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On game value for a pursuit-evasion differential game with state and integral constraints

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Abstract
A pursuit-evasion differential game of countably many pursuers and one evader is investigated. Integral constraints are imposed on the control functions of the players. Duration of the game is fixed, and the payoff functional is the greater lower bound of distances between the pursuers and evader when the game is completed. The pursuers want to minimize, and the evader to maximize the payoff. In this paper, we find the value of the game and construct optimal strategies for the players.

Keywords Pursuit-evasion games · Dynamical systems · The value of the game

1 Introduction and preliminaries

Differential game theory comes into play when one wants to study procedures in which others pursue one controlled object. There are several types of differential games, and one type is called the pursuit-evasion game. Many books developed mathematical foundations for the theory of differential games [1–6]. Pursuit-evasion games have several applications in robotics, such as motion planning in adversarial

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settings (e.g., playing hide-and-seek) or defining the requirements to achieve a goal in the worst-case performance of robotic systems [7].

Designing the player's optimal strategies and finding the value of the game are of specific interest in studying differential games with various types of constraints on control functions of the players, see, for example, [8–22]. Some other investigations were dedicated to study of differential games with integral constraints on control functions of the players in which the main result is the value of the game, see, for example, [8, 10, 16–18, 21–24]. These works are most relevant to the study of this paper.

In [24] and [17] pursuit-evasion differential games in which many pursuers chased a single evader are studied. All players perform simple motions with the duration of the game fixed. The controls of a group of pursuers are subject to integral constraints, whereas the controls of the remaining pursuers and that of the evader are subject to geometric constraints. The payoff of the game is the distance between the evader and closest pursuer at the instant the game is over. In both papers, optimal strategies of the players are constructed, and the value of the game is found using different approaches. In the latter paper, countably many number of pursuers were considered in place of a finite number in the former.

Salimi and Ferrara [16] and Ibragimov and Kuchkarov [20] all studied the pursuit-evasion game of plenty pursuers and one evader. Players dynamics obey simple motion with control functions of players subject to integral constraints. In these two studies, optimal strategies of the players are constructed, and the value of the game is obtained under different conditions.

Ibragimov and Salimi [10] studied a pursuit-evasion game problem of fixed duration and infinitely many pursuers and one evader. Players' dynamic equations are given by second-order differential equations of a specific type in the Hilbert space l_2. The players' control functions are subject to integral constraints. They obtained sufficient conditions for finding the game's value and constructed optimal strategies for the players. Ibragimov et al. [18] improved the result obtained in [10] by eliminating a condition under which the value of the game is obtained in the former paper. Furthermore, the game problem studied in [10] but with geometric constraints on control functions of the players is studied in [19] and obtained the game value.

Badakaya studied differential game problems involving a countable number of pursuer and one evader in [22]. In this work, players' motions obey first-order differential equations with some functions contained in the homogeneous terms, and integral constraints on control functions of the players are considered. The payoff functional is the greatest lower bound of distances between pursuers and the evader. The paper's main results are a formula for computing the value of the game and players' optimal strategies.

The paper [21] is also concerned with the problem of finding the value of the game and the construction of optimal strategies of players. The problem considered in this paper consist of countably many pursuers and one evader. The dynamic of each of the pursuers is governed by the first-order and that of the evader by a second-order differential equation. The control function for each of the players satisfies an integral constraint. The distance between the evader and the closest pursuer at the

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stoppage time of the game is the payoff of the game. The goal of the pursuers is to minimize the distance to the evader, and that of the evader is the opposite.

Badakaya and his coauthors [25] studied a pursuit-evasion differential game problem with countable number pursuers and one evader in the Hilbert space l_2. Players' dynamic equations described by certain nth order ordinary differential equations. Control functions of the players are subject to integral constraints. They obtained the value of the game and constructed optimal strategies for the players. This problem but with geometric constraints on control functions of the players is studied in [26].

This present work discusses an optimal pursuit-evasion differential game with countably many pursuers and one evader in Hilbert space l_2. The control function of the pursuers and the evader has integral constraints. The game is completed at time θ. We obtain a sufficient condition to find the game value and make an optimal strategy for the pursuer, which guarantees to capture the evader. We also show the admissibility of the suggested strategy.

2 Formulation of the problem and result

In l_2 consisting of components $a = (a_1, a_2, \dots, a_k, \dots)$, with $\sum_{k=1}^{\infty} a_k^2 < \infty$, and inner product $(a, b) = \sum_{k=1}^{\infty} a_k b_k$, the movements of the pursuer $P_i, i \in I = \{1, 2, 3, \dots\}$ and the evader E are introduced by the hybrid system of differential equations

$$\begin{aligned} P_i &: \frac{d^n p_i}{dt^n} = \mu_i(t), \quad p_i(0) = p_i^0, \quad \frac{dp_i}{dt}(0) = p_i^1, \quad \dots, \quad \frac{dp_i^{n-1}}{dt^{n-1}}(0) = p_i^{n-1}, \\ E &: \frac{d^m e}{dt^m} = \eta(t), \quad e(0) = e^0, \quad \frac{de}{dt}(0) = e^1, \quad \dots, \quad \frac{de^{m-1}}{dt^{m-1}}(0) = e^{m-1}, \end{aligned} \tag{2.1}$$

where $n < m$ and $p_i, p_i^0, \dots p_i^{n-1}, \mu_i, e, e^0, \dots e^{m-1}, \eta \in l_2$, $\mu = (\mu_{i1}, \mu_{i2}, \dots)$ and $\eta = (\eta_1, \eta_2, \dots)$ are the control variables for the pursuer P and the evader E, respectively. Let the duration of the game be denoted by a fixed positive number θ.

We define $B(p_0, \delta) = \{p \in l_2 : \|p - p_0\| \le \delta\}$ as a ball with radius δ centered at p_0.

Definition 2.1 An *admissible control of the pursuer* is a function $\mu_i(\cdot)$, $\mu_i : [0, \theta] \to l_2$, where $\mu_{ik} : [0, \theta] \to R^1, k = 1, 2, \dots$, are measurable functions and

$$\|\mu_i(\cdot)\|_2^2 = \int_0^{\theta} \|\mu_i(s)\|^2 \, ds \le \rho_i^2, \quad \|\mu_i\|^2 = \sum_{k=1}^{\infty} \mu_{ik}^2,$$

where ρ_i is a fixed positive number.

Definition 2.2 An *admissible control of the evader* is a function $\eta(\cdot), \eta : [0, \theta] \to l_2$, where $\eta_k : [0, \theta] \to R^1, k = 1, 2, \dots$, are measurable functions and

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$$\|\eta(\cdot)\|_2^2 = \int_0^\theta \|\eta(s)\|^2\, ds \le \sigma^2, \quad \|\eta\|^2 = \sum_{k=1}^{\infty} \eta_k^2,$$

where σ is a fixed positive number.

For admissible controls $\mu_i(\cdot)$ and $\eta(\cdot)$ of pursuer and evader, the corresponding movements $p_i(\cdot)$ and $e(\cdot)$ are defined as

$$p_i(t) = (p_{i1}(t), p_{i2}(t), \dots, p_{ik}(t), \dots), \quad e(t) = (e_1(t), e_2(t), \dots, e_k(t), \dots),$$

$$\begin{aligned} p_{ik}(t) &= p_{ik}^0 + tp_{ik}^1 + \frac{t^2}{2!}p_{ik}^2 \\ &\quad + \cdots + \frac{t^{n-1}}{(n-1)!}p_{ik}^{n-1} + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mu_{ik}(s)\ dt_{n-1}\ \cdots\ dt_1\ ds, \\ e_k(t) &= e_k^0 + te_k^1 + \frac{t^2}{2!}e_k^2 + \cdots + \frac{t^{m-1}}{(m-1)!}e_k^{m-1} \\ &\quad + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \eta_k(s)\ dt_{m-1}\ \dots\ dt_1\ ds. \end{aligned}$$

One could observe that $p_i(\cdot),\ e(\cdot) \in C(0,\theta;l_2)$, such that $C(0,\theta;l_2)$ is the space of functions

$$F(t) = (F_1(t), F_2(t), \dots, F_k(t), \dots) \in l_2, \quad t \ge 0,$$

in which the following properties are valid

(I) $F_k(t), 0 \le t \le \theta, k = 1, 2, \dots,$ are absolutely continuous functions;
(II) $F(t),\ 0 \le t \le \theta,$ is a continuous function with norm of l_2.

Definition 2.3 A *strategy of the pursuer P* is a function $\chi_i(t, p_i, e, \eta)$, $\chi_i : [0,\infty) \times l_2 \times l_2 \times l_2 \to l_2$, in which the system

$$\begin{aligned} &\frac{d^n p_i}{dt^n} = \chi_i(t, p_i, e, \eta), \quad p_i(0) = p_i^0, \quad \frac{dp_i}{dt}(0) = p_i^1, \quad \dots \quad \frac{dp_i^{n-1}}{dt^{n-1}}(0) = p_i^{n-1}, \\ &\frac{d^m e}{dt^m} = \eta, \quad e(0) = e^0, \quad \frac{de}{dt}(0) = e^1, \quad \dots \quad \frac{de^{m-1}}{dt^{m-1}}(0) = e^{m-1}, \end{aligned}$$

has a unique answer $(p_i(\cdot), e(\cdot))$, with $p_i(\cdot), e(\cdot) \in C(0,\theta;l_2)$, for an arbitrary admissible control $\eta = \eta(t), 0 \le t \le \theta$, of the evader E. A strategy χ_i is *admissible* if each control obtained by this strategy is admissible too.

For the admissible control $\mu_i(t) = (\mu_{i1}(t), \mu_{i2}(t), \dots),\ 0 \le t \le \theta$, of the pursuer P_i, according to (2.1) we have

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$$
\begin{aligned}
p_i(\theta) &= p_{i0} + \int_0^{\theta}\int_0^{t_1}\dots\int_0^{t_{n-1}} \mu_i(s)\ ds\ dt_{n-1}\ \dots\ dt_1 \\
&= p_{i0} + \int_0^{\theta} \frac{(\theta - t)^{n-1}}{(n-1)!}\mu_i(t)\ dt,
\end{aligned}
$$

where $p_{i0} = p_i^0 + \theta p_i^1 + \dots + \frac{\theta^{(n-1)}}{(n-1)!}p_i^{(n-1)}$. Also, for the evader the same argument can be written in this way

$$
\begin{aligned}
e(\theta) &= e_0 + \int_0^{\theta}\int_0^{t_1}\dots\int_0^{t_{m-1}} \eta(s)\ dt_{m-1}\ \dots\ dt_1 \\
&= e_0 + \int_0^{\theta} \frac{(\theta - t)^{m-1}}{(m-1)!}\eta(t)\ dt,
\end{aligned}
$$

where $e_0 = e^0 + \theta e^1 + \dots + \frac{\theta^{(m-1)}}{(m-1)!}e^{(m-1)}$. Therefore, rather than the game (2.1) one can use an equivalent game, with same payoff function, as the following:

$$
\begin{aligned}
P_i :\ \ \dot{p}_i(t) &= \frac{(\theta - t)^{n-1}}{(n-1)!}\mu_i(t), \quad p_i(0) = p_{i0}, \\
E :\ \ \dot{e}(t) &= \frac{(\theta - t)^{m-1}}{(m-1)!}\eta(t), \quad e(0) = e_0.
\end{aligned}
\tag{2.2}
$$

The main idea of the above estimations is borrowed from [2], page 9.

Proposition 2.0.1 *The closed ball* $B\left(p_{i0}, \left(\frac{\theta^{2n-1}}{2n-1}\right)^{1/2}\frac{\rho_i}{(n-1)!}\right)$ *is the obtainability domain of the pursuer* P_i *at time* θ *from the initial position* p_{i0}.

Proof By Cauchy-Schwartz inequality we obtain

$$
\begin{aligned}
\|p_i(\theta) - p_{i0}\| &= \left\| \int_0^{\theta} \frac{(\theta - s)^{n-1}}{(n-1)!}\mu_i(s)\,ds \right\| \\
&\le \int_0^{\theta} \left\| \frac{(\theta - s)^{n-1}}{(n-1)!}\mu_i(s) \right\| ds \\
&\le \left(\int_0^{\theta} \left(\frac{(\theta - s)^{n-1}}{(n-1)!} \right)^2 ds \right)^{1/2} \cdot \left(\int_0^{\theta} \|\mu_i(s)\|^2\,ds \right)^{1/2} \\
&\le \left(\frac{\theta^{2n-1}}{2n-1} \right)^{1/2} \frac{\rho_i}{(n-1)!}.
\end{aligned}
$$

Let $\bar{p} \in B\left(p_{i0}, \left(\frac{\theta^{2n-1}}{2n-1}\right)^{1/2}\frac{\rho}{(n-1)!}\right)$. If the pursuer P_i uses the control

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$$\mu_i(t) = \frac{(2n-1)(n-1)!}{\theta^{2n-1}}(\theta - t)^{n-1}(\bar{p} - p_{i0}), \quad 0 \le t \le \theta,$$

then we obtain

$$\begin{aligned} p_i(\theta) &= p_{i0} + \int_0^\theta \frac{(\theta - t)^{n-1}}{(n-1)!}\mu_i(t)\, dt, \\ &= p_{i0} + \int_0^\theta \frac{(2n-1)(\theta - t)^{2n-2}}{\theta^{2n-1}}(\bar{p} - p_{i0})\, dt, \\ &= p_{i0} + \frac{2n-1}{\theta^{2n-1}}(\bar{p} - p_{i0}) \int_0^\theta (\theta - t)^{2n-2}\, dt = \bar{p}. \end{aligned}$$

The above pursuer's control is admissible. Indeed,

$$\begin{aligned} \int_0^\theta \|\mu_i(t)\|^2\, dt &= \int_0^\theta \left\| \frac{(2n-1)(n-1)!}{\theta^{2n-1}}(\theta - t)^{n-1}(\bar{p} - p_{i0}) \right\|^2 dt \\ &\le \left(\frac{(2n-1)(n-1)!}{\theta^{2n-1}} \frac{\rho_i}{(n-1)!} \right)^2 \frac{\theta^{2n-1}}{2n-1} \int_0^\theta (\theta - t)^{2n-2} dt = \rho_i^2. \end{aligned}$$

□

Proposition 2.0.2 *The closed ball* $B\left(e_0, \left(\frac{\theta^{2m-1}}{2m-1}\right)^{1/2} \frac{\sigma}{(m-1)!}\right)$ *is the obtainability domain of the evader E at time* θ *from the initial position* e_0.

Proof We have

$$\begin{aligned} \|e(\theta) - e_0\| &= \left\| \int_0^\theta \frac{(\theta - s)^{m-1}}{(m-1)!}\eta(s)\, ds \right\| \\ &\le \int_0^\theta \left\| \frac{(\theta - s)^{m-1}}{(m-1)!}\eta(s) \right\| ds \\ &\le \left(\int_0^\theta \left(\frac{(\theta - s)^{m-1}}{(m-1)!} \right)^2 ds \right)^{1/2} \cdot \left(\int_0^\theta \|\eta(s)\|^2\, ds \right)^{1/2} \\ &\le \left(\frac{\theta^{2m-1}}{2m-1} \right)^{1/2} \frac{\sigma}{(m-1)!}. \end{aligned}$$

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Let $\bar{e} \in B\left(e_0, \left(\frac{\theta^{2m-1}}{2m-1}\right)^{1/2} \frac{\sigma}{(m-1)!}\right)$. If the evader E uses the control

$$\eta(t) = \frac{(2m-1)(m-1)!}{\theta^{2m-1}}(\theta - t)^{m-1}(\bar{e} - e_0), \quad 0 \le t \le \theta,$$

then we obtain

$$\begin{aligned} e(\theta) &= e_0 + \int_0^\theta \frac{(\theta - t)^{m-1}}{(m-1)!}\eta(t)\,dt \\ &= e_0 + \int_0^\theta \frac{(2m-1)(\theta - t)^{2m-2}}{\theta^{2m-1}}(\bar{e} - e_0)\,dt \\ &= e_0 + \frac{2m-1}{\theta^{2m-1}}(\bar{e} - e_0)\int_0^\theta (\theta - t)^{2m-2}\,dt = \bar{e}. \end{aligned}$$

The above evader's control is admissible. Indeed,

$$\begin{aligned} \int_0^\theta \|\eta(t)\|^2\,dt &= \int_0^\theta \left\| \frac{(2m-1)(m-1)!}{\theta^{2m-1}}(\theta - t)^{m-1}(\bar{e} - e_0)\right\|^2 dt \\ &\le \left(\frac{(2m-1)(m-1)!}{\theta^{2m-1}}\frac{\sigma}{(m-1)!}\right)^2 \frac{\theta^{2m-1}}{2m-1}\int_0^\theta (\theta - t)^{2m-2}dt = \sigma^2. \end{aligned}$$

□

The problem is to make a winning strategy for the pursuer in the game (2.1) with $p_{i0} \ne e_0$. That is to find winning strategy of the pursuer P_i that guarantees the equality $p_i(\theta) = e(\theta)$, for any admissible control of the rival. Let

$$X_i = \left\{ q \in l_2 \,:\, 2(e_0 - p_{i0}, q) \le \beta_j + \|e_0\|^2 - \|p_{i0}\|^2, \;\; e_0 \ne p_{i0} \right\}, \qquad (2.3)$$

where $\beta_j = \begin{cases} \frac{\theta^{2(m-n)}}{(n!)^2}(\rho_i^2 - \theta^{2m}\sigma^2), & \theta \ge 1, \\ \frac{\theta^{2n-1}}{(n!)^2}(\rho_i^2 - \sigma^2), & \theta < 1. \end{cases}$

Theorem 1 *If $e(\theta) \in X$ (phase constraint), then the pursuer has a winning strategy.*

Proof Let's define the following strategy as a winning strategy for the pursuer

$$\chi(t) = \frac{(e_0 - p_{i0})n!}{\theta^n} + \frac{(n-1)!}{(m-1)!}(\theta - t)^{m-n}\eta(t), \;\; 0 \le t \le \theta.$$

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We show that the above strategy is admissible. Since the evader is satisfied to the phase constraint, we have

$$2\big(e_0 - p_{i0}, e(\theta)\big) \le \beta_j + \|e_0\|^2 - \|p_{i0}\|^2.$$

Using the above inequality we have the following:

$$2\left(e_0 - p_{i0}, \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!}\eta(t)\,dt\right) \le \beta_j - \|e_0 - p_{i0}\|^2.$$

Indeed,

$$\begin{aligned}
2\left(e_0 - p_{i0}, \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!}\eta(t)\,dt\right) =& 2\big(e_0 - p_{i0}, e(\theta) - e_0\big)\\
=& 2\big(e_0 - p_{i0}, e(\theta)\big) - 2\big(e_0 - p_{i0}, e_0\big)\\
=& 2\big(e_0 - p_{i0}, e(\theta)\big) - 2\|e_0\|^2 + 2(e_0, p_{i0})\\
\le& \beta_j + \|e_0\|^2 - \|p_{i0}\|^2 - 2\|e_0\|^2 + 2(e_0, p_{i0})\\
=& \beta_j - \|e_0\|^2 - \|p_{i0}\|^2 + 2(e_0, p_{i0})\\
=& \beta_j - \|e_0 - p_{i0}\|^2.
\end{aligned}$$

Thus, taking contribution of above inequality we have,

$$\begin{aligned}
\int_0^\theta \|\chi(t)\|^2\,dt =& \int_0^\theta \left\| \frac{(e_0 - p_{i0})n!}{\theta^n} + \frac{(n-1)!}{(m-1)!}(\theta-t)^{m-n}\eta(t)\right\|^2 dt\\
=& \int_0^\theta \left\|\frac{(e_0 - p_{i0})n!}{\theta^n}\right\|^2 dt + 2\int_0^\theta \left(\frac{(e_0 - p_{i0})n!}{\theta^n}, \frac{(n-1)!}{(m-1)!}(\theta-t)^{m-n}\eta(t)\right)dt\\
&+ \int_0^\theta \left\|\frac{(n-1)!}{(m-1)!}(\theta-t)^{m-n}\eta(t)\right\|^2 dt\\
\le& \frac{\|e_0 - p_{i0}\|^2}{\theta^{2n-1}}(n!)^2 + \frac{2n!(n-1)!}{\theta^n}\left(e_0 - p_{i0}, \int_0^\theta \frac{(\theta-t)^{m-n}}{(m-1)!}\eta(t)\,dt\right)\\
&+ \int_0^\theta (\theta-t)^{2(m-n)}\|\eta(t)\|^2 dt\\
\le& \frac{\|e_0 - p_{i0}\|^2}{\theta^{2n-1}}(n!)^2 + \frac{(n!)^2}{\theta^n}\big(\beta_j - \|e_0 - p_{i0}\|^2\big) + \theta^{2m}\sigma^2.
\end{aligned} \tag{2.4}$$

Note that for this deduction we use the fact that (noting that $t \in [0, \theta]$)

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$$\int_0^\theta (\theta - t)^{2(m-n)} \|\eta(t)\|^2 dt \le \int_0^\theta \theta^{2m} \|\eta(t)\|^2 dt < \theta^{2m}\sigma^2.$$

For the case $\theta \ge 1$ and noting that $n < m$, then from inequality (2.4) we achieve the admissibility of the wining strategy of the pursuer as follows:

$$\begin{aligned}\int_0^\theta \|\chi(t)\|^2\, dt \le& \frac{\|e_0 - p_{i0}\|^2}{\theta^{2n-1}}(n!)^2 + \frac{(n!)^2}{\theta^n}\left(\frac{\theta^{2(m-n)}}{(n!)^2}(\rho_i^2 - \theta^{2m}\sigma^2) - \|e_0 - p_{i0}\|^2\right) + \theta^{2m}\sigma^2\\ \le& \frac{\|e_0 - p_{i0}\|^2}{\theta^{2(n-m)}}(n!)^2 + \frac{(n!)^2}{\theta^{2(n-m)}}\left(\frac{\theta^{2(m-n)}}{(n!)^2}(\rho_i^2 - \theta^{2m}\sigma^2) - \|e_0 - p_{i0}\|^2\right) + \theta^{2m}\sigma^2\\ =&\rho_i^2.\end{aligned}$$

In the other hand, for the case $\theta < 1$ then we the admissibility of the wining strategy of the pursuer from the following

$$\begin{aligned}\int_0^\theta \|\chi(t)\|^2\, dt \le& \frac{\|e_0 - p_{i0}\|^2}{\theta^{2n-1}}(n!)^2 + \frac{(n!)^2}{\theta^n}\left(\frac{\theta^{2n-1}}{(n!)^2}(\rho_i^2 - \sigma^2) - \|e_0 - p_{i0}\|^2\right) + \theta^{2m}\sigma^2\\ \le& \frac{\|e_0 - p_{i0}\|^2}{\theta^{2n-1}}(n!)^2 + \frac{(n!)^2}{\theta^{2n-1}}\left(\frac{\theta^{2n-1}}{(n!)^2}(\rho_i^2 - \theta^{2m}\sigma^2) - \|e_0 - p_{i0}\|^2\right) + \theta^{2m}\sigma^2\\ =&\rho_i^2.\end{aligned}$$

Therefore the strategy χ is admissible.

Now we show that χ is a winning strategy for the pursuer. Indeed,

$$\begin{aligned}p(\theta) =& p_{i0} + \int_0^\theta \frac{(\theta - t)^{n-1}}{(n-1)!}\left(\frac{(e_0 - p_{i0})n!}{\theta^n} + \frac{(n-1)!}{(m-1)!}(\theta - t)^{m-n}\mu(t)\right) dt\\ =& p_{i0} + \frac{(e_0 - p_{i0})n!}{\theta^n (n-1)!}\int_0^\theta (\theta - t)^{n-1}\, dt + \int_0^\theta \frac{(\theta - t)^{m-1}}{(m-1)!}\eta(t)\, dt\\ =& p_{i0} + e_0 - p_{i0} + \int_0^\theta \frac{(\theta - t)^{m-1}}{(m-1)!}\eta(t)\, dt = e(\theta).\end{aligned}$$

□

3 Main result

Now consider the game (2.1). We will solve the optimal problem under the following assumption.

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Assumption 1 There exits a nonzero vector z such that $(e_0 - p_{i0}, z) \geq 0$ for all $i \in I$.
Let

$$\gamma = \inf\left\{ l \geq 0 : \quad B\left(e_0, \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{\frac{1}{2}}\right) \subset \bigcup_{i=1}^{\infty} B\left(p_{i0}, \frac{\rho_i}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + l\right)\right\}. \tag{3.1}$$

Theorem 2 *If Assumption 1 is true for all $i \in I$, then the number γ given by (3.1) is the value of the game (2.1).*

Proof of the above theorem relies on the following lemmas. Consider the sphere $S(e_0, r)$ and finitely or countably many balls $B(p_{i0}, R_i)$ and $B(e_0, r)$, where $p_{i0} \neq e_0$ and r and R_i, $i \in I$ are positive numbers.

Lemma 1 *See [8]. Let*

$$X_i = \left\{ z \in l_2 : \quad 2(e_0 - p_{i0}, z) \leq R_i^2 - r^2 + \|e_0\|^2 - \|p_{i0}\|^2 \right\}. \tag{3.2}$$

If Assumption 1 is valid and

$$B(e_0, r) \subset \bigcup_{i \in I} B(p_{i0}, R_i), \tag{3.3}$$

then $B(e_0, r) \subset \bigcup_{i \in I} X_i$.

Lemma 2 *Let $\inf_{i \in I} R_i = R_0 > 0$. If Assumption 1 is true and for any $0 < \varepsilon < R_0$ the set $\bigcup_{i \in I} B(p_{i0}, R_i - \varepsilon)$ does not contain the ball $B(e_0, r)$, then there exists a point $\bar{e} \in S(e_0, r)$such that $\|\bar{e} - p_{i0}\| \geq R_i$.*

Proof of Theorem 2. We prove this Theorem in three parts.

(1) *Pursuers' Strategies construction.* We introduce dummy pursuer z_i, whose motions obey the equations

$$\dot{z}_i(t) = \frac{(\theta - t)^{n-1}}{(n-1)!} w_i^{\varepsilon}, \qquad z_i(0) = p_{i0},$$

$$\left(\int_0^{\theta} \|w_i^{\varepsilon}(s)\|^2 ds \right)^{\frac{1}{2}} \leq \bar{\rho}_i(\varepsilon) = \rho_i + \gamma(n-1)!\left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}} + \frac{\varepsilon(n-1)!}{k_i}\left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}}, \tag{3.4}$$

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where $k_i = \max\{1, \rho_i\}$, ε is positive number such that $\varepsilon \in (0, 1)$. It is clear that the attainability domain of the dummy pursuer z_i at time θ from an initial state p_{i0} is the ball $B(p_{i0}, r^*)$, where $r^* = \frac{\bar{\rho}_i(\varepsilon)}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} = \frac{\rho_i}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + \gamma + \frac{\varepsilon}{k_i}$.

The strategies of the dummy pursuers $z_i, i \in I$ are defined as follows:

$$w_i^\varepsilon(t) = \begin{cases} \frac{(e_0 - p_{i0})n!}{\theta^n} + \frac{(n-1)!}{(m-1)!}(\theta - t)^{m-n}\eta(t), & 0 \le t \le \tau_i^\varepsilon, \\ 0, & \tau_i^\varepsilon < t \le \theta, \end{cases} \tag{3.5}$$

where $\tau_i^\varepsilon \in [0,\ \theta]$, is the time for which the dummy z_i exhausted its energy. That is

$$\int_0^{\tau_i^\varepsilon} \|w_i^\varepsilon(s)\|^2 ds = \bar{\rho}_i^{\,2}(\varepsilon), \tag{3.6}$$

if such a time exists. Using the strategy of the ith dummy pursuer, we define strategy of the real pursuer p_i as follows.

$$\mu_i(t) = \frac{\rho_i}{\bar{\rho}_i} w_i(t), \qquad 0 \le t \le \theta, \tag{3.7}$$

where $\bar{\rho}_i = \bar{\rho}_i(0) = \rho_i + \gamma(n-1)!\left(\frac{2n-1}{\theta^{2n-1}}\right)^{\frac{1}{2}}$ and

$$w_i(t) = \begin{cases} \frac{(e_0 - p_{i0})n!}{\theta^n} + \frac{(n-1)!}{(m-1)!}(\theta - t)^{m-n}\eta(t), & 0 \le t \le \tau_i, \\ 0, & \tau_i < t \le \theta, \end{cases} \tag{3.8}$$

where $\tau_i \in [0,\ \theta]$, is the time for which

$$\int_0^{\tau_i} \|w_i(s)\|^2 ds = \bar{\rho}_i^{\,2}. \tag{3.9}$$

That is, $w_i(t)$ is obtained from (3.5) at $\varepsilon = 0$. Since $\bar{\rho}_i(\varepsilon) > \bar{\rho}_i$, then

$$\int_0^{\tau_i^\varepsilon} \|w_i^\varepsilon(s)\|^2 ds = \bar{\rho}_i^{\,2}(\varepsilon) > \rho_i^2 = \int_0^{\tau_i} \|w_i(s)\|^2 ds. \tag{3.10}$$

Therefore $\tau_i^\varepsilon > \tau_i$.

(2) *The game value γ is true for the pursuers*. We shall show that strategies (3.7) of pursuer ensure that

$$\sup_{\eta(\cdot)} \inf_{i \in I} \|e(\theta) - p_i(\theta)\| \le \gamma. \tag{3.11}$$

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In accordance with the definition of the number γ, we have

$$B\left(e_0, \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{1/2}\right) \subset \bigcup_{i=1}^{\infty} B\left(p_{i0}, \frac{\rho_i}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + \gamma + \frac{\varepsilon}{k_i}\right), \tag{3.12}$$

Denote

$$X_i^\varepsilon = \left\{ q : \ 2(e_0 - p_{i0}, q) \le \left(A_1 + \gamma + \varepsilon/k_i\right)^2 - A_2^2 + \|e_0\|^2 - \|p_{i0}\|^2 \right\}, \tag{3.13}$$

where $A_1 = \frac{\rho_i}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}$ and $A_2 = \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{\frac{1}{2}}$.

According to the condition of the theorem $(e_0 - p_{i0},\ q) \ge 0$ for all $i \in I$. Then by Lemma 1 it follows from (3.12) that

$$B\left(e_0, \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{\frac{1}{2}}\right) \subset \bigcup_{i=1}^{\infty} X_i^\varepsilon. \tag{3.14}$$

Consequently, for the point $e(\theta) \in B\left(e_0, \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{\frac{1}{2}}\right)$ we have the inclusion $e(\theta) \in X_s^\varepsilon$ for some at some $s \in I$. If $e(\theta) \in X_s \cap X_s^\varepsilon$ and for $p_{i0} \ne e_0$ we have by the Theorem 1 and for the strategies (3.5) of dummy pursuers, $z_s(\theta) = e(\theta)$. Then taking into account of (3.4) and (3.7), we get

$$\begin{aligned} \|e(\theta) - p_s(\theta)\| &= \|z_s(\theta) - p_s(\theta)\| \\ &= \left\| \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\left(w_s^\varepsilon(t) - \frac{\rho_s}{\bar{\rho}_s} w_s(t)\right)\right\| \\ &\le \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_s^\varepsilon(t) - w_s(t)\|dt + \int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_s(t) - \frac{\rho_s}{\bar{\rho}_s} w_s(t)\|dt. \end{aligned} \tag{3.15}$$

We now estimate right-hand side of the above inequality. Firstly, we show that

$$\int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_i^\varepsilon(t) - w_i(t)\|dt \le K\sqrt{\varepsilon}, \tag{3.16}$$

for all $i \in I$ and some constant K. Indeed, as we noted above that $\tau_i^\varepsilon > \tau_i$ and accordance to (3.5) and (3.8) $w_i^\varepsilon(t) = w_i(t)$ for $0 \le t \le \tau_i : w_i(t) = 0$ for $t > \tau_i$, $w_i^\varepsilon(t) = 0$ for $t > \tau_i^\varepsilon$, then we have

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$$\int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_i^\varepsilon(t)-w_i(t)\|dt = \int_0^{\tau_i} \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_i^\varepsilon(t)-w_i(t)\|dt$$

$$+\int_{\tau_i}^{\tau_i^\varepsilon} \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_i^\varepsilon(t)-w_i(t)\|dt + \int_{\tau_i^\varepsilon}^{\theta} \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_i^\varepsilon(t)-w_i(t)\|dt$$

$$= \int_{\tau_i}^{\tau_i^\varepsilon} \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_i^\varepsilon(t)\|dt$$

$$\le \frac{1}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\left(\int_0^{\tau_i^\varepsilon}\|w_i^\varepsilon(t)\|^2dt - \int_0^{\tau_i}\|w_i^\varepsilon(t)\|^2dt\right)^{\frac{1}{2}}$$

$$= \frac{1}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\left(\bar\rho_i^2(\varepsilon)-\bar\rho_i^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}$$

$$\cdot\left[\frac{\varepsilon(n-1)!}{k_i}\left(\frac{(2n-1)}{\theta^{2n-1}}\right)^{\frac{1}{2}}\left(2\rho_i + 2\gamma(n-1)!\left(\frac{(2n-1)}{\theta^{2n-1}}\right)^{\frac{1}{2}} + \frac{\varepsilon(n-1)!}{k_i}\left(\frac{(2n-1)}{\theta^{2n-1}}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$$

$$\le K\sqrt{\varepsilon},$$

where K is some positive number and does not depend on i ($k_i \ge 1$ and $0 < \varepsilon < 1$). Thus, we have shown the inequality (3.16). For the second integral of (3.15) we have

$$\left\|\int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\left(1-\frac{\rho_s}{\bar\rho_s}\right)w_s(t)\right\| \le \left(1-\frac{\rho_s}{\bar\rho_s}\right)\int_0^\theta \frac{(\theta-t)^{n-1}}{(n-1)!}\|w_s(t)\|dt$$

$$\le \left(1-\frac{\rho_s}{\bar\rho_s}\right)\left(\int_0^\theta\left(\frac{(\theta-t)^{n-1}}{(n-1)!}\right)^2dt\right)^{\frac{1}{2}}\left(\int_0^\theta\|w_s(t)\|^2dt\right)^{\frac{1}{2}}$$

$$= \left(1-\frac{\rho_s}{\bar\rho_s}\right)\frac{1}{(n-1)!}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}}\bar\rho_s = \gamma.$$

where $\bar\rho_s = \rho_s + \gamma\left(\frac{(2n-1)}{\theta^{2n-1}}\right)^{\frac{1}{2}}$ or $\bar\rho_s - \rho_s = \gamma\left(\frac{(2n-1)}{\theta^{2n-1}}\right)^{\frac{1}{2}}$.

It follows then from (3.15) that

$$\|e(\theta)-p_s(\theta)\| \le \gamma + K\sqrt{\varepsilon}.$$

Thus, if pursuers use strategies (3.7), then inequality (3.11) holds. So the result γ is guaranteed for the pursuers.

(3) *The game value γ is true for the evader.* Let us constant the evader's strategy ensuring that

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$$\inf_{\mu_1(\cdot),\dots,\mu_m(\cdot),\dots} \inf_{i\in I} \|e(\theta)-p_i(\theta)\| \geq \gamma, \tag{3.17}$$

where $\mu_1(\cdot),\dots,\mu_m(\cdot),\dots$ are arbitrary admissible controls of the pursuers. If $\gamma = 0$, then inequality (3.17) is obviously valid for any admissible control of the evader. By the definition of γ and for any $\varepsilon^* \in (0,a)$, where $a = \inf_{i\in I}\left\{ \frac{\rho_i}{(n-1)}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + \gamma \right\}$, the set

$$\bigcup_{i=1}^{\infty} B\left(p_{i0}, \frac{\rho_i}{(n-1)}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + \gamma - \varepsilon^*\right), \tag{3.18}$$

does not contain the ball $B\left(e_0, \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{\frac{1}{2}}\right)$. Then, by Lemma 2 there exists a point $\bar{e} \in S\left(e_0, \frac{\sigma}{(m-1)!}\left(\frac{\theta^{2m-1}}{2m-1}\right)^{\frac{1}{2}}\right)$, such that $\|\bar{e}-p_{i0}\| \geq \frac{\rho_i}{(n-1)}\left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + \gamma$. In view of this, we have

$$\begin{aligned} \|\bar{e}-p_i(\theta)\| &\geq \|\bar{e}-p_{i0}\| - \|p_i(\theta)-p_{i0}\| \\ &\geq \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} + \gamma - \left(\frac{\theta^{2n-1}}{2n-1}\right)^{\frac{1}{2}} = \gamma. \end{aligned} \tag{3.19}$$

The control

$$\eta(t) = \frac{(2m-1)(m-1)!}{\theta^{2m-1}}(\theta-t)^{m-1}(\bar{e}-e_0), \quad 0 \leq t \leq \theta,$$

supplies validity of the inequality (3.17), since for this control we have

$$\begin{aligned} e(\theta) &= e_0 + \int_0^\theta \frac{(\theta-t)^{m-1}}{(m-1)!}\eta(t)\,dt \\ &= e_0 + \int_0^\theta \frac{(2m-1)(\theta-t)^{2m-2}}{\theta^{2m-1}}(\bar{e}-e_0)\,dt \\ &= e_0 + \frac{2m-1}{\theta^{2m-1}}(\bar{e}-e_0)\int_0^\theta (\theta-t)^{2m-2}\,dt = \bar{e}. \end{aligned}$$

Then the game value is not less than γ, and inequality (3.17) holds. The proof of the theorem is complete.

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4 Conclusion

This paper discussed a fixed duration pursuit-evasion problem with countably many pursuers and one evader in Hilbert space l_2. Dynamic equation of each pursuer considered to be a certain nth order differential equation and that of the evader as mth order differential equation (where $n < m$). The controls of pursuers and the evader are subject to integral constraints. An admissible strategy for the pursuer that guarantees to capture the evader was constructed. Besides, we took into account a contribution from an auxiliary differential game and guessed the value of the game, and then we proved the accuracy of our guess. Furthermore, finding the admissible strategy and value of the game for hybrid system differential equations where $n > m$ may constitute the object of future studies.

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