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REAL TIME FRACTIONAL ORDER CONTROL OF ROTARY INVERTED PENDULUM

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Abstract. This paper proposes advanced control strategy for a rotary inverted pendulum (RIP). RIP is an underactuated mechanical system because it has only one control input and two degrees of freedom. Because of its complex nonlinear dynamics, RIP is usually used to test performance of different control algorithms. The mathematical model for the RIP is derived using the Rodriguez method. Control problem is divided and implemented in two different steps: swing-up and stabilization routines. Here, a new algorithm of PID control is suggested based on fractional calculus (FC), i.e.fractional PD^{α} controller as well as classical PID control in the control of RIP. The effectiveness of the proposed control method is tested in Matlab Simulink environment.

1. Introduction

Underactuated systems have more degrees of freedom than actuators,[1-3]. A rotational inverted pendulum, also known as Furuta pendulum, is an example of such a system,[4-10]. Almost all dynamic systems are nonlinear by its nature, therefore a lot of research is done in the area of nonlinear control. On the other hand, fractional calculus (FC) has the potential to accomplish what integer-order calculus cannot. In most cases, our objective ofusing FC is to apply the fractional order controller to enhance thesystem control performance compared to the traditional controllers,[11]. Unlike conventional PID controller, there is no systematic and rigor design or tuning method existing for fractional order controller.The aim of this paper is to develop a nonlinear control system for the rotational pendulum. It means that it has to bring the pendulum in vertical upright position and balance it there. Also, the position of the actuated arm has to be controlled. First, a description of the Furuta pendulum will be given. Then, a mathematical model of the system will be derived. The control strategy consists of two parts, a swing up and a balancing phase. At the defined moment, the swing up controller switches to the balancing PD/PD^{α} controller and stabilize the pendulum. The theory of inverse dynamic control will be used for the latter. However, the resulting zero dynamics of the actuated arm shows unstable behavior. Hence, a control feedback law will be extended in order to stabilize the horizontal arm.

2. Dynamic equations of rotary inverted pendulum

In Fig. 1 a schematic ofFuruta pendulum is shown and it is a mechanical system with two degrees of freedom, where angular position of the arm and the pendulumare denoted as θ

and φ , respectively. The arm is driven with a torque, while no torque is applied directly to the pendulum. Hence, it is an underactuated mechanical system because it has only one control input and two degrees of freedom.

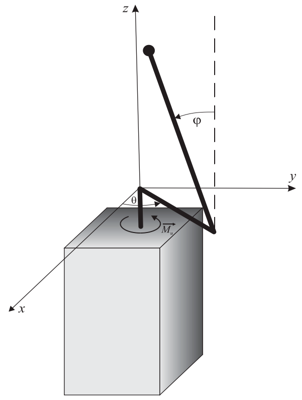


Figure 1.A schematic of the rotational inverted pendulum

Parameters ofthe system are: m_1 - mass of the arm, m_2 - mass of the pendulum, R_1 - distance of the arm's pivot point to the pendulum's pivot point, R_2 - distance of the pendulum's pivot point to its end (extreme), $2r_1$, $2r_2$ - total length of the arm, andpendulum respectively, $J_{\varsigma 1}$ - moment of inertia of the arm with respect to its center of mass, $J_{\xi 2}, J_{\eta 2}, J_{\zeta 2}$ -axial moments of inertia of the pendulum with respect to its center of mass.Here, the Rodriguez method is proposed for modeling the dynamics of the system whereconfiguration of the mechanical model can be defined by generalized coordinates q_1 and q_2 represent by θ and φ , respectively. The equations of motion of the inverted pendulum can be expressed in a covariant form of Langrange's equation of second kindas follows [1,2]:

$$\sum_{\alpha=1}^{n} a_{\gamma\alpha}\ddot{q}_{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \Gamma_{\alpha\beta,\gamma}\dot{q}_{\alpha}\dot{q}_{\beta} = Q_{\gamma} \quad \gamma = 1,2 \tag{1}$$

where the coefficients $a_{\alpha\beta}$ are the covariant coordinates of the basic metric tensor $\left[a_{\gamma\alpha}\right] \in R^{2\times 2}$ and $\Gamma_{\alpha\beta,\gamma}$ $\alpha,\beta,\gamma = 1,2$ presentsChristoffel symbols of the first kind. The generalized forces Q_{γ} can be presented in the following expression (3) where $Q_{\gamma}^{g}, Q_{\gamma}^{a}$ denote the generalized gravitational and control forces, respectively.

$$Q_{\gamma} = Q_{\gamma}^{g} + Q_{\gamma}^{a}, \quad \gamma = 1,2 \tag{2}$$

The equations of motion of our system can be rewritten in compact matrix form:

$$A(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Q}^{a} \tag{3}$$

where

$\boldsymbol{q} = (\theta\ \varphi)^{T}$, $A(\boldsymbol{q}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is basic metric tensor, $C(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^{2}$ - vector which takes care of configuration of inverted pendulum system and velocity dependent effects,

$\boldsymbol{Q}^g(q) = -\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^2$ - vector of generalized gravitational forces, and $\boldsymbol{Q}^a = (M \quad 0)^T \in \mathbb{R}^2$ - vector of generalized control forces. Finally, Eq. (3) written in full form becomes:

$$a_{11}\ddot{\theta} + a_{12}\ddot{\varphi} + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{22,1}\dot{\varphi}^2 = M \tag{4}$$

$$a_{12}\ddot{\theta} + a_{22}\ddot{\varphi} - \Gamma_{12,1}\dot{\theta}^2 = Q_2^g \tag{5}$$

where are

$$\begin{aligned}
a_{11} &= J_{\varsigma 1} + J_{\eta 2}\sin^2(\varphi) + J_{\varsigma 2}\cos^2(\varphi) + m_2 R_1^2 + m_1(R_1 - r_1)^2 + m_2(R_2 - r_2)^2\sin^2(\varphi) \\
a_{12} &= -m_2 R_1(R_2 - r_2)\cos(\varphi) = -K_3\cos(\varphi), \quad a_{22} = J_{\xi 2} + m_2(R_2 - r_2)^2 = K_4 \\
\Gamma_{12,1} &= 0.5\left(m_2(R_2 - r_2)^2 + J_{\eta 2} - J_{\varsigma 2}\right)\sin(2\varphi) = K_2\sin(2\varphi) \\
\Gamma_{22,1} &= m_2 R_1(R_2 - r_2)\sin(\varphi) = K_3\sin(\varphi), \quad Q_2^g = m_2 g(R_2 - r_2)\sin(\varphi) = K_1\sin(\varphi)
\end{aligned} \tag{6}$$

For simplicity, we introduce physical parameters K_1, K_2, K_3, K_4 which are defined as shown above.

3. Controller design

In this section a control strategy is developed to stabilize the pendulum in upright position. As mentioned before, there are two different control problems. The first one is swinging the pendulum up from down to the upright position. Once the system is close to the desired position, with a simple change in the controller, it is possible to bring the pendulum in the desired equilibrium.

3.1. Swing up controller

There are many ways to bring the pendulum to the upper half plane, when $|\varphi| < \pi/2$. One of the most popular is based on energy control [3,4]. The goal of this paper is not to build an accurate swing up controller, but to bring the pendulum close enough so the stabilizing controller can stabilize it in the upright position. Hence, the swing up strategy will only be described here in short.The equation of motion for the pendulum is:

$$\left(J_{\xi 2} + m_2(R_2 - r_2)^2\right)\ddot{\varphi} - m_2 g(R_2 - r_2)\sin\varphi + m_2 a(R_2 - r_2)\cos\varphi = 0 \tag{7}$$

where a represents the acceleration of the pendulum's pivot point. Friction has been neglected. For the sake of clarity, let us introduce $J_2 = J_{\xi 2} + m_2(R_2 - r_2)^2$. The energy of the uncontrolled pendulum (without the rotating arm) is:

$$E = \frac{1}{2}J_2\dot{\varphi}^2 + m_2 g(R_2 - r_2)(\cos\varphi - 1) \tag{8}$$

The energy is defined so that it is zero in upright rest position. Now, it is necessary to understand how the energy is influenced by the acceleration of the pivot. We can find it by computing the time derivative of E,[5]:

$$\frac{dE}{dt} = J_2\dot{\varphi}\ddot{\varphi} - m_2 g(R_2 - r_2)\dot{\varphi}\sin\varphi = -m_2 a(R_2 - r_2)\dot{\varphi}\cos\varphi \tag{9}$$

where Eq. (6) has been used to obtain the last equality. Equation (8) implies that system is simply an integrator with varying gain. To increase energy the acceleration of the pivot a should be positive when the quantity $\dot{\varphi}\cos\varphi$ is negative. With the Lyapunov function:

$$\upsilon = \frac{(E_0 - E)^2}{2} \tag{10}$$

andthe control law $u(t) = a(t) = k(E - E_0)\dot{\varphi}\cos\varphi,\ k = const > 0$, we find that

$$\dot{\upsilon} = -km_2(R_2 - r_2)((E_0 - E)\dot{\varphi}\cos\varphi)^2 \tag{11}$$

This control law drives the energy towards its desired value $E_0 = 0$, except when $\dot{\varphi}\cos\varphi = 0$. To change the energy as fast as possible we introduce the following control law:

$$u(t) = \mathrm{sat}_k\left((E - E_0)\dot{\varphi}\cos\varphi\right) \tag{12}$$

where sat_k denotes a linear function which saturates at k.

3.2. Stabilizing controller

Now we can design a controller that stabilizes the pendulum in upright position. For this purpose, we will use nonlinear control technique known as *inverse dynamic control*. It is basically *a partial feedback linearization* procedure[6], which simplifies the control design. The first step of this procedure is to calculate $\ddot{\theta}$ from Eq. (6) and plug it into Eq. (5). After rearranging, Eq. (5) now reads (see [7,8]):

$$\frac{a_{11}}{a_{12}}\left(Q_2^g + \Gamma_{12,1}\dot{\theta}^2\right) + \left(a_{12} - \frac{a_{11}a_{22}}{a_{12}}\right)\ddot{\varphi} + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{12,1}\dot{\varphi}^2 = M \tag{13}$$

We can see that $\ddot{\theta}$ has been canceled out in (12). Control input M can be chosen as follows:

$$M = \frac{a_{11}}{a_{12}}\left(Q_2^g + \Gamma_{12,1}\dot{\theta}^2\right) + \left(a_{12} - \frac{a_{11}a_{22}}{a_{12}}\right)M_R + 2\Gamma_{12,1}\dot{\theta}\dot{\varphi} + \Gamma_{12,1}\dot{\varphi}^2 \tag{14}$$

where M_R is new control input. Now, Eq. (5) and (6) become:

$$\ddot{\theta} = -\frac{K_1}{K_3}\tan(\varphi) - 2\frac{K_2}{K_3}\sin(\varphi)\dot{\theta}^2 + \frac{K_4}{K_3}\frac{M_R}{\cos(\varphi)} \tag{15}$$

$$\ddot{\varphi} = M_R \tag{16}$$

where physical parameters K_1, K_2, K_3, K_4 are defined in Eqs. (6). Because of the cosine term in term a_{12} in the denominator of Eq. (13), the control signal is defined in every position of the pendulum except for the horizontal, i.e. $|\varphi| < \pi/2$. To achieve asymptotic stability for the $(\varphi, \dot{\varphi})$, a PD controller can be used:

$$M_R = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi} \tag{17}$$

The PD controller stabilizes the inverted pendulum for every $K_{P\varphi}, K_{D\varphi} > 0$, but does not stabilize the arm [9]. This can be seen by observing the zero dynamics of the system. Substituting $\varphi = 0,\ \dot{\varphi} = 0$ into Eq. (14), it follows:

$$\ddot{\theta} = 0 \Rightarrow \dot{\theta} = const \tag{18}$$

So, underactuated mechanical systems like inverted pendulum are not fully feedback linearisable, and control techniques developed for a fully actuated systems cannot be applied here [10]. The new goal is to improve M_R so that asymptotic stability for $\left(\varphi, \dot{\varphi}, \theta, \dot{\theta}\right)$ can be accomplished. To achieve this, control feedback law will be extended as follows:

$$M_R = -K_{P\varphi}\varphi - K_{D\varphi}\dot{\varphi} - K_{P\theta}\theta\cos(\varphi) - K_{D\theta}\dot{\theta}\cos(\varphi) + \frac{K_1}{K_4}\sin(\varphi) \tag{19}$$

After substituting Eq. (18) into Eq. (14) and (15), we obtain:

$$\ddot{\theta} + \frac{K_4}{K_3}K_{D\theta}\dot{\theta} + \frac{K_4}{K_3}K_{P\theta}\theta = -\frac{K_4}{K_3\cos(\varphi)}\left(K_{D\varphi}\dot{\varphi} + K_{P\varphi}\varphi\right) - 2\frac{K_2}{K_3}\sin(\varphi)\dot{\theta}^2 \tag{20}$$

$$\ddot{\varphi} + K_{D\varphi}\dot{\varphi} + K_{P\varphi}\varphi - \frac{K_1}{K_4}\sin(\varphi) = -\cos(\varphi)\left(K_{D\theta}\dot{\theta} + K_{P\theta}\theta\right) \tag{21}$$

We can notice that the last term on the right side of Eq. (18) is introduced to cancel out term which contains $\tan(\varphi)$ in Eq. (14). Now, we can linearize system described with Eqs. (19)-(20) around equilibrium point $\left(\theta, \dot{\theta}, \varphi, \dot{\varphi}\right) = \left(0,0,0,0\right)$. A controller derived from a linearized system will work for a nonlinear system, provided region of attraction is not too large. Under this condition, linearization allows us to neglect nonlinear, quadratic term $\dot{\theta}^2$ in Eq. (19). So, linearization around desired equilibrium point leads to:

$$\ddot{\theta} + \frac{K_4}{K_3}K_{D\theta}\dot{\theta} + \frac{K_4}{K_3}K_{P\theta}\theta = -\frac{K_4}{K_3}K_{D\varphi}\dot{\varphi} - \frac{K_4}{K_3}K_{P\varphi}\varphi \tag{22}$$

$$\ddot{\varphi} + K_{D\varphi}\dot{\varphi} + \left(K_{P\varphi} - \frac{K_1}{K_4}\right)\varphi = -K_{D\theta}\dot{\theta} - K_{P\theta}\theta \tag{23}$$

Choosing the following values for PD parameters:

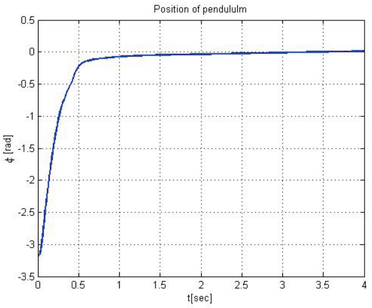
$$K_{P\theta} = -2;\ K_{D\theta} = -1.6;\ K_{P\varphi} = 250;\ K_{D\varphi} = 30; \tag{24}$$

where $K_1 = 6.514e-2,\ K_2 = 9.186e-4,\ K_3 = 1.428e-3,$ and $K_4 = 1.837e-3$ are system parameters taken from the real laboratory model of inverted pendulum, eigenvalues of the linearized system are:

$$s_{1,2}^{*} = -13.8 \pm 3.5j \qquad s_{3,4}^{*} = -0.14 \pm 0.65j \tag{25}$$

Conditions for asymptotic stability of linearized system are fulfilled. Simulation studies are performed inMatlab Simulink environment to illustrate the performance of the designed controller.Particularly,we investigated here use of fractional order PD control ${}_0D_t^{\alpha}\varphi(t), D_t^{\alpha}\theta(t),$ where one can used suitable form for fractional derivative, [11].Some experimental simulations show that the best results are obtained with $\alpha = 0.9$.For calculation of fractional derivatives the Crone approximation of second order was used.

Figure2 below shows results for the change of the pendulum and arm angle, with respect to time. Initial conditions are $\left(\theta, \dot{\theta}, \varphi, \dot{\varphi}\right) = \left(0,0,-\pi,0\right)$. A change from swing up to stabilizing controller happens when $\left|\varphi\right| < \pi/6$.



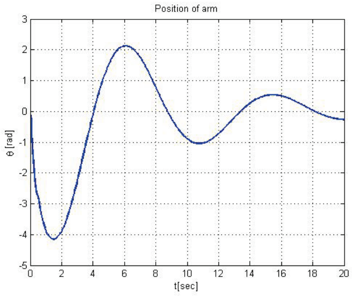


Figure 2. Change of the pendulum and arm angle

4. Conclusion

In this paper a control algorithm for rotational inverted pendulum is provided. The control strategy consists of two parts, a swing up controller and stabilizing controller. A stabilization algorithm is based on partial feedback linearization, which made it possible to compensate some of nonlinearities of the pendulum. Control feedback law is designed to achieve local asymptotic stability for both the pendulum and the driven arm. Results have been supported by means of the using PD/PD^{α} fractional order control as well as computer simulation. For future research, an improvement of the proposed method is to be considered, based on Lyapunov's direct method. Also, transfer from simulation to real laboratory inverted pendulum will be a subject of future investigations.

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