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DYNAMIC FRAGMENTATION: GEOMETRIC APPROACH

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Abstract:

Dynamic fragmentation is a complex and common phenomenon in nature and technological systems. The main task in fragmentation modeling is determination of the fragment size (or mass) distribution law. There are several approaches to the fragmentation problem – empirical, probabilistic, energetic, approach based on fracture mechanics, etc. In the present paper we consider a general approach based on the simple assumption of random geometric partition of a body, following early Lineau [1], Mott [2] and well-known Grady-Kipp work [3]. Starting from the Poisson distribution of fracture sites (points, lines or planes), size distribution laws are derived for 1D, 2D and 3D geometries. Geometric fragmentation models based on the Mott and Grady-Kipp approaches are discussed, as well as the model originated from the Voronoi diagrams. The results of presented models are compared with numerical simulations and experimental data, showing significant compatibility with certain limitations.

Key words: dynamic fragmentation, geometric statistics, fragment size distribution, Voronoi diagrams

1. Introduction

Fragmentation is a common phenomenon in nature and engineering systems that take place on different size and time scales (expansion of galaxies, asteroid impacts, explosively driven fragmentation, fragmentation induced by impact of nuclei, etc.). Fragmentation modeling is extremely difficult problem, involving complex physics dependent on loading conditions, material characteristics and problem geometry. In this paper, the fragmentation problem has been considered from a purely geometric viewpoint. Following the approach of Lineau [1], as the model for brittle fragmentation of a solid, random segmentation of a line, area and volume is analyzed.

2. Fragmentation modeling by geometric statistics

2.1. One-dimensional fragmentation

Let us first consider probabilistic fragmentation of a line produced by random selection of "fracture" sites, using the approach of Lineau [1], Mott [2] and Grady [3, 4]. Assuming break points randomly distributed on the line of infinite length, the probability of finding exactly k "fracture" points on a line segment of length l is determined by the Poisson distribution of the form

$$P(k,l) = e^{-\lambda l}\frac{(\lambda l)^k}{k!}, \tag{1}$$

where λ is the number of points per unit length. Now, using the Poisson distribution (1), the probability dp of finding fragment of the length in the interval [l, l+dl] can be calculated as follows. The probabilities that there is no point on the line segment of length l, and that there is exactly one point in the adjacent segment of length dl are determined by

$$P(0,l) = \mathrm{e}^{-\lambda l} \text{ and } P(1,\mathrm{d}l) = \lambda \mathrm{d}l. \tag{2}$$

Finally, the probability dp can be determined as

$$\mathrm{d}p = P(0,l)P(1,\mathrm{d}l) = \lambda \mathrm{e}^{-\lambda l}\mathrm{d}l. \tag{3}$$

Therefore, the probability density function of fragment length is

$$f(l) = \frac{\mathrm{d}p}{\mathrm{d}l} = \lambda \mathrm{e}^{-\lambda l}. \tag{4}$$

The cumulative probability distribution function $P(>l)=P(l)$ (that the fragment length is greater than l) has the exponential form

$$P(l) = \int_l^{\infty} f(l)dl = \mathrm{e}^{-\lambda l}. \tag{5}$$

The exponential distribution (5) successfully describes random 1D fragmentation; it is also the framework for several advanced fragmentation models.

Similar approach to the random geometric fragmentation is based on Voronoi-Dirichlet diagrams. The Voronoi-Dirichlet decomposition of a space (1D, 2D or 3D) with randomly generated initial points imply partitioning of a space in a certain number of subspaces such that each subspace contains exactly one generating point and every point in a given subspace is closer to its generating point than to any other. Voronoi-Dirichlet algorithm may be used as a model for different physical processes, including fragmentation [4]. In 1D case, Voronoi segmentation (i.e. random fragmentation) of a line is defined by the midpoints of each pair of adjacent randomly distributed initial points. Using similar procedure as in the previous case [4, 5], the probability density function of fragment length and cumulative distribution can be determined as

$$f(l) = 4\lambda^2 l \mathrm{e}^{-2\lambda l}, \; P(l) = (1 + 2\lambda l)\mathrm{e}^{-2\lambda l}. \tag{6}$$

Normalized (λ=1) exponential (Lineau) and Voronoi distribution are plotted in Fig. 1a.

2.2. Two-dimensional fragmentation

Two-dimensional fragmentation is of the much more practical importance, as it involves fragmentation of thin plates and shells. As a paradigmatic example, random fragmentation of a plane will be analyzed first. The simplest method for plane fragmentation is by random generation of horizontal and vertical lines. If we assume that the two sets of lines are independent, with the same density λ, then Lineau distribution can be applied to both horizontal and vertical set of lines, yielding the cumulative distribution

$$P(a) = \lambda^2 \iint_{xy>a} \mathrm{e}^{-\lambda(x+y)}\mathrm{d}x\mathrm{d}y = 2\lambda\sqrt{a}\,\mathrm{K}_1(2\lambda\sqrt{a}), \tag{7}$$

where $\mathrm{K}_1(\bullet)$ is the first order modified Bessel function of the second kind and a is a fragment area.

Mott [2] derived eq. (7), but proposed different fragment area distribution law. His analysis of fragments produced by detonation of high-explosive shells suggested distribution of the form:

$$P(a) = \mathrm{e}^{-\sqrt{\alpha a}} \,. \tag{8}$$

In the well-known Mott distribution law, eq. (8), the parameter α is related to the average fragment area by $\alpha = 2/\bar{a}$. Justification for this distribution is the fact that it is analogous to the Lineau exponential law, having in mind that for 2D fragmentation fragment length $l \sim a^{1/2}$.

Cohen [6] and Grady and Kipp [3] offered another postulate: all fragment area distributions have the same probability, provided constant sum of fragments' area. This is equivalent to the Lineau 1D distribution, so the fragment area distribution law has the form

$$P(a) = \mathrm{e}^{-\alpha a} \,, \tag{9}$$

where α is reciprocal to the average fragment area.

Finally, Kiang [5] suggested (without the proof) approximate generalization of fragment size distribution, eq. (6), generated by Voronoi diagrams in the form:

$$P(s) = \Gamma(n, \mu n s) / \Gamma(n) \tag{10}$$

where $\Gamma(\bullet)$ and $\Gamma(\bullet,\bullet)$ are the complete and upper incomplete gamma function, s is fragment size (length, area or volume), μ is the reciprocal to the average fragment size, and n=2, 4 and 6, corresponds to 1D, 2D and 3D Voronoi distribution, respectively. For 2D case (n=4), the cumulative distribution reads:

$$P(a) = \Gamma(4, 4\alpha a) / \Gamma(4) = \mathrm{e}^{-4\alpha a} \sum_{k=0}^{3} \frac{(4\alpha a)^k}{k!} \,. \tag{11}$$

The Bessel, Mott, Grady and Voronoi distributions for the same average fragment area are shown in Fig. 1b.

2.3. Three-dimensional fragmentation

The three-dimensional fragmentation implies fractures through all three dimensions of a fragmented body. This is the most complex and the most important case of fragmentation from the application aspect. The main example is fragmentation of a space with three sets of parallel and mutually orthogonal planes. Supposing that the three sets of planes are independently distributed with the same density λ, the Lineau approach leads to the cumulative distribution:

$$P(v) = \lambda^3 \iiint_{xyz>v} \mathrm{e}^{-\lambda(x+y+z)} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \lambda\sqrt{\lambda v}\; \mathrm{G}_{0,\,3}^{3,\,0}\left(\lambda^3 v \,\middle|\, \begin{matrix} - \\ -\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \end{matrix} \right), \tag{12}$$

where G is the Meijer G-function [7].

Mott's formula for 3D case, having in mind that fragment length $l \sim v^{1/3}$, has the form

$$P(v) = \mathrm{e}^{-\sqrt[3]{\gamma v}} \,. \tag{13}$$

Parameter γ is defined by $\gamma = 6/\bar{v}$, where $\bar{v}$ is the average fragment volume.
Following the same argument as in eq. (9), Grady's cumulative fragment distribution is

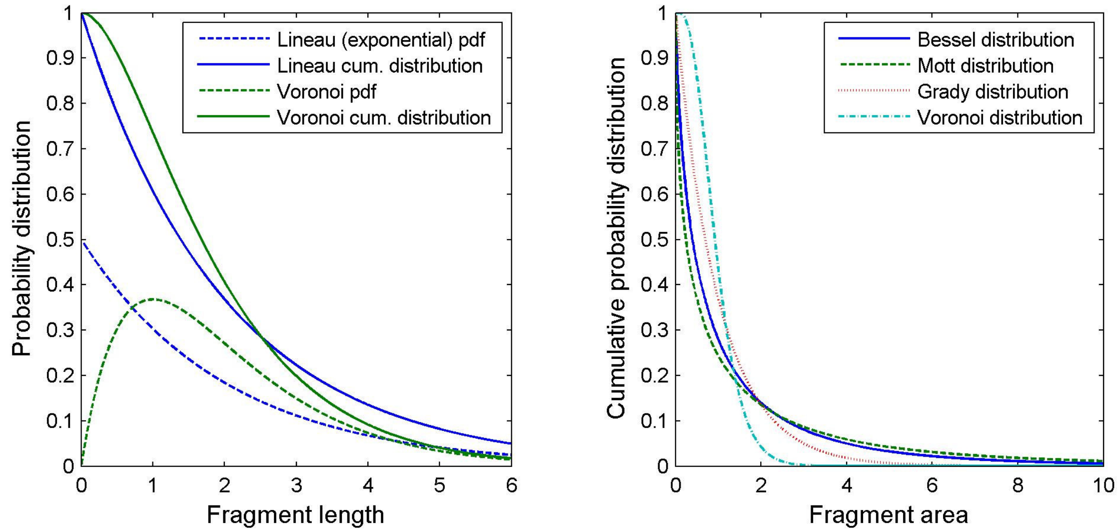
$$P(v) = \mathrm{e}^{-\gamma v} \,, \tag{14}$$

where γ is reciprocal to the average fragment volume.

Applying n=6 to eq. (10), the Voronoi distribution for 3D case becomes:

$$P(v) = \Gamma(6, 6\gamma v) / \Gamma(6) = \mathrm{e}^{-6\gamma v} \sum_{k=0}^{5} \frac{(6\gamma v)^k}{k!} \,. \tag{15}$$

Four analyzed cumulative fragment distributions for 3D case (constant average fragment volume) are compared in Fig. 1c.



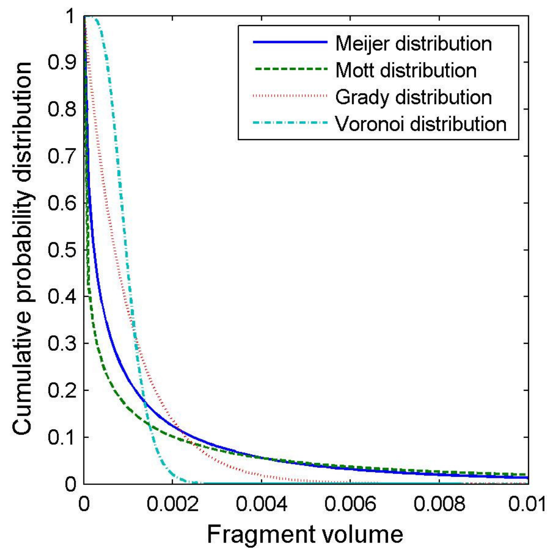


Fig. 1. Comparison of different probability distributions for (a) one-dimensional, (b) two-dimensional and (c) three-dimensional random fragmentation; the average fragment sizes (length for 1D, area for 2D and volume for 3D case) are the same for each distribution

3. Comparison with experiments and discussion

Applicability of analyzed random fragmentation models will be illustrated through comparison with experimental results.

For 1D case, the Lineau and Voronoi distribution are fitted to the experimental data from Grady and Benson [8] (Fig. 2a) and Weisenberg and Sagartz [9] (Fig. 2b). Both experiments treat fragmentation of aluminum rings by impulsive electromagnetic loads. The Lineau distribution fails to describe experimental data, but the Voronoi distribution (especially for the first case) provides the fair agreement with experiments. Possible explanation is that the Voronoi (gamma) distribution is approximation of the advanced physically based Mott's fragmentation model [10] thoroughly analyzed by the present authors [11].

Numerically generated fragmentation of a unit square by randomly chosen vertical and horizontal lines (inset, Fig. 3a) is compared with analyzed theoretical distributions (with the same average fragment area). This is a possible simulation of the 2D fragmentation of a brittle material. As expected, only the Bessel distribution excellently describes numerical data (Fig. 3a). However, fragmentation of the near-spherical ductile metal shell [4] is fairly approximated by the Voronoi distribution (Fig. 3b).

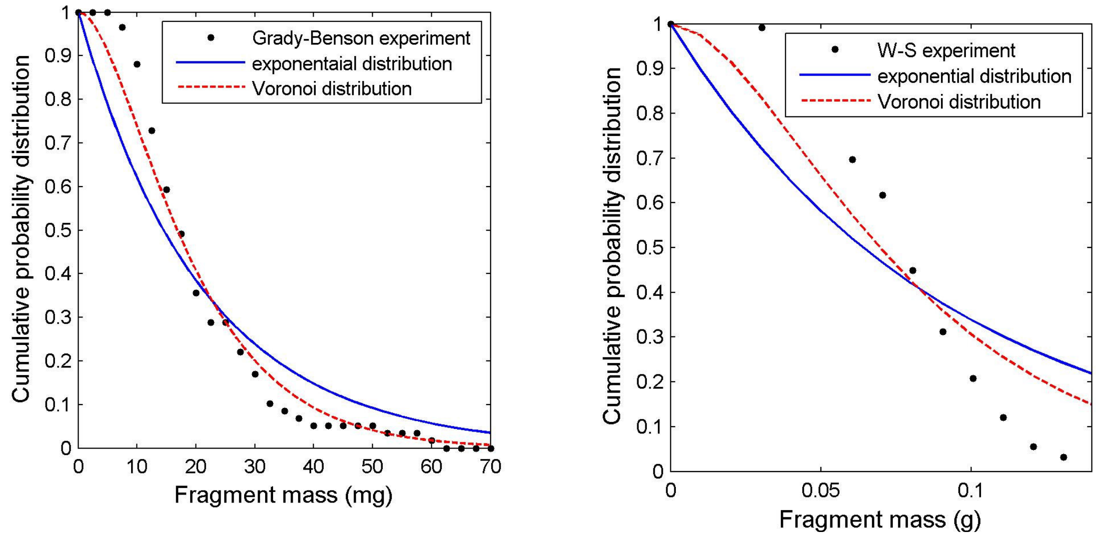


Fig. 2. One-dimensional fragmentation: comparison of theoretical distributions with (a) Grady-Benson [8] and (b) Weisenberg-Sagartz [9] experimental data

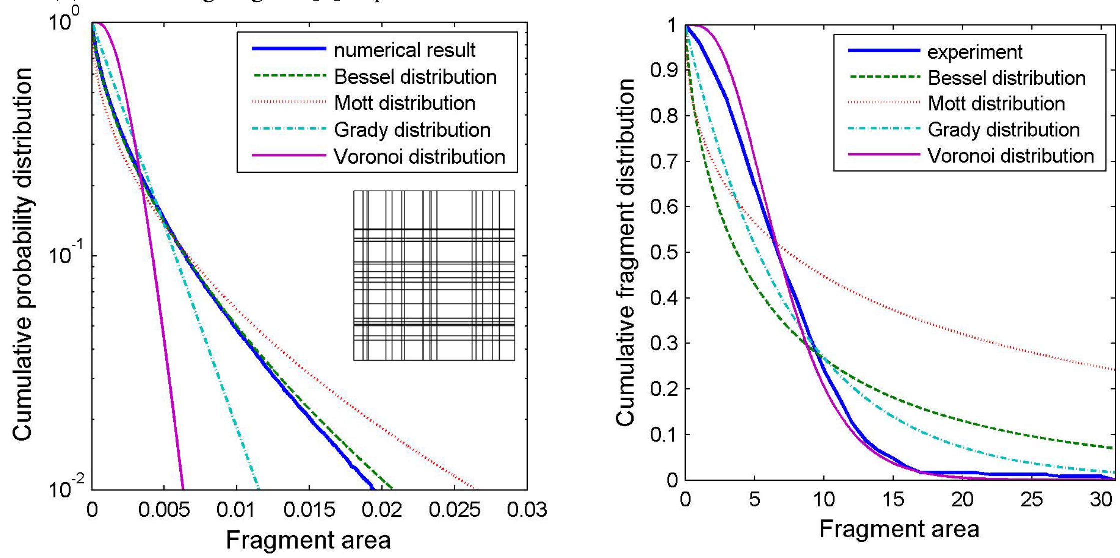


Fig. 3. Two-dimensional fragmentation: comparison of different theoretical models with: (a) numerically determined area distribution obtained by random segmentation of a unit square (inset); (b) experimental data from Grady [4]

As 3D fragmentation examples, the explosively driven steel cylinder [12] (Fig. 4a) and the fragmentation projectile [13] (Fig. 4b) are analyzed. In the first case, 3D Mott distribution obtains the best fit to the experimental data. In the latter, the Meijer and 2D Mott distribution have reasonable accordance with the experiment. The complexity of 3D fragmentation, including tension and adiabatic shearing mechanisms, leads to a variety of fragment sizes and shapes and prevents unimodal Voronoi distribution to successfully describe fragment mass distribution.

4. Conclusions

Fragment size distribution from a purely geometric aspect has been considered. We rederived the Lineau fragment distribution formula starting from the random segmentation of a line. This approach is generalized to the 2D and 3D case using the Bessel and Meijer distributions. The Mott and Grady paradigms are explaind and corresponding fragment size distributions are also presented. Finally, geometric fragmentation model based on the Voronoi diagrams is introduced.

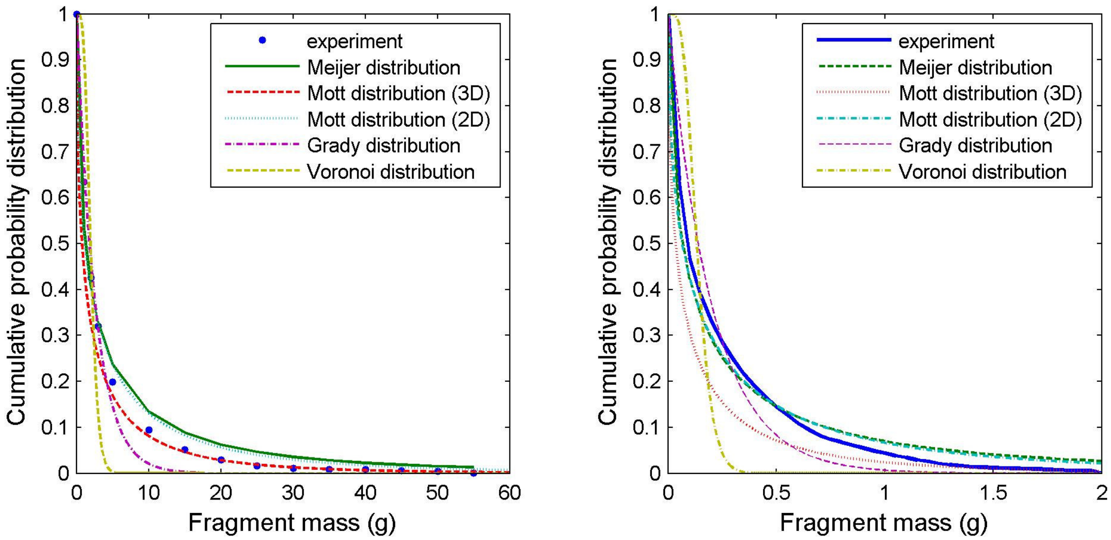


Fig. 4. Three-dimensional fragmentation: theoretical distribution laws compared with fragment distribution data for: (a) explosively driven cylinder [12] and (b) fragmentation projectile A from [13]

The analyzed models are compared with limited experimetal results. The conclusion for brittle materials is that fragment mass distribution can be approximated by the Lineau, Bessel and Meijer distributions for 1D, 2D and 3D case. For ductile materials, 1D and 2D fragmentation can be described by the Voronoi distribution, and in 3D case the Mott distribution is the best choice.

Having in mind that material characteristics, problem geometry and applied loads are not considered, the results are surprisingly good. However, application of geometrically based fragmentation models always requires clear physical justification.

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