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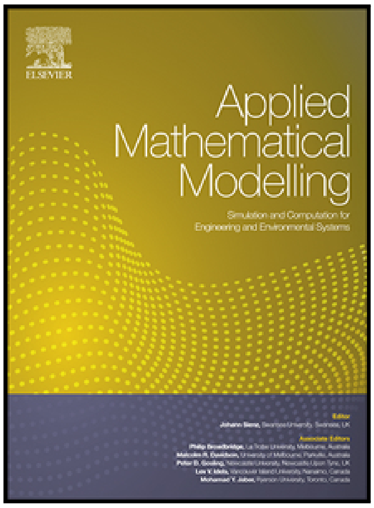

Closed-form solution for the free axial-bending vibration problem of structures composed of rigid bodies and elastic beam segments

Aleksandar Tomović, Slaviša Šalinić, Aleksandar Obradović, Aleksandar Grbović, Milorad Milovančević

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Highlights

- The coupled axial-bending vibration of planar serial frame structures is considered
- The mode shapes orthogonality conditions for planar serial frame structures are derived
- The system responses to initial excitations are analyzed
- The case of distinct as well as repeated natural frequencies are considered

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Closed-form solution for the free axial-bending vibration problem of structures composed of rigid bodies and elastic beam segments

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Abstract

In this paper, the coupled axial-bending vibration of planar serial frame structures composed of rigid bodies and Euler-Bernoulli beam segments is considered. The corresponding mode orthogonality conditions for this kind of structures are derived. It is assumed that the mass centers of rigid bodies have both the transverse and the axial eccentricity with respect to beam neutral axes and that the mass centers are located in the plane in which the rigid bodies perform planar motion. The system responses to initial excitation in the case of distinct as well as repeated natural frequencies are considered. Theoretical considerations are accompanied by four numerical examples.

Keywords: Axial-bending vibration, Initial excitation, Mode orthogonality condition, Planar frames, Crossing phenomenon

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1. Introduction

This paper represents a continuation of investigations initiated in our paper [1]. Namely, paper [1] employed a modified conventional continuous-mass transfer matrix method in forming the frequency equation for planar frame structures composed of rigid bodies and homogeneous elastic beam segments modeled as the Euler-Bernoulli beams. In addition, it was assumed that rigid bodies have both the transverse and the axial eccentricity of their mass centers with respect to neutral axes of the elastic beam segments.

Such systems can embrace both planar frame structures without attached masses [2–5] and planar frame structures carrying lumped masses and rigid bodies [6–9, 11–13]. For this type of structure it is important to determine not only the values of natural frequencies but also the responses of structures to various initial excitations. For that purpose, it is necessary to create the corresponding orthogonality conditions for mode shapes. Thus, in [2, 5] the mode orthogonality conditions were derived for planar frame structures without attached masses. On the other hand,

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the available mode shape orthogonality conditions in the literature regarding structures with attached masses mainly include only simpler cases of structures such as a beam with attached either end lumped masses or end rigid bodies with only axial eccentricity [14, 15]. An exception to this conclusion are papers [7–9]. Namely, the mode orthogonality condition in [7] is derived for an arbitrarily oriented two-member open frames with tip lumped mass, whereas in [8] for frame structures with lumped masses or rotatory inertias at the joints of beam segments. The boundary conditions derived in [9] hold for the system consisting of two parallel cantilever beams joined by a rigid body at the free ends of beams. Paper [10] should be also mentioned, where the mode orthogonality conditions were derived for a free-free beam with rigid bodies connected to the beam ends by torsion springs. But, the continuous system considered in [10] has completely different nature from the system considered in our paper. Note that herein the literature review does not consider the papers where vibration analysis of planar structures is based on discrete models obtained by using approximate methods (such as the finite element method, the assumed-modes method, the Rayleigh-Ritz method, etc.) because the orthogonality conditions for the modes of discrete systems are well-known. Taking into account above mentioned, the objective of this paper is to derive the orthogonality condition for mode shapes in the case of the general system considered in [1]. This orthogonality condition will comprise, as special cases, the existing orthogonality conditions in the literature. On the basis of the orthogonality condition derived, the procedures will be created for determining the system responses to initial excitation for both the case of distinct natural frequencies and the case of repeated natural frequencies.

2. Formulation of the problem

Paper [1] considered a multibody system, shown in Fig. 1, composed of rigid bodies $(V_i)(i = 1, \ldots, n)$ interconnected by homogeneous elastic beam segments $(BS_i)(i = 1, \ldots, n)$. Without loss of generality let the segment (BS_1) be clamped at the left end $B_{1,L}$. The elastic beam segments are positioned in a plane where rigid bodies are performing planar motion. Also, in Fig. 1 by C_i, α_i, and O_i are denoted, respectively, the mass center of body V_i, the angle made by the longitudinal axes of undeformed adjacent segments (BS_i) and (BS_{i+1}), the intersection point of the longitudinal axes of undeformed adjacent segments (BS_i) and (BS_{i+1}).

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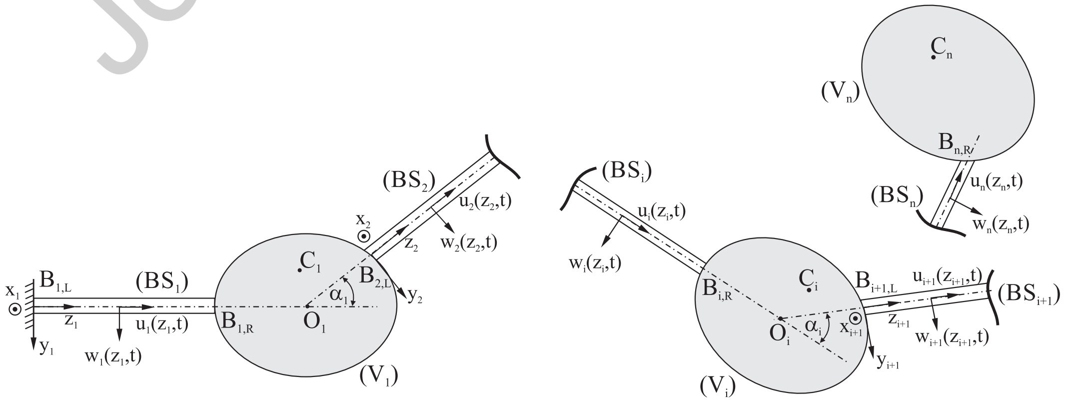

Figure 1: System of interconnected rigid bodies and elastic beam segments [1]

The bending and axial vibrations of the segments $(BS_i)(i = 1, \ldots, n)$ are described with the following partial differential equations [14–16]:

$$E_i I_{x(i)} w_i''''(z_i, t) + \varrho_i A_i \ddot{w}_i(z_i, t) = 0, \ i = 1, \ldots, n, \tag{1}$$

$$E_i A_i u_i''(z_i, t) - \varrho_i A_i \ddot{u}_i(z_i, t) = 0, \ i = 1, \ldots, n, \tag{2}$$

where the segments $(BS_i)(i = 1, \ldots, n)$ have the following material and geometric characteristics: E_i is the modulus of elasticity, $I_{x(i)}$ is the cross-sectional area moment of inertia about axis x_i passing through the center of the cross-section, A_i is the cross-sectional area, L_i is the length of the i-th beam segment, ϱ_i is the mass density.

Using the method of separation of variables [14–16], transverse and axial displacements of any point of the neutral axes of the beam segments, $w_i(z_i, t)(i = 1, \ldots, n)$ and $u_i(z_i, t)(i = 1, \ldots, n)$, can be written as:

$$w_i(z_i, t) = W_i(z_i) T(t), \tag{3}$$

$$u_i(z_i, t) = U_i(z_i) T(t), \tag{4}$$

where $W_i(z_i)(i = 1, \ldots, n)$ and $U_i(z_i)(i = 1, \ldots, n)$ are the mode shapes in free bending and axial vibrations, respectively, and $T(t)$ is a function of time t describing how the amplitudes of mode shapes varies with time t [14–16]. Now a system of $2n + 1$ ordinary differential equations can be formed as follows:

$$W_i''''(z_i) - k_i^4 W_i(z_i) = 0, \ i = 1, \ldots, n, \tag{5}$$

$$U_i''(z_i) + p_i^2 U_i(z_i) = 0, \ i = 1, \ldots, n, \tag{6}$$

$$\ddot{T}(t) + \omega^2 T(t) = 0, \tag{7}$$

where ω is the natural frequency of vibration of the entire system and

$$k_i^4 = \frac{\varrho_i A_i}{E_i I_{x(i)}} \omega^2, \qquad p_i^2 = \frac{\varrho_i}{E_i} \omega^2, \quad i = 1, \ldots, n. \tag{8}$$

Taking now $i = 1$ in Eq. (8) and putting:

$$k_1 = k, \tag{9}$$

yields:

$$\omega^2 = \frac{E_1 I_{x(1)}}{\varrho_1 A_1} k^4. \tag{10}$$

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Finally, substituting Eq. (10) into Eq. (8) yields:

$$k_i = \sqrt[4]{\frac{E_1 I_{x(1)} \varrho_i A_i}{\varrho_1 A_1 E_i I_{x(i)}}} k, \quad p_i = \sqrt{\frac{E_1 I_{x(1)} \varrho_i}{\varrho_1 A_1 E_i}} k^2, \quad i = 1, \ldots, n. \tag{11}$$

The general solutions of Eqs. (5)-(7) read (see [14–16] for details):

$$W_i(z_i) = C_{1(i)} \cos(k_i z_i) + C_{2(i)} \sin(k_i z_i) + C_{3(i)} \cosh(k_i z_i) + C_{4(i)} \sinh(k_i z_i), \; i = 1, \ldots, n, \tag{12}$$

$$U_i(z_i) = C_{5(i)} \cos(p_i z_i) + C_{6(i)} \sin(p_i z_i), \; i = 1, \ldots, n, \tag{13}$$

$$T(t) = K \cos(\omega t) + S \sin(\omega t), \tag{14}$$

where $C_{1(i)}, \ldots, C_{6(i)}$, K, and S are constants of integration.

Further considerations will employ boundary conditions derived in [1]. Namely, one has boundary conditions at the left end of the beam segment (BS_1)

$$w_1(0, t) = 0, \qquad w_1'(0, t) = 0, \qquad u_1(0, t) = 0, \tag{15}$$

boundary conditions influenced by the bodies $(V_i)(i = 1, \ldots, n-1)$ placed between adjacent beam segments (see Fig.2):

$$w_i'(L_i, t) = w_{i+1}'(0, t) \tag{16}$$

$$u_i(L_i, t) = u_{i+1}(0, t) \cos\alpha_i + \left[w_{i+1}(0, t) - \ell_{i(2)} w_{i+1}'(0, t)\right] \sin\alpha_i, \tag{17}$$

$$w_i(L_i, t) + \ell_{i(1)} w_i'(L_i, t) = -u_{i+1}(0, t) \sin\alpha_i + \left[w_{i+1}(0, t) - \ell_{i(2)} w_{i+1}'(0, t)\right] \cos\alpha_i, \tag{18}$$

$$J_i \varepsilon_i = M_{f(i)} - M_{f(i+1)} + F_{t(i)} e_i + F_{a(i)} d_i + F_{t(i+1)} a_i - F_{a(i+1)} b_i, \tag{19}$$

$$m_i \left(\ddot{u}_{i+1}(0, t) + b_i \varepsilon_i\right) = F_{a(i+1)} - F_{a(i)} \cos\alpha_i + F_{t(i)} \sin\alpha_i, \tag{20}$$

$$m_i \left(\ddot{w}_{i+1}(0, t) - a_i \varepsilon_i\right) = F_{t(i+1)} - F_{a(i)} \sin\alpha_i - F_{t(i)} \cos\alpha_i, \tag{21}$$

and boundary conditions influenced by the body (V_n) at the right end of the segment (BS_n):

$$J_n \varepsilon_n = M_{f(n)} + F_{t(n)} e_n + F_{a(n)} d_n, \tag{22}$$

$$m_n \left(\ddot{u}_n(L_n, t) + d_n \varepsilon_n\right) = -F_{a(n)}, \tag{23}$$

$$m_n \left(\ddot{w}_n(L_n, t) + e_n \varepsilon_n\right) = -F_{t(n)}, \tag{24}$$

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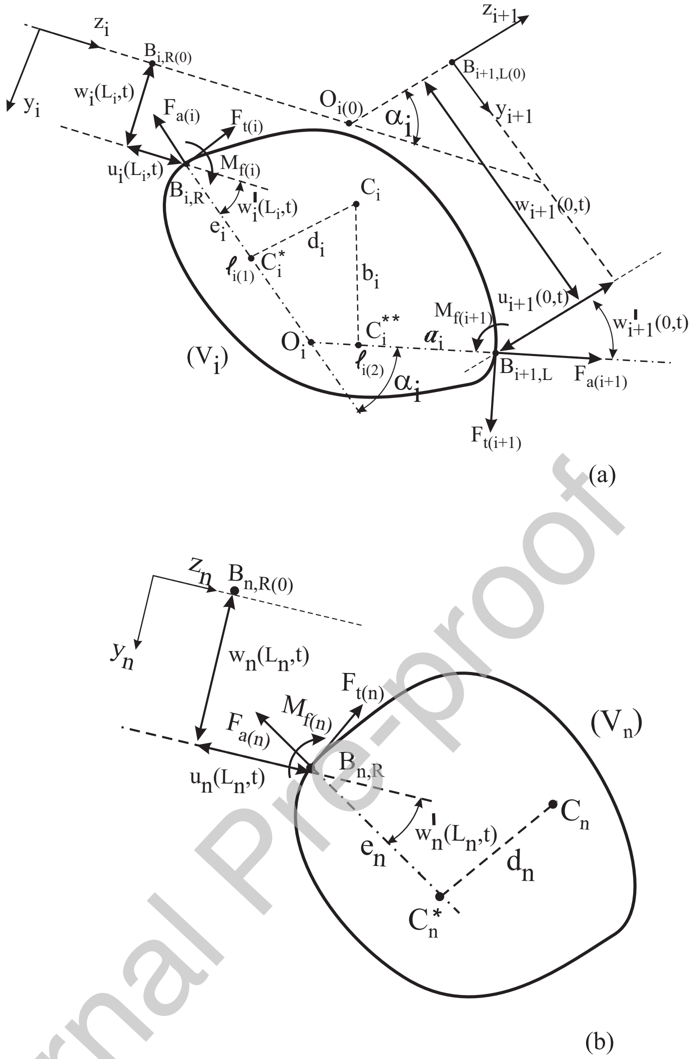

Figure 2: (a) Free-body diagram of the body (V_i) [1]; (b)Free-body diagram of the body (V_n) [1]

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where $\varepsilon_i = \ddot{w}_i'(L_i, t)$, m_i, and J_i are the angular acceleration, the mass, and the mass moment of inertia about centroidal axis of the body (V_i), respectively. Also, by C_i^* and C_i^{**} are denoted orthogonal projections of the mass center C_i onto the directions $O_iB_{i,R}$ and $O_iB_{i+1,L}$, respectively, where $\overline{B_{i,R}C_i^*} = e_i$, $\overline{C_i^{**}B_{i+1,L}} = a_i$, $\overline{C_iC_i^*} = d_i$, $\overline{C_iC_i^{**}} = b_i$, $\overline{O_iB_{i,R}} = \ell_{i(1)}$, and $\overline{O_iB_{i+1,L}} = \ell_{i(2)}$. Further, $F_{t(i)}$ and $F_{t(i+1)}$ represent the shear forces of beam segments (BS_i) and (BS_{i+1}), respectively, defined as:

$$F_{t(i)} = -E_iI_{x(i)}w_i'''(L_i, t), \quad F_{t(i+1)} = -E_{i+1}I_{x(i+1)}w_{i+1}'''(0, t) \tag{25}$$

$F_{a(i)}$ and $F_{a(i+1)}$ are the axial forces of beam segments (BS_i) and (BS_{i+1}), respectively, defined as:

$$F_{a(i)} = E_iA_iu_i'(L_i, t), \quad F_{a(i+1)} = E_{i+1}A_{i+1}u_{i+1}'(0, t) \tag{26}$$

and, finally, $M_{f(i)}$ and $M_{f(i+1)}$ are the bending moments of beam segments (BS_i) and (BS_{i+1}), respectively, defined as:

$$M_{f(i)} = -E_i I_{x(i)} w_i''(L_i, t), \quad M_{f(i+1)} = -E_{i+1} I_{x(i+1)} w_{i+1}''(0, t). \tag{27}$$

In accordance with determined boundary conditions, the following relations between the coefficients $C_{1(i)}, \ldots, C_{6(i)}$ can be established (see [1]):

$$\mathbf{C}_1 = \mathbf{T}_0 \mathbf{C}_0, \tag{28}$$

$$\mathbf{C}_{i+1} = \mathbf{T}_i \mathbf{C}_i, \quad i = 1, \ldots, n-1, \tag{29}$$

where $\mathbf{C}_0 = [C_{1(1)}\ C_{2(1)}\ C_{6(1)}]^T$, $\mathbf{C}_i = [C_{1(i)} \ldots C_{6(i)}]^T \quad (i = 1, \ldots, n)$,

$$\mathbf{T}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

and the entries of matrices $\mathbf{T}_1, \ldots, \mathbf{T}_n$ are given in [1]. From the previous relations it follows that:

$$\mathbf{T}\mathbf{C}_0 = \mathbf{0}_{3\times 1}, \tag{31}$$

where $\mathbf{T} = \mathbf{T}_n \mathbf{T}_{n-1} \cdots \mathbf{T}_1 \mathbf{T}_0$, and consequently the frequency equation:

$$\det \mathbf{T} = 0. \tag{32}$$

In [1] determination of natural frequencies and corresponding mode shapes was made, whereas in this paper the problem of determining the responses of the considering system to initial excitations will be solved. Namely, in the upcoming sections the orthogonality condition will be derived for mode shapes and the procedure for determining the coefficients K and S for the case of distinct frequencies as well as for the case of repeated frequencies will be presented. Note that this paper does not consider the combinations of boundary conditions (for example, the structure free at both ends) that lead to the occurrence of rigid-body vibration modes (the natural frequency is equal to zero).

3. Derivation of orthogonality conditions for mode shapes

For the needs of further considerations, based on Eqs. (3) and (4), the boundary conditions (15)-(24) can be written in the following form:

$$W_1(0) = 0, \quad W_1'(0) = 0, \quad U_1(0) = 0, \tag{33}$$

$$W_i'(L_i) = W_{i+1}'(0), \tag{34}$$

$$U_{i+1}(0) = U_i(L_i)\cos\alpha_i - (\ell_{i(1)} W_i'(L_i) + W_i(L_i))\sin\alpha_i, \tag{35}$$

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$$W_{i+1}(0) = \left(\ell_{i(1)}\cos\alpha_i + \ell_{i(2)}\right) W_i'(L_i) + U_i(L_i)\sin\alpha_i + W_i(L_i)\cos\alpha_i, \tag{36}$$

$$U_{i+1}'(0) = \frac{E_i I_{x(i)} W_i'''(L_i)\sin\alpha_i}{A_{i+1}E_{i+1}} + \frac{E_i A_i U_i'(L_i)\cos\alpha_i}{A_{i+1}E_{i+1}} + \frac{\omega^2 m_i W_i(L_i)\sin\alpha_i}{A_{i+1}E_{i+1}} + \frac{m_i\omega^2(e_i\sin\alpha_i - d_i\cos\alpha_i)W_i'(L_i)}{A_{i+1}E_{i+1}} - \frac{\omega^2 m_i U_i(L_i)\cos\alpha_i}{A_{i+1}E_{i+1}}, \tag{37}$$

$$W_{i+1}''(0) = \frac{E_i I_{x(i)}(\ell_{i(1)} + \ell_{i(2)}\cos\alpha_i)W_i'''(L_i)}{E_{i+1}I_{x(i+1)}} + \frac{E_i I_{x(i)} W_i''(L_i)}{E_{i+1}I_{x(i+1)}} - \frac{E_i A_i \ell_{i(2)} U_i'(L_i)\sin\alpha_i}{E_{i+1}I_{x(i+1)}} + \frac{\omega^2\left[-J_i - m_i d_i^2 + m_i e_i(\ell_{i(1)} - e_i) + m_i d_i \ell_{i(2)}\sin\alpha_i + m_i e_i \ell_{i(2)}\cos\alpha_i\right]W_i'(L_i)}{E_{i+1}I_{x(i+1)}} + \frac{m_i\omega^2(\ell_{i(2)}\sin\alpha_i - d_i)U_i(L_i)}{E_{i+1}I_{x(i+1)}} + \frac{m_i\omega^2(\ell_{i(1)} - e_i + \ell_{i(2)}\cos\alpha_i)W_i(L_i)}{E_{i+1}I_{x(i+1)}}, \tag{38}$$

$$W_{i+1}'''(0) = \frac{E_i I_{x(i)} W_i'''(L_i)\cos\alpha_i}{E_{i+1}I_{x(i+1)}} - \frac{E_i A_i U_i'(L_i)\sin\alpha_i}{E_{i+1}I_{x(i+1)}} + \frac{m_i\omega^2 U_i(L_i)\sin\alpha_i}{E_{i+1}I_{x(i+1)}} + \frac{m_i\omega^2(d_i\sin\alpha_i + e_i\cos\alpha_i)W_i'(L_i)}{E_{i+1}I_{x(i+1)}} + \frac{m_i\omega^2 W_i(L_i)\cos\alpha_i}{E_{i+1}I_{x(i+1)}}, \tag{39}$$

$$W_n''(L_n) = \frac{\omega^2\left[J_n + (d_n^2 + e_n^2)m_n\right]W_n'(L_n)}{E_n I_{x(n)}} + \frac{\omega^2 m_n d_n U_n(L_n)}{E_n I_{x(n)}} + \frac{\omega^2 m_n e_n W_n(L_n)}{E_n I_{x(n)}}, \tag{40}$$

$$W_n'''(L_n) = -\frac{\omega^2 e_n m_n W_n'(L_n)}{E_n I_{x(n)}} - \frac{\omega^2 m_n W_n(L_n)}{E_n I_{x(n)}}, \tag{41}$$

$$U_n'(L_n) = \frac{\omega^2 d_n m_n W_n'(L_n)}{E_n A_n} + \frac{\omega^2 m_n U_n(L_n)}{E_n A_n}. \tag{42}$$

where in the derivation of Eqs. (37)-(39) it is taken into account that $a_i = \ell_{i(1)}\cos\alpha_i + \ell_{i(2)} - e_i\cos\alpha_i - d_i\sin\alpha_i$ and $b_i = \ell_{i(1)}\sin\alpha_i + d_i\cos\alpha_i - e_i\sin\alpha_i$ (see Fig.2).

Using standard procedures for the determination of the orthogonality condition for mode shapes of a beam in axial vibration (see [14–16]) one obtains:

$$(\omega_\alpha^2 - \omega_\beta^2)\int_0^{L_i} \rho_i A_i U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i)dz_i = \left[E_i A_i U_{i(\beta)}'(z_i)U_{i(\alpha)}(z_i) - E_i A_i U_{i(\alpha)}'(z_i)U_{i(\beta)}(z_i)\right]\Big|_0^{L_i}. \tag{43}$$

Similarly, for a beam in bending vibration it stands (see [14–16]):

$$(\omega_\alpha^2 - \omega_\beta^2)\int_0^{L_i} \rho_i A_i W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)dz_i = \left[E_i I_{x(i)} W_{i(\alpha)}'''(z_i)W_{i(\beta)}(z_i) - E_i I_{x(i)} W_{i(\alpha)}''(z_i)W_{i(\beta)}'(z_i) - E_i I_{x(i)} W_{i(\beta)}'''(z_i)W_{i(\alpha)}(z_i) + E_i I_{x(i)} W_{i(\beta)}''(z_i)W_{i(\alpha)}'(z_i)\right]\Big|_0^{L_i}. \tag{44}$$

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Summing Eqs. (43) and (44) yields:

$$(\omega_\alpha^2-\omega_\beta^2)\sum_{i=1}^{n}\int_0^{L_i}\rho_i A_i\left[U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i)+W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)\right]dz_i=\sum_{i=1}^{n}\left[\mathbf{\Phi}_{i(\alpha)}^T(z_i)\mathbf{P}_i^{(1)}\mathbf{\Phi}_{i(\beta)}(z_i)\right]\Big|_0^{L_i}, \tag{45}$$

where

$$\mathbf{\Phi}_{i(r)}(z_i)=\left[W_{i(r)}(z_i)\ \ W'_{i(r)}(z_i)\ \ W''_{i(r)}(z_i)\ \ W'''_{i(r)}(z_i)\ \ U_{i(r)}(z_i)\ \ U'_{i(r)}(z_i)\right]^T,\quad r=\alpha,\beta, \tag{46}$$

$$\mathbf{P}_i^{(1)}=\begin{bmatrix}0&0&0&-E_iI_{x(i)}&0&0\\0&0&E_iI_{x(i)}&0&0&0\\0&-E_iI_{x(i)}&0&0&0&0\\E_iI_{x(i)}&0&0&0&0&0\\0&0&0&0&0&E_iA_i\\0&0&0&0&-E_iA_i&0\end{bmatrix}. \tag{47}$$

Further, one has that:

$$\sum_{i=1}^{n}\left[\mathbf{\Phi}_{i(\alpha)}^T(z_i)\mathbf{P}_i^{(1)}\mathbf{\Phi}_{i(\beta)}(z_i)\right]\Big|_0^{L_i}=-\mathbf{\Phi}_{1(\alpha)}^T(0)\mathbf{P}_1^{(1)}\mathbf{\Phi}_{1(\beta)}(0)$$

$$+\sum_{i=1}^{n-1}\left[\mathbf{\Phi}_{i(\alpha)}^T(L_i)\mathbf{P}_i^{(1)}\mathbf{\Phi}_{i(\beta)}(L_i)-\mathbf{\Phi}_{i+1(\alpha)}^T(0)\mathbf{P}_{i+1}^{(1)}\mathbf{\Phi}_{i+1(\beta)}(0)\right]+\mathbf{\Phi}_{n(\alpha)}^T(L_n)\mathbf{P}_n^{(1)}\mathbf{\Phi}_{n(\beta)}(L_n). \tag{48}$$

Based on the boundary conditions (33)-(42) it follows that

$$\mathbf{\Phi}_{1(\alpha)}^T(0)\mathbf{P}_1^{(1)}\mathbf{\Phi}_{1(\beta)}(0)=0, \tag{49}$$

$$\mathbf{\Phi}_{n(\alpha)}^T(L_n)\mathbf{P}_n^{(1)}\mathbf{\Phi}_{n(\beta)}(L_n)=$$

$$-(\omega_\alpha^2-\omega_\beta^2)\Big\{J_nW'_{n(\alpha)}(L_n)W'_{n(\beta)}(L_n)+m_n(d_n^2+e_n^2)W'_{n(\alpha)}(L_n)W'_{n(\beta)}(L_n)$$

$$+m_nd_n\left[W'_{n(\beta)}(L_n)U_{n(\alpha)}(L_n)+W'_{n(\alpha)}(L_n)U_{n(\beta)}(L_n)\right]+m_n\left[U_{n(\alpha)}(L_n)U_{n(\beta)}(L_n)+W_{n(\alpha)}(L_n)W_{n(\beta)}(L_n)\right]$$

$$+m_ne_n\left[W'_{n(\beta)}(L_n)W_{n(\alpha)}(L_n)+W'_{n(\alpha)}(L_n)W_{n(\beta)}(L_n)\right]\Big\}, \tag{50}$$

$$\mathbf{\Phi}_{i+1(\alpha)}(0)=\mathbf{P}_{i(\alpha)}^{(2)}\mathbf{\Phi}_{i(\alpha)}(L_i), \tag{51}$$

where entries of the matrix $\mathbf{P}_{i(\alpha)}^{(2)}$ are given in Appendix A. Now, from the above relations it can be obtained that:

$$\sum_{i=1}^{n-1}\left[\mathbf{\Phi}_{i(\alpha)}^T(L_i)\mathbf{P}_i^{(1)}\mathbf{\Phi}_{i(\beta)}(L_i)-\mathbf{\Phi}_{i+1(\alpha)}^T(0)\mathbf{P}_{i+1}^{(1)}\mathbf{\Phi}_{i+1(\beta)}(0)\right]=\sum_{i=1}^{n-1}\mathbf{\Phi}_{i(\alpha)}^T(L_i)\left[\mathbf{P}_i^{(1)}-\mathbf{P}_{i(\alpha)}^{(2)T}\mathbf{P}_{i+1}^{(1)}\mathbf{P}_{i(\beta)}^{(2)}\right]\mathbf{\Phi}_{i(\beta)}(L_i)=$$

$$= -(\omega_\alpha^2 - \omega_\beta^2)\sum_{i=1}^{n-1}\Big\{J_i W'_{i(\alpha)}(L_i)W'_{i(\beta)}(L_i) + m_i(d_i^2 + e_i^2)W'_{i(\alpha)}(L_i)W'_{i(\beta)}(L_i)$$

$$+m_i d_i\left[W'_{i(\beta)}(L_i)U_{i(\alpha)}(L_i) + W'_{i(\alpha)}(L_i)U_{i(\beta)}(L_i)\right] + m_i\left[U_{i(\alpha)}(L_i)U_{i(\beta)}(L_i) + W_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]$$

$$+m_i e_i\left[W'_{i(\beta)}(L_i)W_{i(\alpha)}(L_i) + W'_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]\Big\}. \quad (52)$$

Note that the conditions (49) are also valid for the pined left end ($U_{1(\alpha)}(0) = 0, W_{1(\alpha)}(0) = 0, W''_{1(\alpha)}(0) = 0$) of segment ($BS_1$). Also in the case when end $B_{n,R}$ is pinned ($U_{n(\alpha)}(L_n) = 0$, $W_{n(\alpha)}(L_n) = 0$, $W''_{n(\alpha)}(L_n) = 0$) or clamped ($U_{n(\alpha)}(L_n) = 0$, $W_{n(\alpha)}(L_n) = 0$, $W'_{n(\alpha)}(L_n) = 0$) , the condition (50) is reduced to:

$$\mathbf{\Phi}^T_{n(\alpha)}(L_n)\mathbf{P}_n^{(1)}\mathbf{\Phi}_{n(\beta)}(L_n) = 0. \quad (53)$$

Finally, Eq.(45) becomes:

$$(\omega_\alpha^2 - \omega_\beta^2)\sum_{i=1}^{n}\Big\{\int_0^{L_i}\rho_i A_i\left[U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i) + W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)\right]dz_i$$

$$+J_i W'_{i(\alpha)}(L_i)W'_{i(\beta)}(L_i) + m_i(d_i^2 + e_i^2)W'_{i(\alpha)}(L_i)W'_{i(\beta)}(L_i)$$

$$+m_i d_i\left[W'_{i(\beta)}(L_i)U_{i(\alpha)}(L_i) + W'_{i(\alpha)}(L_i)U_{i(\beta)}(L_i)\right] + m_i\left[U_{i(\alpha)}(L_i)U_{i(\beta)}(L_i) + W_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]$$

$$+m_i e_i\left[W'_{i(\beta)}(L_i)W_{i(\alpha)}(L_i) + W'_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]\Big\} = 0, \quad (54)$$

so that, for $\omega_\alpha^2 \neq \omega_\beta^2$, Eq. (54) can be satisfied only if

$$\sum_{i=1}^{n}\Big\{\int_0^{L_i}\rho_i A_i\left[U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i) + W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)\right]dz_i$$

$$+J_i W'_{i(\alpha)}(L_i)W'_{i(\beta)}(L_i) + m_i(d_i^2 + e_i^2)W'_{i(\alpha)}(L_i)W'_{i(\beta)}(L_i)$$

$$+m_i d_i\left[W'_{i(\beta)}(L_i)U_{i(\alpha)}(L_i) + W'_{i(\alpha)}(L_i)U_{i(\beta)}(L_i)\right] + m_i\left[U_{i(\alpha)}(L_i)U_{i(\beta)}(L_i) + W_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]$$

$$+m_i e_i\left[W'_{i(\beta)}(L_i)W_{i(\alpha)}(L_i) + W'_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]\Big\} = 0. \quad (55)$$

The relation (55) represents the orthogonality condition for mode shapes. Note that this orthogonality condition does not depend on angles $\alpha_i (i = 1, \ldots, n-1)$. When the considered system does not contain rigid bodies ($m_i = 0$, $J_i = 0$, $d_i = e_i = 0$), or if the rigid bodies are massless ($m_i = 0$, $J_i = 0$, $d_i \neq 0$, $e_i \neq 0$), then the condition (55) reduces to:

$$\sum_{i=1}^{n}\int_0^{L_i}\rho_i A_i\left[U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i) + W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)\right]dz_i = 0. \quad (56)$$

The orthogonality condition (56) is characteristic of the planar frame structures considered in [2–5]. If the rigid bodies are replaced with point masses ($J_i = 0$, $d_i = e_i = 0$), then one has:

$$\sum_{i=1}^{n}\Big\{\int_0^{L_i}\rho_i A_i\left[U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i) + W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)\right]dz_i$$

$$+m_i\left[U_{i(\alpha)}(L_i)U_{i(\beta)}(L_i)+W_{i(\alpha)}(L_i)W_{i(\beta)}(L_i)\right]\Big\}=0. \tag{57}$$

Finally, if only the rigid body (V_n) is taken into account, then it holds that:

$$\sum_{i=1}^{n}\int_0^{L_i}\rho_i A_i\left[U_{i(\alpha)}(z_i)U_{i(\beta)}(z_i)+W_{i(\alpha)}(z_i)W_{i(\beta)}(z_i)\right]dz_i$$

$$+J_nW'_{n(\alpha)}(L_n)W'_{n(\beta)}(L_n)+m_n(d_n^2+e_n^2)W'_{n(\alpha)}(L_n)W'_{n(\beta)}(L_n)$$

$$+m_nd_n\left[W'_{n(\beta)}(L_n)U_{n(\alpha)}(L_n)+W'_{n(\alpha)}(L_n)U_{n(\beta)}(L_n)\right]+m_n\left[U_{n(\alpha)}(L_n)U_{n(\beta)}(L_n)+W_{n(\alpha)}(L_n)W_{n(\beta)}(L_n)\right]$$

$$+m_ne_n\left[W'_{n(\beta)}(L_n)W_{n(\alpha)}(L_n)+W'_{n(\alpha)}(L_n)W_{n(\beta)}(L_n)\right]=0. \tag{58}$$

Note that the relations (56)-(58) represents only some of the special cases contained in the orthogonality condition (55).

4. Response to initial excitations

In this section, the coefficients K_α and S_α will be determined in the following expressions:

$$w_i(z_i,\,t)=\sum_{\alpha=1}^{\infty}W_{i(\alpha)}(z_i)\left(K_\alpha\cos(\omega_\alpha t)+S_\alpha\sin(\omega_\alpha t)\right),\quad i=1,\ldots,n, \tag{59}$$

$$u_i(z_i,\,t)=\sum_{\alpha=1}^{\infty}U_{i(\alpha)}(z_i)\left(K_\alpha\cos(\omega_\alpha t)+S_\alpha\sin(\omega_\alpha t)\right)\quad i=1,\ldots,n. \tag{60}$$

These expressions describe the system response to initial excitations. Note that in forthcoming considerations the Greek indices take the values $1,\ldots,\infty$.

4.1. The case of distinct natural frequencies

Assume that ω_α is not a repeated natural frequency. After introducing ω_α into (31) one has that rank $\mathbf{T}=2$ which means that the equation system (31) contains only two independent equations. Without loss of generality, take that those are the first two equations. Solving these equations for $C^{(\alpha)}_{6(1)}$ and $C^{(\alpha)}_{2(1)}$ in term of $C^{(\alpha)}_{1(1)}$ it is obtained that:

$$C^{(\alpha)}_{6(1)}=c^{(\alpha)}_{6(1)}C^{(\alpha)}_{1(1)},\qquad C^{(\alpha)}_{2(1)}=c^{(\alpha)}_{2(1)}C^{(\alpha)}_{1(1)} \tag{61}$$

where $C^{(\alpha)}_{1(1)}$ is free and where $c^{(\alpha)}_{6(1)}$ and $c^{(\alpha)}_{2(1)}$ are known constants. Now, using the relations (28) and (29) yields:

$$C^{(\alpha)}_{j(i)}=c^{(\alpha)}_{j(i)}C^{(\alpha)}_{1(1)},\quad i=1,\ldots,n;\ j=1,\ldots,6 \tag{62}$$

where $c^{(\alpha)}_{j(i)}$ are known constants determined by:

$$\left[c^{(\alpha)}_{1(1)}\ldots\ c^{(\alpha)}_{6(1)}\right]^T=\left[1\ c^{(\alpha)}_{2(1)}\ -1\ -c^{(\alpha)}_{2(1)}\ 0\ c^{(\alpha)}_{6(1)}\right]^T, \tag{63}$$

$$\left[c_{1(\ell)}^{(\alpha)} \dots c_{6(\ell)}^{(\alpha)}\right]^T = \mathbf{T}_{\ell-1}\mathbf{T}_{\ell-2}\cdots\mathbf{T}_1\left[1 \; c_{2(1)}^{(\alpha)} \; -1 \; -c_{2(1)}^{(\alpha)} \; 0 \; c_{6(1)}^{(\alpha)}\right]^T, \; \ell = 2,\dots,n. \tag{64}$$

Inserting (62) into (12) and (13) gives

$$W_{i(\alpha)}(z_i) = C_{1(1)}^{(\alpha)} W_{i(\alpha)}^*(z_i), \qquad U_{i(\alpha)}(z_i) = C_{1(1)}^{(\alpha)} U_{i(\alpha)}^*(z_i) \tag{65}$$

where

$$W_{i(\alpha)}^*(z_i) = c_{1(i)}^{(\alpha)} \cos(k_{i(\alpha)}z_i) + c_{2(i)}^{(\alpha)} \sin(k_{i(\alpha)}z_i) + c_{3(i)}^{(\alpha)} \cosh(k_{i(\alpha)}z_i) + c_{4(i)}^{(\alpha)} \sinh(k_{i(\alpha)}z_i), \tag{66}$$

$$U_{i(\alpha)}^*(z_i) = c_{5(i)}^{(\alpha)} \cos(p_{i(\alpha)}z_i) + c_{6(i)}^{(\alpha)} \sin(p_{i(\alpha)}z_i). \tag{67}$$

Similarly as in [15], applying $\alpha = \beta$ in Eq.(54) produces that the value of the sum may be arbitrarily chosen and thus the coefficient $C_{1(1)}^{(\alpha)}$ can be determined by taking that

$$C_{1(1)}^{(\alpha)2} \sum_{i=1}^{n} \left\{ \int_0^{L_i} \varrho_i A_i \left(U_{i(\alpha)}^{*2}(z_i) + W_{i(\alpha)}^{*2}(z_i)\right) dz_i + W_{i(\alpha)}^{*\prime 2}(L_i)\left[J_i + m_i(d_i^2 + e_i^2)\right]\right.$$

$$\left. +2m_i e_i W_{i(\alpha)}^*(L_i) W_{i(\alpha)}^{*\prime}(L_i) + 2m_i d_i W_{i(\alpha)}^{*\prime}(L_i) U_{i(\alpha)}^*(L_i) + m_i\left(U_{i(\alpha)}^{*2}(L_i) + W_{i(\alpha)}^{*2}(L_i)\right)\right\} = 1. \tag{68}$$

Let the initial excitation of the system be given in the form of the initial displacement and initial velocity functions as follows:

$$f_{wi}(z_i) \equiv w_i(z_i, 0) = \sum_{\alpha=1}^{\infty} W_{i(\alpha)}(z_i) K_\alpha, \quad i = 1,\dots,n, \tag{69}$$

$$f_{ui}(z_i) \equiv u_i(z_i, 0) = \sum_{\alpha=1}^{\infty} U_{i(\alpha)}(z_i) K_\alpha, \quad i = 1,\dots,n, \tag{70}$$

$$h_{wi}(z_i) \equiv \dot{w}_i(z_i, 0) = \sum_{\alpha=1}^{\infty} W_{i(\alpha)}(z_i) \omega_\alpha S_\alpha, \quad i = 1,\dots,n, \tag{71}$$

$$h_{ui}(z_i) \equiv \dot{u}_i(z_i, 0) = \sum_{\alpha=1}^{\infty} U_{i(\alpha)}(z_i) \omega_\alpha S_\alpha, \quad i = 1,\dots,n, \tag{72}$$

where the partial derivatives of the functions (69) and (71) with respect to z_i read:

$$f'_{wi}(z_i) \equiv w'_i(z_i, 0) = \sum_{\alpha=1}^{\infty} W'_{i(\alpha)}(z_i) K_\alpha, \quad i = 1,\dots,n, \tag{73}$$

$$h'_{wi}(z_i) \equiv \dot{w}'_i(z_i, 0) = \sum_{\alpha=1}^{\infty} W'_{i(\alpha)}(z_i) \omega_\alpha S_\alpha, \quad i = 1,\dots,n. \tag{74}$$

Journal Pre-proof

Multiplying both sides of Eq. (69) by $\varrho_i A_i W_{i(\beta)}(z_i)$ and Eq. (70) by $\varrho_i A_i U_{i(\beta)}(z_i)$ and integrating over the lengths of the beam segments, multiplying both sides of Eq.(73) evaluated at $z_i = L_i$ by $J_i W'_{i(\beta)}(L_i)$ and by $m_i(d_i^2 + e_i^2)W'_{i(\beta)}(L_i)$, multiplying both sides of Eq.(70) evaluated at $z_i = L_i$ by $m_i d_i W'_{i(\beta)}(L_i)$ and by $m_i U_{i(\beta)}(L_i)$, multiplying both sides of Eq. (73) by $m_i d_i U_{i(\beta)}(L_i)$ and by $m_i e_i W_{i(\beta)}(L_i)$, multiplying both sides of Eq. (69) evaluated at $z_i = L_i$ by $m_i W_{i(\beta)}(L_i)$ and by $m_i e_i W'_{i(\beta)}(L_i)$, adding the results obtained for all beam segments and taking into account the orthogonality condition (55) and Eq. (68) yields:

$$K_\alpha = C_{1(1)}^{(\alpha)} \sum_{i=1}^{n} \left\{ \int_0^{L_i} \varrho_i A_i \left(f_{wi}(z_i) W^*_{i(\alpha)}(z_i) + f_{ui}(z_i) U^*_{i(\alpha)}(z_i) \right) dz_i \right.$$

$$+ m_i d_i \left(f_{ui}(L_i) W^{*\prime}_{i(\alpha)}(L_i) + f'_{wi}(L_i) U^*_{i(\alpha)}(L_i) \right) + m_i e_i \left(f'_{wi}(L_i) W^*_{i(\alpha)}(L_i) + f_{wi}(L_i) W^{*\prime}_{i(\alpha)}(L_i) \right)$$

$$\left. + \left[J_i + m_i(d_i^2 + e_i^2) \right] f'_{wi}(L_i) W^{*\prime}_{i(\alpha)}(L_i) + m_i \left(f_{ui}(L_i) U^*_{i(\alpha)}(L_i) + f_{wi}(L_i) W^*_{i(\alpha)}(L_i) \right) \right\}, \quad \alpha = 1, \ldots, \infty. \tag{75}$$

Also, applying similar steps as in the case of determination of the coefficients K_α to Eqs. (71), (72), and (74) produces:

$$S_\alpha = \frac{C_{1(1)}^{(\alpha)}}{\omega_\alpha} \sum_{i=1}^{n} \left\{ \int_0^{L_i} \varrho_i A_i \left(h_{wi}(z_i) W^*_{i(\alpha)}(z_i) + h_{ui}(z_i) U^*_{i(\alpha)}(z_i) \right) dz_i \right.$$

$$+ m_i d_i \left(h_{ui}(L_i) W^{*\prime}_{i(\alpha)}(L_i) + h'_{wi}(L_i) U^*_{i(\alpha)}(L_i) \right) + m_i e_i \left(h'_{wi}(L_i) W^*_{i(\alpha)}(L_i) + h_{wi}(L_i) W^{*\prime}_{i(\alpha)}(L_i) \right)$$

$$\left. + \left[J_i + m_i(d_i^2 + e_i^2) \right] h'_{wi}(L_i) W^{*\prime}_{i(\alpha)}(L_i) + m_i \left(h_{ui}(L_i) U^*_{i(\alpha)}(L_i) + h_{wi}(L_i) W^*_{i(\alpha)}(L_i) \right) \right\}, \quad \alpha = 1, \ldots, \infty. \tag{76}$$

Finally, introducing Eqs. (75) and (76) into Eqs. (59) and (60), the response to the initial excitation (69)-(72) is completely determined.

4.2. The case of repeated natural frequencies

The occurrence of repeated (coincident) frequencies in distributed-parameter systems (continuous systems) is known as the crossing phenomenon [3, 17–19]. This phenomenon may occur for some combinations of values of the physical and geometrical parameters of a continuous system. The experimental analysis of the crossing phenomenon was shown in [18, 19], whereas their occurrence in the frame structures was analyzed in [3, 17], and in the case of rotating flexible structures in [20].

Let us assume that ω_γ is a repeated natural frequency such that $\omega_\gamma = \omega_{\gamma+1}$. For this case the rank of the matrix $\mathbf{T}$ will be 1. This means that in the system of equations (31) only one equation

Journal Pre-proof

is independent. Without loss of generality, let it be the first equation. Taking $C^{(\gamma)}_{1(1)}$ and $C^{(\gamma)}_{2(1)}$ as free coefficients and solving the chosen equation for $C^{(\gamma)}_{6(1)}$ yields:

$$C^{(\gamma)}_{6(1)} = c^{(\gamma)}_{6(1),1} C^{(\gamma)}_{1(1)} + c^{(\gamma)}_{6(1),2} C^{(\gamma)}_{2(1)}, \tag{77}$$

where $c^{(\gamma)}_{6(1),1}$ and $c^{(\gamma)}_{6(1),2}$ are known constants. Using the relations (28) and (29) one has that

$$C^{(\gamma)}_{j(i)} = C^{(\gamma+1)}_{j(i)} = c^{(\gamma)}_{j(i),1} C^{(\gamma)}_{1(1)} + c^{(\gamma)}_{j(i),2} C^{(\gamma)}_{2(1)}, \quad j = 1, \ldots, 6;\ i = 1, \ldots, n \tag{78}$$

where $c^{(\gamma)}_{j(i),1}$ and $c^{(\gamma)}_{j(i),2}$ are known constants determined by:

$$\left[c^{(\gamma)}_{1(1),1} \ldots\ c^{(\gamma)}_{6(1),1}\right]^T = \left[1\ 0\ -1\ 0\ 0\ c^{(\gamma)}_{6(1),1}\right]^T, \tag{79}$$

$$\left[c^{(\gamma)}_{1(1),2} \ldots\ c^{(\gamma)}_{6(1),2}\right]^T = \left[0\ 1\ 0\ -1\ 0\ c^{(\gamma)}_{6(1),2}\right]^T, \tag{80}$$

$$\left[c^{(\gamma)}_{1(\ell),1} \ldots\ c^{(\gamma)}_{6(\ell),1}\right]^T = \mathbf{T}_{\ell-1}\mathbf{T}_{\ell-2}\cdots\mathbf{T}_1 \left[1\ 0\ -1\ 0\ 0\ c^{(\gamma)}_{6(1),1}\right]^T,\ \ell = 2, \ldots, n, \tag{81}$$

$$\left[c^{(\gamma)}_{1(\ell),2} \ldots\ c^{(\gamma)}_{6(\ell),2}\right]^T = \mathbf{T}_{\ell-1}\mathbf{T}_{\ell-2}\cdots\mathbf{T}_1 \left[0\ 1\ 0\ -1\ 0\ c^{(\gamma)}_{6(1),2}\right]^T,\ \ell = 2, \ldots, n. \tag{82}$$

Introducing (78) into (12) and (13) gives

$$W_{i(\gamma)}(z_i) = W_{i(\gamma+1)}(z_i) = C^{(\gamma)}_{1(1)} W^*_{i(\gamma)}(z_i) + C^{(\gamma)}_{2(1)} W^*_{i(\gamma+1)}(z_i), \quad i = 1, \ldots, n, \tag{83}$$

$$U_{i(\gamma)}(z_i) = U_{i(\gamma+1)}(z_i) = C^{(\gamma)}_{1(1)} U^*_{i(\gamma)}(z_i) + C^{(\gamma)}_{2(1)} U^*_{i(\gamma+1)}(z_i), \quad i = 1, \ldots, n, \tag{84}$$

where

$$W^*_{i(\gamma)}(z_i) = c^{(\gamma)}_{1(i),1} \cos(k_{i(\gamma)} z_i) + c^{(\gamma)}_{2(i),1} \sin(k_{i(\gamma)} z_i) + c^{(\gamma)}_{3(i),1} \cosh(k_{i(\gamma)} z_i) + c^{(\gamma)}_{4(i),1} \sinh(k_{i(\gamma)} z_i), \tag{85}$$

$$W^*_{i(\gamma+1)}(z_i) = c^{(\gamma)}_{1(i),2} \cos(k_{i(\gamma)} z_i) + c^{(\gamma)}_{2(i),2} \sin(k_{i(\gamma)} z_i) + c^{(\gamma)}_{3(i),2} \cosh(k_{i(\gamma)} z_i) + c^{(\gamma)}_{4(i),2} \sinh(k_{i(\gamma)} z_i), \tag{86}$$

$$U^*_{i(\gamma)}(z_i) = c^{(\gamma)}_{5(i),1} \cos(p_{i(\gamma)} z_i) + c^{(\gamma)}_{6(i),1} \sin(p_{i(\gamma)} z_i), \tag{87}$$

$$U^*_{i(\gamma+1)}(z_i) = c^{(\gamma)}_{5(i),2} \cos(p_{i(\gamma)} z_i) + c^{(\gamma)}_{6(i),2} \sin(p_{i(\gamma)} z_i). \tag{88}$$

Based on above relations the following holds:

$$W_{i(\gamma)}(z_i) \Big(K_\gamma \cos(\omega_\gamma t) + S_\gamma \sin(\omega_\gamma t)\Big) + W_{i(\gamma+1)}(z_i) \Big(K_{\gamma+1} \cos(\omega_{\gamma+1} t) + S_{\gamma+1} \sin(\omega_{\gamma+1} t)\Big)$$

Journal Pre-proof

$$= W_{i(\gamma)}(z_i)\left(K_{\gamma,\gamma+1}\cos(\omega_\gamma t) + S_{\gamma,\gamma+1}\sin(\omega_\gamma t)\right), \tag{89}$$

as well as:

$$U_{i(\gamma)}(z_i)\left(K_\gamma\cos(\omega_\gamma t) + S_\gamma\sin(\omega_\gamma t)\right) + U_{i(\gamma+1)}(z_i)\left(K_{\gamma+1}\cos(\omega_{\gamma+1} t) + S_{\gamma+1}\sin(\omega_{\gamma+1} t)\right)$$

$$= U_{i(\gamma)}(z_i)\left(K_{\gamma,\gamma+1}\cos(\omega_\gamma t) + S_{\gamma,\gamma+1}\sin(\omega_\gamma t)\right) \tag{90}$$

where $K_{\gamma,\gamma+1} = K_\gamma + K_{\gamma+1}$ and $S_{\gamma,\gamma+1} = S_\gamma + S_{\gamma+1}$. Applying the same steps as in the case of determination of the coefficients $K_\alpha(\alpha = 1, \ldots, \infty)$ in the previous section to the functions $W_{i(\gamma)}(z_i)$ and $U_{i(\gamma)}(z_i)$ and taking into account Eqs. (89) and (90) as well as the orthogonality condition (55) yields:

$$\sum_{i=1}^{n}\left\{\int_0^{L_i}\varrho_i A_i\left(f_{wi}(z_i)W_{i(\gamma)}(z_i) + f_{ui}(z_i)U_{i(\gamma)}(z_i)\right)dz_i + \left[J_i + m_i(d_i^2 + e_i^2)\right]f'_{wi}(L_i)W'_{i(\gamma)}(L_i)\right.$$

$$+m_i d_i\left(f_{ui}(L_i)W'_{i(\gamma)}(L_i) + f'_{wi}(L_i)U_{i(\gamma)}(L_i)\right) + m_i e_i\left(f'_{wi}(L_i)W_{i(\gamma)}(L_i) + f_{wi}(L_i)W'_{i(\gamma)}(L_i)\right)$$

$$\left.+ m_i\left(f_{ui}(L_i)U_{i(\gamma)}(L_i) + f_{wi}(L_i)W_{i(\gamma)}(L_i)\right)\right\} = \sum_{i=1}^{n}\left\{\int_0^{L_i}\rho_i A_i\left(U^2_{i(\gamma)}(z_i) + W^2_{i(\gamma)}(z_i)\right)dz_i\right.$$

$$+J_i W'^2_{i(\gamma)}(L_i) + m_i(d_i^2 + e_i^2)W'^2_{i(\gamma)}(L_i) + m_i\left(U^2_{i(\gamma)}(L_i) + W^2_{i(\gamma)}(L_i)\right)$$

$$\left.+2m_i d_i W'_{i(\gamma)}(L_i)U_{i(\gamma)}(L_i) + 2m_i e_i W'_{i(\gamma)}(L_i)W_{i(\gamma)}(L_i)\right\}K_{\gamma,\gamma+1}. \tag{91}$$

Substituting the expressions (83) and (84) as well as $K_{\gamma(R)} = C^{(\gamma)}_{1(1)}K_{\gamma,\gamma+1}$, and $K_{\gamma+1(R)} = C^{(\gamma)}_{2(1)}K_{\gamma,\gamma+1}$ into Eq.(91) and taking into account the fact that the coefficients $C^{(\gamma)}_{1(1)}$ and $C^{(\gamma)}_{2(1)}$ are free, the following equation system in unknowns $K_{\gamma(R)}$ and $K_{\gamma+1(R)}$ is obtained:

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} K_{\gamma(R)} \\ K_{\gamma+1(R)} \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{92}$$

where the expressions for the quantities g_{11}, g_{12}, g_{21}, g_{22}, g_1, and g_2 are given in Appendices B and C. Solving the equation system (92) for $K_{\gamma(R)}$ and $K_{\gamma+1(R)}$ gives:

$$\begin{bmatrix} K_{\gamma(R)} \\ K_{\gamma+1(R)} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{93}$$

Applying the same procedure as above to Eqs. (71), (72), and (74) yields:

Journal Pre-proof

$$\begin{bmatrix} S_{\gamma(R)} \\ S_{\gamma+1(R)} \end{bmatrix} = \frac{1}{\omega_\gamma} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \begin{bmatrix} g_3 \\ g_4 \end{bmatrix} \tag{94}$$

where $S_{\gamma(R)} = C_{1(1)}^{(\gamma)} S_{\gamma,\gamma+1}$, $S_{\gamma+1(R)} = C_{2(1)}^{(\gamma)} S_{\gamma,\gamma+1}$, and where the quantities g_3 and g_4 are given in Appendix C. Now, Eqs. (89) and (90) take the following form:

$$W_{i(\gamma)}(z_i)\left(K_\gamma \cos(\omega_\gamma t) + S_\gamma \sin(\omega_\gamma t)\right) + W_{i(\gamma+1)}(z_i)\left(K_{\gamma+1} \cos(\omega_{\gamma+1} t) + S_{\gamma+1} \sin(\omega_{\gamma+1} t)\right)$$

$$= \left(K_{\gamma(R)} W_{i(\gamma)}^*(z_i) + K_{\gamma+1(R)} W_{i(\gamma+1)}^*(z_i)\right)\cos(\omega_\gamma t) + \left(S_{\gamma(R)} W_{i(\gamma)}^*(z_i) + S_{\gamma+1(R)} W_{i(\gamma+1)}^*(z_i)\right)\sin(\omega_\gamma t), \tag{95}$$

$$U_{i(\gamma)}(z_i)\left(K_\gamma \cos(\omega_\gamma t) + S_\gamma \sin(\omega_\gamma t)\right) + U_{i(\gamma+1)}(z_i)\left(K_{\gamma+1} \cos(\omega_{\gamma+1} t) + S_{\gamma+1} \sin(\omega_{\gamma+1} t)\right)$$

$$= \left(K_{\gamma(R)} U_{i(\gamma)}^*(z_i) + K_{\gamma+1(R)} U_{i(\gamma+1)}^*(z_i)\right)\cos(\omega_\gamma t) + \left(S_{\gamma(R)} U_{i(\gamma)}^*(z_i) + S_{\gamma+1(R)} U_{i(\gamma+1)}^*(z_i)\right)\sin(\omega_\gamma t) \tag{96}$$

5. Numerical examples

5.1. Example 1: distinct natural frequencies

Consider a system shown in Fig. 3 consisting of a rigid body (V_1) supported by a single elastic beam segment (BS_1) of circular cross-section. As in [13], the following values of the parameters of the system described are used: $e_1 = 0$, $d_1 = 0.4\,\mathrm{m}$, $m_1 = 7.8917\,\mathrm{kg}$, $J_1 = 0.4209\,\mathrm{kg\,m^2}$, $\varrho_1 = 7850\,\mathrm{kg/m^3}$, $E_1 = 2.068 \times 10^{11}\,\mathrm{N/m^2}$, $D_1 = 0.02\,\mathrm{m}$ (diameter of the cross-section), and $L_1 = 1.0\,\mathrm{m}$.

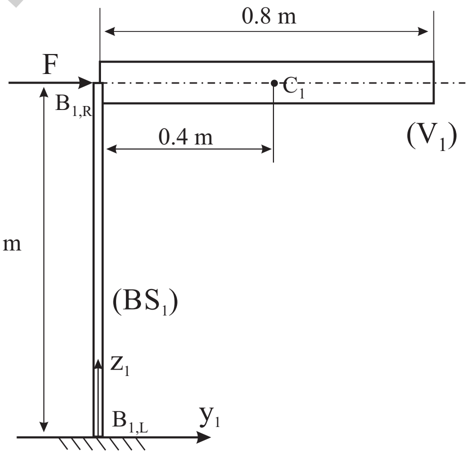

Figure 3: A rigid body supported by a single flexible beam

A force of horizontal direction and constant magnitude F is applied at the end $B_{1,R}$ of the beam segment. Accordingly, the initial displacement and velocity functions read:

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$$f_{w1}(z_1) = \frac{F}{E_1 I_{x(1)}}\left(\frac{L_1 z_1^2}{2} - \frac{z_1^3}{6}\right), \qquad f_{u1}(z_1) = 0, \tag{97}$$

$$h_{w1}(z_1) = 0, \quad h_{u1}(z_1) = 0. \tag{98}$$

The aim is to find the resulting vibrations that occur after the force F is suddenly removed. Now, the functions $W^*_{1(\alpha)}(z_1)$ and $U^*_{1(\alpha)}(z_1)$ have the following forms:

$$W^*_{1(\alpha)}(z_1) = \cos(k_{1(\alpha)}z_1) + c^{(\alpha)}_{2(1)}\sin(k_{1(\alpha)}z_1) - \cosh(k_{1(\alpha)}z_1) - c^{(\alpha)}_{2(1)}\sinh(k_{1(\alpha)}z_1), \tag{99}$$

$$U^*_{1(\alpha)}(z_1) = c^{(\alpha)}_{6(1)}\sin(p_{1(\alpha)}z_1), \tag{100}$$

where $c^{(\alpha)}_{2(1)} = r^{(\alpha)}_{2(1)}/q^{(\alpha)}_{2(1)}$, $c^{(\alpha)}_{6(1)} = r^{(\alpha)}_{6(1)}/q^{(\alpha)}_{6(1)}$, and:

$$r^{(\alpha)}_{2(1)} = \left[\left(d_1^2 m + J_1\right)A_1 E_1 p_{1(\alpha)}\omega_\alpha^2\cos(p_{1(\alpha)}L_1) - m_1 J_1\omega_\alpha^4\sin(p_{1(\alpha)}L_1)\right]\left(\sin(k_{1(\alpha)}L_1) + \sinh(k_{1(\alpha)}L_1)\right)$$

$$-E_1 I_{x(1)}k_{1(\alpha)}\left(A_1 E_1 p_{1(\alpha)}\cos(p_{1(\alpha)}L_1) - m_1\omega_\alpha^2\sin(p_{1(\alpha)}L_1)\right)\left(\cos(k_{1(\alpha)}L_1) + \cosh(k_{1(\alpha)}L_1)\right), \tag{101}$$

$$q^{(\alpha)}_{2(1)} = \left[\left(d_1^2 m + J_1\right)A_1 E_1 p_{1(\alpha)}\omega_\alpha^2\cos(p_{1(\alpha)}L_1) - m_1 J_1\omega_\alpha^4\sin(p_{1(\alpha)}L_1)\right]\left(\cos(k_{1(\alpha)}L_1) - \cosh(k_{1(\alpha)}L_1)\right)$$

$$+E_1 I_{x(1)}k_{1(\alpha)}\left(A_1 E_1 p_{1(\alpha)}\cos(p_{1(\alpha)}L_1) - m_1\omega_\alpha^2\sin(p_{1(\alpha)}L_1)\right)\left(\sin(k_{1(\alpha)}L_1) + \sinh(k_{1(\alpha)}L_1)\right), \tag{102}$$

$$r^{(\alpha)}_{6(1)} = -2d_1 E_1 I_{x(1)}k^2_{1(\alpha)}m_1\omega_\alpha^2\sec(p_{1(\alpha)}L_1)\sin(k_{1(\alpha)}L_1)\sinh(k_{1(\alpha)}L_1), \tag{103}$$

$$q^{(\alpha)}_{6(1)} = \left[\left(d_1^2 m_1 + J_1\right)A_1 E_1 p_{1(\alpha)}\omega_\alpha^2\cos(p_{1(\alpha)}L_1) - m_1 J_1\omega_\alpha^4\tan(p_{1(\alpha)}L_1)\right]\left(\cos(k_{1(\alpha)}L_1) - \cosh(k_{1(\alpha)}L_1)\right)$$

$$+E_1 I_{x(1)}k_{1(\alpha)}\left(A_1 E_1 p_{1(\alpha)} - m_1\omega_\alpha^2\tan(p_{1(\alpha)}L_1)\right)\left(\sin(k_{1(\alpha)}L_1) + \sinh(k_{1(\alpha)}L_1)\right). \tag{104}$$

Further, based on Eqs.(68), (75), and (76) one has:

$$C^{(\alpha)2}_{1(1)} = \left(\int_0^{L_1}\varrho_1 A_1\left(U^{*2}_{1(\alpha)}(z_1) + W^{*2}_{1(\alpha)}(z_1)\right)dz_1 + (J_1 + m_1 d_1^2)W^{*\prime 2}_{1(\alpha)}(L_1)\right.$$

$$\left.+2m_1 d_1 W^{*\prime}_{1(\alpha)}(L_1)U^*_{1(\alpha)}(L_1) + m_1(U^{*2}_{1(\alpha)}(L_1) + W^{*2}_{1(\alpha)}(L_1))\right)^{-1}, \tag{105}$$

$$K_\alpha = \frac{FC^{(\alpha)}_{1(1)}}{E_1 I_{x(1)}} \left\{ \int_0^{L_1} \varrho_1 A_1 \left(\frac{L_1 z_1^2}{2} - \frac{z_1^3}{6} \right) W^*_{1(\alpha)}(z_1) dz_1 + m_1 d_1 U^*_{1(\alpha)}(L_1) \frac{L_1^2}{2} \right.$$

$$\left. + (J_1 + m_1 d_1^2) W^{*\prime}_{1(\alpha)}(L_1) \frac{L_1^2}{2} + m_1 W^*_{1(\alpha)}(L_1) \frac{L_1^3}{3} \right\}, \tag{106}$$

$$S_\alpha = 0. \tag{107}$$

Finally, the response of the system to the initial excitation defined by (97) and (98) is determined based on Eqs. (59) and (60) as follows:

$$w_1(z_1, t) = \frac{F}{E_1 I_{x(1)}} \sum_{\alpha=1}^{\infty} W^*_{1(\alpha)}(z_1) K^*_\alpha \cos(\omega_\alpha t), \tag{108}$$

$$u_1(z_1, t) = \frac{F}{E_1 I_{x(1)}} \sum_{\alpha=1}^{\infty} U^*_{1(\alpha)}(z_1) K^*_\alpha \cos(\omega_\alpha t), \tag{109}$$

where $K^*_\alpha = E_1 I_{x(1)} K_\alpha C^{(\alpha)}_{1(1)} / F$. In Table 1, the lowest six natural frequencies, $\omega_i (i = 1, \dots, 6)$, are shown.

Table 1: The lowest six values of natural frequencies for a rigid body supported by a flexible beam

Methods	Natural frequencies (rad/s)					
	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
Our approach	19.62512	73.93647	591.75208	1599.01501	3118.01091	4775.03631
FEM	19.713	73.752	590.179	1590.588	3090.573	4771.765

The values in Table 1 obtained by means of the finite element method (FEM) are calculated by using ANSYS beam element BEAM188. Namely, the segment (BS_1) is modeled by 17 BEAM188 elements, while the body (V_1) is modeled by one BEAM188 element with the Young's modulus equaling to $2.0 \times 10^{20}\,\mathrm{N/m^2}$. Based on the natural frequencies obtained, the values of the coefficients $K^*_\alpha (i = 1, \dots, 6)$ are calculated as follows:

$$K^*_\alpha = -0.5669105,\ -0.0245278,\ 0.000221,\ -0.0000195,\ 3.6947 \times 10^{-6},\ -2.5081 \times 10^{-8}. \tag{110}$$

The obtained values suggest that truncation at the forth term in the displacement series (108) and (109) would be adequate. Using the same approach as in [15], this observation can be confirmed by comparing an initial displacement field obtained from a truncated form of the displacement series to the actual initial displacement field determined by the function $f_{w1}(z_1)$ as depicted in Fig. 4. Figure 5 shows the displacements $w_1(L_1,\ t)$ and $u_1(L_1,\ t)$ scaled by $F/(E_1 I_{x(1)})$ and represented through the first four terms in the displacement series. At the end, note that in [13] the authors performed incorrectly the numbering of natural frequencies, that is, the first two frequencies are left out (see our results in Table 1).

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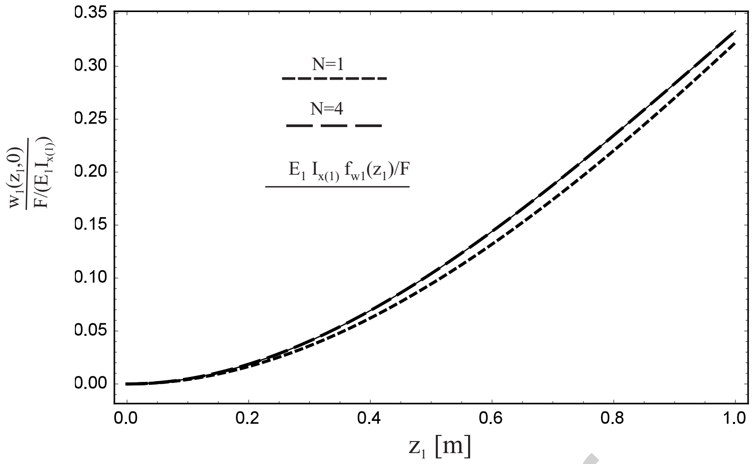


Figure 4: The numerical convergence of the solution of the problem considered in Section 5.1 for various numbers of terms, N, in the displacement series (107)

5.2. Example 2: repeated natural frequencies

Consider an angled-beam structure with clamped-clamped ends shown in Fig. 6. It is assumed that the elastic beam segments, (BS_1) and (BS_2), with circular cross-section of diameter D have the same parameters as follows: $A_1 = A_2 = \pi D^2/4$, $I_{x(1)} = I_{x(2)} = \pi D^4/64$, $L_1 = L_2 = 1$m, $E_1 = E_2 = 2.069 \times 10^{11}$N/m^2, $\varrho_1 = \varrho_2 = 7.8367 \times 10^3$kg/m^3. In addition, the following initial displacement and velocity functions are used:

$$f_{w1}(z_1) = f_{w2}(z_2) = 0, \tag{111}$$

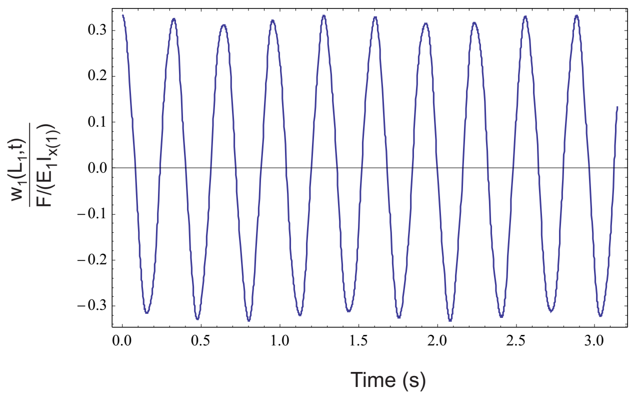
$$f_{u1}(z_1) = f_{u2}(z_2) = 0, \tag{112}$$

$$h_{w1}(z_1) = V_0 \sin\left(\frac{\pi z_1}{L_1}\right), \quad h_{w2}(z_2) = 0, \tag{113}$$

$$h_{u1}(z_1) = h_{u2}(z_2) = 0. \tag{114}$$

This type of structures was used in [3] for the purpose of the investigating of the occurrence of repeated frequencies (the crossing phenomenon). Unlikely [3] that studied the possibility of the crossing phenomenon occurrence caused by change of the angle α_1 value, in our paper the value of diameter D changes. By changing diameter D from 0.01m to 0.10m, where the diameter values satisfy the condition $D/L \leq 0.1$ (the Euler-Bernoulli beam theory [16]), the effect of the diameter D on the lowest six dimensionless frequency coefficients, $\lambda_r = k_r L_1 = \sqrt[4]{\omega_r^2 \varrho_1 A_1 L_1^4/(E_1 I_{x(1)})}, (r = 1, \ldots, 6)$, of the system is shown in Fig. 7.

As can be seen in Fig.7, for specified boundaries of change of the diameter D values, there is the crossing phenomenon for the third and forth modes as well as for the fifth and sixth modes. Further, we will determine the system response to the initial excitation defined by the relations



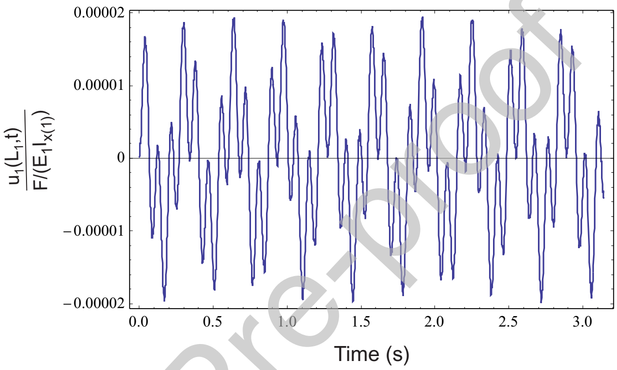


Figure 5: System response to the initial excitation

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(111)-(114) for the case of $\lambda_3 = \lambda_4$. The values of diameter D corresponding to this case can be determined by solving a nonlinear equations system of the form:

$$\det \mathbf{T} = 0, \qquad \frac{\partial}{\partial k}(\det \mathbf{T}) = 0, \tag{115}$$

for unknowns k and D yielding $D = 0.056904350279688015\text{m}$ and $k = k_3 = k_4 = 7.06111126\,\text{m}^{-1}$. Based on the boundary conditions (111) and (112), by using the relations (75) and (93) it is obtained that:

$$K_\alpha = 0(\alpha = 1, \dots, \infty \wedge \alpha \neq 3, 4), \quad K_{3(R)} = K_{4(R)} = 0 \tag{116}$$

and thus, in accordance to Eqs. (59), (60), (95), (96), and (113), the response of the system to the initial excitation has the following form:

$$w_i(z_i,\ t) = V_0 \left[S^*_{3(R)} W^*_{i(3)}(z_i) + S^*_{4(R)} W^*_{i(4)}(z_i) \right] \sin(\omega_3 t) + V_0 \sum_{\substack{\alpha=1 \\ \alpha\neq 3,4}}^{\infty} W^*_{i(\alpha)}(z_i) S^*_\alpha C^{(\alpha)2}_{1(1)} \sin(\omega_\alpha t),\ i = 1, 2, \tag{117}$$

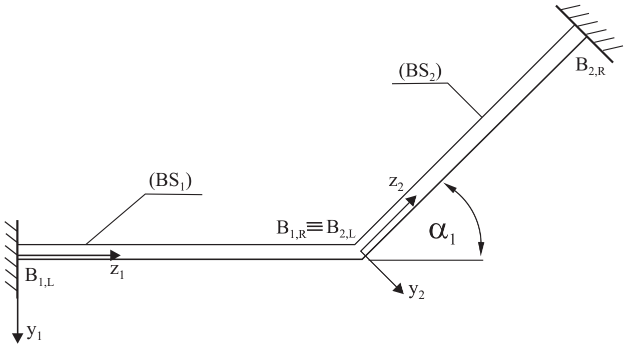

Figure 6: An angled-beam structure

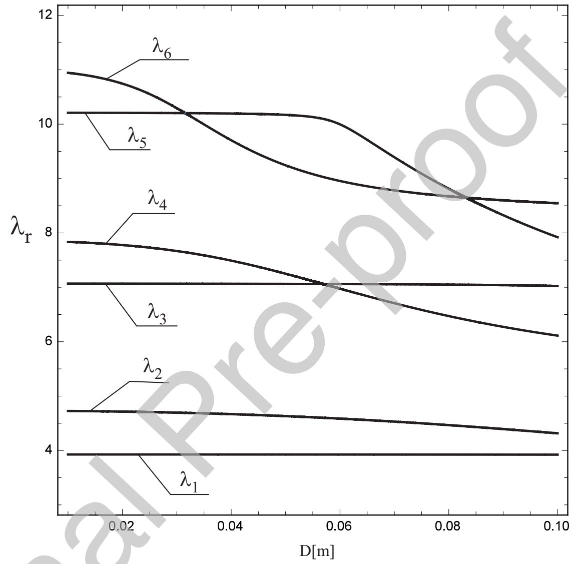

Figure 7: The effect of diameter D on the lowest six dimensionless frequency coefficients.

$$u_i(z_i,\ t) = V_0\left[S^*_{3(R)}U^*_{i(3)}(z_i) + S^*_{4(R)}U^*_{i(4)}(z_i)\right]\sin(\omega_3 t) + V_0\sum_{\substack{\alpha=1\\ \alpha\neq 3,4}}^{\infty} U^*_{i(\alpha)}(z_i)S^*_{\alpha}C^{(\alpha)2}_{1(1)}\sin(\omega_\alpha t),\ i = 1, 2, \tag{118}$$

where $S^*_\alpha = S_\alpha/(V_0 C^{(\alpha)}_{1(1)})$, $S^*_{3(R)} = S_{3(R)}/V_0$, and $S^*_{4(R)} = S_{4(R)}/V_0$. Further considerations will be restricted to the first eight modes of vibration. In addition, values of the first eight frequency coefficients, $k_i(i = 1, \ldots, 8)$ are: $k_1 = 3.92559175\,\mathrm{m}^{-1}$, $k_2 = 4.60131309\,\mathrm{m}^{-1}$, $k_3 = k_4 = 7.06111126\,\mathrm{m}^{-1}$, $k_5 = 9.02911756\,\mathrm{m}^{-1}$, $k_6 = 10.09725026\,\mathrm{m}^{-1}$, $k_7 = 10.57691221\,\mathrm{m}^{-1}$, $k_8 = 11.62268127\,\mathrm{m}^{-1}$. The numerical convergence of the solution can be shown similarly as in the previous example by comparing an initial velocity field obtained from a truncated form of the velocity series of the segment (BS_1) to the actual initial velocity field determined by the function $h_{w1}(z_1)$ as depicted in Fig. 8.

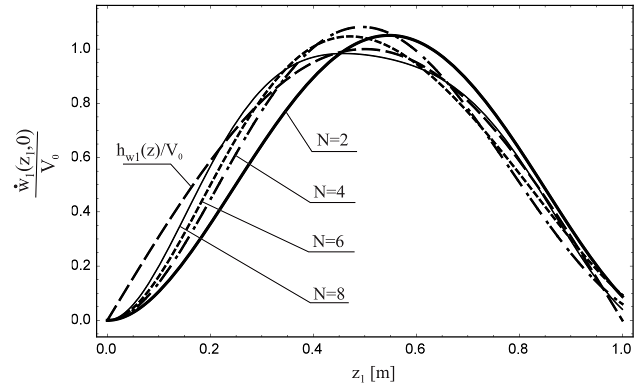

Figure 8: The numerical convergence of the solution of the problem considered in Section 5.2 for various numbers of terms, N, in the velocity series of the segment (BS_1)

5.3. Example 3: Portal frame carrying a concentrated mass

A portal frame with clamped ends carrying a concentrated mass m_2 is shown in Fig. 9.

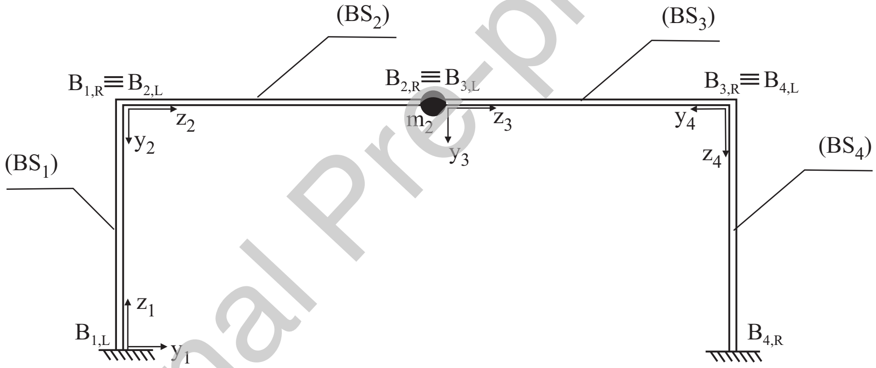

Figure 9: A portal frame carrying a concentrated mass

The beam segments $(BS\,i)(i = 1,\ldots,4)$ with circular cross-section of diameter $D = 0.02\,\mathrm{m}$ have the following parameters: $A_i = \pi D_i^2/4\,(i = 1,\ldots,4)$, $I_{x(i)} = \pi D_i^4/64\,(i = 1,\ldots,4)$, $E_i = 2.068\times10^{11}\mathrm{N/m^2}(i = 1,\ldots,4)$, $\varrho_i = 7.850\times10^3\mathrm{kg/m^3}(i = 1,\ldots,4)$, $L_1 = L_4 = 1\mathrm{m}$, $L_2 = L_3 = 2\mathrm{m}$. In this numerical example the following initial displacement and velocity functions are used:

$$f_{wi}(z_i) = 0(i = 1,\ldots,4), \quad f_{ui}(z_i) = 0(i = 1,\ldots,4), \tag{119}$$

$$h_{w1}(z_1) = V_0 \sin\left(\frac{\pi z_1}{L_1}\right), \quad h_{wj}(z_j) = 0(i = 1,\ldots,4), \tag{120}$$

$$h_{ui}(z_i) = 0(i = 1,\ldots,4). \tag{121}$$

For various values of the ratio $m_2/(\varrho_1 A_1 L_1) = \bar{m}_2$, the values of the first five dimensionless frequency coefficients, $\lambda_r = k_r L_1 = \sqrt[4]{\omega_r^2 \varrho_1 A_1 L_1^4/(E_1 I_{x(1)})}, (r = 1,\ldots,6)$, of the frame are shown in

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Table 2. Note that it can be observed in Table 3 that increasing the mass m_2 decreases the natural frequencies of the considered portal frame.

Table 2: The lowest six values of dimensionless natural frequencies $\lambda_i (i = 1, \ldots, 6)$ of the portal frame considered for various values of the parameter $\bar{m}_2$

$\bar{m}_2$	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.2	1.04799	1.20509	1.84255	2.49955	3.28031	3.86722
0.5	1.01037	1.18848	1.84028	2.44407	3.2803	3.81409
1.0	0.959766	1.16309	1.83711	2.38486	3.28027	3.75934

Further, for the initial excitation given by (119)-(121) one has:

$$K_\alpha = 0,\ \alpha = 1, \ldots, \infty, \tag{122}$$

$$S_\alpha = \frac{C_{1(1)}^{(\alpha)} \varrho A V_0}{\omega_\alpha} \int_0^{L_1} W_{1(\alpha)}^*(z_1) \sin\left(\frac{\pi z_1}{L_1}\right) dz_1 \equiv C_{1(1)}^{(\alpha)} V_0 S_\alpha^*,\ \alpha = 1, \ldots, \infty. \tag{123}$$

and, based on this, the response of the frame to the initial excitation reads:

$$w_i(z_i,\ t) = V_0 \sum_{\alpha=1}^{\infty} C_{1(1)}^{(\alpha)2} S_\alpha^* W_{i(\alpha)}^*(z_i) \sin(\omega_\alpha t),\ \ i = 1, \ldots, 4, \tag{124}$$

$$u_i(z_i,\ t) = V_0 \sum_{\alpha=1}^{\infty} C_{1(1)}^{(\alpha)2} S_\alpha^* U_{i(\alpha)}^*(z_i) \sin(\omega_\alpha t),\ \ i = 1, \ldots, 4. \tag{125}$$

Finally, using the truncation at the fifth term in the displacement series (124) and (125), the corresponding transverse and axial displacements of the concentrated mass are shown in Fig. 10.

5.4. Example 4: Planar gabled frame with four members

In this example a planar gabled four-member frame with clamped ends is shown in Fig. 11.

The ends of the frame are clamped and the beam segments $(BS\,i)(i = 1, \ldots, 4)$ have circular cross section. In the further considerations the following characteristics of the frame are taken: $D_i = 0.02\,\text{m}\,(i = 1, \ldots, 4)$, $A_i = \pi D_i^2/4\,(i = 1, \ldots, 4)$, $I_{x(i)} = \pi D_i^4/64\,(i = 1, \ldots, 4)$, $E_i = 2.068 \times 10^{11}\text{N/m}^2 (i = 1, \ldots, 4)$, $\varrho_i = 7.850 \times 10^3\text{kg/m}^3 (i = 1, \ldots, 4)$, $L_1 = L_4 = 1\text{m}$, $L_2 = L_3 = 2\text{m}$. Also, the initial excitation of the frame defined with the relations (119)-(121) is used. Based on this, the response of the gabled frame is defined by the expressions (124) and (125). The first sixth dimensionless frequency coefficients, $\lambda_r = k_r L_1 = \sqrt[4]{\omega_r^2 \varrho_1 A_1 L_1^4/(E_1 I_{x(1)})}$, $(r = 1, \ldots, 6)$, are shown in Table 3.

Note that, in Table 3, the frequency coefficients obtained in [21] under the assumption of ignored effect of axial deformations are also shown. Based on the determined frequency coefficients and using the truncation at the fifth term in the displacement series (124) and (125), the transverse and axial displacements at the middle of the beam segment $(BS\,4)$ are shown in Fig. 12.

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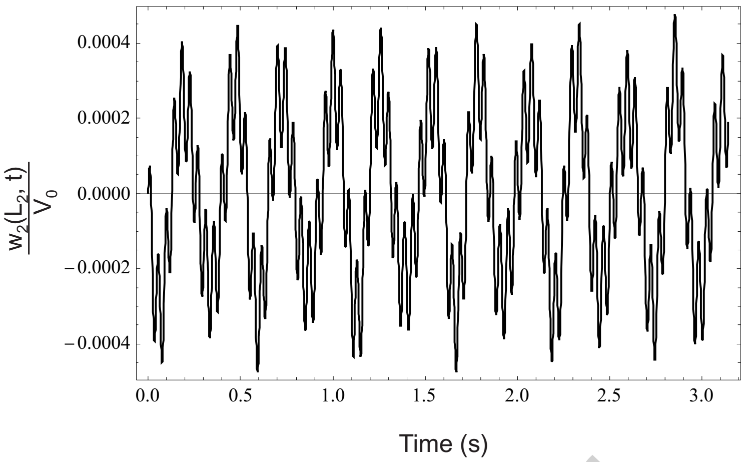

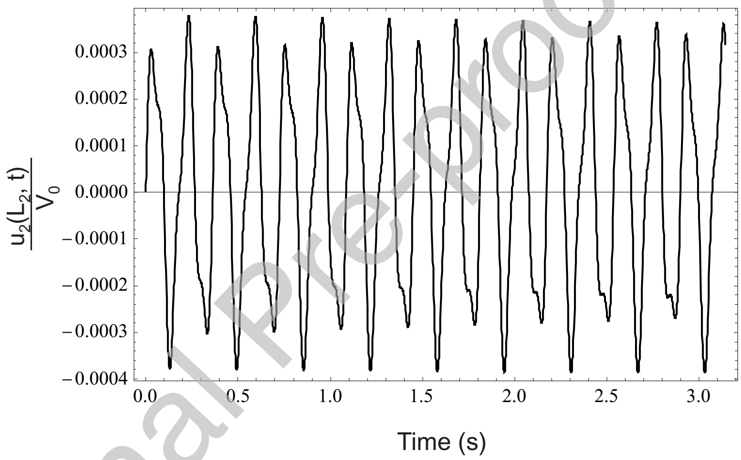

Figure 10: The portal frame response to the initial excitation

6. Conclusions

In this paper, the orthogonality conditions of mode shapes of the structures composed of rigid bodies and elastic beam segments have been derived. This paper along with our previous paper [1] allows for a complete solution for the free vibration problem of the considered kind of structures. The known orthogonality conditions in the literature dealing with the considered kind of structures arise as special cases of the orthogonality conditions derived in this study. The study provides a foundation for the forced vibration analysis of structures. The responses of the structure to the initial excitations obtained by our approach may be used for testing the accuracy of various approximate methods. The fact that all relations derived in this paper and [1] are given in the symbolic form makes vibration analysis of structures simpler, because it is much easier to examine the effect of some parameter on the system oscillatory behavior when symbolic expressions are available, where that parameter occurs (see, for example, the crossing phenomenon analysis shown in Fig. 8 in [3]).

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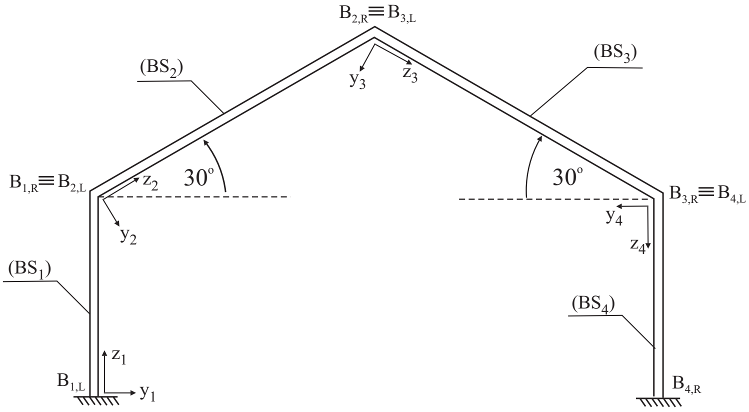

Figure 11: A planar gabled frame with four members

Table 3: The lowest six values of dimensionless natural frequencies of the gabled frame

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
This study	1.14372	1.49553	2.03539	2.2255	3.2799	3.7195
[21]	1.1437	1.4956	2.0355	2.2261	-	-

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Appendix A. Components of the matrix $\mathbf{P}^{(2)}_{i(\alpha)}$

$$P^{(2)}_{i(\alpha)11} = \cos\alpha_i, \quad P^{(2)}_{i(\alpha)12} = \ell_{i(1)}\cos\alpha_i + \ell_{i(2)}, \tag{A.1}$$

$$P^{(2)}_{i(\alpha)13} = P^{(2)}_{i(\alpha)14} = P^{(2)}_{i(\alpha)16} = 0, \quad P^{(2)}_{i(\alpha)15} = \sin\alpha_i, \tag{A.2}$$

$$P^{(2)}_{i(\alpha)21} = P^{(2)}_{i(\alpha)23} = P^{(2)}_{i(\alpha)24} = P^{(2)}_{i(\alpha)25} = P^{(2)}_{i(\alpha)26} = 0, \quad P^{(2)}_{i(\alpha)22} = 1, \tag{A.3}$$

$$P^{(2)}_{i(\alpha)31} = \frac{m_i\omega_\alpha^2(\ell_{i(1)} - e_i + \ell_{i(2)}\cos\alpha_i)}{E_{i+1}I_{x(i+1)}}, \tag{A.4}$$

$$P^{(2)}_{i(\alpha)32} = \frac{\omega_\alpha^2\left[-J_i - m_i d_i^2 + m_i e_i(\ell_{i(1)} - e_i) + m_i\ell_{i(2)}(d_i\sin\alpha_i + e_i\cos\alpha_i)\right]}{E_{i+1}I_{x(i+1)}}, \tag{A.5}$$

$$P^{(2)}_{i(\alpha)33} = \frac{E_iI_{x(i)}}{E_{i+1}I_{x(i+1)}}, \quad P^{(2)}_{i(\alpha)34} = \frac{E_iI_{x(i)}(\ell_{i(1)} + \ell_{i(2)}\cos\alpha_i)}{E_{i+1}I_{x(i+1)}}, \tag{A.6}$$

$$P^{(2)}_{i(\alpha)35} = \frac{m_i\omega_\alpha^2(\ell_{i(2)}\sin\alpha_i - d_i)}{E_{i+1}I_{x(i+1)}}, \quad P^{(2)}_{i(\alpha)36} = -\frac{E_iA_i\ell_{i(2)}\sin\alpha_i}{E_{i+1}I_{x(i+1)}}, \tag{A.7}$$

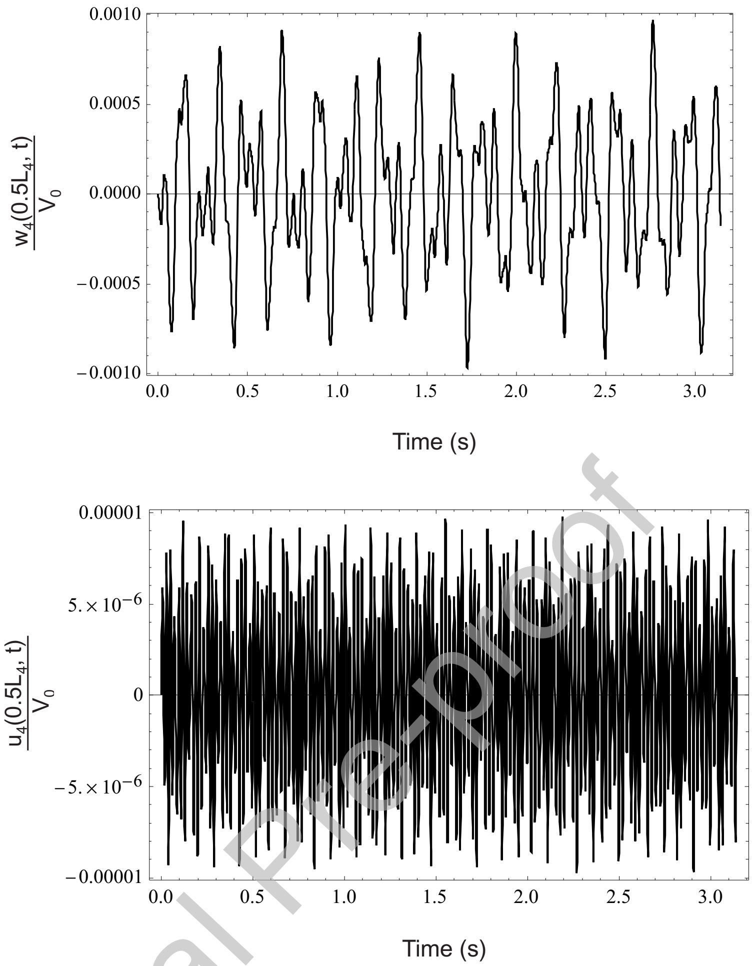

Figure 12: The gabled frame response to the initial excitation

$$P^{(2)}_{i(\alpha)41} = \frac{m_i\omega_\alpha^2 \cos\alpha_i}{E_{i+1}I_{x(i+1)}}, \quad P^{(2)}_{i(\alpha)42} = \frac{m_i\omega_\alpha^2(d_i \sin\alpha_i + e_i \cos\alpha_i)}{E_{i+1}I_{x(i+1)}}, \tag{A.8}$$

$$P^{(2)}_{i(\alpha)43} = 0, \quad P^{(2)}_{i(\alpha)44} = \frac{E_i I_{x(i)} \cos\alpha_i}{E_{i+1}I_{x(i+1)}}, \quad P^{(2)}_{i(\alpha)45} = \frac{m_i\omega_\alpha^2 \sin\alpha_i}{E_{i+1}I_{x(i+1)}}, \quad P^{(2)}_{i(\alpha)46} = -\frac{E_i A_i \sin\alpha_i}{E_{i+1}I_{x(i+1)}}, \tag{A.9}$$

$$P^{(2)}_{i(\alpha)51} = -\sin\alpha_i, \quad P^{(2)}_{i(\alpha)52} = -\ell_{i(1)} \sin\alpha_i, \tag{A.10}$$

$$P^{(2)}_{i(\alpha)53} = P^{(2)}_{i(\alpha)54} = P^{(2)}_{i(\alpha)56} = 0, \quad P^{(2)}_{i(\alpha)55} = \cos\alpha_i, \tag{A.11}$$

$$P^{(2)}_{i(\alpha)61} = \frac{\omega_\alpha^2 m_i \sin\alpha_i}{A_{i+1}E_{i+1}}, \quad P^{(2)}_{i(\alpha)62} = \frac{m_i\omega_\alpha^2(e_i \sin\alpha_i - d_i \cos\alpha_i)}{A_{i+1}E_{i+1}}, \quad P^{(2)}_{i(\alpha)63} = 0, \tag{A.12}$$

$$P^{(2)}_{i(\alpha)64} = \frac{E_i I_{x(i)} \sin\alpha_i}{A_{i+1}E_{i+1}}, \quad P^{(2)}_{i(\alpha)65} = -\frac{\omega_\alpha^2 m_i \cos\alpha_i}{A_{i+1}E_{i+1}}, \quad P^{(2)}_{i(\alpha)66} = \frac{E_i A_i \cos\alpha_i}{A_{i+1}E_{i+1}}, \tag{A.13}$$

Appendix B. Expressions for elements g_{ij} ($i = 1, 2$; $j = 1, 2$)

$$g_{11} = \sum_{i=1}^{n}\left\{\int_0^{L_i} \varrho_i A_i \left(U^{*2}_{i(\gamma)}(z_i) + W^{*2}_{i(\gamma)}(z_i)\right) dz_i + \left[J_i + m_i(d_i^2 + e_i^2)\right] W^{*\prime 2}_{i(\gamma)}(L_i)\right.$$
$$\left. +2m_i W^{*\prime}_{i(\gamma)}(L_i)\left(d_i U^{*}_{i(\gamma)}(L_i) + e_i W^{*}_{i(\gamma)}(L_i)\right) + m_i\left(U^{*2}_{i(\gamma)}(L_i) + W^{*2}_{i(\gamma)}(L_i)\right)\right\}, \tag{B.1}$$

$$g_{12} = g_{21} = \sum_{i=1}^{n}\left\{\int_0^{L_i} \varrho_i A_i \left(U^{*}_{i(\gamma)}(z_i)U^{*}_{i(\gamma+1)}(z_i) + W^{*}_{i(\gamma)}(z_i)W^{*}_{i(\gamma+1)}(z_i)\right) dz_i + \left[J_i + m_i(d_i^2 + e_i^2)\right] W^{*\prime}_{i(\gamma)}(L_i)W^{*\prime}_{i(\gamma+1)}(L_i)\right.$$
$$+m_i e_i\left(W^{*}_{i(\gamma)}(L_i)W^{*\prime}_{i(\gamma+1)}(L_i) + W^{*}_{i(\gamma+1)}(L_i)W^{*\prime}_{i(\gamma)}(L_i)\right) + m_i\left[W^{*}_{i(\gamma)}(L_i)W^{*}_{i(\gamma+1)}(L_i) + U^{*}_{i(\gamma)}(L_i)U^{*}_{i(\gamma+1)}(L_i)\right]$$
$$\left. +m_i d_i\left[W^{*\prime}_{i(\gamma)}(L_i)U^{*}_{i(\gamma+1)}(L_i) + W^{*\prime}_{i(\gamma+1)}(L_i)U^{*}_{i(\gamma)}(L_i)\right]\right\}, \tag{B.2}$$

$$g_{22} = \sum_{i=1}^{n}\left\{\int_0^{L_i} \varrho_i A_i \left(U^{*2}_{i(\gamma+1)}(z_i) + W^{*2}_{i(\gamma+1)}(z_i)\right) dz_i + \left[J_i + m_i(d_i^2 + e_i^2)\right] W^{*\prime 2}_{i(\gamma+1)}(L_i)\right.$$
$$\left. +2m_i W^{*\prime}_{i(\gamma+1)}(L_i)\left(d_i U^{*}_{i(\gamma+1)}(L_i) + e_i W^{*}_{i(\gamma+1)}(L_i)\right) + m_i\left(U^{*2}_{i(\gamma+1)}(L_i) + W^{*2}_{i(\gamma+1)}(L_i)\right)\right\}. \tag{B.3}$$

Appendix C. Expressions for elements g_i ($i = 1, \ldots, 4$)

$$g_1 = \sum_{i=1}^{n}\left\{\int_0^{L_i} \varrho_i A_i \left(f_{wi}(z_i)W^{*}_{i(\gamma)}(z_i) + f_{ui}(z_i)U^{*}_{i(\gamma)}(z_i)\right) dz_i + \left[J_i + m_i(d_i^2 + e_i^2)\right] f'_{wi}(L_i)W^{*\prime}_{i(\gamma)}(L_i)\right.$$
$$+m_i d_i\left(f_{ui}(L_i)W^{*\prime}_{i(\gamma)}(L_i) + f'_{wi}(L_i)U^{*}_{i(\gamma)}(L_i)\right) + m_i e_i\left(f'_{wi}(L_i)W^{*}_{i(\gamma)}(L_i) + f_{wi}(L_i)W^{*\prime}_{i(\gamma)}(L_i)\right)$$
$$\left. +m_i\left(f_{ui}(L_i)U^{*}_{i(\gamma)}(L_i) + f_{wi}(L_i)W^{*}_{i(\gamma)}(L_i)\right)\right\}, \tag{C.1}$$

$$g_2 = \sum_{i=1}^{n}\left\{\int_0^{L_i} \varrho_i A_i \left(f_{wi}(z_i)W^{*}_{i(\gamma+1)}(z_i) + f_{ui}(z_i)U^{*}_{i(\gamma+1)}(z_i)\right) dz_i + \left[J_i + m_i(d_i^2 + e_i^2)\right] f'_{wi}(L_i)W^{*\prime}_{i(\gamma+1)}(L_i)\right.$$

Journal Pre-proof

$$+m_i d_i \left(f_{ui}(L_i) W^{*\prime}_{i(\gamma+1)}(L_i) + f'_{wi}(L_i) U^{*}_{i(\gamma+1)}(L_i) \right) + m_i e_i \left(f'_{wi}(L_i) W^{*}_{i(\gamma+1)}(L_i) + f_{wi}(L_i) W^{*\prime}_{i(\gamma+1)}(L_i) \right)$$

$$+m_i \left(f_{ui}(L_i) U^{*}_{i(\gamma+1)}(L_i) + f_{wi}(L_i) W^{*}_{i(\gamma+1)}(L_i) \right) \Big\} . \tag{C.2}$$

$$g_3 = \sum_{i=1}^{n} \left\{ \int_0^{L_i} \varrho_i A_i \left(h_{wi}(z_i) W^{*}_{i(\gamma)}(z_i) + h_{ui}(z_i) U^{*}_{i(\gamma)}(z_i) \right) dz_i + \left[J_i + m_i (d_i^2 + e_i^2) \right] h'_{wi}(L_i) W^{*\prime}_{i(\gamma)}(L_i) \right.$$

$$+m_i d_i \left(h_{ui}(L_i) W^{*\prime}_{i(\gamma)}(L_i) + h'_{wi}(L_i) U^{*}_{i(\gamma)}(L_i) \right) + m_i e_i \left(h'_{wi}(L_i) W^{*}_{i(\gamma)}(L_i) + h_{wi}(L_i) W^{*\prime}_{i(\gamma)}(L_i) \right)$$

$$+m_i \left(h_{ui}(L_i) U^{*}_{i(\gamma)}(L_i) + h_{wi}(L_i) W^{*}_{i(\gamma)}(L_i) \right) \Big\} , \tag{C.3}$$

$$g_4 = \sum_{i=1}^{n} \left\{ \int_0^{L_i} \varrho_i A_i \left(h_{wi}(z_i) W^{*}_{i(\gamma+1)}(z_i) + h_{ui}(z_i) U^{*}_{i(\gamma+1)}(z_i) \right) dz_i + \left[J_i + m_i (d_i^2 + e_i^2) \right] h'_{wi}(L_i) W^{*\prime}_{i(\gamma+1)}(L_i) \right.$$

$$+m_i d_i \left(h_{ui}(L_i) W^{*\prime}_{i(\gamma+1)}(L_i) + h'_{wi}(L_i) U^{*}_{i(\gamma+1)}(L_i) \right) + m_i e_i \left(h'_{wi}(L_i) W^{*}_{i(\gamma+1)}(L_i) + h_{wi}(L_i) W^{*\prime}_{i(\gamma+1)}(L_i) \right)$$

$$+m_i \left(h_{ui}(L_i) U^{*}_{i(\gamma+1)}(L_i) + h_{wi}(L_i) W^{*}_{i(\gamma+1)}(L_i) \right) \Big\} . \tag{C.4}$$

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