrature operator $\hat{\boldsymbol{R}}=\left(\hat{q}_{a}, \hat{p}_{a}, \hat{q}_{b}, \hat{p}_{b}\right)^{T}$ is transformed via

$$
\hat{\boldsymbol{R}}^{\prime}=\hat{\boldsymbol{B}}(\mathcal{T}) \hat{\boldsymbol{R}}, \text { with } \hat{\boldsymbol{B}}(\mathcal{T})=\left(\begin{array}{cc}
\sqrt{1-\mathcal{T}} \hat{\boldsymbol{I}} & \sqrt{\mathcal{T}} \hat{\boldsymbol{I}} \hat{\mathrm{T}} \hat{\boldsymbol{I}} \tag{1.4.25}
\end{array}\right)
$$

then, following the diagram depicted by Fig. 1, we associate a pair of orthogonal quadrature operators $\hat{x}_{1}, \hat{p_{1}}$ and $\hat{x_{2}}, \hat{p_{2}}$ to the inputs 1 and 2 respectively; these pairs are given by means of Eqs. (1.4.19) and (1.4.20) with the squeezing factors $r_{1}$ and $r_{2}$ adjusted according to the diagram of figure 1. Then, we have the quadratures vector $\boldsymbol{R}=\left(\hat{x}_{1}, \hat{p}_{1}, \hat{x}_{2}, \hat{p}_{2}\right)^{T}$. Hence, following the linear transformation, Eq. (1.4.25), we have the quadrature operators $\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}$ and $\hat{x}_{a}^{\prime}, \hat{p_{a}}$ for the outputs 1 and 2 as

$$
\begin{align*}
& \hat{x}_{1}^{\prime}=\sqrt{1-\mathcal{T}_{1}} \hat{x}_{1}+\sqrt{\mathcal{T}_{1}} \hat{x}_{2},  \tag{1.4.26}\\
& \hat{p}_{1}^{\prime}=\sqrt{1-\mathcal{T}_{1}} \hat{p}_{1}+\sqrt{\mathcal{T}_{1}} \hat{p}_{2},  \tag{1.4.27}\\
& \hat{x}_{a}^{\prime}=\sqrt{1-\mathcal{T}_{1}} \hat{x}_{2}-\sqrt{\mathcal{T}_{1}} \hat{x}_{1},  \tag{1.4.28}\\
& \hat{p}_{a}^{\prime}=\sqrt{1-\mathcal{T}_{1}} \hat{p}_{2}-\sqrt{\mathcal{T}_{1}} \hat{p}_{1} . \tag{1.4.29}
\end{align*}
$$

Now, we take the output 1 of the first beam splitter and we combine it with another vacuum squeezed state with associated quadratures $\hat{x}_{3}, \hat{p}_{3}$; the orientation of squeezing is equal to the presented by input 2 . Then, we have the following quadratures for the outputs 3 and 4 respectively as

$$
\begin{align*}
& \hat{x}_{b}=\sqrt{1-\mathcal{T}_{2}} \hat{x}_{3}-\sqrt{\mathcal{T}_{2}}\left(\sqrt{1-\mathcal{T}_{1}} \hat{x}_{1}+\sqrt{\mathcal{T}_{1}} \hat{x}_{2}\right),  \tag{1.4.30}\\
& \hat{p}_{b}=\sqrt{1-\mathcal{T}_{2}} \hat{p}_{3}-\sqrt{\mathcal{T}_{2}}\left(\sqrt{1-\mathcal{T}_{1}} \hat{p}_{1}+\sqrt{\mathcal{T}_{1}} \hat{p}_{2}\right),  \tag{1.4.31}\\
& \hat{x}_{c}=\sqrt{1-\mathcal{T}_{2}}\left(\sqrt{1-\mathcal{T}_{1}} \hat{x}_{1}+\sqrt{\mathcal{T}_{1}} \hat{x}_{2}\right)+\sqrt{\mathcal{T}_{2}} \hat{x}_{3},  \tag{1.4.32}\\
& \hat{p}_{c}=\sqrt{1-\mathcal{T}_{2}}\left(\sqrt{1-\mathcal{T}_{1}} \hat{p}_{1}+\sqrt{\mathcal{T}_{1}} \hat{p}_{2}\right)+\sqrt{\mathcal{T}_{2}} \hat{p}_{3}, \tag{1.4.33}
\end{align*}
$$

where we have left the explicit dependency in the $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ parameters. We recall that the initial quadratures $\hat{x}_{j}$ and $\hat{p}_{j}$ of each squeezed vacuum state, are given by Eqs. (1.4.19) and (1.4.20) with the squeezing factors $r_{j}$ properly adjusted. In the limit of infinite squeezing: $r_{j} \longrightarrow \infty$, the GHZ-states have the non-normalizable representation $\int d x_{i}\left|x_{1}\right\rangle\left|x_{2}\right\rangle\left|x_{3}\right\rangle$, which is a simultaneous eigenstate of the total momentum of the system: $\hat{p}_{1}+\hat{p}_{2}+\hat{p}_{3}$ and a simultaneous eigenstates of all relative positions $\hat{x}_{i}-\hat{x}_{j}$ with zero eigenvalues. [81].

The GHZ-states are a particular example of three-mode Gaussian states which are invariant under the exchange of any two modes in the covariance matrix; that is, they are fully symmetric. Therefore, they are bisymmetric for any $(1 \times 2)$-mode bipartition, which implies that each bipartite entanglement is locally equivalent to two-mode entanglement. In particular, the pure GHZ-states have a covariance matrix of the form

$$
\boldsymbol{\sigma}_{s}^{p}=\left(\begin{array}{ccc}
\boldsymbol{\alpha} & \varepsilon & \varepsilon  \tag{1.4.34}\\
\varepsilon^{T} & \boldsymbol{\alpha} & \varepsilon \\
\varepsilon^{T} & \varepsilon^{T} & \boldsymbol{\alpha}
\end{array}\right)
$$

where the local symplectic quantities $\boldsymbol{\alpha}=a \mathbb{I}_{2 \times 2}$ (with $a$ given by Eq. (1.4.9)) are the same for all the three modes and we have the limit situation of infinite squeezing $(a \longrightarrow \infty)$. Besides, we have $\boldsymbol{\varepsilon}=\operatorname{diag}\left\{e^{+}, e^{-}\right\}$, where

$$
\begin{equation*}
e^{ \pm}=\frac{a^{2}-1 \pm \sqrt{\left(a^{2}-1\right)\left(9 a^{2}-1\right)}}{4 a} . \tag{1.4.35}
\end{equation*}
$$

In the finite squeezing situation, the minimum residual Gaussian contangle, quantifying the tripartite entanglement (see Section 1.4.4.2 for a review of this measure of tripartite entanglement), takes the simpler form

$$
\begin{equation*}
G_{\tau}^{\mathrm{res}}=\operatorname{arcsinh}^{2}\left[\sqrt{a^{2}-1}\right]-\frac{1}{2} \ln ^{2}\left[\frac{3 a^{2}-1-\sqrt{9 a^{4}-10 a^{2}+1}}{2}\right] . \tag{1.4.36}
\end{equation*}
$$

It is interesting to note that the (with finite and infinite squeezing) pure GHZ-states are completely inseparable for the three $1 \times 2$-mode bipartitions; therefore, they exhibit genuine tripartite entanglement [82, 83, 84]; see Section 1.4.2.

The GHZ states have been created experimentally following the schematic of Figure 1. The three independent squeezed vacuum states are
generated through three subthreshold degenerate parametric oscillators. The detection of tripartite entanglement is made by recurring to the van Loock-Furusawa inequalities [83] through measurements with homodyne detection. Even, notably, it is possible to extend the scheme of Figure 1 to create $N$-mode entanglement for a network of $N-1$ beamsplitters with $N$ input [77].

### 1.4.3.2 Continuous variable Einstein-Podolsky-Rosen (EPR) states

We review the configuration for the generation of the tripartite version of the continuous variable (EPR)-type states [60]; this carries a modification for the one employed by the GHZ-states in Figure 1: in this scheme, the third input vacuum state is substituted by a coherent state, which we obtain by displacing the vacuum state by the single-mode displacement operator, Eq. (1.1.17). Therefore, we have the quadratures for the third input as

$$
\begin{align*}
& \hat{x}_{3}=\hat{x}_{0}^{(3)}+x_{0}  \tag{1.4.37}\\
& \hat{p}_{3}=\hat{p}_{0}^{(3)}+p_{0} \tag{1.4.38}
\end{align*}
$$

being $x_{0}$ and $p_{0}$ the centres of the position and momentum variables; then, it is trivial to verify the mean values $\left\langle x_{0}\right\rangle$ and $\left\langle p_{0}\right\rangle$. It is important to note that the treatment for the two-beam splitter transformations explained in Section 1.4.3.1 remains valid in this scheme; therefore, the general results of Eqs. (1.4.30) to (1.4.33) describe the quadratures of the outputs for the sketch of Figure 2 with the proper substitution of Eqs. (1.4.37) and (1.4.38). The entanglement involving $N>3$ modes using these states has been studied in References [77, 78]. Like the GHZ states, the tripartite version of the EPR states exhibits genuine tripartite entanglement between the three modes of the system [60].


Figure 2: Scheme sketching the generation of the continuous variable tripartite version of the EPR-type states. In the diagram, BS1 and BS2 represent two (phasefree) beam splitters with reflectivities $\mathcal{R}_{1}=2 / 3\left(\mathcal{T}_{1}=1 / 3\right)$ and $\mathcal{R}_{2}=1 / 2$ $\left(\mathcal{T}_{2}=1 / 2\right)$. We have the first two orthogonal quadrature phase amplitudes $\left(\hat{x}_{1,2}, \hat{p}_{1,2}\right)$ for each optical mode, which are given by Eqs. (1.4.19) and (1.4.20). The $x-p$ axis and the ellipses depict the orientation of squeezing.

### 1.4.4 Entanglement measures

Here we review, in particular, two adequate entanglement measures developed for three-mode Gaussian states: (1) the minimum residual Gaussian contangle [35] and (2) the residual tripartite Rényi-2 entanglement [3]. These measures derive from the essential constraint of monogamy which we will specifically review in Section 1.4.4.1. Besides this condition, these entanglement quantifiers obey the following necessary conditions for a bona fide entanglement measure [85]:
(i) They are zero if and only if the state is non-entangled.
(ii) They are monotone functions under local operations and classical communications (LOCC).
(iii) They no increase, on average, under LOCC.

The condition ( $i$ ) derives from the fact that separable (pure) states do not contain some degree of entanglement; specifically, they can not be purified employing LOCC to maximally entangled states [85]. The condition (ii) comes from the fact that unitary transformations on the individual subsystems only represent a change of basis, leaving the existent quantum correlations invariant. The condition (iii) comes from the reason that any increase in correlations under LOCC should be classical; hence the entanglement is not increased.

### 1.4.4.1 The Coffman-Kundu-Wootters (CKW) inequality

Quantum correlations can not be freely distributed among the multiple parts of a multipartite system. This fact is commonly called monogamy constraint, and it has first demonstrated by the seminal work of Coffman, Kundu, and Wooters for the discrete case of three qubits $A, B$ and $C$ [86]. Through the squared concurrence (tangle) as entanglement measure, they obtain the inequality:

$$
\begin{equation*}
\mathcal{C}_{A \mid B C}^{2} \geq \mathcal{C}_{A \mid B}^{2}+\mathcal{C}_{B \mid C}^{2} \tag{1.4.39}
\end{equation*}
$$

which essentially means that the entanglement (quantified by the tangle) between the qubit $A$ and the system composed by $B$ and $C$ (denoted as $B C$ ) is greater or equal than the sum of the entanglements of the reduced qubit systems $A \mid B$ and $B \mid C$. This monogamy constraint was subsequently proved valid for any pure or mixed systems with $N$-qubits [87]. Then, we refer to the monogamous inequality or monogamy of an $N$ partite system in a general context as

$$
\begin{equation*}
E^{j_{i} \mid\left(j_{1} \cdots j_{N-1}\right)} \geq \sum_{l=2}^{N-1} E^{j_{i} \mid j_{l}}, \tag{1.4.40}
\end{equation*}
$$

where $E$ stands as an adequate entanglement measure for any pure or mixed, and discrete or continuous variable system; besides, the inequality, Eq. (1.4.40), is taked for a reference partie $j_{i}$ from the whole set $\left\{j_{1}, j_{l}, \cdots, j_{N-1}\right\}$, where each $j_{i}$ contains a single system. The monogamy constraint given by Eq. (1.4.40) is naturally extended for infinitely dimensional continuous variable systems; in particular, the inequality of Eq. (1.4.40) has been adapted to the CV context of three-mode Gaussian states $[34,35]$ sing the analog of the tangle, that is the "contangle", defined as the square of the logarithmic negativity, Eq. (1.3.27). Besides, the generalization to the general case of N -mode CV Gaussian states has been made based on the symplectic analysis of covariances matrices and the properties of Gaussian monotones of entanglement [37].

The verification for the monogamy inequality in the continuous variable Gaussian context is vitally important to define correct entanglement monotones, for example, the definition of minimum residual Gaussian contangle (see section 1.4.4.2) or the residual tripartite Rényi-2 entanglement (see section 1.4.4.3) for entanglement quantification of three-mode Gaussian states.

### 1.4.4.2 The minimum residual Gaussian contangle

Let us define some definitions to put in context the minimum residual Gaussian contangle as a genuine tripartite entanglement quantifier measure.

First, let us define the continuous-variable version of the tangle $E_{\tau}$ for a generic pure state $|\psi\rangle$ composed by $(1+N)$-mode CV subsystems, then,

$$
\begin{equation*}
E_{\tau}(\psi) \equiv \ln ^{2}\|\tilde{\hat{\rho}}\|, \hat{\rho}=|\psi\rangle\langle\psi|, \tag{1.4.41}
\end{equation*}
$$

hence, it is defined in terms of the squared logarithmic negativity, Eq. (1.3.27).

For a pure Gaussian state $|\psi\rangle$ with covariance matrix $\boldsymbol{\sigma}^{p}$, we have

$$
\begin{equation*}
E_{\tau}\left(\boldsymbol{\sigma}^{p}\right)=\arcsin ^{2}\left(\frac{\sqrt{1-\mu_{1}^{2}}}{\mu_{1}}\right) \tag{1.4.42}
\end{equation*}
$$

where $\mu_{1}=1 / \sqrt{\sigma_{1}}$ is the local single purity of the mode 1 , which has associated covariance matrix $\boldsymbol{\sigma}_{1}$; see Eq. (1.4.10). The definition, Eq. (1.4.41), can be extended to mixed states $\hat{\rho}$ of $(n+1)$-mode CV systems through the convex-roof formalism [88], as

$$
\begin{equation*}
E_{\tau}(\hat{\rho}) \equiv \inf _{\left\{p_{i}, \psi_{i}\right\}} \sum_{i} p_{i} E_{\tau}\left(\psi_{i}\right) \tag{1.4.43}
\end{equation*}
$$

where the infimum is taken over all convex decompositions of $\hat{\rho}$ in terms of pure states $\left\{\left|\psi_{i}\right\rangle\right\}$. If the index $i$ is continuous, the sum in q. (1.4.43) is replaced by an integral, and the probabilities by a probability distribution $\varrho(\psi)$.

Then, restricting us to the regime of continuous variable Gaussian states, any mixed Gaussian state with covariance matrix $\boldsymbol{\sigma}$, admits a decomposition in terms of pure Gaussian states. Then, the infimum of the average contangle in all pure Gaussian state decompositions defines the Gaussian contangle $G_{\tau}$

$$
\begin{equation*}
G_{\tau}(\boldsymbol{\sigma}) \equiv \inf _{\left\{\varrho\left(d \boldsymbol{\sigma}^{P}\right), \boldsymbol{\sigma}^{P}\right\}} \int \varrho\left(d \boldsymbol{\sigma}^{P}\right) E_{\tau}\left(d \boldsymbol{\sigma}^{P}\right) \tag{1.4.44}
\end{equation*}
$$

it can be showed that $G_{\tau}(\boldsymbol{\sigma})$ is a bipartite entanglement monotone under Gaussian local operations and classical communications (GLOCC) [65, 73]. Notably, all three-mode Gaussian states satisfy the CKW inequality (see Section 1.4.4.1) with the Gaussian contangle as a quantifier entanglement measure [34, 35].

Then, the CKW constraint gives place to the definition of residial contangle as a quantifier of genuine tripartite entanglement of continuousvariable three-mode Gaussian states. This measure is dependent on the chosen bipartition according to the reference mode, except for fully symmetric states. Now, a bona fide entanglement measure for continu-
ous variable tripartite entanglement is given by the minimum residual contangle [34], which is defined as

$$
\begin{equation*}
E_{\tau}^{i|j| k} \equiv \min _{(i, j, k)}\left[E_{\tau}^{i \mid j k}-E_{\tau}^{i \mid j}-E_{\tau}^{i \mid k}\right], \tag{1.4.45}
\end{equation*}
$$

where the symbol $(i, j, k)$ represents all the permutations of the threemode indexes $i, j, k \in\{1,2,3\}$. This condition ensuring that $E_{\tau}^{i|j| k}$ is invariant under all permutations of the modes. The quantity $E_{\tau}^{i \mid j k}$ represents the contangle characterizing the global bipartition $i \mid j k$, while $E_{\tau}^{i \mid k}$ is the contangle associated with the reduced bipartition comprising the modes $i$ and $j$.

Adapting the above definition for continuous variable Gaussian states [35], we have the minimum residual Gaussian contangle according to

$$
\begin{equation*}
G_{\tau}^{\min } \equiv G_{\tau}^{i|j| k} \equiv \min _{(i, j, k)}\left[G_{\tau}^{i \mid j k}-G_{\tau}^{i \mid j}-G_{\tau}^{i \mid k}\right], \tag{1.4.46}
\end{equation*}
$$

where $G_{\tau}^{i j j k}$ represents the Gaussian contangle characterizing the global bipartition $i \mid j k$, while $G_{\tau}^{i \mid k}$ is the Gaussian contangle associated with the reduced bipartition comprising the modes $i$ and $j$. See Figure 2 of [35] for a pictorial representation of Eq. (1.4.46). It must be worthy to note that the minimum in Eqs. (1.4.45) and (1.4.45) is a necessary requirement for the bona fide condition as entanglement measures: if one choose a reference bipartition for which the minimum is not fulfilled, the resulting quantifier is not monotone under LOCC (GLOCC); therefore, it is not a valid entanglement measure [35]. The minimum residual contangle, Eq. (1.4.45), as well as its Gaussian version, Eq. (1.4.46), have been proved monotone under LOCC and GLOCC; besides, they are non-increasing under probabilistic operations [35]. Therefore, both quantifiers represent trusty tripartite entanglement measures.

The main problem tackled through this thesis work involves only pure three-mode Gaussian states; therefore, we will focus in recover the recipe to determine the quantitative entanglement properties in these classes of systems; hence, we recover the step by step application procedure presented in Reference [35] for the compute of the minimum residual Gaussian contangle:
(1) Determine the local purities. The state is globally pure, then it satisfy Eq. (1.4.2); therefore, we only need the three local
purities given by Eq. (1.4.10), hence we only use the three reduced covariance matrices $\boldsymbol{\sigma}_{i}, i=1,2,3$.
(2) Find the minimum. The minimum in Eq. (1.4.46) is attained in the bipartition whose single-mode has the smallest reciprocal local single-mode purity, that is, the smallest quantity given by Eq. (1.4.9).
(3) Verify range and compute. Once determined the smallest reciprocal local single-mode purity; say, for example $a_{\text {min }}=a_{l}$; we need to compute the following parameters

$$
\begin{align*}
& s=\frac{a_{2}+a_{3}}{2}  \tag{1.4.47}\\
& d=\frac{a_{2}-a_{3}}{2} \tag{1.4.48}
\end{align*}
$$

if $a_{\text {min }}=1$, then mode 1 is uncorrelated from the others and we have $G_{\tau}^{\mathrm{res}}=0$. Instead, if $a_{\min }>1$ then we have the minimum residual Gaussian contangle as

$$
\begin{equation*}
G_{\tau}^{\mathrm{res}}=\arcsin ^{2}\left[\sqrt{a_{\min ^{2}-1}}\right]-Q\left(a_{\mathrm{min}, s, d}\right) \tag{1.4.49}
\end{equation*}
$$

The first term on the right hand side of Eq. (1.4.49) represent the Gaussian contangle $G_{\tau}^{l \mid j k}$ of the $l \mid j k$-mode bipartition and the quantity $Q\left(a_{\min }, s, d\right)=G_{\tau}^{l \mid j}+G_{\tau}^{l \mid k}$ represent the sum of the Gaussian contangles of the $l \mid j$ and $l \mid k$ reduced mode bipartitions; such term is defined by

$$
\begin{align*}
Q\left(a_{\min }, s, d\right)=\operatorname{arcsinh}^{2} & {\left[\sqrt{\eta^{2}\left(a_{\min }, s, d\right)-1}\right] }  \tag{1.4.50}\\
& +\operatorname{arcsinh}^{2}\left[\sqrt{\eta^{2}\left(a_{\min }, s,-d\right)-1}\right]
\end{align*}
$$

being $\eta=\eta_{+}$if $D>0$ and $\eta=\eta_{-}$if $D \leq 0\left(\eta_{+}=\eta_{-}\right.$if $\left.D=0\right)$, where

$$
\eta_{+}=\frac{\sqrt{2\left[2 a_{\min }^{2}\left(1+2 s^{2}+2 d^{2}\right)-\left(4 s^{2}-1\right)\left(4 d^{2}-1\right)-a_{\min }^{4}-\sqrt{\delta}\right]}}{4(s-d)}
$$

$$
\begin{equation*}
\eta_{-}=\frac{\left|k_{-}\right|}{(s-d)^{2}-1} \tag{1.4.51}
\end{equation*}
$$

$$
\begin{gather*}
D=2(s-d)-\sqrt{2\left[k_{-}^{2}+2 k_{+}+\left|k_{-}\right|\left(k_{-}^{2}+8 k_{+}\right)^{1 / 2}\right] / k_{+}}  \tag{1.4.53}\\
k_{ \pm}=a_{\min }^{2} \pm(s+d)^{2} \tag{1.4.54}
\end{gather*}
$$

and

$$
\begin{aligned}
\delta= & \left(a_{\min }-2 d-1\right)\left(a_{\min }-2 d+1\right)\left(a_{\min }+2 d-1\right)\left(a_{\min }+2 d+1\right) \\
& \left(a_{\min }-2 s-1\right)\left(a_{\min }-2 s+1\right)\left(a_{\min }+2 s-1\right)\left(a_{\min }+2 s+1\right) .
\end{aligned}
$$

Each quantity of the left hand side of Eqs. (1.4.51) to (1.4.55) has been omitted the dependence on the parameters $a_{\min }, s$ and $d$ for brevity. Be aware that the second term in the right hand side of Eq. (1.4.50) must be calculated replacing $d$ by $-d$ in each quantity of Eqs. (1.4.51) to (1.4.54).

In section 3.5.3, we will recover the definitions for the minimum residual contangle to quantify the tripartite quantum entanglement generated in a full dynamical scheme of the Arthurs-Kelly measurement process of position ad momentum observables.

### 1.4.4.3 The residual tripartite Rényi-2 entropy

In this section, we review the residual tripartite Rényi-2 entropy, which is a bona fide measure of entanglement for three-mode Gaussian states. First, we will briefly take some basic concepts which will be helpful to understand the basis of this entanglement quantifier. We follow the approach given in Reference [3].

In quantum information, the degree of purity, i.e., the amount of information at our disposal about a quantum state $\hat{\rho}$ is often measured through the von Neumann entropy, that is,

$$
\begin{equation*}
\mathcal{S}(\hat{\rho})=-\operatorname{Tr}(\hat{\rho} \log [\hat{\rho}]) \tag{1.4.56}
\end{equation*}
$$

in fact, entanglement in pure bipartite states is commonly measured by the von Neumann entropy of the reduced density matrix of any of the two subsystems

$$
\begin{equation*}
\mathcal{S}\left(\hat{\rho}_{A: B}\right)=\mathcal{S}\left(\hat{\rho}_{A}\right)=\mathcal{S}\left(\hat{\rho}_{B}\right) \tag{1.4.57}
\end{equation*}
$$

being $\hat{\rho}_{A, B}=\operatorname{Tr}_{B, A}\left[\hat{\rho}_{A: B}\right]$ the reduced density matrix of the system $A, B$; however, there exist other entropic measures which can help us to
measure the information about a quantum system. In specific, we have the family of Rényi- $\alpha$ entropies [52] which satisfy the strong subadditivity inequality [89, 90, 91]. Entropic measures quantify, in general, the degree of ignorance about the preparation of a quantum state. In particular, the Rényi- $\alpha$ entropies are defined as

$$
\begin{equation*}
\mathcal{S}_{\alpha}(\hat{\rho})=\frac{1}{1-\alpha} \log \left[\operatorname{Tr}\left(\hat{\rho}^{\alpha}\right)\right], \tag{1.4.58}
\end{equation*}
$$

and reproduce the von Neumann entropy in the limit $\alpha \longrightarrow 1$. Focusing on the continuous-variable regime, the Rényi entropies can be evaluated for a generic $N$-mode Gaussian state with a density matrix $\hat{\rho}$ in terms of its covariance matrix $\boldsymbol{\sigma}$ [72] according to

$$
\begin{equation*}
\mathcal{S}_{\alpha}(\hat{\rho})=\frac{\sum_{k=1}^{N} \ln \left[g_{\alpha}\left(\nu_{k}\right)\right]}{1-\alpha}, \tag{1.4.59}
\end{equation*}
$$

where $\left\{\nu_{k}\right\}$ represent the set of symplectic eigenvalues of $\boldsymbol{\sigma}$, and

$$
\begin{equation*}
g_{\alpha}(x)=\frac{2^{\alpha}}{(x+1)^{\alpha}-(x-1)^{\alpha}} . \tag{1.4.60}
\end{equation*}
$$

Notably, in Reference [36], it has been proved that the particular case of $\alpha=2$ represents a natural measure of information for any multimode Gaussian state. Therefore, using Eq. (1.4.58) we have

$$
\begin{equation*}
\mathcal{S}_{\alpha}(\hat{\rho})=-\log \left[\operatorname{Tr}\left(\hat{\rho}^{2}\right)\right]=\frac{1}{2} \ln (\operatorname{det} \boldsymbol{\sigma}), \tag{1.4.61}
\end{equation*}
$$

where in the last term of the above equation we have made use of the definition for purity of a N -mode Gaussian state, Eq. (1.3.12). Then, the Rényi-2 entropy is directly related to the purity of the Gaussian state. Notably, in the Gaussian scenario (and only here), the Rényi-2 entropy satisfies the strong subadditivity inequality for all tripartite Gaussian states $\hat{\rho}_{A B C}$ whose subsystems encompass an arbitrary number of modes; see Reference [3] for a proof.

We can establish a measure of bipartite entanglement $\mathcal{E}_{2}$ for Gaussian states based on the Rényi-2 entropy for mixed states through the convex roof formalism [3]. In the particular scenario of pure states, we have for a bipartite Gaussian state with covariance matrix $\boldsymbol{\sigma}_{A: B}^{\text {pure }}$

$$
\begin{equation*}
\mathcal{E}_{2}\left(\boldsymbol{\sigma}_{A: B}^{\text {pure }}\right)=\mathcal{S}_{2}\left(\boldsymbol{\sigma}_{A}\right)=\frac{1}{2} \ln \left(\operatorname{det} \boldsymbol{\sigma}_{A}\right), \tag{1.4.62}
\end{equation*}
$$

where $\sigma_{A}$ is the reduced covariance matrix of the subsystem $A$. This expression will be useful for future computations.

It must be noted that not all entanglement measures which satisfy the monotonicity condition can obey the monogamy constraint of Eq. (1.4.40) [3]. Notably, the Rényi-2 entropy for all $N$-mode Gaussian states are monogamous [36], which is a crucial property to define trusty entanglement measures in this class of states; therefore, for a $N$-mode Gaussian state $\hat{\rho}_{A_{1}, A_{2} \cdots A_{n}}$ we have

$$
\begin{equation*}
\mathcal{E}_{2}^{\left(A_{1} \mid A_{2} \cdots A_{n}\right)}-\sum_{j=2}^{N-1} \mathcal{E}_{2}^{\left(A_{1} \mid A_{j}\right)} \geq 0, \tag{1.4.63}
\end{equation*}
$$

where $\mathcal{E}_{2}\left(\hat{\rho}_{A_{1} \mid A_{2} \cdots A_{n}}\right)$ is the Rényi- 2 entropy characterizing the entanglement of the global bipartition of $A_{1}$ vs the set $\left(A_{2} \cdots A_{n}\right)$, while $\mathcal{E}_{2}\left(\hat{\rho}_{A_{1} \mid A_{j}}\right)$ characterizes the entanglement of the reduced bipartition $A_{1} \mid A_{j}$. Each partie $A_{j}$ comprises one mode only.

Now, focusing on three-mode Gaussian states, we recall that their covariance matrix $\sigma_{A_{1} A_{2} A_{3}}$ can be putted in the standard form of the Eq. (1.4.12), with the local symplectic quantities $a_{i}$ given by Eq. (1.4.9) constraint to vary according to the inequality Eq. (1.4.11). Then, we recover the definition for the residual tripartite Rényi-2 entropy, $\mathcal{E}_{2}^{\left(A_{i}\left|A_{j}\right| A_{k}\right)}$, with respect to the focus mode $A_{i}$, as [3]

$$
\begin{align*}
\mathcal{E}_{2}^{\left(A_{i}\left|A_{j}\right| A_{k}\right)} & =\mathcal{E}_{2}^{\left(A_{i} \mid A_{j} A_{k}\right)}-\mathcal{E}_{2}^{\left(A_{i} \mid A_{j}\right)}-\mathcal{E}_{2}^{\left(A_{i} \mid A_{k}\right)} \\
& =\frac{1}{2} \ln \left(\frac{a_{i}^{2}}{g_{j} g_{k}}\right), \quad i, j, k \in\{1,2,3\}, i \neq j \neq k, \tag{1.4.64}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{E}_{2}^{\left(A_{i} \mid A_{j}\right)}=\frac{1}{2} \ln g_{k}, \tag{1.4.65}
\end{equation*}
$$

with

$$
g_{k}= \begin{cases}1, & \text { if } a_{k} \geq \sqrt{a_{i}^{2}+a_{j}^{2}-1}  \tag{1.4.66}\\ \frac{\beta}{8 a_{k}^{2}}, & \text { if } \alpha_{k}<a_{k}<\sqrt{a_{i}^{2}+a_{j}^{2}-1} \\ \left(\frac{a_{i}^{2}-a_{j}^{2}}{a_{k}^{2}-1}\right)^{2}, & \text { if } a_{k} \leq \alpha_{k}\end{cases}
$$

besides, we have the quantities

$$
\begin{equation*}
\alpha_{k}=\sqrt{\frac{2\left(a_{i}^{2}+a_{j}^{2}\right)+\left(a_{i}^{2}-a_{j}^{2}\right)^{2}+\left|a_{i}^{2}-a_{j}^{2}\right| \sqrt{\left(a_{i}^{2}-a_{j}^{2}\right)^{2}+8\left(a_{i}^{2}+a_{j}^{2}\right)}}{2\left(a_{i}^{2}+a_{j}^{2}\right)}} \tag{1.4.67}
\end{equation*}
$$

$$
\begin{aligned}
\beta= & 2 a_{1}^{2}+2 a_{2}^{2}+2 a_{3}^{2}+2 a_{1}^{2} a_{2}^{2}+2 a_{1}^{2} a_{3}^{2}+2 a_{2}^{2} a_{3}^{2} \\
& -a_{1}^{4}-a_{2}^{4}-a_{3}^{4}-\sqrt{\delta}-1,
\end{aligned}
$$

and

$$
\begin{align*}
\delta & =\left[\left(a_{1}+a_{2}+a_{3}\right)^{2}-1\right]\left[\left(a_{1}-a_{2}+a_{3}\right)^{2}-1\right]  \tag{1.4.68}\\
& \times\left[\left(a_{1}+a_{2}-a_{3}\right)^{2}-1\right]\left[\left(a_{1}-a_{2}-a_{3}\right)^{2}-1\right] .
\end{align*}
$$

Besides, using Eq. (1.4.62) and the condition, (1.4.5), for three-mode Gaussian states, we have the Rényi-2 entropy for the bipartitions $\left(A_{i} \mid A_{j} A_{k}\right)$ as

$$
\begin{align*}
\mathcal{E}_{2}^{\left(A_{i} \mid A_{j} A_{k}\right)} & =\frac{1}{2} \ln \left[\operatorname{det}\left(\boldsymbol{\sigma}_{i}\right)\right] \\
& =\frac{1}{2} \ln \left[\operatorname{det}\left(\boldsymbol{\sigma}_{j k}\right)\right] \tag{1.4.69}
\end{align*}
$$

being $\boldsymbol{\sigma}_{i}$ and $\boldsymbol{\sigma}_{j k}$ the reduced covariance matrices associated with the systems comprising the single-mode $i$ and the two-mode $j k$, which can be obtaining by removing the entries of the excluded modes.

In general, the expression Eq. (1.4.64) is dependent on the choice for the mode playing as focus; this is a good characteristic that can throw light on the entanglement structure of a tripartite system. We will take advantage of this characteristic in Section 2.4 to study the features of the tripartite entanglement in the strong coupling regime of the Arthurs-Kelly measurement process.

THE ARTHURS-KELLY SIMULTANEOUS MEASUREMENT PROCESS

"[..] Then what will happen tomorrow? Nobody knows."<br>Yuval Noah Harari

### 2.1 THEORETICAL BACKGROUND

### 2.1.1 The standard model of quantum measurement

In quantum mechanics, there exists a model of quantum measurement that has been played a crucial role in understanding and develop fundamental aspects of the theory; such a model is usually called the standard model of the quantum measurement [92]. In this mechanism, a quantum observable (which we pretend to know) enters in interaction with a pointer observable (also called prove) through an interaction Hamiltonian to create quantum correlations among them; later, the proof is read (i.e. measured), and the output carries information about the original observable of the system. The first mathematically rigorous and physically oriented mechanism of this model was proposed by von Neumann in the last three pages of his book: Mathematical foundations of quantum mechanics [93]; in that, he defines an interaction Hamiltonian between the position $\hat{q}$ of an observed system and the momentum $\hat{p}$ of a probe according to

$$
\begin{equation*}
\hat{H}=\hat{q} \otimes \hat{p}, \tag{2.1.1}
\end{equation*}
$$

it is important to note that von Neumann assumes the masses of the systems are big enough to discard their free energies. Hence is the
interaction energy part is the only which plays a decisive aspect for the measurement creating strong correlations between the observables of the prove and the system. In this sense, the time dependent Schrödinger equations becomes

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t} \Psi_{t}(q, r)=-i \hbar q \frac{\partial \Psi_{t}(q, r)}{\partial r} \tag{2.1.2}
\end{equation*}
$$

where we use $\hat{p}=-i \hbar \frac{\partial}{\partial r}$. In this manner, we have

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+q \frac{\partial}{\partial r}\right) \Psi_{t}(q, r) \tag{2.1.3}
\end{equation*}
$$

with the solution

$$
\begin{equation*}
\Psi_{t}(q, r)=f(q, r-t q) \tag{2.1.4}
\end{equation*}
$$

this result can be verified more simply by recurring to the time evolution operator method [94]: $\Psi_{t}(q, r)=e^{-i t \hat{H} / \hbar} f(q, r)$, with $f_{0}(q, r)$ the initial wave function describing the probe and the system; then we have

$$
\begin{align*}
\Psi_{t}(q, r) & =e^{-i t \hat{H} / \hbar} f(q, r) \\
& =e^{-t q \frac{\partial}{\partial r}} f(q, r) \\
& =f(q, r-t q) \tag{2.1.5}
\end{align*}
$$

In particular, if the initial wave function of the system and the prove is a product, i. e., $f(q, r)=\phi(q) \xi(r)$, we have

$$
\begin{equation*}
\Psi_{t}(q, r)=\phi(q) \xi(r-q t) \tag{2.1.6}
\end{equation*}
$$

therefore, by projectively measuring the position of the probe at time $t=1$ ${ }^{1}$ we will find that statistically (i.e. in a large number of experimental measurements made on identically prepared systems) the probability distribution of the outputs will be centered at the position value $q$ of the observed system. For seeing a more complete description of the standard model of measurement, the reader can see [95], pages 59-61.

[^2]
### 2.1.2 Retrodictive and predictive aspects of accuracy

Measurements in physics constitute the bridge between the theory and its predictions. Classically the act of measuring involves the comparison of certain property of a physical object with another acting as a meter; thus, the extracted information constitutes the exact result of the measurement. Nevertheless, quantum-mechanically speaking, this notion of accuracy does not make sense. The first to beat these classical deterministic conceptions was Heisenberg with him inaccuracy relation derived through their famous think experiment of $\gamma$-ray microscope [96]. However, the widely celebrated uncertainty relation appearing in almost all elementary treaty of quantum mechanics:

$$
\begin{equation*}
\delta_{q} \delta_{p} \geq \frac{1}{2}, \tag{2.1.7}
\end{equation*}
$$

was mathematically derived by Kennard [97] from the quantum mechanical formalism. Nowadays, there is a consensus that ineq. (2.1.7) has nothing to do with the simultaneous measurement of complementary observables. Instead, the first quantum-mechanically description where a measuring device interacts with a (pure) system to simultaneously measure its position and momentum observables was proposed by Arthurs and Kelly [98]. They conceived their model as a generalization of the Von Neumann measurement process (see Sec 2.1.1), extending the model to the non-commuting pair of position and momentum observables. As we will see in section 2.2 , derive the following uncertainty relation

$$
\begin{equation*}
\delta_{q} \delta_{p} \geq 1, \tag{2.1.8}
\end{equation*}
$$

which has the double value in the lower bound of the Ineq. (2.1.7); therefore, they establish a fundamental difference between their associated measurement process. Instead, the adequate operational interpretation of the Ineq. (2.1.7) is: take a set of a big number of identically prepared quantum systems and carry out a single measurement of the position observable on one-half of the systems; then, takes the independent single measurement for the momentum observable on the other half of the systems. By evaluating the corresponding standard deviations of the associated probability distributions of the measurements, one finds that their product is bounded from below according to Ineq. (2.1.7) [99]. Then, that
inequality refers to ideal separate single measurements applied to separate members of the ensemble [100]. On the other hand, the quantities $\delta_{q}$ and $\delta_{p}$ in the Ineq. (2.1.8) denote the standard deviations characterizing the widths of the probability distributions of the simultaneous measurement outputs of position and momentum observables carried by independent systems.Besides, the double value in the lower bound of the Ineq. (2.1.8) indicate that in a simultaneous measurement process for the canonical pair, there exists an unavoidable noise that affects the measurement results. As we will see in the following sections, this fact implies the well-known statute that we cannot simultaneously measure both the position and momentum observables with arbitrary accuracy; however, even though the Ineq. (2.1.7) certainly provides knowledge about the accuracy of the simultaneous measurement, it only refers to one aspect of the accuracy; besides, the Ineq. (2.1.7) does not quantify operationally the error concept, which is a key to understanding the accuracy of the simultaneous measurement process. Hence, to evaluate the straiten aspects of accuracy of that model, we define the corresponding concepts based on the theory developed by Appleby [101, 102, 103].

Let us consider a quantum system in the position or momentum representation $\left|\psi_{1}\left(\xi_{1}\right)\right\rangle$ such that the variable $\xi_{1}$ label any of these representations, that is $\xi_{1} \in\left\{x_{1}, p_{1}\right\}$. From this state, we pretend to know its position and momentum observables ( $\hat{x}_{1}$ and $\hat{p}_{1}$ ) (which only is possible in a statistical sense in the quantum mechanical regime) . Besides, let us take a measuring device with two commuting and independent degrees of freedom acting as pointers (also called probes), each represented also by (independent) one-dimensional continuous-variable states $\left|\psi_{2}\left(\xi_{2}\right)\right\rangle$ and $\left|\psi_{3}\left(\xi_{3}\right)\right\rangle$. The global state (the quantum system plus the pointers) is defined as the tensor product of the individual states $|\psi\rangle=\left|\psi_{1}\left(\xi_{1}\right)\right\rangle\left|\psi_{2}\left(\xi_{2}\right)\right\rangle\left|\psi_{3}\left(\xi_{3}\right)\right\rangle$ on a tensor Hilbert space $\mathcal{H}=\otimes_{i=1}^{3} \mathcal{H}_{i}$. Then, at the time $t$, the position and momentum observables of the quantum system are linked with the observables $\hat{\xi}_{2} \in\left\{\hat{x}_{2}, \hat{p}_{2}\right\}$ and $\hat{\xi}_{3} \in\left\{\hat{x}_{3}, \hat{p}_{3}\right\}$ of the pointers through a coupling $\hat{U}$ (the measurement interaction), which of course is an unitary: $\hat{U}^{\dagger} \hat{U}=\hat{\mathbb{I}}$ and a self-adjoint operator: $\hat{U}^{\dagger}=\hat{U}$. After a time of interaction $\Delta t$, local simultaneous projective measurements are made on the pointers to know their observables $\hat{x}_{1}$ and $\hat{x}_{2}$. Then, the measurement outputs are recorded (without affecting
it in any way) by the measuring apparatus; in a statistical sense, such measurement outputs should have information about the canonical pair of the system as a direct consequence of the unitary coupling governing the dynamics.

For that process, we define the concept of accuracy in the simultaneous measurement of our (maybe oversimplified) measurement model based on the definitions of Ozawa [104] for the measurement of a single observable:

Theorem 1. The measuring apparatus precisely measures the position or momentum observable of the quantum system if, independently, the recorded probability distribution of each pointer output exactly matches the probability distribution of their corresponding linked canonical observable of the quantum system; this means, rigorously:

$$
\begin{equation*}
P\left(x_{i}^{\prime} \in \Delta\right)=\langle\psi| E^{\xi_{1}}(\Delta)|\psi\rangle, \tag{2.1.9}
\end{equation*}
$$

where $P\left(x_{i}^{\prime} \in \Delta\right)$ with $i=2,3$, represents the probability distribution for the record value $x_{i}^{\prime}$ of the pointer $i$ in an interval $\Delta$, and $E^{\hat{\xi}_{i}}(\Delta)$ stands as the spectral projection of the canonical observable $\hat{\xi}_{1} \in\left\{\hat{x}_{1}, \hat{p}_{1}\right\}$ in the same interval $\Delta$. As we will soon see, the canonical probability distribution against which the distribution of the pointer record is compared, can refer to before or after the measurement process indicating two distinct aspects of accuracy in the simultaneous measurement of the canonical pair.

Therefore, to quantify the accuracy in a simultaneous measurement process, we recover the so-called error observables [101, 102, 103]. We first switch the Heisenberg picture, and we define the "final" observables as those what evolve under the dynamics associated with the measurement interaction

$$
\begin{equation*}
\hat{X}^{\prime}=\hat{U}^{\dagger} \hat{X} \hat{U} \tag{2.1.10}
\end{equation*}
$$

Therefore, we recover the so-called retrodictive error observables

$$
\begin{align*}
& \hat{\varepsilon}_{\hat{x}_{1}}^{R}=\hat{\xi}_{2}^{\prime}-\hat{x}_{1},  \tag{2.1.11}\\
& \hat{\varepsilon}_{\hat{p}_{1}}^{R}=\hat{\xi}_{3}^{\prime}-\hat{p}_{1} . \tag{2.1.12}
\end{align*}
$$

Then, the observables $\varepsilon_{\hat{x}_{1}, \hat{p}_{1}}^{R}$, are defined as the difference between the final pointers observables and the initial position and momentum observables of the system; then, through a measure of dispersion, they give a notion about the degree of match between the probability distributions of the registered pointer outputs and the canonical observables of the system before the measurement, i.e., they give a notion of the retrodictive accuracy of the simultaneous measurement, according to the definition established by the Theorem 1.

Now we recover the so-called predictive error observables

$$
\begin{align*}
& \hat{\varepsilon}_{\hat{x}_{1}^{\prime}}^{P}=\hat{\xi}_{2}^{\prime}-\hat{x}_{1}^{\prime}  \tag{2.1.13}\\
& \hat{\varepsilon}_{\hat{p}_{1}^{\prime}}^{P}=\hat{\xi}_{3}^{\prime}-\hat{p}_{1}^{\prime} \tag{2.1.14}
\end{align*}
$$

The observables $\varepsilon_{\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}}^{P}$ are defined as the difference between the final pointers observables and the final position and momentum observables of the system, hence through a measure of dispersion they gives a notion about the degree of match between the probability distributions of the registered pointer outputs and the canonical observables of the system just after the simultaneous measurement.

The necessity for the distinction of two aspects of accuracy in any quantum measurement process has been pointed out by several authors, emphasizing the preparative (predictive) and a determinative (retrodictive) nature of a quantum measurement [105, 106, 107, 108]. The distinction for these two aspects has been materialized in the error observables by Appleby $[101,102,103]$, appealing to the need to quantify the recoil effects on the measured system due to the measurement process; in fact, the definitions given by Eqs. (2.1.11) to (2.1.14) allows defining disturbance observables to quantify the degree of perturbation on the system due to the measurement process, that is,

$$
\begin{align*}
\delta_{\hat{x}_{1}}^{D} & =\hat{\varepsilon}_{\hat{x}}^{R}-\hat{\varepsilon}_{\hat{x}^{\prime}}^{P}=\hat{x}_{1}^{\prime}-\hat{x}_{1},  \tag{2.1.15}\\
\delta_{\hat{p}_{1}}^{D} & =\hat{\varepsilon}_{\hat{p}}^{R}-\hat{\varepsilon}_{\hat{p}_{1}^{\prime}}^{P}=\hat{p}_{1}^{\prime}-\hat{p}_{1}, \tag{2.1.16}
\end{align*}
$$

which through a measure of dispersion they give a notion about the degree of change in the statistic of the observables of position and momentum
due to the unitary dynamics of the measurement process. It must be note that the root-mean-square of the retrodictive error observables, that is, the Eqs. (2.1.11) and (2.1.11), and the disturbance observables, that is, the Eqs. (2.1.15) and (2.1.16), exactly coincide with the noise and disturbance observables given by Ozawa [104]. Hence the operational formulation (in units of $\hbar=1$ ) about the well-known statute that it is impossible to simultaneous measure the position and momentum observables with arbitrary accuracy for the retrodictive and predictive aspects can be captured by the uncertainty relation [101, 102, 103]:

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{x}}^{X}}^{2} \sigma_{\hat{\varepsilon}_{\hat{p}}^{X}}^{2} \geq \frac{1}{4} \tag{2.1.17}
\end{equation*}
$$

where this time the superscript $X$ refers to any of the superscripts $R$ or $P$ to refer to any of the retrodictive or predictive error operators and

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{x}, \hat{p}}^{X}}=\sqrt{\langle\psi|\left(\hat{\varepsilon}_{\hat{x}, \hat{p}}^{X}\right)^{2}|\psi\rangle-\langle\psi|\left(\hat{\varepsilon}_{\hat{x}, \hat{p}}^{X}\right)|\psi\rangle}, \tag{2.1.18}
\end{equation*}
$$

stands as the standard deviation of the error observable; of course, the squared quantity representing the variance. The Ineq. (2.1.17) implies that the statistic of the recorded measurement outputs of the pointers never coincides with the (before or after) statistic of the canonical observables of the system. The reason for this is that the simultaneous measurement process of the canonical pair carries an intrinsic noise related to the dynamics of the pointer's observables at the time of the measurement. To show this argument, let us go more quantitatively.

First, we compute explicitly the variances for the error retrodictive observables, therefore, we have

$$
\begin{equation*}
\langle\psi|\left(\hat{\varepsilon}_{\hat{x}, \hat{p}}^{R}\right)^{2}|\psi\rangle=\langle\psi|\left\{\left(\hat{\xi}_{2,3}^{\prime}\right)^{2}-\hat{\xi}_{2,3}^{\prime} \times\left(\hat{x}_{1}, \hat{p}_{1}\right)-\left(\hat{x}_{1}, \hat{p}_{1}\right) \times \hat{\xi}_{2,3}^{\prime}+\left(\hat{x}_{1}, \hat{p}_{1}\right)^{2}\right\}|\psi\rangle, \tag{2.1.19}
\end{equation*}
$$

and

$$
\begin{gather*}
\left(\langle\psi| \hat{\varepsilon}_{\hat{x}, \hat{p}}^{R}|\psi\rangle\right)^{2}=\left(\langle\psi| \hat{\xi}_{2,3}^{\prime}|\psi\rangle\right)^{2}-2\left(\langle\psi| \hat{\xi}_{2,3}^{\prime}|\psi\rangle\right)\left(\langle\psi|\left(\hat{x}_{1}, \hat{p}_{1}\right)|\psi\rangle\right) \\
+\left(\langle\psi|\left(\hat{x}_{1}, \hat{p}_{1}\right)|\psi\rangle\right)^{2} \tag{2.1.20}
\end{gather*}
$$

hence we recover the variance

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{q}, \hat{p}}^{R}}^{2}=\sigma_{\hat{\xi}_{2,3}^{\prime}}^{2}+\sigma_{\hat{x}_{1}, \hat{p}_{1}}^{2}-\operatorname{Cov}\left[\hat{\xi}_{2,3}^{\prime},\left(\hat{x}_{1}, \hat{p}_{1}\right)\right], \tag{2.1.21}
\end{equation*}
$$

where $\sigma_{\hat{\tilde{k}}_{2,3}^{\prime}}^{2}$ and $\sigma_{\hat{x}_{1}, \hat{p}_{1}}^{2}$ represent the variances of the final pointer outputs and the initial position and momentum observables, and

$$
\begin{align*}
\operatorname{Cov}\left[\hat{\xi}_{2,3}^{\prime},\left(\hat{x}_{1}, \hat{p}_{1}\right)\right]= & \langle\psi| \hat{\xi}_{2,3}^{\prime} \times\left(\hat{x}_{1}, \hat{p}_{1}\right)+\left(\hat{x}_{1}, \hat{p}_{1}\right) \times \hat{\xi}_{2,3}^{\prime}|\psi\rangle \\
& -2\left(\langle\psi| \hat{\xi}_{2,3}^{\prime}|\psi\rangle\right)\left(\langle\psi|\left(\hat{x}_{1}, \hat{p}_{1}\right)|\psi\rangle\right), \tag{2.1.22}
\end{align*}
$$

is the covariance between the observables $\hat{\xi}_{2,3}^{\prime}$ and $\left(\hat{x}_{1}, \hat{p}_{1}\right)$. Notably, it must be noted that Eq. (2.1.21) can be written as

$$
\begin{equation*}
\sigma_{\hat{\tilde{q}}_{\hat{2}, \hat{p}}^{R}}^{2}=\sigma_{\hat{x}_{1}, \hat{p}_{1}}^{2}+\delta_{\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}}^{R} \tag{2.1.23}
\end{equation*}
$$

where $\delta_{\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}}^{R}=\sigma_{\hat{\xi}_{2,3}^{\prime}}^{2}-\operatorname{Cov}\left[\hat{\xi}_{2,3}^{\prime},\left(\hat{x}_{1}, \hat{p}_{1}\right)\right]$ represents all noise affecting the retrodictive accuracy in the simultaneous measurement of the position and momentum observables. From that term we can verify that the retrodictive inaccuracy affecting the simultaneous measurement depends directly on the initial fluctuations of the probability distributions of the pointers observables at the time of the measurement, and the covariance of that observables with the initial canonical pair under inspection.

By a similar procedure, the variance of the predictive error observable is

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{q}, \hat{p}}^{P}}^{2}=\sigma_{\hat{\xi}_{2,3}^{\prime}}^{2}+\sigma_{\hat{x}_{1}^{1}, \hat{p}_{1}^{\prime}}^{2}-\operatorname{Cov}\left[\hat{\xi}_{2,3}^{\prime},\left(\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}\right)\right], \tag{2.1.24}
\end{equation*}
$$

with

$$
\begin{align*}
\operatorname{Cov}\left[\hat{\xi}_{2,3}^{\prime},\left(\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}\right)\right]= & \langle\psi| \hat{\xi}_{2,3}^{\prime} \times\left(\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}\right)+\left(\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}\right) \times \hat{\xi}_{2,3}^{\prime}|\psi\rangle \\
& -2\left(\langle\psi| \hat{\xi}_{2,3}^{\prime}|\psi\rangle\right)\left(\langle\psi|\left(\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}\right)|\psi\rangle\right) ; \tag{2.1.25}
\end{align*}
$$

therefore, in an equivalent way to the retrodictive aspect the variance given by Eq. (2.1.24) can be written as

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{q}, \hat{p}}^{P}}^{2}=\sigma_{\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}}^{2}+\delta_{\hat{q}^{\prime}, \hat{p}^{\prime}}^{P} \tag{2.1.26}
\end{equation*}
$$

where $\delta_{\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}}^{P}=\sigma_{\hat{\xi}_{2,3}^{\prime}}^{2}-\operatorname{Cov}\left[\hat{\xi}_{2,3}^{\prime},\left(\hat{x}_{1}^{\prime}, \hat{p}_{1}^{\prime}\right)\right]$ represents all noise affecting the predictive accuracy in the simultaneous measurement of the position and momentum observables. Therefore, as in the retrodictive case, the fluctuations of the probability distributions of the pointers observables at the time of the measurement play a role in the inaccuracy for the predictive aspect of the simultaneous measurement, together with the covariance of those observables with the canonical pair just after the measurement.

### 2.1.3 Unbiased measurements

In the context of a unbiased condition as was assumed by Arthurs and Kelly (see Sec. 5.2), there are not systematic error of retrodiction:

$$
\begin{equation*}
\langle\psi| \hat{\varepsilon}_{\hat{x}, \hat{p}}^{R}|\psi\rangle=0 ; \tag{2.1.27}
\end{equation*}
$$

besides, it is natural to accept the unbiased predictive condition of the measurement

$$
\begin{equation*}
\langle\psi| \hat{\varepsilon}_{\hat{x}, \hat{p}}^{P}|\psi\rangle=0 \tag{2.1.28}
\end{equation*}
$$

We recall that in the Heisenberg picture, the state $|\psi\rangle$ refers to the tensor product of the individual states of the pointers and system under measurement; this together the Eqs. (2.1.27) and (2.1.28) implies that the variance of the error observable are simplified to

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{x}, \hat{p}}^{R, P}}^{2}=\langle\psi|\left(\hat{\varepsilon}_{\hat{x}, \hat{p}}^{R, P}\right)^{2}|\psi\rangle . \tag{2.1.29}
\end{equation*}
$$

Now, if $\left|\psi^{\prime}\right\rangle=\left(\hat{x}_{1}, \hat{p}_{1}\right)|\psi\rangle$ and the simultaneous measurement is retrodictively unbiased, it can be proved that (see appendix of [101, 102, 103] or preposition 2.4.3 of [109]):

$$
\begin{equation*}
\langle\psi|(\hat{x}, \hat{p}) \times\left(\hat{\varepsilon}_{\hat{x}, \hat{p}}^{R, P}\right)|\psi\rangle=\langle\psi|\left(\hat{\varepsilon}_{\hat{x}, \hat{p}}^{R, P}\right) \times(\hat{x}, \hat{p})|\psi\rangle=0 \tag{2.1.30}
\end{equation*}
$$

with this condition, it is straightforward to verify the following conditions

$$
\begin{align*}
&\langle\psi| \hat{\xi}_{2}^{\prime} \hat{x}_{1}|\psi\rangle=\langle\psi| \hat{x}_{1} \hat{\xi}_{2}^{\prime}|\psi\rangle=\langle\psi| \hat{x}_{1}^{2}|\psi\rangle,  \tag{2.1.31}\\
&\langle\psi| \hat{\xi}_{2}^{\prime} \hat{p}_{1}|\psi\rangle=\langle\psi| \hat{x}_{1} \hat{p}_{1}|\psi\rangle,  \tag{2.1.32}\\
&\langle\psi| \hat{p}_{1} \hat{\xi}_{2}^{\prime}|\psi\rangle=|\psi\rangle \hat{p}_{1} \hat{x}_{1}|\psi\rangle,  \tag{2.1.33}\\
&\langle\psi| \hat{\xi}_{3}^{\prime} \hat{x}_{1}|\psi\rangle=\langle\psi| \hat{p}_{1} \hat{x}_{1}|\psi\rangle,  \tag{2.1.34}\\
&\langle\psi| \hat{x}_{1} \hat{\xi}_{3}^{\prime}|\psi\rangle=\langle\psi| \hat{x}_{1} \hat{p}_{1}|\psi\rangle,  \tag{2.1.35}\\
&\langle\psi| \hat{\xi}_{3}^{\prime} \hat{p}_{1}|\psi\rangle=\langle\psi| \hat{p}_{1} \hat{\xi}_{3}^{\prime}|\psi\rangle=\langle\psi| \hat{p}_{1}^{2}|\psi\rangle, \tag{2.1.36}
\end{align*}
$$

from which is easy to verify the following commutation relations

$$
\begin{align*}
\langle\psi|\left[\hat{x}_{1}, \hat{\xi}_{2}^{\prime}\right]|\psi\rangle & =0  \tag{2.1.37}\\
\langle\psi|\left[\hat{\xi}_{2}^{\prime}, \hat{p}_{1}\right]|\psi\rangle & =i \hbar \tag{2.1.38}
\end{align*}
$$

$$
\begin{align*}
\langle\psi|\left[\hat{x}_{1}, \hat{\xi}_{3}^{\prime}\right]|\psi\rangle & =i \hbar  \tag{2.1.39}\\
\langle\psi|\left[\hat{\xi}_{3}^{\prime}, \hat{p}_{1}\right]|\psi\rangle & =0 \tag{2.1.40}
\end{align*}
$$

Let us take the expected value of the commutator between the retrodictive error observables, Eqs. (2.1.11) and (2.1.12); then,

$$
\begin{gather*}
{\left[\varepsilon_{\hat{x}_{1}}^{R}, \varepsilon_{\hat{p}_{1}}^{R}\right]=\langle\psi|\left(\hat{\xi}_{2}^{\prime}-\hat{x}_{1}\right)\left(\hat{\xi}_{3}^{\prime}-\hat{p}_{1}\right)-\left(\hat{\xi}_{3}^{\prime}-\hat{p}_{1}\right)\left(\hat{\xi}_{2}^{\prime}-\hat{x}_{1}\right)|\psi\rangle} \\
=\langle\psi|\left(\left[\hat{\xi}_{2}^{\prime}, \hat{\xi}_{3}^{\prime}\right]+\left[\hat{\xi}_{3}^{\prime}, \hat{x}_{1}\right]+\left[\hat{p}_{1}, \hat{\xi}_{2}^{\prime}\right]+\left[\hat{x}_{1}, \hat{p}_{1}\right]\right)|\psi\rangle \\
=-i \hbar, \tag{2.1.41}
\end{gather*}
$$

where we have used $\left[\hat{\xi}_{2}^{\prime}, \hat{\xi}_{3}^{\prime}\right]=0$ (i.e., the pointer obervables commute even through the measurement process), $\left[\hat{x}_{1}, \hat{p}_{1}^{\prime}\right]=i \hbar$, and Eqs. (2.1.38) and (2.1.39). Therefore, given the above commutator, we have in units $\hbar=1$ :

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{x}_{1}}^{R}}^{2} \sigma_{\hat{\varepsilon}_{\hat{p}_{1}}^{R}}^{2} \geq \frac{\left|\left\langle\left[\hat{\varepsilon}_{\hat{x}}^{R}, \hat{\varepsilon}_{\hat{p}}^{R}\right]\right\rangle\right|^{2}}{4}=\frac{1}{4} \tag{2.1.42}
\end{equation*}
$$

Hence we can derive Eq. (2.1.17) for the retrodictive aspect of accuracy under the basis of the unbiased nature of the joint measurement.

The proof for the uncertainty relation of the predictive aspect is straightforward:

Taking commutator

$$
\begin{gather*}
{\left[\varepsilon_{\hat{x}_{1}}^{P}, \varepsilon_{\hat{p}_{1}}^{P}\right]=\langle\psi|\left(\hat{\xi}_{2}^{\prime}-\hat{x}_{1}^{\prime}\right)\left(\hat{\xi}_{3}^{\prime}-\hat{p}_{1}^{\prime}\right)-\left(\hat{\xi}_{3}^{\prime}-\hat{p}_{1}^{\prime}\right)\left(\hat{\xi}_{2}^{\prime}-\hat{x}_{1}^{\prime}\right)|\psi\rangle} \\
=\langle\psi|\left(\left[\hat{\xi}_{2}^{\prime}, \hat{\xi}_{3}^{\prime}\right]+\left[\hat{\xi}_{3}^{\prime}, \hat{x}_{1}^{\prime}\right]+\left[\hat{p}_{1}, \hat{\xi}_{2}^{\prime}\right]+\left[\hat{x}_{1}, \hat{p}_{1}^{\prime}\right]\right)|\psi\rangle \\
=i \hbar, \tag{2.1.43}
\end{gather*}
$$

therefore, in units of $\hbar=1$

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{\hat{x}_{1}}^{P}}^{2} \sigma_{\hat{\varepsilon}_{\hat{p}_{1}}^{P}}^{2} \geq \frac{\left|\left\langle\left[\hat{\varepsilon}_{\hat{x}}^{P}, \hat{\varepsilon}_{\hat{p}}^{P}\right]\right\rangle\right|^{2}}{4}=\frac{1}{4} . \tag{2.1.44}
\end{equation*}
$$

It must be note that we made no use of the unbiased condition of the measurement, which means that the predictive uncertainty relation holds generally.

### 2.1.4 Interaction Hamiltonian

In 1965 was published on the now extinct Bell system technical journal a paper whose title reads "On the Simultaneous Measurement of a Pair of Conjugate Ohservables" signed by E. Arthurs and J.L Kelly Jr [98]. In that work, they proposed a direct extension of the standard model of measurement of a single observable raised by von Neumann to a simultaneous measurement process for the conjugate observables of position and momentum of a quantum system; therefore, they allow the interaction of this quantum system with a measuring device equipped with two independent degrees of freedom; that is, the pointers from which is extracted the information about the position and momentum; then, they allow a coupling between these variables according to the following interaction Hamiltonian

$$
\begin{equation*}
\hat{H}_{\text {int }}=\kappa_{1} \hat{x}_{3} \hat{p}_{1}+\kappa_{2} \hat{p}_{3} \hat{p}_{2}, \tag{2.1.45}
\end{equation*}
$$

where the $\hat{x}_{3}$ and $\hat{p}_{3}$ quantities correspond to the conjugate observables 'position' and 'momentum' of the system; they can represent, for example, the field amplitude oscillating out of phase by $\phi=\pi / 2$ of an electromagnetic field, or the position and momentum of a free and heavy particle, or the position and momentum of a mechanically resonant bar antenna. The observables $\hat{p}_{1}$ and $\hat{p}_{2}$ correspond to the momentum operators of the first and second pointer of the measuring device, respectively; then, in the sense of the Hamiltonian Eq. (2.1.1), the interaction Eq. (2.1.45) will cause a correlation of the position and momentum of the system with the momentum of the first and second probe of the measuring device respectively. The $\kappa_{1,2}$ are positive constants that represent the strength of coupling between these observables; then, in a mathematical sense, they represent the degree to which the positions of the pointers are displaced by the canonical observables of the system. It must be noted that in the interaction Hamiltonian of Eq. (2.1.45) Arthurs and Kelly invoke (based on the essence of the von Neumann measurement process) the fact of discarding the free energies in the Hamiltonian governing the measurement process, arguing that the interaction is stronger enough to discarding the contributions of that terms. In the following, we discuss one possible physical meaning of the Interaction Eq. (2.1.45) from a perspective of quantum optics, as well as its implications.

### 2.1.4.1 Possible implementations of the Arthurs-Kelly interaction Hamiltonian

As we said before, the role of canonical pair in the quantum optics regime could be played by the quadratures of one mode of the electromagnetic field, which can be mathematically defined from the bosonic creation and annihilation operators as stated by Eq. (1.1.2). For this optical regime, we follow the interpretation given by Stenholm [110], in which the Hamiltonian interaction Eq. (2.1.45) is carried out into a non-linear region where directly enters another two signals acting as the probes, i.e., the detectors of the measuring device. The realization of this scheme, without any involvement with the spread of the wave functions characterizing the individual measurement components due to the free energies, can be achieved, for example, with the use of ultrashort optical signals (whose temporal lapse is of the order of pico or femtoseconds) within the rotating wave approximation. Such optical implementations are relegated to the so-called mode-locking technique [111]. Besides, to avoid mixing the prove signals, we need to supply a strong field to pump the non-linear interaction region. According to Stenholm, the dynamics of such interaction is given by the following Hamiltonian

$$
\begin{align*}
\hat{H}_{\mathrm{int}} & =\frac{i}{2}\left[\hat{a}_{3}^{\dagger}\left(\hat{a}_{1}^{\dagger}+i \hat{a}_{2}^{\dagger}\right) \hat{b}^{2}-\hat{a}_{3}\left(\hat{a}_{1}^{\dagger}-i \hat{a}_{2}^{\dagger}\right)\left(\hat{b}^{\dagger}\right)^{2}\right. \\
& \left.+\hat{a}_{3}\left(\hat{a}_{1}^{\dagger}-i \hat{a}_{2}^{\dagger}\right) \hat{b}^{\dagger} \hat{b}-\hat{a}_{3}\left(\hat{a}_{1}^{\dagger}-i \hat{a}_{2}^{\dagger}\right) \hat{b}^{\dagger} \hat{b}\right] \tag{2.1.46}
\end{align*}
$$

where the subscript $1,2,3$ refers to the first and second probe, and the system under investigation respectively, and $\hat{b}$ being the annihilation operator of the pumped field, which if it is strong enough, we can replace it by a real number $\beta$; then, by grouping terms in the above expression we get

$$
\begin{align*}
\hat{H}_{\mathrm{int}} & =\beta^{2}\left[\frac{\left(\hat{a}_{3}+\hat{a}_{3}^{\dagger}\right)}{\sqrt{2}} \frac{\left(\hat{a}_{1}-\hat{a}_{1}^{\dagger}\right)}{i \sqrt{2}}+\frac{\left(\hat{a}_{3}-\hat{a}_{3}^{\dagger}\right)}{i \sqrt{2}} \frac{\left(\hat{a}_{3}-\hat{a}_{3}^{\dagger}\right)}{i \sqrt{2}}\right] \\
& =\beta^{2}\left(\hat{x}_{3} \hat{p}_{1}+\hat{p}_{3} \hat{p}_{2}\right) \tag{2.1.47}
\end{align*}
$$

Therefore choosing $\kappa=\kappa_{1}=\kappa_{2}$ in Eq. (2.1.45), it is apparent that the coupling constant $\kappa$ is related with the field amplitude of the pumped signal, i.e., the average photon number in the field (see page 46 of [4]).

Another scheme for the interaction Hamiltonian, Eq. (2.1.45), is theoretically proposed by Power et. al. [112]; in that, a free traveling atom in its electronic ground state is subject to two cavity fields prepared in coherent states by monochromatic laser fields of equal frequency, one by a standing-wave laser field and the other by a traveling-wave laser field, both propagating along the same direction but polarized orthogonally. The interaction with the atom causes changes in the phase of the fields, which can be measured using some phase measurement techniques such as a balanced homodyne or a heterodyne detection scheme. In the rotating wave approximation and within the considerations of a very large atom-field detuning, a large period of the standing-wave field, and in the Raman-Nath regime of short interaction (allowing discard the free evolution of the system), the authors get an interaction Hamiltonian of the form

$$
\begin{equation*}
\hat{H}_{\mathrm{int}}=-\hbar\left(\lambda_{1} \hat{x}^{\dagger} \hat{a}+\lambda_{2} \hat{p} \hat{b}^{\dagger} \hat{b}\right), \tag{2.1.48}
\end{equation*}
$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators of the atom and $\hat{a}^{\dagger} \hat{a}$ and $\hat{b}^{\dagger} \hat{b}$ represent the photon number operators of the standing and traveling cavity fields respectively; then, the role of coupling constants is played by the $\lambda_{1}$ and $\lambda_{2}$ constants, which are

$$
\begin{equation*}
\lambda_{1}=\frac{\tilde{q} \tilde{\kappa}}{\delta}, \quad \lambda_{2}=-\frac{q \kappa^{2}}{M \delta^{2}}, \tag{2.1.49}
\end{equation*}
$$

where, $\delta$ is the atom-field detuning, $M$ the mass of the atom, $\tilde{q}$ and $q$ are the wavenumbers of the travelling and standing cavity fields respectively and $\tilde{\kappa}$ and $\kappa$ their coupling parameters given by $\kappa=d / \hbar$ and $\tilde{\kappa}=\kappa \sqrt{2}$, where $d$ represent the dipole matrix element, as is usually found in atom-field interactions [4], pag. 75-76. In the Arthurs-Kelly interaction Hamiltonian, the momentum operators of the pointers act as displacement operators in the position of the pointers in proportion to the position and momentum of the investigated system. Instead, in Power's proposal, the canonical pair of the atom is coupled with the number operators of two modes of the field; then, these operators act as phase shifts in the two-cavity fields, which can be measured to infer the position and momentum of the atom. Then, In the context of the proposal of Power, the coupling terms directly depend on the atomic dipole matrix element of the atom and the wavenumbers of the two fields; also, it depends inversely proportional to the atom field detuning.

Another more physical idea in perspective, is analyzed by Törmä et. al. [113] where they tackle a scheme of the simultaneous measurement from the perspective of interferometry using non-classical light; in their work, the system under inspection is given by a simple boson mode represented by a creation operator $\hat{a}^{\dagger}$; besides, there are two independent meter signals, each described by creation operators $\hat{a}_{1}$ and $\hat{a}_{2}$; in the rotating wave approximation the coupling between the three systems is

$$
\begin{equation*}
\hat{H}=\lambda\left[\hat{a}\left(\hat{a}_{1}^{\dagger}+\hat{a}_{2}^{\dagger}\right)+\hat{a}^{\dagger}\left(\hat{a}_{1}+\hat{a}_{2}\right)\right] . \tag{2.1.50}
\end{equation*}
$$

Where such interaction describes photon transfer from the investigated system into two meter channels, and the coupling constant $\lambda$ is in frequency units. According to the authors, the implementation of the interaction, Eq. (2.1.50), is possible through boson coupling in optical fibers.

### 2.1.5 The state of the detectors

In the original proposal of Arthurs and Kelly, they define particular states to represent the proves of the measuring device, in particular, they choice the following Gaussian states

$$
\begin{align*}
\phi_{1}\left(x_{1}\right) & =\left(\frac{2}{\pi b}\right)^{\frac{1}{4}} e^{-x^{2} / b}  \tag{2.1.51}\\
\phi_{2}\left(x_{2}\right) & =\left(\frac{2 b}{\pi}\right)^{\frac{1}{4}} e^{-b y^{2}} \tag{2.1.52}
\end{align*}
$$

being $b$ the so called balance parameter which is apparently related with the variances of the probability distributions of the functions, Eqs. (2.1.51) and (2.1.52). In fact, with the handling of this parameter, it is possible to manipulate the accuracy in the measurement of position or momentum in the simultaneous measurement. However, the wave function of the detectors should be narrow enough such that their possible positions are distinguishable, avoiding overlaps among them; besides, the width of the wave packet neither should be narrow than necessary to prevent the excess of dispersion in the measurement process due to the uncertainty principle [95] pag. 59-61.

To show this argument in more quantitative terms, take into account that the dynamics of the measurement process induce by itself a spread in the position observable of the proves, which, of course, will affect the result of the measurement. The free evolution is governed by the Hamiltonian $\hat{H}=\hat{p}^{2}(0) / 2 m$, where $p(0)$ is the initial momentum of the prove after a preparation procedure; hence in the Heisenberg picture we get the temporal evolution of the position of the prove, according to

$$
\begin{equation*}
\hat{x}(t)=e^{i t \hat{H} / h} \hat{x}(0) e^{-i t \hat{H} / h} \tag{2.1.53}
\end{equation*}
$$

then, using the relation

$$
\begin{equation*}
\left(e^{f_{i}(\zeta) \hat{X}} \hat{Y} e^{-f_{i}(\zeta) \hat{X}}\right)=\hat{Y}+f_{i}[\hat{X}, \hat{Y}]+\frac{f_{i}^{2}}{2!}[\hat{X},[\hat{X}, \hat{Y}]]+\cdots \tag{2.1.54}
\end{equation*}
$$

and the commutator $\left[\hat{p}^{2}, \hat{x}\right]=-2 i \hbar \hat{p}$, we obtain

$$
\begin{equation*}
\hat{x}(t)=\hat{x}(0)+\hat{p}(0) t / m \tag{2.1.55}
\end{equation*}
$$

therefore, the variance of that position observable at time $t$ is

$$
\begin{align*}
(\Delta x)^{2}(t)= & (\Delta \hat{x})^{2}(0)+\left(\frac{t}{m}\right)^{2}(\Delta \hat{p})^{2}(0) \\
& +\left(\frac{t}{m}\right)(\langle\hat{x}(0) \hat{p}(0)+\hat{p}(0) \hat{x}(0)\rangle-2\langle\hat{x}(0)\rangle\langle\hat{p}(0)\rangle) \tag{2.1.56}
\end{align*}
$$

It is important to note that for the minimum uncertainty Gaussian states of Eqs. (2.1.51) and (2.1.52), the covariance term of Eq. (2.1.56) is zero. Then, using the saturation of the Heisenberg uncertainty relation:

$$
\begin{equation*}
(\Delta \hat{p})^{2}(0)=\frac{\hbar^{2}}{4(\Delta \hat{x})^{2}(0)} \tag{2.1.57}
\end{equation*}
$$

in Eq. (2.1.56), we have

$$
\begin{equation*}
(\Delta x)^{2}(t)=(\Delta \hat{x})^{2}(0)+\frac{1}{(\Delta \hat{x})^{2}(0)}\left(\frac{\hbar t}{2 m}\right)^{2} \tag{2.1.58}
\end{equation*}
$$

The above expression takes the possible minimum value for the value

$$
\begin{equation*}
\Delta x(0)=\sqrt{\frac{\hbar t}{2 m}} \tag{2.1.59}
\end{equation*}
$$

which gives the well-known standard quantum limit (SQL)

$$
\begin{equation*}
\Delta x(t)_{\mathrm{SQL}}=\sqrt{\frac{\hbar t}{m}} \tag{2.1.60}
\end{equation*}
$$

The above equation governs the spreading of a free wave packet. And this result states that in two successive measurements of the position of a free mass, the result of the second measurement cannot be predicted with less uncertainty than the SQL [114]. However, it must be noted that this result is partially correct. Yuen [115] has shown that the correlation term in Eq. (2.1.56) can be negative, therefore allowing variances smaller than the SQL for certain contractive states that can be obtained after the first measurement. Notably, as can be graphically verified, for values $\Delta x(0)<\sqrt{\frac{h t}{2 m}}$, the Eq. (2.1.58) rapidly increases, however, for values $\Delta x(0)>\sqrt{\frac{h t}{2 m}}$ it increases slowly, therefore the initial width of the wave packet should acquire values according to $\Delta x(0) \geq \sqrt{\frac{h t}{2 m}}$ in order to avoid excess of spreading in the wave packet.

Interestingly, the states, Eqs. (2.1.51) and (2.1.52) can be summarized by the following expression

$$
\begin{equation*}
\phi_{i}\left(x_{i}\right)=\frac{\sqrt{S_{i}}}{\pi^{\frac{1}{4}}} \exp \left[-\left(x_{i} S_{i}\right)^{2} / 2\right], \quad i=1,2 \tag{2.1.61}
\end{equation*}
$$

where the variances of the probability distributions associated with the wave functions, Eq. (2.1.51) and (2.1.52), are given by $\sigma_{i}^{2}=\left(S_{i}\right)^{-2} / 2$, with $S_{1}=(2 / b)^{\frac{1}{2}}, S_{2}=(2 b)^{\frac{1}{2}}$. However, for the minimum uncertainty Gaussian states given by Eqs. (2.1.51) and (2.1.52) we can rest assure that the SQL holds be true implying that (units of $\hbar=1$ )

$$
\begin{equation*}
\frac{1}{2 S_{i}^{2}} \geq \frac{t}{2 m} \tag{2.1.62}
\end{equation*}
$$

implying that the balance parameter should be bounded as

$$
\begin{equation*}
\frac{2 t}{m} \leq b \leq \frac{m}{2 t} \tag{2.1.63}
\end{equation*}
$$

in order to avoid noise excess due to the free evolution. Notably, the wave function given by Eq. (2.1.61) represent a particular class of minimum uncertainty states known as squeezed vacuum states with
squeezing factor $S_{i}$ [80]; as we will see, these states contribute their noises to the measurement process, hence affecting statistically speaking the measurement outputs.

### 2.2 Derivation of the results of arthurs and kelly

In this section, we derive directly the results given by Arthurs and Kelly. Besides, we analyze the dynamics through the time evolution operator method instead of solving the Schrödinger equation as in their original work. The temporal evolution of the system is obtained as

$$
\begin{equation*}
\Psi\left(x_{1}, x_{2}, x_{3}, t\right)=\exp [-i \hat{H} t] \psi\left(x_{1}, x_{2}, x_{3}, t=0\right) \tag{2.2.1}
\end{equation*}
$$

where the initial wave function $\psi\left(x_{1}, x_{2}, x_{3}, t=0\right)$ is given as a product of pure states according to

$$
\begin{equation*}
\psi\left(x_{1}, x_{2}, x_{3}, t=0\right)=\phi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) \phi_{3}\left(x_{3}\right), \tag{2.2.2}
\end{equation*}
$$

where the states $\phi_{1,2}\left(x_{1,2}\right)$ are the wave functions of the proves of the measuring device, given by Eqs. (2.1.51) and (2.1.52) and the state $\phi_{3}\left(x_{3}\right)$ represent the wave function of the system to be measured, which is assumed arbitrary. The resulting dynamics for the system plus the proves is (see Sec. 5.1 for a derivation step-by-step)

$$
\begin{align*}
\Psi\left(x_{1}, x_{2}, x_{3}, t\right) & =(2 \pi)^{-\frac{1}{2}} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}-\alpha_{2} p_{3}-\frac{\alpha_{1} \alpha_{2} p_{1}}{2}\right) \\
& \times \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \tag{2.2.3}
\end{align*}
$$

which is presented by Arthurs and Kelly and other authors [110, 116]. That state describes the temporal evolution of the whole system while the measurement is in progress and its principal feature lies in the entanglement among the three position variables of the proves and the system. The nature of the entanglement in the complete model that includes the free energies is analyzed in chapter 3 .

Taking the change of variable $\ell=p_{3}+\alpha_{1} p_{1}$, the Eq. (2.2.3) is transformed to

$$
\begin{align*}
\Psi\left(x_{1}, x_{2}, x_{3}, t\right)=\left(\alpha_{1} \sqrt{2 \pi}\right)^{-1} \int \phi_{1}\left(\frac{\ell-p_{3}}{\alpha_{1}}\right) & \phi_{2}\left(x_{2}-\frac{\alpha}{2}\left[p_{3}+\ell\right]\right) \phi_{3}(\ell) \\
& \times e^{-i x_{1}\left(\frac{\ell-p_{3}}{\alpha_{1}}\right)} d \ell \tag{2.2.4}
\end{align*}
$$

taking the squared modulus, the whole probability distribution is

$$
\begin{array}{r}
\left|\Psi\left(x_{1}, x_{2}, x_{3}, t\right)\right|^{2}=\left(2 \pi \alpha_{1}^{2}\right)^{-1} \int \phi_{1}\left(\frac{\ell-p_{3}}{\alpha_{1}}\right) \phi_{1^{*}}\left(\frac{\ell^{\prime}-p_{3}}{\alpha_{1}}\right) \\
\times \phi_{2}\left(x_{2}-\frac{\alpha}{2}\left[p_{3}+\ell\right]\right) \phi_{2}^{*}\left(x_{2}-\frac{\alpha}{2}\left[p_{3}+\ell^{\prime}\right]\right) f(\ell) f^{*}\left(\ell^{\prime}\right) e^{-i x_{1}\left(\frac{\ell-\ell^{\prime}}{\alpha_{1}}\right)} \tag{2.2.5}
\end{array}
$$

Integrating the above expression in $x_{1}$ variable and using the integral formulation of the delta function: $\int e^{-i x_{1}\left(\left[\ell-\ell^{\prime}\right] / \alpha_{1}\right)}\left(\alpha_{1}\right)^{-1} d x_{1}=2 \pi \delta\left(\ell-\ell^{\prime}\right)$, we obtain the probability distribution $P\left(x_{2}, p_{3}, t\right)$

$$
\begin{equation*}
P\left(x_{2}, p_{3}, t\right)=\left(\alpha_{1}\right)^{-1} \int\left|\phi_{1}\left(\frac{\ell-p_{3}}{\alpha_{1}}\right) \phi_{2}\left(x_{2}-\frac{\alpha}{2}\left[p_{3}+\ell\right]\right) \phi_{3}(\ell)\right|^{2} \tag{2.2.6}
\end{equation*}
$$

then, by simple integration, we compute the expected values $\left\langle x_{2}\right\rangle(t)$ and $\left\langle x_{2}^{2}\right\rangle(t)$ which are

$$
\begin{gather*}
\left\langle x_{2}\right\rangle(t)=\int x_{2} P\left(x_{2}, p_{3}, t\right) d x_{2} d p_{3}=\alpha_{2}\left\langle p_{3}\right\rangle  \tag{2.2.7}\\
\left\langle x_{2}^{2}\right\rangle(t)=\int\left(x_{2}\right)^{2} P\left(x_{2}, p_{3}, t\right) d x_{2} d p_{3}=\frac{\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)}{4 b}+\left(\alpha_{2}\right)^{2}\left\langle p_{3}^{2}\right\rangle \tag{2.2.8}
\end{gather*}
$$

Where we recall that the time dependence of the expected values is contained in the $\alpha_{i}$ quantities. By expressing the position representation for the wave function of the measured system, and expressing the wave function of the second prove through a Fourier transform, and using the factorization given by Eq. (5.1.4) we obtain the expected values $\left\langle x_{1}\right\rangle(t)$ and $\left\langle x_{1}^{2}\right\rangle(t)$ according to

$$
\begin{gather*}
\left\langle x_{1}\right\rangle(t)=\alpha_{1}\left\langle x_{3}\right\rangle  \tag{2.2.9}\\
\left\langle x_{1}^{2}\right\rangle(t)=\frac{b}{4}\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)+\left(\alpha_{1}\right)^{2}\left\langle x_{3}^{2}\right\rangle \tag{2.2.10}
\end{gather*}
$$

Hence the expected values, Eqs. (2.2.7) and (2.2.9), indicate that the mean value of the measurement outputs at time $t$ proportionally match the initial mean values (i.e. at time $t=0$ ) of position and momentum observables for any inspected system; this argument goes under the name of joint unbiasedness condition [104]; see Appendix sec. 5.2, and Subsection 2.1.3.

With the help of Eqs. (2.2.7) to (2.2.10), the time-dependent variances of the probability distributions of the position proves are respectively

$$
\begin{align*}
\sigma_{1}^{2}(t) & =\left\langle x_{1}^{2}\right\rangle-\left\langle x_{1}\right\rangle^{2}=\frac{b}{4}\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)+\alpha_{1}^{2} \sigma_{x_{3}}^{2},  \tag{2.2.11}\\
\sigma_{2}^{2}(t) & =\left\langle x_{2}^{2}\right\rangle-\left\langle x_{2}\right\rangle^{2}=\frac{1}{4 b}\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)+\alpha_{1}^{2} \sigma_{p_{3}}^{2}, \tag{2.2.12}
\end{align*}
$$

with $\sigma_{x_{3}}=\left\langle x_{3}^{2}\right\rangle-\left\langle x_{3}\right\rangle^{2}$ and $\sigma_{p_{3}}=\left\langle p_{3}^{2}\right\rangle-\left\langle p_{3}\right\rangle^{2}$ the initial variances of the probability distributions of the position and momentum observables of the system under measurement. Taking the product between the variances (2.2.11) and (2.2.12) and minimizing it respect to $b$, we find the ideal value for the balance parameter which comes as

$$
\begin{equation*}
b=\frac{\alpha_{1} \sigma_{x_{3}}}{\alpha_{2} \sigma_{p_{3}}} . \tag{2.2.13}
\end{equation*}
$$

Assuming that the system under measurement saturates the Heisenberg uncertainty relation (which implies that the system is a minimum uncertainty state), that is $\sigma_{x_{3}}^{2} \sigma_{p_{3}}^{2}=1 / 4$, and using together the balance parameter established by Eq. (2.2.13), we obtain the lower bound for the product of Eqs. (2.2.11) and (2.2.12)

$$
\begin{equation*}
\left(\sigma_{1}^{2}(t) \sigma_{2}^{2}(t)\right)_{\min }=\frac{\left(\alpha_{1} \alpha_{2}\right)^{2}}{4}+\frac{\alpha_{1} \alpha_{2}\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)}{4}+\frac{\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)^{2}}{16} \tag{2.2.14}
\end{equation*}
$$

consequently

$$
\begin{equation*}
\sigma_{1}^{2}(t) \sigma_{2}^{2}(t) \geq \frac{\left(\alpha_{1} \alpha_{2}\right)^{2}}{4}+\frac{\alpha_{1} \alpha_{2}\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)}{4}+\frac{\left(1+\alpha_{1}^{2} \alpha_{2}^{2}\right)^{2}}{16} . \tag{2.2.15}
\end{equation*}
$$

It is important to note that the above inequality represents the uncertainty relation associated with the proves positions at time $t$; that is, while the measurement interaction is in progress. At the time $t=\tau$, we carry out projective measurements on the proves to extract the information about the conjugate pair of the investigated system; in this specific time, the inequality Eq. (2.2.15) constitute the uncertainty relation for the simultaneous measurement.

Now, the measurement process of Arthurs and Kelly is constructed on the joint unbiasedness condition (see Appendix 5.2 and subsection
2.1.3), which implies that the mean values of the outputs of the detectors match on average the theoretical values of the system under inspection, this condition implies that we need $\alpha_{i}=1, i=1,2$ in Eqs. (2.2.7) and (2.2.9), implying the following two physical conditions:
(i) Setting $\alpha_{1}=\alpha_{2}=1$, implying that the coupling constants $\kappa_{i}, i=1,2$ must be equal; therefore, the time of the joint measurement is $\tau=\frac{1}{\kappa}$. Hence the positions of the pointers are just displaced by the canonical pair of the system.
(ii) Setting one time of the measurement $\tau=\frac{1}{\kappa_{i}}$ implying that the coupling constants $\kappa_{i}, i=1,2$ are different; then, it is necessary to adjust the measurement of the pointer $j$ at the rate $\frac{\kappa_{j}}{\kappa_{i}}$.

At time $\tau$ for any of the two previous situations, the Ineq. (2.2.15) is reduced to the uncertainty relation obtained by Arthurs and Kelly:

$$
\begin{equation*}
\sigma_{1}(\tau) \sigma_{2}(\tau) \geq 1 \tag{2.2.16}
\end{equation*}
$$

### 2.3 ACCURACY IN THE SIMULTANEOUS MEASUREMENT PROCESS

In this section, we employ the full artillery developed in Sec. 2.1.2 to characterize the retrodictive and predictive aspects of accuracy in the simultaneous measurement process raised by Arthurs and Kelly.

To start the analysis, we recall that the labels $\{1,2,3\}$ refers to the first and second detectors, and to the system under measurement. Then, let us switch the Heisenberg scenario to compute the dynamics of the measurement process under the interaction Hamiltonian, Eq. (2.1.45).

Hence, by using the Baker-Campbell-Hausdorf factorization, Eqs. (5.1.3) and (5.1.4) we have

$$
\begin{align*}
& \hat{x}_{1}^{\prime}=\hat{x}_{1}+\alpha_{1} \hat{x}_{3}+\frac{\alpha_{1} \alpha_{2}}{2} \hat{p}_{2}, \\
& \hat{x}_{2}^{\prime}=\hat{x}_{2}+\alpha_{2} \hat{p}_{3}-\frac{\alpha_{1} \alpha_{2}}{2} \hat{p}_{1}, \\
& \hat{x}_{3}^{\prime}=\hat{x}_{3}+\alpha_{2} \hat{p}_{2},  \tag{2.3.1}\\
& \hat{p}_{1}^{\prime}=\hat{p}_{1}, \\
& \hat{p}_{2}^{\prime}=\hat{p}_{2}, \\
& \hat{p}_{3}^{\prime}=\hat{p}_{3}-\alpha_{1} \hat{p}_{1} .
\end{align*}
$$

In the subsequent, we choose the $\kappa_{1}=\kappa_{2}=1$, which implies $\alpha_{1}=$ $\alpha_{2}=1$ as in the original Arthurs-Kelly proposal.

### 2.3.1 Retrodictive aspect

Using the definitions for the detector proves, Eqs. (2.1.51) and (2.1.52), and the definitions for the retrodictive error operators, Eqs. (2.1.11) and (2.1.12), together the dynamics (2.3.1), we obtain for the retrodictive error observables

$$
\begin{align*}
& \hat{\varepsilon}_{\hat{x}_{3}}^{R}=\hat{x}_{1}^{\prime}-\hat{x}_{3}=\hat{x}_{1}+\frac{\hat{p}_{2}}{2}  \tag{2.3.2}\\
& \hat{\varepsilon}_{\hat{p}_{3}}^{R}=\hat{x}_{2}^{\prime}-\hat{p}_{3}=\hat{x}_{2}-\frac{\hat{p}_{1}}{2} \tag{2.3.3}
\end{align*}
$$

from which is straightforward verify the completion of Eq. (2.1.42), since $\left[\hat{\varepsilon}_{\hat{x}_{3}}^{R}, \hat{\varepsilon}_{\hat{p}_{3}}^{R}\right]=-i \hbar$. Then, by computing the variances of Eqs. (2.3.2) and (2.3.3) by means of the initial state, Eq. (2.2.2), we have

$$
\begin{align*}
& \sigma_{\hat{\varepsilon}_{\hat{x}_{3}}^{R}}^{2}=\sigma_{\hat{x}_{1}}^{2}+\frac{\sigma_{\hat{p}_{2}}^{2}}{4}=\frac{b}{2},  \tag{2.3.4}\\
& \sigma_{\hat{\epsilon}_{\hat{p}_{3}}^{R}}^{2}=\sigma_{\hat{x}_{2}}^{2}+\frac{\sigma_{\hat{p}_{1}}^{2}}{4}=\frac{1}{2 b}, \tag{2.3.5}
\end{align*}
$$

where we have used $\sigma_{\hat{x}_{1}}^{2}=(b / 4), \sigma_{\hat{p}_{1}}^{2}=(1 / b), \sigma_{\hat{x}_{2}}^{2}=(4 b)^{-1}$ and $\sigma_{\hat{p}_{2}}^{2}=$ $b$ directly verifiable from the definitions Eqs. (2.1.51) and (2.1.52). Therefore, the variances, Eqs. (2.3.4) and (2.3.5), shows that the noise directly affecting the the retrodictive accuracy of the joint measurement of
the canonical pair, comes from the noise of the detectors itself. Evidently, one can manipulate the balance parameter in the scheme preparation of the state proves, of course, before the measurement process, to improve the retrodictive accuracy in the measurement of any of the two canonical observables; however, it must be noted from Eqs. (2.3.8) and (2.4.1) that a simultaneous measurement of position and momentum observables never will be precise in the retrodictive aspect, since by improving the resolution for some of the two conjugate observables, one is necessarily obliged to increase the error in the corresponding conjugate observable. The maximum retrodictive accuracy in the simultaneous measurement of the canonical pair of a quantum system (with independent detector proves) comes with the saturation of the retrodictive error uncertainty relation, Eq. (2.1.42).

### 2.3.2 Predictive aspect

Using the definitions for the detector proves, Eqs. (2.1.11) and (2.1.12), and the definitions for the predictive error operators, Eqs. (2.1.13) and (2.1.14), together the dynamics (2.3.1), we obtain for the predictive error observables the following expressions

$$
\begin{align*}
& \hat{\varepsilon}_{\hat{x}_{3}}^{P}=\hat{x}_{1}^{\prime}-\hat{x}_{3}^{\prime}=\hat{x}_{1}-\frac{\hat{p}_{2}}{2}  \tag{2.3.6}\\
& \hat{\varepsilon}_{\hat{p}_{3}}^{P}=\hat{x}_{2}^{\prime}-\hat{p}_{3}=\hat{x}_{2}+\frac{\hat{p}_{1}}{2} \tag{2.3.7}
\end{align*}
$$

Then, we also verify the commutator $\left[\hat{\varepsilon}_{\hat{x}_{3}}^{P}, \hat{\varepsilon}_{\hat{p}_{3}}^{P}\right]=i \hbar$, expressed by Eq. (2.1.43) and consequently the uncertainty relation, Eq. (2.1.44). Then, by computing the variances of Eqs. (3.2.1) and (3.2.2) by means of the initial state, Eq. (2.2.2), we have

$$
\begin{align*}
& \sigma_{\hat{\varepsilon}_{\hat{x}_{3}}^{P}}^{2}=\sigma_{\hat{x}_{1}}^{2}+\frac{\sigma_{\hat{p}_{2}}^{2}}{4}=\frac{b}{2}  \tag{2.3.8}\\
& \sigma_{\hat{\varepsilon}_{\hat{p}_{3}}^{P}}^{2}=\sigma_{\hat{x}_{2}}^{2}+\frac{\sigma_{\hat{p}_{1}}^{2}}{4}=\frac{1}{2 b} \tag{2.3.9}
\end{align*}
$$

hence, we verify directly that the predictive error in the Arthurs-Kelly scheme for the simultaneous measurement of the canonical pair is exactly the same that the retrodictive error described in the subsection 2.3.2; therefore we have the same conclusions explained for that aspect.

### 2.4 The Simultaneous measurement process As a QuanTUM ENTANGLEMENT GENERATOR

### 2.4.1 Genuine tripartite entanglement

In this section, we pose the original Arthurs-Kelly measurement process as a multipartite, in particular, tripartite entanglement generator in the regime of continuous variable Gaussian states. It is a well-known fact that any quantum measurement process employs the quantum entanglement as a resource to transfer the information of the physical properties of interest to the quantum state of a measuring device; we explain with more details this argument in section 2.4.

Nowadays, there exists considerable interest in the entanglement of multipartite systems, that is, those which consider more than two subsystems. Its usefulness lies in the potential benefits for the quantum information processing; for example, the creation of entanglement between many atoms or ions [117, 118, 119] is useful for fault-tolerant quantum computing [120, 121, 122], high-precision measurements using matter-wave interference [123] and the fundamental investigation of the quantum to classical regime transition [124, 125, 126]. In the case of many entangled photons [127, 128, 129, 130, 131, 132, 133], it allows entanglement verification in quantum networks [134], generation of cluster states [135] and error correction protocols [136, 137, 138]. Also, we can generate multipartite entanglement in the continuous variable regime by using modes of the electromagnetic field [76, 77, 139], which find applicability in teleportation networks [7] and multiuser quantum channel for telecloning [140].

In particular, the multipartite entanglement in continuous-variable systems constitutes a valuable resource for fundamental studies and practical applications. The archetypical paradigm is represented by the Gaussian states, which are those who present a Gaussian profile in the quantum phase space. The Gaussian states are referent from the optical domain, like the coherent, squeezed, thermal, and vacuum states, however, also living in other physical scenarios as in trapped ions [141, 142, 143], nanomechanical resonators [144, 145] and optical cavities $[146,147]$. The kind properties of Gaussian states include easy experimental generation, simple mathematical description, and accessible
resources for creating quantum entanglement; therefore, they are the favored systems for CV quantum informational tasks. Besides, the Gaussian states are the test-bed for fundamental studies in CV systems as non-separability [9], entanglement sharing [34, 37], and entanglement and correlation measures $[3,34,35]$.

The first case of a multipartite Gaussian entangled system is given by the three-partie scenario. The establishment of genuine tripartite entanglement in Gaussian states gives a wide variety of applications in quantum communication processes, offering the direct possibility to improve the new wave of quantum technologies which will make up the second quantum revolution [148]. In this section, we pose the original Arthurs-Kelly measurement process as a multipartite, in particular, tripartite entanglement generator in the regime of continuous variable Gaussian states. It is a well-known fact that any quantum measurement process employs the quantum entanglement as a resource to transfer the information of the physical properties of interest to the quantum state of a measuring device; we explain with more details this argument in section 2.4. Therefore, in this section, we determine the qualitative and quantitative entanglement properties develop in the measurement process in a complete Gaussian measurement configuration. It is important to note that through this section, we will use units of $\hbar=2$.

### 2.4.2 Measurement configuration

The original measurement set-up consider a couple of pointer detectors which are represented by balanced centred Gaussian states with finite squeezing; see Section 2.1.5. In particular we establish the relation with the states presented in Reference [79]; then we define the state of the detectors, Eqs. (2.1.51) and (2.1.52) in Dirac notation as

$$
\begin{equation*}
\left|0, V_{j}\right\rangle_{p}=\left(\pi V_{j}\right)^{-\frac{1}{4}} \int d x_{j} e^{-x_{j}^{2} / 2 V_{j}}\left|x_{j}\right\rangle, \quad j=1,2 \tag{2.4.1}
\end{equation*}
$$

with $V_{j}<1$; therefore, being both detectors squeezed in $x_{j}$ direction. The label $x$ remind us that we define the state in position space through the superposition of the position quadrature basis $\left\{\left|x_{j}\right\rangle\right\}_{q \in \mathbb{R}}$ with the

Gaussian wave packet $\phi\left(x_{j}\right)=\left(\pi V_{j}\right)^{-1 / 4} e^{-x_{j}^{2} / 2 V_{j}}$, being $\delta_{\hat{x}_{j}}^{2}=V_{j} / 2$ its variance. By a Fourier transform, the representation in position space is

$$
\begin{equation*}
\left|0, V_{j}\right\rangle_{p}=\left(\frac{V_{j}}{4 \pi}\right)^{\frac{1}{4}} \int d p_{j} e^{-V_{j} p_{j}^{2} / 8}\left|p_{j}\right\rangle, \quad j=1,2, \tag{2.4.2}
\end{equation*}
$$

expanded in the momentum quadrature basis $\left\{\left|p_{j}\right\rangle\right\}_{p_{j} \in \mathbb{R}}$ and variance $\delta_{\hat{p}_{j}}^{2}=2 / V_{j}$. Therefore, the quadrature basis of the states, Eqs. (2.4.1) and (2.4.2), are connected through

$$
\begin{align*}
|q\rangle & =\frac{1}{2 \sqrt{\pi}} \int d p e^{-i q p / 2}|p\rangle  \tag{2.4.3}\\
|p\rangle & =\frac{1}{2 \sqrt{\pi}} \int d p e^{i q p / 2}|q\rangle \tag{2.4.4}
\end{align*}
$$

For the Arthurs-Kelly pointers we have $V_{1}=b, V_{2}=1 / b$, hence the states, Eq. (2.4.1) and (2.4.2), are related via $V_{1}=\left(V_{2}\right)^{-1}$. Besides, it is easy to verify the saturation of the Heisenberg uncertainty relation $\delta_{\hat{x}_{j}}^{2} \delta_{\hat{p}_{j}}^{2} \geq 1$; therefore, they are minimum uncertainty states, which is a necessary characteristic to reduce the inaccuracy affecting the statistics of the measurement outputs since the detectors contribute its noises to the measurement process; see Sections 2.3.1 and 2.3.2.

For the system under measurement, we choose the most general pure one-mode Gaussian state, that is, a rotated, displaced, and squeezed vacuum state, $|\alpha, \theta, r\rangle$, which is defined in terms of the vacuum state as [23]

$$
\begin{equation*}
|\alpha, \theta, r\rangle=\hat{D}(\alpha) \hat{R}(\theta) \hat{S}(r)|0\rangle \tag{2.4.5}
\end{equation*}
$$

with the unitary operators

$$
\begin{gather*}
\hat{D}(\alpha) \equiv \exp \left[\alpha \hat{a}^{\dagger}-\alpha^{*} \hat{a}\right],  \tag{2.4.6}\\
\hat{S}(r) \equiv \exp \left[r\left(\hat{a}^{2}-\hat{a}^{\dagger 2}\right) / 2\right], \quad r \in \mathbb{R},  \tag{2.4.7}\\
\hat{R}(\theta) \equiv \exp \left[-i \theta \hat{a}^{\dagger} \hat{a}\right], \quad 0 \leq \theta \leq 2 \pi \tag{2.4.8}
\end{gather*}
$$

being $\hat{D}(\alpha), \hat{S}(r)$ and $\hat{R}(\theta)$ the displacement, the one-mode squeezing, and the rotation operators respectively. The squared modulus of the complex amplitude $\alpha=(q+i p / 2)$ is related to the average energy of the single-mode system, the squeezing parameter $r$ governs its degree of
squeezing and the angle $\theta$ represents a phase displacement with respect to a local oscillator. Using the operators, Eqs. (2.4.6) to (2.4.8), the quadratures in the Heisenberg picture for this system are

$$
\begin{gather*}
\hat{q}_{3}=(\hat{q}+q) e^{-r} \cos \theta+(\hat{p}+p) e^{r} \sin \theta  \tag{2.4.9}\\
\hat{p}_{3}=-(\hat{q}+q) e^{-r} \sin \theta+(\hat{p}+p) e^{r} \cos \theta \tag{2.4.10}
\end{gather*}
$$

where $\hat{q}$ and $\hat{p}$ is the initial canonical pair of the vacuum state $|0\rangle$. It is straightforward verify the mean values

$$
\begin{gather*}
\left\langle\hat{q}_{3}\right\rangle=q e^{-r} \cos \theta+p e^{r} \sin \theta  \tag{2.4.11}\\
\left\langle\hat{p}_{3}\right\rangle=-q e^{-r} \sin \theta+p e^{r} \cos \theta \tag{2.4.12}
\end{gather*}
$$

and the variances

$$
\begin{align*}
& \delta_{\hat{q}_{3}}^{2}=\frac{(\cos \theta)^{2}}{e^{2 r}}+e^{2 r}(\sin \theta)^{2}  \tag{2.4.13}\\
& \delta_{\hat{p}_{3}}^{2}=\frac{(\sin \theta)^{2}}{e^{2 r}}+e^{2 r}(\cos \theta)^{2} \tag{2.4.14}
\end{align*}
$$

Then, within this considerations, the system under measurement does not necessarily represent a minimum uncertainty state, unless that $r=0$ or $\theta=n \pi / 2, n=0,1,2, \cdots$; in this case, the system under measurement is given by a symmetric Gaussian quasi-probability distribution (Wigner function) in the phase space. Hence we have the initial state of the measurement setting as the product

$$
\begin{equation*}
|\psi\rangle_{t=0}=\left|0, V_{1}\right\rangle_{q, p}\left|0, V_{2}\right\rangle_{q, p}|\alpha, \theta, r\rangle \tag{2.4.15}
\end{equation*}
$$

where the label $q, p$ in the state of the detectors means that we can express them in the position or momentum representation. Therefore, we take each subsystem as a continuous variable mode with a associated bosonic operator $\hat{a}_{k}, k=1,2,3$ and quadratures $\hat{q}_{k}=\left(\hat{a}_{k}+\hat{a}_{k}\right)$ and $\hat{p}_{k}=i\left(\hat{a}_{k}^{\dagger}-\hat{a}_{k}\right)$, just as those of a quantum harmonic oscillator or a single-mode of the quantum electromagnetic field. Therefore, within this considerations we have an entirely pure continuous variable Gaussian measurement setting.

### 2.4.3 Problem statement

The original proposal raised by Arthurs and Kelly involves three quantum mechanical systems, i.e., the two detectors of the measuring device, together with the system from which we desire simultaneously to know its conjugate pair. As we will explain in Section 2.4, it is the measurement process that causes the loss of individuality for each subsystem, hence creating quantum correlations among the three variables of each subsystem. To give a more detailed explanation for this argument, let us analyze the general dynamics of the measurement process given by Eqs. (2.3.1). These equations suggest the existence of quantum correlations among the three CV modes constituting the measurement setting. By inspecting the $\hat{x}_{1}^{\prime}$ and $\hat{x}_{3}^{\prime}$ variables we observe correlations among $\hat{x}_{1}, \hat{x}_{3}$ and $\hat{p}_{2}$, and through $\hat{x}_{2}^{\prime}$ and $\hat{p}_{3}^{\prime}$, we observe correlations between $\hat{p}_{1}, \hat{x}_{2}$ and $\hat{p}_{3}$; then, both relations implying a tripartite entanglement relation between three distinct observables of the measurement configuration; see Fig. 3. In the subsequent we will focus on to prove the existence of genuine tripartite entanglement in the measurement setting as Eqs. (2.3.1) suggest.


Figure 3: Tripartite entanglement implied by the Arthurs-Kelly measurement process. The dynamics (see Eq. (2.3.1)) of $x_{1}^{\prime}$ and $\hat{x}_{3}^{\prime}$ shows an tripartite correlation among the $\hat{x}_{1}, \hat{x}_{3}$ and $\hat{p}_{2}$ variables (dashed lines). The dynamics of $x_{2}^{\prime}$ and $\hat{p}_{3}^{\prime}$ shows an tripartite correlation among the $\hat{p}_{1}, \hat{x}_{2}$ and $\hat{p}_{3}$ variables (dotted lines). The red circle represents the first detector, the green one the second detector, and the blue circle constitutes the system under inspection, in this case a Gaussian state.

### 2.4.4 Separability properties

To prove genuine tripartite entanglement in the measurement set-up, we recur to the biseparability condition expressed by Eq. (1.3.18). Therefore, focusing on the regime of pure states, the biseparability condition implies that by neglecting the three possible bipartitions of (1 vs 2$)$-modes of the system we certify genuine tripartite entanglement. This statute also will be employed in Section 3.5.1 to determine the separability properties of the completely dynamical Arthurs-Kelly measurement process. Since the PPT criterion is a necessary a sufficient condition to determine the separability of any ( 1 vs N )-mode Gaussian system (see Section 1.3.5.2), we use it to test each of the bipartitions in the measurement set-up.

Using the dynamics exposed by Eqs. (2.3.1), we define the following three quadrature vectors

$$
\begin{align*}
& \hat{\mathbf{R}}_{1}=\left(\hat{\mathbf{C}}_{1}, \hat{\mathbf{C}}_{2}, \hat{\mathbf{C}}_{3}\right)^{T}  \tag{2.4.16}\\
& \hat{\mathbf{R}}_{2}=\left(\hat{\mathbf{C}}_{3}, \hat{\mathbf{C}}_{1}, \hat{\mathbf{C}}_{2}\right)^{T}  \tag{2.4.17}\\
& \hat{\mathbf{R}}_{3}=\left(\hat{\mathbf{C}}_{2}, \hat{\mathbf{C}}_{3}, \hat{\mathbf{C}}_{1}\right)^{T} \tag{2.4.18}
\end{align*}
$$

with $\hat{\mathbf{C}}_{i}=\left(\hat{x}_{i}^{\prime}, \hat{p}_{i}^{\prime}\right)$. Then, by employing the definition for the generic element of the covariance matrix (units free $\hbar=2$ ): $(1 / 2) V_{i j}$ (see Eq. (1.2.9)), we obtain the following three $(6 \times 6)$-dimensional covariance matrices in block form

$$
\begin{align*}
& \boldsymbol{\sigma}_{1 \mid 23}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{1} & \varepsilon_{1,2} & \varepsilon_{1,3} \\
\boldsymbol{\varepsilon}_{1,2}^{T} & \boldsymbol{\sigma}_{2} & \varepsilon_{2,3} \\
\boldsymbol{\varepsilon}_{1,3}^{T} & \varepsilon_{2,3}^{T} & \boldsymbol{\sigma}_{3}
\end{array}\right)  \tag{2.4.19}\\
& \boldsymbol{\sigma}_{3 \mid 12}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{3} & \boldsymbol{\varepsilon}_{1,3}^{T} & \varepsilon_{2,3}^{T} \\
\boldsymbol{\varepsilon}_{1,3} & \boldsymbol{\sigma}_{1} & \varepsilon_{1,2} \\
\boldsymbol{\varepsilon}_{2,3} & \boldsymbol{\varepsilon}_{1,2}^{T} & \boldsymbol{\sigma}_{2}
\end{array}\right)  \tag{2.4.20}\\
& \boldsymbol{\sigma}_{2 \mid 31}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{2} & \boldsymbol{\varepsilon}_{2,3} & \boldsymbol{\varepsilon}_{1,2}^{T} \\
\boldsymbol{\varepsilon}_{2,3}^{T} & \boldsymbol{\sigma}_{3} & \boldsymbol{\varepsilon}_{1,3}^{T} \\
\boldsymbol{\varepsilon}_{1,2} & \boldsymbol{\varepsilon}_{1,3} & \boldsymbol{\sigma}_{1}
\end{array}\right) \tag{2.4.21}
\end{align*}
$$

where the $\boldsymbol{\sigma}_{i}$ and $\boldsymbol{\varepsilon}_{j, k}$ elements are $(2 \times 2)$-dimensional matrices defined in the Appendix 5.8. The covariance matrices, Eqs. (2.4.19) to (2.4.21),
are associated with a pure state since they satisfy Eqs. (1.4.2), (1.4.3) and (1.4.5).

Using the definition, Eq. (1.3.19), through the software Mathematica, we partially transpose the covariance matrices, Eqs. (2.4.19) to (2.4.21); then, we compute the symplectic eigenvalues $\tilde{\nu}_{l}^{i \mid j k}, i, j, k, l=1,2,3, i \neq$ $j \neq k$, as we explain in Eq. (1.3.8). In particular, Mathematica does not provide close expressions for the symplectic eigenvalues; instead, these quantities are given in terms of the roots of a cubic polynomial whose coefficients are functions of $r$ and $\theta$. Notably, we find

$$
\begin{equation*}
\tilde{\nu}_{i}^{(1 \mid 23)}=\tilde{\nu}_{i}^{(2 \mid 31)}, i=1,2,3, \tag{2.4.22}
\end{equation*}
$$

which is an indicative of certain symmetry arround the mode associated with the system under measurement; we will analyze this fact in Sec. 2.4.6.

Now, in the following, we will focus on a rectangular window in the plane $r$ and $\theta$

$$
\begin{equation*}
\mathcal{R}=\{(r, \theta) \mid-5 \leq r \leq 5,0 \leq \theta \leq 2 \pi\}, \tag{2.4.23}
\end{equation*}
$$

in order to explore numerically the maximum and minimum values of the computed symplectic eigenvalues inside this region; we infer the following three cases

$$
\begin{align*}
& \tilde{\nu}_{1}^{\{(1 \mid 23),(3 \mid 12),(2 \mid 31)\}}>1,  \tag{2.4.24}\\
& \tilde{\nu}_{2}^{\{(1 \mid 23),(3 \mid 12),(2 \mid 31)\}}=1,  \tag{2.4.25}\\
& \tilde{\nu}_{3}^{\{(1 \mid 23),(3 \mid 12),(2 \mid 31)\}}<1 ; \tag{2.4.26}
\end{align*}
$$

in particular, we find the following maximum values: $\max \left[\tilde{\nu}_{3}^{(1 \mid 23)}\right]=$ $\max \left[\tilde{\nu}_{3}^{(2 \mid 31)}\right] \approx 0.268$ and $\max \left[\tilde{\nu}_{3}^{(3 \mid 12)}\right] \approx 0.171$, obtained when the Gaussian system under measurement is a minimum uncertainty state. In Fig. 4 we show the plots and density plots for the symplectic eigenvalues, $\tilde{\nu}_{3}^{i \mid j k} \forall r, \theta \in \mathcal{R}$, of the three transposed covariance matrices, Eqs. (2.4.19) to (2.4.21); the existence of values less than 1 for these quantities is equivalent to the violation of the uncertainty relation, Eq. (1.2.11), for each partially transposed covariance matrix; hence, this negates all (1 vs 2 )-mode bipartitions of the system; see Fig. 5. Therefore, we certify the generation of genuine tripartite entanglement in the simultaneous measurement process of Arthurs and Kelly when the system under


Figure 4: Plots (left figures) and density plots (right figures) for the symplectic eigenvalues: a) $\tilde{\nu}_{3}^{\{(1 \mid 23),(2 \mid 31)\}}$ and b) $\tilde{\nu}_{3}^{(3 \mid 12)}$ as functions of the squeezing parameter $r$ and the rotation angle $\theta$ of the Gaussian system under measurement. Values less than 1 for these quantities confirm the unphysical validity of the three partially transposed CMs given in Eqs. (2.4.19) to (2.4.21); hence this neglect the corresponding bipartitions, therefore, this implies genuine tripartite entanglement in the measurement setting of Arthurs and Kelly when the system under observation is the most general pure single-mode Gaussian state, that is, a rotated, displaced, and squeezed vacuum state.


Figure 5: Sketch for the three $i \mid j k$-mode bipartitions in the Arthurs-Kelly measurement setting. The green rectangle assumes entanglement between the first detector and the Gaussian system but not from them with the second detector (bipartition $2 \mid 31$ ). The brown rectangle implies entanglement between the second detector and the Gaussian system but not from them with the first detector (bipartition $1 \mid 23$ ). The purple semicircles encompass entanglement among the two detectors but not with the system under measurement (bipartition $3 \mid 12)$. The three depicted bipartitions are not valid due to the tripartite entanglement relation (dotted and dashed lines) between the three conjugate pairs of observables in the whole system.
observation is the most general one-mode Gaussian state, being this fact independent of the squeezing $r$ and the rotation angle $\theta$ of the Gaussian system under observation.

### 2.4.5 Qualitative properties

In section 2.4.4, we prove the non-separability of the three ( 1 vs 2 )-mode bipartitions of the measurement configuration of the Arthurs-Kelly model when the system under measurement is the most general single-mode Gaussian state. Notably, this fact is independent of the properties of the Gaussian system under observation. Therefore, according to these results, we catalog the generated entanglement in the category ( $C 1$ ) of fully inseparable three-mode Gaussian states (see Section 1.4.2). Placing the Gaussian tripartite entanglement of the strong coupling regime of the measurement model of Arthurs and Kelly together the GHZ states [10, 76] (see section 1.4.3.1), and the tripartite version of the CV Einstein-Podolsky-Rosen (EPR) states (see Section 1.4.3.2), generated in [77, 78].

### 2.4.6 Quantitative properties

In this section, we use the residual tripartite Rényi-2 entanglement as a quantifier measure to examine the amount of tripartite entanglement generated in our specific Gaussian measurement set-up; see Section 2.4.2. The use of this measure will allow us to analyze the entanglement of all global and reduced mode bipartitions $\mathcal{E}_{2}^{\left(A_{i} \mid A_{j} A_{k}\right)}$ and $\mathcal{E}_{2}^{\left(A_{i} \mid A_{j}\right)}$, allowing us to explore the nature of the entanglement structure of the system.

We recall that in our tripartite system, $A_{1}$ represents the partie containing the mode of the first detector, $A_{2}$ contains the mode of the second detector and $A_{3}$ comprises the mode of the Gaussian system under observation. Then, according to the mathematical requirements of Section 1.4.4.3 and the standard form 1.4.12, we only need the local single-mode symplectic invariants $a_{i}$ of the covariance matrices, Eqs. (2.4.19) to (2.4.21); therefore using Eq. (1.4.9) togheter the definitions for the block matrices of Appendix 5.8, we determine the following quantities for our Gaussian tripartite system

$$
\begin{align*}
& a_{1}=\sqrt{2+2 \sqrt{\cosh ^{2}(2 r)-\cos ^{2}(2 \theta) \sinh ^{2}(2 r)}}  \tag{2.4.27}\\
& a_{2}=\sqrt{2+2 \sqrt{\cosh ^{2}(2 r)-\cos ^{2}(2 \theta) \sinh ^{2}(2 r)}}  \tag{2.4.28}\\
& a_{3}=\sqrt{5+4 \sqrt{\cosh ^{2}(2 r)-\cos ^{2}(2 \theta) \sinh ^{2}(2 r)}} \tag{2.4.29}
\end{align*}
$$

These quantities have an oscillatory behavior with a period of $\pi / 2$; besides, they are increasing functions as the magnitude of the squeeze parameter also increases. Their maximum values are attained at $\theta=(2 n+$ 1) $\pi / 4, n=0,1,2, \cdots$ for a determined $r \neq 0$. Their absolute minimums are reached when the Gaussian system under measurement is a minimum uncertainty state, that is when $r=0$ for any $\theta$, or when $\theta=n \pi / 2$ for any $r$. From the above and Eq. (1.4.10), we deduce that the reduced singlemode systems decrease their purity as the magnitude of the squeezing of the Gaussian system under measurement grows, which is indicative that the global tripartite entanglement also increases at increasing $|r|$. This is in concordance with the fact (in an optical context) that protocols using entanglement as a resource (like teleportation) allow carry out the transfer of information with increasing fidelity as the squeezing of the
involved optical modes (usually squeezed states) is higher. In all the following, we omit the $r$ and $\theta$ dependence in all pertinent quantities for simplicity. We begin with the Rényi-2 entanglements $\mathcal{E}_{2}^{\left(A_{i} \mid A_{j}\right)}$ for the two-mode reduced states to examine the one-to-one entanglement relations in our Arthurs-Kelly measurement setting. First, we carry out a numerical minimization for the difference $a_{3}-\sqrt{a_{1}^{2}+a_{2}^{2}-1}, \forall r, \theta$ obtaining 0 , which implies $a_{3} \geq \sqrt{a_{1}^{2}+a_{2}^{2}-1}$; therefore, according to Eq. (1.4.66) we conclude that $g_{3}=1$; then, due to Eq. (1.4.65), the above means that

$$
\begin{equation*}
\mathcal{E}_{2}^{\left(A_{1} \mid A_{2}\right)}=0 . \tag{2.4.30}
\end{equation*}
$$

For the reduced entanglements $\mathcal{E}_{2}^{\left(A_{1} \mid A_{3}\right)}$ and $\mathcal{E}_{2}^{\left(A_{2} \mid A_{3}\right)}$ it must be noticed from Eqs. (2.4.27) and (2.4.28) that $a_{1}=a_{2}=a$; then, from Eq. (1.4.67), this implies that $\alpha_{1}=\alpha_{2}=\alpha$; hence, from Eq. (1.4.66), this means $g_{1}=g_{2}=g$; using this values in Eqs. (1.4.65) to (1.4.68) we deduce

$$
\begin{equation*}
\mathcal{E}_{2}^{D S}=\mathcal{E}_{2}^{\left(A_{1} \mid A_{3}\right)}=\mathcal{E}_{2}^{\left(A_{2} \mid A_{3}\right)} . \tag{2.4.31}
\end{equation*}
$$

Where the superscript $D S$ in Eq. (2.4.31) means 'Detector-System'; it is used to denote the Rényi-2 entanglement of the reduced bipartition, including the mode of any detector and the Gaussian system under measurement.

We carry out a numerical minimization for the differences $\sqrt{a^{2}+a_{3}^{2}-1}-$ $a$ and $\sqrt{a^{2}+a_{3}^{2}-1}-\alpha, \forall r, \theta$ obtaining respectively $\approx 1.464$ and 1.428; this implies that $\sqrt{a^{2}+a_{3}^{2}-1}>\{a, \alpha\}$; therefore, according to Eqs. (1.4.65) and (1.4.66), we have $\mathcal{E}_{2}^{D S} \neq 0$. On the other hand, we analyze the difference $a-\alpha$ as a function of $r$ and $\theta$, finding the two cases: $\alpha<a$ and $a \leq \alpha$; therefore, according to Eqs. (1.4.65) and (1.4.66) we have On the other hand, we analyze the difference $a-\alpha$ as a function of $r$ and $\theta$, finding the two cases: $\alpha<a$ and $a \leq \alpha$; therefore, according to Eqs. (1.4.65) and (1.4.66) we have

$$
\begin{equation*}
\mathcal{E}_{2}^{D S}=\frac{1}{2} \ln g, \tag{2.4.32}
\end{equation*}
$$

with

$$
g= \begin{cases}\frac{\beta}{8 a^{2}}, & \text { if } \alpha<a  \tag{2.4.33}\\ \left(\frac{a^{2}-a_{3}^{2}}{a^{2}-1}\right)^{2}, & \text { if } a \leq \alpha\end{cases}
$$



Figure 6: Plot for the reduced entanglement $\mathcal{E}_{2}^{D S}$ as a function of $r$ and $\theta$. This quantity represents the amount of entanglement contained in the reduced bipartition containing the mode of any detector and that of the Gaussian system under inspection. It has an oscillatory behavior with period $\pi / 2$. The maximum value is $\ln (5 / 3)$ attained when the Gaussian system under measurement is a minimum uncertainty state. The minimum values are at $\theta=(2 n+1) \pi / 4, n=0,1,2, \cdots$ for $r \neq 0$, tending asymptotically to the absolute minimum of $(1 / 2) \ln (2)$ in the limit situation of $|r| \longrightarrow \infty$.

What essentially Eq. (2.4.30) means is that the partie $A_{3}$, which contains the mode of the Gaussian system under measurement, is the principal support of the tripartite entanglement relationship between $A_{1}, A_{2}$ and $A_{3}$; that is, without the partie $A_{3}$, there is no possibility to see any entanglement in the reduced system containing the parties $A_{1}$ and $A_{2}$ only, which is an expected fact since, under the interaction Hamiltonian, Eq. (2.1.45), one of the two conjugate observables of the detectors is directly linked with one of the two belonging to the Gaussian system under inspection, just as we sketch in Fig. 3. Therefore, the entanglement relation between the detectors in the tripartite system is an inherent consequence of their coupling with the Gaussian system under observation. On the other hand, Eq. (2.4.31) imply that the two reduced bipartitions containing the modes of any detector and the Gaussian system under measurement present the same amount of entanglement
for the same $r$ and $\theta$. Notably, these reduced entanglements are distinct from zero, which is a necessary fact for the information transfer of the canonical pair on study toward the quantum state of the detectors; this behavior is again a consequence of the symmetry of the Hamiltonian, Eq. (2.1.45). In Fig. 6 we plot $\mathcal{E}_{2}^{D S}$ as a function of $r$ and $\theta$.

Taking into account Eqs. (1.4.9) and (1.4.69) and the considerations for the reduced two-mode Rényi-2 entanglements explained before, we have the following expressions for all $\mathcal{E}_{2}^{\left(A_{i} \mid A_{j} A_{k}\right)}$ of the system

$$
\begin{gather*}
\mathcal{E}_{2}^{D D S}=\mathcal{E}_{2}^{\left(A_{1,2} \mid A_{2,1} A_{3}\right)}=\frac{1}{2} \ln \left(a^{2}\right),  \tag{2.4.34}\\
\mathcal{E}_{2}^{\left(A_{3} \mid A_{1} A_{2}\right)}=\frac{1}{2} \ln \left(a_{3}^{2}\right), \tag{2.4.35}
\end{gather*}
$$

where the superscript $D D S$ in Eq. (2.4.34) stand as 'Detector, DetectorSystem'; it is used to denote the Rényi-2 entanglements of the global bipartitions containing any mode detector as focus. Hence, Eq. (2.4.34), imply that any of the two global bipartitions focused on the partie containing the mode of any detector, present the same amount of entanglement for the same $r$ and $\theta$ parameters; this entanglement is ever distinct from zero since the minimum value of $a_{1}$ and $a_{2}$ (consequently of $a$ ) is 2 as can be directly verified from Eqs. (2.4.27) and (2.4.28). This symmetry is again a consequence of the linear interaction, Eq. (2.1.45), linking symmetrically the system of the two detectors with the Gaussian system under observation. In Fig. 7 we plot $\mathcal{E}_{2}^{D D S}$ and $\mathcal{E}_{2}^{\left(A_{3} \mid A_{1} A_{2}\right)}$ as functions of $r$ and $\theta$. From Eqs. (2.4.27) to (2.4.29) we can verify $\left\{a_{1}, a_{2}, a_{3}\right\}>1$ and $a_{3}>\left\{a_{1}, a_{2}\right\}$ for a determined $r$ and $\theta$; then, from Eqs. (2.4.34) and (2.4.35) we deduce $\mathcal{E}_{2}^{\left(A_{3} \mid A_{1} A_{2}\right)}>\mathcal{E}_{2}^{D D S}$; see Fig 8.

For the full tripartite entanglements, we need to consider the three focus options for the expression, Eq. (1.4.64); however, by taking into account the same considerations that for the reduced two-mode Rényi-2 entanglements, it is straightforward verify that

$$
\begin{equation*}
\mathcal{E}_{2}^{\prime}=\mathcal{E}_{2}^{\left(A_{1}\left|A_{2}\right| A_{3}\right)}=\mathcal{E}_{2}^{\left(A_{2}\left|A_{1}\right| A_{3}\right)} ; \tag{2.4.36}
\end{equation*}
$$

then, using Eqs. (1.4.64) and (1.4.66) we have

$$
\begin{equation*}
\mathcal{E}_{2}^{\prime}=\frac{1}{2} \ln \left(\frac{a^{2}}{g}\right) . \tag{2.4.37}
\end{equation*}
$$



Figure 7: Plots for the Rényi-2 entanglements $\mathcal{E}_{2}^{D D S}$ (bottom figure) and $\mathcal{E}_{2}^{A_{3} \mid A_{1} A_{2}}$ (upper figure) as functions of $r$ and $\theta$. Such quantities dictate the amount of entanglement for the global bipartitions focused on the partie containing the mode of any detector and that of the Gaussian system under measurement respectively. They have an oscillatory behaviour with a period of $\pi / 2$. Their maximum values are reached at $\theta=(2 n+1) \pi / 4, \quad n=0,1,2, \cdots$ for any $r \neq 0$. Besides, they are increasing functions of $|r|$. The minimum values are $\min \left[\mathcal{E}_{2}^{D D S}\right]=(1 / 2) \ln (4)$ and $\min \left[\mathcal{E}_{2}^{A_{3} \mid A_{1} A_{2}}\right]=(1 / 2) \ln (9)$, reached when the Gaussian system under measurement is a minimum uncertainty state.


Figure 8: Plots for the Rényi-2 entanglements $\mathcal{E}_{2}^{D D S}$ and $\mathcal{E}_{2}^{A_{3} \mid A_{1} A_{2}}$ (bottom and upper curves respectively) at various fixed rotation angles. The gray dashed lines represent the possible minimum values for these quantities. From this, we can see that $\mathcal{E}_{2}^{A_{3} \mid A_{1} A_{2}}>\mathcal{E}_{2}^{D D S}$.

In a similar procedure we get

$$
\begin{equation*}
\mathcal{E}_{2}^{\left(A_{3}\left|A_{1}\right| A_{2}\right)}=\frac{1}{2} \ln \left(\frac{a_{3}^{2}}{g^{2}}\right) \tag{2.4.38}
\end{equation*}
$$

According to Eqs. (2.4.37) and (2.4.38), the amount of tripartite entanglement is the same when the focus is the partie containing the mode of any detector and distinct from the case when is focused on the mode of the Gaussian system under examination. In Fig. 9 we plot $\mathcal{E}_{2}^{\prime}$ and $\mathcal{E}_{2}^{\left(A_{3}\left|A_{1}\right| A_{2}\right)}$ as functions of $r$ and $\theta$. Both tripartite entanglements grow as the magnitude of the squeezing of the Gaussian system under observation increases, in concordance with a loss of purity of the reduced single-mode systems, i.e., a loss of information about the knowledge of its initial quantum states. Notably, the tripartite entanglement focused on any detector is greater than the one focused on the Gaussian system under inspection for fixed $r$ and $\theta$ values, then $\mathcal{E}_{2}^{\prime}>\mathcal{E}_{2}^{(A 1|A 2| A 3)}$; see Fig. 10.

This behavior is consistent with the description of a user locally localized in the partie (focus) containing a single mode with associated symplectic invariant $a_{i}$.


Figure 9: Plot for the residual tripartite entanglements $\mathcal{E}_{2}^{\prime}$ (top figure) and $\mathcal{E}_{2}^{\left(A_{3}\left|A_{1}\right| A_{2}\right)}$ (bottom figure) as functions of $r$ and $\theta$. These quantities quantify the complete amount of tripartite entanglement in our Gaussian Arthurs-Kelly measurement setting. They are focused respectively on any detector and the Gaussian system under examination. They have an oscillatory behavior with period $\pi / 2$. They have a maximum value for a determined $r \neq 0$ at $\theta=$ $(2 n+1) \pi / 4, n=0,1,2, \cdots$. Besides, the global tripartite entanglement grows as the squeezing factor $r$ also grows. The minimum values are, respectively, $\ln (6 / 5)$ and $\ln (27 / 25)$, attained when the system under measurement is a minimum uncertainty state.


Figure 10: Plots for the tripartite entanglements $\mathcal{E}_{2}^{\prime}$ and $\mathcal{E}_{2}^{A_{3}\left|A_{1}\right| A_{2}}$ (top and lower curves respectively) at various fixed rotation angles. The gray dashed lines represent the possible minimum values for these quantities. From this, we can see that $\mathcal{E}_{2}^{\prime}>\mathcal{E}_{2}^{A_{3}\left|A_{1}\right| A_{2}}$.

Then, to close this section and chapter, we summarize the findings of the quantification of the tripartite entanglement. In essence, we find one symmetric entanglement structure in the mode bipartitions of the system, in particular:
(a) The reduced bipartition containing only the modes of the two detectors has zero amount of entanglement; this means that the partie associated with the mode of the Gaussian system under measurement is the principal support of the tripartite entanglement relation in the measurement setting. That is, as a consequence of the symmetry associated with the interaction Hamiltonian, Eq. (2.1.45), the canonical pair of the Gaussian system under measurement gets linked with the canonical set composed by an observable of the first detector and the corresponding conjugate variable of the second detector; see Fig. 3. Also, as a consequence of the symmetry in the unitary dynamics governing the measurement process, the two reduced bipartitions containing the mode of any detector present the same quantity of entanglement, which implies that both detectors are equally entangled with the Gaussian system under observation; this is the fact allowing the transfer of information
about the canonical pair under measurement to the quantum state of the detectors.
(b) The two global bipartitions focused on any detector contain an equal quantity of entanglement, which comes again from the symmetry of the interaction Hamiltonian given by Eq. (1.1.1). Notably, the global bipartition focused on the mode of the Gaussian system under observation presents a higher quantity of entanglement from those focused on the detectors; see Figure 8. Therefore, the amount of entanglement in any global bipartition of the system will depend logarithmically on the symplectic invariant $a_{i}$ associated with the mode chosen as focus, as Eqs. (2.4.34) and (2.4.35) suggests.
(c) The full tripartite entanglements quantified by the residual tripartite Rényi-2 entanglement depend, in general, on the partie chosen as the focus. We find the same amount of entanglement when the focus is the partie containing the mode of any detector, which is again a consequence of the symmetry of the interaction Hamiltonian governing the dynamics of the measurement process. Besides, the tripartite entanglement focused on any detector is greater than the one focused on the Gaussian system under examination. This behovior is consistent with the deseription of an observer loealized on the partie chesien as foeus, where the entanclement from the peropective of thin observer will be lilke 'at night' of the reciproealh purities associated with the other two parties; in this manner, the entanglement foeused on the Goussion system under measurement will be bigher.

## 3

## THE FULL DYNAMICAL ARTHURS-KELLY MEASUREMENT PROCESS

$$
\frac{\text { "History is written by victors." }}{\text { unknown origin }}
$$

### 3.1 JUSTIFICATION

The seminal model raised by Arthurs and Kelly constitutes the direct generalization of the standard von Neumann measurement scheme [93] for the position and momentum observables of an arbitrary system. However, the Arthurs-Kelly model is attained only for the particular situation in which the detectors are, concretely, infinitely coupled with the system under measurement. Besides, the joint unbiasedness condition (see Appendix 5.2 and subsection 2.1.3) formally demands that the expected values of the marginal measurement outputs match with the theoretical mean values of the position and momentum observables of the system under examination; this implies that the time of the measurement must be chosen as the reciprocal of the coupling constant (see Section 2.2); this fact results in that the time-lapse for the measurement process must be instantaneous, [110] which becomes a non-plausible situation for experimental situations in which it could not be possible to achieve the strong coupling regime.

The phenomenon of spreading is an inherent characteristic of a quantum particle which is described by a wave packet, i.e., by a continuous superposition of plane waves, where the whole group runs with constant motion. Then, the effect of spreading is given by the temporal increase
of the associated probability distribution together with the diminishing of the amplitude of the associated wave packet [149]. Then, with this in mind, the Arthurs-Kelly measurement process boards a situation in which the wave functions representing each component of the measurement setting do not come under spreading, which allows reaching the fundamental limits of predictive and retrodictive accuracy of the measurement (see sections 2.1.2, 2.3.1, 2.3.2). Therefore, the consideration of a model that includes the free evolution of each measurement component allows studying the impact of the free propagation in the accuracy aspects of the simultaneous measurement process.

On the other hand, related with the entanglement generation, there exists a close relationship between the entanglement of composed systems and the spreading of the wave packet describing the dynamics of the system; for example, in the coincidence and single-particle schemes of photoionization [150] and spontaneous emission of a photon [151]. Therefore, in this Chapter, we board the complete dynamics of the Arthurs-Kelly measurement model to investigate the effects of free propagation of the system in its entanglement properties in the particular regime of Gaussian states.

### 3.2 DYNAMICS

The full (closed) dynamics of the Arthurs-Kelly measurement process is given by the following Hamiltonian

$$
\begin{equation*}
\hat{H}=\left(\kappa_{1} \hat{x}_{3} \hat{p}_{1}+\kappa_{2} \hat{p}_{3} \hat{p}_{2}\right)+\hat{H}_{\text {free }} \tag{3.2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{H}_{\text {free }}=\frac{\hat{p}_{1}^{2}}{2 m_{1}}+\frac{\hat{p}_{2}^{2}}{2 m_{2}}+\frac{\hat{p}_{3}^{2}}{2 m_{3}} \tag{3.2.2}
\end{equation*}
$$

represents the free energy operator of each subsystem in the measurement setting. It must be noted that the Hamiltonian, Eq. (3.2.1), does not appeal to a particular coupling regime; therefore, we have no more an instantaneous time interval for the measurement process.

We recur to the time evolution operator method to compute the dynamics of the wave function through the measurement process as Eq.
(2.2.1) establishes. Then, with the Hamiltonian, Eq. (3.2.1), the time evolution operator becomes

$$
\begin{equation*}
e^{-i \hat{H} t}=e^{-\frac{i t}{2 m_{1}} \hat{p}_{1}^{2}} e^{-\frac{i t}{2 m_{2}} \hat{p}_{2}^{2}} e^{-i t\left(\frac{\hat{p}_{3}^{2}}{2 m_{3}}+\kappa \hat{x}_{3} \hat{p}_{1}+\kappa \hat{p}_{3} \hat{p}_{2}\right)} . \tag{3.2.3}
\end{equation*}
$$

Notably, we have the quadratic momentum operator $\hat{p}_{3}^{2}$ which does not commute with the position $\hat{x}_{3}$ of the system under observation; therefore, the factorization of operator, Eq. (3.2.3), cannot carried out through the conventional Baker-Campbell-Hausdorff formula; instead, we proceed with the general method of factorization explained in Reference [94]; then, for simplicity of the process, we assume the same coupling constants $\kappa_{1}=\kappa_{2}=\kappa$ in the unitary operator, Eq. (3.2.3); we give a step-by-step derivation for the factorization process in the Appendix, Section 5.4; the result is

$$
\begin{equation*}
e^{-i \hat{H} t}=e^{\Delta x_{1} \hat{p}_{1}^{2}} e^{-\frac{i t}{2 m_{2}} \hat{p}_{2}^{2}} e^{-\frac{i t}{4 m_{3}} \hat{p}_{3}^{2}} e^{-\frac{i t \kappa}{2} \hat{p}_{3} \hat{p}_{2}} e^{-i t \kappa \hat{x}_{3} \hat{p}_{1}} e^{-\frac{i t}{4 m_{3} \hat{p}_{3}^{2}}} e^{-\frac{i t \kappa}{2} \hat{p}_{3} \hat{p}_{2}}, \tag{3.2.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\Delta x_{1}=-\left(i t / 2 m_{1}\right)+\left(i t^{3} \kappa^{2} / 12 m_{3}\right) . \tag{3.2.5}
\end{equation*}
$$

One of the principal advantages of using the time evolution operator method to calculate the dynamical evolution is that it allows us to avoid the difficult task of solving the Schrödinger equation with the Hamiltonian, Eq. (3.2.1). Comparing the full dynamical time evolution operator, Eq. (3.2.4), with that of the strong coupling regime, Eqs. (5.1.3) and (5.1.4), we observe the presence of operators of the form $e^{c \hat{p}_{i}^{2}}$ as a difference; these terms appears only as a consequence of the consideration of the free Hamiltonian, Eq. (3.2.2), in the dynamics of the measurement process.

Since the original proposal of Arthurs and Kelly consider an arbitrary state as the system under measurement, let us consider this assumption for the application of the free energy operator on an arbitrary wave function in the position representation; that is

$$
\begin{align*}
e^{-i \hat{H} t} \psi(x)= & \exp \left[-i \hat{p}^{2} t / 2 m\right] \psi(x) \\
& =\exp \left[\frac{i t}{2 m} \frac{d^{2}}{d x^{2}}\right] \psi(x)  \tag{3.2.6}\\
= & \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i t}{2 m}\right)^{k} \frac{d^{2 k} \psi(x)}{d x^{2 k}}
\end{align*}
$$

where we have expanded the exponential in McLaurin series.
Notably, as Blinder indicates [152], there no exist a closed expression for the series in the last line of Eq. (3.2.6); then, we can not infer information about the exact analytical expression for an arbitrary wave packet under free dynamics; consequently, it is difficult to obtain exact expressions for the expected values of polynomials of any order in the quadrature operators. However, let us recall that we have the two following conditions
(i) We need to satisfy the joint unbiasedness condition.
(ii) We need to explore the fundamental limits of retrodictive and predictive accuracy aspects of the simultaneous measurement under the consideration of a full dynamics of the measurement process.

Therefore, we need to restrict ourselves to a kind of wave function which satisfies the two above conditions and allows us to apply the free energy operators of the unitary operator of Eq. (3.2.6) to obtain a closed expression for the resulting wave function governing the dynamics of the system and the expected values of polynomials in the quadrature operators of the system. In this thesis work, we appeal to a Gaussian function to represent the initial state of the system under measurement.

The Gaussian functions are contained in the well-known Schwartz space, $\mathcal{S}(\mathbb{R})$, which is the linear space of all functions $f: \mathbb{R} \longrightarrow \mathbb{C}$ which have derivatives of all orders and satisfy the condition [153]:

$$
\begin{equation*}
P_{a, b}(f) \equiv \sup _{x \in \mathbb{R}}\left|x^{a} f^{b}(x)\right|<\infty \tag{3.2.7}
\end{equation*}
$$

for $a, b \in W=\{0,1,2, \cdots\}$. The finiteness condition for all $a \geq 1$ and $b \in W$, implies that $x^{a} f^{b}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ for all $a, b \in W$; so they are commonly refered as rapidly decreasing functions.

By using a Gaussian wave function, $\psi(X) \in \mathcal{S}(\mathbb{R})$, for the system under measurement we can satisfy $(i)$, because under the unitary operator, Eq. (3.2.4), the wave function remains Gaussian through the temporal evolution; this implies that the expected values of the linear quadrature
position and momentum operators are vanishing at any time $t$ (including $t=0$ ), that is,

$$
\begin{equation*}
\langle X\rangle=\int_{X \in \mathbb{R}} \psi^{*}(X, t) X \psi(X, t) d X=0, \quad X \in\{x, p\} \tag{3.2.8}
\end{equation*}
$$

therefore, the joint unbiasedness condition will be trivially satisfied by using a Gaussian wave function to represent the system under inspection.

On the other hand, the Gaussian states are minimum uncertainty states, which implies that they saturate the Heisenberg uncertainty relation, and the retrodictive and predictive uncertainty relations summarized by Eq. (2.1.17). Therefore, a Gaussian function representing the system under measurement will allow inspecting the effect of the free propagation of the measurement setting on the fundamental accuracy aspects of the simultaneous measurement, i.e., fulfill the condition established in (ii).

Therefore, in the following, we use a Gaussian wave function to describe the system under measurement; this is one of the central facts of this thesis work, allowing us to apply the Gaussian mathematical background developed through Chapter 1 to a measurement setting described entirely by continuous variable Gaussian wave functions. In particular, and without any preferential reason, we choose the same mathematical structure as that of the pointers of the measuring device, Eqs. (2.1.51) and (2.1.52), to represent the system under observation; then, we characterize it by a squeezed vacuum state with a squeeze factor of $S_{3}=1 /\left(2^{\frac{1}{2}} \delta_{q}\right)$, being $\delta_{q}$ the standard deviation of the associated initial position probability distribution; by using Eq. (2.1.61) and a Fourier transform, it is trivial to verify that the initial momentum probability distribution have variance of $\delta_{p}=1 / 2 \delta_{q}$. Then, following Eq. (3.2.4), we have a product of three squeezed vacuum states as the initial wave function of the measurement setting.

Using Eqs. (2.2.1) and (2.2.2) together the time evolution operator, Eq. (3.2.4), to compute the wave function of the system at any time $t$, the result in compact form is

$$
\begin{align*}
\Psi(\chi, t)=\mathcal{N}(t) \exp [- & \left\{\varepsilon_{1}(t) x_{1}^{2}+\varepsilon_{2}(t) x_{2}^{2}+\varepsilon_{3}(t) x_{3}^{2}+\varepsilon_{4}(t) x_{1} x_{2}\right. \\
& \left.\left.+\varepsilon_{5}(t) x_{1} x_{3}+\varepsilon_{6}(t) x_{2} x_{3}\right\}\right] \tag{3.2.9}
\end{align*}
$$

where $\mathcal{N}(t)$ and $\varepsilon_{j}(t)$ are the complex time-dependent functions explicitly defined in the Appendix, Sec. 5.5. Besides, we have condensed the
spatial dependence on the $x_{i}$ variables in $\chi$. The application of the time evolution operator of Eq. (3.2.4) to the initial state involves one and two-dimensional Fourier transforms; we give a step-by-step derivation of Eq. (3.2.9) in Appendix, Sec. 5.6.

The wave function, Eq. (3.2.9), has associated a three-variable Gaussian probability distribution; it can be verified that it is normalized for all $t$. Besides, it describes the dynamics of the pointers plus the Gaussian system under examination in the position space while the measurement is in progress, and since it cannot be expressed as a product of individual functions, it is entangled [58] between the pointers and system variables. In the particular situation of the Arthur-Kelly model where the system under inspection is a Gaussian state, the entanglement generated constitutes an example of three-mode Gaussian entanglement [11]; we will study the entanglement properties of this system in Section 3.5.

### 3.3 ACCURACY IN THE FULL DYNAMICAL MEASUREMENT PROCESS

In this section, we employ the concepts developed in Sec. 2.1.2 to characterize the retrodictive and predictive accuracy aspects in the completely dynamical Arthurs-Kelly simultaneous measurement process for the specific Gaussian configuration of squeezed vacuum states.

In all the following we will compute the temporal dynamics of the mean value for any position and momentum function, $Q(\hat{\chi}, \hat{\varrho}, t)$, through

$$
\begin{equation*}
\langle Q(\hat{\chi}, \hat{\varrho}, t)\rangle=\int \Psi^{*}(\chi, t) Q(\hat{\chi}, \hat{\varrho}, t) \Psi(\chi, t) d \chi \tag{3.3.1}
\end{equation*}
$$

with $d \chi=\prod_{i=1}^{3} d x_{i}$. For the remaining of this section, we take the mean value of the $n$-th moment of some observable at the time in which the simultaneous measurement is carried out in the strong coupling regime; as we explain in Sec. 2.2, this time is the reciprocal of the coupling constant: $t=\tau=1 / \kappa[98]^{1}$, this election superbly simplifies the definitions and calculations. We further choose the variance $\sigma_{\hat{A}}^{2}=\left\langle\hat{A}^{2}\right\rangle-\langle\hat{A}\rangle^{2}$, as the measure of dispersion for the probability distribution of any observable

[^3]$\hat{A}$. We emphasize that each marginal probability distribution associated with the wave function of Eq. (3.2.9) have a Gaussian shape with zero mean value; therefore, their variances coincide with the squared root-mean-square (rms) error, which is a reasonable measure of dispersion that quantifies the degree at which the probability distribution of an observable deviates from another which is pretended to estimate [154, 155].

### 3.3.1 Retrodictive aspect

First we recall that indexes 1,2,3 label the first and second detector, as well as the system under observation for our measurement setting respectively. Then, using the definitions for the retrodictive error operators, Eqs. (2.1.11) and (2.1.12), together the unbiased condition of the retrodictive aspect of accuracy, Eq. (2.1.27), and the condition, (2.1.30), we deduce the following variances

$$
\begin{align*}
& \sigma_{\hat{\varepsilon}_{x_{3}}^{R}}^{2}=\sigma_{\hat{x}_{1}}^{2}-\delta_{q_{3}}^{2},  \tag{3.3.2}\\
& \sigma_{\hat{\varepsilon}_{p_{3}}^{R}}^{2}=\sigma_{\hat{x}_{2}}^{2}-\delta_{p_{3}}^{2} . \tag{3.3.3}
\end{align*}
$$

Now using the wave function, Eq. (3.2.9), together the condition, Eq. (3.3.1), we compute the variances quantifying the widths of the marginal probability distributions of the position detectors at the time $t=\kappa^{-1}$ :

$$
\begin{gather*}
\sigma_{\hat{x}_{1}}^{2}=\delta_{q_{3}}^{2}+\frac{b}{2}+\eta_{1}(\kappa)+\eta_{2}(\kappa),  \tag{3.3.4}\\
\sigma_{\hat{x}_{2}}^{2}=\delta_{p_{3}}^{2}+\frac{1}{2 b}+\eta_{3}(\kappa) . \tag{3.3.5}
\end{gather*}
$$

Where $\delta_{q_{3}}^{2}\left(\delta_{p_{3}}^{2}\right)$ represent the variance of the initial probability distribution of position (momentum) of the Gaussian system under measurement. Comparing the last two expressions with Eqs. (2.2.11) and (2.2.12) at the time $t=\kappa^{-1}$ (therefore $\alpha_{1}=\alpha_{2}=1$ ), we can note the extra terms $\eta_{i}$; they are given by the following expressions

$$
\begin{gather*}
\eta_{1}(\kappa)=\frac{\left(m_{1}-6 m_{3}\right)^{2}}{36 b m_{1}^{2} m_{3}^{2} \kappa^{2}}  \tag{3.3.6}\\
\eta_{2}(\kappa)=\frac{1}{16 \delta_{q}^{2} m_{3}^{2} \kappa^{2}} \tag{3.3.7}
\end{gather*}
$$



Figure 11: Behavior of the product $\sigma_{\hat{x}_{1}}^{2} \sigma_{\hat{x}_{2}}^{2}$ given by Eq. (3.3.10), versus the coupling constant $\kappa$, for the values $\delta_{x_{3}}=1, m_{i}=1$. As $\kappa$ decreases, the product shift upwards from its minimal value of 1 due to the contribution of the $\Delta_{1}(\kappa)$-function; this expression quantifies the spreading contribution of the wave function to the product of variances characterizing the marginal probability distributions of the pointer positions at the time of the simultaneous measurment.

$$
\begin{equation*}
\eta_{3}(\kappa)=\frac{b}{m_{2}^{2} \kappa^{2}} \tag{3.3.8}
\end{equation*}
$$

The Eqs. (3.3.6) to (3.3.8) appears only with the consideration of the free Hamiltonian, Eq. (3.2.2); they quantify the spread in the variances of the marginal probability distributions of positions of the pointers as a consequence of the free evolution of the wave function governing the dynamics of the measurement process; see Section 3.4.

Besides, the Eqs. (3.3.6) to (3.3.8) are vanishing as the coupling between the Gaussian system and pointers becomes larger, that is,

$$
\begin{equation*}
\lim _{\kappa \longrightarrow \infty} \eta_{j}(\kappa)=0, j=1,2,3 \tag{3.3.9}
\end{equation*}
$$

therefore, in this situation, we recover the variances, Eqs. (2.2.11) and (2.2.12), for the strong coupling regime at the time $t=\kappa^{-1}$. If we take the product of Eqs. (3.3.4) and (3.3.5) with the ideal balance parameter
expressed by Eq. (2.2.13) (with $\alpha_{1}=\alpha_{2}=1$ ) as $b=2 \delta_{x_{3}}^{2}$, together the lower bound of the Heisenberg uncertainty relation, we obtain the ArthursKelly uncertainty product associated with the completely dynamical measurement process for our specific Gaussian configuration:

$$
\begin{equation*}
\sigma_{\hat{x}_{1}}^{2} \sigma_{\hat{x}_{2}}^{2}=\left[\sigma_{\hat{x}_{1}}^{2} \sigma_{\hat{x}_{2}}^{2}\right]_{\min }+\Delta_{1}(\kappa), \tag{3.3.10}
\end{equation*}
$$

where $\left[\sigma_{\hat{x}_{1}}^{2} \sigma_{\hat{x}_{2}}^{2}\right]_{\text {min }}=1$ is the lower bound of the uncertainty relation derived by Arthurs and Kelly [98]; this term represent the minimum value that the product $\sigma_{\hat{x}_{1}}^{2} \sigma_{\hat{x}_{2}}^{2}$ can be reach. Besides, the $\Delta_{1}(\kappa)$-function is given by

$$
\begin{equation*}
\Delta_{1}(\kappa)=\frac{11 m_{1}^{2}\left(4 \delta_{q}^{4}+\kappa^{2} m_{2}^{2}\right)-24 m_{1}\left(4 \delta_{q}^{4}+\kappa^{2} m_{2}^{2}\right) m_{3}+72\left(4 \delta_{q}^{4}+\kappa^{2}\left[16 \delta_{q}^{8} m_{1}^{2}+m_{2}^{2}\right]\right) m_{3}^{2}}{288\left(\delta_{q}^{2} \kappa^{2} m_{1} m_{2} m_{3}\right)^{2}} . \tag{3.3.11}
\end{equation*}
$$

Since the $\Delta_{1}$-function reciprocally depends on the square of the coupling constant $\kappa$, the lower bound $\left[\sigma_{\hat{x}_{1}}^{2} \sigma_{\hat{x}_{2}}^{2}\right]_{\text {min }}$ shift upwards as $\kappa$ decreases; see Fig. 11, in fact we have

$$
\begin{equation*}
\lim _{\kappa \rightarrow \infty} \Delta_{1}(\kappa)=0, \tag{3.3.12}
\end{equation*}
$$

which is an indicative that a weak coupling between the pointers and the Gaussian system causes a loss in the retrodictive accuracy of the simultaneous measurement.

However, although the product given by Eq. (3.3.10), gives a quantitative notion of the retrodictive accuracy in the simultaneous measurement, the variances $\sigma_{\hat{x}_{1}}^{2}$ and $\sigma_{\hat{x}_{2}}^{2}$ cannot be directly interpretable as experimental errors [101, 102, 103]. Instead, the variances of the retrodictive error operators positively quantify the noises deviating the probability distributions of the measurement outputs from the initial probability distributions of the canonical pair under examination. Substituting Eqs. (3.3.4) and (3.3.5) in Eqs. (3.3.2) and (3.3.3), we trivially find

$$
\begin{gather*}
\sigma_{\hat{\varepsilon}_{x_{3}}^{R}}^{2}=\frac{b}{2}+\eta_{1}(\kappa)+\eta_{2}(\kappa),  \tag{3.3.13}\\
\sigma_{\hat{\varepsilon}_{p_{3}}^{R}}^{2}=\frac{1}{2 b}+\eta_{3}(\kappa) . \tag{3.3.14}
\end{gather*}
$$



Figure 12: Behavior of the product $\sigma_{\hat{\varepsilon}_{x_{3}}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}}^{2}$ given by Eq. (3.3.15) versus the coupling constant $\kappa$, for the values $\delta_{x_{3}}=1, m_{i}=1$. As $\kappa$ decreases, the product shift hyperbolically upwards from its minimal value of $1 / 4$ due to the contribution of the $\Delta_{2}(\kappa)$-function. This plot describes the behaviour of the uncertainty product associated with the retrodictive accuracy in the full dynamical simultaneous measurement process of position and momentum observables; the lower bound is reached in the strong coupling regime described by Arthurs and Kelly.

The Eqs. (3.3.13) and (3.3.14) quantify all error in the retrodictive accuracy of the simultaneous measurement of the position and momentum observables; the terms $b$-dependent characterize the intrinsic noise induced by the detectors to the measurement outputs in the strong coupling regime, as Eqs. (2.3.4) and (2.3.5) show. Comparing Eqs. (3.3.13) and (3.3.14) with Eqs. (2.3.4) and (2.3.5), we see that the expressions $\eta_{i}(\kappa)$ given by Eqs. (3.3.6) to (3.3.8) adds extra noise to the retrodictive aspect of the simultaneous measurement.

Taking the product of Eqs. (3.3.13) and (3.3.14) with the ideal balance parameter $b=2 \delta_{x_{3}}^{2}$ and using the lower bound of the Heisenberg uncertainty relation, we obtain the uncertainty product of the retrodictive accuracy of the simultaneous measurement

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{x_{3}}^{R}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}^{R}}^{2}=\left[\sigma_{\hat{\varepsilon}_{x_{3}}^{R}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}^{R}}^{2}\right]_{\min }+\Delta_{2}(\kappa) ; \tag{3.3.15}
\end{equation*}
$$

where the term $\left[\sigma_{\hat{x}_{x_{3}}^{R}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}^{R}}^{2}\right]_{\min }=1 / 4$ represents the lower bound of the retrodictive uncertainty relation given by Eq. (2.1.42); besides, the $\Delta_{2}(\kappa)$-function is given by

$$
\begin{equation*}
\Delta_{2}(\kappa)=\frac{11 m_{1}^{2}\left(8 \delta_{q}^{4}+\kappa^{2} m_{2}^{2}\right)-24 m_{1}\left(8 \delta_{q}^{4}+\kappa^{2} m_{2}^{2}\right) m_{3}+72\left(8 \delta_{q}^{4}+\kappa^{2}\left[16 \delta_{q}^{8} m_{1}^{2}+m_{2}^{2}\right]\right) m_{3}^{2}}{\left(24 \delta_{q}^{2} \kappa^{2} m_{1} m_{2} m_{3}\right)^{2}} ; \tag{3.3.16}
\end{equation*}
$$

therefore, again, the lower bound of the uncertainty product associated with the retrodictive error operators is hyperbolically displaced upwards by the $\Delta_{2}(\kappa)$-function as the coupling constant $\kappa$ goes smaller; this is because the expression, Eq. (3.3.16), reciprocally depends on the square of $\kappa$; see Figure 12.

Hence, we can conclude that the consideration of the free evolution operators in the dynamics of the measurement process negatively affects the retrodictive accuracy in the simultaneous measurement of the position and momentum observables. The extra fluctuations go beyond the intrinsic noise induced by the pointers of the measuring apparatus; they result from the spreading of the global Gaussian wave function characterizing the measurement configuration through the measurement process. Then, that spreading noise is reduced only by increasing the coupling between the pointers and the system under observation.

### 3.3.2 Predictive aspect

Considering the definitions for the predictive error operators, Eqs. (2.1.13) and (2.1.14), together the predictive unbiased condition for the simultaneous measurement, Eq. (2.1.28), we derive the following variances

$$
\begin{align*}
\sigma_{\hat{\varepsilon}_{x_{3}}}^{2} & =\sigma_{\hat{x}_{1}}^{2}+\sigma_{\hat{x}_{3}}^{2}-2\left\langle\hat{U}^{\dagger} \hat{x}_{1} \hat{x}_{3} \hat{U}\right\rangle,  \tag{3.3.17}\\
\sigma_{\hat{e}_{p_{3}}}^{2} & =\sigma_{\hat{x}_{2}}^{2}+\sigma_{\hat{p}_{3}}^{2}-2\left\langle\hat{U}^{\dagger} \hat{x}_{2} \hat{p}_{3} \hat{U}\right\rangle, \tag{3.3.18}
\end{align*}
$$

where the last term in the last two expressions express the double covariance between the observables defining the predictive error operators; we compute it by using the definition, Eq. (3.3.1); hence we obtain

$$
\begin{equation*}
2\left\langle\hat{U}^{\dagger} \hat{x}_{1} \hat{x}_{3} \hat{U}\right\rangle=2 \delta_{q_{3}}^{2}+b+2 \eta_{2}(\kappa), \tag{3.3.19}
\end{equation*}
$$



Figure 13: Behavior of the product $\sigma_{\hat{\varepsilon}_{x_{3}}^{P}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}}^{2}$ given by Eq. (3.3.25) versus the coupling constant $\kappa$, for the values $\delta_{x_{3}}=1, m_{i}=1$. As $\kappa$ decreases, the product shift hyperbolically upwards from its minimal value of $1 / 4$ due to the contribution of the $\Delta_{3}(\kappa)$-function. This plot describes the behaviour of the uncertainty product associated with the predictive accuracy in the full dynamical simultaneous measurement process of position and momentum observables; the lower bound is reached in the strong coupling regime described by Arthurs and Kelly.

$$
\begin{equation*}
2\left\langle\hat{U}^{\dagger} \hat{x}_{2} \hat{p}_{3} \hat{U}\right\rangle=\frac{1}{b}+2 \delta_{p_{3}}^{2} \tag{3.3.20}
\end{equation*}
$$

In the same way we compute the variances $\sigma_{\hat{x}_{3}}^{2}$ and $\sigma_{\hat{p}_{3}}^{2}$, obtaining

$$
\begin{gather*}
\sigma_{\hat{x}_{3}}^{2}=\delta_{q_{3}}^{2}+b+\eta_{2}(\kappa)  \tag{3.3.21}\\
\sigma_{\hat{p}_{3}}^{2}=\delta_{p_{3}}^{2}+\frac{1}{b} \tag{3.3.22}
\end{gather*}
$$

Then, using Eqs. (3.3.4) and (3.3.5), together Eqs. (3.3.19) to (3.3.22) in Eqs. (3.3.17) to (3.3.18), we obtain the following variances

$$
\begin{align*}
\sigma_{\hat{\varepsilon}_{x_{3}}}^{2} & =\frac{b}{2}+\eta_{1}(\kappa)  \tag{3.3.23}\\
\sigma_{\hat{\varepsilon}_{p_{3}}}^{2} & =\frac{1}{2 b}+\eta_{3}(\kappa) \tag{3.3.24}
\end{align*}
$$

The Eqs. (3.3.23) and (3.3.24) quantify all the noise deviating the probability distributions of the positions of the pointers at the time of the measurement from the probability distributions of the canonical pair under observation at the same time. Comparing Eqs. (3.3.23) and (3.3.24) with Eqs. (2.3.8) and (2.4.1), we observe the extra terms $\eta_{1,3}(\kappa)$, which, as we explain in Section 3.3.1, result from the spreading in the global Gaussian wave function associated with the measurement configuration. Therefore, we recover the fundamental noise inherently coming from the measurement process in the strong coupling limit, i.e., in the situation mathematically described by Eq. (3.3.9).

Taking the product of variances, Eqs. (3.3.23) and (3.3.24) and using the ideal balance parameter $b=2 \delta_{x_{3}}^{2}$ together the lower bound of Heisenberg uncertainty relation, we obtain the uncertainty product associated with the predictive accuracy of our specific Gaussian measurement process

$$
\begin{equation*}
\sigma_{\hat{\varepsilon}_{x_{3}}^{P}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}^{P}}^{2}=\left[\sigma_{\hat{\varepsilon}_{x_{3}}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}^{P}}^{2}\right]_{\min }+\Delta_{3}(\kappa) ; \tag{3.3.25}
\end{equation*}
$$

where $\left[\sigma_{\hat{\varepsilon}_{x_{3}}^{P}}^{2} \sigma_{\hat{\varepsilon}_{p_{3}}}^{2}\right]_{\min }=1 / 4$, represent the lower bound of the predictive uncertainty relation given by Eq. (2.1.44); besides, the $\Delta_{3}(\kappa)$-function is given by

$$
\begin{equation*}
\Delta_{3}(\kappa)=\frac{m_{1}^{2}\left(8 \delta_{q}^{4}+\kappa^{2} m_{2}^{2}\right)-12 m_{1}\left(8 \delta_{q}^{4}+\kappa^{2} m_{2}^{2}\right) m_{3}+36\left(8 \delta_{q}^{4}+\kappa^{2}\left[16 \delta_{q}^{8} m_{1}^{2}+m_{2}^{2}\right]\right) m_{3}^{2}}{288\left(\delta_{q}^{2} \kappa^{2} m_{1} m_{2} m_{3}\right)^{2}} ; \tag{3.3.26}
\end{equation*}
$$

therefore, as happened with the retrodictive aspect of accuracy reviewed in Section 3.3.1, the lower bound of the uncertainty product associated with the predictive error operators is hyperbolically displaced upwards by the $\Delta_{3}(\kappa)$-function as the coupling constant $\kappa$ goes smaller; this is because this expression reciprocally depends on the square of $\kappa$; see Figure 13.

Therefore, we conclude that the free energy operators negatively affect the predictive accuracy in the simultaneous measurement of the canonical pair. that is, the spreading induced by the free evolution adds extra fluctuations to the intrinsic noise of the measurement process, resulting in the variances of the predictive error operators increasing as the coupling constant $\kappa$ diminishes.

### 3.4 The Role of The free Evolution in The accuracy OF THE MEASUREMENT

We explain in subsections (3.3.1) and (3.3.2) that the consideration of the free energy operators in the dynamics of the simultaneous measurement process results in a spreading of the global Gaussian wave function characterizing the measurement configuration; this process adds extra fluctuations to the variances of the probability distributions of the observables of each component of the measurement setting, which derives in the reduction of the retrodictive and predictive accuracy of the simultaneous measurement. To give a more detailed explanation for the relation between the spreading of the Gaussian wave function of the measurement setting and the retrodictive and predictive accuracy in the simultaneous measurement, let us analyze the process of free evolution of a Gaussian wave packet in one dimension.

Let us consider a particle whose potential energy is zero at every point of space. The particle is not subject to any force; therefore, it is free. Under this assumption, the Schrödinger equation in one dimension is

$$
\begin{equation*}
i \hbar \frac{\partial \Psi(x, t)}{\partial t}=-\frac{\hbar^{2}}{2 m} \frac{\partial^{2} \Psi(x, t)}{\partial x^{2}} \tag{3.4.1}
\end{equation*}
$$

this equation is satisfied by solutions of the form

$$
\begin{equation*}
\Psi(x, t)=A \exp [i(k x-\omega t)] \tag{3.4.2}
\end{equation*}
$$

being $A$ a normalization constant; besides, $k$ and $\omega$ satisfy the relation $\omega=\hbar k / 2 m$. Now, the superposition principle establishes that every linear combination of plane waves is a solution of the Schrödinger equation given in Eq. (3.4.1); then we have

$$
\begin{equation*}
\Psi(x, t)=(2 \pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} g(k) \exp [i(k x-\omega t)] d k \tag{3.4.3}
\end{equation*}
$$

We choose the $g(k)$-function as a Gaussian centred at $k_{0}$, that is,

$$
\begin{equation*}
g(k)=\frac{\sqrt{a}}{(2 \pi)^{\frac{1}{4}}} e^{-\frac{a^{2}}{4}\left(k-k_{0}\right)^{2}}, a \in \mathbb{R}, \tag{3.4.4}
\end{equation*}
$$

then, we have

$$
\begin{equation*}
\Psi(x, t)=\frac{\sqrt{a}}{(2 \pi)^{\frac{3}{4}}} \int_{-\infty}^{+\infty} e^{-\frac{a^{2}}{4}\left(k-k_{0}\right)^{2}} \exp [i(k x-\omega t)] d k \tag{3.4.5}
\end{equation*}
$$



Figure 14: Plots of the probability density given by Eq. (3.4.9) at the subsequent times: a) $t_{1}=0.1$, b) $t_{2}=0.7$, c) $t_{3}=2.0$. As the time elapses, the width of the probability density of the Gaussian wave packet increases according to Eq. (3.4.11), while the amplitude diminishes. This phenomenon describes the spreading of the wave packet.
by grouping the $k$-dependent terms in the exponentials, and completing a perfect square, we have the solution

$$
\begin{equation*}
\Psi(x, t)=\left(\frac{2 a^{2}}{\pi}\right)^{1 / 4} \frac{e^{i \varphi}}{\left(a^{4}+\frac{4 \hbar^{2} t^{2}}{m^{2}}\right)^{1 / 4}} e^{i k_{0} x} \exp \left[-\frac{\left(x-\frac{\hbar k_{0}}{m} t\right)^{2}}{a^{2}+\frac{2 i \hbar t}{m}}\right] \tag{3.4.6}
\end{equation*}
$$

with

$$
\begin{align*}
\varphi & =-\left(\theta-\frac{\hbar k_{0}^{2}}{2 m} t\right)  \tag{3.4.7}\\
\theta & =\frac{1}{2} \arctan \left[\frac{2 \hbar t}{m a^{2}}\right] \tag{3.4.8}
\end{align*}
$$

Taking the squared modulus of $\Psi(x, t)$ we have the probability density for the Gaussian wave packet:

$$
\begin{equation*}
|\Psi(x, t)|^{2}=\frac{1}{\sigma_{0}(t)} \sqrt{\frac{2}{\pi a^{2}}} \exp \left[-\frac{2\left(x-\frac{\hbar k_{0}}{m} t\right)^{2}}{a^{2} \sigma_{0}^{2}(t)}\right], \tag{3.4.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{0}(t)=\sqrt{1+\frac{4 \hbar^{2} t^{2}}{m^{2} a^{4}}} \tag{3.4.10}
\end{equation*}
$$

It can be trivially verify that the variance, $\left[\delta_{x}(t)\right]^{2}$, associated with the Gaussian distribution, Eq. (3.4.9), is

$$
\begin{equation*}
\left[\delta_{x}(t)\right]^{2}=\frac{a^{2}}{4}\left[\sigma_{0}(t)\right]^{2} \tag{3.4.11}
\end{equation*}
$$

therefore, the width of the Gaussian probability distribution quadratically increases with increasing time, while the amplitude decreases; see Fig. (14). This is the phenomenon of spreading of the Gaussian wave packet.

With the last results in mind, let us go back to the dynamics expressed by Eq. (3.2.1). The first two terms between parentheses are the interaction Hamiltonian which links the observables of the position of the pointers with the canonical pair under observation. This dynamics is a generalization to the position and momentum observables of the standard Von Neumann measurement process for a single observable [93].


Figure 15: Set of 4000 points generated from the joint probability distribution $\rho\left(x_{1}, x_{2}, t=\kappa^{-1}\right)=\int_{-\infty}^{+\infty}\left|\Psi\left(\chi, t=\kappa^{-1}\right)\right|^{2} d x_{3}$ with the wave function, Eq. (3.2.9), at time of measurement $t=\kappa^{-1}$ using the values $\delta_{q}=1, b=2 \delta_{q}^{2}$, and $m_{i}=1$. Each point constitutes a joint record of the pointers, which is registered in the $x_{1} x_{2}$-plane. To graphically inspect the notion of retrodictive and predictive accuracy, the whole set of records is compared versus the contour plots of the Wigner function (with $\delta_{q}=1$ ) associated to the Gaussian state under inspection before (Figs. (a), (b)) and after (Figs. (c), (d)) the measurement respectively, for the cases of strong coupling ( $\kappa=100$, left figures.) and weak coupling ( $\kappa=0.5$, right figures.). The deviation from the Wigner functions becomes larger in the weak coupling regime.

Arthurs and Kelly find that the dynamics involving only these Hamiltonian carries an unavoidable noise, which is proportional to the fluctuations of the probability distributions of the pointers; this noise raises the lower bound of the Heisenberg uncertainty relation by the extra factor of $\hbar / 2$. Notably, the transfer of information from the system under observation
to the state of the pointers is possible due to the quantum entanglement created by the interaction, Eq. (2.1.45).

Instead, the terms of free energy given by Eq. (3.2.2) cause the spread of the three-dimensional Gaussian wave function as the time elapses (consequently, of its probability distribution), as we explained before. It must be noted that we choose $t=\kappa^{-1}$ as that time in which the simultaneous reading of the pointers is carried out. If we adjust the beginning of the interaction at $t=0$, the time interval for the measurement process is

$$
\begin{equation*}
\Delta \tau=t_{\text {final }}-t_{\text {initial }}=\kappa^{-1} \tag{3.4.12}
\end{equation*}
$$

which implies that the strong coupling assumption carries an instantaneous measurement process, while a weak coupling carries a long time. From the results analyzed for the spreading phenomena of the Gaussian wave packet, we deduce that the wave function of the measurement configuration goes on a higher spread in the weak coupling regime, which implies an increasing noise affecting the variances of the probability distributions of the system's variables; then, this noise negatively affects the retrodictive and predictive accuracy aspect of the simultaneous measurement; then, both aspects will be higher as the coupling between the detectors and the system under measurement is bigger. Therefore, to see this argument, in Fig. 15 we plot a set of 4000 points coming from the joint probability density

$$
\begin{equation*}
\rho\left(x_{1}, x_{2}, t=\kappa^{-1}\right)=\int_{-\infty}^{+\infty}\left|\Psi\left(\chi, t=\kappa^{-1}\right)\right|^{2} d x_{3} \tag{3.4.13}
\end{equation*}
$$

and we compare it versus both, the Wigner distribution of the Gaussian system before (retrodictive aspect) the measurement:

$$
\begin{equation*}
W\left(x_{3}, p_{3}, t=0\right)=\pi^{-1} \int_{-\infty}^{+\infty} \phi_{3}^{*}\left(x_{3}+\xi\right) \phi_{3}\left(x_{3}-\xi\right) e^{2 i p_{3} \xi} d \xi \tag{3.4.14}
\end{equation*}
$$

and the Wigner distribution of the Gaussian system after (predictive aspect) the measurement:

$$
\begin{gather*}
W\left(x_{3}, p_{3}, t=\kappa^{-1}\right)=\pi^{-1} \int_{-\infty}^{+\infty} \Psi^{*}\left(x_{1}, x_{2}, x_{3}+\xi, \tau\right) \Psi\left(x_{1}, x_{2}, x_{3}-\xi, \tau\right) \\
\times e^{2 i p_{3} \xi} d x_{1} d x_{2} d \xi \tag{3.4.15}
\end{gather*}
$$

we recall that $\phi_{3}\left(x_{3}\right)$ is the initial vacuum state of the Eq. (2.1.61) that represents the Gaussian system under measurement, and $\Psi(\chi, t)$
is given by Eq. (3.2.9) which represent the global measurement setting through the measurement process. Then, for a determined coupling $\kappa$, the degree of accuracy for the retrodictive and predictive aspects of the simultaneous measurement will be stipulated by the variances of the error operators, Eqs. (3.3.4) and (3.3.5), and (3.3.23) and (3.3.24).

### 3.5 GENERATION OF QUANTUM Entanglement

Quantum entanglement is the fundamental resource for the development of quantum technologies, like quantum computers, quantum-enhanced sensors, and efficient communications schemes; these advances build up the so-called second quantum revolution [148]. The fundamental nature for entangled systems was put in context by Schrödinger [55], who noticed that for the two-particle system described by EPR [54], it is not possible to give an individual description for each particle. Then, he concludes that until some observation does not happen, one only disposes of a global account for the whole system; this is the well-known statute that an entangled quantum system can not be expressed as a product of the individual functions until some interaction with the global system takes place.

One of the seminal manifestations of the quantum entanglement as a resource was posed by Von Neumann in his standard model of a quantum measurement of a single observable [93]; see Section 2.1.1. In this scheme, an interaction Hamiltonian links the quantum state of a "detector" with another system that we desire to measure. Then, due to the superposition principle, the states of the system under examination becomes correlated with the states of the measuring apparatus through the unitary dynamics; therefore, we have a global quantum state which can not be factorizable; that is, there exists a lack of individuality for each subsystem. This entangling process allows transferring the information from the system under examination to the quantum state of the detector. Hence, just one way to theoretically arm ourselves with quantum entanglement is through the so-called meter-system interactions, which is nothing more than an entanglement generator process between


Figure 16: Schematic representation of the Arthurs-Kelly measurement process which is divided into two stages: a) an interaction (starting at $t=0$ ) between the system and the pointers, described by a unitary operator $\hat{U}=e^{-i t \hat{H}}$, which creates entanglement between the components of the measurement setting, and b ), the readout process (carried out at $t=\tau=\kappa^{-1}$ ), where projective measurements take place in each pointer variable. The statistics of the measurement outputs carry information about the statistics of the initial conjugate pair of the system under observation. The dotted line between these two stages represents the entanglement between all components of the whole configuration. Besides, the joint unbiasedness condition establishes that the mean values of the measurement outputs must match with the mean values of the initial position and momentum observables that we pretend to know. The post-measurement state $\left|\phi_{3}^{\prime}\left(x_{3}\right)\right\rangle$ is normalized projection of the state before the measurement, $\left|\phi_{3}\left(x_{3}\right)\right\rangle$, on the eigenspace of the observed eigenvalue, according to the postulate of a (projective) quantum measurement [5].
one or more pointer variables (representing the degrees of freedom of the measuring device) and a physical observable (represented by a hermitian operator) of some system which we pretend to know. This process is the same line of thinking presented by Arthurs and kelly in their measurement process.

In this section, we pose the Arthurs-Kelly model as a process of generation of entanglement between the three systems building the measurement configuration. Given our assumption of a Gaussian system as the system under observation, we pose the measurement process as a generator of continuous-variable tripartite entanglement in the particular regime of Gaussian states. Hence we study the effects of free evolution of the dynamics of the measurement process in the qualitative and quantitative entanglement properties developed in the system. Then, we show that
the full dynamical Arthurs-Kelly measurement model represents one more mechanism for the three-mode entanglement generation artillery [156].
It must be noted that any measurement process can be divided into two stages [157, 158]: ( $i$ ) an entangling interaction, where the observables of the system under measurement come in the form of quantum correlations with the observables of the measuring device, and (ii) a readout process, where the measurement output takes place (here is where lies the quantum measurement problem [159], with which we will not get involved). In the Arthurs-Kelly measurement model, the transfer of information of the canonical pair to the state of the pointers is possible due to the quantum entanglement induced by the interaction of the Eq. (2.1.45), in the first stage, and finished in the second stage with the reading of the pointers, see Figure 16.

Notably, the non-commutative part of the right-hand side of Eq. (3.2.3) involves the quadratic momentum operator $\hat{p}_{3}^{2}$ (which only appears with the consideration of the free energy of the system under examination) which does not commute with the position operator $\hat{x}_{3}$ which implies that the entanglement properties of the global system should be affected by this dispersive operator. Therefore, we formulate the hypothesis that the spreading (i. e., the coupling between the Gaussian system and the pointers of the measuring device) directly affects the tripartite entanglement properties of the system.

### 3.5.1 Separability properties

The state given by Eq. (3.2.9) describes the system throughout the measurement process; it is entangled (at any time $t$ ) between the three CV modes building the measurement setting, that is, the two pointers of the measurement device and the Gaussian system under observation; therefore, it constitutes the first trivial case of multipartite entanglement. Then, focusing on $N=3$ subsystems, one can establish genuinely tripartite entanglement appealing to Werner's condition expressed by Eq. (1.3.17). Then, by recurring to this definition, the state expressed by Eq. (3.2.9) cannot be written as a product of individual functions, that is, $\Psi\left(x_{1}, x_{2}, x_{3}, t\right) \neq f_{1}\left(x_{1}, t\right) f_{2}\left(x_{2}, t\right) f_{3}\left(x_{3}, t\right)$. Even by assuming the time of the simultaneous measurement $t=\kappa^{-1}$, and taking the limit situation
of the strong coupling described by Arthurs and Kelly, the wave function of the system is not a product of individual functions. In that case, we recover the situation where the pointers and the Gaussian system interact instantly; therefore, it is natural to expect that the system is still entangled between the three CV modes.

Alternatively, one can establish genuine tripartite entanglement by means of the biseparability condition (see Section 1.3.5.1) [59, 60, 61]:

Let $\hat{\rho}$ the density operator describing a composite CV system consisting of $N=3$ modes, labeled respectively by the indexes $1,2,3$. That state is genuinely tripartite entangled if the density operator can not be written in the biseparable form, that is, by Eq. (1.3.18).

Then, by neglecting the three possible mixtures of pure biseparable states for a tripartite system, we certify genuine tripartite entanglement. This general statute implies, only within the particular regime of pure states [60], that by neglecting the three possible bipartitions of (1 vs 2 )-subsystems of a tripartite system, we certify genuine tripartite entanglement.

In our particular measurement scheme, which involves only pure Gaussian states, it is possible to prove genuine tripartite entanglement through the PPT criterion since, as we explain in Section 1.3.5.2, it is a necessary and sufficient condition for determining the separability of all " 1 vs various" mode-bipartitions of the system [63, 64]. Then, we test the separability for each of the three possible mode-bipartitions of the system described by Eq. (3.2.9). We proceed as follows.

First, we recall that for our specific Arthurs-Kelly measurement configuration, the indexes $1,2,3$ label the modes associated with the first and the second pointer and the Gaussian system under measurement, respectively; then, we compute the three CMs associated with the three mode-bipartitions: $1|23,3| 12$ and $2 \mid 31$. For this process, we array the canonical pairs of each mode in a vector operator like the one given in Eq. (1.1.6), that is,

$$
\begin{equation*}
\hat{\mathbf{R}}=\left(\hat{q}_{1}, \hat{p}_{1}, \hat{q}_{2}, \hat{p}_{2}, \hat{q}_{3}, \hat{p}_{3}\right)^{T} \tag{3.5.1}
\end{equation*}
$$

$$
\begin{align*}
& \hat{\mathbf{R}}=\left(\hat{q}_{3}, \hat{p}_{3}, \hat{q}_{1}, \hat{p}_{1}, \hat{q}_{2}, \hat{p}_{2}\right)^{T}  \tag{3.5.2}\\
& \hat{\mathbf{R}}=\left(\hat{q}_{2}, \hat{p}_{2}, \hat{q}_{3}, \hat{p}_{3}, \hat{q}_{1}, \hat{p}_{1}\right)^{T} . \tag{3.5.3}
\end{align*}
$$

Then we use the definition for the generic element of the covariance matrix, Eq. (1.2.9), to compute each of the 36 elements of the $6 \times 6$ dimensional covariance matrices associated with the three mode-bipartitions of the system. For the compute of the expected values, we use the definition, Eq. (3.3.1) together the wave function, Eq. (3.2.9), at the time $t=\kappa^{-1}$ and the balance parameter tuned at the ideal value for maximal accuracy in the simultaneous measurement as we explain in section 2.2. Then, we obtain the following three covariance matrices which we express in block form

$$
\begin{align*}
& \boldsymbol{\sigma}_{1 \mid 23}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{1} & \boldsymbol{\lambda}_{1,2} & \boldsymbol{\lambda}_{1,3} \\
\boldsymbol{\lambda}_{1,2}^{T} & \boldsymbol{\sigma}_{2} & \boldsymbol{\lambda}_{2,3} \\
\boldsymbol{\lambda}_{1,3}^{T} & \boldsymbol{\lambda}_{2,3}^{T} & \boldsymbol{\sigma}_{3}
\end{array}\right),  \tag{3.5.4}\\
& \boldsymbol{\sigma}_{3 \mid 12}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{3} & \boldsymbol{\lambda}_{1,3}^{T} & \boldsymbol{\lambda}_{2,3}^{T} \\
\boldsymbol{\lambda}_{1,3} & \boldsymbol{\sigma}_{1} & \boldsymbol{\lambda}_{1,2} \\
\boldsymbol{\lambda}_{2,3} & \boldsymbol{\lambda}_{1,2}^{T} & \boldsymbol{\sigma}_{2}
\end{array}\right),  \tag{3.5.5}\\
& \boldsymbol{\sigma}_{2 \mid 31}=\left(\begin{array}{ccc}
\boldsymbol{\sigma}_{2} & \boldsymbol{\lambda}_{2,3} & \boldsymbol{\lambda}_{1,2}^{T} \\
\boldsymbol{\lambda}_{2,3}^{T} & \boldsymbol{\sigma}_{3} & \boldsymbol{\lambda}_{1,3}^{T} \\
\boldsymbol{\lambda}_{1,2} & \boldsymbol{\lambda}_{1,3} & \boldsymbol{\sigma}_{1}
\end{array}\right), \tag{3.5.6}
\end{align*}
$$

where each entry in the last three matrices represents a $(2 \times 2)$-dimensional matrix and the superscript $T$ denotes its transpose. We define the block matrices composing the CMs, Eqs. (3.5.4) to (3.5.6), in the Appendix 5.7. The diagonal elements represent the single-mode covariance matrices, and the off-diagonal codify the correlation between them. Each term in the entries of the block matrices depends on the masses of each subsystem, the coupling constant, and the standard deviation of the initial position probability distribution of the Gaussian system under examination; in what follows, we omit this dependence for brevity. The covariance matrices, Eqs. (3.5.4) to (3.5.6), are associated with a pure state since they satisfy Eqs. (1.4.2), (1.4.3) and (1.4.5).

To analyze the separability of each bipartition, we partially transpose the corresponding CM with respect to the single-mode partie; then, we test its physical validity by checking the satisfaction of the uncertainty
relation, Eq. (1.2.11). This process implies testing the positive semidefiniteness of three $(6 \times 6)$-dimensional Hermitian matrices; therefore, we use the so-called Silvester's criterion [160]:

## Silvester's criterion.

A Hermitian matrix $\boldsymbol{A}$ is positive semidefinite if every principal minor (including $\operatorname{Det} \boldsymbol{A}$ ) is non-negative.

Then, by using the software Mathematica and Silvester's criterion, we find that each of the three partially transposed covariance matrices, Eqs. (3.5.4) to (3.5.6), do not obey the uncertainty relation $\tilde{\boldsymbol{\sigma}}_{j \mid k l}+i \boldsymbol{\Omega} \geq 0$ since each corresponding matrix has at least one negative principal minor, which is given by the determinant of the $(5 \times 5)$-dimensional leading submatrix of each matrix. Therefore, this implies the non-physical validity of each partially transposed covariance matrix associated with each bipartition, which means the certification of genuine tripartite entanglement in our specific Gaussian configuration of the full dynamical Arthurs-Kelly measurement process.

It is important to note that this non-separability is maintained for any value of the physical parameters involved in the wave function describing the dynamics, Eq. (3.2.9). The non-separability of each bipartition is maintained in the strong coupling regime described by Arthurs and Kelly; that is to say, if we take the limit situation for each of the partially transposed covariance matrices:

$$
\begin{equation*}
\lim _{\kappa \longrightarrow \infty} \boldsymbol{\sigma}_{i \mid j k}, \text { for } i, j, k \in\{1,2,3\}, i \neq j \neq k \tag{3.5.7}
\end{equation*}
$$

and we compute the left-hand-side of the uncertainty relation $\boldsymbol{\sigma}_{\tilde{i} \mid j k}+i \boldsymbol{\Omega}$, we obtain the following matrices

$$
\boldsymbol{\sigma}_{1 \mid 23}+i \boldsymbol{\Omega}=\left(\begin{array}{cccccc}
4 \delta_{q_{3}}^{2} & i & 0 & 2 \delta_{q_{3}}^{2} & 4 \delta_{q_{3}}^{2} & 0  \tag{3.5.8}\\
-i & \frac{1}{\delta_{q_{3}}^{2}} & \frac{1}{2 \delta_{q_{3}}^{2}} & 0 & 0 & \frac{1}{\delta_{q_{3}}^{2}} \\
0 & \frac{1}{2 \delta_{q_{3}}^{2}} & \frac{1}{\delta_{q_{3}}^{2}} & i & 0 & \frac{1}{\delta_{q_{3}}^{2}} \\
2 \delta_{q_{3}}^{2} & 0 & -i & 4 \delta_{q_{3}}^{2} & 4 \delta_{q_{3}}^{2} & 0 \\
4 \delta_{q_{3}}^{2} & 0 & 0 & 4 \delta_{q_{3}}^{2} & 6 \delta_{q_{3}}^{2} & i \\
0 & \frac{1}{\delta_{q_{3}}^{2}} & \frac{1}{\delta_{q_{3}}^{2}} & 0 & -i & \frac{3}{2 \delta_{q_{3}}^{2}}
\end{array}\right),
$$

$$
\begin{gather*}
\boldsymbol{\sigma}_{3 \mid 12}+i \boldsymbol{\Omega}=\left(\begin{array}{cccccc}
6 \delta_{q_{3}}^{2} & i & 4 \delta_{q_{3}}^{2} & 0 & 0 & 4 \delta_{q_{3}}^{2} \\
-i & \frac{3}{2 \delta_{q_{3}}^{2}} & 0 & \frac{1}{\delta_{q_{3}}^{2}} & -\frac{1}{\delta_{q_{3}}^{2}} & 0 \\
4 \delta_{q_{3}}^{2} & 0 & 4 \delta_{q_{3}}^{2} & i & 0 & 2 \delta_{q_{3}}^{2} \\
0 & \frac{1}{\delta_{q_{3}}^{2}} & -i & \frac{1}{\delta_{q_{3}}^{2}} & -\frac{1}{2 \delta_{q_{3}}^{2}} & 0 \\
0 & -\frac{1}{\delta_{q_{3}}^{2}} & 0 & -\frac{1}{2 \delta_{q_{3}}^{2}} & \frac{1}{\delta_{q_{3}}^{2}} & i \\
4 \delta_{q_{3}}^{2} & 0 & 2 \delta_{q_{3}}^{2} & 0 & -i & 4 \delta_{q_{3}}^{2}
\end{array}\right),  \tag{3.5.9}\\
\boldsymbol{\sigma}_{2 \mid 31} \tilde{0}+i \boldsymbol{\Omega}=\left(\begin{array}{cccccc}
\frac{1}{\delta_{q_{3}}^{2}} & i & 0 & \frac{1}{\delta_{q_{3}}^{2}} & 0 & -\frac{1}{2 \delta_{q_{3}}^{2}} \\
-i & 4 \delta_{q_{3}}^{2} & -4 \delta_{q_{3}}^{2} & 0 & -2 \delta_{q_{3}}^{2} & 0 \\
0 & -4 \delta_{q_{3}}^{2} & 6 \delta_{q_{3}}^{2} & i & 4 \delta_{q_{3}}^{2} & 0 \\
\frac{1}{\delta_{q_{3}}^{2}} & 0 & -i & \frac{3}{2 \delta_{q_{3}}^{2}} & 0 & -\frac{1}{\delta_{q_{3}}^{2}} \\
0 & -2 \delta_{q_{3}}^{2} & 4 \delta_{q_{3}}^{2} & 0 & 4 \delta_{q_{3}}^{2} & i \\
-\frac{1}{2 \delta_{q_{3}}^{2}} & 0 & 0 & -\frac{1}{\delta_{q_{3}}^{2}} & -i & \frac{1}{\delta_{q_{3}}^{2}}
\end{array}\right), \tag{3.5.10}
\end{gather*}
$$

for which it is trivial to verify the following determinants of the corresponding upper $(5 \times 5)$-dimensional leading sub matrices

$$
\begin{align*}
& \operatorname{Det}\left[\boldsymbol{\sigma}_{\tilde{1} \mid 23}+i \boldsymbol{\Omega}\right]_{(5 \times 5)}=-8 \delta_{q_{3}}^{2},  \tag{3.5.11}\\
& \operatorname{Det}\left[\boldsymbol{\sigma}_{\tilde{3} \mid 12}+i \boldsymbol{\Omega}\right]_{(5 \times 5)}=-8 / \delta_{q_{3}}^{2},  \tag{3.5.12}\\
& \operatorname{Det}\left[\boldsymbol{\sigma}_{\tilde{2} \mid 31}+i \boldsymbol{\Omega}\right]_{(5 \times 5)}=-32 \delta_{q_{3}}^{2} ; \tag{3.5.13}
\end{align*}
$$

then, since $\delta_{q_{3}}^{2}>0$, the principal minors expressed by Eqs. (3.5.11) to (3.5.13) are negative; hence, this verify the non-physical validity of the partially transposed covariance matrices associated with each modebipartition in the strong coupling regime; therefore, the condition of genuine tripartite entanglement remains valid even in the original model of Arthurs and Kelly.

### 3.5.2 Qualitative properties

We have proved in the last section the non-physical validity of each of the three mode-bipartitions in our specific full dynamical Gaussian model of the Arthurs-Kelly measurement process. As we explain in Section 3.5.1, the biseparability condition (see Eq. (1.3.18)) implies that the negation of the three bipartitions of the pure tripartite system means genuine tripartite entanglement. Therefore, given the extensive classification
for three-mode Gaussian states explained in Section 1.4.2, we conclude that our system is categorized in the class ( C 1 ) ; that is, there exists inseparability for each of the three bipartitions of the system.

### 3.5.3 Quantitative properties

In this section, we study the quantification of the tripartite entanglement generated in our full dynamical Arthurs-Kelly measurement process; for this purpose, we recur to the minimum residual Gaussian contangle as quantifier measure; then, we follow the steps developed for pure tripartite Gaussian states explained in Section 1.4.4.2.

From Eq. (1.4.9) and the CMs, Eqs. (3.5.4) to (3.5.6), we identify the local invariants $a_{i}$ corresponding to the modes 1,2 and 3 , that is, to the first and second pointer, and the Gaussian system under observation for our measurement configuration; then we have

$$
\begin{gather*}
a_{j}=\sqrt{4+\frac{1}{\ell}}  \tag{3.5.14}\\
a_{l}=a_{\min }=2  \tag{3.5.15}\\
a_{k}=\sqrt{9+\frac{1}{\ell}} \tag{3.5.16}
\end{gather*}
$$

where $\ell=\left(2 \sqrt{2} \kappa m_{3} \delta_{\hat{q}_{3}}^{2}\right)^{2}$. Then, according to Eqs. (1.4.9) and (1.4.10), the local invariants $a_{j}$ and $a_{k}$ are increasing as the quantity $\ell$ becomes smaller; this implies the loss of purity of the single-mode systems associated with the first pointer and the Gaussian system under examination as the coupling constant $\kappa$, the mass $m_{3}$, and the variance $\delta_{\hat{q}_{3}}^{2}$ are small. This fact is a consequence of the free evolution of the Gaussian system under measurement since as $\kappa \longrightarrow \infty$ (i.e., the situation without any free dynamics), the quantities $a_{j}$ and $a_{k}$ acquire punctual values which


Figure 17: a) Region of allowed values delimited by the saturation of the triangle inequality, Eq. (1.4.11), for the local invariants given by Eq. (3.5.14) to (3.5.16); they tend to vary hyperbolically (dotted line) within this region according to the relation $a_{k}=\sqrt{a_{j}^{2}+5}$. This fact verify the physical validity of the covariance matrices representing the three bipartitions of the system.


Figure 18: Plot of the minimum residual Gaussian contangle $G_{\tau}^{\mathrm{res}}(\ell)$ for the entangled three-mode Gaussian system associated with our specific Gaussian measurement configuration of the full dynamical Arthurs-Kelly measurement process. This plot is associated with symplectic invariants, Eqs. (3.5.14) to (3.5.16). The blue curve represents the quantity of tripartite entanglement contained in the measurement configuration as a function of the parameter $\ell$. The increase in the amount of entanglement happens as this parameter decreases and vice versa.
are independent of the initial preparation of the measured Gaussian system. In this situation, we recover a specific value for the tripartite entanglement developed in the measurement configuration, which is correspondent with the strong coupling regime described by Arthurs and Kelly.

At the value of $a_{l}=2$, the local invariants given by Eqs. (3.5.14) and (3.5.16), are constrained to hyperbolically vary in the form $a_{k}^{2}-a_{j}^{2}=5$, inside the region of allowed values delimited from the saturation of the triangular inequality of Eq. (1.4.11) with the constraints $2<a_{j}<\infty$ and $3<a_{k}<\infty$; see Fig. 17; this verify the validity of the three CMs given by Eqs. (3.5.4) to (3.5.6). Since the local symplectic invariants of Eqs. (3.5.14) and (3.5.16), depend on the mass of the Gaussian system under measurement and completely determine the amount of
entanglement of the three-partite Gaussian system, we can conclude that the free energy operator $e^{\hat{p}_{3}^{2} / 2 m_{3}}$ appearing in the time evolution operator of the Eq. (3.2.4), is the only one that affects the quantitative entanglement properties of the system as we hypothesize in Section 3.5. This is because the free energy operators $e^{\hat{p}_{1,2}^{2} / 2 m_{1,2}}$ merely act as dispersive operators without some involvement in the correlation between the $\hat{x}_{3}$ and $\hat{p}_{3}$ observables appearing in the time evolution operator given in Eq. (3.2.4).
Now, in the following, we systematically follow the three steps explained in Section 1.4.4.2 for the determination of the quantitative entanglement properties of our Gaussian three-mode state describing the full dynamical Arthurs-Kelly simultaneous measurement process at the time $t=\kappa^{-1}$.

Then, following the steps (1) and (2) and using Eqs. (3.5.14) to (3.5.16), we determine the smallest reciprocal single-mode purity; that is, $a_{l}=2$, which is associated with the state of the second detector of the measurement setting. Hence, following the step (3), we compute the parameters $s$ and $d$ according to Eqs. (1.4.47) and (1.4.48); then, we compute the quantities given in Eqs. (1.4.51) to (1.4.55), to finally substitute in Eqs. (1.4.50) and (1.4.49) in order to get the quantifier $G_{\tau}^{\text {res }}$ for the tripartite entanglement of the system. Finally, we plot the minimum residual Gaussian contangle of our measurement setting at the time $t=\kappa^{-1}$ as a function of $\ell$ in Fig. 18. Hence, let us analyze that plot in order to establish the role of the free evolution of the system in the entanglement properties. Specifically, we can observe the following three points:
(a) We have an increasing amount of tripartite entanglement as the $\ell$-parameter diminishes; the maximum is $\operatorname{arcsinh}^{2}[\sqrt{3}]$, reached in the limit situation $\lim _{\ell \rightarrow 0} G_{\tau}^{\text {res }}(\ell)$.
(b) We have an decreasing amount of tripartite entanglement as the $\ell$-parameter grows, following an asymptotic regularization up to the minimum value of $\operatorname{arcsinh}^{2}[\sqrt{3}]-5 \ln ^{2}(3) / 4$ reached in the limit situation: $\lim _{\ell \rightarrow \infty} G_{\tau}^{\text {res }}(\ell)$.
(c) Since the quantity of tripartite entanglement of the system depends on the $\ell$-parameter, we conclude that in the full dynamical mea-
surement process, the initial preparation of the Gaussian system under observation plays a role in the quantitative entanglement properties of the system.

From point (a), we can associate an increasing amount of tripartite entanglement in the measurement configuration as the mass $m_{3}$ and the variance $\delta_{\hat{q}_{3}}^{2}$ diminishes; notably at fixed values for these preparation parameters we deduce that a weak coupling between pointers and Gaussian system also causes an increase in the amount of tripartite entanglement of the system. From point (b) we deduce the conversely; that is, as the mass $m_{3}$ and the variance $\delta_{\hat{q}_{3}}^{2}$ increases, the tripartite entanglement of the system follows an asymptotic decreasing up to the minimum value; hence, for fixed values of these parameters, we infer that strong coupling between pointers and system carries less development in the tripartite entanglement of the system. Therefore, we have the bound

$$
\begin{equation*}
\operatorname{arcsinh}^{2}[\sqrt{3}]-5 \ln ^{2}(3) / 4 \leq G_{\tau}^{\mathrm{res}}(\ell) \leq \operatorname{arcsinh}^{2}[\sqrt{3}] \tag{3.5.17}
\end{equation*}
$$

for the tripartite entanglement developed in the full dynamical ArthursKelly measurement process in our specific Gaussian measurement configuration. Then, notably, the lower bound of $G_{\tau}^{\mathrm{res}}(\ell)$ is reached in the strong coupling scheme described by Arthurs and Kelly, which, as we will explain just bellow, is a consequence of an instantaneous time-lapse of the measurement process.

The variation of the tripartite entanglement in dependency of the $\ell$-parameter (for fixed values of $m_{3}$ and $\delta_{\hat{q}_{3}}^{2}$ ) is only a consequence of the spread of the global wave function through the measurement process. We recall that in a weak coupling regime, we have a larger time-lapse for the measurement process (see Section 3.4); hence, as a consequence, we have an increasing spreading in the global wave function describing the measurement setting, as well as in each marginal wave function associated with each system. Therefore, this phenomenon carries a increasing loss information process about the initial quantum state of each subsystem. Then, the spreading affects only the single-mode purity of the first pointer and that of the Gaussian system under examination, as can be verified from Eqs. (3.5.14) and (3.5.16) and the definition for purity, Eq. (1.4.10). Notably, this behavior is analogous to that of a free propagating Gaussian wave packet entangled with another subsystem; for example, in hybrid
continuous-discrete variable schemes $[161,162,163]$ where the external (the position) and internal (the spin) degrees of freedom of a spin particle are entangled. In such settings, the entanglement becomes increasing as the spreading of the Gaussian wave function (which characterizes the external degrees of freedom of the system) increases; hence reflecting the loss of purity of the system, therefore, the loss of information initially associated with the quantum state of the particle. Even such spreading of the Gaussian wave function has been associated with the quantum steerability and the non-local features of the system [164, 165]. Then, the minimum residual Gaussian contangle correctly quantifies the loss of purity in the three-mode Gaussian system describing the measurement process, which is a natural fact since this entanglement quantifier is based on the logarithmic negativity, Eq. (1.4.41), which, in principle, is equivalent to the entropy of entanglement for pure states [35].

On the other hand, the point (c) highlights the fact that only with the consideration of the free evolution of the measurement configuration through the measurement process, the initial preparation of the Gaussian system under inspection gets involved in the entanglement properties of the system, since the $\ell$-parameter depends on its mass $m_{3}$ and its initial variance $\delta_{\tilde{q}_{3}}^{2}$. The squeezing of the Gaussian system under measurement gets involved in the amount of entanglement of the system since we have $S_{3}=\left(\sqrt{2} \delta_{q}\right)^{-1}$ and the $\ell$-parameter depends directly proportional to $\delta_{\hat{q}_{3}}^{2}$. Then, by increasing the squeezing of the Gaussian system under measurement, the entanglement of the system grows and vice versa.

## 4

## SUMMARY AND CONCLUSIONS

### 4.1 SCOPE AND PROPOSALS OF THIS THESIS WORK CONCERNING WITH THE ARTHURS-KELLY MEASUREMENT PROCESS AS A CONTINUOUS-VARIABLE SCHEME FOR QUANTUM INFORMATION PROCESSING

> "La esperanza es el peor de los males, pues prolonga el tormento del hombre"

One of the most recurrent treaties of the theory of quantum mechanics is the famous statute that is impossible to know with arbitrary accuracy the position and momentum of a quantum object. The birth of this principle comes originally from Heisenberg [96], who, through the famous thought gamma-ray experiment, intuitively arrives at the particular position-momentum indeterminacy relation: $q_{1} p_{1} \sim h$; then, in this expression $q_{1}$ represents the precision at which the position is known (quantified by some dispersive measure) and $p_{1}$ the accuracy at which the value of momentum is determinable. However, the first mathematically formal attempt to associate the Heisenberg uncertainty relation for the canonical pair with a measurement process comes from the hand of Arthurs and Kelly [98] using the standard model of a quantum measurement proposed by von Neumann [93]. Then, the model joins two conceptual cornerstones of the theory of quantum mechanics: the indeterminacy of conjugate variables of a quantum system and the first measurement theory of von Neumann proposed as soon after the birth of quantum mechanics.

One of the goals of this thesis work is to identify the model of simultaneous measurement as a continuous-variable scheme from which it is possible to highlight genuinely quantum features. Concerning with the field of quantum information with continuous variables [19, 166], we point out the model as a specific generator of multipartite, and, particularly, tripartite continuous-variable Gaussian entanglement either in the full dynamical or the strong coupling model; see Sections 4.3 and 4.4. We prove this statute in the particular regime of Gaussian states which is the more extensive regime for the develop of quantum information research field [23]. However, we conjecture that the generation of tripartite entanglement for the Arthurs-Kelly model must be maintained for any quantum state since the original proposal (which uses an arbitrary state as the system under observation) demonstrates the transfer of the statistical information about the canonical pair for this more general case, which in principle is an indirect proof of the existence of quantum entanglement, even, in this more general scenario.

The seminal model proposed by Arthus and Kelly [98] pretends to show the fundamental limits of (retrodictive) accuracy in the measurement of position and momentum variables; because of this, they discard the free evolution of the system and take only into account the interaction Hamiltonian, which links the three subsystems of the measurement setting. In this scenario, they show the existence of an inherent and unavoidable noise carried by the measurement process, which directly comes from the fluctuations of the detectors. Besides, this extra noise is a mathematical consequence of keeping commuting the observables of the detectors even through the measurement process [110]; in other words, to maintain the simultaneous measurability of the canonical pair from the system of the detectors, it is necessary to add extra fluctuations to the initial statistics of the observables under examination; however that noise is balanced, then, it prevents to measure with arbitrary accuracy both canonical observables.

We demonstrate that the original proposal for the simultaneous measurement is a particular case of the fully dynamical model. In this more general case, we show the emergence of an extra noise that affects the accuracy of the simultaneous measurement as the system under examination spreads in time. Therefore, the free evolution of the wave function of the measurement configuration has a role in the accuracy of the si-
multaneous measurement. Also, we study the concepts of retrodictive and predictive accuracy $[101,102,103]$ of the simultaneous measurement process in a complete dynamics regime; we find that the free evolution of the system has an impact on the determination of the statistics of the position and momentum variables for both aspects of accuracy. In particular, we show that the coupling between the detectors and the Gaussian system under measurement determines the quantity of spreading of the wave function describing the system through the measurement process; based on this fact, we conclude that the coupling between the three subsystems of the measurement set-up has a direct influence on both aspects of accuracy of the simultaneous measurement.

The future steps concerning with the research of this thesis can involve direct applications of the entanglement developed by the simultaneous measurement process; for example, the task of teleportation [167], quantum communication and tomography [168], the research field of one-way quantum computation using continuous variables [169] or the link with the quantum information using continuous-variable cluster states [79]. Besides, we can board fundamental studies like quantum non-locality [170], quantum steering [60], and consider the effects of decoherence [35] in the more realistic case where the system is not closed. Also, we can try to demonstrate the three-mode entanglement in the most general case of an arbitrary system under observation using, for example, entanglement witnesses for continuous-variable systems [171, 172].

In the following sections, we will give the conclusions about the three principal research topics of this thesis work, that is: 1. the accuracy in the full dynamical measurement process of the position and momentum observables; see Sections 3.3 and 3.4. 2. The tripartite entanglement generated in the simultaneous measurement process for the position and momentum in the original regime of strong coupling; see Section 2.4 and 3. the tripartite entanglement generated in the simultaneous measurement process in the full dynamical model; see Section 3.5.

### 4.2 CONCLUSIONS ABOUT THE ACCURACY IN THE FULL DYNAMICAL SIMULTANEOUS MEASUREMENT PROCESS OF POSITION AND MOMENTUM OBSERVABLES OF A GAUSSIAN SYSTEM

We consider the complete dynamics of the simultaneous measurement process of the position and momentum; therefore, we include the free energy operator of each subsystem in the measurement set-up. Besides, to examine the effect of the free evolution in the accuracy of the simultaneous measurement, we assume a quantum system represented by a Gaussian state; in particular, we choose a squeezed vacuum state like the state representing the detectors of the measurement apparatus. With these assumptions, we implicitly assume an arbitrary coupling constant in a complete Gaussian configuration, and the whole system becomes under temporal spread through the measurement process.

As a mathematical consequence of the joint unbiasedness condition (see Section 5.2 and Subsection 2.1.3), the time required for the simultaneous reading of the pointers is the reciprocal of the coupling constant; this fact implies that a weak coupling entails a long time-lapse of the measurement process and, conversely, the strong coupling implicitly carries a short time of measurement. Then, due to these facts, the effect of the spreading in all marginal and global density probabilities becomes higher in the weak coupling regime; consequently, the extra noise due to the spreading results in an increase in the fluctuations affecting the marginal statistics collected by the detectors of the measurement device; see Section 3.3.

From the last arguments and considerations, we arrive to the the following conclusions:
(1.) The accuracy of the simultaneous measurement of position and momentum observables increases by increasing the coupling between the detectors and the Gaussian system under examination.

Conversely,
(2.) The accuracy of the simultaneous measurement of position and momentum observables decreases by decreasing the coupling between the detectors and the Gaussian system under observation.

In particular, the conclusions (1.) and (2.) are valid for the two aspects of accuracy in the simultaneous measurement [101, 102, 103]. We study two aspects of accuracy: (i) the retrodictive, which essentially quantifies the degree to which the statistics of the marginal measurements of the pointers coincides with the initial (before the measurement) statistics of the observables of position and momentum of the system under inspection and (ii) the predictive, which compares the match between the statistics of the marginal measurements collected by the pointers and the final statistics of position and momentum just after the time of measurement. Notably, this last aspect of accuracy allows the definition for a disturbance observable which quantifies the degree of perturbation on the canonical observables of the system under examination due to the measurement process [101, 102, 103].

Finally, we establish the fact that the strength of the conclusions (1.) and (2.) must be maintained for any other system that is not Gaussian for which the measurement is being carried out. This argument comes from the fact that the free energy operators $e^{a(t) \hat{p}_{i}^{2}}$ appearing in the time evolution operator of the Eq. (3.2.4) will cause the free propagation of the wavefunction of the whole system ${ }^{1}$, i.e. the temporal dispersion of its associated probability distribution [149, 173].

### 4.3 Conclusions about the entanglement generated in the simultaneous measurement process in the regime of strong coupling

In general, a quantum measurement process is divided into two stages [157, 158]: (i) an entangling interaction, where all subsystems of the measurement configuration become correlated, and (ii) a readout process, where the observable property appears in the subsystem acting as the

[^4]detector; in this stage, all quantum correlations disappear, and the subsystems are no more entangled.

In particular, we study the entanglement properties generated in the original model of simultaneous measurement proposed by Arthurs and Kelly; then, we focus on the interaction stage of this process. We consider a rotated, displaced, and squeezed vacuum state as the system under measurement; that is, we have the most general continuous-variable Gaussian single-mode state [23]; therefore, with this consideration, we have a complete Gaussian measurement configuration since the detectors of the measurement apparatus are represented by squeezed vacuum states. Essentially, the mechanism to transfer the statistical information about the position and momentum observables of the system under examination is possible due to the quantum entanglement between the three subsystems. Then, under these considerations, we conclude that
(1.) The entanglement generated in the simultaneous measurement process of position and momentum observables proposed by Arthurs and Kelly constitutes a particular case of tripartite Gaussian entanglement when the system under measurement is Gaussian.

We prove the previous conclusion by testing the physical validity of each of the three partially transposed covariance matrices describing the three (1 vs 2)-mode bipartitions of the three-mode system. We carry out this process through the PPT criterion [62] in the language of the covariance matrices [63]; see Section 1.3.5.2. In particular, we find that each partially transposed covariance matrix does not obey the Heisenberg uncertainty relation for multiple continuous-variable modes, which implies that the three subsystems that compose the measurement configuration are entangled between them. This last fact allows to classify the generated entanglement in the category of genuinely threemode entangled states [9] just like other systems as the tripartite versions of the continuous variable GHZ-states $[10,76]$ and the analogous version of the continuous-variable Einstein-Podolsky-Rosen (EPR) states [77, 78]. We conjecture that the conclusion (1.) must remain valid for any other system under examination since the original model of simultaneous measurement of Arthurs and Kelly shows the transfer of information
about the canonical pair of an arbitrary system, which implicitly means generation of quantum entanglement in this more general assumption. Besides, the strength of this conjecture is supported by the fact that our analysis shows that the entangled nature of the system is independent of the initial preparation of the Gaussian system under observation. Moreover, the general dynamics displayed by the Eqs. (2.3.1) shows a tripartite relation between three distinct variables, each belonging to a different mode of the measurement configuration, which implies quantum correlations between these observables; therefore, this also supports the validity of our conjecture.

On the other hand, we study the entanglement structure developed by the measurement process. Then, we use the residual tripartite Rényi-2 entanglement as a quantifier measure; see Section 1.4.4.3. In particular, we examine all global and reduced mode-bipartitions as well as the entanglement between the three modes of the system.

For the one-to-one entanglements, that is, for the reduced bipartitions including only two modes, we conclude the following
(2a.) The reduced bipartition containing the modes of each detector has zero entanglement.
(2b.) The reduced bipartition containing the modes of any detector and the one of the system under measurement presents the same amount of entanglement.

The conclusion (2a.) shows that the mode associated with the Gaussian system under measurement is the principal support of the tripartite relation. Then, by tracing this system, we cannot see direct entanglement relation between the canonical observables of the detectors. These variables are related to just one distinct canonical variable of the system under examination; see Fig. 3. On the other hand, the conclusion (2b.) implies that each detector is equally entangled with the Gaussian system under measurement, which is a crucial and necessary fact for the transfer of information about the canonical pair to the state of the detectors. Notably, the entanglement contained in this reduced bipartition depends on the squeezing parameter $r$ and the rotation angle $\theta$ of the Gaussian system under examination; besides, that entanglement
is bounded according to $(1 / 2) \ln (2)$ and $\ln (5 / 3)$, where the maximum happens when the system is in a minimum uncertainty state.

For the reduced bipartitions of ( 1 vs 2 ) modes, we conclude that
(3.) The global bipartitions focused on any detector contain the same amount of entanglement for the same squeezing parameter $r$ and rotation angle $\theta$ of the Gaussian system under examination. Besides, the entanglement of the global bipartition focused on the Gaussian system under inspection is ever greater than those bipartitions focused on the detectors for the same squeezing parameter $r$ and rotation angle $\theta$.

The conclusion (3.) is a direct consequence of the interaction Hamiltonian governing the measurement process; that is, it creates a symmetric entanglement structure around the Gaussian system under examination. Besides, the dynamic of the measurement process principally affects the purity of the system under examination, as can be verified from Eqs. (3.5.16) and (1.4.10); then, since the quantifier of residual Rényi-2 entanglement for the global bipartitions directly depends on the inverse of the single-mode purities, hence, it is natural to expect a higher amount of entanglement in the global bipartition focused on the system under examination. Therefore, in essence, the nature of the conclusion (3.) is a consequence of the symmetry in the unitary dynamics of the measurement process and how it affects the single-mode systems. Notably, the entanglement in the global bipartitions focused on any detector and the Gaussian system under observation are bounded below by $(1 / 2) \ln (4)$ and $(1 / 2) \ln (9)$ respectively; besides, these entanglements have an oscillatory behavior with a period of $\pi / 2$, and they increase indefinitely as the magnitude of the squeezing parameter $r$ increases.

For the full tripartite entanglements, we conclude that
(4.) The tripartite entanglements focused on any detector contain the same amount of entanglement for the same squeezing parameter $r$ and rotation angle $\theta$ of the Gaussian system under examination. Besides, the tripartite entanglement focused on the Gaussian system under inspection is ever smaller than the tripartite entanglement focused on the detectors for the same squeezing parameter $r$ and rotation angle $\theta$ of the Gaussian system under examination.

The conclusion (4.) is again a consequence of the symmetry and nature of the unitary dynamics of the simultaneous measurement process. Besides, the behavior of the full tripartite entanglements is consistent with the description of an observer localized on the mode chosen as focus, where the entanglement from the perspective of this observer will be like 'at sight' of the reciprocals purities associated with the other two parties; in this manner, the tripartite entanglement focused on the Gaussian system under measurement will be higher for the same squeezing and rotation angle parameters. Notably, the tripartite entanglements have an equivalent behavior to the ones presented in the global bipartitions of ( 1 vs 2 )-modes; that is, they are oscillatory functions of $r$ and $\theta$ with a period of $\pi / 2$, and they increase indefinitely as the magnitude of the squeezing parameter $r$ also increases; besides, they are bounded below according to $\ln (6 / 5)$ and $\ln (27 / 25)$ for the focus of any detector and the Gaussian system under measurement respectively.

### 4.4 CONCLUSIONS ABOUT THE ENTANGLEMENT GENERATED IN THE SIMULTANEOUS MEASUREMENT PROCESS IN THE COMPLETELY DYNAMICAL MODEL

As we explain in the last section, all measurement process includes an entangling interaction, where all observable properties are in correlation, and the measurement read-out time, where the desired property under examination takes form in the detector. Then, we study the first stage in this process. In this case, we board the completely dynamical process; therefore, we include the free evolution of each subsystem of the measure-
ment configuration. Under this fact, we have the free energy operator $\left(\hat{p}_{3}\right)^{2} / 2 m$ which gets involved with the non-commutativity of the $\hat{x}_{3}$ variable; then, from this fact, is deduced that the spreading phenomena of the wave function will be involved in the entanglement properties of the system.

To consider the involvement of the free evolution in the entanglement generated in the simultaneous measurement process, we assume a particular state as the system under exploration; this is because we do have not an explicit mathematical expression that gives us the temporal evolution of an arbitrary wave function in the time evolution operator method [152]; see Eq. (3.2.6) and the text bellow. Therefore, we choose a Gaussian state as the system under measurement; in particular, we select a squeezed vacuum state like the state of the detectors in order to represent the system under examination. Therefore, with this assumption, we have an entire Gaussian measurement configuration, and all frameworks [3, 9, 35] to test and quantify entanglement in these systems are valid.

First, we compute the temporal evolution employing the time evolution operator method [94]; hence, we use the wave function governing the dynamics to obtain the covariance matrices describing the three (1 vs 2)mode bipartition of the system. Then, by testing the separability of each partially transposed covariance matrix, we arrive at the following conclusion,
(1.) The entanglement that arises in the completely dynamical simultaneous measurement process of Arthurs-Kelly for the position and momentum observables constitutes a particular example of tripartite Gaussian entanglement when the system under measurement is represented by a Gaussian state.

The last conclusion is based on the fact that the three partially covariance matrices of the ( 1 vs 2)-mode bipartitions do not obey the Heisenberg uncertainty relation for multiple continuous-variable modes. Then, we can classify the generated entanglement of the category of genuinely three-mode entangled states [9] as we establish for the model of strong coupling. Notably, the non-separability of these matrices holds for the limit situation of strong coupling, which means that our completely dynamical model of measurement proves the entanglement generation
for any proportionally-dependent coupling between the detectors and the Gaussian system under examination. Again, we conjecture that the strength of the conclusion (1.) must maintain for any quantum state under examination since the unitary dynamics governing the measurement process includes the interaction Hamiltonian, which, inherently entangle the three subsystems defining the measurement setting. This aspect is indirectly shown by Arthurs and Kelly [98] who show the transfer of information of the statistics of the canonical pair on interest to the system of the detectors. Besides, the completely dynamical model includes the free energy operator $\left(\hat{p}_{3}\right)^{2} / 2 m$, which does not commute with the position observable $\hat{x}_{3}$, which is an independent fact of the system under examination. Moreover, the conclusion (1.) is independent of the initial preparation of the Gaussian system under observation, which is an indirect indication for their validity and extension on any quantum system under measurement.

On the other hand, we determine the quantitative entanglement properties developed in the system through the measurement process as a proportionally-dependent function of the coupling between the detectors and the system under observation. In particular, we use the minimum residual Gaussian contangle as a quantifier; see Section 1.4.4.2. First of all, we find the loss of purity of the single-mode systems associated with the first detector and the Gaussian system under observation as the coupling between the detectors and the system under measurement is small. This fact comes directly from the non-commutativity of the $\left(\hat{p}_{3}\right)^{2}$ and $\hat{x}_{3}$ variables. Then, analyzing the tripartite entanglement of the system as a proportionally-dependent function of the coupling between the detectors and the Gaussian system, we arrive at the following conclusions
(2a.) The tripartite Gaussian entanglement developed by the simultaneous measurement process is increasing as the coupling between detectors and the Gaussian system under observation diminishes.

## Conversely

(2b.) The tripartite Gaussian entanglement developed by the simultaneous measurement process diminishes as the coupling between detectors and the Gaussian system under observation increases.

Notably, the tripartite entanglement of the system (quantified by the minimum residual Gaussian contangle) is bounded between the margins stipulated by the conclusions (2a.) and (2b.) according to $\operatorname{arcsinh}^{2}[\sqrt{3}]-5 \ln ^{2}(3) / 4 \leq G_{\tau}^{\text {res }}(\ell) \leq \operatorname{arcsinh}^{2}[\sqrt{3}]$. Hence, the tripartite entanglement of the system is a proportionally-dependent function of the coupling between the detectors and system under examination, where the proportionality factor depends on the mass of the system under measurement $m_{3}$ and its initial variance $\delta_{\hat{q}_{3}}^{2}$ (consequently also depends on its squeezing). Therefore, in this case, the initial preparation of the Gaussian system under examination plays a role in the entanglement properties developed in the system.

Besides, the conclusions (2a.) and (2b.) imply that the free evolution of the system under measurement plays a role in the tripartite entanglement properties of the system; This is because the minimum residual Gaussian contangle proportionally depends on (of $m_{3}$ and $\delta_{\hat{q}_{3}}^{2}$ ) the coupling between the detectors and the Gaussian system under measurement. Hence, since the time of the simultaneous read-out of the detectors is the reciprocal of the coupling constant, we deduce that a long time-lapse (weak coupling) of the measurement process carries a further development of the entanglement in the system. Conversely, a short time-lapse (strong coupling) of measurement takes less amount of entanglement in the system. Notably, the behavior of this process is correctly captured by the minimum residual Gaussian contangle since it is an adequate entropic tripartite entanglement quantifier for three-mode Gaussian states [35]; therefore, we conclude that
(2c.) The behavior of the tripartite Gaussian entanglement developed by the simultaneous measurement process describes a phenomenon of loss of the initial information about the Gaussian measurement configuration due to the spreading phenomena.

The conclusion (2c.) is well supported, for example, by the fact that in hybrid continuous-discrete variable schemes [161, 162, 163] the entanglement becomes increasing as the spreading of the Gaussian wave function increases; hence reflecting the loss of purity of the system through the unitary dynamics; therefore, describing the loss of the initial information associated with the quantum state of the particle.

Once again, we conjecture that the validity of the conclusions (2a.), (2b.) and (2c.) must be maintained for any system under measurement because the free energy operator $\left(\hat{p}_{3}\right)^{2} / 2 m$ will cause the spreading of any state, since that phenomena is in inherent characteristic of the free evolution of wave packets.

## APPENDICES

### 5.1 Derivation of the arthurs-kelly dynamics through the time evolution operator method

In this Appendix we derive the wave function governing the dynamics of the Arthurs-Kelly simultaneous measurement process.

To apply correctly the operator $e^{-i \hat{H} t}$ to the initial wave function, Eq. (2.2.2), it is necessary to factorize it; then, we resort to the disentangling theorem (see pag. 49 [4]):

$$
\begin{align*}
e^{\hat{A}+\hat{B}} & =e^{-\frac{1}{2}[\hat{A}, \hat{B}]} e^{\hat{A}} e^{\hat{B}} \\
& =e^{\frac{1}{2}[\hat{A}, \hat{B}]} e^{\hat{B}} e^{\hat{A}} . \tag{5.1.1}
\end{align*}
$$

From Eq. (2.1.45) we identify $\hat{A}=-i t \kappa_{1} \hat{x}_{3} \hat{p}_{1}$ and $\hat{B}=-i t \kappa_{2} \hat{p}_{3} \hat{p}_{2}$, and

$$
\begin{align*}
{[\hat{A}, \hat{B}] } & =-\alpha_{1} \alpha_{2} \hat{p}_{1} \hat{p}_{2}\left[\hat{x}_{3}, \hat{p}_{3}\right] \\
& =-i \alpha_{1} \alpha_{2} \hat{p}_{1} \hat{p}_{2}, \tag{5.1.2}
\end{align*}
$$

where we take $\alpha_{i}=\kappa_{i} t, i=1,2$, omitting the time dependence for brevity. Then, according to Eq. (5.1.1) we can factorize the ArthursKelly interaction Hamiltonian as

$$
\begin{align*}
& e^{-i t \hat{H}}=e^{\frac{i}{2} \alpha_{1} \alpha_{2} \hat{p}_{1} \hat{p}_{2}} e^{-i \alpha_{1} \hat{x}_{3} \hat{p}_{1}} e^{-i \alpha_{2} \hat{p}_{3} \hat{p}_{2}}  \tag{5.1.3}\\
& e^{-i t \hat{H}}=e^{-\frac{i}{2} \alpha_{1} \alpha_{2} \hat{p}_{1} \hat{p}_{2}} e^{-i \alpha_{2} \hat{p}_{3} \hat{p}_{2}} e^{-i \alpha_{1} \hat{x}_{3} \hat{p}_{1}} \tag{5.1.4}
\end{align*}
$$

Taking the momentum representation for the wave function of the observed system $\phi_{3}\left(x_{3}\right) \longrightarrow \phi_{3}\left(p_{3}\right)$, and expressing the wave function of the
first prove through a Fourier transform: $\phi_{1}\left(x_{1}\right)=(2 \pi)^{-1 / 2} \int e^{-i x_{1} p_{1}} \phi_{1}\left(p_{1}\right) d p_{1}$, the initial wave function, Eq. (2.2.2), is

$$
\begin{equation*}
\psi\left(x_{1}, x_{2}, x_{3}, t=0\right)=(2 \pi)^{-1 / 2} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}\right) \phi_{3}\left(p_{3}\right) e^{-i x_{1} p_{1}} d p_{1} \tag{5.1.5}
\end{equation*}
$$

using the factorization given by Eq. (5.1.4), we apply the first exponential operator to the initial wave function:

$$
\begin{align*}
e^{-i \alpha_{1} \hat{x}_{3} \hat{p}_{1}} & \psi\left(x_{1}, x_{2}, x_{2}, t=0\right) \\
& =\sum_{n} \frac{\left(-i \alpha_{1} \hat{x}_{3} \hat{p}_{1}\right)^{n}}{n!} \psi\left(x_{1}, x_{2}, x_{2}, t=0\right) \\
& =(2 \pi)^{-1 / 2} \int \sum_{n} \frac{\left(-i \alpha_{1}\right)^{n}}{n!} \hat{p}_{1}^{n} \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}\right) \hat{x}_{3}^{n} \phi_{3}\left(p_{3}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-1 / 2} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}\right) \sum_{n}^{\infty} \frac{\left(-i \alpha_{1} p_{1}\right)^{n}}{n!} \hat{x}_{3}^{n} \phi_{3}\left(p_{3}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-1 / 2} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}\right) \sum_{n}^{\infty} \frac{\left(\alpha_{1} p_{1}\right)^{n}}{n!}\left(\frac{\partial \phi_{3}\left(p_{3}\right)}{\partial p_{3}}\right)^{n} e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-1 / 2} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \tag{5.1.6}
\end{align*}
$$

where we have used the McLaurin series of the exponential

$$
\begin{equation*}
e^{\hat{A}} \Psi=\sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^{n} \Psi \tag{5.1.7}
\end{equation*}
$$

and the condition [152]

$$
\begin{equation*}
\exp \left(\beta \partial_{z}\right) f(z)=\sum_{n=0}^{\infty} \frac{\beta^{n}}{n!}\left(\frac{\partial}{\partial z}\right)^{n} f(z)=f(z+\beta) \tag{5.1.8}
\end{equation*}
$$

Applying the second exponential operator of Eq. (2.2.6) to the result of Eq. (5.1.6), we have

$$
\begin{align*}
& e^{-i \alpha_{2} \hat{p}_{3} \hat{p}_{2}} e^{-i \alpha_{1} \hat{x}_{3} \hat{p}_{1}} \psi\left(x_{1}, x_{2}, x_{2}, t=0\right) \\
& =(2 \pi)^{-1 / 2} \int \sum_{n=0}^{\infty} \frac{\left(-i \alpha_{2} \hat{p}_{3} \hat{p}_{2}\right)^{n}}{n!} \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-1 / 2} \int \sum_{n=0}^{\infty} \frac{\left(-i \alpha_{2}\right)^{n}}{n!} \phi_{1}\left(p_{1}\right) \hat{p}_{2}^{n} \phi_{2}\left(x_{2}\right) \hat{p}_{3}^{n} \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-1 / 2} \int \phi_{1}\left(p_{1}\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) \sum_{n=0}^{\infty} \frac{\left(-\alpha_{2} p_{3}\right)^{n}}{n!}\left(\frac{\partial \phi_{2}\left(x_{2}\right)}{\partial x_{2}}\right)^{n} e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-1 / 2} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}-\alpha_{2} p_{3}\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \tag{5.1.9}
\end{align*}
$$

Finally, applying the last operator of Eq. (2.2.6) to the resulting wave function, Eq. (5.1.9), we have

$$
\begin{align*}
& e^{-\frac{i}{2} \alpha_{1} \alpha_{2} \hat{p}_{1} \hat{p}_{2}} e^{-i \alpha_{2} \hat{p}_{3} \hat{p}_{2}} e^{-i \alpha_{1} \hat{x}_{3} \hat{p}_{1}} \psi\left(x_{1}, x_{2}, x_{2}, t=0\right) \\
& =(2 \pi)^{-\frac{1}{2}} \int \sum_{n=0}^{\infty} \frac{\left(-\frac{i}{2} \alpha_{1} \alpha_{2} \hat{p}_{1} \hat{p}_{2}\right)^{n}}{n!} \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}-\alpha_{2} p_{3}\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-\frac{1}{2}} \int \sum_{n=0}^{\infty} \frac{\left(-\frac{i}{2} \alpha_{1} \alpha_{2}\right)^{n}}{n!} \hat{p}_{1}^{n} \phi_{1}\left(p_{1}\right) \hat{p}_{2}^{n} \phi_{2}\left(x_{2}-\alpha_{2} p_{3}\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-\frac{1}{2}} \int \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2} \alpha_{1} \alpha_{2} p_{1}\right)^{n}}{n!} \phi_{1}\left(p_{1}\right)\left(\frac{\partial \phi_{2}}{\partial x_{2}}\right)^{n} \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \\
& =(2 \pi)^{-\frac{1}{2}} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}-\alpha_{2} p_{3}-\alpha_{1} \alpha_{2} p_{1} / 2\right) \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} . \tag{5.1.10}
\end{align*}
$$

Therefore, the wave function describing the dynamics of the Arthurs-Kelly simultaneous measurement is

$$
\begin{align*}
\Psi\left(x_{1}, x_{2}, x_{3}, t\right) & =(2 \pi)^{-\frac{1}{2}} \int \phi_{1}\left(p_{1}\right) \phi_{2}\left(x_{2}-\alpha_{2} p_{3}-\frac{\alpha_{1} \alpha_{2} p_{1}}{2}\right) \\
& \times \phi_{3}\left(p_{3}+\alpha_{1} p_{1}\right) e^{-i x_{1} p_{1}} d p_{1} \tag{5.1.11}
\end{align*}
$$

which is entangled between the three positions $x_{i}$ of the proves and the investigated system.

## 5.2 mathematical formulation of the joint unbiasedNESS CONDITION

In this appendix we analyze the mathematical definitions for the joint unbiasedness condition which constitute the mathematical basis for the so called Heisenberg uncertainty relation for joint measurements [104]. This analysis goes under the theory presented by Roy [174].

Lets Assume a measuring device like one presented by Arthurs and Kelly; then, there is a system with two independent degrees of freedom where each is linked with an self-adjoint observable of an arbitrary system, such observables being canonically conjugate. Each system have associated an own Hilbert space $\mathcal{H}_{i}$, such that the whole configurations
live on the global Hilbert space $\mathcal{H}=\otimes_{i=1}^{3} \mathcal{H}$. With out any specific interaction, the time evolution of the global system is

$$
\begin{equation*}
|\Psi(t)\rangle=\hat{U}(t)|\psi(0)\rangle, \tag{5.2.1}
\end{equation*}
$$

with the initial state

$$
\begin{equation*}
|\psi(0)\rangle=\left|\phi_{1}(0)\right\rangle\left|\phi_{2}(0)\right\rangle, \tag{5.2.2}
\end{equation*}
$$

being a pure and factorizable product of the detectors states; the labels $i=1,2$ identifying the first and second detector respectively. In the more general case including mixed states, the dynamics of the measuring detectors and the system is given through the density operator

$$
\begin{equation*}
\hat{\rho}(t)=\hat{U}^{\dagger}(t) \hat{\rho}(0) \hat{U}(t), \text { with } \hat{\rho}(0)=\hat{\rho}_{1} \otimes \hat{\rho}_{2} \tag{5.2.3}
\end{equation*}
$$

In the Heisenberg representation, any observable $\hat{X}$ evolves according to

$$
\begin{equation*}
\hat{X}(t)=\hat{U}(t) X(0) \hat{U}^{\dagger} . \tag{5.2.4}
\end{equation*}
$$

by using Eqs. (5.2.3) and (5.2.4) and the cyclic property of the trace, we have

$$
\begin{equation*}
\langle\hat{X}(t)\rangle=\operatorname{Tr}[\hat{\rho}(T) \hat{X}(0)]=\operatorname{Tr}[\hat{\rho}(0) \hat{X}(t) .] \tag{5.2.5}
\end{equation*}
$$

Then, the observables of the detectors after the interaction, $\hat{X}(t)$ 'tracks' the initial system observables $Y(0)$, if

$$
\begin{equation*}
\langle X(t)\rangle=\langle Y(0)\rangle \tag{5.2.6}
\end{equation*}
$$

hence by using Eq. (5.2.5), we have

$$
\begin{equation*}
\operatorname{Tr}[\hat{\rho}(0)(\hat{X}(T)-\hat{Y}(0))]=0 \tag{5.2.7}
\end{equation*}
$$

The above result constitutes the so called joint unbiasedness condition, which implies that the experimental outputs of the detectors match on average the theoretical values of the inspected system.

### 5.3 Prove for expression given by eq. (1.4.5)

In this appendix we give an explicit prove for Eq. (1.4.6). To complete this mission, expand the symplectic invariant, Eq. (1.4.3), for $N=3$, that is

$$
\begin{equation*}
\Delta_{1,2,3}=\operatorname{Det} \boldsymbol{\sigma}_{1}+\operatorname{Det} \boldsymbol{\sigma}_{2}+\operatorname{Det} \boldsymbol{\sigma}_{3}+2\left(\text { Det } \boldsymbol{\varepsilon}_{12}+\operatorname{Det} \boldsymbol{\varepsilon}_{13}+\operatorname{Det} \boldsymbol{\varepsilon}_{23}\right) ; \tag{5.3.1}
\end{equation*}
$$

then expand the symplectic invariant, Eq. (1.4.3), for each of the reduced two-mode states of the tripartite system, that is

$$
\begin{align*}
& \Delta_{1,2}=\operatorname{Det} \boldsymbol{\sigma}_{1}+\operatorname{Det} \boldsymbol{\sigma}_{2}+2 \operatorname{Det} \boldsymbol{\varepsilon}_{12},  \tag{5.3.2}\\
& \Delta_{1,3}=\operatorname{Det} \boldsymbol{\sigma}_{1}+\operatorname{Det} \boldsymbol{\sigma}_{3}+2 \operatorname{Det} \boldsymbol{\varepsilon}_{13},  \tag{5.3.3}\\
& \Delta_{2,3}=\operatorname{Det} \boldsymbol{\sigma}_{2}+\operatorname{Det} \boldsymbol{\sigma}_{3}+2 \operatorname{Det} \boldsymbol{\varepsilon}_{23}, \tag{5.3.4}
\end{align*}
$$

hence, using Eqs. (5.3.2) to (5.3.4) and (1.4.5), it easy to show that

$$
\begin{align*}
& \Delta_{1,2}-\operatorname{Det} \boldsymbol{\sigma}_{12}=\Delta_{1,2}-\text { Det } \boldsymbol{\sigma}_{3},  \tag{5.3.5}\\
& \Delta_{1,3}-\operatorname{Det} \boldsymbol{\sigma}_{13}=\Delta_{1,3}-\text { Det } \boldsymbol{\sigma}_{2},  \tag{5.3.6}\\
& \Delta_{2,3}-\operatorname{Det} \boldsymbol{\sigma}_{23}=\Delta_{2,3}-\operatorname{Det} \boldsymbol{\sigma}_{1} . \tag{5.3.7}
\end{align*}
$$

Therefore, taking the sum of the right hand side of Eqs. (5.3.5) to (5.3.7), and the definitions Eqs. (5.3.2) to (5.3.4)

$$
\begin{align*}
\left(\Delta_{12}-\operatorname{Det} \boldsymbol{\sigma}_{12}\right)+\left(\Delta_{13}-\operatorname{Det} \boldsymbol{\sigma}_{13}\right)+\left(\Delta_{23}-\operatorname{Det} \boldsymbol{\sigma}_{23}\right) & =\Delta_{1,2,3} \\
& =3 . \tag{5.3.8}
\end{align*}
$$

### 5.4 Factorization of the time evolution operator

In this appendix, we show the process for factorizing the time evolution operator, Eq. (3.2.3); then we follow the methodology exposed in [94].

We have the following time evolution operator

$$
\begin{equation*}
e^{-i \hat{H} t}=e^{-\frac{i t}{2 m_{1}} \hat{p}_{1}^{2}} e^{-\frac{i t}{2 m_{2}} \hat{p}_{2}^{2}} e^{-i t\left(\frac{\hat{p}_{3}^{2}}{2 m_{3}}+\kappa \hat{x}_{3} \hat{p}_{1}+\kappa \hat{p}_{3} \hat{p}_{2}\right)} ; \tag{5.4.1}
\end{equation*}
$$

then, it is necessary to factor the last exponential in the above equation because $\hat{x}_{3}$ do not commute with $\hat{p}_{3}$ and $\hat{p}_{3}^{2}$. Take the operators

$$
\begin{gather*}
\hat{A}=-i t\left(\frac{\hat{p}_{3}^{2}}{2 m_{3}}+\kappa \hat{p}_{3} \hat{p}_{2}\right),  \tag{5.4.2}\\
\hat{B}=-i t \kappa \hat{x}_{3} \hat{p}_{1} \tag{5.4.3}
\end{gather*}
$$

then, the following commutators are obtained through a test function $f\left(x_{3}\right)$

$$
\begin{gather*}
{[\hat{A}, \hat{B}]=i t^{2}\left(\frac{\kappa}{m_{3}} \hat{p}_{1} \hat{p}_{3}+\kappa^{2} \hat{p}_{1} \hat{p}_{2}\right)}  \tag{5.4.4}\\
{[\hat{A},[\hat{A}, \hat{B}]]=0}  \tag{5.4.5}\\
{[\hat{B},[\hat{A}, \hat{B}]]=\frac{i t^{3} \kappa^{2}}{m_{3}} \hat{p}_{1}^{2}} \tag{5.4.6}
\end{gather*}
$$

We now define an auxiliary function $F(\zeta)$ in terms of the exponential to factor

$$
\begin{equation*}
F(\zeta)=e^{\zeta(\hat{A}+\hat{B})} \tag{5.4.7}
\end{equation*}
$$

and its derivative

$$
\begin{equation*}
F^{\prime}(\zeta)=(\hat{A}+\hat{B}) F(\zeta) \tag{5.4.8}
\end{equation*}
$$

where $\zeta$ is an auxiliary parameter which we take as 1 after finish all calculations; we then set a generic factorization that allows factorizing the last exponential of Eq. (5.4.1) in the most convenient manner; that is

$$
\begin{equation*}
F(\zeta)=e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{N}_{1}} e^{f_{2}(\zeta) \hat{N}_{2}} e^{f_{3}(\zeta) \hat{N}_{3}} \tag{5.4.9}
\end{equation*}
$$

besides, its derivative is

$$
\begin{align*}
& F^{\prime}(\zeta)=f_{0}^{\prime}(\zeta) F(\zeta)+f_{1}^{\prime}(\zeta) \hat{N}_{1} F(\zeta)+f_{2}^{\prime}(\zeta) e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{N}_{1}} \hat{N}_{2} e^{f_{2}(\zeta) \hat{N}_{2}} e^{f_{3}(\zeta) \hat{N}_{3}} \\
&+f_{3}^{\prime}(\zeta) e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{N}_{1}} e^{f_{2}(\zeta) \hat{N}_{2}} \hat{N}_{3} e^{f_{3}(\zeta) \hat{N}_{3}} \tag{5.4.10}
\end{align*}
$$

In this work we consider $\hat{N}_{1}=\hat{A}, \hat{N}_{2}=\hat{B}, \hat{N}_{3}=\hat{A}$; therefore, Eq. (5.4.10) is

$$
\begin{align*}
F^{\prime}(\zeta)=f_{0}^{\prime}(\zeta) F(\zeta)+f_{1}^{\prime}(\zeta) & \hat{A} F(\zeta)+f_{2}^{\prime}(\zeta) e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{A}} e^{f_{2}(\zeta) \hat{B}} e^{f_{3}(\zeta) \hat{A}} \\
& +f_{3}^{\prime}(\zeta) e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{A}} e^{f_{2}(\zeta) \hat{B}} \hat{A} e^{f_{3}(\zeta) \hat{A}} \tag{5.4.11}
\end{align*}
$$

then, according to Eq. (5.4.8), it is necessary to factorize the function $F(\zeta)$ to the right in the right-hand side of the above equation, but the third and fourth terms prevent it. The third term of Eq. (5.4.11) is solved as

$$
\begin{equation*}
f_{2}^{\prime}(\zeta) e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{A}} \hat{B} e^{f_{2}(\zeta) \hat{B}} e^{f_{3}(\zeta) \hat{A}}=f_{2}^{\prime}(\zeta)\left(\hat{B}+f_{1}(\zeta)[\hat{A}, \hat{B}]\right) F(\zeta) \tag{5.4.12}
\end{equation*}
$$

where we have used the fact that for self-adjoint operators $\hat{X}$ and $\hat{Y}$ it is satisfied that $e^{f_{i}(\zeta) \hat{X}} \hat{Y}=\left(e^{f_{i}(\zeta) \hat{X}} \hat{Y} e^{-f_{i}(\zeta) \hat{X}}\right) e^{f_{i}(\zeta) \hat{X}}$, together the commutators, Eqs. (5.4.4) and (5.4.5), and the condition [175]:

$$
\begin{equation*}
\left(e^{f_{i}(\zeta) \hat{X}} \hat{Y} e^{-f_{i}(\zeta) \hat{X}}\right)=\hat{Y}+f_{i}[\hat{X}, \hat{Y}]+\frac{f_{i}^{2}}{2!}[\hat{X},[\hat{X}, \hat{Y}]]+\cdots, \tag{5.4.13}
\end{equation*}
$$

following the same procedure and using the commutators Eqs. (5.4.4) and (5.4.6), the fourth therm of Eq. (5.4.11) is given by

$$
\begin{array}{r}
f_{3}^{\prime}(\zeta) e^{f_{0}(\zeta)} e^{f_{1}(\zeta) \hat{A}} e^{f_{2}(\zeta) \hat{B}} \hat{A} e^{f_{3}(\zeta) \hat{A}} \\
=f_{3}^{\prime}(\zeta)\left(\hat{A}-f_{2}(\zeta)[\hat{A}, \hat{B}]-\frac{\left(f_{2}(\zeta)\right)^{2}}{2}[\hat{B},[\hat{A}, \hat{B}]]\right) F(\zeta) ; \tag{5.4.14}
\end{array}
$$

therefore the derivative, Eq. (5.4.11), is expressed as

$$
\begin{array}{r}
F^{\prime}(\zeta)=\left\{f_{0}^{\prime}(\zeta)+f_{1}^{\prime}(\zeta) \hat{A}+f_{2}^{\prime}(\zeta)\left(\hat{B}+f_{1}(\zeta)[\hat{A}, \hat{B}]\right)\right. \\
\left.+f_{3}^{\prime}(\zeta)\left(\hat{A}-f_{2}(\zeta)[\hat{A}, \hat{B}]-\left(f_{2}^{2}(\zeta) / 2\right)[\hat{B},[\hat{A}, \hat{B}]]\right)\right\} F(\zeta) \tag{5.4.15}
\end{array}
$$

equalling it with Eq. (5.4.8), we obtain the following set of differential equations

$$
\begin{gather*}
f_{1}^{\prime}(\zeta)+f_{3}^{\prime}(\zeta)=1  \tag{5.4.16}\\
f_{2}^{\prime}(\zeta)=1  \tag{5.4.17}\\
f_{0}^{\prime}(\zeta)-\frac{f_{3}^{\prime}(\zeta) f_{2}^{2}(\zeta)}{2}[\hat{B},[\hat{A}, \hat{B}]]=0  \tag{5.4.18}\\
f_{2}^{\prime}(\zeta) f_{1}(\zeta)-f_{3}^{\prime}(\zeta) f_{2}(\zeta)=0 \tag{5.4.19}
\end{gather*}
$$

which are subject to the conditions

$$
\begin{equation*}
f_{0}(0)=f_{1}(0)=f_{2}(0)=f_{3}(0)=0 \tag{5.4.20}
\end{equation*}
$$

the solutions are given by

$$
\begin{gather*}
f_{0}(\zeta)=\frac{[\hat{B},[\hat{A}, \hat{B}]]}{12} \zeta^{3},  \tag{5.4.21}\\
f_{1}(\zeta)=\frac{\zeta}{2}  \tag{5.4.22}\\
f_{2}(\zeta)=\zeta \tag{5.4.23}
\end{gather*}
$$

$$
\begin{equation*}
f_{3}(\zeta)=\frac{\zeta}{2} \tag{5.4.24}
\end{equation*}
$$

Substituting Eqs. (5.4.21) to (5.4.24) in Eq. (5.4.9) with $\hat{N}_{1}=\hat{A}$, $\hat{N}_{2}=\hat{B}, \hat{N}_{3}=\hat{A}$ and doing $\zeta=1$, the time evolution operator, Eq. (5.4.1), is expressed as

$$
\begin{equation*}
e^{-i \hat{H} t}=e^{\Delta x_{1} \hat{p}_{1}^{2}} e^{-\frac{i t}{2 m_{2}} \hat{p}_{2}^{2}} e^{-\frac{i t}{4 m_{3}} \hat{p}_{3}^{2}} e^{-\frac{i t \kappa}{2} \hat{p}_{3} \hat{p}_{2}} e^{-i t \kappa \hat{x}_{3} \hat{p}_{1}} e^{-\frac{i t}{4 m_{3}} \hat{p}_{3}^{2}} e^{-\frac{i t \kappa}{2} \hat{p}_{3} \hat{p}_{2}} \tag{5.4.25}
\end{equation*}
$$

where we have grouped the operators with power $\hat{p}_{1}^{2}$ into one, and we have taken

$$
\begin{equation*}
\Delta x_{1}=-\left(i t / 2 m_{1}\right)+\left(i t^{3} \kappa^{2} / 12 m_{3}\right) \tag{5.4.26}
\end{equation*}
$$

### 5.5 TIME-DEPENDENT COEFFICIENTS OF THE WAVE FUNCTION DESCRIBING THE FULL DYNAMICS OF THE SIMULTANEOUS MEASUREMENT PROCESS

In this appendix, we define the form of the time-dependent functions $\mathcal{N}(t)$ and $\varepsilon_{j}(t)$ appearing in the wave function, Eq. (3.2.9), which dictates the temporal evolution through the completely dynamic measurement process of the measurement configuration.

The $\varepsilon_{j}(t)$ functions are defined as

$$
\begin{equation*}
\varepsilon_{j}(t)=\frac{\Gamma_{j}(t)}{\Theta(t)}, \quad j=1,2,3,4,5,6 \tag{5.5.1}
\end{equation*}
$$

with

$$
\begin{gather*}
\Gamma_{1}(t)=3 m_{1} m_{3}\left(t\left(-2 b t+m_{2}\left(i+4 b \kappa^{2} m_{3} t\right)\right)+4 m_{3}\left(m_{2}+2 i b t\right) \delta_{q}^{2}\right),  \tag{5.5.2}\\
\Gamma_{2}(t)=b m_{2}\left(3 b m_{1} m_{3}+i\left(-6 m_{3} t+\kappa^{2} m_{1} t^{3}\right)\right)\left(i t+4 m_{3} \delta_{q}^{2}\right),  \tag{5.5.3}\\
\Gamma_{3}(t)=\quad m_{3}\left(6 i b^{2} m_{1} m_{3} t+2 m_{2} t\left(2 i \kappa^{2} m_{1} t^{2}+m_{3}\left(-3 i+6 \kappa^{2} m_{1} t \delta_{q}^{2}\right)\right)\right. \\
 \tag{5.5.4}\\
\left.+b\left(12 m_{3} t^{2}+m_{1}\left(3 m_{2}\left(m_{3}+\kappa^{4} m_{3} t^{4}\right)-8 \kappa^{2} t^{3}\left(t-3 i m_{3} \delta_{q}^{2}\right)\right)\right)\right),
\end{gather*}
$$

$$
\begin{gather*}
\Gamma_{4}(t)=6 b \kappa^{2} m_{1} m_{2} m_{3} t^{2}\left(-t+4 i m_{3} \delta_{q}^{2}\right) \\
\Gamma_{5}(t)=-6 \kappa m_{1} m_{3} t\left(t\left(i m_{2}+2 b\left(-1+\kappa^{2} m_{2} m_{3}\right) t\right)+4 m_{3}\left(m_{2}+2 i b t\right) \delta_{q}^{2}\right)
\end{gather*}
$$

$\Gamma_{6}(t)=2 b \kappa m_{2} m_{3} t\left(-6 i b m_{1} m_{3}+t\left(5 k^{2} m_{1} t^{2}+m_{3}\left(-12-12 i \kappa^{2} m_{1} t \delta_{q}^{2}\right)\right)\right)$,
and

$$
\begin{align*}
\Theta(t)= & i b t\left(3 m_{1} m_{2} m_{3}+12 m_{3}\left(1-2 \kappa^{2} m_{2} m_{3}\right) t^{2}+\kappa^{2} m_{1}\left(-2+7 \kappa^{2} m_{2} m_{3}\right) t^{4}\right) \\
& +4 b m_{3}\left(3 m_{1} m_{2} m_{3}+12 m_{3} t^{2}+\kappa^{2} m_{1}\left(-2+3 \kappa^{2} m_{2} m_{3}\right) t^{4}\right) \delta_{q}^{2} \\
& +m_{2} t\left(6 m_{3}-\kappa^{2} m_{1} t^{2}\right)\left(t-4 i m_{3} \delta_{q}^{2}\right) \\
& +6 b^{2} m_{1} m_{3} t\left(\left(-1+2 \kappa^{2} m_{2} m_{3}\right) t+4 i m_{3} \delta_{q}^{2}\right) \tag{5.5.8}
\end{align*}
$$

While the $\mathcal{N}(t)$-function is given by

$$
\begin{equation*}
\mathcal{N}(t)=2\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\frac{3 m_{1} m_{2} m_{3}^{2}}{\Theta(t)}\right)^{\frac{1}{2}} \tag{5.5.9}
\end{equation*}
$$

which plays the role of a normalization constant.

### 5.6 APPLICATION OF THE FULLY DYNAMICAL TIME EVOLUTION OPERATOR TO THE INITIAL WAVE FUNCTION

Because the initial wave function of the system is in the canonical position representation, it is convenient to express the time evolution operator, Eq. (3.2.4), also in the same basis, this is done by simply taking the substitution $\hat{p}_{i^{\prime}} \rightarrow-i \partial_{x_{i^{\prime}}}$ for $i^{\prime}=1,2,3$. Thus, the factorization given by Eq. (3.2.4) can be expressed as

$$
\begin{equation*}
e^{-i \hat{H} t}=e^{-\Delta x_{1} \partial_{x_{1}}^{2}} e^{\delta_{2}(t) \partial_{x_{2}}^{2}} e^{\frac{\delta_{3}(t)}{2} \partial_{x_{3}}^{2}} e^{\delta_{\kappa}(t) \partial_{x_{3}} \partial_{x_{2}}} e^{\delta_{k}^{\prime}(t) \hat{x}_{3} \partial_{x_{1}}} e^{\frac{\delta_{3}(t)}{2} \partial_{x_{3}}^{2}} e^{\delta_{\kappa}(t) \partial_{x_{3}} \partial_{x_{2}}} \tag{5.6.1}
\end{equation*}
$$

where we have done

$$
\begin{equation*}
\delta_{i^{\prime}}(t)=\frac{i t}{2 m_{i^{\prime}}}, \quad i^{\prime}=1,2,3 \tag{5.6.2}
\end{equation*}
$$

And

$$
\begin{gather*}
\delta_{\kappa}(t)=i t \kappa / 2  \tag{5.6.3}\\
\delta_{\kappa}^{\prime}(t)=-t \kappa \tag{5.6.4}
\end{gather*}
$$

We will refer to the exponentials of Eq. (5.6.1) from right to left, labeling them from the first to seventh respectively; to apply them, we use the one and two dimensional Fourier transform (FT)

$$
\begin{gather*}
\mathcal{F}[f(q)](p)=g(p)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} f(q) e^{-i q p} d q  \tag{5.6.5}\\
\mathcal{F}_{2 \mathbb{D}}[m(x, y)](u, v)=n(u, v)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} m(x, y) e^{-i(x u+y v)} d x d y \tag{5.6.6}
\end{gather*}
$$

and the one and two dimensional inverse Fourier transform (IFT)

$$
\begin{gather*}
\mathcal{F}^{-1}[g(p)](q)=f(q)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} g(p) e^{i q p} d p  \tag{5.6.7}\\
\mathcal{F}_{2 \mathbb{D}}^{-1}[n(u, v)](x, y)=m(x, y)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} n(u, v) e^{i(x u+y v)} d u d v \tag{5.6.8}
\end{gather*}
$$

besides, we use the expansion in McLaurin series of an exponential operator when we apply it to a wave function, that is,

$$
\begin{equation*}
e^{\hat{A}} \Psi=\sum_{n=0}^{\infty} \frac{1}{n!} \hat{A}^{n} \Psi . \tag{5.6.9}
\end{equation*}
$$

Applying the first operator of Eq. (5.6.1) to the initial function, we have

$$
\begin{equation*}
\psi_{1}=e^{\delta_{\kappa}(t) \partial_{x_{3}} \partial_{x_{2}}} \phi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) \phi_{3}\left(x_{3}\right)=\phi_{1}\left(x_{1}\right) \sum_{n=0}^{\infty} \frac{\left(\delta_{\kappa}(t)\right)^{n}}{n!} \frac{d^{n} \phi_{3}\left(x_{3}\right)}{d x_{3}^{n}} \frac{d^{n} \phi_{2}\left(x_{2}\right)}{d x_{2}^{n}}, \tag{5.6.10}
\end{equation*}
$$

Applying two dimensional FT in $x_{2}$ and $x_{3}$ variables to Eq. (5.6.10), we have

$$
\begin{align*}
\mathcal{F}_{2 \mathrm{D}} & {\left[\psi_{1}\right]\left(x_{1}, p_{2}, p_{3}, t\right) } \\
& =\phi_{1}\left(x_{1}\right) \sum_{n=0}^{\infty} \frac{\left(\delta_{k}(t)\right)^{n}}{n!} \mathcal{F}\left[\frac{d^{n} \phi_{2}\left(x_{2}\right)}{d x_{2}^{n}}\right]\left(p_{2}\right) \mathcal{F}\left[\frac{d^{n} \phi_{3}\left(x_{3}\right)}{d x_{3}^{n}}\right]\left(p_{3}\right) \\
& =\phi_{1}\left(x_{1}\right) \sum_{n=0}^{\infty} \frac{\left(\delta_{\kappa}(t)\right)^{n}}{n!}\left(i p_{2}\right)^{n}\left(i p_{3}\right)^{n} \mathcal{F}\left[\phi_{2}\left(x_{2}\right)\right]\left(p_{2}\right) \mathcal{F}\left[\phi_{3}\left(x_{3}\right)\right]\left(p_{3}\right) \\
& =\phi_{1}\left(x_{1}\right) e^{-\delta_{\kappa}(t) p_{2} p_{3}} \mathcal{F}\left[\phi_{2}\left(x_{2}\right)\right]\left(p_{2}\right) \mathcal{F}\left[\phi_{3}\left(x_{3}\right)\right]\left(p_{3}\right), \tag{5.6.11}
\end{align*}
$$

where in the second line we have used the case of separable functions (see for example [176] pp. 9-10), and in the third line the derivative theorem of the one FT

$$
\begin{equation*}
\mathcal{F}\left[\frac{d^{n} f(q)}{d q^{n}}\right](p)=(i p)^{n} \mathcal{F}[f(q)](p) \tag{5.6.12}
\end{equation*}
$$

Using the definition, Eq. (2.1.61), for $\phi_{3}\left(x_{3}\right)$ and $\phi_{2}\left(x_{2}\right)$ and taking the two-dimensional IFT, the application of the first operator is finished, the result is

$$
\begin{equation*}
\psi_{1}=\left(\frac{\delta_{q}}{(2 \pi)^{1 / 2} \sigma_{1}(t)}\right)^{\frac{1}{2}} \phi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) e^{-\frac{\left(x_{3}-2 b \delta_{\kappa}(t) x_{2}\right)^{2}}{4 \sigma_{1}(t)}}, \tag{5.6.13}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma_{1}(t)=\left(\delta_{q}^{2}-b\left(\delta_{\kappa}(t)\right)^{2}\right) . \tag{5.6.14}
\end{equation*}
$$

As can be seen, the wave function has become entangled between the position variables $x_{2}$ and $x_{3}$; therefore, the initial product structure has been lost due to the unitary dynamics.

Following a similar methodology, we apply the second operator of Eq. (5.6.1) Eq. (5.6.13)

$$
\begin{equation*}
\psi_{2}=e^{\frac{\delta_{3}(t)}{2} \partial_{x_{3}}^{2}} \psi_{1}=\psi_{1}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\frac{\delta_{3}(t)}{2}\right)^{n}}{n!} \partial_{x_{3}}^{2 n}\left\{\left(\sigma_{1}(t)\right)^{-1 / 2} e^{-\frac{\left(x_{3}-2 b_{k}(t) x_{2}\right)^{2}}{4 \sigma_{1}(t)}}\right\} \tag{5.6.15}
\end{equation*}
$$

where we have left inside the brackets the term $\left(\sigma_{1}(t)\right)^{-1 / 2}$, in order to simplify the application of the exponential operator (see for example the methodology exposed in [152]) and we take

$$
\begin{equation*}
\psi_{1}^{\prime}=\left(\frac{\delta_{q}}{(2 \pi)^{1 / 2}}\right)^{\frac{1}{2}} \phi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) . \tag{5.6.16}
\end{equation*}
$$

Applying FT in $x_{3}$ variable to Eq. (5.6.15), we have

$$
\begin{align*}
\mathcal{F}\left[\psi_{2}\right] & \left(x_{1}, x_{2}, p_{3}, t\right) \\
& =\psi_{1}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\frac{\delta_{3}(t)}{2}\right)^{n}}{n!} \mathcal{F}\left[\partial_{x_{3}}^{2 n}\left\{\left(\sigma_{1}(t)\right)^{-1 / 2} e^{-\frac{\left(x_{3}-2 \delta_{k}(t) x_{2}\right)^{2}}{4 \sigma_{1}(t)}}\right\}\right]\left(x_{1}, x_{2}, p_{3}, t\right) \\
& =\psi_{1}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\frac{\delta_{3}(t)}{2}\right)^{n}}{n!}\left(i p_{3}\right)^{2 n} \mathcal{F}\left[\left\{\left(\sigma_{1}(t)\right)^{-1 / 2} e^{-\frac{\left(x_{3}-2 b \delta_{k}(t) x_{2}\right)^{2}}{4 \sigma_{1}(t)}}\right\}\right]\left(x_{1}, x_{2}, p_{3}, t\right) \\
& =\psi_{1}^{\prime} e^{-\left(\frac{\delta_{3}(t)}{2}\right) p_{3}^{2}} \mathcal{F}\left[\left\{\left(\sigma_{1}(t)\right)^{-1 / 2} e^{-\frac{\left(x_{3}-2 \delta_{\left.\delta_{( }(t) x_{2}\right)^{2}}^{4 \sigma_{1}(t)}\right.}{}}\right\}\right]\left(x_{1}, x_{2}, p_{3}, t\right), \tag{5.6.17}
\end{align*}
$$

Taking the IFT in $p_{x_{3}}$ variable to Eq. (5.6.17), the application of the second operator is finished, the result is

$$
\begin{equation*}
\psi_{2}=\left(\frac{\delta_{q}}{(2 \pi)^{1 / 2} \sigma_{2}(t)}\right)^{\frac{1}{2}} \phi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) e^{\frac{-\left(x_{3}-2 \delta_{\kappa}(t) x_{2}\right)^{2}}{4 \sigma_{2}(t)}}, \tag{5.6.18}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma_{2}(t)=\left(\delta_{q}^{2}-b\left(\delta_{\kappa}(t)\right)^{2}+\frac{\delta_{3}(t)}{2}\right) \tag{5.6.19}
\end{equation*}
$$

then, the effect of the second operator is to increase the amplitude of the entangled wave function and to contribute to the dispersion of the Gaussian wave packet associated with the entangled variables $x_{2}$ and $x_{3}$. In general, the operators of the type $e^{C \partial_{x_{i}}^{2}}$ have this dispersive effect when they act on Gaussian wave packets; see [152].

We now apply the third operator

$$
\begin{equation*}
\psi_{3}=e^{\delta_{k}^{\prime}(t) \hat{x}_{3} \partial_{x_{1}}} \psi_{2}=\sum_{n=0}^{\infty} \frac{\left(\delta_{k}^{\prime}(t)\right)^{n}}{n!} \hat{x}_{3}^{n} \partial_{x_{1}}^{n} \psi_{2}=\psi_{2}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{k}^{\prime}(t)\right)^{n}}{n!} x_{3}^{n}\left\{\partial_{x_{1}}^{n} \phi_{1}\left(x_{1}\right)\right\}, \tag{5.6.20}
\end{equation*}
$$

where we have done

$$
\begin{equation*}
\psi_{2}^{\prime}=\left(\frac{\delta_{q}}{(2 \pi)^{1 / 2} \sigma_{2}(t)}\right)^{\frac{1}{2}} \phi_{2}\left(x_{2}\right) e^{\frac{-\left(x_{3}-2 b \delta_{K}(t) x_{2}\right)^{2}}{4 \sigma_{2}(t)}} . \tag{5.6.21}
\end{equation*}
$$

Taking the FT of Eq. (5.6.20) in $x_{1}$ variable

$$
\begin{align*}
\mathcal{F}\left[\psi_{3}\right] & \left(p_{1}, x_{2}, x_{3}, t\right) \\
& =\psi_{2}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{k}^{\prime}(t) x_{3}\right)^{n}}{n!} \mathcal{F}\left[\partial_{x_{1}}^{n} \phi_{1}\left(x_{1}\right)\right]\left(p_{1}, x_{2}, x_{3}, t\right) \\
& =\psi_{2}^{\prime} \sum_{n=0}^{\infty} \frac{\left(i \delta_{k}^{\prime}(t) x_{3} p_{1}\right)^{n}}{n!} \mathcal{F}\left[\phi_{1}\left(x_{1}\right)\right]\left(p_{1}, x_{2}, x_{3}, t\right) \\
& =\psi_{2}^{\prime} e^{i \delta_{k}^{\prime}(t) x_{3} p_{1}} \mathcal{F}\left[\phi_{1}\left(x_{1}\right)\right]\left(p_{1}, x_{2}, x_{3}, t\right), \tag{5.6.22}
\end{align*}
$$

Taking the IFT on Eq. (5.6.22) in $p_{1}$ variable, the application of the third operator is finished; thus we have

$$
\begin{equation*}
\psi_{3}=\left(\frac{\delta_{q}}{(2 \pi)^{1 / 2} \sigma_{2}(t)}\right)^{\frac{1}{2}} \phi_{1}\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right) \phi_{2}\left(x_{2}\right) e^{\frac{-\left(x_{3}-2 b \delta_{k}(t) x_{2}\right)^{2}}{4 \sigma_{2}(t)}} \tag{5.6.23}
\end{equation*}
$$

The effect of the third operator is proportionally to displace the $x_{1}$ variable by the $x_{3}$ variable or equivalently to entangled them; thus, this operator acts as a displacement operator.

It must be noted that all operators in Eq. (5.6.1) that are still to apply commute, hence the order of application of they does not matter from this point. Following this idea, we apply the seventh operator of Eq. (5.6.1) to Eq. (5.6.23)

$$
\begin{equation*}
\psi_{7}=e^{-\Delta x_{1}(t) \partial_{x_{1}}^{2}} \psi_{3}=\psi_{3}^{\prime} \sum_{n=0}^{\infty} \frac{\left(-\Delta x_{1}(t)\right)^{n}}{n!} \partial_{x_{1}}^{2 n} \phi_{1}\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right) \tag{5.6.24}
\end{equation*}
$$

where we have taken

$$
\begin{equation*}
\psi_{3}^{\prime}=\left(\frac{\delta_{q}}{(2 \pi)^{1 / 2} \sigma_{2}(t)}\right)^{\frac{1}{2}} \phi_{2}\left(x_{2}\right) e^{\left.\frac{-\left(x_{3}-2 b \delta_{\delta_{2}(t)}\right.}{4 \sigma_{2}(t)}\right)^{2}} . \tag{5.6.25}
\end{equation*}
$$

Taking the FT of Eq. (5.6.24) in $x_{1}$ variable, we have

$$
\begin{align*}
\mathcal{F}\left[\psi_{7}\right] & \left(p_{1}, x_{2}, x_{3}, t\right) \\
& =\psi_{3}^{\prime} \sum_{n=0}^{\infty} \frac{\left(-\Delta x_{1}(t)\right)^{n}}{n!} \mathcal{F}\left[\partial_{x_{1}}^{2 n} \phi_{1}\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)\right]\left(p_{1}, x_{2}, x_{3}, t\right) \\
& =\psi_{3}^{\prime} \sum_{n=0}^{\infty} \frac{\left(-\Delta x_{1}(t)\right)^{n}\left(i p_{1}\right)^{2 n}}{n!} \mathcal{F}\left[\phi_{1}\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)\right]\left(p_{1}, x_{2}, x_{3}, t\right) \\
& =\psi_{3}^{\prime} \Delta^{\Delta x_{1}(t) p_{1}^{2}} \mathcal{F}\left[\phi_{1}\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)\right]\left(p_{1}, x_{2}, x_{3}, t\right) \tag{5.6.26}
\end{align*}
$$

Taking the definition for $\phi_{1}\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)$ and the IFT in $p_{1}$ variable; we finish the application of the seventh operator, the result is

$$
\begin{equation*}
\psi_{7}=\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{2}(t) \sigma_{3}(t)\right)^{-\frac{1}{2}} e^{-b x_{2}^{2}} e^{-\frac{\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)^{2}}{\sigma_{3}(t)}} e^{\frac{-\left(x_{3}-2 b \delta_{k}(t) x_{2}\right)^{2}}{4 \sigma_{2}(t)}} \tag{5.6.27}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma_{3}(t)=b+4 \Delta x_{1}(t) \tag{5.6.28}
\end{equation*}
$$

The seventh operator contributes to the amplitude and the dispersion of the wave function associated with $x_{1}$ and $x_{3}$ variables. Before applying the next operator, we complete the square binomial in the power of exponentials for the $x_{2}$ variable; thus, the wave function, Eq. (5.6.27), can be rewritten as

$$
\begin{equation*}
\psi_{7}=\left(\frac{2 b^{2} \sigma^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{2}(t) \sigma_{3}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\beta(t) x_{2}-\alpha(t) x_{3}\right)^{2}}{4 \sigma_{2}(t)}} e^{-\frac{\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)^{2}}{\sigma_{3}(t)}} e^{-\frac{x_{3}^{2}\left(1-(\alpha(t))^{2}\right)}{4 \sigma_{2}(t)}} \tag{5.6.29}
\end{equation*}
$$

where we have done

$$
\begin{gather*}
\beta(t)=\left[\left(2 b \delta_{\kappa}(t)\right)^{2}+4 \sigma_{2}(t) b\right]^{\frac{1}{2}}  \tag{5.6.30}\\
\alpha(t)=\frac{2 b \delta_{\kappa}(t)}{\beta(t)} \tag{5.6.31}
\end{gather*}
$$

We now apply the sixth operator of Eq. (5.6.1) to Eq. (5.6.29)

$$
\begin{equation*}
\psi_{6}=e^{\delta_{2}(t) \partial_{x_{2}}^{2}} \psi_{7}=\psi_{7}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{2}(t)\right)^{n}}{n!} \partial_{x_{2}}^{2 n}\left\{\left(\sigma_{2}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\beta(t) x_{2}-\alpha(t) x_{3}\right)^{2}}{4 \sigma_{2}(t)}}\right\} \tag{5.6.32}
\end{equation*}
$$

with

$$
\begin{equation*}
\psi_{7}^{\prime}=\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{3}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)^{2}}{\sigma_{3}(t)}} e^{-\frac{x_{3}^{2}\left(1-\alpha^{2}(t)\right)}{4 \sigma_{2}(t)}} \tag{5.6.33}
\end{equation*}
$$

taking the FT on Eq. (5.6.32) in $x_{2}$ variable, we have

$$
\begin{align*}
\mathcal{F}\left[\psi_{6}\right] & \left(x_{1}, p_{2}, x_{3}, t\right) \\
& =\psi_{7}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{2}(t)\right)^{n}}{n!} \mathcal{F}\left[\partial_{x_{2}}^{2 n}\left\{\left(\sigma_{2}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\beta(t) x_{2}-\alpha(t) x_{3}\right)^{2}}{4 \sigma_{2}(t)}}\right\}\right]\left(x_{1}, p_{2}, x_{3}, t\right) \\
& =\psi_{7}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{2}(t)\right)^{n}}{n!}\left(i p_{2}\right)^{2 n} \mathcal{F}\left[\left\{\left(\sigma_{2}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\beta(t) x_{2}-\alpha(t) x_{3}\right)^{2}}{4 \sigma_{2}(t)}}\right\}\right]\left(x_{1}, p_{2}, x_{3}, t\right) \\
& =\psi_{7}^{\prime} e^{-\delta_{2}(t) p_{2}^{2}} \mathcal{F}\left[\left\{\left(\sigma_{2}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\beta(t) x_{2}-\alpha(t) x_{3}\right)^{2}}{4 \sigma_{2}(t)}}\right\}\right]\left(x_{1}, p_{2}, x_{3}, t\right), \tag{5.6.34}
\end{align*}
$$

Taking the IFT on Eq. (5.6.34) in $p_{2}$ variable, we finish the application of the sixth operator; thus we have

$$
\begin{equation*}
\psi_{6}=\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{3}(t) \sigma_{4}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\beta(t) x_{2}-\alpha(t) x_{3}\right)^{2}}{4 \sigma_{4}(t)}} e^{-\frac{\left(x_{1}+\delta_{k}^{\prime}(t) x_{3}\right)^{2}}{\sigma_{3}(t)}} e^{-\frac{x_{3}^{2}\left(1-\alpha^{2}(t)\right)}{4 \sigma_{2}(t)}}, \tag{5.6.35}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma_{4}(t)=\sigma_{2}(t)+(\beta(t))^{2} \delta_{2}(t) \tag{5.6.36}
\end{equation*}
$$

The sixth operator also contributes to the amplitude of the whole wave function and the dispersion of the Gaussian wave packet associate to the $x_{2}$ and $x_{3}$ variables. Before applying the next operator, we complete the square binomial in $x_{3}$ variable in the power of exponentials of Eq. (5.6.35); thus, the wave function, Eq. (5.6.35), is rewrite as

$$
\begin{equation*}
\psi_{6}=\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{3}(t) \sigma_{4}(t)\right)^{-\frac{1}{2}} e^{-\left(\lambda(t) x_{3}+\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}} e^{\left.\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}} e^{-\xi_{2}\left(x_{1}, x_{2}, t\right)}, \tag{5.6.37}
\end{equation*}
$$

with

$$
\begin{gather*}
\lambda(t)=\left(\frac{1-\alpha^{2}(t)}{4 \sigma_{2}(t)}+\frac{\left(\delta_{k}^{\prime}(t)\right)^{2}}{\sigma_{3}(t)}+\frac{(\alpha(t))^{2}}{4 \sigma_{4}(t)}\right)^{\frac{1}{2}}  \tag{5.6.38}\\
\xi_{1}\left(x_{1}, x_{2}, t\right)=\left(\frac{2 \delta_{k}^{\prime}(t)}{\sigma_{3}(t)} x_{1}-\frac{\alpha(t) \beta(t)}{2 \sigma_{4}(t)} x_{2}\right),  \tag{5.6.39}\\
\xi_{2}\left(x_{1}, x_{2}, t\right)=\left(\frac{1}{\sigma_{3}(t)} x_{1}^{2}+\frac{(\beta(t))^{2}}{4 \sigma_{4}(t)} x_{2}^{2}\right) . \tag{5.6.40}
\end{gather*}
$$

Hence we apply the fifth operator of Eq. (5.6.1) to Eq. (5.6.37)

$$
\begin{equation*}
\psi_{5}=\psi_{6}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{3}(t) / 2\right)^{n}}{n!} \partial_{x_{3}}^{2 n} e^{-\left(\lambda(t) x_{3}+\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}} \tag{5.6.41}
\end{equation*}
$$

where we have taken

$$
\begin{equation*}
\psi_{6}^{\prime}=\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{3}(t) \sigma_{4}(t)\right)^{-\frac{1}{2}} e^{\left(\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}} e^{-\xi_{2}\left(x_{1}, x_{2}, t\right)} \tag{5.6.42}
\end{equation*}
$$

taking the FT, Eq. (5.6.5), on Eq. (5.6.41) in $x_{3}$ variable, we have

$$
\begin{align*}
\mathcal{F} & {\left[\psi_{5}\right]\left(x_{1}, x_{2}, p_{x_{3}}, t\right) } \\
& =\psi_{5}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{3}(t) / 2\right)^{n}}{n!} \mathcal{F}\left[\partial_{x_{3}}^{2 n} e^{-\left(\lambda(t) x_{3}+\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}}\right]\left(x_{1}, x_{2}, p_{3}, t\right) \\
& =\psi_{5}^{\prime} \sum_{n=0}^{\infty} \frac{\left(\delta_{3}(t) / 2\right)^{n}\left(i p_{3}\right)^{2 n}}{n!} \mathcal{F}\left[e^{-\left(\lambda(t) x_{3}+\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}}\right]\left(x_{1}, x_{2}, p_{x_{3}}, t\right) \\
& =\psi_{5}^{\prime} e^{-\left(\frac{\delta_{3}(t)}{2}\right) p_{3}^{2}} \mathcal{F}\left[e^{-\left(\lambda(t) x_{3}+\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}}\right]\left(x_{1}, x_{2}, p_{x_{3}}, t\right) \tag{5.6.43}
\end{align*}
$$

Taking the IFT on Eq. (5.6.43) in $p_{3}$ variable, we finish the application of the fifth operator, the result is

$$
\begin{gather*}
\psi_{5}=\left(\frac{2 b^{2} \delta_{q}^{2}}{\pi^{3}}\right)^{\frac{1}{4}}\left(\sigma_{3}(t) \sigma_{4}(t) \sigma_{5}(t)\right)^{-\frac{1}{2}} e^{-\frac{\left(\lambda(t) x_{3}+\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}}{\sigma_{5}(t)}} e^{\left(\frac{\xi_{1}\left(x_{1}, x_{2}, t\right)}{2 \lambda(t)}\right)^{2}} \\
\times e^{-\xi_{2}\left(x_{1}, x_{2}, t\right)} \tag{5.6.44}
\end{gather*}
$$

with

$$
\begin{equation*}
\sigma_{5}(t)=1+2(\lambda(t))^{2} \delta_{3}(t) \tag{5.6.45}
\end{equation*}
$$

then, the fifth operator contributes to the amplitude, and because of the entanglement of all position variables, it also contributes to the temporal dispersion of the whole Gaussian wave function.

Finally, to obtain the final wave function, we apply the fourth operator of Eq. (5.6.1) to Eq. (5.6.44)

$$
\begin{equation*}
\Psi\left(x_{1}, x_{2}, x_{3}, t\right)=e^{\delta_{\kappa}(t) \partial_{x_{3}} \partial_{x_{2}}} \psi_{5}=\sum_{n=0}^{\infty} \frac{\left(\delta_{\kappa}(t)\right)^{n}}{n!} \partial_{x_{3}}^{n} \partial_{x_{2}}^{n} \psi_{5} \tag{5.6.46}
\end{equation*}
$$

Applying the two dimensional FT to Eq. (5.6.46) in $x_{2}$ and $x_{3}$ variables, we have

$$
\begin{align*}
\mathcal{F}_{2 \mathbb{D}}\left[e^{\delta_{\kappa} \partial_{x_{3}} \partial_{x_{2}}} \psi_{5}\right] & \left(x_{1}, p_{2}, p_{3}, t\right) \\
& =\sum_{n=0}^{\infty} \frac{\left(\delta_{\kappa}(t)\right)^{n}}{n!} \mathcal{F}_{2 \mathbb{D}}\left[\partial_{x_{3}}^{n} \partial_{x_{2}}^{n} \psi_{5}\right]\left(x_{1}, p_{2}, p_{3}, t\right) \\
& =\sum_{n=0}^{\infty} \frac{\left(\delta_{\kappa}(t)\right)^{n}}{n!}\left(i p_{2}\right)^{n}\left(i p_{3}\right)^{n} \mathcal{F}_{2 \mathbb{D}}\left[\psi_{5}\right]\left(x_{1}, p_{2}, p_{3}, t\right) \\
& =e^{-\delta_{\kappa}(t) p_{2} p_{3}} \mathcal{F}_{2 \mathbb{D}}\left[\psi_{5}\right]\left(x_{1}, p_{2}, p_{3}, t\right) \tag{5.6.47}
\end{align*}
$$

where in the second line we use the derivative theorem of the two dimensional FT

$$
\begin{equation*}
\mathcal{F}_{2 \mathbb{D}}\left[\partial_{x}^{n} \partial_{y}^{n} f(x, y)\right](u, v)=(i u)^{n}(i v)^{n} \mathcal{F}_{2 \mathbb{D}}[f(x, y)](u, v) . \tag{5.6.48}
\end{equation*}
$$

Taking the two dimensional IFT of Eq. (5.6.47) in $p_{2}$ and $p_{3}$ variables, we finish the application of the fourth operator, we present the final result as

$$
\begin{align*}
\Psi(\hat{\chi}, t)=\mathcal{N}(t) \exp [- & \left\{\varepsilon_{1}(t) x_{1}^{2}+\varepsilon_{2}(t) x_{2}^{2}+\varepsilon_{3}(t) x_{3}^{2}+\varepsilon_{4}(t) x_{1} x_{2}\right. \\
& \left.\left.+\varepsilon_{5}(t) x_{1} x_{3}+\varepsilon_{6}(t) x_{2} x_{3}\right\}\right], \tag{5.6.49}
\end{align*}
$$

The wave function, Eq. (5.6.49), is clearly a Gaussian wave function with temporal dispersion, that is entangled between the three position variables $x_{1}, x_{2}$ and $x_{3}$.

### 5.7 Definition of the block matrices for the covariance matrices given in eqs. (3.5.4) to (3.5.6)

In this appendix, we define the $(2 \times 2)$-dimensional block matrices composing the covariance matrices given in Eqs. (3.5.4) to (3.5.6); they are

$$
\begin{gather*}
\boldsymbol{\sigma}_{1}=\left(\begin{array}{cc}
4 \delta_{\hat{q}}^{2}+\alpha_{11} & \alpha_{12} \\
\alpha_{12} & 1 / \delta_{\hat{q}}^{2}
\end{array}\right)  \tag{5.7.1}\\
\boldsymbol{\sigma}_{2}=\left(\begin{array}{cc}
\left(1 / \delta_{\hat{q}}^{2}\right)+\alpha_{33} & \alpha_{34} \\
\alpha_{34} & 4 \delta_{\hat{q}}^{2}
\end{array}\right)  \tag{5.7.2}\\
\boldsymbol{\sigma}_{3}=\left(\begin{array}{cc}
6 \delta_{\hat{q}}^{2}+\alpha_{55} & \alpha_{56} \\
\alpha_{56} & \frac{3}{2 \delta_{q}^{2}}
\end{array}\right) \tag{5.7.3}
\end{gather*}
$$

$$
\begin{gather*}
\boldsymbol{\lambda}_{1,2}=\left(\begin{array}{cc}
\alpha_{13} & 2 \delta_{\hat{q}}^{2} \\
-\frac{1}{2 \delta_{\hat{q}}^{2}} & 0
\end{array}\right)  \tag{5.7.4}\\
\boldsymbol{\lambda}_{1,3}=\left(\begin{array}{cc}
4 \delta_{\hat{q}}^{2}+\alpha_{15} & \alpha_{16} \\
0 & -1 / \delta_{\hat{q}}^{2}
\end{array}\right)  \tag{5.7.5}\\
\boldsymbol{\lambda}_{2,3}=\left(\begin{array}{cc}
\alpha_{35} & \frac{1}{\delta_{\hat{q}}^{2}} \\
4 \delta_{\hat{q}}^{2} & 0
\end{array}\right) \tag{5.7.6}
\end{gather*}
$$

where the functions $\alpha_{i j}$ (with $\alpha_{i j}=\alpha_{j i}$ ) appearing in Eqs. (5.7.1) to (5.7.6) represent the contribution of the spreading in the global system caused by the free energies appearing in the Hamiltonian, Eq. (3.2.1), of the measurement process; these functions are given by

$$
\begin{array}{r}
\alpha_{11}=\frac{11 m_{1}^{2}-24 m_{1} m_{3}+72 m_{3}^{2}}{72 \kappa^{2} m_{1}^{2} m_{3}^{2} \delta_{q}^{2}}, \quad \alpha_{12}=\frac{m_{1}-6 m_{3}}{6 \kappa m_{1} m_{3} \delta_{q}^{2}}, \quad \alpha_{13}=\frac{\frac{3}{m_{1}}+\frac{1}{m_{3}}+\frac{12 \delta_{q}^{4}}{m_{2}}}{6 \kappa \delta_{q}^{2}} \\
\alpha_{15}=\frac{1}{8 \kappa^{2} m_{3}^{2} \delta_{q}^{2}}, \quad \alpha_{16}=\frac{m_{1}+12 m_{3}}{12 \kappa m_{1} m_{3} \delta_{q}^{2}}, \quad \alpha_{33}=\frac{4 \delta_{q}^{2}}{\kappa^{2} m_{2}^{2}}, \quad \alpha_{34}=\kappa \alpha_{33}(5.7 .7) \\
\alpha_{35}=\frac{\frac{1}{m_{3}}+\frac{16 \delta_{q}^{4}}{m_{2}}}{4 \kappa \delta_{q}^{2}}, \quad \alpha_{55}=\alpha_{15}, \quad \alpha_{56}=\frac{1}{4 \kappa m_{3} \delta_{q}^{2}}
\end{array}
$$

it is straightforward to verify that the functions given by Eqs. (5.7.7) are vanishing in the strong coupling regime, that is, when

$$
\begin{equation*}
\lim _{k \longrightarrow \infty} \alpha_{i j}=0 \tag{5.7.8}
\end{equation*}
$$

therefore, recovering the situation of an instantaneous measurement interaction raised by Arthurs and Kelly for our particular measurement configuration.

### 5.8 BLOCK COMPONENTS OF THE COVARIANCES MATRICES OF EQS. (2.4.19) TO (2.4.21)

In this appendix, we define the $2 \times 2$ dimensional block matrices composing the CMs, Eqs. (2.4.19) to (2.4.21). All terms appearing depend on the parameters $r$ and $\theta$; in the subsequent, we omit this dependence for brevity. Then, we have $\left(\alpha_{1}=\alpha_{2}=1\right)$

$$
\boldsymbol{\sigma}_{1}=\left[\begin{array}{cc}
\delta_{\hat{q}_{1}}^{2}+\left(\delta_{\hat{p}_{2}} / 2\right)^{2}+\delta_{\hat{q}_{3}}^{2} & 0  \tag{5.8.1}\\
0 & \delta_{\hat{p}_{1}}^{2}
\end{array}\right]
$$

$$
\begin{gather*}
\boldsymbol{\sigma}_{2}=\left[\begin{array}{cc}
\left(\delta_{\hat{p}_{1}} / 2\right)^{2}+\delta_{\hat{q}_{2}}^{2}+\delta_{\hat{p}_{3}}^{2} & 0 \\
0 & \delta_{\hat{p}_{2}}^{2}
\end{array}\right]  \tag{5.8.2}\\
\boldsymbol{\sigma}_{3}=\left[\begin{array}{cc}
\delta_{\hat{p}_{2}}^{2}+\delta_{\hat{q}_{3}}^{2} & \operatorname{Cov} \\
\operatorname{Cov} & \delta_{\hat{p}_{1}}^{2}+\delta_{\hat{p}_{3}}^{2}
\end{array}\right]  \tag{5.8.3}\\
\boldsymbol{\varepsilon}_{1,2}=\left[\begin{array}{cc}
\operatorname{Cov} & \frac{\delta_{\hat{p}_{2}}^{2}}{2} \\
-\frac{\delta_{\hat{p}_{1}}^{2}}{2} & 0
\end{array}\right]  \tag{5.8.4}\\
\boldsymbol{\varepsilon}_{1,3}=\left[\begin{array}{cc}
\frac{\delta_{\hat{p}_{2}}^{2}}{2}+\delta_{\hat{q}_{3}}^{2} & \operatorname{Cov} \\
0 & -\delta_{\hat{p}_{1}}^{2}
\end{array}\right]  \tag{5.8.5}\\
\boldsymbol{\varepsilon}_{2,3}=\left[\begin{array}{cc}
\operatorname{Cov} & \frac{\delta_{\hat{p}_{1}}^{2}+\delta_{\hat{p}_{3}}^{2}}{2} \\
\delta_{\hat{p}_{2}}^{2} & 0
\end{array}\right] \tag{5.8.6}
\end{gather*}
$$

where the variances $\delta_{\hat{X}}^{2}$ with $\hat{X} \in\left\{\hat{x}_{1}, \hat{x}_{2}, \hat{x}_{3}, \hat{p}_{1}, \hat{p}_{2}, \hat{p}_{3}\right\}$. On the other hand, the term Cov represent the covariance of the Gaussian system under observation, this is

$$
\begin{equation*}
\operatorname{Cov}=\sin (2 \theta)\left(e^{2 r}-e^{-2 r}\right) \tag{5.8.7}
\end{equation*}
$$

Besides, we recall that the balance parameter appearing in $\delta_{\hat{x}_{j}}^{2}$ and $\delta_{\hat{p}_{j}}^{2}$ with $j=1,2$, is fixed at the rate $b=\delta_{\hat{x}_{3}} / \delta_{\hat{p}_{3}}$ as we explain in Eq. (2.2.13).

### 5.9 MATHEMATICAL EQUIVALENCE BETWEEN THE FRAMES OF SCHRÖDINGER AND FREE-PROPAGATOR IN THE METHOD OF GREEN'S FUNCTIONS FOR THE CASE OF A FREE PARTICLE

Take the solution to the Schrödinger equation:

$$
\begin{equation*}
\Psi(x, t)=e^{-i t \hat{H} / \hbar} \psi^{\prime}(x, 0) \tag{5.9.1}
\end{equation*}
$$

where $\psi^{\prime}(x, 0)$ is an arbitrary initial wave function at time $t=0$ and represent $\Psi(x, t)$ the associated wave function at any time $t$. Let us assume a free particle in one dimension; then: $\hat{H}=\hat{p}_{x}^{2} / 2 m$, where $\hat{p}_{x}$ is the momentum operator in $x$-direction defined as $\hat{p}_{x}=-i \hbar \partial_{x}$. Therefore, the Eq. (5.9.1) can be written as

$$
\begin{equation*}
\Psi(x, t)=e^{a \partial_{x}^{2}} \psi^{\prime}(x, 0)=\sum_{k=0}^{\infty} \frac{a^{k}}{k!} \partial_{x}^{2 k} \psi^{\prime}(x, 0) \tag{5.9.2}
\end{equation*}
$$

where $a=i t \hbar / 2 m$. Now, apply a Fourier transform on both sides of Eq. (5.9.2)

$$
\begin{align*}
F\left[e^{a \partial_{x}^{2}} \psi^{\prime}(x, t)\right]\left(p_{x}\right) & =\sum_{k=0}^{\infty} \frac{a^{k}}{k!} F\left[\partial_{x}^{2 k} \psi^{\prime}(x, 0)\right]\left(p_{x}\right) \\
& =\sum_{k=0}^{\infty} \frac{a^{k}}{k!}\left(i p_{x}\right)^{2 k} F\left[\psi^{\prime}(x, 0)\right]\left(p_{x}\right)  \tag{5.9.3}\\
& =e^{-a p_{x}^{2}} F\left[\psi^{\prime}(x, 0)\right]\left(p_{x}\right)
\end{align*}
$$

Besides, in the second line we use the derivative property of the Fourier transform: $F\left[\frac{d^{n} f(q)}{d q^{n}}\right](p)=(i p)^{n} F[f(q)](p)$. It must be noted that the exponential $e^{-a p_{x}^{2}}$ depends on the variable $p_{x}$; therefore, it can be expressed by means of a Fourier transform of a function $f(x)$; that is,

$$
\begin{equation*}
F[f(x)]\left(p_{x}\right)=e^{-a p_{x}^{2}} . \tag{5.9.4}
\end{equation*}
$$

Using the definition of the inverse Fourier transform: $F^{-1}[M(b)]=$ $\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} M(b) e^{+i b a} d b$, we trivially verify

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{2 a}} e^{-\frac{x^{2}}{4 a}} \tag{5.9.5}
\end{equation*}
$$

Now, using Eq. (5.9.4) the last line of Eq. (5.9.3) is

$$
\begin{equation*}
F\left[e^{a \partial_{x}^{2}} \psi^{\prime}(x, t)\right]\left(p_{x}\right)=F[f(x)]\left(p_{x}\right) \cdot F\left[\psi^{\prime}(x, 0)\right]\left(p_{x}\right) \tag{5.9.6}
\end{equation*}
$$

Take the inverse Fourier transform on both sides of the last equation

$$
\begin{equation*}
e^{a \partial_{x}^{2}} \psi^{\prime}(x, t)=F^{-1}\left[F[f(x)]\left(p_{x}\right) \cdot F\left[\psi^{\prime}(x, 0)\right]\left(p_{x}\right)\right] ; \tag{5.9.7}
\end{equation*}
$$

then, take the convolution theorem for a product of Fourier transforms: $f * g=F^{-1}[F[f] \cdot F[g]]=(\sqrt{2 \pi})^{-1} \int_{-\infty}^{\infty} f(\eta) g(x-\eta, 0) d \eta$; therefore, the Eq. (5.9.7) is

$$
\begin{align*}
e^{a \partial_{x}^{2}} \psi^{\prime}(x, t)=f(x) * \psi^{\prime}(x, 0) & =(\sqrt{2 \pi})^{-1} \int_{-\infty}^{\infty} f(\eta) \psi(x-\eta, 0) d \eta \\
& =(\sqrt{2 \pi})^{-1} \int_{-\infty}^{\infty} f(x-\eta) \psi(\eta, 0) d \eta \tag{5.9.8}
\end{align*}
$$

where in the second line of the last equation we use the commutative property of the convolution theorem. Now, we use the Eq. (5.9.5)
with the explicit definition $a=i t h / 2 m$ in the last line of Eq. (5.9.8), obtaining

$$
\begin{equation*}
\Psi(x, t)=e^{a \partial_{x}^{2}} \psi^{\prime}(x, t)=\sqrt{\frac{m}{i 2 \pi \hbar t}} \int_{-\infty}^{+\infty} e^{-\frac{m(x-\eta)^{2}}{2 i h t}} \psi(\eta, 0) d \eta . \tag{5.9.9}
\end{equation*}
$$

Then take into account that the dynamics of an arbitrary system $\Psi(x, t)$ can be obtained from its initial $(t=0)$ wave function $\psi(x, 0)$ through propagators (Green's functions), that is [173],

$$
\begin{equation*}
\Psi(x, t)=\int_{-\infty}^{+\infty} K\left(x, x^{\prime} ; t, 0\right) \psi\left(x^{\prime}, 0\right) d x^{\prime} \tag{5.9.10}
\end{equation*}
$$

For a free particle, a simple form of the propagator $K\left(x, x^{\prime} ; t, 0\right)$ is

$$
\begin{equation*}
K\left(x, x^{\prime} ; t, 0\right)=\sqrt{\frac{m}{i 2 \pi \hbar t}} e^{i m\left(x-x^{\prime}\right)^{2} / 2 h t} ; \tag{5.9.11}
\end{equation*}
$$

therefore, using Eqs. (5.9.10) and (5.9.11) we have, for a free particle:

$$
\begin{equation*}
\Psi(x, t)=\sqrt{\frac{m}{i 2 \pi \hbar t}} \int_{-\infty}^{+\infty} e^{-\frac{m(x-\eta)^{2}}{2 h \hbar t}} \psi(\eta, 0) d \eta \tag{5.9.12}
\end{equation*}
$$

which is equal to Eq. (5.9.9).

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[^0]:    1 Are improper eigenstat \#ice are non-normalizable; consequently, they live outside the Hilbert space of the system. See [2], 屃. 100-104.

[^1]:    2 When we have a quantum system formed by multiple subsystems, one can distribute them to various parties; each is restricted to act locally on their respective subsystem by doing quantum operation, for example, quantum measurements. Then, the parties are free to communicate any classical data. Quantum operations following this line are known as local operations with classical communication (LOCC) $[70,71]$.

[^2]:    1 there is nothing special about choosing this measurement time; experimentally speaking we can adjust the displacement scale of the probe to get at any time a displacement equal to $q$.

[^3]:    1 In this time the joint readout takes place. Therefore, projective measurements are done in the pointers leaving the post-measurement Gaussian state as the normalized projection on the eigenspace of the observed eigenvalues according to the postulate of a projective quantum measurement [5].

[^4]:    1 In Appendix 5.9 it is proved through the convolution theorem for inverse Fourier transform, that the application of this operator to an arbitrary state is equivalent to the dynamics with the free particle-propagator in the method of Green's functions.

