Free vibration of structures composed of rigid bodies and elastic beam segments

Aleksandar Obradović[a], Slaviša Šalinić[b,*], Dragan R. Trifković[c], Nemanja Zorić[a], Zoran Stokić[a]

[a] *University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade 35, Serbia*
[b] *University of Kragujevac, Faculty of Mechanical and Civil Engineering, Dositejeva 19, 36000 Kraljevo, Serbia*
[c] *University of Defense (Serbia), Military Academy, Serbia*

Abstract

Free vibration of structures composed of rigid bodies and elastic beam segments are considered, assuming that the mass centers of rigid bodies are not located on the neutral axes of undeformed elastic beam segments. It is assumed that the rigid bodies of the system perform planar motion in the same plane and that their mass centers are located in that plane. The elastic beam segments are treated as the Euler-Bernoulli beams. In order to determine natural frequencies of the system, modification of the conventional continuous-mass transfer matrix method has been done. The order of the overall transfer matrix has been reduced by this modification. Theoretical considerations are accompanied by two numerical examples.

Keywords: Free vibration, Natural frequency, Euler-Bernoulli theory, Transfer matrix method

---

*Corresponding author
Tel.: +381 36 383269; fax: +381 36 383269
Email addresses: salinic.s@ptt.rs (Slaviša Šalinić), salinic.s@mfkv.kg.ac.rs (Slaviša Šalinić)

1. Introduction

Studies of free oscillations of systems composed of elastic beam segments and rigid bodies are of critical importance in structural system modeling. In the literature, the most frequently analyzed case is that of the systems composed of a single rigid body and two elastic beam segments (see papers [1–7]) as well as of the systems that have the shape of elastic cantilever beam with a rigid body attached to its free end (see papers [8–13]). Using two different approaches (transfer matrix and direct approaches), two dimensional structures composed of two-part elastic beam-rigid body elements are analyzed in [14]. The transfer matrix method is also applied in [15] to analyze the oscillations of complex-form shafts. Further, [16] analyzes vibration characteristics of so-called hybrid elastic beam carrying several elastic-supported rigid bodies. All above mentioned references use analytical approach in treating the problems considered. On the other hand, in reference [17], within the framework of finite elements theory, two-dimensional frame structures with arbitrarily distributed rigid beam segments are analyzed using elastic and rigid combined beam elements. The common characteristic of references above given is that considerations in these references are based on the assumption that the mass centers of the rigid bodies are located on the neutral axes of elastic beams.

The objective of our paper is to extend the existing results to the case of free vibration of structures composed of elastic beam segments and rigid bodies, where mass centers of rigid bodies are not located on the neutral axes of elastic beam segments. In this case, all elastic segments are in the same plane where, while the system is oscillating, rigid bodies are performing planar motion. To the authors' best knowledge of the literature, a special case of thus described system was considered in references [18, 19]. Specifically, these studies consider the case of a single rigid body which is fixed at the ends of two elastic rods, whose axes are parallel in undeformed state of the rods. Studies of free oscillations of thus defined system employ the method that is a modification of the continuous-mass transfer matrix method (CTMM) from [20]. Our modification of the CTMM gives the coefficient of lower order determinant as compared to the determinant obtained by using the original CTMM from paper [20]. This fact is of importance in terms of numerical procedures in systems with a large number of elastic beam segments and rigid bodies.

2. Description of the system and equations of motion

Figure 1 shows a system of rigid bodies (V_i) $(i = 1, \ldots, n)$ interconnected by homogeneous elastic beam segments (BS_i) $(i = 1, \ldots, n)$ where C_i represents the mass center of body (V_i), α_i is the angle made by the longitudinal axes of undeformed adjacent segments (BS_i) and (BS_{i+1}), and O_i is the point of body (V_i) representing the intersection point of the longitudinal axes of undeformed adjacent segments (BS_i) and (BS_{i+1}). Elastic segments are positioned in a plane where rigid bodies are performing planar motion. The left and right ends of the beam segment (BS_i) are denoted by $B_{i,L}$ and $B_{i,R}$, respectively. In Fig. 1, the beam segments $(BS_i)(i = 1, \ldots, n)$ are shown in their undeformed states. The stationary inertial coordinate frames $\{x_i y_i z_i\}(i = 1, \ldots, n)$ are introduced and positioned in a such way that, in the undeformed state of the segments $(BS_i)(i = 1, \ldots, n)$, the left ends $B_{i,L}(i = 1, \ldots, n)$ are positioned at the origins of the frames $\{x_i y_i z_i\}(i = 1, \ldots, n)$, respectively, the axes $z_i(i = 1, \ldots, n)$ coincide with the neutral axes of the segments $(BS_i)(i = 1, \ldots, n)$, respectively, and the coordinate planes $y_i z_i$ $(i = 1, \ldots, n)$ coincide with the plane of planar motion of the rigid bodies. Besides, $w_i(z_i,\, t)$ represents the transverse displacement in the y_i direction, $u_i(z_i,\, t)$ is the axial displacement in the z_i direction, and $\overrightarrow{i}_i$, $\overrightarrow{j}_i$, and $\overrightarrow{k}_i$ are the unit vectors of the axes x_i, y_i, and z_i, respectively.

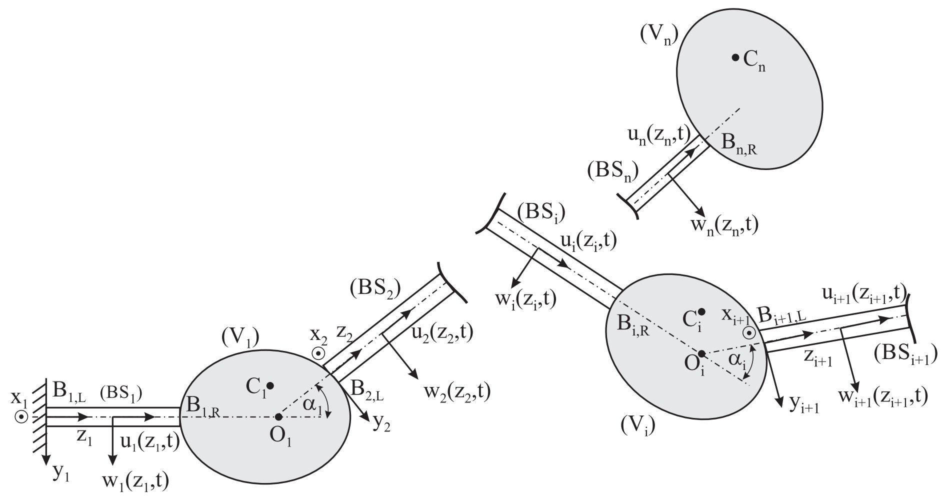

Figure 1: System of interconnected rigid bodies and elastic beam segments

In further considerations the following quantities will be used to describe the material and geometric characteristics of the segments (BS_i) $(i = 1, \ldots, n)$: E_i is the modulus of elasticity, $I_{x(i)}$ is the cross-sectional area moment of inertia about axis x_i passing through the center of the cross section, A_i is the cross-sectional area, L_i is the length of the ith beam segment, ϱ_i is the mass density. It is assumed that the beam segments are treated as the Euler-Bernoulli beams (rotary inertia and shear effects are ignored) [21] and that deformations $u_i(z_i, t)$ $(i = 1, \ldots, n)$ and $w_i(z_i, t)$ $(i = 1, \ldots, n)$ as well as rotations $w_i'(z_i, t)$ $(i = 1, \ldots, n)$ are small.

The partial differential equations for bending and axial vibrations of the beam segments $(BS_i)\,(i=1,\ldots,n)$ read [21]:

$$E_i I_{x(i)} w''''(z_i,t) + \varrho_i A_i \ddot{w}_i(z_i,t) = 0,\ i=1,\ldots,n, \tag{1}$$

$$\varrho_i A_i \ddot{u}_i(z_i,t) - E_i A_i u''(z_i,t) = 0,\ i=1,\ldots,n, \tag{2}$$

where primes denote differentiation with respect to z and dots with respect to time t.

Using the method of separation of variables, the displacements $w_i(z_i,t)$ and $u_i(z_i,t)$ can be written as

$$w_i(z_i,t) = W_i(z_i)T(t), \tag{3}$$

$$u_i(z_i,t) = U_i(z_i)T(t), \tag{4}$$

where $W_i(z_i)\,(i=1,\ldots,n)$ and $U_i(z_i)\,(i=1,\ldots,n)$ are the normal modes in free bending and axial vibrations, respectively. Based on Eqs. (3) and (4), Eqs. (1) and (2) can be rewritten as the following system of $2n+1$ ordinary differential equations

$$W_i''''(z_i) - k_i^4 W_i(z_i) = 0,\ i=1,\ldots,n, \tag{5}$$

$$U_i''(z_i) + p_i^2 U_i(z_i) = 0,\ i=1,\ldots,n, \tag{6}$$

$$\ddot{T}(t) + \omega^2 T(t) = 0, \tag{7}$$

where ω is the natural frequency of vibration of the entire system and

$$k_i^4 = \frac{\varrho_i A_i}{E_i I_{x(i)}}\omega^2, \qquad p_i^2 = \frac{\varrho_i}{E_i}\omega^2, \quad i = 1, \ldots, n. \tag{8}$$

From Eq. (8) it is obvious that the following relation can be established between the quantities k_i and p_i:

$$p_i = \sqrt{\frac{I_{x(i)}}{A_i}} k_i^2, \quad i = 1, \ldots, n. \tag{9}$$

Taking that $k_1 = k$ and $p_1 = \sqrt{I_{x(1)}/A_1} k^2$, from Eqs. (8) and (9) it follows that:

$$k_i = \sqrt[4]{\frac{E_1 I_{x(1)} \varrho_i A_i}{\varrho_1 A_1 E_i I_{x(i)}}} k, \qquad p_i = \sqrt{\frac{E_1 I_{x(1)} \varrho_i}{\varrho_1 A_1 E_i}} k^2, \quad i = 2, \ldots, n, \tag{10}$$

and

$$\omega = \sqrt{\frac{E_1 I_{x(1)}}{\varrho_1 A_1}} k^2. \tag{11}$$

The general solutions of Eqs. (5) and (6) can be expressed as [21]:

$$W_i(z_i) = C_{1(i)} \cos(k_i z_i) + C_{2(i)} \sin(k_i z_i) + C_{3(i)} \cosh(k_i z_i) + C_{4(i)} \sinh(k_i z_i), \ i = 1, \ldots, n, \tag{12}$$

$$U_i(z_i) = C_{5(i)} \cos(p_i z_i) + C_{6(i)} \sin(p_i z_i), \ i = 1, \ldots, n. \tag{13}$$

3. Boundary conditions

3.1. Boundary conditions at the left end of beam segment (BS_1)

Let the segment (BS_1) be clamped at the left end $B_{1,L}$. Based on this, the following boundary conditions hold:

$$w_1(0,\, t) = 0, \qquad w_1'(0,\, t) = 0, \qquad u_1(0,\, t) = 0, \tag{14}$$

which, taking into account Eqs. (3), (4), (12), and (13), can be written in the developed form as follows:

$$C_{1(1)} + C_{3(1)} = 0, \tag{15}$$

$$k_1 C_{2(1)} + k_1 C_{4(1)} = 0, \tag{16}$$

$$C_{5(1)} = 0. \tag{17}$$

Based on Eqs. (15), (16), and (17) the following matrix relation can be formed:

$$\mathbf{C}_1 = \mathbf{T}_0 \mathbf{C}_0 \tag{18}$$

where:

$$\mathbf{C}_1 = \left[C_{1(1)}\; C_{2(1)}\; \ldots\; C_{6(1)} \right]^T, \tag{19}$$

$$\mathbf{C}_0 = \left[C_{1(1)}\; C_{2(1)}\; C_{6(1)} \right]^T, \tag{20}$$

$$\mathbf{T}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

Note that, in the case of pinned left end $B_{1,L}$, the following boundary conditions hold:

$$w_1(0,\, t) = 0, \qquad w_1''(0,\, t) = 0, \qquad u_1(0,\, t) = 0, \tag{22}$$

and, in this case, the matrices $\mathbf{C}_0$ and $\mathbf{T}_0$ have the following components:

$$\mathbf{C}_0 = \left[C_{2(1)}\; C_{4(1)}\; C_{6(1)} \right]^T, \tag{23}$$

$$\mathbf{T}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

3.2. Boundary conditions influenced by the body (V_i)

In Fig. 2, the free-body diagram of the body (V_i) is shown where C_i^* and C_i^{**} represent the perpendicular projections of the mass center C_i to the directions $B_{i,R}O_i$ and $B_{i+1,L}O_i$, respectively.

In further considerations the following quantities will be used to describe the material and geometric characteristics of the rigid bodies (V_i) $(i = 1, \ldots, n)$: body mass m_i, mass moment of inertia about centroidal axis J_i, $\overline{B_{i,R}C_i^*} = e_i$, $\overline{C_i^{**}B_{i+1,L}} = a_i$, $\overline{C_iC_i^*} = d_i$, $\overline{C_iC_i^{**}} = b_i$, $\overline{O_iB_{i,R}} = \ell_{i(1)}$, $\overline{O_iB_{i+1,L}} = \ell_{i(2)}$. The slopes of the displacements at the ends $B_{i,R}$ and $B_{i+1,L}$ of the segments (BS_i) and (BS_{i+1}) equal the angle of rotation of the body (V_i) (see Fig. 2), that is,

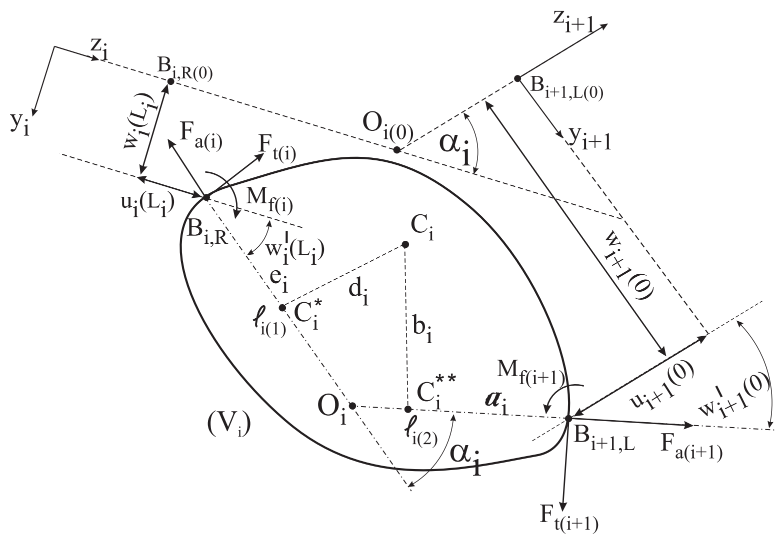

Figure 2: Free-body diagram of the body (V_i)

$$w_i'(L_i,\, t) = w_{i+1}'(0,\, t) \tag{25}$$

or in developed form:

$$k_i\left(-C_{1(i)}\sin k_iL_i + C_{2(i)}\cos k_iL_i + C_{3(i)}\sinh k_iL_i + C_{4(i)}\cosh k_iL_i\right)$$

$$= k_{i+1}\left(C_{2(i+1)} + C_{4(i+1)}\right). \tag{26}$$

Further, according to the assumption on small elastic deformations of beam segments, the displacement vector of point O_i determined based on the displacement of point $B_{i,R}$ and the slope $w_i'(L_i,\, t)$ reads:

$$\overrightarrow{(O_i)_0O_i} = \left(w_i(L_i,\, t) + \overline{B_{i,R}O}_i w_i'(L_i,\, t)\right)\vec{j}_i + u_i(L_i,\, t)\vec{k}_i. \tag{27}$$

Also, the displacement vector of point O_i can be expresed through the displacement of point $B_{i+1,L}$ and the slope $w_{i+1}'(0,\, t)$ as follows:

$$\overrightarrow{(O_i)_0 O_i} = \left(w_{i+1}(0,\, t) - \overline{B_{i+1,L}O}_i w'_{i+1}(0,\, t)\right) \overrightarrow{j}_{i+1} + u_{i+1}(0,\, t) \overrightarrow{k}_{i+1}. \quad (28)$$

Equating the expressions (27) and (28) and taking dot product of such obtained expression by the vectors $\overrightarrow{j}_i$ and $\overrightarrow{k}_i$, respectively, yields:

$$u_i(L_i,\, t) = u_{i+1}(0,\, t)\cos\alpha_i + \left[w_{i+1}(0,\, t) - \ell_{i(2)} w'_{i+1}(0,\, t)\right]\sin\alpha_i, \quad (29)$$

$$w_i(L_i,\, t) + \ell_{i(1)} w'_i(L_i,\, t) = -u_{i+1}(0,\, t)\sin\alpha_i + \left[w_{i+1}(0,\, t) - \ell_{i(2)} w'_{i+1}(0,\, t)\right]\cos\alpha_i, \quad (30)$$

or in developed form:

$$C_{5(i)}\cos p_i L_i + C_{6(i)}\sin p_i L_i = C_{5(i+1)}\cos\alpha_i$$

$$+ \left[C_{1(i+1)} + C_{3(i+1)} - \ell_{i(2)} k_{i+1}(C_{2(i+1)} + C_{4(i+1)})\right]\sin\alpha_i, \quad (31)$$

$$C_{1(i)}\cos k_i L_i + C_{2(i)}\sin k_i L_i + C_{3(i)}\cosh k_i L_i + C_{4(i)}\sinh k_i L_i$$

$$+\ell_{i(1)} k_i \left(-C_{1(i)}\sin k_i L_i + C_{2(i)}\cos k_i L_i + C_{3(i)}\sinh k_i L_i + C_{4(i)}\cosh k_i L_i\right)$$

$$= -C_{5(i+1)}\sin\alpha_i + \left[C_{1(i+1)} + C_{3(i+1)} - \ell_{i(2)} k_{i+1}\left(C_{2(i+1)} + C_{4(i+1)}\right)\right]\cos\alpha_i. \quad (32)$$

The angular acceleration and the acceleration of the mass center C_i of the body (V_i), respectively, read:

$$\varepsilon_i = \ddot{w}'_{i+1}(0,\, t) = -\omega^2 k_{i+1} \left(C_{2(i+1)} + C_{4(i+1)}\right) T(t), \tag{33}$$

$$\overrightarrow{a}_{C_i} = \overrightarrow{a}_{B_{i+1,L}} + \overrightarrow{\varepsilon}_i \times \overrightarrow{B_{i+1,L}C_i} \tag{34}$$

where $\overrightarrow{a}_{B_{i+1,L}}$ is the acceleration of point $B_{i+1,L}$ and $\overrightarrow{\varepsilon}_i = \varepsilon_i \overrightarrow{i}_{i+1}$. In Eq. (34), on account of the assumption about small deformations of the segments, the term $\overrightarrow{\omega}_i \times \overrightarrow{\omega}_i \times \overrightarrow{B_{i+1,L}C_i}$ representing normal acceleration of the mass center C_i is ignored. In that case $\overrightarrow{\omega}_i = \dot{w}'_{i+1}(0,t)\overrightarrow{i}_{i+1}$ is the vector of angular velocity of the body (V_i). Now, Newton-Euler differential equations of motion [22] of the body (V_i) read:

$$J_i\varepsilon_i = M_{f(i)} - M_{f(i+1)} + F_{t(i)}e_i + F_{a(i)}d_i + F_{t(i+1)}a_i - F_{a(i+1)}b_i, \tag{35}$$

$$m_i\left(\ddot{u}_{i+1}(0,t) + b_i\varepsilon_i\right) = F_{a(i+1)} - F_{a(i)}\cos\alpha_i + F_{t(i)}\sin\alpha_i, \tag{36}$$

$$m_i\left(\ddot{w}_{i+1}(0,t) - a_i\varepsilon_i\right) = F_{t(i+1)} - F_{a(i)}\sin\alpha_i - F_{t(i)}\cos\alpha_i, \tag{37}$$

where $F_{t(i)}$ and $F_{t(i+1)}$ are the shear forces of beam segments (BS_i) and (BS_{i+1}), respectively, defined as:

$$F_{t(i)} = -E_iI_{x(i)}w'''_i(L_i,t), \tag{38}$$

$$F_{t(i+1)} = -E_{i+1}I_{x(i+1)}w'''_{i+1}(0,t), \tag{39}$$

$F_{a(i)}$ and $F_{a(i+1)}$ are the axial forces of beam segments (BS_i) and (BS_{i+1}), respectively, defined as:

$$F_{a(i)} = E_i A_i u_i'(L_i, t), \tag{40}$$

$$F_{a(i+1)} = E_{i+1} A_{i+1} u_{i+1}'(0, t), \tag{41}$$

and, finally, $M_{f(i+1)}$ and $M_{f(i)}$ are the bending moments of beam segments (BS_i) and (BS_{i+1}), respectively, defined as:

$$M_{f(i)} = -E_i I_{x(i)} w_i''(L_i, t), \tag{42}$$

$$M_{f(i+1)} = -E_{i+1} I_{x(i+1)} w_{i+1}''(0, t). \tag{43}$$

Based on above relations, Eqs. (35)-(37) can be written in developed form as follows:

$$-\omega^2 J_i k_{i+1} \left(C_{2(i+1)} + C_{4(i+1)}\right) = -E_i I_{x(i)} k_i^2 \left[-C_{1(i)} \cos k_i L_i - C_{2(i)} \sin k_i L_i\right.$$

$$\left.+C_{3(i)} \cosh k_i L_i + C_{4(i)} \sinh k_i L_i\right] + E_{i+1} I_{x(i+1)} k_{i+1}^2 \left(-C_{1(i+1)} + C_{3(i+1)}\right)$$

$$-E_i I_{x(i)} e_i k_i^3 \left(C_{1(i)} \sin k_i L_i - C_{2(i)} \cos k_i L_i + C_{3(i)} \sinh k_i L_i + C_{4(i)} \cosh k_i L_i\right)$$

$$+E_i A_i d_i p_i \left(-C_{5(i)} \sin p_i L_i + C_{6(i)} \cos p_i L_i\right) - E_{i+1} I_{x(i+1)} a_i k_{i+1}^3 \left(-C_{2(i+1)} + C_{4(i+1)}\right)$$

$$-E_{i+1} A_{i+1} b_i p_{i+1} C_{6(i+1)}, \tag{44}$$

$$-m_i \omega^2 \left[C_{5(i+1)} + b_i k_{i+1} \left(C_{2(+1)} + C_{4(i+1)}\right)\right] = E_{i+1} A_{i+1} p_{i+1} C_{6(i+1)}$$

$$-E_i A_i p_i \left[-C_{5(i)} \sin p_i L_i + C_{6(i)} \cos p_i L_i\right] \cos\alpha_i$$

$$-E_i I_{x(i)} k_i^3 \left[C_{1(i)} \sin k_i L_i - C_{2(i)} \cos k_i L_i + C_{3(i)} \sinh k_i L_i + C_{4(i)} \cosh k_i L_i\right] \sin\alpha_i, \tag{45}$$

$$m_i \omega^2 \left[-C_{1(i+1)} - C_{3(i+1)} + a_i k_{i+1} \left(C_{2(i+1)} + C_{4(i+1)}\right)\right]$$

$$= E_{i+1} I_{x(i+1)} k_{i+1}^3 \left(C_{2(i+1)} - C_{4(i+1)}\right) + E_i A_i p_i \left(C_{5(i)} \sin p_i L_i - C_{6(i)} \cos p_i L_i\right) \sin\alpha_i$$

$$+E_i I_{x(i)} k_i^3 \left[C_{1(i)} \sin k_i L_i - C_{2(i)} \cos k_i L_i + C_{3(i)} \sinh k_i L_i + C_{4(i)} \cosh k_i L_i\right] \cos\alpha_i. \tag{46}$$

Equations (26), (31), (32), (44), (45), and (46) can be written in the matrix form as follows:

$$\mathbf{T}_{iL}\mathbf{C}_i = \mathbf{T}_{iR}\mathbf{C}_{i+1} \tag{47}$$

where $C_i = \left[C_{1(i)}\ C_{2(i)}\ \ldots C_{6(i)}\right]^T$, $C_{i+1} = \left[C_{1(i+1)}\ C_{2(i+1)}\ \ldots C_{6(i+1)}\right]^T$, and where components of the matrices $\mathbf{T}_{iL} \in R^{6\times6}$ and $\mathbf{T}_{iR} \in R^{6\times6}$ are given in Appendices A and B. Finally, based on Eq. (47), the following recurrence relation can be written:

$$\mathbf{C}_{i+1} = \mathbf{T}_i \mathbf{C}_i, \qquad i = 1, \ldots, n-1 \tag{48}$$

where $\mathbf{T}_i \in R^{6\times6}(i = 1, , \ldots, n-1)$ is the transfer matrix between the integration constants for beam segments BS_i and BS_{i+1} determined as:

$$\mathbf{T}_i = \mathbf{T}_{iR}^{-1}\mathbf{T}_{iL}, \;\; i = 1, \ldots, n-1. \tag{49}$$

After $n-1$ successive application of the recurrence relation (48) it is obtained:

$$\mathbf{C}_n = \mathbf{T}_{n-1}\mathbf{T}_{n-2}\cdots\mathbf{T}_1\mathbf{T}_0\mathbf{C}_0. \tag{50}$$

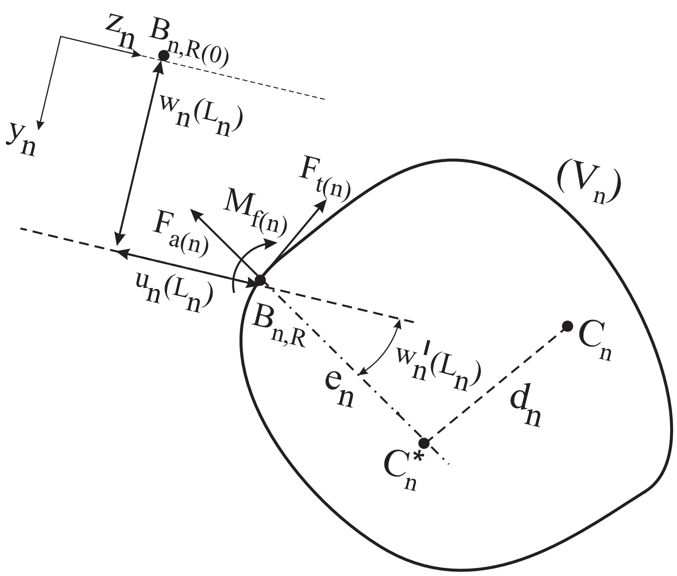

Figure 3: Free-body diagram of the body (V_n)

3.3. Boundary conditions at the right end of the beam segment (BS_n)

Now, let us assume that the rigid body (V_n) is attached to the end of elastic beam segment (BS_n). Based on the free-body diagram of body (V_n) (see Fig. 3), Newton-Euler differential equations of motion of the body (V_n) read:

$$J_n\varepsilon_n = M_{f(n)} + F_{t(n)}e_n + F_{a(n)}d_n, \tag{51}$$

$$m_n\left(\ddot{u}_n(L_n,t)+d_n\varepsilon_n\right)=-F_{a(n)}, \tag{52}$$

$$m_n\left(\ddot{w}_n(L_n,t)+e_n\varepsilon_n\right)=-F_{t(n)}, \tag{53}$$

where

$$\varepsilon_n=\ddot{w}_n'(L_n,\,t)=-\omega^2k_n\left(-C_{1(n)}\sin k_nL_n+C_{2(n)}\cos k_nL_n\right.$$

$$\left.+C_{3(n)}\sinh k_nL_n+C_{4(n)}\cosh k_nL_n\right)T(t). \tag{54}$$

Developed form of Eqs. (51)-(53) reads:

$$-\omega^2J_nk_n\left(-C_{1(n)}\sin k_nL_n+C_{2(n)}\cos k_nL_n+C_{3(n)}\sinh k_nL_n+C_{4(n)}\cosh k_nL_n\right)$$

$$=-E_nI_{x(n)}k_n^2\left(-C_{1(n)}\cos k_nL_n-C_{2(n)}\sin k_nL_n+C_{3(n)}\cosh k_nL_n+C_{4(n)}\sinh k_nL_n\right)$$

$$-E_nI_{x(n)}k_n^3e_n\left(C_{1(n)}\sin k_nL_n-C_{2(n)}\cos k_nL_n+C_{3(n)}\sinh k_nL_n+C_{4(n)}\cosh k_nL_n\right)$$

$$+E_nA_nd_np_n\left(-C_{5(n)}\sin p_nL_n+C_{6(n)}\cos p_nL_n\right), \tag{55}$$

$$m_n\omega^2\left[C_{5(n)}\cos p_nL_n+C_{6(n)}\sin p_nL_n+d_nk_n\left(-C_{1(n)}\sin k_nL_n+C_{2(n)}\cos k_nL_n\right.\right.$$

$$\left.\left.+C_{3(n)}\sinh k_nL_n+C_{4(n)}\cosh k_nL_n\right)\right]=E_nA_np_n\left(-C_{5(n)}\sin p_nL_n+C_{6(n)}\cos p_nL_n\right), \tag{56}$$

$$-m_n\omega^2\left[C_{1(n)}\cos k_nL_n+C_{2(n)}\sin k_nL_n+C_{3(n)}\cosh k_nL_n+C_{4(n)}\sinh k_nL_n\right.$$

$$+e_n k_n \left(-C_{1(n)} \sin k_n L_n + C_{2(n)} \cos k_n L_n + C_{3(n)} \sinh k_n L_n + C_{4(n)} \cosh k_n L_n\right)\Big]$$

$$= E_n I_{x(n)} k_n^3 \left(C_{1(n)} \sin k_n L_n - C_{2(n)} \cos k_n L_n + C_{3(n)} \sinh k_n L_n + C_{4(n)} \cosh k_n L_n\right). \tag{57}$$

The last three equations can be represented by the following matrix equation:

$$\mathbf{T}_n \mathbf{C}_n = \mathbf{0}_{3\times 1} \tag{58}$$

where the components of the matrix $\mathbf{T}_n \in R^{3\times 6}$ are given in Appendix C. For the case when the particle is attached to the end $B_{n,R}$ or the end $B_{n,R}$ is free, it should be taken in previous relations, respectively, that:

$$J_n = 0, \qquad e_n = 0, \qquad d_n = 0, \tag{59}$$

or

$$J_n = 0, \qquad e_n = 0, \qquad d_n = 0, \qquad m_n = 0. \tag{60}$$

Finally, in the case of pinned or clamped end $B_{n,R}$, instead of Newton-Euler equations (51)-(53) the following relations are used, respectively:

$$u_n(L_n, t) = 0, \qquad w_n(L_n, t) = 0, \qquad w_n''(L_n, t) = 0, \tag{61}$$

$$u_n(L_n, t) = 0, \qquad w_n(L_n, t) = 0, \qquad w_n'(L_n, t) = 0. \tag{62}$$

For these boundary conditions, the corresponding components of matrix $\mathbf{T}_n$ are given in Appendices D and E.

3.4. Frequency equation and mode shapes

Taking into account Eq. (50), it follows from Eq. (58) that:

$$\mathbf{T}\mathbf{C}_0 = \mathbf{0}_{3\times 1} \tag{63}$$

where $\mathbf{T} \in R^{3\times 3}$ represents overall transfer matrix determined by the following expression:

$$\mathbf{T} = \mathbf{T}_n \mathbf{T}_{n-1} \cdots \mathbf{T}_1 \mathbf{T}_0. \tag{64}$$

Equation (63) represents a matrix form of the homogeneous system of equations for unknown components of the matrix $\mathbf{C}_0$. In order that this system can have non-trivial solutions, it is needed to hold that:

$$\det \mathbf{T} = 0. \tag{65}$$

The last relation represents the frequency equation for the problem analyzed. Once the natural frequencies $\omega_i (i = 1, 2, 3, \ldots)$ are obtained, the components of the matrix $\mathbf{C}_0$ corresponding to each natural frequency one may obtain from Eq. (63). After that, applying Eq. (18) and the recurrence relation (48) yields the components of remaining matrices $\mathbf{C}_i (i = 1, \ldots, n)$. Finally, introducing values of the constants $C_{1(i)}, \ldots, C_{6(i)}$ $(i = 1, \ldots, n)$ into Eqs. (12) and (13), the mode shapes are determined.

Based on above considerations, it is now possible to explain what our modification of CTMM from reference[20] consists of. Namely, unlike the approach in [20], our approach uses straightaway boundary conditions at the end $B_{1,L}$, whereby in forming further relations, instead of the matrix $\mathbf{C}_1$ using the matrix $\mathbf{C}_0$ is achievable, which has two times smaller dimension than the matrix $\mathbf{C}_1$. This further implies that the dimension of overall transfer matrix $\mathbf{T}$, too, and therefore the coefficient determinant det $\mathbf{T}$ are reduced by two times as compared to the dimension that would be obtained by using standard CTMM from [20]. For the case of the systems considered in references [1, 2, 4, 7, 14, 16, 17], when only bending vibrations are considered and where $\alpha_1 = 0$, $d_1 = b_1 = 0$, and $C_1^* \equiv C_1^{**} \equiv O_1 = C_1$ holds, the application of our approach should be restricted only to Eqs. (25), (30), (35), (37), (51), and (53), and in this case we have that $\mathbf{T} \in R^{2\times 2}$.

4. Numerical examples

4.1. Free vibration analysis of a elastic beam carrying a rigid body

Figure 4 shows an elastic uniform hybrid beam composed of two elastic beam segments and one rigid body. The beam segments have circular cross sections. Vibrations characteristics of thus described system are studied in [7]. In [7] analysis was carried out on the effects of various boundary conditions as well as parameters of the rigid body such as dimensions, mass, mass moment of inertia, and mass center position along neutral axis of the undeformed beam segment on the values of the first three frequencies.

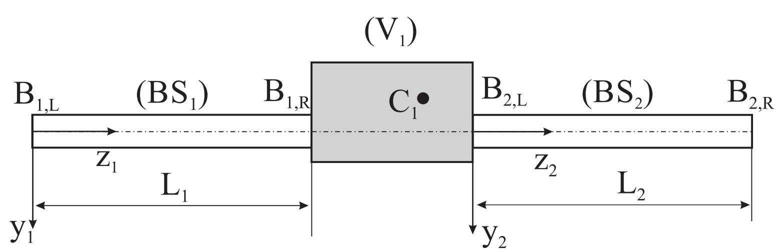

Figure 4: Elastic beam carrying a rigid body

The aim of this subsection is to show the effect of the positions of the mass center of the rigid body on the values of frequencies for the case when the mass center of the body is not located on the neutral axes of undeformed beam segments. For that purpose, the following values of the parameters of the system are used: Young's modulus $E_1 = E_2 = 2.069 \times 10^{11}\,\mathrm{N/m^2}$, mass density $\varrho_1 = \varrho_2 = 7.8367 \times 10^3\,\mathrm{kg/m^3}$, diameters of the beam segments $D_1 = D_2 = 0.05\,\mathrm{m}$, reference length $L_R = 2\,\mathrm{m}$, lengths of the beam segments $L_1 = 0.4L_R$ and $L_2 = 0.6L_R$, the rigid body mass $m_1 = 0.5\varrho_1 A_1 L_R$, moment of inertia $J_1 = 0.1\varrho_1 A_1 L_R^3$. Note that, for the considered example one has that: $C_1^\star \equiv C_1^{\star\star} \equiv O_1$, $d_1 = b_1$, $e_1 = \ell_{1(1)} = 0.2L_R$, $\alpha_1 = 0$, and $a_1 = \ell_{1(2)} = 0.1L_R$. For various values of the parameter d_1 and three type of boundary conditions (clamped-clamped, clamped-pinned, and clamped-free), the values of the first three dimensionless frequency coefficients $\lambda_r = k_r L_R = \sqrt[4]{\omega^2 \varrho_1 A_1 L_R^4/(E_1 I_{x(1)})}$ of the system are determined and presented in Table 1. Based on data from Table 1, it can be deduced that as the values of the parameter d_1 increase, the values of the first three dimensionless frequency coefficients decrease.

Table 1: The lowest three dimensionless frequency coefficients

d_1	Methods	clamped-clamped			clamped-pinned			clamped-free		
		λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.0	**Present**	3.49611	4.7166	8.25012	2.8207	4.7166	7.00126	1.42212	3.80242	4.72232
	Ref. [7]	3.4961	4.7166	8.2501	2.8207	4.7166	7.0013	1.4221	3.8024	4.7223
0.2	**Present**	3.48254	4.68626	8.24532	2.81093	4.68603	6.99522	1.41935	3.79667	4.68999
0.4	**Present**	3.44168	4.60772	8.23324	2.78216	4.60473	6.98017	1.41119	3.77885	4.60539
0.6	**Present**	3.37411	4.50858	8.21849	2.73616	4.49633	6.96207	1.39805	3.74765	4.49667

4.2. Free vibration analysis of a two-member open frame carrying two rigid bodies

Figure 5 shows a two-member open frame composed of two elastic beam segments (BS_1) and (BS_2) and two rigid bodies (V_1) and (V_2). The beam segment (BS_1) is horizontal, while the beam segment (BS_2) is inclined from the horizontal by angle α. The rigid bodies (V_1) and (V_2) represent, respectively, homogeneous thin circular and rectangular plates.

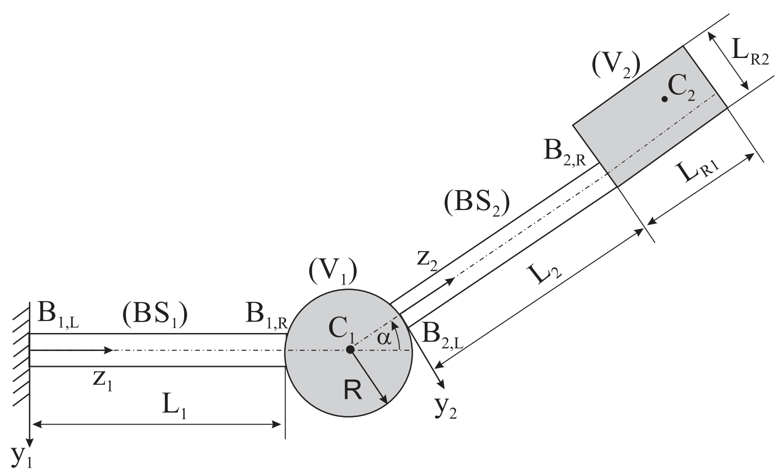

Figure 5: Two-member open frame with rigid bodies

The aim of this subsection is to show the effect of the value of angle α on the values of the first five frequencies of the system considered. A special case of this system are systems considered in references [23, 24] where there is not a body (V_1), and instead of the body (V_2) there is a particle. Furthermore, the same values are used for material and geometric characteristics of the beam segments as those of the previous subsection, whereas for the rigid bodies (V_1) and (V_2) one has that: $R = 0.14\,\text{m}$, $m_1 = 1\,\text{kg}$, $J_1 = m_1R^2/2$, $C_1 \equiv O_1$, $d_1 = b_1 = 0$, $e_1 = a_1 = \ell_{1(1)} = \ell_{1(2)} = R$, $m_2 = 5\,\text{kg}$, $L_{R1} = 0.7\,\text{m}$, $L_{R2} = 0.35\,\text{m}$, $J_2 = m_2(L_{R1}^2 + L_{R2}^2)/12$, $e_2 = 0.35\,\text{m}$, $d_2 = 0.15\,\text{m}$.

For certain values of the angle α, the values of the first five dimensionless frequency coefficients $\lambda_r = k_r L_R = \sqrt[4]{\omega^2 \varrho_1 A_1 L_R^4/(E_1 I_{x(1)})}$ of the considered open frame structure are determined and presented in Table 2 where $L_R = 2\,\mathrm{m}$.

The effect of the angle α on the dimensionless frequency coefficients is shown in Fig. 6. This effect is considered on the interval $[-5\pi/6,\, 5\pi/6]$. For $\alpha = \pi/3$, corresponding mode shapes are presented in Fig. 7. By observing Fig. 6 it can be concluded that in the neighborhood of $\alpha = 0$ the second, third, fourth and fifth dimensionless frequency coefficients achieve maximum value, while the first frequency coefficient achieves minimum value. In increasing and decreasing the angle α with respect to the value at which the second, third, fourth and fifth dimensionless frequency coefficients achieve the maximum value, the values of these frequency coefficients are decreased. This trend exists up to a certain value of the angle α and afterwards there is a regrowth of the values of these coefficients. On the other hand, the value of the first frequency coefficient is permanently growing. In the fourth frequency coefficient, the change of angle α in the interval $[-2,\, 2]$ causes very small changes of its values. Also, it is noticeable in Fig. 6 that the graphs of changes of the first, second, third and fourth frequency coefficients are roughly symmetric in character. The reason for this lies in the fact that for $d_2 = 0$ positive values of the angle α are corresponded by the system configurations that are, relative to the axis z_1, symmetrical to the system configurations corresponding to the negative values of the angle α. For $d_2 \neq 0$ this symmetry is disturbed as can be seen in Fig. 6.

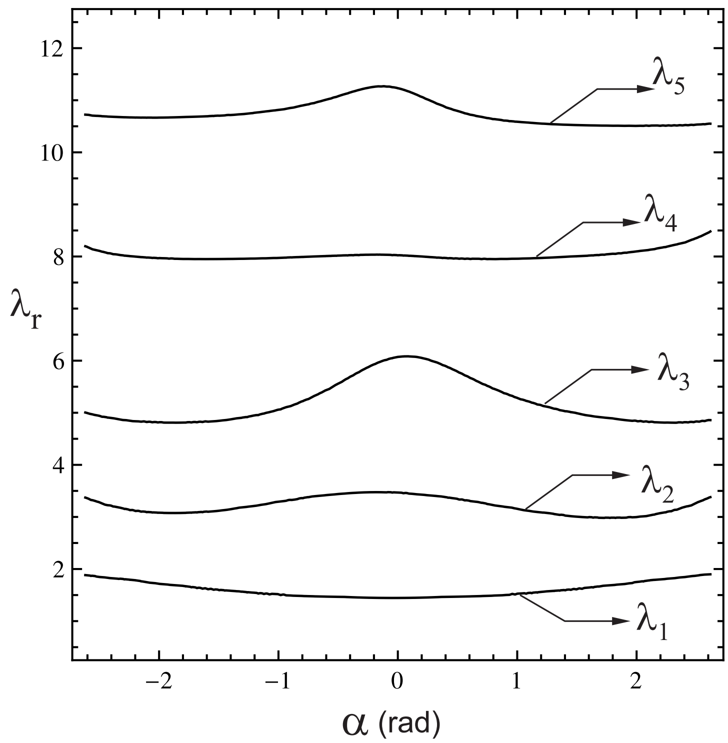

Figure 6: The effect of angle α on the lowest five dimensionless frequency coefficients

5. Conclusions

The paper considers free vibrations of structures composed of rigid bodies and elastic beam segments. It is assumed that the axes of undeformed beam segments are not collinear and that the mass centers of rigid bodies are not located on them. To meet the needs of analysis on the vibrations of considered system, CTMM modification from reference [20] has been done. Utilization of this modification allows for reducing the overall transfer matrix dimension by two times with respect to the dimension that would be obtained by using the original CTMM from [20]. The transfer matrix $\mathbf{T}$ can be easily formed by using some of the programming environments for symbolic calculations such as Mathematica, Maple, Maxima etc. Using the automated procedure, developed in this paper, the left side of Eq. (65) can be obtained in analytical form that provides the possibility to analyze dependencies of frequencies of any of the parameters that they depend on. The obtained results represent generalization of the results from references [7, 14, 16, 17]. Also, our approach provides the possibility to obtain the results that can be used for testing the accuracy of different numerical methods in the analysis of vibrations of various structures composed of elastic segments and rigid bodies. Note that considerations in this paper do not include the cases for parallel axes of adjacent beam segments, however these cases, too, can be treated by slight changes of relations (29) and (30) in forming matrices $\mathbf{T}_{iL}$ and $\mathbf{T}_{iR}$.

Acknowledgment This research was supported under Grant No. TR35006 by the Ministry of Education, Science and Technological Development of Serbia. This support is gratefully acknowledged.

Appendix A. COMPONENTS OF THE MATRIX $\mathbf{T}_{iL}$ $(i = 1, \ldots, n-1)$

$$T_{11(iL)} = -k_i \sin k_i L_i, \quad T_{12(iL)} = k_i \cos k_i L_i, \quad T_{13(iL)} = k_i \sinh k_i L_i, \tag{A.1}$$

$$T_{14(iL)} = k_i \cosh k_i L_i, \quad T_{15(iL)} = T_{16(iL)} = 0, \tag{A.2}$$

$$T_{21(iL)} = T_{22(iL)} = T_{23(iL)} = T_{24(iL)} = 0, \quad T_{25(iL)} = \cos p_i L_i, \quad T_{26(iL)} = \sin p_i L_i, \tag{A.3}$$

$$T_{31(iL)} = \cos k_i L_i - \ell_{i(1)} k_i \sin k_i L_i, \quad T_{32(iL)} = \sin k_i L_i + \ell_{i(1)} k_i \cos k_i L_i, \tag{A.4}$$

$$T_{33(iL)} = \cosh k_i L_i + \ell_{i(1)} k_i \sinh k_i L_i, \quad T_{34(iL)} = \sinh k_i L_i + \ell_{i(1)} k_i \cosh k_i L_i, \tag{A.5}$$

$$T_{35(iL)} = T_{36(iL)} = 0, \tag{A.6}$$

$$T_{41(iL)} = -I_{x(i)} k_i^2 \cos k_i L_i + I_{x(i)} e_i k_i^3 \sin k_i L_i, \tag{A.7}$$

$$T_{42(iL)} = -I_{x(i)} k_i^2 \sin k_i L_i - I_{x(i)} e_i k_i^3 \cos k_i L_i, \tag{A.8}$$

$$T_{43(iL)} = I_{x(i)} k_i^2 \cosh k_i L_i + I_{x(i)} e_i k_i^3 \sinh k_i L_i, \tag{A.9}$$

$$T_{44(iL)} = I_{x(i)} k_i^2 \sinh k_i L_i + I_{x(i)} e_i k_i^3 \cosh k_i L_i, \tag{A.10}$$

$$T_{45(iL)} = A_i d_i p_i \sin p_i L_i, \quad T_{46(iL)} = -A_i d_i p_i \cos p_i L_i, \tag{A.11}$$

$$T_{51(iL)} = I_{x(i)} k_i^3 \sin k_i L_i \sin \alpha_i, \quad T_{52(iL)} = -I_{x(i)} k_i^3 \cos k_i L_i \sin \alpha_i, \tag{A.12}$$

$$T_{53(iL)} = I_{x(i)} k_i^3 \sinh k_i L_i \sin \alpha_i, \quad T_{54(iL)} = I_{x(i)} k_i^3 \cosh k_i L_i \sin \alpha_i, \quad \text{(A.13)}$$

$$T_{55(iL)} = -A_i p_i \sin p_i L_i \cos \alpha_i, \quad T_{56(iL)} = A_i p_i \cos p_i L_i \cos \alpha_i, \quad \text{(A.14)}$$

$$T_{61(iL)} = -I_{x(i)} k_i^3 \sin k_i L_i \cos \alpha_i, \quad T_{62(iL)} = I_{x(i)} k_i^3 \cos k_i L_i \cos \alpha_i, \quad \text{(A.15)}$$

$$T_{63(iL)} = -I_{x(i)} k_i^3 \sinh k_i L_i \cos \alpha_i, \quad T_{64(iL)} = -I_{x(i)} k_i^3 \cosh k_i L_i \cos \alpha_i, \quad \text{(A.16)}$$

$$T_{65(iL)} = -A_i p_i \sin p_i L_i \sin \alpha_i, \quad T_{66(iL)} = A_i p_i \cos p_i L_i \sin \alpha_i. \quad \text{(A.17)}$$

Appendix B. COMPONENTS OF THE MATRIX $\mathbf{T}_{iR}$ $(i = 1, \ldots, n-1)$

$$T_{11(iR)} = T_{13(iR)} = T_{15(iR)} = T_{16(iR)} = 0, \quad T_{12(iR)} = T_{14(iR)} = k_{i+1}, \quad \text{(B.1)}$$

$$T_{21(iR)} = T_{23(iR)} = \sin \alpha_i, \quad T_{22(iR)} = T_{24(iR)} = -\ell_{i(2)} k_{i+1} \sin \alpha_i, \quad \text{(B.2)}$$

$$T_{25(iR)} = \cos \alpha_i, \quad T_{26(iR)} = 0, \quad T_{31(iR)} = T_{33(iR)} = \cos \alpha_i, \quad \text{(B.3)}$$

$$T_{32(iR)} = T_{34(iR)} = -\ell_{i(2)} k_{i+1} \cos\alpha_i, \quad T_{35(iR)} = -\sin\alpha_i, \quad T_{36(iR)} = 0, \tag{B.4}$$

$$T_{41(iR)} = -\frac{E_{i+1}}{E_i} I_{x(i+1)} k_{i+1}^2, \quad T_{42(iR)} = \frac{\omega^2 J_i k_{i+1}}{E_i} + \frac{E_{i+1}}{E_i} I_{x(i+1)} a_i k_{i+1}^3, \tag{B.5}$$

$$T_{43(iR)} = \frac{E_{i+1}}{E_i} I_{x(i+1)} k_{i+1}^2, \quad T_{44(iR)} = \frac{\omega^2 J_i k_{i+1}}{E_i} - \frac{E_{i+1}}{E_i} I_{x(i+1)} a_i k_{i+1}^3, \tag{B.6}$$

$$T_{45(iR)} = 0, \quad T_{46(iR)} = -\frac{E_{i+1}}{E_i} A_{i+1} b_i p_{i+1}, \quad T_{51(iR)} = T_{53(iR)} = 0, \tag{B.7}$$

$$T_{52(iR)} = T_{54(iR)} = \frac{m_i \omega^2 b_i k_{i+1}}{E_i}, \quad T_{55(iR)} = m_i \omega^2, \quad T_{56(iR)} = \frac{E_{i+1}}{E_i} A_{i+1} p_{i+1}, \tag{B.8}$$

$$T_{61(iR)} = T_{63(iR)} = \frac{m_i \omega^2}{E_i}, \quad T_{62(iR)} = -\frac{m_i \omega^2 a_i k_{i+1}}{E_i} + \frac{E_{i+1}}{E_i} I_{x(i+1)} k_{i+1}^3, \tag{B.9}$$

$$T_{64(iR)} = -\frac{m_i \omega^2 a_i k_{i+1}}{E_i} - \frac{E_{i+1}}{E_i} I_{x(i+1)} k_{i+1}^3, \quad T_{65(iR)} = T_{66(iR)} = 0. \tag{B.10}$$

Appendix C. COMPONENTS OF THE MATRIX $\mathbf{T}_n$ FOR THE CASE OF A RIGID BODY FIXED TO END $B_{n,R}$

$$T_{11(n)} = \left(\frac{\omega^2 J_n k_n}{E_n} + I_{x(n)} k_n^3 e_n \right) \sin k_n L_n - I_{x(n)} k_n^2 \cos k_n L_n, \tag{C.1}$$

$$T_{12(n)} = -\left(\frac{\omega^2 J_n k_n}{E_n} + I_{x(n)} k_n^3 e_n \right) \cos k_n L_n - I_{x(n)} k_n^2 \sin k_n L_n, \tag{C.2}$$

$$T_{13(n)} = \left(I_{x(n)} k_n^3 e_n - \frac{\omega^2 J_n k_n}{E_n} \right) \sinh k_n L_n + I_{x(n)} k_n^2 \cosh k_n L_n, \tag{C.3}$$

$$T_{14(n)} = \left(I_{x(n)} k_n^3 e_n - \frac{\omega^2 J_n k_n}{E_n} \right) \cosh k_n L_n + I_{x(n)} k_n^2 \sinh k_n L_n, \tag{C.4}$$

$$T_{15(n)} = A_n d_n p_n \sin p_n L_n, \quad T_{16(n)} = -A_n d_n p_n \cos p_n L_n, \tag{C.5}$$

$$T_{21(n)} = -\frac{m_n \omega^2 d_n k_n}{E_n} \sin k_n L_n, \quad T_{22(n)} = \frac{m_n \omega^2 d_n k_n}{E_n} \cos k_n L_n, \tag{C.6}$$

$$T_{23(n)} = \frac{m_n \omega^2 d_n k_n}{E_n} \sinh k_n L_n, \quad T_{24(n)} = \frac{m_n \omega^2 d_n k_n}{E_n} \cosh k_n L_n, \tag{C.7}$$

$$T_{25(n)} = \frac{m_n \omega^2}{E_n} \cos p_n L_n + A_n p_n \sin p_n L_n, \tag{C.8}$$

$$T_{26(n)} = \frac{m_n \omega^2}{E_n} \sin p_n L_n - A_n p_n \cos p_n L_n, \tag{C.9}$$

$$T_{31(n)} = -\frac{m_n\omega^2}{E_n}\cos k_nL_n + \left(\frac{m_n\omega^2 e_n k_n}{E_n} - I_{x(n)}k_n^3\right)\sin k_nL_n, \quad \text{(C.10)}$$

$$T_{32(n)} = -\frac{m_n\omega^2}{E_n}\sin k_nL_n - \left(\frac{m_n\omega^2 e_n k_n}{E_n} - I_{x(n)}k_n^3\right)\cos k_nL_n, \quad \text{(C.11)}$$

$$T_{33(n)} = -\frac{m_n\omega^2}{E_n}\cosh k_nL_n - \left(\frac{m_n\omega^2 e_n k_n}{E_n} + I_{x(n)}k_n^3\right)\sinh k_nL_n, \quad \text{(C.12)}$$

$$T_{34(n)} = -\frac{m_n\omega^2}{E_n}\sinh k_nL_n - \left(\frac{m_n\omega^2 e_n k_n}{E_n} + I_{x(n)}k_n^3\right)\cosh k_nL_n, \quad \text{(C.13)}$$

$$T_{35(n)} = T_{36(n)} = 0. \quad \text{(C.14)}$$

Appendix D. COMPONENTS OF THE MATRIX $\mathbf{T}_n$ FOR THE CASE OF PINNED END $B_{n,R}$

$$T_{11(n)} = T_{12(n)} = T_{13(n)} = T_{14(n)} = 0, \quad T_{15(n)} = \cos p_nL_n, \quad T_{16(n)} = \sin p_nL_n, \quad \text{(D.1)}$$

$$T_{21(n)} = \cos k_nL_n, \quad T_{22(n)} = \sin k_nL_n, \quad T_{23(n)} = \cosh k_nL_n, \quad \text{(D.2)}$$

$$T_{24(n)} = \sinh k_nL_n, \quad T_{25(n)} = T_{26(n)} = 0, \quad \text{(D.3)}$$

$$T_{31(n)} = -k_n^2 \cos k_n L_n, \quad T_{32(n)} = -k_n^2 \sin k_n L_n, \quad T_{33(n)} = k_n^2 \cosh k_n L_n, \tag{D.4}$$

$$T_{34(n)} = k_n^2 \sinh k_n L_n, \quad T_{35(n)} = T_{36(n)} = 0. \tag{D.5}$$

Appendix E. COMPONENTS OF THE MATRIX $\mathbf{T}_n$ FOR THE CASE OF CLAMPED END $B_{n,R}$

$$T_{11(n)} = T_{12(n)} = T_{13(n)} = T_{14(n)} = 0, \quad T_{15(n)} = \cos p_n L_n, \quad T_{16(n)} = \sin p_n L_n, \tag{E.1}$$

$$T_{21(n)} = \cos k_n L_n, \quad T_{22(n)} = \sin k_n L_n, \quad T_{23(n)} = \cosh k_n L_n, \tag{E.2}$$

$$T_{24(n)} = \sinh k_n L_n, \quad T_{25(n)} = T_{26(n)} = 0, \tag{E.3}$$

$$T_{31(n)} = -k_n \sin k_n L_n, \quad T_{32(n)} = k_n \cos k_n L_n, \quad T_{33(n)} = k_n \sinh k_n L_n, \tag{E.4}$$

$$T_{34(n)} = k_n \cosh k_n L_n, \quad T_{35(n)} = T_{36(n)} = 0. \tag{E.5}$$

References

[1] O. Kopmaz, S. Telli, On the eigenfrequencies of a two-part beam-mass system. Journal of Sound and Vibration 252 (2002) 370-384.

[2] S. Ilanko, Transcendental dynamic stability functions for beams carrying rigid bodies. Journal of Sound and Vibration 279 (2005) 1195-1202.

[3] S. Naguleswaran, Comments on "On the eigenfrequencies of a two-part beam-mass system". Journal of Sound and Vibration 265 (2003) 897-898.

[4] J. Banerjee, A.J. Sobey, Further investigation into the eigenfrequencies of a two-part beam-mass system. Journal of Sound and Vibration 265 (2003) 899-908.

[5] S. Ilanko, Comments on "On the eigenfrequencies of a two-part beam-mass system". Journal of Sound and Vibration 265 (2003) 909-910.

[6] O. Kopmaz, S. Telli, Authors' reply. Journal of Sound and Vibration 265 (2003) 911-916.

[7] S. Naguleswaran, Vibration of an Euler-Bernoulli stepped beam carrying a non-symmetrical rigid body at the step. Journal of Sound and Vibration 271 (2004) 1121-1132.

[8] B.R. Bhat, H. Wagner, Natural frequencies of a uniform cantilever with a tip mass slender in the axial direction. Journal of Sound and Vibration 45(2) (1976) 304-307.

[9] M. Gurgoze, H. Batan, A note on the vibrations of a restrained can-

tilever beam carrying a heavy tip body. Journal of Sound and Vibration 106(3) (1986) 533-536.

[10] W.H. Liu, C.C. Huang, Vibrations of a constrained beam carrying a heavy tip body. Journal of sound and Vibration 123 (1988) 15-29.

[11] S.A. Mousavi Lajimi, G.R. Heppler, Comments on "Natural frequencies of a uniform cantilever with a tip mass slender in the axial direction". Journal of Sound and Vibration 331 (2012) 2964-2968.

[12] D.C.D. Oguamanam, Free vibration of beams with finite mass rigid tip load and flexural–torsional coupling. International Journal of Mechanical Sciences 45 (2003) 963-979.

[13] H. Salarieh, M. Ghorashi, Free vibration of Timoshenko beam with finite mass rigid tip load and flexural–torsional coupling. International Journal of Mechanical Sciences 48 (2006) 763-779.

[14] H. Su, J.R. Banerjee, Exact natural frequencies of structures consisting of two-part beam-mass systems. Structural Engineering and Mechanics 19 (2005) 551-566.

[15] D.R. Trifković, Theoretical and experimental analysis of oscillations of systems with long shafts, PhD Thesis, University of Belgrade, Faculty of mechanical engineering, 2012.

[16] H.Y. Lin, C.Y. Wang, Free vibration analysis of a hybrid beam composed of multiple elastic beam segments and elastic-supported rigid bodies. Journal of Marine Science and Technology 20 (2012) 525-533.

[17] J.J. Wu, Use of the elastic-and-rigid-combined beam element for dynamic analysis of a two-dimensional frame with arbitrarily distributed

rigid beam segments. Applied Mathematical Modelling 35 (2011) 1240-1251.

[18] G.L. Anderson, Natural frequencies of two cantilevers joined by a rigid connector at their free ends. Journal of Sound and Vibration 57 (1978) 403-412.

[19] H.D. Kwon, Y.P. Park, Dynamic characteristics of stepped cantilever beams connected with a rigid body. Journal of Sound and Vibration 255 (2002) 701-717.

[20] J.S. Wu, C.T. Chen, A continuous-mass TMM for free vibration analysis of a non-uniform beam with various boundary conditions and carrying multiple concentrated elements. Journal of Sound and Vibration 311 (2008) 1420-1430.

[21] L. Meirovitch, Fundamentals of Vibrations, McGraw-Hill, New York, 2001.

[22] A.A. Shabana, Computational Dynamics, John Wiley & Sons, New York, 2010.

[23] D.C.D. Oguamanam, J.S. Hansen, G.R. Heppler, Vibration of arbitrarily oriented two-member open frames with tip mass. Journal of Sound and Vibration 209 (1998) 651-669.

[24] M. Gurgoze, On the dynamic analysis of a flexible L-shaped structure. Journal of Sound and Vibration 211 (1998) 683-688.

Table 2: The lowest five dimensionless frequency coefficients and corresponding values of the natural frequencies $\omega_i (rad/s)$ $(i = 1, \ldots, 5)$

α	λ_1	λ_2	λ_3	λ_4	λ_5
	(ω_1)	(ω_2)	(ω_3)	(ω_4)	(ω_5)
$-5\pi/6$	1.885	3.37409	5.00522	8.2001	10.7292
	(57.054)	(182.8)	(402.263)	(1079.697)	(1848.41)
$-2\pi/3$	1.75603	3.09807	4.83066	7.9815	10.6675
	(49.5139)	(154.115)	(374.694)	(1022.9)	(1827.21)
$-\pi/2$	1.62183	3.11272	4.84134	7.95696	10.6977
	(42.2352)	(155.576)	(376.353)	(1016.62)	(1837.57)
$-\pi/3$	1.52429	3.27755	5.03102	7.97492	10.8027
	(37.3077)	(172.489)	(406.421)	(1021.21)	(1873.82)
$-\pi/6$	1.469	3.44422	5.53163	8.01644	11.0629
	(34.6503)	(190.478)	(491.326)	(1031.87)	(1965.18)
0	1.45297	3.47102	6.07214	8.02761	11.2333
	(33.8982)	(193.454)	(592.035)	(1034.75)	(2026.18)
$\pi/6$	1.47487	3.34107	5.76586	7.96294	10.7787
	(34.9278)	(179.24)	(533.816)	(1018.15)	(1865.51)
$\pi/3$	1.53663	3.14453	5.25283	7.96652	10.5789
	(37.9142)	(158.772)	(443.048)	(1019.06)	(1796.986)
$\pi/2$	1.64121	3.0043	4.96526	8.01541	10.5252
	(43.2506)	(144.927)	(395.865)	(1031.61)	(1778.79)
$2\pi/3$	1.77972	3.03394	4.83305	8.12743	10.5144
	(50.8589)	(147.801)	(375.065)	(1060.65)	(1775.14)
$5\pi/6$	1.9015	3.38361	4.86668	8.48105	10.5532
	(58.0572)	(183.833)	(380.303)	(1154.95)	(1788.27)

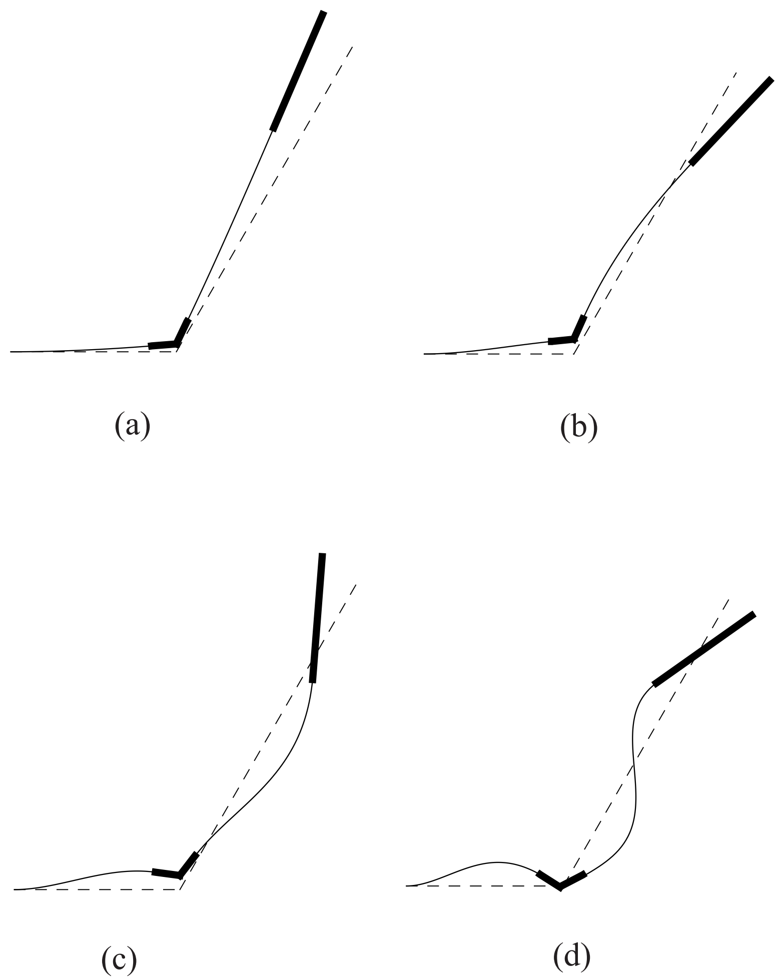

Figure 7: The lowest four mode shapes of the system shown in Fig. 5: (a) First mode, (b) Second mode, (c) Third mode, and (d) Fourth mode.