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Cohen-Macaulay Type of Weighted Path Ideals

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Abstract

In this dissertation we give a combinatorial characterization of all the weighted r-path suspensions for which the f-weighted r-path ideal is Cohen-Macaulay. In particular, it is shown that the f-weighted r-path ideal of a weighted r-path suspension is Cohen-Macaulay if and only if it is unmixed. Type is an important invariant of a Cohen-Macaulay homogeneous ideal in a polynomial ring R with coefficients in a field. We compute the type of R/I when I is any Cohen-Macaulay f-weighted r-path ideal of any weighted r-path suspension, for some chosen function f. In particular, this computes the type for all weighted trees T_ω such that the corresponding ideal is Cohen-Macaulay.

I am dedicating this dissertation to my parents Manlong Wei and Jihua Pu, with love.

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Contents

Chapter 1

Introduction

Combinatorial commutative algebra is a branch of mathematics that uses combinatorics and graph theory to understand algebraic constructions; it also uses algebra to understand objects in combinatorics and graph theory. Richard Stanley was the first to strongly leverage commutative algebra techniques to study combinatorial objects in his proof of the upper bound conjecture for simplicial spheres [12]. His focus was on square-free monomials ideals. Since then, the study of square-free monomial ideals has become a very active area of research in commutative algebra.

1.1 Graphs and Ideals

In this dissertation, we explore aspects of this area via path ideals of graphs and edge-weighted graphs.

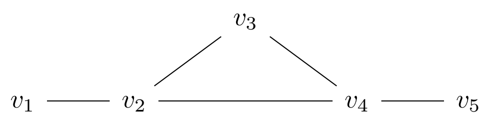
On the graph-theoretic side, let G be a (finite simple) graph with vertex set $V = V(G) = \{v_1, \ldots, v_d\}$ and edge set $E = E(G)$. To the graph G, one can attach a positive integer-valued function $\omega : E \to \mathbb{N}$, producing an edge-weighted graph G_ω.

On the algebraic side, let $\mathbb{K}$ be a field, and consider the polynomial ring $R = \mathbb{K}[X_1, \ldots, X_d]$. Villarreal [13] defined the *edge ideal* of G to be the ideal $I(G)$ that is "generated by the edges of G":

$$I(G) = (X_i X_j \mid v_i v_j \in E)R.$$

By definition, the edge ideal $I(G)$ is square-free.

Example 1.1.1. Let G be the following graph and $R = \mathbb{K}[X_1, X_2, X_3, X_4, X_5]$.



The edge ideal of G is

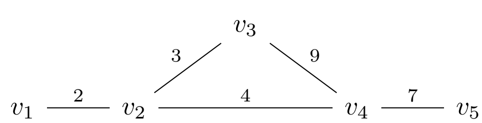
$$I(G) = (X_1X_2, X_2X_3, X_2X_4, X_3X_4, X_4X_5)R.$$

Paulsen and Sather-Wagstaff [10] introduced the weighted edge ideal of G_ω to be the ideal $I(G_\omega)$ which is generated by all monomials of the form $X_i^{\omega(v_iv_j)}X_j^{\omega(v_iv_j)}$ such that $v_iv_j \in E$:

$$I(G_\omega) = \left(X_i^{\omega(v_iv_j)}X_j^{\omega(v_iv_j)} \mid v_iv_j \in E\right) R.$$

In particular, if ω is the constant function defined by $\omega(v_iv_j) = 1$ for $v_iv_j \in E$, then $I(G_\omega) = I(G)$. By definition, the weighted edge ideal $I(G_\omega)$ is usually not square-free.

Example 1.1.2. Let G_ω be the following edge-weighted graph and $R = \mathbb{K}[X_1, X_2, X_3, X_4, X_5]$.



The weighted edge ideal of G_ω is

$$I(G_\omega) = (X_1^2X_2^2, X_2^3X_3^3, X_2^4X_4^4, X_3^9X_4^9, X_4^7X_5^7)R.$$

$I(G_\omega)$ has the same number of generators as $I(G)$.

Let $r \geq 1$. Building from Villarreal's work, Conca and De Negri [3] defined the *r-path ideal* associated to G to be the ideal $I_r(G) \subseteq R$ that is "generated by the paths in G of length r":

$$I_r(G) = (X_{i_1} \cdots X_{i_{r+1}} \mid v_{i_1} \cdots v_{i_{r+1}} \text{ is a path in } G)R.$$

In particular, if $r = 1$, then $I_1(G) = I(G)$.

Example 1.1.3. Consider the graph G from Example 1.1.1 with $r = 2$. Then the 2-path ideal of

G is

$$\begin{aligned} I_2(G) &= (X_1X_2X_3, X_1X_2X_4, X_2X_3X_4, X_2X_4X_3, X_3X_2X_4, X_2X_4X_5, X_3X_4X_5)R \\ &= (X_1X_2X_3, X_1X_2X_4, X_2X_3X_4, X_2X_4X_5, X_3X_4X_5)R. \end{aligned}$$

Expanding this to the edge-weighted context, Kubik and Sather-Wagstaff [7] defined the *weighted r-path ideal* of G_ω to be the ideal $I_r(G_\omega)$ that is "generated by the max-weighted paths in G of length r":

$$I_r(G_\omega) = \left(X_{i_1}^{e_{i_1}} \cdots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \cdots v_{i_{r+1}} \text{ is a path in } G \text{ with } e_{i_1} = \omega(v_{i_1}v_{i_2}), \\ e_{i_j} = \max(\omega(v_{i_{j-1}}v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})) \text{ for } 1 < j \le r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r}v_{i_{r+1}}) \end{array} \right) R.$$

In particular, if $r = 1$, then $I_1(G_\omega) = I(G_\omega)$.

Example 1.1.4. Consider the edge-weighted graph G_ω from Example 1.1.2. Then the weighted 2-path ideal of G_ω is

$$I_2(G_\omega) = (X_1^2X_2^3X_3^3, X_1^2X_2^4X_4^4, X_2^3X_3^9X_4^9, X_2^4X_4^9X_3^9, X_3^3X_2^4X_4^4, X_2^4X_4^7X_5^7, X_3^9X_4^9X_5^7)R.$$

Kubik and Sather-Wagstaff also consider a much more general situation. For reasonable functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ they defined the *f-weighted r-path ideal* associated to G_ω to be the ideal $I_{r,f}(G_\omega) \subseteq R$ that is "generated by the f-weighted paths in G of length r":

$$I_{r,f}(G_\omega) = \left(X_{i_1}^{e_{i_1}} \cdots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \cdots v_{i_{r+1}} \text{ is a path in } G \text{ with } e_{i_1} = \omega(v_{i_1}v_{i_2}), \\ e_{i_j} = f(\omega(v_{i_{j-1}}v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})) \text{ for } 1 < j \le r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r}v_{i_{r+1}}) \end{array} \right) R.$$

So f in the definition of $I_{r,f}(G_\omega)$ replaces the max in the definition of $I_r(G_\omega)$.

Example 1.1.5. Consider the edge-weighted graph G_ω from Example 1.1.2. Let $r = 3$ and let $f : \mathbb{N}^2 \to \mathbb{N}$ be the gcd function. Then the gcd-weighted 3-path ideal of G_ω is

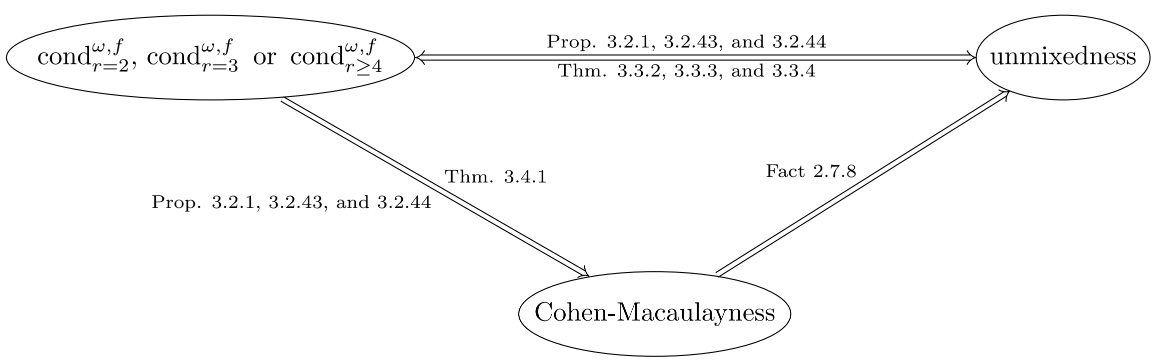
$$I_{3,\text{gcd}}(G_\omega) = (X_1^2X_2X_3^3X_4^9, X_1^2X_2^2X_4X_5^7, X_3^3X_2X_4X_5^7)R.$$

In this dissertation, we investigate two important notions in commutative algebra for these ideals: the Cohen-Macaulay property and the type, discussed next. Specifically, we show how properties of G_ω yield information about these notions.

1.2 Cohen-Macaulayness

An important concept in commutative algebra is the "Cohen-Macaulay" property; see Section 2.4 in chapter 2. The definition of Cohen-Macaulayness is somewhat technical. For now, the reader should understand that Cohen-Macaulay ideals in polynomial rings are particularly nice. If G is a tree, a theorem of Villarreal [13] characterizes when $R/I(G)$ is Cohen-Macaulay, and a theorem of Paulsen and Sather-Wagstaff [10] characterizes when $R/I(G_\omega)$ is Cohen-Macaulay. Theorems of Campos, et al. [2] and of Kubik and Sather-Wagstaff [7] characterize the Cohen-Macaulay property for $R/I_r(G)$ and $R/I_r(G_\omega)$, respectively, again for trees. These characterizations are purely graph-theoretical. In particular, they are independent of the choice of the ground field $\mathbb{K}$.

Cohen-Macaulay trees can be characterized in terms of suspensions (see [10, Definition 5.4]) when the edge ideals are considered. One of our focus area is on the path ideals of weighted trees. Let G be an r-path suspension (see Definition 3.1.14). The first goal of this dissertation is to characterize when $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay: for each $r \geq 2$, find all combinations of $\omega : E \to \mathbb{N}$ and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay. Let $\mathrm{cond}^{\omega,f}_{r=2}$, $\mathrm{cond}^{\omega,f}_{r=3}$ and $\mathrm{cond}^{\omega,f}_{r\geq 4}$ be the constraints from Propositions 3.2.1, 3.2.43, and 3.2.44, respectively. The diagram of proof is as follows.



Thus, given that G is an r-path suspension, $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay if and only if $\mathrm{cond}^{\omega,f}_{r=2}$, $\mathrm{cond}^{\omega,f}_{r=3}$ or $\mathrm{cond}^{\omega,f}_{r\geq 4}$ is satisfied. Furthermore, we use this to characterize the Cohen-Macaulay property for $R/I_{r,f}(G_\omega)$ when G is a tree. The results are in Theorems 3.5.5 and 3.5.6.

We use similar techniques to study certain non-square-free monomial ideals as in studying square-free monomials. However, we observe that the useful polarization technique used in [7, 10] fails in studying the Cohen-Macaulayness of $R/I_{r,f}(G_\omega)$. We solved this by combining commutative algebraic techniques and combinatorial analysis. It is reflected in the theorems and propositions from the above proof diagram.

1.3 Cohen-Macaulay Type

If I is a Cohen-Macaulay ideal in R, the "type" of R/I defined by

$$\mathrm{r}_R(R/I) = \dim_{\mathbb{K}}(\mathrm{Ext}^n_R(\mathbb{K}, R/I))$$

roughly measures how nice the ideal is, where $n = \mathrm{depth}(R/I)$ (see e.g., [11]). For instance, some of the nicest Cohen-Macaulay ideals are the "Gorenstein" ideals, which end up being the Cohen-Macaulay ideals of type 1. In Chapter 4, we compute the type of $R/I_{r,f}(G_\omega)$ when $f = \max$ and the ring is Cohen-Macaulay. We accomplish this in Theorem 4.2.25. As with Villarreal's results, this computation is purely graph-theoretical. As a sample, we state the special case $r = 1$ of the result.

Theorem 1.3.1. *Let $G = \Sigma H$ be a suspension of H (see Definition 3.1.12) and $\omega : E(G) \to \mathbb{N}$ such that $\omega(v_i v_j) \leq \omega(v_i w_i)$ and $\omega(v_i v_j) \leq \omega(w_j v_j)$ for each $v_i v_j \in E(H)$. Then the Cohen-Macaulay type of $R/I(G_\omega)$ is*

$$\mathrm{r}_R(R/I(G_\omega)) = \sharp\{\textit{minimal weighted vertex covers of } H_{\omega'}\}, \ \textit{where } \omega' = \omega|_{E(H)},$$

where the definition for "minimal weighted vertex covers" can be found in [10, Definitions 1.4 and 1.9], or it can be regarded a special case of Definitions 3.1.3 and 3.1.5.

The classification of Cohen-Macaulay path ideals and type computing are the main results of this dissertation. They form the bulk of Chapter 3 and 4. Necessary background information is collected in Chapter 2 and Section 3.1. See also p.102 and p.105.

Chapter 2

Background

This chapter covers the necessary algebraic details for understanding the definitions, theorems, and techniques used in the subsequent chapters. Section 2.1 begins with the definition of local rings and then treats localizations of modules. Section 2.2 is devoted to monomial ideals, our main subject, and their irredundant m-irreducible decompositions. Section 2.3 contains a brief discussion of regular sequences. Section 2.4 introduces some material from homological algebra needed to define **Ext** modules. The notions of depth, type, and Cohen-Macaulayness occupy Section 2.5. Section 2.6 contains an account of graded rings and modules, and closes with a fact that a polynomial ring $R = A[X_1, \ldots, X_d]$ behaves like a local ring with the (homogeneous) maximal ideal $\mathfrak{X} = (X_1, \ldots, X_d)$ when R is regarded as a graded ring over A when A is a local ring, e.g., a field. Section 2.7 provides a way to compute the type of R/I when I is a monomial ideal in R and has an irredundant parametric decomposition.

Convention. In this chapter, let d be a positive integer, R a commutative ring with identity, M an R-module, $I \subseteq R$ an ideal, and $\mathfrak{p} \subseteq R$ a prime ideal.

2.1 Local Rings

There are several invariants defined in terms of local ring, so we first recall the definition of local rings and some relevant properties to be used later.

Definition 2.1.1. We say R is *local* if it has a unique maximal ideal $\mathfrak{m}$, also known as *"quasi-local"*, that is, R has finitely many maximal ideals. The *residue field* of R is $R/\mathfrak{m}$.

"Assume $(R, \mathfrak{m}, k)$ is local" or "assume $(R, \mathfrak{m})$ is local", means that R is a local ring and $\mathfrak{m}$ is the unique maximal ideal of R and $k = R/\mathfrak{m}$.

Example 2.1.2. Let $\mathfrak{k}$ be a field.

(a) $\mathfrak{k}$ is local with the maximal ideal (0).

(b) Let $n \geq 1$ and p be prime in $\mathbb{Z}$. Then $\mathbb{Z}/(p^n)$ is local with the maximal ideal $(p)/(p^n)$.

(c) Let $R = \mathfrak{k}[X_1, \dots, X_d]/(X_1^{a_1}, \dots, X_d^{a_d})$, where $a_i \geq 1$ for $i = 1, \dots, d$. Then R is local with $\mathfrak{m} = (X_1, \dots, X_d)/(X_1^{a_1}, \dots, X_d^{a_d})$.

Definition 2.1.3. Let $U \subseteq R$ be multiplicatively closed and $1 \in U$. The *localization of M with respect to U* is defined to be

$$U^{-1}M = \{\text{equivalence classes from } M \times U \text{ under } \sim\},$$

where $(m, u) \sim (n, u)$ if there exists $w \in U$ such that $w(vm - un) = 0$. Denote the equivalence class of (m, u) as $\frac{m}{u}$ or m/u.

Localization is a useful technique of reducing many problems in commutative algebra to those about local rings.

Notation 2.1.4. By the definition of prime ideals, we have that $R \smallsetminus \mathfrak{p}$ is multiplicatively closed. Set

$$M_\mathfrak{p} = (R \smallsetminus \mathfrak{p})^{-1}M.$$

In particular, we have that $R_\mathfrak{p} = (R \smallsetminus \mathfrak{p})^{-1}R$, $I_\mathfrak{p} = (R \smallsetminus \mathfrak{p})^{-1}I = (R \smallsetminus \mathfrak{p})^{-1}RI = IR_\mathfrak{p}$, and $(R/I)_\mathfrak{p} = (R \smallsetminus \mathfrak{p})^{-1}(R/I)$.

Fact 2.1.5. $(R_\mathfrak{p}, \mathfrak{p}_\mathfrak{p})$ is a local ring.

Fact 2.1.6. Let $\pi : R \to R/I$ be the natural surjection. We have that $(R/I)_\mathfrak{p} \cong R_\mathfrak{p}/I_\mathfrak{p}$. So $(R/I)_\mathfrak{p} \neq 0$ if and only if $I_\mathfrak{p} \subsetneq R_\mathfrak{p}$ if and only if $1/1 \notin I_\mathfrak{p}$. In fact, $(R/I)_\mathfrak{p} \neq 0$ if and only if $I \subseteq \mathfrak{p}$. Indeed, if $I \not\subseteq \mathfrak{p}$, then $I \cap (R \smallsetminus \mathfrak{p}) \neq \emptyset$, so there exists $x \in I$ such that $x \in (R \smallsetminus \mathfrak{p})$ and $1/1 \sim x/x \in I_\mathfrak{p}$. On the other hand, if $(R/I)_\mathfrak{p} = 0$, then $\bar{1}/u = 0$ for any $u \in (R \smallsetminus \mathfrak{p})$, implying that there exists $u'' \in (R \smallsetminus \mathfrak{p})$ such that $\overline{u''} = u''\bar{1} = 0$, implying that $u'' \in I$, it follows that $I \cap (R \smallsetminus \mathfrak{p}) \neq \emptyset$, therefore $I \not\subseteq \mathfrak{p}$. Thus, in summary, $(R/I)_\mathfrak{p} \neq 0$ if and only if $I \cap (R \smallsetminus \mathfrak{p}) = \emptyset$ if and only if $I \subseteq \mathfrak{p}$.

2.2 Monomial Ideals

In this section, we introduce monomial ideals and their irredundant m-irreducible decompositions, and most of the definitions can be found in [9]. Let A be a non-zero commutative ring with identity and $R = A[X_1, \ldots, X_d]$ unless otherwise stated. Set $\mathfrak{X} = (X_1, \ldots, X_d)R$, the ideal generated by all variables in R and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Definition 2.2.1. A *monomial* in elements $X_1, \ldots, X_d \in R$ is an element of the form $X_1^{n_1} \cdots X_d^{n_d}$ in R, where $n_1, \ldots, n_d \in \mathbb{N}_0$. For short, we write $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ and $\underline{X}^{\underline{n}} = X_1^{n_1} \ldots X_d^{n_d}$.

Definition 2.2.2. Denote the set of monomials in R by

$$[\![R]\!] = \{\underline{X}^{\underline{n}} \mid \underline{n} \in \mathbb{N}_0^d\}.$$

Definition 2.2.3. A *monomial ideal* I in R is an ideal generated by monomials in $X_1, \ldots, X_d$, i.e., elements of the form $\underline{X}^{\underline{n}}$ with $\underline{n} \in \mathbb{N}_0^d$.

Remark. The trivial ideals 0 and R are monomial ideals since $0 = (\emptyset)R$ and $R = (1)R = (X_1^0 \cdots X_d^0)R$.

Definition 2.2.4. A monomial $\underline{X}^{\underline{n}}$ with $\underline{n} \in \mathbb{N}_0^d$ is *square-free* if $n_i = 0$ or 1 for $i = 1, \ldots, d$. A monomial ideal I of R is *square-free* if it is generated by square-free monomials.

Example 2.2.5. We have that $I_1 = (X_1X_2, X_3)R$ and $I_2 = (X_1^2, X_2, X_1X_3^3)R$ are monomial ideals in $R = A[X_1, X_2, X_3]$, but only I_1 is a square-free monomial ideal.

Assumption. For the remainder of this section, let $I \subseteq R$ be a monomial ideal.

Fact 2.2.6 (Dickson's Lemma [9, Theorem 1.3.1])**.** I is finitely generated by a set of monomials.

Definition 2.2.7. Denote the set of monomials in I by

$$[\![I]\!] = \{\underline{X}^{\underline{n}} \in I \mid \underline{n} \in \mathbb{N}_0^d\} = I \cap [\![R]\!].$$

Fact 2.2.8. [9, Lemma 1.1.10] For each $f \in I$, each monomial occurring in f is in I.

Definition 2.2.9. Let $f = \underline{X}^{\underline{n}} \in [\![R]\!]$. The *support* of f is the set of variables that appear in f:

$$\operatorname{Supp}(f) = \{i \in \{1, \ldots, d\} : n_i \geq 1\} = \{i \in \{1, \ldots, d\} : X_i \mid f\}.$$

The *reduction* of f is the monomial achieved by reducing all non-zero exponents down to 1:

$$\mathrm{red}(f) = \prod_{i \in \mathrm{Supp}(f)} X_i = \prod_{X_i \mid f} X_i.$$

Example 2.2.10. $\mathrm{Supp}(X_1^5 X_3^4) = \{1, 3\}$ and $\mathrm{red}(X_1^5 X_3^4) = X_1 X_3$.

Definition 2.2.11. Define the *monomial radical* of I by

$$\text{m-rad}(I) = \big(\mathrm{rad}(I) \cap [\![R]\!]\big)R,$$

where $\mathrm{rad}(I)$ is the *radical* of I, defined by

$$\mathrm{rad}(I) = \sqrt{I} = \{x \in R \mid x^n \in I,\ \forall\, n \gg 0\} = \{x \in R \mid x^n \in I \text{ for some } n \geq 1\}.$$

Remark. Example 2.2.13 shows that $\mathrm{rad}(I)$ may not be a monomial ideal. This is due to the fact that the ring A may have nilpotents. See Section 2.4 [9] for more details about this phenomenon.

Fact 2.2.12. [9, Theorem 2.3.7] Assume $I = (S)R$ for some $S \subseteq [\![R]\!]$, then we have that $\text{m-rad}(I) = \big(\mathrm{red}(s) \mid s \in S\big)R$.

It is important to note that you can use the generators.

Example 2.2.13. The monomial ideal $I = (X^3Y^2, XY^3, Y^5)R$ in $R = A[X, Y]$ has

$$\text{m-rad}(I) = \big(\mathrm{red}(X^3Y^2), \mathrm{red}(XY^3), \mathrm{red}(Y^5)\big)R = (XY, XY, Y)R = (Y)R.$$

If $A = \mathbb{Z}/4\mathbb{Z}$, then $\mathrm{rad}(I) = (2, Y)R \neq \text{m-rad}(I)$.

Definition 2.2.14. I is *m-reducible* if there exist monomial ideals $J, K \subseteq R$ such that $I = J \cap K$ and $J \neq I$ and $K \neq I$. I is *m-irreducible* if it is not m-reducible and $I \neq R$.

Fact 2.2.15. [9, Theorem 3.1.4] The zero ideal $I = (0)$ is m-irreducible. A non-zero I is m-irreducible if and only if it can be generated by "pure powers", i.e., if and only if $I = (X_{i_1}^{a_1}, \ldots, X_{i_t}^{a_t})R$ for some $t \geq 1$ and $a_i \geq 1$ for $i = 1, \ldots, t$.

Example 2.2.16. The monomial ideal $(X^3, X^2Y^2, Y^4)R$ in $R = A[X, Y]$ is m-reducible because it cannot be generated by "pure powers". One can also see this from the non-trivial decomposition

$(X^3, X^2Y^2, Y^4)R = (X^2, Y^4)R \cap (X^3, Y^2)R.$

Definition 2.2.17. An *m-irreducible decomposition* of I is an expression $I = \bigcap_{i=1}^n J_i$ with $n \geq 1$ such that monomial ideals $J_1, \ldots, J_n \subseteq R$ are m-irreducible.

Example 2.2.18. The monomial ideal $I = (X^2, XY, Y^3)R$ in $R = A[X, Y]$ has an m-irreducible decomposition $I = (X, Y^3)R \cap (X^2, Y)R$. The intersection in Example 2.2.16 is another m-irreducible decomposition for the corresponding ideal.

Fact 2.2.19. [9, Theorem 3.3.3] If $I \neq R$, then I has an m-irreducible decomposition.

Definition 2.2.20. An m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$ is *redundant* if $I = \bigcap_{i \neq k} J_i$ for some $k \in \{1, \ldots, n\}$. An m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$ is *irredundant* if it is not redundant, that is, if every $k \in \{1, \ldots, n\}$ satisfies $I \neq \bigcap_{i \neq k} J_i$. As $I = \bigcap_{i=1}^n J_i \subseteq \bigcap_{i \neq k} J_i$ holds automatically, the given decomposition is irredundant if and only if every $k \in \{1, \ldots, d\}$ satisfies $I \subsetneq \bigcap_{i \neq k} J_i$.

Example 2.2.21. The m-irreducible decompositions in Examples 2.2.16 and 2.2.18 are irredundant.

The following two facts provide existence and uniqueness for irredundant m-irreducible decompositions.

Fact 2.2.22. [9, Corollary 3.3.8] If $I \neq R$, then I has an irredundant m-irreducible decomposition.

Fact 2.2.23. [9, Theorem 3.3.9] If I has two irredundant m-irreducible decompositions $I = \bigcap_{i=1}^n I_i$ and $I = \bigcap_{j=1}^m J_j$, then $n = m$ and there exists $\sigma \in S_m$ such that $I_i = J_{\sigma(i)}$ for $i = 1, \ldots, n$, where S_n is the permutation group.

An important concept for Cohen-Macaulayness is next.

Definition 2.2.24. The *prime spectrum* of R is

$$\operatorname{Spec}(R) = \{\text{prime ideals of } R\}.$$

By convention, we have that $R \notin \operatorname{Spec}(R)$. Let $\mathrm{V}(I)$ denote the *set of prime ideals in R containing I*:

$$\mathrm{V}(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}.$$

Let M be an R-module. The *support* of M is the set

$$\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}.$$

Fact 2.2.25. It is straightforward to show that

$$\mathrm{Supp}_R(R) = \mathrm{Spec}(R),$$

and by Fact 2.1.6, we have

$$\mathrm{Supp}_R(R/I) = \mathrm{V}(I).$$

Definition 2.2.26. Let M be an R-module. The *Krull dimension* of M is

$$\dim_R(M) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \mathrm{Supp}_R(M)\}.$$

Set $\dim(R) = \dim_R(R)$.

Based on Fact 2.2.25, we have the following Krull dimension computations for rings and quotient rings.

Fact 2.2.27. (a) The *Krull dimension* of R is

$$\dim(R) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \mathrm{Spec}(R)\}.$$

(b) The *Krull dimension* of R/I is

$$\dim(R/I) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \mathrm{V}(I)\}.$$

Fact 2.2.28. [9, Theorem 5.1.2] Let A be a field and $I = \bigcap_{i=1}^m J_i$ an m-irreducible decomposition. Then the Krull dimension is $\dim(R/I) = d - n$, where n is the smallest number of generators needed for one of the J_i's.

Fact 2.2.28 provides us a simple formula to compute $\dim(R/I)$.

Example 2.2.29. The monomial ideal $I = (X_1X_2, X_2^2X_3^2, X_3X_4)$ in $R = A[X_1, X_2, X_3, X_4]$ has an (irredundant) m-irreducible decomposition

$$I = (X_1, X_3)R \cap (X_2, X_3)R \cap (X_1, X_2^2, X_4)R \cap (X_2, X_4)R \cap (X_1, X_3^2, X_4)R.$$

Therefore, by Fact 2.2.28, we have that $\dim(R/I) = 4 - 2 = 2$.

Definition 2.2.30. Let $I = \bigcap_{i=1}^m J_i$ be an irredundant m-irreducible decomposition. Let n_i be the smallest number of generators needed for J_i for $i = 1, \ldots, m$. We say that I is *m-unmixed* if $n_1 = \cdots = n_m$. We say that I is *m-mixed* if it is not m-unmixed, i.e., there exist $i, j \in \{1, \ldots, m\}$ such that $n_i \neq n_j$.

Fact 2.2.31. If A is a field, then I is m-unmixed if and only if I is unmixed.

Example 2.2.32. The monomial ideals in Examples 2.2.16 and 2.2.18 are m-unmixed. The monomial ideal in Example 2.2.29 is m-mixed.

Definition 2.2.33. A *parameter ideal* in R is an ideal of the form $(X_1^{a_1}, \ldots, X_d^{a_d})$ with $a_1, \ldots, a_d \geq 1$. For $\underline{X}^{\underline{n}} = X_1^{n_1} \cdots X_d^{n_d} \in [\![R]\!]$ with $\underline{n} \in \mathbb{N}_0^d$, set

$$\mathrm{P}_R(\underline{X}^{\underline{n}}) = \left(X_1^{n_1+1}, \ldots, X_d^{n_d+1}\right)R.$$

Note that

$$\text{m-rad}(P_R(\underline{X}^{\underline{n}})) = (\mathrm{red}(X_1^{n_1+1}), \ldots, \mathrm{red}(X_d^{n_d+1}))R = (X_1, \ldots, X_d)R = \mathfrak{X}.$$

Example 2.2.34. The monomial ideal $I = (X_1^2, X_2, X_3^8)R$ is a parameter ideal in $R = A[X_1, X_2, X_3]$ but not in $R = A[X_1, X_2, X_3, X_4]$.

Definition 2.2.35. A *parametric decomposition* of I is an m-irreducible decomposition of I of the form $I = \bigcap_{i=1}^n \mathrm{P}_R(f_i)$ with $f_i \in [\![R]\!]$.

Example 2.2.36. The m-irreducible decompositions in Examples 2.2.16 and 2.2.18 are irredundant parametric decompositions.

Fact 2.2.37. [9, Exercise 2.4.5, Theorem 6.1.5 and Exercise 5.1.7] I has a parametric decomposition if and only if $\text{m-rad}(I) = \mathfrak{X}$. Furthermore, if A is a field, then $\text{m-rad}(I) = \mathfrak{X}$ is equivalent to $\dim(R/I) = 0$.

We end this section by exhibiting one technique for computing m-irreducible decompositions of arbitrary monomial ideals.

Fact 2.2.38. [9, Theorem 7.5.1] Let $I = \big(\underline{X}^{\underline{a}_1}, \ldots, \underline{X}^{\underline{a}_n}\big)R$ with $\underline{a}_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}_0^d$ for $i = 1, \ldots, n$. Then

$$I = \bigcap_{i_1=1}^{d} \cdots \bigcap_{i_n=1}^{d} (X_{i_1}^{a_{1,i_1}}, \ldots, X_{i_n}^{a_{n,i_n}}).$$

Example 2.2.39. Let $R = A[X_1, X_2]$ and $I = (X_1^2 X_2, X_1 X_3)R$. Then by Fact 2.2.38,

$$\begin{aligned}
I &= (X_1^2, X_1)R \cap (X_1^2, X_2^0)R \cap (X_1^2, X_3)R \\
&\quad \cap (X_2, X_1)R \cap (X_2, X_2^0)R \cap (X_2, X_3)R \\
&\quad \cap (X_3^0, X_1)R \cap (X_3^0, X_2^0)R \cap (X_3^0, X_3)R \\
&= (X_1)R \cap R \cap (X_1^2, X_3)R \cap (X_1, X_2)R \cap R \cap (X_2, X_3)R \cap R \cap R \cap R \\
&= (X_1)R \cap (X_1^2, X_3)R \cap (X_2, X_3)R.
\end{aligned}$$

2.3 Regular Sequences

The interplay between regular sequences and certain homological invariants is one of the key techniques used to compute the type of a module.

Definition 2.3.1. An element $x \in R$ is a *non-zero divisor* on M if the multiplication by x map $M \xrightarrow{\cdot x} M$ is 1-1; equivalently, for $m \in M$, if $xm = 0$, then $m = 0$. Set

$$\mathrm{NZD}_R(M) = \{a \in R \mid a \text{ is a non-zero divisor on } M\}.$$

Definition 2.3.2. An element $x \in R$ is *weakly M-regular* if $x \in \mathrm{NZD}_R(M)$. A weakly M-regular element $x \in R$ is *M-regular* if $xM \neq M$.

Definition 2.3.3. A sequence $a_1, \ldots, a_n \in R$ is *weakly M-regular* if

(a) a_1 is weakly M-regular and

(b) a_i is weakly $\frac{M}{(a_1,\ldots,a_{i-1})M}$-regular for $i = 2, \ldots, n$.

A sequence $a_1, \ldots, a_n \in R$ is *M-regular* if

(a) $a_1, \dots, a_n$ is weakly M-regular and

(b) $(a_1, \dots, a_n)M \neq M$.

Example 2.3.4. A list of variables $X_1, \dots, X_n$ is $A[X_1, \dots, X_n]$-regular for any non-zero commutative ring A.

Remark. Note that for $a_1, \dots, a_i \in R$, we have

$$\frac{M}{(a_1, \dots, a_i)M} \overset{①}{\cong} \frac{M/(a_1, \dots, a_{i-1})M}{(a_1, \dots, a_i)M/(a_1, \dots a_{i-1})M} \cong \frac{M/(a_1, \dots, a_{i-1})M}{a_iM/(a_1, \dots, a_{i-1})M},$$

where ① is from an isomorphism theorem for modules. Thus, we have that $a_iM/(a_1, \dots, a_{i-1})M \neq M/(a_1, \dots, a_{i-1})M$ if and only if $M/(a_1, \dots, a_i)M \neq 0$. This observation justifies the following equivalent definition for M-regular sequences.

Definition 2.3.5. A sequence $a_1, \dots, a_n \in R$ is *weakly M-regular* if

(a) $a_1 \in \mathrm{NZD}_R(M)$, and

(b) $a_i \in \mathrm{NZD}_R(M/(a_1, \dots, a_{i-1})M)$ for $i = 2, \dots, n$.

Remark. If $(R, \mathfrak{m})$ is local with $a_1, \dots, a_n \in \mathfrak{m}$, and M is non-zero and finitely generated, then by Nakayama's lemma, we have that $(a_1, \dots, a_n)M \subseteq \mathfrak{m}M \subsetneq M$. So $a_1, \dots, a_n$ is M-regular if and only if it is weakly M-regular.

Definition 2.3.6. A sequence $a_1, \dots, a_n \in R$ is a *maximal M-regular sequence* if $a_1, \dots, a_n$ is an M-regular sequence such that for all $b \in R$, the longer sequence $a_1, \dots, a_n, b$ is not M-regular.

2.4 Ext via Projective Resolutions

In this section, let N be another R-module. We will present some definitions and facts from homological algebra leading to the definition of Ext.

Definition 2.4.1. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of R-module homomorphisms is *exact* at B if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. Note that $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$ if and only if $g \circ f = 0$.

More generally, a sequence of R-module homomorphism

$$\cdots \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} \cdots$$

is *exact* if $\mathrm{Im}(d_{i+1}) = \mathrm{Ker}(d_i)$ for all $i \in \mathbb{Z}$.

Fact 2.4.2. We have the following facts:

(a) The sequence $0 \to A \xrightarrow{f} A'$ of R-module homomorphisms is exact (at A) if and only if f is 1-1.

(b) The sequence $B' \xrightarrow{g} B \to 0$ of R-module homomorphisms is exact (at B) if and only if g is onto.

(c) The sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R-module homomorphisms is exact if and only if f is 1-1, g is onto and $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Definition 2.4.3. A *short exact sequence* is an exact sequence of R-module homomorphisms of the form

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0.$$

Definition 2.4.4. A *homomorphism of short exact sequences* of $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ is a triple (α, β, γ) of R-module homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0. \end{array}$$

Fact 2.4.5 (The Short Five Lemma [5, Proposition 10.24])**.** Let (α, β, γ) be a homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0. \end{array}$$

(a) If α and γ are 1-1, then so is β.

(b) If α and γ are onto, then so is β.

(c) If α and γ are isomorphisms, then so is β.

Definition 2.4.6. A short exact sequence of R-module homomorphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is *split* if and only if it is equivalent to the canonical exact sequence $0 \to A \xrightarrow{\epsilon} A \oplus C \xrightarrow{\rho} C \to 0$, i.e., if and only if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle =}\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow[\epsilon]{} & A \oplus C & \xrightarrow[\rho]{} & C & \longrightarrow & 0. \end{array}$$

In this event, β is an isomorphism by the short five lemma, so $B \cong A \oplus C$.

Notation 2.4.7.

$$\mathrm{Hom}_R(M,N) := \{R\text{-module homomorphisms } f : M \to N\},$$

which is an R-module because R is commutative.

Let A, B be R-modules. For each $f \in \mathrm{Hom}_R(A,B)$, define

$$\begin{aligned} f^* = \mathrm{Hom}_R(f,N) : \mathrm{Hom}_R(B,N) &\longrightarrow \mathrm{Hom}_R(A,N) \\ \phi &\longmapsto \phi \circ f. \end{aligned}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \overset{f^*(\phi)}{\searrow} & \downarrow \phi \\ & & N \end{array}$$

Then f^* is an R-module homomorphism.

Fact 2.4.8. $\mathrm{Hom}_R(-,N)$ is a *contravariant functor*, i.e.,

(a) it respects identity maps: $\mathrm{Hom}_R(\mathrm{id}_M, N) = \mathrm{id}_{\mathrm{Hom}_R(M,N)}$, and

(b) it respects compositions: for all R-module homomorphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$,

$$\mathrm{Hom}_R(\beta \circ \alpha, N) = \mathrm{Hom}_R(\alpha, N) \circ \mathrm{Hom}_R(\beta, N).$$

Or equivalently, $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$, i.e., the following diagram commutes:

$$\mathrm{Hom}(-,N): \quad \begin{array}{ccc} \mathrm{Hom}_R(A,N) & \xleftarrow{\mathrm{Hom}_R(\alpha,N)} & \mathrm{Hom}_R(B,N) \\ & \overset{\mathrm{Hom}_R(\beta\circ\alpha,N)}{\nwarrow} & \uparrow \mathrm{Hom}_R(\beta,N) \\ & & \mathrm{Hom}_R(C,N). \end{array}$$

Fact 2.4.9 (Left Exactness of $\mathrm{Hom}(-,N)$ [5, Theorem 10.33])**.** Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be exact. Then the induced sequence $0 \to \mathrm{Hom}(C,N) \xrightarrow{\beta^*} \mathrm{Hom}(B,N) \xrightarrow{\alpha^*} \mathrm{Hom}(A,N)$ is exact.

Remark. The functor $\mathrm{Hom}(N,-)$ is defined similarly with notation $f_* = \mathrm{Hom}(N,f)$. This functor is *covariant.* It is left exact, i.e., if $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of R-module homomorphisms, then the induced sequence $0 \to \mathrm{Hom}(N, A) \xrightarrow{f_*} \mathrm{Hom}(N, B) \xrightarrow{g_*} \mathrm{Hom}(N, C)$ is exact.

Fact 2.4.10. [5, Theorem 10.30] The following conditions are equivalent.

(i) $\mathrm{Hom}_R(N, -)$ transforms R-module epimorphisms into R-module epimorphisms.

(ii) $\mathrm{Hom}_R(N, -)$ transforms short exact sequences of R-module homomorphisms into short exact sequences of R-module homomorphisms.

(iii) $\mathrm{Hom}_R(N, -)$ transforms exact sequences of R-module homomorphisms into exact sequences R-module homomorphisms.

(iv) Every short exact sequence $0 \to A \to B \to N \to 0$ of R-module homomorphisms splits.

(v) If the sequence $B \xrightarrow{\beta} C \to 0$ of R-module homomorphisms is exact, then every R-module homomorphism from N to C lifts to an R-module homomorphism into B, i.e., given $\phi \in \mathrm{Hom}_R(N, C)$, there is a map $\psi \in \mathrm{Hom}_R(N, B)$ making the following diagram commute:

$$\begin{array}{ccccc} & & N & & \\ & \overset{\exists\, \psi}{\swarrow} & \downarrow{\scriptstyle \phi} & & \\ B & \xrightarrow{\beta} & C & \longrightarrow & 0. \end{array}$$

(vi) There exists an R-module N' such that $N \oplus N'$ is free, i.e., N is a summand of a free R-module.

Definition 2.4.11. An R-module P is called *projective* if it satisfies any of the equivalent conditions of Fact 2.4.10.

Definition 2.4.12. A *chain complex* or *R-complex* is a sequence of R-module homomorphisms

$$M_\bullet = \cdots \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \xrightarrow{\partial^M_{i-1}} \cdots$$

such that $\partial^M_{i-1} \circ \partial^M_i = 0$ for all $i \in \mathbb{Z}$. We say M_i is the module in (homological) *degree i* in the R-complex $M_\bullet$.

The i^{th} *homology module* of an R-complex $M_\bullet$ is the R-module

$$\mathrm{H}_i(M_\bullet) = \mathrm{Ker}(\partial^M_i)/\,\mathrm{Im}(\partial^M_{i+1}).$$

Definition 2.4.13. A *projective resolution* of M over R or an *R-projective resolution* of M is an exact sequence of R-module homomorphisms

$$P_\bullet^+ = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \to 0$$

such that each P_i is a projective R-module.

The *truncated projective resolution* of M associated to $P_\bullet^+$ is the R-complex

$$P_\bullet = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \to 0.$$

By convention, we have that $P_i = 0$ for all $i \leq -1$ and $\partial_i^P = 0$ for all $i \leq 0$. Define the R-complex $\operatorname{Hom}(P_\bullet^+, N)$ as follows:

$$\operatorname{Hom}(P_\bullet^+, N) = \qquad 0 \to M^* \xrightarrow{\tau^*} P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} \cdots \xrightarrow{(\partial_{i-1}^P)^*} P_{i-1}^* \xrightarrow{(\partial_i^P)^*} P_i^* \xrightarrow{(\partial_{i+1}^P)^*} \cdots,$$

where we set $P_i^* = \operatorname{Hom}(P_i, N)$ and $(\partial_i^P)^* = \operatorname{Hom}(\partial_i^P, N)$ for $i \geq 0$. Define the R-complex $P_\bullet^*$ as follows:

$$P_\bullet^* = \operatorname{Hom}(P_\bullet, N) = \qquad 0 \to P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} \cdots \xrightarrow{(\partial_{i-1}^P)^*} P_{i-1}^* \xrightarrow{(\partial_i^P)^*} P_i^* \xrightarrow{(\partial_{i+1}^P)^*} \cdots.$$

Let P_i^* be in degree $-i$, i.e., $P_i^* = (P^*)_{-i}$ for $i \in \mathbb{Z}$. Then

$$\begin{array}{ccccccccccccc}
P_\bullet^* = 0 \longrightarrow & P_0^* & \xrightarrow{(\partial_1^P)^*} & P_1^* & \xrightarrow{(\partial_2^P)^*} & \cdots & \xrightarrow{(\partial_{i-1}^P)^*} & P_{i-1}^* & \xrightarrow{(\partial_i^P)^*} & P_i^* & \xrightarrow{(\partial_{i+1}^P)^*} & \cdots \\
 & \| & & \| & & & & \| & & \| & & \\
P_\bullet^* = 0 \to & (P^*)_0 & \xrightarrow{\partial_0^{P^*}} & (P^*)_{-1} & \xrightarrow{\partial_{-1}^{P^*}} & \cdots & \xrightarrow{\partial_{-i+2}^{P^*}} & (P^*)_{-i+1} & \xrightarrow{\partial_{-i+1}^{P^*}} & (P^*)_{-i} & \xrightarrow{\partial_{-i}^{P^*}} & \cdots.
\end{array}$$

So

$$\partial_i^{P^*} = \left(\partial_{-i+1}^P\right)^*, \ \forall\, i \in \mathbb{Z}.$$

By convention, we have that $(P^*)_i = P_{-i}^* = 0^* = 0$ and $\partial_i^{P^*} = \left(\partial_{-i+1}^P\right)^* = 0^* = 0$ for all $i \geq 1$.

Remark. Because of the condition $\partial_i^P \circ \partial_{i+1}^P = 0$ for $i \geq 1$, by Fact 2.4.8, we have

$$\left(\partial_{i+1}^P\right)^* \circ \left(\partial_i^P\right)^* = \left(\partial_i^P \circ \partial_{i+1}^P\right)^* = 0^* = 0, \ \forall\, i \geq 1.$$

Thus, $\operatorname{Hom}(P_\bullet, N)$ and similarly $\operatorname{Hom}(P_\bullet^+, N)$ are R-complexes. However, these are not exact in general.

Definition 2.4.14 (Ext via projective resolutions)**.** Let $P_\bullet^+$ be a projective resolution of M. Define the Ext module by

$$\operatorname{Ext}_R^i(M,N) := \mathrm{H}_{-i}(P_\bullet^*) = \operatorname{Ker}\big(\partial_{-i}^{P^*}\big)/\operatorname{Im}\big(\partial_{-i+1}^{P^*}\big) = \operatorname{Ker}\big((\partial_{i+1}^P)^*\big)/\operatorname{Im}\big((\partial_i^P)^*\big).$$

Fact 2.4.15. Let $P_\bullet^+$ be a projective resolution of M. By the left exactness of Hom, we have an exact sequence:

$$0 \longrightarrow M^* \xrightarrow{\tau^*} P_0^* \xrightarrow{(\partial_1^P)^*} P_1^*.$$

Then we have

$$\operatorname{Ext}_R^0(M,N) = \operatorname{Ker}\big((\partial_1^P)^*\big)/\operatorname{Im}(0) \cong \operatorname{Ker}\big((\partial_1^P)^*\big) = \operatorname{Im}(\tau^*) \cong M^* = \operatorname{Hom}_R(M,N),$$

$$\operatorname{Ext}_R^i(M,N) = \operatorname{Ker}(\partial_{-i}^{P^*})/\operatorname{Im}(\partial_{-i+1}^{P^*}) = 0/\operatorname{Im}(\partial_{-i+1}^{P^*}) = 0,\ \forall\, i \leq -1.$$

Remark. $\operatorname{Ext}_R^i(M,N)$ is well-defined, i.e., independent of the choices of projective resolution of M, by [11, Theorem VIII.5.2].

Remark. We can also define the Ext module via *injective modules*, but this is not needed for this dissertation.

2.5 Depth, Type, and Cohen-Macaulayness

In this section, we define the depth and the type of M when $M \neq 0$, M is finitely generated, and $(R, \mathfrak{m})$ is local.

Assumption. For this section, we assume that R is Noetherian, I is an ideal of R, and M is a finitely generated R-module.

The next fact is due to Rees.

Fact 2.5.1. [1, Theorem 1.2.5] If $IM \neq M$, then all maximal M-regular sequences in I have the same length, namely

$$\inf\big\{i \geq 0 \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\big\}.$$

Through Fact 2.5.1, we have the following definition:

Definition 2.5.2. [1, Definition 1.2.11] If $IM \neq M$, we define the *grade* of M on I by

$$\operatorname{grade}_R(I; M) = \inf\bigl\{i \geq 0 \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\bigr\}.$$

If $IM = M$, then set $\operatorname{grade}_R(I; M) = \infty$.

Remark. (a) By Fact 2.4.15 we also have $\operatorname{grade}_R(I; M) = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$.

(b) If $(R, \mathfrak{m})$ is local and $M \neq 0$, then by Nakayama's lemma, $IM \subseteq \mathfrak{m}M \subsetneq M$, so $IM \neq M$.

Definition 2.5.3. [1, Definition 1.2.8] If $(R, \mathfrak{m}, k)$ is local and $M \neq 0$, we define the *depth* of M by

$$\operatorname{depth}(M) = \operatorname{grade}_R(\mathfrak{m}; M) = \inf\bigl\{i \geq 0 \mid \operatorname{Ext}^i_R(k, M) \neq 0\bigr\}.$$

Fact 2.5.4. By Fact 2.5.1, the depth can be calculated as the maximum length among all M-regular sequences in $\mathfrak{m}$.

Fact 2.5.5. [1, Theorem 1.2.10] If $f_1, \ldots, f_r \in R$ is an R/I-regular sequence, then

$$\dim(R/(I + (f_1, \ldots, f_r)R)) = \dim(R/I) - r,$$

$$\operatorname{depth}(R/(I + (f_1, \ldots, f_r)R)) = \operatorname{depth}(R/I) - r.$$

Definition 2.5.6. [1, Definition 1.2.15] Let $(R, \mathfrak{m}, k)$ be local and $M \neq 0$. Assume $\operatorname{depth}_R(M) = n$. The *type* of M is the positive integer

$$\mathrm{r}_R(M) = \dim_k\bigl(\operatorname{Ext}^n_R(k, M)\bigr).$$

Definition 2.5.7. [1, Definition 2.1.1] Let $(R, \mathfrak{m}, k)$ be local and $M \neq 0$. Then M is a *Cohen-Macaulay module* if $\operatorname{depth}_R(M) = \dim_R(M)$. If R itself is a Cohen-Macaulay module, then it is also called a *Cohen-Macaulay ring*.

2.6 Graded rings and modules

The rings we mainly work on are polynomial rings. They form an important class of graded rings. In this section, we exhibit a series of definitions and conclusions, most of which can be found in Section 1.5 of [1].

Definition 2.6.1. A *graded ring* is a ring R together with a decomposition $R = \bigoplus_{i \in \mathbb{Z}} R_i$ (as a $\mathbb{Z}$-module) such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Assumption. For the remainder of this section, we assume that R be a graded ring.

Definition 2.6.2. A *graded module* is an R-module M together with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (as a $\mathbb{Z}$-module) such that $R_i M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. The elements $x \in R_i$ are called *homogeneous* (of degree i). One calls M_i the i^{th} *homogeneous* (or *graded*) *component* of M.

Definition 2.6.3. Let M be a graded R-module. An arbitrary element $x \in M$ has a unique presentation $x = \sum_i x_i$ as a sum of homogeneous elements $x_i \in M_i$. The elements x_i are called the *homogeneous components* of x.

Definition 2.6.4. An ideal I of R is *homogeneous* if I is generated by homogeneous elements of I.

Definition 2.6.5. Let M be a graded R-module and $i \in \mathbb{Z}$. Let $M(i)$ denote the *shifted R-module* M with grading given by $M(i)_n = M_{i+n}$. One can also read $M(i)$ as "M twisted by i".

Definition 2.6.6. Let M and N be graded R-modules, and $n \in \mathbb{Z}$. An R-module homomorphism $\varphi : M \to N$ is called *homogeneous of degree* n if $\varphi(M_i) \subseteq N_{i+n}$ for all $i \in \mathbb{Z}$. Denote by $\mathrm{Hom}_n(M, N)$ the group of homogeneous R-module homomorphisms of degree n. In particular, if $\varphi \in \mathrm{Hom}_0(M, N)$, we call it a homogeneous R-module homomorphism.

If $\varphi \in \mathrm{Hom}_n(M, N)$, then $\varphi \in \mathrm{Hom}_0(M, N(n))$ and $\varphi \in \mathrm{Hom}_0(M(-n), N)$ since $\varphi(M_i) \subseteq N_{i+n}$ for all $i \in \mathbb{Z}$ if and only if $\varphi(M_{-n+i}) \subseteq N_i$ for all $i \in \mathbb{Z}$.

Definition 2.6.7. Let M and N be graded R-modules. Define ${}^*\mathrm{Hom}_R(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_i(M, N)$, which is a submodule of $\mathrm{Hom}_R(M, N)$. If $P_\bullet$ is a graded projective resolution of M, then

$$ {}^*\mathrm{Ext}^i_R(M, N) \cong \mathrm{H}^i({}^*\mathrm{Hom}_R(P_\bullet, N)), \ \forall\ i \geq 0. $$

Fact 2.6.8. [1, p.33] If R is Noetherian and M is a finitely generated R-module, then ${}^*\mathrm{Ext}^i_R(M, N) = \mathrm{Ext}^i_R(M, N)$ for all $i \geq 0$.

Definition 2.6.9. The *homogeneous prime spectrum* of R is

$$^*\mathrm{Spec}(R) = \{\text{homogeneous prime ideals of } R\}.$$

Let $^*\mathrm{V}(I)$ denote the *set of homogeneous prime ideals in R containing I*:

$$^*\mathrm{V}(I) = \{\mathfrak{p} \in {}^*\mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\} = \mathrm{V}(I) \cap {}^*\mathrm{Spec}(R).$$

The **Krull dimension* of R is

$$^*\dim(R) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } {}^*\mathrm{Spec}(R)\}.$$

The **Krull dimension* of R/I can be computed as

$$^*\dim(R/I) = \sup\{n \geq 0 \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } {}^*\mathrm{V}(I)\}.$$

Definition 2.6.10. [1, Definition 1.5.13] Let R be a graded ring. A homogeneous ideal $\mathfrak{m}$ of R is called **maximal* if every homogeneous ideal that properly contains $\mathfrak{m}$ equals R. The ring R is called *local if it has a unique *maximal ideal $\mathfrak{m}$. A *local ring with *maximal ideal $\mathfrak{m}$ will be denoted by $(R, \mathfrak{m})$.

Remark. With respect to its finitely generated graded R-modules, a *local ring $(R, \mathfrak{m})$ behaves like a local ring.

2.7 Graded Cohen-Macaulay Rings

Let A be a field, set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$ and let $I \subsetneq R$ be an ideal generated by homogeneous polynomials. In this section, we define Cohen-Macaulayness and see how to compute the type of R/I, when R/I is Cohen-Macaulay.

Remark. The graded ring $(R, \mathfrak{X})$ with the natural grading is a *local ring, where $\mathfrak{X} = \bigoplus_{i \geq 1} R_i$, is called the *irrelevant ideal* of R.

We have already defined depth and type in the local setting. Now we define them in the *local setting. See Fact 2.7.2 for a comparison.

Definition 2.7.1. The **depth* of R/I is

$$^*\mathrm{depth}(R/I) = \text{the length of a maximal homogeneous } R/I\text{-regular sequence in } \mathfrak{X}.$$

The *type* of R/I is

$$\mathrm{r}_R(R/I) = \dim_A({}^*\mathrm{Ext}^n_R(A, R/I)) = \dim_A(\mathrm{Ext}^n_R(A, R/I)),$$

where $n = {}^*\mathrm{depth}(R/I)$.

Fact 2.7.2. [1, Theorem 1.5.8 and Proposition 1.5.15] The polynomial ring R is Noetherian and R/I is a finitely generated graded R-module. We have

$$^*\mathrm{depth}(R/I) = \mathrm{depth}(R/I),$$

$$^*\dim(R/I) = \dim(R/I).$$

From Fact 2.5.5 and 2.7.2, we get the following fact directly.

Fact 2.7.3. [1, Theorem 1.2.10] If $f_1, \ldots, f_r \in \mathfrak{X}$ is a homogeneous R/I-regular sequence, then

$$^*\dim(R/(I + (f_1, \ldots, f_r)R)) = {}^*\dim(R/I) - r,$$

$$^*\mathrm{depth}(R/(I + (f_1, \ldots, f_r)R)) = {}^*\mathrm{depth}(R/I) - r.$$

For the rest of the dissertation, in light of Fact 2.7.2, we will not write * for notations used in *local ring.

Cohen-Macaulay rings, defined next in the *local setting, have been shown in the literature to be extremely nice. See the discussion in [1, p.57] for more about this.

Definition 2.7.4. The quotient R/I is *Cohen-Macaulay* if $\mathrm{depth}(R/I) = \dim(R/I)$.

Remark. We can either regard the quotient R/I as an R-module, or regard $(R/I, \mathfrak{X}/I)$ as a local ring with the residue field $(R/I)/(\mathfrak{X}/I) \cong A$. So Definition 2.7.4 can be deduced from Definition 2.5.7.

Definition 2.7.5. We say that I is *Cohen-Macaulay* if the quotient R/I is Cohen-Macaulay.

Fact 2.7.6. [1, Lemma 3.1.16] If R/I is Cohen-Macaulay and $f_1, \dots, f_n \in \mathfrak{X}$ is a homogeneous R/I-regular sequence, then with $S = R/(f_1, \dots, f_n)$, we have that

$$\mathrm{r}_R(R/I) = \mathrm{r}_S(R/(I + (f_1, \dots, f_n))),$$

Fact 2.7.7. [14, Fact 2.93(b)] If I is a monomial ideal and has an irredundant parametric decomposition $I = \bigcap_{i=1}^t Q_i$, then $\mathrm{r}_R(R/I) = t$.

Fact 2.7.8. [9, Theorem 5.3.16] Let I be a monomial ideal. If R/I is Cohen-Macaulay, then I is unmixed.

In practice, when we compute the Cohen-Macaulay type of R/I, we will try to find a maximal homogeneous R/I-regular sequence $f_1, \dots, f_n$ with $n = \operatorname{depth}(R/I) = \dim(R/I)$, such that $R/(I + (f_1, \dots, f_n))$ has dimension 0 by Fact 2.5.5. We can usually simplify $R/(I + (f_1, \dots, f_n))$ as, say S/J. Since A is a field and $\dim(S/J) = 0$, we have that J has an irredundant parametric decomposition by Fact 2.2.37. Thus, we utilize Fact 2.7.7 to compute the type of S/J. Finally, Fact 2.7.6 tells us it is also the type of R/I.

Example 2.7.9. Consider the monomial ideal

$$I = (X_1X_2, X_2X_3, X_3X_4)R = (X_1, X_3)R \cap (X_2, X_3)R \cap (X_2, X_4)R$$

in $R = A[X_1, X_2, X_3, X_4]$. One can check that $\dim(R/I) = 2$ and that $X_1 - X_2, X_3 - X_4$ is an R/I-regular sequence. Thus, R/I is Cohen-Macaulay. Note that

$$R/(I + (X_1 - X_2, X_3 - X_4)R) \cong S/(X_2^2, X_2X_3, X_3^2)S,$$

where $S = A[X_1, X_2]$. We have an irredundant parametric decomposition $(X_2^2, X_2X_3, X_3^2)S = (X_2, X_3^2)S \cap (X_2^2, X_3)S$. So by Facts 2.7.3 and 2.7.7, we have

$$\begin{aligned}\mathrm{r}_R(R/I) &= \mathrm{r}_R(R/(I + (X_1 - X_2, X_3 - X_4)R)) \\ &= \mathrm{r}_S(S/(X_2^2, X_2X_3, X_3^2)S) \\ &= 2.\end{aligned}$$

Chapter 3

Cohen-Macaulayness of f-Weighted r-Path Ideals

Let $\mathbb{K}$ be a field, $d \geq 2$, $R = \mathbb{K}[X_1, \ldots, X_d]$ and $\mathfrak{m} = (X_1, \ldots, X_d)R$. Let $G = (V, E)$ be a (finite simple) graph with vertex set $V = \{v_1, \ldots, v_d\}$ and edge set E. Let $r \geq 2$ be a positive integer and $R' = \mathbb{K}\big[\{X_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\}\big]$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that $f(a, b) = f(b, a)$ for all $a, b \in \mathbb{N}$. For example, f may be max, min, gcd, or lcm, etc.

In this chapter, we classify all weighted r-path suspensions $G'_{\omega'}$ (see Definition 3.1.16) for which the f-weighted r-path ideal of $G'_{\omega'}$ (see Definition 3.1.2) is Cohen-Macaulay. In particular, we classify all weighted trees for which the f-weighted r-path ideal is Cohen-Macaulay. These results are in Theorems 3.5.5 and 3.5.6.

3.1 Background

In this section, we give some background information needed for classifying Cohen-Macaulay weighted r-path suspensions.

We first list the definitions for paths and cycles from Diestel [4].

Definition 3.1.1. An *r-path* in G is a non-empty graph $P = (V', E')$ of the form $V' = \{x_1, \ldots, x_{r+1}\}$ and $E' = \{x_1x_2, x_2x_3, \ldots, x_rx_{r+1}\}$, where x_i are all distinct. We denote an r-path by $P_r = \big(\ x_1 \text{ —— } x_2 \text{ —— } \cdots \text{ —— } x_{r+1}\ \big)$ or $x_1 \ldots x_{r+1}$ for simplicity. Note that there are $r+1$ vertices and r edges in P_r.

If $P_r = \big(\, x_1 \;\text{——}\; x_2 \;\text{——}\; \cdots \;\text{——}\; x_r \, \big)$ is an $(r-1)$-path, then the graph $C_r := P_{r-1} + x_r x_1$ is called an *r-cycle*. Note that there are r vertices and r edges in C_r.

Definition 3.1.2. We have the following definitions:

(a) [7, Definition 2.1] The *f-weighted r-path ideal* associated to G_ω is the ideal $I_{r,f}(G_\omega) \subseteq R$ that is "generated by the f-weighted paths in G of length r":

$$I_{r,f}(G_\omega) = \left(X_{i_1}^{e_{i_1}} \dots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \dots v_{i_{r+1}} \text{ is a path in } G \text{ with } e_{i_1} = \omega(v_{i_1} v_{i_2}), \\ e_{i_j} = f(\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})) \text{ for } 1 < j \le r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r} v_{i_{r+1}}) \end{array} \right) R.$$

(b) [7, Definition 2.5] For $V' \subseteq V$ and $\delta' : V' \to \mathbb{N}$, we write

$$P(V', \delta') = \left(X_i^{\delta'(v_i)} \mid v_i \in V' \right) R.$$

Remark. When $f = \max$, we write that $I_r(G_\omega) := I_{r,\max}(G_\omega)$, which is the *weighted r-path ideal* associated to G_ω.

Definition 3.1.3. [7, Definition 1.5] An *f-weighted r-path vertex cover* of G_ω is an ordered pair (V', δ') with $V' \subseteq V$ and $\delta' : V \to \mathbb{N}$ such that V' is an r-path vertex cover of G and such that for any r-path $P_r := v_{i_1} \dots v_{i_{r+1}}$ in G at least one of the following holds:

(a) $\delta'(v_{i_1}) \le \omega(v_{i_1} v_{i_2})$;

(b) $\delta'(v_{i_{r+1}}) \le \omega(v_{i_r} v_{i_{r+1}})$; or

(c) $\delta'(v_{i_j}) \le f\big\{\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j} v_{i_{j+1}})\big\}$ for some $j \in \{2, \dots, r\}$.

The number $\delta'(v_i)$ is the *weight* of v_{i_j}. We say that a vertex $v_i \in V'$ *weighted-covers* the r-path P_r with respect to (V', δ') if v_i satisfies one of the 3 conditions above.

Remark. When $f = \max$, we write that (V', δ') is a *weighted r-path vertex cover* of G_ω.

Notation 3.1.4. For an f-weighted r-path vertex cover (V', δ') of G_ω, we also use $\big\{ v_i^{\delta'(v_i)} \mid v_i \in V' \big\}$ to denote it, especially when we depict an f-weighted r-path vertex cover of G_ω in sketches.

Definition 3.1.5. [7, Definition 1.7] Given two f-weighted r-path vertex covers (V_1', δ_1') and (V_2', δ_2') of G_ω, we write $(V_2', \delta_2') \leq (V_1', \delta_1')$ if $V_2' \subseteq V_1'$ and $\delta_2'(v_i) \geq \delta_1'(v_i)$ for all $v_i \in V_2'$. An f-weighted r-path vertex cover (V', δ') is *minimal* if there does not exist another f-weighted r-path vertex cover (V'', δ'') such that $(V'', \delta'') < (V', \delta')$.

Fact 3.1.6. [7, Lemma 1.11] For every f-weighted r-path vertex cover (V', δ') of G_ω, there is a minimal f-weighted r-path vertex cover (V'', δ'') of G_ω such that $(V'', \delta'') \leq (V', \delta')$.

Fact 3.1.7. [7, Theorem 2.7] The f-weighted r-path ideal $I_{r,f}(G_\omega)$ has the following decomposition:

$$I_{r,f}(G_\omega) = \bigcap_{(V',\delta') \text{ } f\text{-w. } r\text{-path v. cover}} P(V', \delta') = \bigcap_{(V',\delta') \text{ min. } f\text{-w. } r\text{-path v. cover}} P(V', \delta'),$$

where the first intersection is taken over all f-weighted r-path vertex covers of G_ω, and the second intersection is taken over all minimal f-weighted r-path vertex covers of G_ω. The second intersection is irredundant.

Remark. The second decomposition of $I_{r,f}(G_\omega)$ is much more intensive than the first one.

Fact 3.1.8. [7, Lemma 2.11] If $I_{r,f}(G_\omega)$ is unmixed, then $I_r(G)$ is also unmixed.

Definition 3.1.9. [7, Definition 3.1] Let v_i be a vertex of degree 1 in G that is not a part of any r-path in G. We write that v_i is an *r-pathless leaf* of G_ω. Let H_λ be the subgraph of G_ω induced by the vertex subset $V \smallsetminus \{v_i\}$. We write that H_λ is obtained by *pruning* an r-pathless leaf from G_ω. A subgraph $\Gamma_{\lambda'}$ of G_ω is obtained by *pruning a sequence of r-pathless leaves* from G_ω if there exists a sequence of graphs $G_\omega = G^{(0)}_{\omega^{(0)}}, G^{(1)}_{\omega^{(1)}}, \ldots, G^{(l)}_{\omega^{(l)}} = \Gamma_{\lambda'}$ such that each $G^{(i+1)}_{\omega^{(i+1)}}$ is obtained by pruning an r-pathless leaf from $G^{(i)}_{\omega^{(i)}}$.

Fact 3.1.10. [7, Lemma 3.3] Let H_λ be a weighted graph obtained by pruning a single r-pathless leaf v_i from G_ω.

(a) The set of r-paths in G is the same as the set of r-paths in H.

(b) The minimal f-weighted r-path vertex covers of G_ω are the same as the minimal f-weighted r-path vertex covers of H_λ.

Lemma 3.1.11. Let H_λ be a weighted graph obtained by pruning a sequence of r-pathless leaves from G_ω.

(a) The ideals $I_{r,f}(G_\omega)$ and $I_{r,f}(H_\lambda)$ have the same generators.

(b) The ideal $I_{r,f}(G_\omega)$ is unmixed if and only if $I_{r,f}(H_\lambda)$ is so.

(c) The ideal $I_{r,f}(G_\omega)$ is Cohen-Macaulay if and only if $I_{r,f}(H_\lambda)$ is so.

Proof. (a) By Fact 3.1.10(a), the set of r-paths in G is the same as the set of r-paths in H and $\lambda(e) = \omega(e)$ for each edge $e \in E(H) \subseteq E(G)$. Then the claim about the generators now follows directly.

(b) It follows from Theorem 3.1.7 and Lemma 3.1.10(b).

(c) Part (a) implies $(S'/I_{r,f}(H_\lambda))[X] \cong R/I_{r,f}(G_\omega)$, where $S' = \mathbb{K}[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d]$. It follows that $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay if and only if $S'/I_{r,f}(H_\lambda)$ is so. $\square$

Definition 3.1.12. The *suspension* of G is the graph ΣG with vertex set

$$V(\Sigma G) = V \sqcup \{w_1, \ldots, w_d\} = \{v_1, \ldots, v_d, w_1, \ldots, w_d\}$$

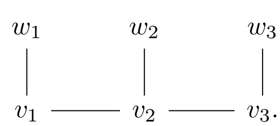
and edge set

$$E(\Sigma G) = E(G) \sqcup \{v_1 w_1, \ldots, v_d w_d\}.$$

This is also known as the K_1-*corona* of G.

Remark. The term "suspension" is due to Villarreal [13]. It is not related to the suspension of a topological space.

Example 3.1.13. The suspension ΣP_2 of the 2-path $G = P_2 = \left(v_1 \text{ —— } v_2 \text{ —— } v_3 \right)$ is



Definition 3.1.14. The *r-path suspension* of G is the graph $\Sigma_r G$ obtained by adding a new path of length r to each vertex of G such that the vertex set is

$$V(\Sigma_r G) = \{v_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\} \text{ with } v_{i,0} = v_i, \ \forall\, i = 1, \ldots, d.$$

The new r-paths are called *r-whiskers*.

Example 3.1.15. The 2-path suspension $\Sigma_2 P_2$ of the 2-path $G = P_2 = \big(v_1 \text{ —— } v_2 \text{ —— } v_3 \big)$ is

$$\begin{array}{ccccc} v_1 & \text{——} & v_{1,1} & \text{——} & v_{1,2} \\ | & & & & \\ v_2 & \text{——} & v_{2,1} & \text{——} & v_{2,2} \\ | & & & & \\ v_3 & \text{——} & v_{3,1} & \text{——} & v_{3,2}. \end{array}$$

We will classify all the weighted trees H_μ such that $I_{r,f}(H_\mu)$ is Cohen-Macaulay in terms of the weighted r-path suspension, define below.

Definition 3.1.16. [7, Definition 3.4] A *weighted r-path suspension* of G_ω is a weighted graph $(\Sigma_r G)_\lambda$ with weight function $\lambda : \Sigma_r G \to \mathbb{N}$ such that the underlying graph $\Sigma_r G$ is an r-path suspension of G and $\lambda(v_i v_j) = \omega(v_i v_j)$ for all $v_i v_j \in E(G)$, i.e., $\lambda|_{E(G)} = \omega$.

Remark. If $r = 1$, then $(\Sigma_1 G)_\lambda = (\Sigma G)_\lambda$ is a *weighted suspension* of G_ω [10, Definition 5.6].

Example 3.1.17. A weighted 2-path suspension $(\Sigma_2 P_2)_\lambda$ of

$$G_\omega := (P_2)_\omega = \big(v_1 \xrightarrow{\;1\;} v_2 \xrightarrow{\;2\;} v_3 \big)$$

is

$$\begin{array}{ccccc} v_1 & \overset{4}{\text{——}} & v_{1,1} & \overset{3}{\text{——}} & v_{1,2} \\ 1\,| & & & & \\ v_2 & \overset{3}{\text{——}} & v_{2,1} & \overset{3}{\text{——}} & v_{2,2} \\ 2\,| & & & & \\ v_3 & \overset{2}{\text{——}} & v_{3,1} & \overset{5}{\text{——}} & v_{3,2}. \end{array}$$

Based on the convention that $v_{i,0} = v_i$ for $i = 1, \ldots, d$, we have that $X_{i,0} = X_i$ for $i = 1, \ldots, d$.

Definition 3.1.18. Define a ring homomorphism p by

$$p : R' \longrightarrow R$$
$$a \longrightarrow a, \ \forall\, a \in \mathbb{K},$$
$$X_{ij} \longmapsto X_i \ \forall\, i = 1, \ldots, d, \ j = 0, \ldots, r.$$

One can think of p as a "projection".

Remark. Let $I \subseteq R'$ be a monomial ideal and set

$$IR = p(I)R = \left(X_{i_1}^{a_1} \dots X_{i_n}^{a_n} \in R \mid \exists\, X_{i_1,j_1}^{a_1} \cdots X_{i_n,j_n}^{a_n} \in [\![I]\!]\right)R.$$

In words, IR is the monomial ideal of R obtained from I by setting $X_{i,j} = X_i$ for all i, j. It is straightforward to show that if $f_1, \dots, f_m$ is a monomial generating sequence for I, then $p(f_1), \dots, p(f_m)$ is a monomial generating sequence for IR.

Example 3.1.19. Let $I = (X_{1,1}X_{1,2}^2X_{1,3}^3, X_{1,0}^4X_{2,0}X_{3,0}^2)R'$ be an ideal of the polynomial ring $R' = \mathbb{K}\big[\{X_{i,j} \mid i = 1, \dots, 3, j = 0, \dots, 3\}\big]$. Then

$$IR = (X_1X_1^2X_1^3, X_1^4X_2X_3^2)R = (X_1^6, X_1^4X_2X_3^2)R.$$

In Section 3.2, we will prove that if $I = I_{r,f}((\Sigma_r G)_\lambda)$ for some weighted r-path suspension $(\Sigma_r G)_\lambda$, then in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$ with $\underline{n} \in \mathbb{N}^d$, variables in each component have different first indexes when certain conditions for r, f and λ are satisfied. This result then will be used in finding regular sequences for R'/I in Theorem 3.4.1 and be used in Propositions 3.2.1, 3.2.43, and 3.2.44 to prove that $I = I_{r,f}((\Sigma_r G)_\lambda)$ is unmixed.

Definition 3.1.20. Let $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$. Define a ring homomorphism $p_{\underline{n}}$ by

$$\begin{aligned} p_{\underline{n}} : R' &\longrightarrow A[X_{1,0}, \dots, X_{1,\min\{n_1-1,r\}}, \dots, X_{d,0}, \dots, X_{d,\min\{n_d-1,r\}}] =: S \\ a &\longmapsto a, \ \forall\, a \in A, \\ X_{i,j} &\longmapsto X_{i,n_i-1}, \ \forall\, i = 1, \dots, d, \ j = n_i, \dots, r. \end{aligned}$$

Let $I \subseteq R'$ be a monomial ideal. Then $p_{\underline{n}}(I)S$ is the monomial ideal of S obtained from I by setting $X_{i,j} = X_{i,n_i-1}$ for any $X_{i,j} \in I$ such that $n_i \leq j \leq r$. It is straightforward to show that if $f_1, \dots, f_m$ is a monomial generating sequence for I, then $p_{\underline{n}}(f_1), \dots, p_{\underline{n}}(f_m)$ is a monomial generating sequence for $p_{\underline{n}}(I)S$.

Remark. (a) Let $\underline{1} = (1, \dots, 1) \in \mathbb{N}^d$, then $p_{\underline{1}} = p$, where p is from Definition 3.1.18.

(b) If $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ such that $n_1, \dots, n_d > r$, then $S = R'$ and $p_{\underline{n}}(I)S = I$ for any monomial ideal $I \subseteq R'$.

Example 3.1.21. Consider the following graph $(\Sigma_2 P_2)_\lambda$ with $G_\omega := (P_2)_\omega = (\ v_1 \overset{1}{\text{---}} v_2 \overset{2}{\text{---}} v_3\ )$.

$$\begin{array}{ccccc} v_1 & \overset{4}{\text{———}} & v_{1,1} & \overset{3}{\text{———}} & v_{1,2} \\ 1 \vert & & & & \\ v_2 & \overset{3}{\text{———}} & v_{2,1} & \overset{3}{\text{———}} & v_{2,2} \\ 2 \vert & & & & \\ v_3 & \overset{2}{\text{———}} & v_{3,1} & \overset{5}{\text{———}} & v_{3,2} \end{array}$$

Let

$$\begin{aligned} I := I_{2,\min}\big((\Sigma_2 P_2)_\lambda\big) = \big(&X_{1,2}^3 X_{1,1}^3 X_1^4, X_{1,1}^4 X_1 X_2, X_1 X_2 X_{2,1}^3, X_1 X_2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3, \\ &X_{2,1}^3 X_2^2 X_3^2, X_2^2 X_3^2 X_{3,1}^2, X_{3,2}^5 X_{3,1}^2 X_3^2\big) R'. \end{aligned}$$

Let $\underline{n} = (2,3,1) \in \mathbb{N}^3$. Then $S = R[X_{1,0}, X_{1,1}, X_{2,0}, X_{2,1}, X_{2,2}, X_{3,0}]$ and setting $X_{1,2} = X_{1,1}$, $X_{3,1} = X_3$, and $X_{3,2} = X_3$ in I we have

$$\begin{aligned} p_{\underline{n}}(I)S = \big(&X_{1,1}^3 X_{1,1}^3 X_1^4, X_{1,1}^4 X_1 X_2, X_1 X_2 X_{2,1}^3, X_1 X_2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3, \\ &X_{2,1}^3 X_2^2 X_3^2, X_2^2 X_3^2 X_3^2, X_3^5 X_3^2 X_3^2\big) R' \\ = \big(&X_{1,1}^6 X_1^4, X_{1,1}^4 X_1 X_2, X_1 X_2 X_{2,1}^3, X_1 X_2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3, \\ &X_{2,1}^3 X_2^2 X_3^2, X_2^2 X_3^4, X_3^9\big) R'. \end{aligned}$$

Definition 3.1.22. Let $(\Sigma_r G)_\lambda$ be a weighted r-suspension of G_ω, $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$, $\underline{n} \in \mathbb{N}^d$ and P_r an r-path in $(\Sigma_r G)_\lambda$ with the corresponding generator $\underline{X}^{\alpha}$ in I. We write

$$P_r \overset{\underline{n}}{\rightsquigarrow} v_{i_1,j_1} \cdots v_{i_m,j_m} =: \wp$$

if the reduction is $\operatorname{red}\big(p_{\underline{n}}(\underline{X}^{\alpha})\big) = X_{i_1,j_1} \ldots X_{i_m,j_m}$. We call that $\wp$ is a *path* in $p_{\underline{n}}(I)$.

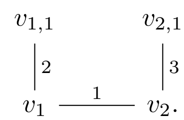
Remark. If $\underline{n}$ is known from context, we usually write $P_r \rightsquigarrow \wp$ instead of $P_r \overset{\underline{n}}{\rightsquigarrow} \wp$.

Example 3.1.23. In Example 3.1.21, $P_2 := v_{1,2} v_{1,1} v_{1,0} \overset{\underline{n}}{\rightsquigarrow} v_{1,1} v_{1,0} =: \wp$ since $X_{1,2}^3 X_{1,1}^3 X_{1,0}^4$ is the corresponding generator of P_r in I and $\operatorname{red}\big(p_{\underline{n}}(X_{1,2}^3 X_{1,1}^3 X_{1,0}^4)\big) = \operatorname{red}(X_{1,1}^3 X_{1,1}^3 X_{1,0}^4) = X_{1,1} X_{1,0}$.

Definition 3.1.24. Let $(\Sigma_r G)_\lambda$ be a weighted r-suspension of G_ω, $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$, $\underline{n} \in \mathbb{N}^d$, P_r an r-path in $(\Sigma_r G)_\lambda$ with the corresponding generator $\underline{X}^{\alpha}$ in I and $P_r \overset{\underline{n}}{\rightsquigarrow} \wp$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. Denote $v_{i,j} \smile (P_r \overset{\underline{n}}{\rightsquigarrow} \wp, \mathfrak{P})$ if $v_{i,j} \in V'' \cap V(\wp)$ and $X_{i,j}^{\delta''(v_{i,j})} \mid p_{\underline{n}}(\underline{X}^{\underline{\alpha}})$ and denote $v_{i,j} \not\smile (P_r \overset{\underline{n}}{\rightsquigarrow} \wp, \mathfrak{P})$ otherwise. In particular, if $P_r = \wp$, then denote $v_{i,j} \smile (\wp, \mathfrak{P})$ or $v_{i,j} \not\smile (\wp, \mathfrak{P})$ if $v_{i,j} \in V'' \cap V(\wp)$ and $X_{i,j}^{\delta''(v_{i,j})} \mid p_{\underline{n}}(\underline{X}^{\underline{\alpha}})$.

Remark. If $\underline{n}$ and $\mathfrak{P}$ are known from context, we write $v_{i,j} \smile (P_r \rightsquigarrow \wp)$ or $v_{i,j} \not\smile (P_r \rightsquigarrow \wp)$. In particular, if $P_r = \wp$, then write $v_{i,j} \smile \wp$ or $v_{i,j} \not\smile \wp$.

Example 3.1.25. A weighted suspension $(\Sigma G)_\lambda$ of $G_\omega := (P_1)_\omega = \big(\ v_1 \overset{1}{\text{———}} v_2\ \big)$ is



Let $I := I_{2,\min}\big((\Sigma P_1)_\lambda\big)$ and $\underline{n} := (1,1)$. Then $p_{\underline{n}}(I)$ is obtained from I by setting $X_{1,1} = X_{1,0}$ and $X_{2,1} = X_{2,0}$ in I. We have that $P_2 := v_{1,1}v_1v_2 \overset{\underline{n}}{\rightsquigarrow} v_1v_2$. Let $\underline{X}^{\underline{\alpha}} := X_{1,1}^2 X_{1,0} X_{2,0}$ be the corresponding generator of $v_{1,1}v_1v_2$ in I. Then $p_{\underline{n}}(\underline{X}^{\underline{\alpha}}) = X_{1,0}^3 X_{2,0}$. Let $\mathfrak{P} = \{v_{1,0}^2, v_{2,0}^2, v_{1,1}\}$. Then $v_{1,0} \smile (v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}, \mathfrak{P})$ since $X_{1,0}^2 \mid X_{1,0}^3 X_{2,0}$, $v_{2,0} \not\smile (v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}, \mathfrak{P})$ since $X_{2,0}^2 \nmid X_{1,0}^3 X_{2,0}$, $v_{1,1} \not\smile (v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}, \mathfrak{P})$ since $v_{1,1} \notin \{v_{1,0}, v_{2,0}\}$.

Lemma 3.1.26. Let $(\Sigma_r G)_\lambda$ be a weighted r-suspension of G_ω, $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$, and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$. Then for any path $\wp$ in $p_{\underline{n}}(I)$ such that $P_r \rightsquigarrow \wp$, we have that $v_{k,l} \smile (P_r \rightsquigarrow \wp)$ for some $v_{k,l} \in V''$.

Proof. Assume that $\wp := v_{i_1,j_1} \dots v_{i_m,j_m}$ and $\underline{X}^{\underline{\alpha}}$ is the corresponding generator of the r-path P_r in I. Then $\operatorname{red}\big(p_{\underline{n}}(\underline{X}^{\underline{\alpha}})\big) = X_{i_1,j_1} \dots X_{i_m,j_m}$ and $p_{\underline{n}}\big(\underline{X}^{\underline{\alpha}}\big) \in p_{\underline{n}}(I) \subseteq P(V'', \delta'')$. So there exists some $v_{k,l} \in V''$ such that $v_{k,l} \in V(\wp)$ and $X_{k,l}^{\delta''(v_{k,l})} \mid p_{\underline{n}}\big(\underline{X}^{\underline{\alpha}}\big)$. Hence $v_{k,l} \smile (P_r \rightsquigarrow \wp)$. □

Remark. One can think of $p_{\underline{n}}(\underline{X}^{\underline{\alpha}})$ as the corresponding generator of $(P_r \rightsquigarrow \wp)$.

Definition 3.1.27. Let $(\Sigma_r G)_\lambda$ be a weighted r-suspension of G_ω, $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$, and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. For $v_{i,j} \in V''$, set

$$\mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big) := \big\{P_r \overset{\underline{n}}{\rightsquigarrow} \wp \mid v_{i,j} \smile (P_r \rightsquigarrow \wp) \text{ but } v_{k,l} \not\smile (P_r \rightsquigarrow \wp)\ \forall\ v_{k,l} \in V'' \setminus \{v_{i,j}\}\big\}.$$

If $(P_r \rightsquigarrow \wp) \in \mathfrak{P}\big(p_{\underline{n}}(I)\big)$ such that $P_r = \wp$, then we write $P_r \in \mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big)$.

Remark. If $p_{\underline{n}}(I)$ is known from context, we usually write $\mathfrak{P}_{i,j}$ instead of $\mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big)$. If $P_r \rightsquigarrow \wp$ is such that $P_r = \wp$, we simplify $P_r \rightsquigarrow \wp$ as P_r. For example,

$$\mathfrak{P}_{i,j}(I) = \big\{P_r \mid P_r \text{ an } r\text{-path in } (\Sigma_r G)_\lambda \text{ such that } v_{i,j} \smile P_r \text{ but } v_{k,l} \not\smile P_r \ \forall\ v_{k,l} \in V'' \smallsetminus \{v_{i,j}\}\big\}.$$

$\mathfrak{P}_{i,j}(I)$ is a set of r-paths in $(\Sigma_r G)_\lambda$ that is uniquely "weighted covered" by $v_{i,j}$ when considering the "covering set" $\mathfrak{P} = (V'', \delta'')$. That's to say, when $\mathfrak{P}_{i,j}(I) \neq \emptyset$, for P_r in $\mathfrak{P}_{i,j}(I)$, we have that $v_{i,j} \in V''$ satisfies one of the constraints in Definition 3.1.3 and other vertices in V'' don't.

For $v_{i,j} \in V''$, we say that $v_{i,j}$ "weighted cover" $(P_r \overset{\underline{n}}{\rightsquigarrow} \wp)$, notationally, $v_{i,j} \smile (P_r \overset{\underline{n}}{\rightsquigarrow} \wp)$, if $v_{i,j} \in V(\wp)$ and $X_{i,j}^{\delta''(v_{i,j})} \mid p_{\underline{n}}(\underline{X}^\alpha)$, where $\underline{X^\alpha}$ is the corresponding generators of P_r in I. Then one can mimic the interpretation of $\mathfrak{P}_{i,j}(I)$ to understand $\mathfrak{P}_{i,j}(p_{\underline{n}}(\underline{X^\alpha}))$.

Example 3.1.28. In Example 3.1.25, we have that $(v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}) \in \mathfrak{P}_{1,0}$. Let $\underline{X^\beta} := X_{1,0}X_{2,0}X_{2,1}^3$ be the corresponding generator of $v_{1,0}v_{2,0}v_{2,1}$ in I. Then $p_{\underline{n}}(\underline{X^\beta}) = X_{1,0}X_{2,0}^4$. We have that $v_{2,0} \smile (v_{1,0}v_{2,0}v_{2,1} \rightsquigarrow v_{1,0}v_{2,0})$ since $X_{2,0}^2 \mid X_{1,0}X_{2,0}^4$. Then $(v_{1,0}v_{2,0}v_{2,1} \rightsquigarrow v_{1,0}v_{2,0}) \notin \mathfrak{P}_{1,0}$. Also, for a fixed $\underline{n}$, there is no other $P_3 \rightsquigarrow \wp$, therefore, $\mathfrak{P}_{1,0} = \{v_{1,1}v_{1,0}v_{2,0} \rightsquigarrow v_{1,0}v_{2,0}\}$.

Proposition 3.1.29. Let $(\Sigma_r G)_\lambda$ be a weighted r-suspension of G_ω and $I = I_{r,f}((\Sigma_r G)_\lambda)$. If $\mathfrak{P} = (V'', \delta'')$ is a minimal f-weighted vertex cover of $(\Sigma_r G)_\lambda$, then $\mathfrak{P}_{i,j}(I) \neq \emptyset$ for any $v_{i,j} \in V''$.

Proof. Suppose that $\mathfrak{P}_{i,j}(I) = \emptyset$ for some $v_{i,j} \in V''$. Then since $I \subseteq P(V'', \delta'')$ and (V'', δ'') is minimal, by Fact 3.1.7, we have that $I \subseteq P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big)$, a contradiction. □

We have a general version of Proposition 3.1.29, state in the following lemma:

Lemma 3.1.30. Let $(\Sigma_r G)_\lambda$ be a weighted r-suspension of G_ω, $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$, and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ doesn't occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$. Then $\mathfrak{P}_{i,j}(p_{\underline{n}}(I)) \neq \emptyset$ for any $v_{i,j} \in V''$. In particular, if $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$, then $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ doesn't occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$.

Proof. Suppose that $\mathfrak{P}_{i,j}\big(p_{\underline{n}}(I)\big) = \emptyset$ for some $v_{i,j} \in V''$. Then since $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$, we have that $p_{\underline{n}}(I) \subseteq P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big)$, a contradiction.

Let $I_k := P(V'', \delta'')$ occur in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I) = \bigcap_{i=1}^{n} I_i$ with $k \in \{1, \dots, n\}$ such that $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ occur in an m-irreducible decomposition of $p_{\underline{n}}(I)$ for some $v_{i,j} \in V''$. Then since $p_{\underline{n}}(I) \subseteq I_k = P(V'', \delta'')$, we have that

$$p_{\underline{n}}(I) \subseteq P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big) \subsetneq P(V'', \delta'') = I_k.$$

So

$$\bigcap_{i=1}^{n} I_i = p_{\underline{n}}(I) = P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big) \cap \bigcap_{i=1}^{n} I_i = P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big) \cap \bigcap_{i=1, i \neq k}^{n} I_i.$$

By Fact 2.2.23, the number of ideals in any irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$ is n, so the above decomposition on the right is also an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$. Then Fact 2.2.23 implies that $I_k = P\big(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}}\big)$, a contradiction. $\square$

3.2 Sufficient Conditions for Unmixedness

In this section, we prove the sufficient conditions for which the f-weighted r-path ideal of a weighted r-path suspension is unmixed. We divide the classification into 3 kinds of cases. We first discuss the sufficient conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed for the case $r = 2$.

Proposition 3.2.1. Let $(\Sigma_2 G)_\lambda$ be a weighted 2-path suspension of G_ω such that

(a) $\lambda(v_i v_j) \leq f(\lambda(v_i v_j), \lambda(v_i v_{i,1})) \leq \lambda(v_i v_{i,1})$ and $\lambda(v_i v_j) \leq f(\lambda(v_i v_j), \lambda(v_j v_{j,1})) \leq \lambda(v_j v_{j,1})$ for all edges $v_i v_j \in E\big((\Sigma_2 G)_\lambda\big)$,

(b) $\lambda(v_i v_{i,1}) \leq f(\lambda(v_i v_{i,1}), \lambda(v_{i,1} v_{i,2}))$ for $i = 1, \dots, d$,

(c) $f(\lambda(v_i v_j), \lambda(v_j v_k)) \leq \min\big\{f(\lambda(v_i v_j), \lambda(v_j v_{j,1})), f(\lambda(v_k v_j), \lambda(v_j v_{j,1}))\big\}$ for all 2-paths $v_i v_j v_k$ in G,

(d) for all 3-paths $v_i v_j v_k v_l$ in G: if $f(\lambda(v_{j,1} v_j), \lambda(v_j v_i)) < \lambda(v_j v_k)$, then $f(\lambda(v_j v_k), \lambda(v_k v_l)) \geq \lambda(v_j v_k)$,

(e) for all 3-cycles $v_i v_j v_k v_i$ in G: if $f(\lambda(v_i v_{i,1}), \lambda(v_i v_j)) < \lambda(v_i v_k)$, then either $f(\lambda(v_k v_i), \lambda(v_k v_j)) \geq \lambda(v_k v_i)$, or, $f(\lambda(v_k v_i), \lambda(v_k v_j)) \geq \lambda(v_k v_j)$ and $f(\lambda(v_j v_i), \lambda(v_j v_k)) \leq \max\{\lambda(v_j v_i), \lambda(v_j v_k)\}$.

Let $I := I_{2,f}\big((\Sigma_r G)_\lambda\big)$ and $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$. Then there exists at most one $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$. Note also that there exists a $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$, so $p_{\underline{n}}(I)$ is unmixed.

Proof. Suppose there exist $v_{i,\alpha}, v_{i,\beta} \in V''$ with $0 \leq \alpha < \beta \leq 2$ for some $i \in \{1, \ldots, d\}$. Then we have the following 3 cases (a), (b), and (c).

(a) Suppose that $\alpha = 1$ and $\beta = 2$. By Lemma 3.1.30, we have $\mathfrak{P}_{i,1} \neq \emptyset \neq \mathfrak{P}_{i,2}$. If $(P_2 \overset{n}{\rightsquigarrow} \wp) \in \mathfrak{P}_{1,2}$, then $v_{1,2} \in V(\wp)$, as $(\Sigma_2 G)_\lambda$ is a 2-path suspension, we have $P_2 \rightsquigarrow \wp$ must be of the unique form $v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,2}v_{i,1}v_{i,0}$. So we have $\mathfrak{P}_{i,2}\big(p_{\underline{n}}(I)\big) = \{v_{i,2}v_{i,1}v_{i,0}\}$. Hence $v_{i,2}v_{i,1}v_{i,0} \notin \mathfrak{P}_{i,1}$. Also, since $v_{i,1}$ is in $V(\wp)$ for any $P_r \rightsquigarrow \wp$, we have that $v_{i,1}v_{i,0}v_{j,0} \in \mathfrak{P}_{i,1}$ for some edge $v_{i,0}v_{j,0} \in E(G)$. So we have that $v_{i,1} \smile v_{i,1}v_{i,0}v_{j,0}$ and then $\delta''(v_{i,1}) \leq \lambda(v_{i,0}v_{i,1}) \leq f(\lambda(v_{i,0}v_{i,1}), \lambda(v_{i,1}v_{i,2}))$ by Condition (b), implying $v_{i,1} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicting the condition $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

(b) Suppose that $\alpha = 0$ and $\beta = 2$. Then $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$ and $\mathfrak{P}_{i,0} \neq \emptyset$ by Lemma 3.1.30. It is straightforward to show that we have the following 3 cases:

(1) Assume that $v_{i,1}v_{i,0}v_{j,0} \in \mathfrak{P}_{i,0}$. Then we have that $v_{i,0} \smile v_{i,1}v_{i,0}v_{j,0}$, and so $\delta''(v_{i,0}) \leq f(\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})) \leq \lambda(v_{i,0}v_{i,1})$ by Condition (a), implying $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicting $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

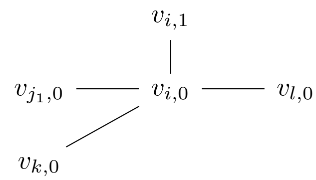
(2) Assume that $v_{i,0}v_{j,0}v_{k,0} \in \mathfrak{P}_{i,0}$ or $(v_{i,0}v_{j,0}v_{j,1} \rightsquigarrow v_{i,0}v_{j,0}) \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq \lambda(v_{i,0}v_{j,0}) \leq f(\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})) \leq \lambda(v_{i,0}v_{i,1})$ by Condition (a). So we have that $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicting $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

(3) Assume that $v_{j,0}v_{i,0}v_{k,0} \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq f(\lambda(v_{j,0}v_{i,0}), \lambda(v_{i,0}v_{k,0})) \leq \lambda(v_i v_{i,1})$ by Conditions (c) and (a), implying $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}$, contradicting $\mathfrak{P}_{i,2} = \{v_{i,2}v_{i,1}v_{i,0}\}$.

(c) Suppose that $\alpha = 0$ and $\beta = 1$. Suppose that $(v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,1}v_{i,0}) \in \mathfrak{P}_{i,1}$, then $v_{i,2}v_{i,1}v_{i,0}$ is not a path in $p_{\underline{n}}(I)$ and similar to Case (b), we have that $v_{i,0} \smile (v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,1}v_{i,0})$, a contradiction. Similarly, we have that $v_{i,2}v_{i,1}v_{i,0} \notin \mathfrak{P}_{i,1}$. So there exists $v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$. Then we have that $(v_{j_1,1}v_{j_1,0}v_{i,0} \rightsquigarrow v_{j_1,0}v_{i,0}) \notin \mathfrak{P}_{i,0}$, $v_{k,0}v_{j_1,0}v_{i,0} \notin \mathfrak{P}_{i,0}$ for any $v_{k,0}v_{j_1,0} \in E(G)$, and $v_{j_1,0}v_{i,0}v_{l,0} \notin \mathfrak{P}_{i,0}$ for any $v_{i,0}v_{l,0} \in E(G)$.

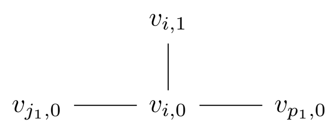
(1) Assume that $v_{k,0}v_{i,0}v_{l,0} \in \mathfrak{P}_{i,0}$ with $k \neq j_1$. (The following drawing shows part of $p_{\underline{n}}(\Sigma_r G)$

after setting $p_{\underline{n}}(v_{i,j}) = v_{i_1,j_1}$ and deleting the corresponding edges whenever $p_{\underline{n}}(X_{i,j}) = X_{i_1,j_1}$.)



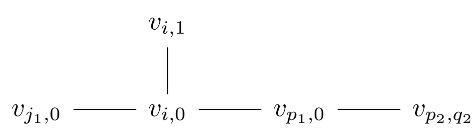
Since $v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$, we have that $v_{j_1,0}, v_{i,0} \not\smile v_{j_1,0}v_{i,0}v_{l,0}$ through Conditions (a) and (c). Then since $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$, we have that $v_{l,0} \smile v_{j_1,0}v_{i,0}v_{l,0}$ by Lemma 3.1.26. So we have that $v_{l,0} \smile v_{k,0}v_{i,0}v_{l,0}$, contradicting $v_{k,0}v_{i,0}v_{l,0} \in \mathfrak{P}_{i,0}$.

(2) Assume that $(v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$.



Similar to Case (c)(1), we have that $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$ and then $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{p_1,0})$, contradicting $(v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$.

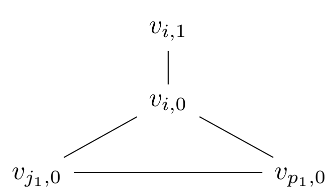
(3) Assume that $v_{i,0}v_{p_1,0}v_{p_2,p_2} \in \mathfrak{P}_{i,0}$ with $p_1 \neq j_1$.



Similar to Case (c)(1), we have $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$. As $v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$, we have $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Then $f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) < \delta''(v_{p_1,0}) \leq \lambda(v_{i,0}v_{p_1,0})$ and hence $q_2 = 0$. As $v_{i,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{i,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}$, we have $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$.

(i) Assume that $j_1 \neq p_2$. Then $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ is a 3-path in G. As $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) \geq \lambda(v_{i,0}v_{p_1,0})$ by Condition (d), contradicting $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$ and $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}$.

(ii) Assume that $j_1 = p_2$.



Then $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$. So by Condition (e), we have the following 2 cases:

A. Assume that $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})) \geq \lambda(v_{p_1,0}v_{i,0})$. Similar to Case (c)(1), we have that $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}$. So we have that $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{j_1,0}$, contradicting $v_{i,0}v_{p_1,0}v_{j_1,0} \in \mathfrak{P}_{i,0}$.

B. Assume now that $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})) \geq \lambda(v_{p_1,0}v_{j_1,0})$ and $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})) \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})\}$. Then since $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{i,1}$ and $v_{j_1,0} \not\smile v_{j_1,0}v_{p_1,0}v_{i,0}$, we have that $v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}$. Since $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{j_1,0}$, we have that $v_{p_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}$, contradicting Lemma 3.1.26.

Note that for $i = 1, \ldots, d$, by definition of $p_{\underline{n}}(I)$, there exists a generator where all variables are of the form X_{i,i_l} with $i_l \in \{0, 1, 2\}$, so there exists a vertex $v_{i,i_j} \in V''$. $\square$

Starting here, we discuss the sufficient conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed if $r = 3$.

Notation 3.2.2. We consider the next conditions on a weighted 3-path suspension $(\Sigma_3 G)_\lambda$ of G_ω.

(a) $\lambda(v_iv_j) \leq f(\lambda(v_iv_j), \lambda(v_iv_{i,1})) \leq \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \leq f(\lambda(v_iv_j), \lambda(v_jv_{j,1})) \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E\big((\Sigma_3 G)_\lambda\big)$,

(b) $\lambda(v_{i,k}v_{i,k+1}) \leq f(\lambda(v_{i,k}v_{i,k+1}), \lambda(v_{i,k+1}v_{i,k+2}))$ for $i = 1, \ldots, d$ and $k = 0, 1$,

(c) $f(\lambda(v_iv_j), \lambda(v_jv_k)) \leq \min\big\{f(\lambda(v_iv_j), \lambda(v_jv_{j,1})), f(\lambda(v_kv_j), \lambda(v_jv_{j,1}))\big\}$ for all 2-paths $v_iv_jv_k$ in G

(d) for all 4-paths $v_iv_jv_kv_lv_m$ in G: if $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) < \lambda(v_jv_k)$ or $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) < \lambda(v_kv_l)$, then $f(\lambda(v_iv_j), \lambda(v_jv_k)) \geq \lambda(v_jv_k)$ or $f(\lambda(v_kv_l), \lambda(v_lv_m)) \geq \lambda(v_kv_l)$,

(e) for all 3-cycles $v_iv_jv_kv_i$ in G:

(1) if $f(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < f(\lambda(v_iv_{i,1}), \lambda(v_iv_k))$ or there exists $v_iv_l \in E(G)$ with $j \neq l \neq k$ such that $f(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < f(\lambda(v_iv_l), \lambda(v_iv_k))$, then $f(\lambda(v_jv_i), \lambda(v_jv_k)) \leq \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$ and $f(\lambda(v_kv_i), \lambda(v_kv_j)) \geq \lambda(v_kv_j)$, and

(2) if $f(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < \lambda(v_iv_k)$, then

(i) $f(\lambda(v_kv_i), \lambda(v_kv_j)) \geq \lambda(v_kv_i)$, and

(ii) for any $v_jv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_k$,

$$\begin{cases} f(\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2})) \leq \max\big\{\lambda(v_jv_i), f(\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2}))\big\}, \text{ and} \\ f(\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2})) \leq \max\big\{f(\lambda(v_jv_i), \lambda(v_jv_k)), f(\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2}))\big\}, \end{cases}$$

and

(iii)

$$\begin{cases} f(\lambda(v_kv_j),\lambda(v_kv_{l_1,l_2}))\leq \max\bigl\{\lambda(v_kv_j), f(\lambda(v_kv_i),\lambda(v_kv_{l_1,l_2}))\bigr\} \\ \qquad\qquad\qquad\qquad \forall\ v_kv_{l_1,l_2}\in E\bigl((\Sigma_3 G)_\lambda\bigr) \text{ with } v_i\neq v_{l_1,l_2}\neq v_k, \\ \text{or } f(\lambda(v_jv_i),\lambda(v_jv_k))\geq \lambda(v_jv_i), \end{cases}$$

(f) for all 4-cycles $v_iv_jv_kv_lv_i$: if $f(\lambda(v_{i,1}v_i),\lambda(v_iv_j))<\lambda(v_iv_l)$, then either

(1)

$$f(\lambda(v_kv_j),\lambda(v_kv_l))\geq\lambda(v_kv_l) \text{ and } f(\lambda(v_jv_i),\lambda(v_jv_k))\geq\lambda(v_jv_i),$$

or

(2) (i) $f(\lambda(v_lv_i),\lambda(v_lv_k))\geq\min\{\lambda(v_lv_i),\lambda(v_lv_k)\}$, and

(ii) $f(\lambda(v_jv_i),\lambda(v_jv_k))\leq\max\{\lambda(v_jv_i),\lambda(v_jv_k)\}$, and

(iii)

$$\begin{cases} \text{either } f(\lambda(v_kv_j),\lambda(v_kv_l))\geq\lambda(v_kv_l), \\ \text{or } f(\lambda(v_lv_i),\lambda(v_lv_k))\geq\lambda(v_lv_k) \text{ and} \\ \quad \text{if } v_jv_l\in E(G),\ \text{then } f(\lambda(v_jv_i),\lambda(v_jv_k))\leq\max\bigl\{\lambda(v_jv_k), f(\lambda(v_jv_i),\lambda(v_jv_l))\bigr\}, \\ \text{or } f(\lambda(v_kv_j),\lambda(v_kv_l))\geq\lambda(v_kv_j) \text{ and } f(\lambda(v_lv_i),\lambda(v_lv_k))\geq\lambda(v_lv_i) \text{ and} \\ \quad \text{if } v_jv_l\in E(G),\ \text{then } f(\lambda(v_jv_i),\lambda(v_jv_k))\leq\max\bigl\{\lambda(v_jv_k), f(\lambda(v_jv_i),\lambda(v_jv_l))\bigr\}, \end{cases}$$

and

(iv)

$$\begin{cases} \text{either } f(\lambda(v_jv_i),\lambda(v_jv_k))\geq\lambda(v_jv_i), \\ \text{or } f(\lambda(v_lv_i),\lambda(v_lv_k))\geq\lambda(v_lv_i), \\ \text{or } f(\lambda(v_lv_i),\lambda(v_lv_k))\geq\min\{\lambda(v_lv_i),\lambda(v_lv_k)\} \text{ and } f(\lambda(v_kv_j),\lambda(v_kv_l))\leq\lambda(v_kv_l), \\ \text{or } \forall\ v_kv_{l_1,l_2}\in E\bigl((\Sigma_3 G)_\lambda\bigr) \text{ with } v_j\neq v_{l_1,l_2}\neq v_l: \\ \quad \text{either } f(\lambda(v_kv_j),\lambda(v_kv_l))\leq\max\bigl\{\lambda(v_kv_j), f(\lambda(v_kv_l),\lambda(v_kv_{l_1,l_2}))\bigr\}, \\ \quad \text{or } f(\lambda(v_kv_j),\lambda(v_kv_{l_1,l_2}))\leq\max\bigl\{\lambda(v_kv_j), f(\lambda(v_kv_l),\lambda(v_kv_{l_1,l_2}))\bigr\}, \end{cases}$$

and

(v)

$$\begin{cases} \text{either } f(\lambda(v_k v_j), \lambda(v_k v_l)) \geq \lambda(v_k v_l), \\ \text{or } f(\lambda(v_l v_i), \lambda(v_l v_k)) \geq \lambda(v_l v_i), \\ \text{or } \forall\ v_j v_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big) \text{ with } v_i \neq v_{l_1,l_2} \neq v_k : \\ \quad \text{either } f(\lambda(v_j v_i), \lambda(v_j v_k)) \leq \max\big\{\lambda(v_j v_k), f(\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2}))\big\}, \\ \quad \text{or } f(\lambda(v_j v_k), \lambda(v_j v_{l_1,l_2})) \leq \max\big\{\lambda(v_j v_k), f(\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2}))\big\}. \end{cases}$$

The next results show that $p_{\underline{n}}(I)$ is unmixed in the setting of Notation 3.2.2. See Proposition 3.2.43 for the full conclusion.

Proposition 3.2.3. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Then for $i = 1, \ldots, d$, we have $v_{i,\alpha} \notin V''$ or $v_{i,\beta} \notin V''$ for any $(\alpha, \beta) \in \{(1,2), (1,3), (2,3)\}$.

Proof. Proof by contradiction: assume that there exist such $v_{i,\alpha} \in V''$ and $v_{i,\beta} \in V''$. Then $\mathfrak{P}_{i,\alpha} \neq \emptyset$ by Lemma 3.1.30. So

$$\delta''(v_{i,\alpha}) \leq \max\{\lambda(v_{i,\alpha} v_{i,\alpha-1}), f\big(\lambda(v_{i,\alpha+1} v_{i,\alpha}), \lambda(v_{i,\alpha} v_{i,\alpha-1})\big)\} = f(\lambda(v_{i,\alpha+1} v_{i,\alpha}), \lambda(v_{i,\alpha} v_{i,\alpha-1})),$$

by Notation 3.2.2(b). Then for any $(P_r \rightsquigarrow \wp) \in \mathfrak{P}_{i,\beta}$ we have that $v_{i,\alpha} \smile (P_r \rightsquigarrow \wp)$, contradiction by the definition of $\mathfrak{P}_{i,\beta}$ and $\alpha \neq \beta$. ☐

Proposition 3.2.4. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Then for $i = 1, \ldots, d$, we have $v_{i,0} \notin V''$ or $v_{i,3} \notin V''$.

Proof. Proof by contradiction: assume that $v_{i,0} \in V''$ and $v_{i,3} \in V''$. Then $\mathfrak{P}_{i,3} = \{v_{i,3} v_{i,2} v_{i,1} v_{i,0}\}$ and $\mathfrak{P}_{i,0} \neq \emptyset$, by Lemma 3.1.30. So we have the following 3 cases:

(a) Assume that $v_{i,2} v_{i,1} v_{i,0} v_{j,0} \in \mathfrak{P}_{i,0}$, $v_{i,1} v_{i,0} v_{j,0} v_{k,l} \in \mathfrak{P}_{i,0}$, or $(v_{i,1} v_{i,0} v_{j,0} v_{j,1} \rightsquigarrow v_{i,1} v_{i,0} v_{j,0}) \in \mathfrak{P}_{i,0}$. Then we have that $\delta''(v_{i,0}) \leq f(\lambda(v_{i,0} v_{j,0}), \lambda(v_{i,0} v_{i,1})) \leq \lambda(v_{i,0} v_{i,1})$ by Notation 3.2.2(a), implying $v_{i,0} \smile v_{i,3} v_{i,2} v_{i,1} v_{i,0}$, contradicting $\mathfrak{P}_{i,3} = \{v_{i,3} v_{i,2} v_{i,1} v_{i,0}\}$.

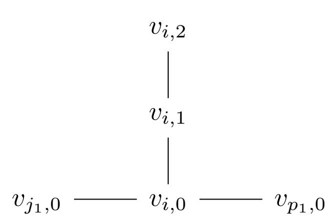
(b) Assume that $v_{i,0}v_{j,0}v_{k,0}v_{l,m} \in \mathfrak{P}_{i,0}$, $v_{i,0}v_{j,0}v_{j,1}v_{j,2} \in \mathfrak{P}_{i,0}$, $(v_{i,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i,0}v_{j,0}v_{j,1}) \in \mathfrak{P}_{i,0}$, or $(v_{i,0}v_{j,0}v_{j,1}v_{j,2} \rightsquigarrow v_{i,0}v_{j,0}) \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq \lambda(v_{i,0}v_{j,0}) \leq f(\lambda(v_{i,0}v_{j,0}), \lambda(v_{i,0}v_{i,1})) \leq \lambda(v_{i,0}v_{i,1})$ by Notation 3.2.2(a). So we have that $v_{i,0} \smile v_{i,3}v_{i,2}v_{i,1}v_{i,0}$, contradicting $\mathfrak{P}_{i,3} = \{v_{i,3}v_{i,2}v_{i,1}v_{i,0}\}$.

(c) Assume that $v_{j,0}v_{i,0}v_{k,0}v_{l,m} \in \mathfrak{P}_{i,0}$ or $(v_{j,0}v_{i,0}v_{k,1}v_{k,0} \rightsquigarrow v_{j,0}v_{i,0}v_{k,0}) \in \mathfrak{P}_{i,0}$. Then $\delta''(v_{i,0}) \leq f(\lambda(v_{j,0}v_{i,0}), \lambda(v_{i,0}v_{k,0})) \leq \lambda(v_iv_{i,1})$ by Notations 3.2.2(c) and (a), implying $v_{i,0} \smile v_{i,3}v_{i,2}v_{i,1}v_{i,0}$, contradicting $\mathfrak{P}_{i,3} = \{v_{i,3}v_{i,2}v_{i,1}v_{i,0}\}$. □

Propositions 3.2.5 to 3.2.15 will be used to prove a result similar to the one in Proposition 3.2.4.

Proposition 3.2.5. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $(v_{i,1}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{i,0}v_{p_1,0} \in E(G)$ such that $p_1 \neq j_1$.

Proof. Proof by contradiction: assume that $(v_{i,1}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $v_{i,0}v_{p_1,0} \in E(G)$ such that $p_1 \neq j_1$.

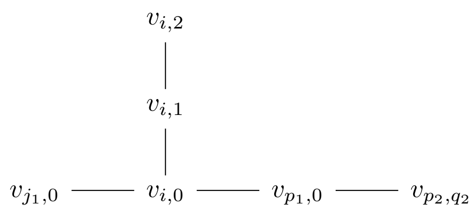


Then we have that $v_{p_1,0} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, through Notation 3.2.2(c) we have that $v_{j_1,0}, v_{i,0} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$, contradicting Lemma 3.1.26. □

Proposition 3.2.6. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any

$v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for a $v_{i,0}v_{j_1,0} \in E(G)$, then $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any 2-path $v_iv_{p_1,0}v_{p_2,q_2}$ in $\Sigma_3 G$ such that $p_1 \neq j_1$ and $v_{p_2,q_2} \neq v_{j_1,0}$.

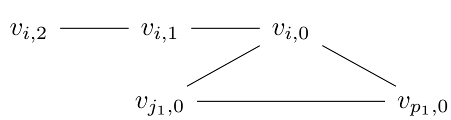
Proof. Proof by contradiction: assume that $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for some 2-path $v_iv_{p_1,0}v_{p_2,q_2}$ in $\Sigma_3 G$ such that $p_1 \neq j_1$ and $v_{p_2,q_2} \neq v_{j_1,0}$.



Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, we have that $v_{j_1,0}, v_{i,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by Notation 3.2.2(c). Note that $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$, so $v_{p_1,0}, v_{p_2,q_2} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, contradicting $p_{\underline{n}}(I) \subseteq P(V'', \delta'')$ and Lemma 3.1.26. □

Proposition 3.2.7. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for a $v_{i,0}v_{j_1,0} \in E(G)$, then $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any 2-path $v_iv_{p_1,0}v_{p_2,q_2}$ in $\Sigma_3 G$ such that $p_1 \neq j_1$ and $v_{p_2,q_2} = v_{j_1,0}$.

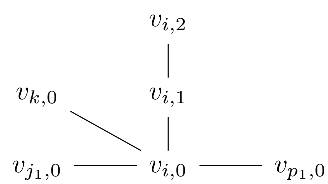
Proof. Proof by contradiction: assume that $v_{i,1}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for some 2-path $v_iv_{p_1,0}v_{p_2,q_2}$ in $\Sigma_3 G$ such that $p_1 \neq j_1$ and $v_{p_2,q_2} = v_{j_1,0}$. Then $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G_ω.



Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, we have that $v_{i,0} \not\succ v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$ and $f(\lambda(v_{i,0}v_{i,1}), \lambda(v_{i,0}v_{j_1,0})) < f(\lambda(v_{i,0}v_{i,1}), \lambda(v_{i,0}v_{p_1,0}))$. Then $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})) \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})\}$ and $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})) \geq \lambda(v_{p_1,0}v_{j_1,0})$ by Notation 3.2.2(e)(1). Hence we can show $v_{j_1,0}, v_{p_1,0} \not\succ v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$ by way of contradiction. By similar to the proof of Proposition 3.2.3, we have that $v_{i,1} \notin V''$, so $v_{i,1} \not\succ v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.8. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any 3-path $v_{k,0}v_{i,0}v_{p_1,0}$ in G such that $k \neq j_1 \neq p_1$.

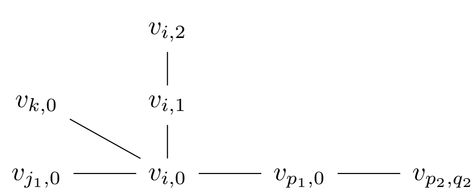
Proof. Proof by contradiction: assume that $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some 3-path $v_{k,0}v_{i,0}v_{p_1,0}$ in G such that $k \neq j_1 \neq p_1$.



So we have that $v_{p_1,0}, v_{p_2,q_2} \not\smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Then it follows that $v_{j_1,0} \smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ or $v_{i,0} \smile (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ by Lemma 3.1.26. Hence $v_{j_1,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ or $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ by Notation 3.2.2(c), contradicting the condition $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$. □

Proposition 3.2.9. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for a $v_{i,0}v_{j_1,0} \in E(G)$, then $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any 3-path $v_{k,0}v_{i,0}v_{p_1,0}p_{p_2,q_2}$ in $\Sigma_3 G$ such that $k \neq j_1 \neq p_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$.

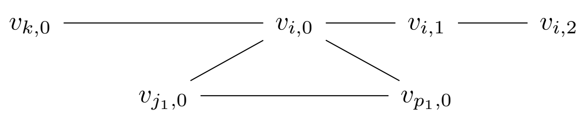
Proof. Proof by contradiction: assume that $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for a 3-path $v_{k,0}v_{i,0}v_{p_1,0}p_{p_2,q_2}$ in $\Sigma_3 G$ such that $k \neq j_1 \neq p_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$.



Then we have that $v_{p_1,0}, v_{p_2,q_2} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. So we have that $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ or $v_{i,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by Lemma 3.1.26. Hence $v_{j_1,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ or $v_{i,0} \smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ by Notation 3.2.2(c), contradicting $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$. □

Proposition 3.2.10. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for a $v_{i,0}v_{j_1,0} \in E(G)$, then $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any 3-path $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ in G such that $k \neq j_1 \neq p_1$ and $v_{j_1,0} = v_{p_2,q_2}$.

Proof. Proof by contradiction: assume that $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for a 3-path $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ in G such that $k \neq j_1 \neq p_1$ and $v_{j_1,0} = v_{p_2,q_2}$. Then $v_{i,0}v_{p_1,0}v_{j_1,0}v_{i,0}$ is a 3-cycle in G_ω.



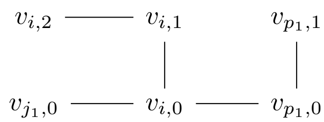
Since $v_{i,0} \not\smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$, we have that $v_{i,0} \not\smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$ by Notation 3.2.2(c). Since $v_{k,0} \not\smile v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have that $v_{k,0} \not\smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$. Since $v_{i,0} \not\smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$ and $v_{i,0} \smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$, we have

$$f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \delta''(v_{i,0}) \leq f(\lambda(v_{k,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})).$$

By Notation 3.2.2(e)(1), we have that $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})) \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_1,p_1})\}$ and $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{i,0})) \geq \lambda(v_{p_1,0}v_{j_1,0})$. Hence similar to the proof of Proposition 3.2.7, $v_{j_1,0}, v_{p_1,0} \not\smile v_{k,0}v_{i,0}v_{j_1,0}v_{p_1,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.11. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,1}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{i,0}v_{p_1,0} \in E(G)$ such that $p_1 \neq j_1$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,1}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $v_{i,0}v_{p_1,0} \in E(G)$ such that $p_1 \neq j_1$.



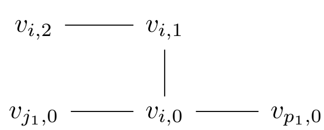
By Notation 3.2.2(b) we have

$$\lambda(v_{p_1,0}v_{p_1,1}) \leq f(\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})) < f(\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})) + \lambda(v_{p_1,1}v_{p_1,2}).$$

So we have that $v_{p_1,1} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$. Also, we have that $v_{i,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$ by Notation 3.2.2(c), and $v_{j_1,0}, v_{p_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$, contradicting Lemma 3.1.26. ☐

Proposition 3.2.12. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{i,0}v_{p_1,0} \in E(G)$ such that $p_1 \neq j_1$.

Proof. Proof by contradiction: assume that we have $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $v_{i,0}v_{p_1,0} \in E(G)$ such that $p_1 \neq j_1$.



By Notation 3.2.2(b) we have that

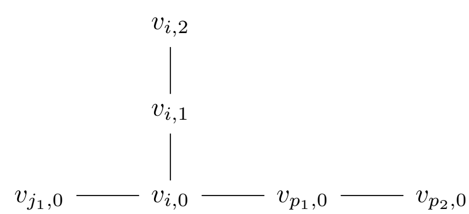
$$\begin{aligned} & f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_1,1})) + \lambda(v_{p_1,0}v_{p_1,1}) \\ \leq\ & f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_1,1})) + f(\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})) \\ <\ & f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_1,1})) + f(\lambda(v_{p_1,0}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,2})) + \lambda(v_{p_1,1}v_{p_1,2}), \end{aligned}$$

we have that $v_{p_1,0} \not\leftarrow (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Also, note that $v_{i,0} \not\leftarrow (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow$

$v_{j_1,0}v_{i,0}v_{p_1,0})$ by Notation 3.2.2(c), and $v_{j_1,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$, contradicting Lemma 3.1.26. $\square$

Proposition 3.2.13. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \notin \mathfrak{P}_{i,0}$ for any 2-path $v_{i,0}v_{p_1,0}v_{p_2,0}$ in G such that $p_1 \neq j_1 \neq p_2$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some 2-path $v_{i,0}v_{p_1,0}v_{p_2,0}$ in G such that $p_1 \neq j_1 \neq p_2$.



By Notation 3.2.2(a),

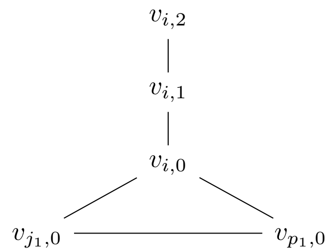
$$\lambda(v_{p_1,0}v_{p_2,0}) \leq f(\lambda(v_{p_1,0}v_{p_2,0}), \lambda(v_{p_2,0}v_{p_2,1})) < f(\lambda(v_{p_1,0}v_{p_2,0}), \lambda(v_{p_2,0}v_{p_2,1})) + \lambda(v_{p_2,0}v_{p_2,1}).$$

So we have that $v_{p_2,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Also, we have that $v_{i,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ by Notation 3.2.2(c), and $v_{j_1,0}, v_{p_1,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicting Lemma 3.1.26. $\square$

Proposition 3.2.14. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \notin \mathfrak{P}_{i,0}$ for any 2-path $v_{i,0}v_{p_1,0}v_{p_2,0}$ in G such that $p_1 \neq j_1 = p_2$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some

2-path $v_{i,0}v_{p_1,0}v_{p_2,0}$ in G such that $p_1 \neq j_1 = p_2$.



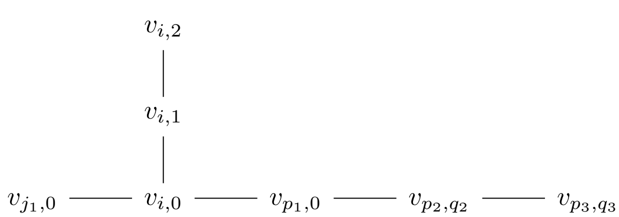
By Notation 3.2.2(a),

$$\lambda(v_{i,0}v_{j_1,0}) \leq f(\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) < f(\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) + \lambda(v_{j_1,1}v_{j_1,2}).$$

So we have that $v_{j_1,0} \not\succ (v_{j_1,1}v_{j_1,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{i,1})$. Additionally, we also have that $v_{i,1}, v_{i,0} \not\succ (v_{j_1,1}v_{j_1,0}v_{i,0}v_{i,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{i,1})$, contradicting Lemma 3.1.26. □

Proposition 3.2.15. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for a $v_{i,0}v_{j_1,0} \in E(G)$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any 3-path $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ in $\Sigma_3 G$ such that $p_1 \neq j_1$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ such that $p_1 \neq j_1$.



Then $v_{p_1,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{p_2,q_2} \not\succ v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$. Since $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$, we have that $v_{j_1,0}, v_{i,0} \not\succ v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by Notation 3.2.2(c). So we have that $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ by Lemma 3.1.26. Hence

$$f(\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})) < \delta''(v_{p_2,q_2}) \leq \lambda(v_{p_1,0}v_{p_2,q_2}).$$

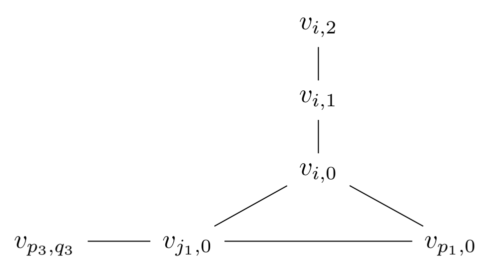
Thus, by Notation 3.2.2(b) we have that $q_2 = 0$ and then $q_3 = 0$ by Notation 3.2.2(a). Since $v_{i,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ and $v_{i,0} \not\smile v_{i,2}v_{i,1}v_{i,0}v_{j_1,0}$, we have that $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$.

(a) Assume that $p_2 \neq j_1 \neq p_3$. Then $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G. So by Notations 3.2.2(d) and (c) we have

$$\lambda(v_{i,0}v_{p_1,0}) \leq f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})) \leq f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})),$$

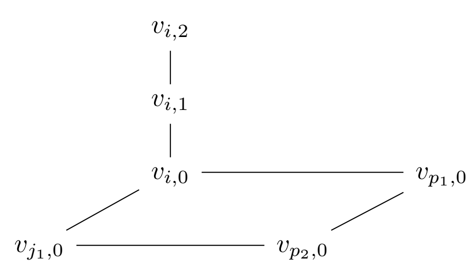
contradicting $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$.

(b) Assume that $j_1 = p_2$.



Then by Notation 3.2.2(e)(2)(i), we have that $v_{p_1,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$. Observe that $v_{p_3,q_3}, v_{i,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$ and $v_{i,1}, v_{i,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$. By Notation 3.2.2(e)(2)(ii), we have that $v_{j_1,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26.

(c) Assume that $j_1 = p_3$. Then $v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}v_{i,0}$ is a 4-cycle in G.



So we have the following 3 cases:

(1) $f(\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})) \geq \lambda(v_{p_2,0}v_{p_1,0})$ by Notation 3.2.2(f)(1) or (f)(2)(iii). Then we have that $v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. But $v_{j_1,0}, v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicting Lemma 3.1.26.

(2) $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})) \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\}$ and

$$f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \lambda(v_{p_1,0}v_{p_2,0}),$$

by Notations 3.2.2(f)(2)(ii) and (f)(2)(iii). Then we have that $v_{j_1,0} \not\succ v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$ and $v_{p_1,0} \not\succ v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$. But $v_{i,0}, v_{p_2,0} \not\succ v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$, contradicting Lemma 3.1.26.

(3) $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})) \leq \max\{\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})\}$ and

$$f(\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})) \geq \lambda(v_{p_2,0}v_{j_1,0}) \text{ and } f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \lambda(v_{p_1,0}v_{i,0}),$$

by Notations 3.2.2(f)(2)(ii) and (f)(2)(iii). Then $v_{j_1,0} \not\succ v_{p_2,0}v_{j_1}v_{i,0}v_{p_1,0}$, $v_{p_2,0} \not\succ v_{p_2,0}v_{j_1}v_{i,0}v_{p_1,0}$ and $v_{p_1,0} \not\succ v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. But $v_{i,0} \not\succ v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.16. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Then for $i = 1, \ldots, d$, we have $v_{i,0} \notin V''$ or $v_{i,2} \notin V''$.

Proof. Proof by contradiction: assume that $v_{i,0} \in V''$ and $v_{i,2} \in V''$. Then similar to the proof of Proposition 3.2.4, $v_{i,3}v_{i,2}v_{i,1}v_{i,0}, (v_{i,3}v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,2}v_{i,1}v_{i,0}) \notin \mathfrak{P}_{i,2}$. Then $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ for some $v_{i,0}v_{j_1,0} \in E(G)$. Then one can check that $(P_r \rightsquigarrow \wp) \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,2} \in V(\wp)$. Combining $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,2}$ with Propositions 3.2.5 to 3.2.15, we have $\mathfrak{P}_{i,0} = \emptyset$, contradicting Lemma 3.1.26. □

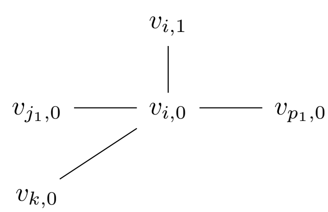
Proposition 3.2.17. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $\mathfrak{P}_{i,0} = \emptyset$.

Proof. One can check that $(P_r \rightsquigarrow \wp) \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,2} \in V(\wp)$. So one can also check that the remaining 11 cases are identical to the ones in Proposition 3.2.5 to 3.2.15 and their corresponding proofs. □

Proposition 3.2.18. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $(v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0}) \in \mathfrak{P}_{i,1}$, then $\mathfrak{P}_{i,0} = \emptyset$.

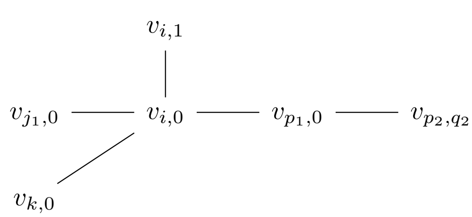
Proof. Note that $\wp \notin \mathfrak{P}_{i,0}$ for any path $\wp$ in $p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,1} \in V(\wp)$.

(a) Assume that $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $k \neq j_1 \neq p_1$.



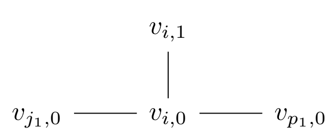
Then $v_{j_1,0}, v_{k,0} \not\vdash (v_{j_1,1}v_{j_1,0}v_{i,0}v_{k,0} \rightsquigarrow v_{j_1,0}v_{i,0}v_{k,0})$. Also, we have that $v_{i,0} \not\vdash (v_{j_1,1}v_{j_1,0}v_{i,0}v_{k,0} \rightsquigarrow v_{j_1,0}v_{i,0}v_{k,0})$ by Notation 3.2.2(c), contradicting Lemma 3.1.26.

(b) Assume that $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for some $k \neq j_1 \neq p_1$.



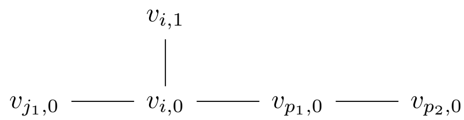
Then this case is similar to Case (a). Note that in this case, we may have that $v_{j_2,k_2} = v_{p_2,q_2}$.

(c) Assume that $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \in \mathfrak{P}_{i,0}$ or $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1$.



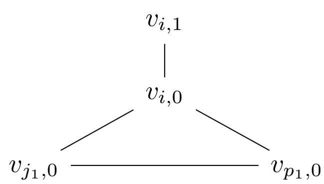
Then $v_{p_1,0} \not\vdash (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by Notation 3.2.2(a). Also, note that $v_{i,0} \not\vdash (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by Notation 3.2.2(c), and that $v_{j_1,0} \not\vdash (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$, contradicting Lemma 3.1.26.

(d) Assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1 \neq p_2$.



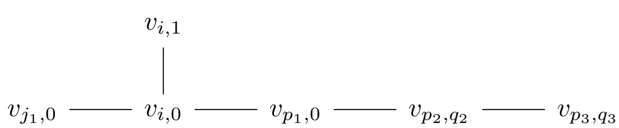
Then we have that $v_{p_2,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ by Notation 3.2.2(a), and $v_{i,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ by Notation 3.2.2(c) and $v_{p_1,0}, v_{j_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicting Lemma 3.1.26.

(e) Assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1 = p_2$.



Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})) \geq \lambda(v_{p_1,0}v_{i,0})$. So we have $v_{p_1,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. Since $v_{j_1,0} \not\leftarrow (v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0})$, we have $v_{j_1,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. Also, we have that $v_{i,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by Notation 3.2.2(c). Hence we have that $v_{p_1,0}, v_{i,0}, v_{j_1,0} \not\leftarrow (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$, contradicting Lemma 3.1.26.

(f) Assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $j_1 \neq p_1$ and $v_{p_2,q_2} \neq v_{j_1,0} \neq v_{p_3,q_3}$.

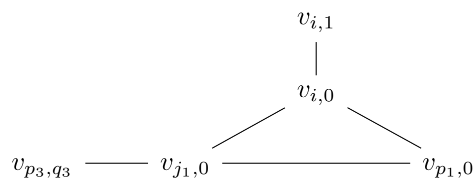


By way of contradiction, we get that $v_{j_1,0}, v_{i,0}, v_{p_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, so we have that $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{p_2,q_2} \not\leftarrow v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have $f(\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})) < \lambda(v_{p_1,0}v_{p_2,q_2})$. So we have that $q_3 = 0$ and then $q_2 = 0$. So we have that $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G. Hence by Notation 3.2.2(d),

$$f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})) \geq \lambda(v_{i,0}v_{p_1,0}) > f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})),$$

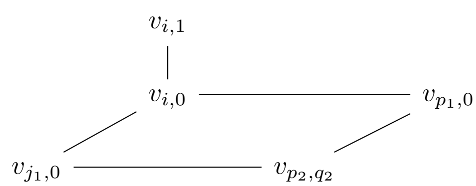
contradicting Notation 3.2.2(c).

(g) Assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_1,0} = v_{p_2,q_2}$.



Then similar to Case (e), we have that $v_{p_1,0}, v_{i,0}, v_{j_1,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$, contradicting Lemma 3.1.26.

(h) Assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_1,0} = v_{p_3,q_3}$.



Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}v_{i,0}$ is a 4-cycle in G, we have the following 2 cases by Notation 3.2.2(f):

(1) $f(\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_1,0})) \geq \lambda(v_{p_2,q_2}v_{p_1,0})$ by Notation 3.2.2(f)(1). Then we have that $v_{p_2,q_2} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$. Since

$$\lambda(v_{j_1,1}v_{j_1,0}) + f(\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) > f(\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) \geq \lambda(v_{i,0}v_{j_1,0}),$$

by Notation 3.2.2(a), we have that $v_{j_1,0} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$. Also, $v_{i,0} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$ by condition (c) and $v_{p_1,0} \not\smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$, contradicting Lemma 3.1.26.

(2) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\}$ by Notation 3.2.2(f)(2)(i). As $v_{j_1,0} \not\smile (v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,1} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0})$, we have that $v_{j_1,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. Also, since $v_{i,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by Notation 3.2.2(c), we have that $v_{p_1,0} \not\smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$. So we have that $v_{p_1,0} \smile (v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{p_1,0}v_{i,0}v_{j_1,0})$ by Lemma 3.1.26. So we have that $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Since

$$\begin{aligned} f(\lambda(v_{i,0}v_{j_1,0}), \lambda(v_{j_1,0}v_{p_2,q_2})) &\leq f(\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) \\ &< \lambda(v_{j_1,1}v_{j_1,0}) + f(\lambda(v_{j_1,1}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) \end{aligned}$$

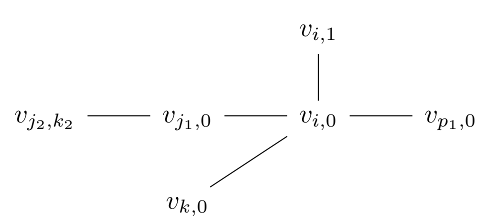
by Notation 3.2.2(c), we have that $v_{j_1,0} \not\sim v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Also, $v_{i,0} \not\sim v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$ by Notation 3.2.2(a) and $v_{p_2,q_2} \not\sim v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$, contradicting Lemma 3.1.26. $\square$

Proposition 3.2.19. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \in \mathfrak{P}_{i,1}$ for some $v_{i,0}v_{j_1,0} \in E(G)$, then $\wp \notin \mathfrak{P}_{i,0}$ for any path $\wp \in p_{\underline{n}}(I)$ with $v_{i,0}, v_{j_1,0} \in V(\wp)$ or with $v_{i,0}, v_{i,1} \in V(\wp)$

Proof. It is straightforward to show this statement. $\square$

Proposition 3.2.20. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} \neq v_{k,0} \neq v_{j_1,0}$ and $j_1 \neq p_1$.

Proof. Proof by contradiction: assume that $(v_{k,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{k,0}v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} \neq v_{k,0} \neq v_{j_1,0}$ and $j_1 \neq p_1$.

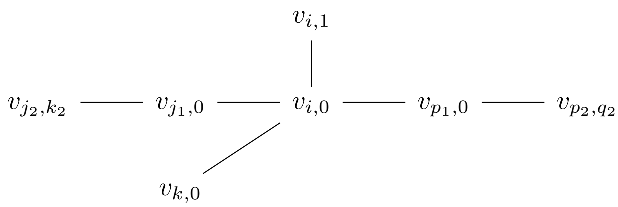


Then $v_{j_2,k_2}, v_{j_1,0}, v_{k,0} \not\sim v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{k,0}$. Also, we have that $v_{i,0} \not\sim v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{k,0}$ by Notation 3.2.2(c), contradicting Lemma 3.1.26. Note that in this case, we may have that $v_{j_2,k_2} = v_{p_1,0}$. $\square$

Proposition 3.2.21. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any

$v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} \neq v_{k,0} \neq v_{j_1,0}$ and $j_1 \neq p_1$.

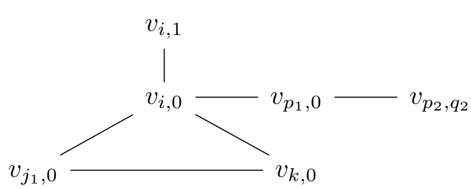
Proof. Proof by contradiction: assume that $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} \neq v_{k,0} \neq v_{j_1,0}$ and $j_1 \neq p_1$.



Then the proof is similar to the proof of Proposition 3.2.20. Note that in this case, we may have that $v_{j_2,k_2} = v_{p_1,0}$, etc. □

Proposition 3.2.22. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{k,0}$ and $j_1 \neq p_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$.

Proof. Proof by contradiction: assume that $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{k,0}$ and $j_1 \neq p_1$ and $v_{j_1,0} \neq v_{p_2,q_2}$.

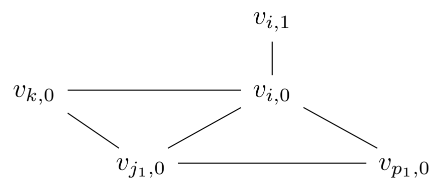


Then by way of contradiction, we get that $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{k,0}$. Since $v_{p_1,0} \not\smile v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have that $f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) < \lambda(v_{i,0}v_{p_1,0})$ and then $q_2 = 0$. So we have that $v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ is a 4-path in G. Hence $f(\lambda(v_{k,0}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) \geq \lambda(v_{j_1,0}v_{i,0})$, contradicting $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ and $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{k,0}$. □

Proposition 3.2.23. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and

$P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{k,0}$ and $j_1 \neq p_1$ and $v_{j_1,0} = v_{p_2,q_2}$.

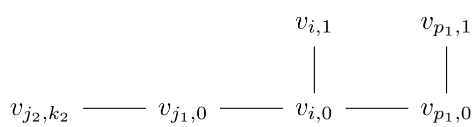
Proof. Proof by contradiction: assume that $v_{k,0}v_{i,0}v_{p_1,0}v_{p_2,q_2} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{k,0}$ and $j_1 \neq p_1$ and $v_{j_1,0} = v_{p_2,q_2}$.



Then $v_{k,0}, v_{j_1,0} \not\leftarrow v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $f(\lambda(v_{k,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})) > f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0}))$ and $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G, we have that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})) \geq \lambda(v_{p_1,0}v_{i,0})$ by Notation 3.2.2(e)(1). So we have that $v_{p_1,0} \not\leftarrow v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have that $v_{i,0} \not\leftarrow v_{k,0}v_{j_1,0}v_{i,0}v_{p_1,0}$ by Notation 3.2.2(c), contradicting Lemma 3.1.26. □

Proposition 3.2.24. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \notin \mathfrak{P}_{i,0}$ for any $p_1 \neq j_1$ and $v_{p_1,0} \neq v_{j_2,k_2}$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1$ and $v_{p_1,0} \neq v_{j_2,k_2}$.

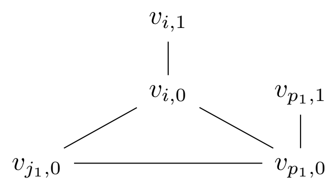


By Notation 3.2.2(a), we have $v_{p_1,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, note that $v_{i,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$ by Notation 3.2.2(c), and that $v_{j_2,k_2}, v_{j_1,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.25. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and

$P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \notin \mathfrak{P}_{i,0}$ for any $p_1 \neq j_1$ and $v_{p_1,0} = v_{j_2,k_2}$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_1,1}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1$ and $v_{p_1,0} = v_{j_2,k_2}$.



By Notation 3.2.2(b), we have

$$\lambda(v_{p_1,1}v_{p_1,0}) \leq f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) < f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) + \lambda(v_{p_1,2}v_{p_1,1}).$$

So we have that $v_{p_1,1} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$ and $v_{p_1,1} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$. Since $v_{i,0}v_{p_1,0}v_{j_1,0}v_{i,0}$ is a 3-cycle in G and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by Notation 3.2.2(e)(2)(iii):

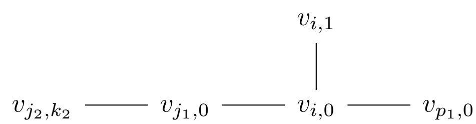
(a) $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1})) \leq \max\{\lambda(v_{p_1,0}v_{j_1,0}), f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1}))\}$. Then $v_{p_1,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$. Also, we have that $v_{i,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$ by Notation 3.2.2(a) and $v_{j_1,0} \not\leftarrow v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1}$, contradicting Lemma 3.1.26.

(b) $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})) \geq \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$. Also, we have that $v_{i,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$ by Notation 3.2.2(c) and $v_{p_1,0} \not\leftarrow v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1}$, contradicting Lemma 3.1.26. □

Proposition 3.2.26. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any $p_1 \neq j_1$ and $v_{p_1,0} \neq v_{j_2,k_2}$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1$

and $v_{p_1,0} \neq v_{j_2,k_2}$.



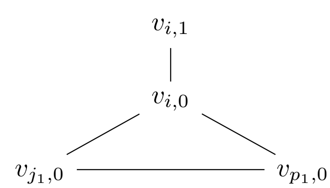
Since

$$\begin{aligned}\lambda(v_{i,0}v_{p_1,0}) &\leq f(\lambda(v_{p_1,1}v_{p_1,0}), \lambda(v_{i,0}v_{p_1,0})) \\ &< f(\lambda(v_{p_1,1}v_{p_1,0}), \lambda(v_{i,0}v_{p_1,0})) + f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) + \lambda(v_{p_1,2}v_{p_1,1}),\end{aligned}$$

we have that $v_{p_1,0} \not\succ v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have that $v_{i,0} \not\succ v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$ by Notation 3.2.2(c), and $v_{j_2,k_2}, v_{j_1,0} \not\succ v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. $\square$

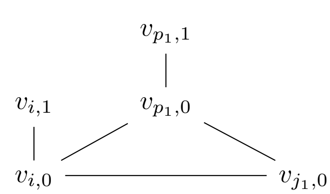
Proposition 3.2.27. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \notin \mathfrak{P}_{i,0}$ for any $p_1 \neq j_1$ and $v_{p_1,0} = v_{j_2,k_2}$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \rightsquigarrow v_{i,0}v_{p_1,0}) \in \mathfrak{P}_{i,0}$ for some $p_1 \neq j_1$ and $v_{p_1,0} = v_{j_2,k_2}$.



Since $v_{i,0}v_{p_1,0}v_{j_1,0}v_{i,0}$ is a 3-cycle in G and $\lambda(v_{i,0}v_{p_1,0}) > f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0}))$, we have the following 2 cases by Notation 3.2.2(e)(2)(iii):

(a) $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1})) \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0}), f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1}))\big\}$.



Assume that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1})) < f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1}))$. Since $v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_1,0}$ is a 3-cycle in G, we have that $f(\lambda(v_{j_1,0}v_{p_1,0}), \lambda(v_{j_1,0}v_{i,0})) \geq \lambda(v_{j_1,0}v_{i,0})$ by Notation 3.2.2(e)(1). Assume that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_1,1})) \geq f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_1,1}))$. Then by Notation 3.2.2(b),

$$\begin{aligned} & f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,1}v_{p_1,0})) + \lambda(v_{p_1,1}v_{p_1,0}) \\ \leq\ & f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})) + f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) \\ <\ & f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})) + f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) + \lambda(v_{p_1,2}v_{p_1,1}). \end{aligned}$$

Thus, $v_{p_1,0} \not\succ (v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{j_1,0}v_{p_1,0})$. Also, $v_{i,0} \not\succ (v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{j_1,0}v_{p_1,0})$ by Notation 3.2.2(a) and $v_{j_1,0} \not\succ (v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{i,0}v_{j_1,0}v_{p_1,0})$, contradicting Lemma 3.1.26.

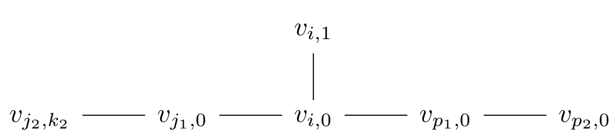
(b) $f(\lambda(v_{j_1,0}v_{i_1,0}), \lambda(v_{j_1,0}v_{p_1,0})) \geq \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_1,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. By Notation 3.2.2(b) we have

$$\begin{aligned} & f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})) + \lambda(v_{p_1,1}v_{p_1,0}) \\ \leq\ & f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})) + f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) \\ <\ & f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,1}v_{p_1,0})) + f(\lambda(v_{p_1,2}v_{p_1,1}), \lambda(v_{p_1,1}v_{p_1,0})) + \lambda(v_{p_1,2}v_{p_1,1}). \end{aligned}$$

So we have that $v_{p_1,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$. Additionally, we also have that $v_{i,0} \not\succ (v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_1,1} \rightsquigarrow v_{j_1,0}v_{i,0}v_{p_1,0})$ by Notation 3.2.2(c), contradicting Lemma 3.1.26. □

Proposition 3.2.28. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} \neq v_{p_1,0} \neq v_{j_1,0}$ and $v_{j_2,k_2} \neq v_{p_2,q_2} \neq v_{j_1,0}$.

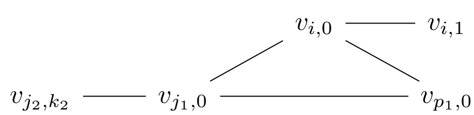
Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} \neq v_{p_1,0} \neq v_{j_1,0}$ and $v_{j_2,k_2} \neq v_{p_2,q_2} \neq v_{j_1,0}$.



By way of contradiction, we get that $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, we have that $f(\lambda(v_{j_2,k_2}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) < \lambda(v_{j_1,0}v_{i,0})$. Then $k_2 = 0$, and so $v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ is a 4-path in G. Since $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have $f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \lambda(v_{i,0}v_{p_1,0})$ by Notation 3.2.2(d). So we have that $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0})$, contradicting $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$. □

Proposition 3.2.29. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \notin \mathfrak{P}_{i,0}$ for any $j_1 = p_2$.

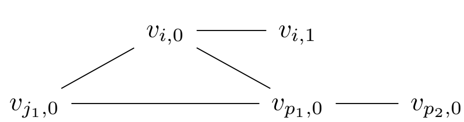
Proof. Proof by contradiction: assume $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some $j_1 = p_2$.



By way of contradiction, we get that $v_{j_2,k_2}, v_{j_1,0}, v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G and $\lambda(v_{i,0}v_{p_1,0}) > f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0}))$, we have that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})) \geq \lambda(v_{p_1,0}v_{i,0})$ by Notation 3.2.2(e)(2)(i). So we have that $v_{p_1,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.30. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_1,0}$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_1,0}$.



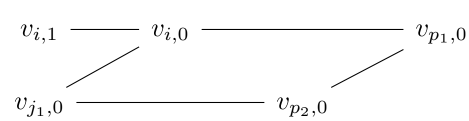
It is straightforward to show $v_{i,0}, v_{j_1,0}, v_{p_2,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$ and $v_{i,0}, v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Then $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by Notation 3.2.2(e)(2)(iii):

(a) $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_2,0})) \leq \max\big\{\lambda(v_{p_1,0}v_{j_1,0}), f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0}))\big\}$. So we have that $v_{p_1,0} \smile v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0}$ or that $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0})$ since $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$, a contradiction.

(b) $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})) \geq \lambda(v_{j_1,0}v_{i,0})$. Then we have that $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So we have that $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So we have that $v_{p_1,0} \smile (v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0})$, a contradiction. ☐

Proposition 3.2.31. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_2,0}$ and $j_1 \neq p_1$.

Proof. Proof by contradiction: assume that $(v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_2,1} \rightsquigarrow v_{i,0}v_{p_1,0}v_{p_2,0}) \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_2,0}$ and $j_1 \neq p_1$.



Then

$$f(\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})) \leq f(\lambda(v_{p_2,0}v_{p_1,0}), \lambda(v_{p_2,0}v_{p_2,1}))$$
$$< f(\lambda(v_{p_2,0}v_{p_1,0}), \lambda(v_{p_2,0}v_{p_2,1})) + \lambda(v_{p_2,0}v_{p_2,1}).$$

So we have that $v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. By way of contradiction, we get that $v_{i,0}, v_{j_1,0}, v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. So we have that $v_{p_1,0} \smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{j_1,0}v_{i,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by Notation 3.2.2(f):

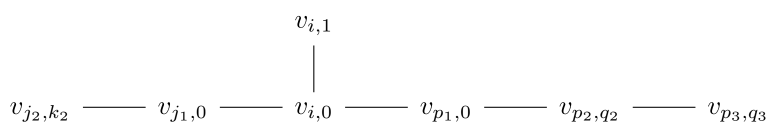
(a) $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})) \geq \lambda(v_{j_1,0}v_{i,0})$ by Notation 3.2.2(f)(1). So $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$.

By way of contradiction, $v_{i,0}, v_{p_1,0}, v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, contradicting Lemma 3.1.26.

(b) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}$ by Notation 3.2.2(f)(2)(i). Then since $v_{p_1,0} \smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$, we have that $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,0}$. But by way of contradiction, we get that $v_{i,0}, v_{j_1,0}, v_{p_2,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.32. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} \neq v_{p_1,0} \neq v_{j_1,0}$, $v_{j_2,k_2} \neq v_{p_2,q_2} \neq v_{j_1,0}$ and $v_{j_2,k_2} \neq v_{p_3,q_3} \neq v_{j_1,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_1,1}v_{p_1,2} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} \neq v_{p_1,0} \neq v_{j_1,0}$, $v_{j_2,k_2} \neq v_{p_2,q_2} \neq v_{j_1,0}$, and $v_{j_2,k_2} \neq v_{p_3,q_3} \neq v_{j_1,0}$.



By way of contradiction, we get that $v_{p_1,0} \smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have that $q_2 = 0$. By way of contradiction, we can only have the two cases $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ or $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$.

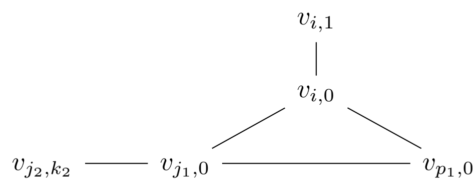
(a) Suppose that $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, we have that $k_2 = 0$ and $f(\lambda(v_{j_2,k_2}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) < \lambda(v_{j_1,0}v_{i,0})$. Then $v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$ is a 4-path in G. Since $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) \geq \lambda(v_{i,0}v_{p_1,0})$ by Notation 3.2.2(d), contradicting $v_{p_1,0} \smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$ and $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$.

(b) Suppose that $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have that $q_3 = 0$ and $f(\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3})) < \lambda(v_{p_1,0}v_{p_2,q_2})$. Then $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path. Since $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})) \geq \lambda(v_{i,0}v_{p_1,0})$ by Notation 3.2.2(d), contradicting $v_{i,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ and $v_{i,0} \not\smile v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. □

Proposition 3.2.33. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$

and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_1,0} = v_{p_2,q_2}$ and $v_{p_3,q_3} = v_{j_2,k_2}$.

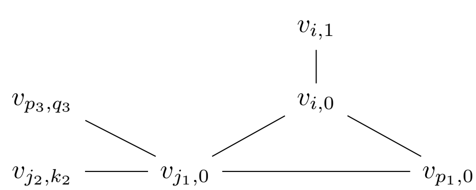
Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_1,0} = v_{p_2,q_2}$ and $v_{p_3,q_3} = v_{j_2,k_2}$.



By way of contradiction, we get that $v_{j_2,k_2}, v_{j_1,0}, v_{i,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})) \geq \lambda(v_{p_1,0}v_{i,0})$ by Notation 3.2.2(e)(2)(i). So we have that $v_{p_1,0} \not\leftarrow v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. □

Proposition 3.2.34. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_1,0} = v_{p_2,q_2}$ and $v_{p_3,q_3} \neq v_{j_2,k_2}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_1,0} = v_{p_2,q_2}$ and $v_{p_3,q_3} \neq v_{j_2,k_2}$.

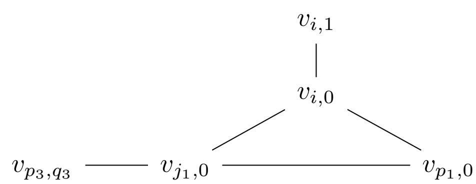


Then the proof is similar to the proof of Proposition 3.2.33. □

Proposition 3.2.35. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and

$P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_1,0} = v_{p_2,q_2}$ and $v_{j_2,k_2} = v_{p_1,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_1,0} = v_{p_2,q_2}$ and $v_{j_2,k_2} = v_{p_1,0}$.



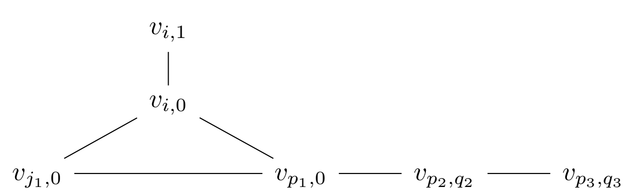
Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{j_1,0})) \geq \lambda(v_{p_1,0}v_{i,0})$ and

$$f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_3,q_3})) \leq \max\big\{f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})), f(\lambda(v_{j_1,0}v_{p_3,q_3}), \lambda(v_{j_1,0}v_{p_1,0}))\big\},$$

by Notation 3.2.2(e)(2)(ii). So we have that $v_{p_1,0}, v_{j_1,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$. Also, we have that $v_{p_3,q_3}, v_{i,0} \not\smile v_{p_3,q_3}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. $\square$

Proposition 3.2.36. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_1,0}$ and $v_{p_3,q_3} \neq v_{j_1,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_1,0}$ and $v_{p_3,q_3} \neq v_{j_1,0}$.



Then by way of contradiction, we get that $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$ and $v_{i,0}, v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$. Suppose that $v_{p_2,q_2} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$. Then as $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have that $q_2 = 0$ by Notation 3.2.2(b). Then $q_3 = 0$ by Notation 3.2.2(a). So we have that

$v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}v_{p_3,0}$ is a 4-path in G. Since $\lambda(v_{p_1,0}v_{p_2,q_2}) > f(\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3}))$, by Notation 3.2.2(d) we have

$$\lambda(v_{i,0}v_{p_1,0}) \leq f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})) \leq f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})),$$

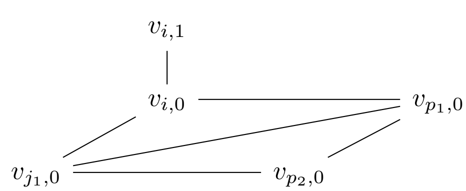
a contradiction. Hence $v_{p_2,q_2} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$ and so $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$. By Notation 3.2.2(e)(2)(iii), we have the following 2 cases:

(a) $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) \leq \max\{\lambda(v_{p_1,0}v_{j_1,0}), f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2}))\}$. Then since $v_{p_1,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,q_2}$, we have that $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, a contradiction.

(b) $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})) \geq \lambda(v_{j_1,0}v_{i,0})$. Then we have that $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Also, since $v_{i,0}, v_{p_2,q_2} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have that $v_{p_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. So we have that $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, a contradiction. □

Proposition 3.2.37. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_1,0}$ and $v_{p_3,q_3} = v_{j_1,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_1,0}$ and $v_{p_3,q_3} = v_{j_1,0}$.



So we have that $q_2 = 0$. Since $v_{i,1}v_{i,0}v_{j_1,0}v_{p_1,0} \in \mathfrak{P}_{i,1}$, we have that $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$ and that $v_{i,0}, v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$. Since $v_{i,0}v_{j_1,0}v_{p_1,0}v_{i,0}$ is a 3-cycle in G, we have that $f(\lambda(v_{p_1,0}v_{j_1,0}), \lambda(v_{p_1,0}v_{p_2,0})) \leq \max\{\lambda(v_{p_1,0}v_{j_1,0}), f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0}))\}$ or that $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0})) \geq \lambda(v_{j_1,0}v_{i,0})$ by Notation 3.2.2(e)(2)(iii). So we have that $v_{p_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$ or $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Hence by way of contradiction, we get that $v_{p_2,0} \smile v_{i,0}v_{j_1,0}v_{p_1,0}v_{p_2,0}$ or $v_{p_2,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. So we have that $\delta''(v_{p_2,0}) \leq \lambda(v_{p_2,0}v_{p_1,0})$. Since

$v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}v_{i,0}$ is a 4-cycle in G and $v_{j,0}v_{l,0} \in E(G)$, we have the following 3 cases by Notation 3.2.2(f):

(a) $f(\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})) \geq \lambda(v_{p_2,0}v_{p_1,0})$ by Notation 3.2.2(f)(1) or (f)(2)(iii). As $\delta''(v_{p_2,0}) \leq \lambda(v_{p_2,0}v_{p_1,0})$, we have that $v_{p_2,0} \smile v_{i,0}v_{p_1,0}v_{p_2,0}v_{j_1,0}$, a contradiction.

(b) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \lambda(v_{p_1,0}v_{p_2,0})$ and

$$f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})) \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0}))\big\},$$

by Notation 3.2.2(f)(2)(iii). Then $v_{p_1,0}, v_{j_1,0} \not\smile v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$. But we have that $v_{i,0}, v_{p_2,0} \not\smile v_{i,0}v_{j_1,0}v_{p_2,0}v_{p_1,0}$, contradicting Lemma 3.1.26.

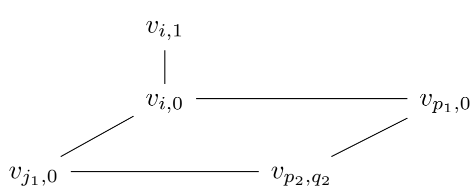
(c) $f(\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})) \geq \lambda(v_{p_2,0}v_{j_1,0})$, $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \lambda(v_{p_1,0}v_{i,0})$, and

$$f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})) \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_1,0}))\big\},$$

by Notation 3.2.2(f)(2)(iii). Then $v_{p_2,0}, v_{p_1,0}, v_{j_1,0}, v_{i,0} \not\smile v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$. But we have that $v_{i,0} \not\smile v_{p_2,0}v_{j_1,0}v_{i,0}v_{p_1,0}$, contradicting Lemma 3.1.26. $\square$

Proposition 3.2.38. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_2,q_2}$ and $v_{p_3,q_3} = v_{j_1,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_2,q_2}$ and $v_{p_3,q_3} = v_{j_1,0}$.



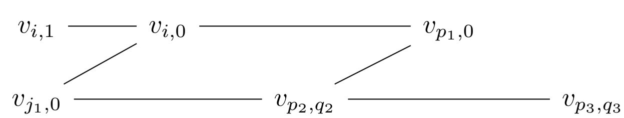
Then by way of contradiction, we get that $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$ and $v_{p_1,0} \smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle in G and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have the following 2 cases by Notation 3.2.2(f):

(a) $f(\lambda(v_{p_2,q_2}v_{j_1,0}),\lambda(v_{p_2,q_2}v_{p_1,0})) \geq \lambda(v_{p_2,q_2}v_{p_1,0})$ and $f(\lambda(v_{j_1,0}v_{i,0}),\lambda(v_{j_1,0}v_{p_2,q_2})) \geq \lambda(v_{j_1,0}v_{i,0})$ by Notation 3.2.2(f)(1). Then $v_{p_2,q_2}, v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. But $v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, contradicting Lemma 3.1.26.

(b) $f(\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,0})) \geq \min\{\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,0})\}$ by Notation 3.2.2(f)(2)(i). Then $v_{p_1,0} \smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, a contradiction. $\square$

Proposition 3.2.39. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_2,q_2}$ and $v_{p_3,q_3} \neq v_{j_1,0} \neq v_{p_1,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_2,q_2}$ and $v_{p_3,q_3} \neq v_{j_1,0} \neq v_{p_1,0}$.



Then $q_2 = 0$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle in G and $f(\lambda(v_{i,1}v_{i,0}),\lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have the following 5 cases by Notation 3.2.2(f):

(a) $f(\lambda(v_{j_1,0}v_{i,0}),\lambda(v_{j_1,0}v_{p_2,q_2})) \geq \lambda(v_{j_1,0}v_{i,0})$ by Notation 3.2.2(f)(1) or (f)(2)(iv). Then we have that $v_{j_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$, we have that $v_{p_2,q_2} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}$. Since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have that $q_3 = 0$ and $\lambda(v_{p_2,q_2}v_{p_1,0}) > f(\lambda(v_{p_2,q_2}v_{p_1,0}),\lambda(v_{p_2,q_2}v_{p_3,q_3}))$. So we have that $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G. Hence by Notation 3.2.2(d), we have the following which provides a contradiction

$$\lambda(v_{i,0}v_{p_1,0}) \leq f(\lambda(v_{i,0}v_{j_1,0}),\lambda(v_{i,0}v_{p_1,0})) \leq f(\lambda(v_{i,1}v_{i,0}),\lambda(v_{i,0}v_{j_1,0})).$$

(b) $f(\lambda(v_{p_1,0}v_{i,0}),\lambda(v_{p_1,0}v_{p_2,q_2})) \geq \lambda(v_{p_1,0}v_{i,0})$ by Notation 3.2.2(f)(2)(iv). Then we have that $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$. But $v_{i,0}, v_{j_1,0}, v_{p_2,q_2} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$, contradicting Lemma 3.1.26.

(c) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})\}$ and

$$f(\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_1,0})) \leq \lambda(v_{p_2,q_2}v_{p_1,0}),$$

by Notation 3.2.2(f)(2)(iv). By way of contradiction, we get that $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$. So we have that $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}$. Since $v_{i,0}, v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}$, we have that $v_{p_2,q_2} \smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}$. Then $v_{p_2,q_2} \smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$. Since $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$, we have that $q_3 = 0$. So we have that $v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$ is a 4-path in G. Since $f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{i,0}v_{p_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have that $\lambda(v_{p_1,0}v_{p_2,q_2}) \leq f(\lambda(v_{p_1,0}v_{p_2,q_2}), \lambda(v_{p_2,q_2}v_{p_3,q_3}))$, contradicting $v_{p_2,q_2} \smile v_{p_2,q_2}v_{p_1,0}v_{i,0}v_{j_1,0}$ and $v_{p_2,q_2} \not\smile v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3}$.

(d) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2})) \geq \min(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,q_2}))$ and

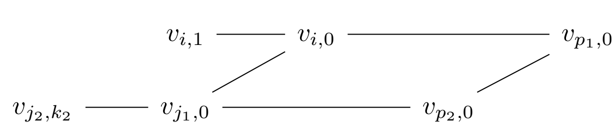
$$f(\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_1,0})) \leq \max\big\{\lambda(v_{p_2,q_2}v_{j_1,0}), f(\lambda(v_{p_2,q_2}v_{p_1,0}), \lambda(v_{p_2,q_2}v_{p_3,q_3}))\big\},$$

by Notations 3.2.2(f)(2)(i) and (f)(2)(iv). Then $v_{p_2,q_2} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. Since $v_{j_1,0}, v_{i,0} \not\smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$, we have that $v_{p_1,0} \smile v_{p_1,0}v_{p_2,q_2}v_{j_1,0}v_{i,0}$. So we have $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$. But $v_{i,0}, v_{j_1,0}, v_{p_2,q_2} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{p_2,q_2}$, contradicting Lemma 3.1.26.

(e) $f(\lambda(v_{p_2,q_2}v_{j_1,0}), \lambda(v_{p_2,q_2}v_{p_3,q_3})) \leq \max\big\{\lambda(v_{p_2,q_2}v_{j_1,0}), f(\lambda(v_{p_2,q_2}v_{p_1,0}), \lambda(v_{p_2,q_2}v_{p_3,q_3}))\big\}$ by Notation 3.2.2(f)(2)(iv). Then $v_{p_2,q_2} \not\smile v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_3,q_3}$. But we have that $v_{i,0}, v_{j_1,0}, v_{p_3,q_3} \not\smile v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_3,q_3}$, contradicting Lemma 3.1.26. □

Proposition 3.2.40. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_1,0} = v_{p_3,q_3}$ and $v_{p_1,0} \neq v_{j_2,k_2} \neq v_{p_2,0}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_1,0} = v_{p_3,q_3}$ and $v_{p_1,0} \neq v_{j_2,k_2} \neq v_{p_2,0}$.



Then by way of contradiction, we get that $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$. Since $v_{i,0}v_{j_1,0}v_{p_2,q_2}v_{p_1,0}$ is a 4-cycle in G_ω and $f(\lambda(v_{i,1}v_{i,0}), \lambda(v_{i,0}v_{j_1,0})) < \lambda(v_{i,0}v_{p_1,0})$, we have the following 4 cases by Notation 3.2.2(f):

(a) $f(\lambda(v_{p_2,0}v_{j_1,0}), \lambda(v_{p_2,0}v_{p_1,0})) \geq \lambda(v_{p_2,0}v_{p_1,0})$ by Notation 3.2.2(f)(1) or (f)(2)(v). Then $v_{p_2,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Note that $v_{i,0}, v_{p_1,0} \not\smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$, so $v_{j_1,0} \smile v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$. Since $v_{j_1,0} \not\smile v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, we have that $k = 0$ and $f(\lambda(v_{j_2,k_2}v_{j_1,0}), \lambda(v_{j_1,0}v_{i,0})) < \lambda(v_{j_1,0}v_{i,0})$. So we have that $v_{j_2,k_2}v_{j_1,0}v_{i,0}v_{p_1,0}v_{p_2,0}$ is a 4-path in G. Hence $\lambda(v_{i,0}v_{p_1,0}) \geq f(\lambda(v_{i,0}v_{p_1,0}), \lambda(v_{p_1,0}v_{p_2,0}))$, contradicting $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$ and $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,0}v_{j_1,0}$.

(b) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0}))$ by Notation 3.2.2(f)(2)(v). Then $v_{p_1,0} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$. But $v_{i,0}, v_{j_1,0}, v_{j_2,k_2} \not\smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$, contradicting Lemma 3.1.26.

(c) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}$ and

$$f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{p_2,0})) \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_2,k_2}))\big\},$$

by Notations 3.2.2(f)(2)(i) and (f)(2)(v). So we have that $v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. By way of contradiction we have that $v_{p_1,0} \smile v_{p_1,0}v_{i,0}v_{j_1,0}v_{j_2,k_2}$. Since $v_{p_1,0} \not\smile v_{i,0}v_{p_1,0}v_{p_2,0}v_{j_1,0}$ we have that $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$. But $v_{i,0}, v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{i,0}$, contradicting Lemma 3.1.26.

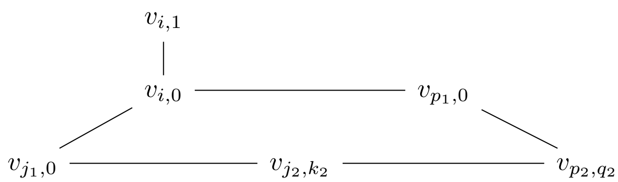
(d) $f(\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})) \geq \min\{\lambda(v_{p_1,0}v_{i,0}), \lambda(v_{p_1,0}v_{p_2,0})\}$ and

$$f(\lambda(v_{j_1,0}v_{p_2,0}), \lambda(v_{j_1,0}v_{j_2,k_2})) \leq \max\big\{\lambda(v_{j_1,0}v_{p_2,0}), f(\lambda(v_{j_1,0}v_{i,0}), \lambda(v_{j_1,0}v_{j_2,k_2}))\big\},$$

by Notations 3.2.2(f)(2)(i) and (f)(2)(v). Then $v_{j_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{j_2,k_2}$. Similar to Case (c), we have that $v_{p_1,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{j_2,k_2}$. But $v_{j_2,k_2}, v_{p_2,0} \not\smile v_{p_1,0}v_{p_2,0}v_{j_1,0}v_{j_2,k_2}$, contradicting Lemma 3.1.26. □

Proposition 3.2.41. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Assume that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \in \mathfrak{P}_{i,1}$, then $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \notin \mathfrak{P}_{i,0}$ for any $v_{j_2,k_2} = v_{p_3,q_3}$ and $v_{p_1,0} \neq v_{j_1,0} \neq v_{p_2,q_2}$.

Proof. Proof by contradiction: assume that $v_{i,0}v_{p_1,0}v_{p_2,q_2}v_{p_3,q_3} \in \mathfrak{P}_{i,0}$ for some $v_{j_2,k_2} = v_{p_3,q_3}$ and $v_{p_1,0} \neq v_{j_1,0} \neq v_{p_2,q_2}$.



Then $k_2 = 0 = q_2$ and the proof is similar to the proof of Proposition 3.2.32. $\square$

Proposition 3.2.42. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Then for $i = 1, \ldots, d$, we have $v_{i,0} \notin V''$ or $v_{i,1} \notin V''$.

Proof. Proof by contradiction: assume that $v_{i,0} \in V''$ and $v_{i,1} \in V''$. Then similar to the proof of Proposition 3.2.16, we have that

$$v_{i,3}v_{i,2}v_{i,1}v_{i,0},\ (v_{i,3}v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,2}v_{i,1}v_{i,0}),\ (v_{i,3}v_{i,2}v_{i,1}v_{i,0} \rightsquigarrow v_{i,1}v_{i,0}) \notin \mathfrak{P}_{i,1}.$$

Proposition 3.2.17 and Lemma 3.1.30 imply that $v_{i,2}v_{i,1}v_{i,0}v_{j_1,0} \notin \mathfrak{P}_{i,1}$ for any $v_{i,0}v_{j_1,0} \in E(G)$ Proposition 3.2.18 and Lemma 3.1.30 imply that $(v_{i,1}v_{i,0}v_{j_1,0}v_{j_1,0} \rightsquigarrow v_{i,1}v_{i,0}v_{j_1,0}) \notin \mathfrak{P}_{i,1}$ for any $v_{i,0}v_{j_1,0} \in E(G)$. Proposition 3.2.19 to 3.2.41 and Lemma 3.1.30 say that $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2} \notin \mathfrak{P}_{i,1}$ for any 3-path $v_{i,1}v_{i,0}v_{j_1,0}v_{j_2,k_2}$ in $\Sigma_3 G$. Thus, we get that $\mathfrak{P}_{i,1} = \emptyset$, contradicting Lemma 3.1.30. $\square$

Proposition 3.2.43. Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω that satisfies the conditions from Notation 3.2.2. Let $I := I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_3 G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an m-irreducible decomposition of $p_{\underline{n}}(I)$ and $P(V'' \smallsetminus \{v_{i,j}\}, \delta''|_{V'' \smallsetminus \{v_{i,j}\}})$ does not occur in any m-irreducible decomposition of $p_{\underline{n}}(I)$ for any $v_{i,j} \in V''$. Then there exists at most one $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$. Note also that there exists a $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$, so $p_{\underline{n}}(I)$ is unmixed.

Proof. It follows from Proposition 3.2.3, 3.2.4, 3.2.16, and 3.2.42 $\square$

We discuss the sufficient conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed for the remaining cases $r \geq 4$.

Proposition 3.2.44. Assume that $r \geq 4$. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \leq f(\lambda(v_iv_j), \lambda(v_iv_{i,1})) \leq \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \leq f(\lambda(v_iv_j), \lambda(v_jv_{j,1})) \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E\big((\Sigma_r G)_\lambda\big)$ and $\lambda(v_{i,k}v_{i,k+1}) \leq f(\lambda(v_{i,k}v_{i,k+1}), \lambda(v_{i,k+1}v_{i,k+2}))$ for $i = 1, \ldots, d$ and $k = 0, \ldots, r-2$, $f(\lambda(v_iv_j), \lambda(v_jv_k)) \leq \lambda(v_jv_{j,1})$ and for all 2-paths $v_iv_jv_k$ in G:

$$f(\lambda(v_iv_j), \lambda(v_jv_k)) \leq f(\lambda(v_iv_j), \lambda(v_jv_{j,1})) = f(\lambda(v_kv_j), \lambda(v_jv_{j,1})).$$

Let $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$ and $\underline{n} \in \mathbb{N}^d$. Let $\mathfrak{P} := (V'', \delta'')$ with $V'' \subseteq V\big((\Sigma_r G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$ be such that $P(V'', \delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}}(I)$. Then there exists a unique $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$. Note also that there exists a $v_{i,i_j} \in V''$ for $i = 1, \ldots, d$, so $p_{\underline{n}}(I)$ is unmixed.

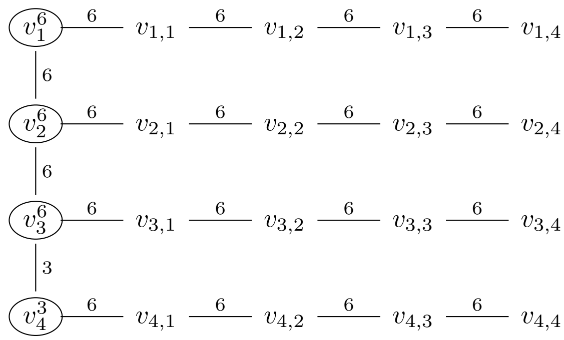
Proof. Suppose there exist $v_{i,\alpha}, v_{i,\beta} \in V''$ with $0 \leq \alpha < \beta \leq r$. Suppose that $v_{i,\beta} \smile (P_r \rightsquigarrow \wp)$ for some r-path P_r and some path $\wp \in p_{\underline{n}}(I)$, then we must have that $v_{i,\alpha} \in V(\wp)$. But since $\mathfrak{P}_{i,\alpha} \neq \emptyset$ by Lemma 3.1.30, it is straightforward to show that $v_{i,\alpha} \smile (P_r \rightsquigarrow \wp)$. So we have that $\mathfrak{P}_{i,\beta} = \emptyset$, contradicting Lemma 3.1.30. Note that for $i = 1, \ldots, d$, by the definition of $p_{\underline{n}}(I)$, we have that there exists a generator where all variables are of the form X_{i,i_l} with $i_l \in \{0, \ldots, r\}$, so there exists a vertex $v_{i,i_j} \in V''$. $\square$

Example 3.2.45. Let $r = 4$ and $f = \min$. Let $(\Sigma_4 G)_\lambda$ be a weighted 4-path suspension of $G_\omega := \; v_1 \overset{6}{\text{———}} v_2 \overset{6}{\text{———}} v_3 \overset{3}{\text{———}} v_4$:

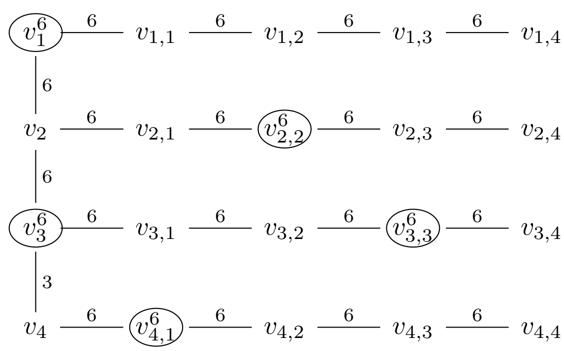
$$\begin{array}{ccccccccc}
v_1 & \overset{6}{\text{——}} & v_{1,1} & \overset{6}{\text{——}} & v_{1,2} & \overset{6}{\text{——}} & v_{1,3} & \overset{6}{\text{——}} & v_{1,4} \\
\big|\,6 & & & & & & & & \\
v_2 & \overset{6}{\text{——}} & v_{2,1} & \overset{6}{\text{——}} & v_{2,2} & \overset{6}{\text{——}} & v_{2,3} & \overset{6}{\text{——}} & v_{2,4} \\
\big|\,6 & & & & & & & & \\
v_3 & \overset{6}{\text{——}} & v_{3,1} & \overset{6}{\text{——}} & v_{3,2} & \overset{6}{\text{——}} & v_{3,3} & \overset{6}{\text{——}} & v_{3,4} \\
\big|\,3 & & & & & & & & \\
v_4 & \overset{6}{\text{——}} & v_{4,1} & \overset{6}{\text{——}} & v_{4,2} & \overset{6}{\text{——}} & v_{4,3} & \overset{6}{\text{——}} & v_{4,4}
\end{array}$$

Then $\mathfrak{P}_1 := \{v_1^6, v_2^6, v_3^6, v_4^3\}$ is a minimal min-weighted 4-path vertex cover of $(\Sigma_4 G)_\lambda$. It is depicted in the following drawing, where $v_i^{i_j} \in \mathfrak{P}_1$ if and only if it is encompassed by a circle. Note that

$|\mathfrak{P}_1| = 4$.

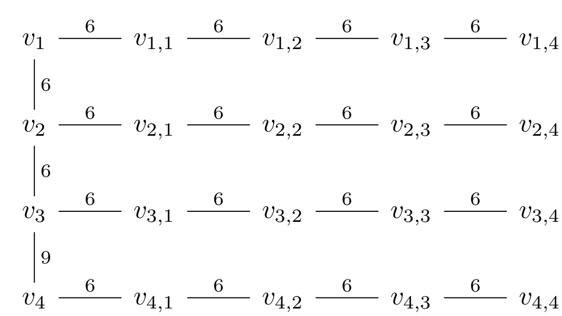


In fact, it is straightforward to show that the cardinality of any minimal min-weighted 4-path vertex cover of $(\Sigma_4 G)_\lambda$ is at least 4. Also, we can see that there always exists a minimal min-weighted 4-path vertex cover of cardinality 4, generated from the min-weighted 4-path vertex cover $\{v_1^1, v_2^1, v_3^1, v_4^1\}$. We see that $\mathfrak{P}_2 := \{v_1^6, v_{2,2}^6, v_3^6, v_{3,3}^6, v_{4,1}^6\}$ is another minimal min-weighted 4-path vertex cover of $(\Sigma_4 G)_\lambda$ depicted in the following sketch.

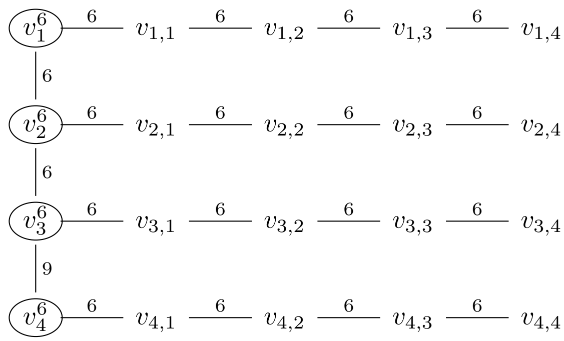


Since $|\mathfrak{P}_2| = 5$, we have that $I_{r,\min}((\Sigma_4 G)_\lambda)$ is mixed by the definition of mixedness and Fact 3.1.7.

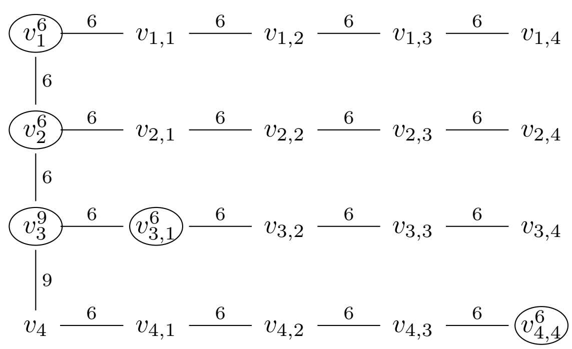
Example 3.2.46. Let $r = 4$ and $f = \min$. Let $(\Sigma_4 G)_\lambda$ be a weighted 4-path suspension of $G_\omega := v_1 \overset{6}{\text{———}} v_2 \overset{6}{\text{———}} v_3 \overset{9}{\text{———}} v_4$:



The only difference between the above graph $(\Sigma_4 G)_\lambda$ and the one from Example 3.2.45 is the weight of v_3v_4. Then $\mathfrak{P}_1 := \{v_1^6, v_2^6, v_3^6, v_4^6\}$ is a minimal min-weighted 4-path vertex cover of $(\Sigma_4 G)_\lambda$ depicted in the following.



In fact, it is straightforward to show that the cardinality of any minimal min-weighted 4-path vertex cover of $(\Sigma_4 G)_\lambda$ is at least 4. Also, we can see that there always exists a minimal min-weighted 4-path vertex cover of cardinality 4, generated from the min-weighted 4-path vertex cover $\{v_1^1, v_2^1, v_3^1, v_4^1\}$. We see that $\mathfrak{P}_2 := \{v_1^6, v_2^6, v_3^9, v_{3,1}^6, v_{4,4}^6\}$ is another minimal min-weighted 4-path vertex cover of $(\Sigma_4 G)_\lambda$ depicted in the following.



Since $|\mathfrak{P}_2| = 5$, we have that $I_{r,\min}((\Sigma_4 G)_\lambda)$ is mixed by the definition of mixedness and Fact 3.1.7.

From Examples 3.2.45 and 3.2.46, we see that there must be some strict constraints on the weights of G to make $I_{r,\min}((\Sigma_4 G)_\lambda)$ be unmixed. We will show that in general when $r \geq 4$, if $I_{r,\min}(\Sigma_4 G_\omega)$ is unmixed, then all edges in G have the same weight, i.e., $\omega = \lambda|_G$ is a constant. This result can be found in Corollary 3.5.3, Proposition 3.2.44, and Theorem 3.3.4.

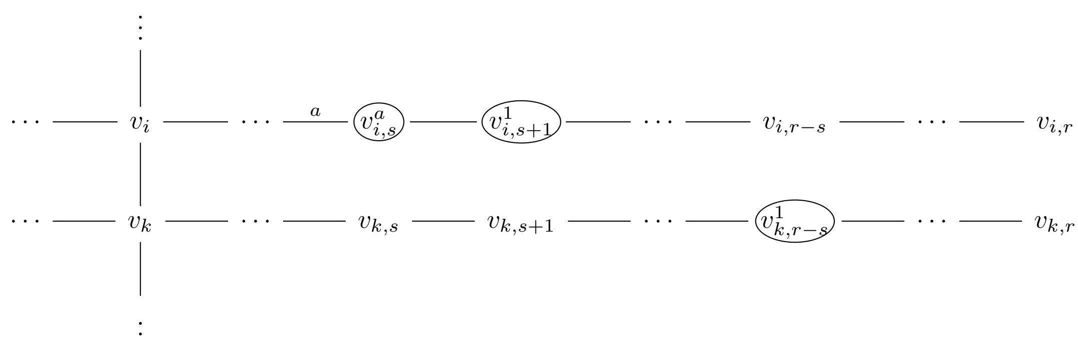
3.3 Necessary Conditions for Unmixedness

In this section, we prove the necessary conditions for which the f-weighted r-path ideal of a weighted r-path suspension is unmixed. We divide the classification into 3 kinds of cases. We first discuss the necessary conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed for the case $r = 2$.

Lemma 3.3.1. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. If $I_{r,f}\big((\Sigma_r G)_\lambda\big)$ is unmixed, then $\lambda(v_i v_j) \leq f(\lambda(v_i v_j), \lambda(v_i v_{i,1})) \leq \lambda(v_i v_{i,1})$ and $\lambda(v_i v_j) \leq f(\lambda(v_i v_j), \lambda(v_j v_{j,1})) \leq \lambda(v_j v_{j,1})$ for all edges $v_i v_j \in E(G)$ and $\lambda(v_{i,k} v_{i,k+1}) \leq f(\lambda(v_{i,k} v_{i,k+1}), \lambda(v_{i,k+1} v_{i,k+2}))$ for $i = 1, \ldots, d$ and $k = 0, \ldots, r-2$, $f(\lambda(v_{i_1} v_{i_2}), \lambda(v_{i_2} v_{i_3})) \leq \lambda(v_{i_2} v_{i_2,1})$, $f(\lambda(v_{i_1} v_{i_2}), \lambda(v_{i_2} v_{i_3})) \leq f(\lambda(v_{i_1} v_{i_2}), \lambda(v_{i_2} v_{i_2,1}))$ and $f(\lambda(v_{i_1} v_{i_2}), \lambda(v_{i_2} v_{i_3})) \leq f(\lambda(v_{i_3} v_{i_2}), \lambda(v_{i_2} v_{i_2,1}))$ for all 2-paths $v_{i_1} v_{i_2} v_{i_3}$ in G.

Proof. Since $\{v_1^1, \ldots, v_d^1\}$ is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$, by Fact 3.1.6, there exists a minimal f-weighted r-path vertex cover (V''', δ''') of $(\Sigma_r G)_\lambda$ such that $(V''', \delta''') \leq \{v_1^1, \ldots, v_d^1\}$. By the minimality of V''', we have that $V''' = \{v_1, \ldots, v_d\}$ and so $|V'''| = d$. Hence by [7, Theorem 2.7], it suffices to show that if Conditions on weights are not satisfied, then there exists an f-weighted r-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$.

(a) Suppose that $a := \lambda(v_{i,s-1} v_{i,s}) > f(\lambda(v_{i,s-1} v_{i,s}), \lambda(v_{i,s} v_{i,s+1}))$ for some $i \in \{1, \ldots, d\}$ and some $s \in \{1, \ldots, r-1\}$. We use the following diagram as a guide for constructing $\mathfrak{P}$, where the column represents G and rows represent the r-whiskers in $\Sigma_r G$. A vertex encompassed by a circle is in V''.



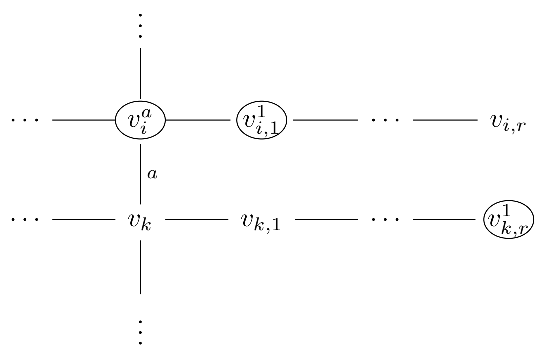
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,s+1}^1, v_{i,s}^a, v_{k,r-s}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i,k\}\right\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d+1$, $v_{i,r} \cdots v_{i,1} v_i \in \mathfrak{P}_{i,s+1}$,

$v_{i,s} \cdots v_{i,1} v_i v_k v_{k,1} \cdots v_{k,r-s-1} \in \mathfrak{P}_{i,s}$, $v_{k,r} \cdots v_{k,1} v_k \in \mathfrak{P}_{k,r-s}$, and $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any t in $\{1, \ldots, d\} \smallsetminus \{i, k\}$.

(b) Suppose that $a := \lambda(v_i v_k) > f(\lambda(v_{i,1} v_i), \lambda(v_i v_k))$ for some $i \in \{1, \ldots, d\}$ and some $v_i v_k \in E(G)$. The following diagram has the same representation as in (a) except for the elements in V''.

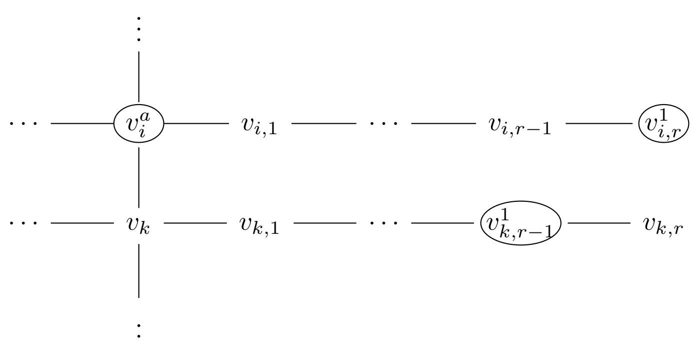


Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{ v_{i,1}^1, v_i^a, v_{k,r}^1 \right\} \sqcup \left\{ v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, k\} \right\}$$

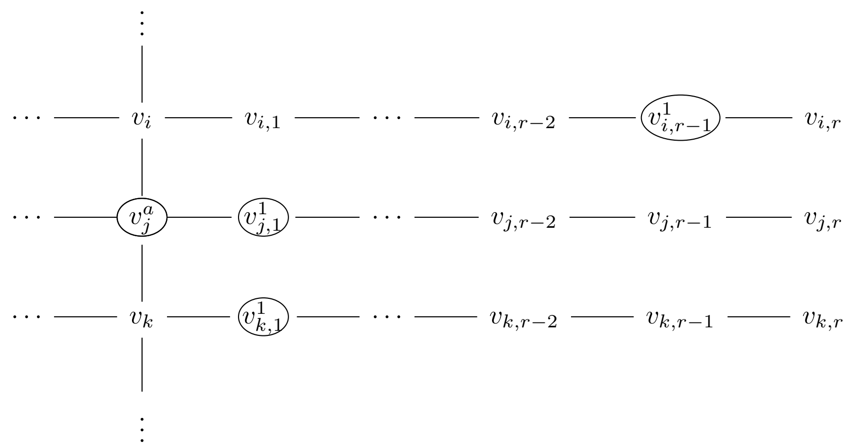
is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d + 1$, $v_{i,1} v_i v_k v_{k,1} \cdots v_{k,r-2} \in \mathfrak{P}_{i,1}$, $v_i v_k v_{k,1} \cdots v_{k,r-1} \in \mathfrak{P}_{i,0}$, $v_{k,r} \cdots v_{k,1} v_k \in \mathfrak{P}_{k,r}$, and $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, k\}$.

(c) Suppose $a := f(\lambda(v_{i,1} v_i), \lambda(v_i v_k)) > \lambda(v_i v_{i,1})$ for some $i \in \{1, \ldots, d\}$ and some $v_i v_k \in E(G)$.



It is straightforward to show that $\mathfrak{P} := (V'', \delta'') := \left\{ v_{i,r}^1, v_i^a, v_{k,r-1}^1 \right\} \sqcup \left\{ v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, k\} \right\}$ is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d + 1$, $v_{i,1} v_i v_k v_{k,1} \cdots v_{k,r-2} \in \mathfrak{P}_{i,0}$, $v_i \cdots v_{i,r} \in \mathfrak{P}_{i,r}$, $v_{k,r} \cdots v_{k,1} v_k \in \mathfrak{P}_{k,r-1}$, and $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, k\}$.

(d) Suppose that $a := f(\lambda(v_jv_i), \lambda(v_jv_k)) > f(\lambda(v_iv_j), \lambda(v_jv_{j,1}))$ for some 2-path $v_iv_jv_k$ in G.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,r-1}^1, v_j^a, v_{j,1}^1, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$ such that $|V''| = d + 1$,

$$v_{i,r} \cdots v_{i,1}v_i \in \mathfrak{P}_{i,r-1},\ v_{i,r-2} \cdots v_iv_jv_k \in \mathfrak{P}_{j,0},\ v_{i,r-2} \cdots v_iv_jv_{j,1} \in \mathfrak{P}_{j,1},\ v_{k,r} \ldots v_{k,1}v_k \in \mathfrak{P}_{k,1},$$

and $v_{t,r} \cdots v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$. $\square$

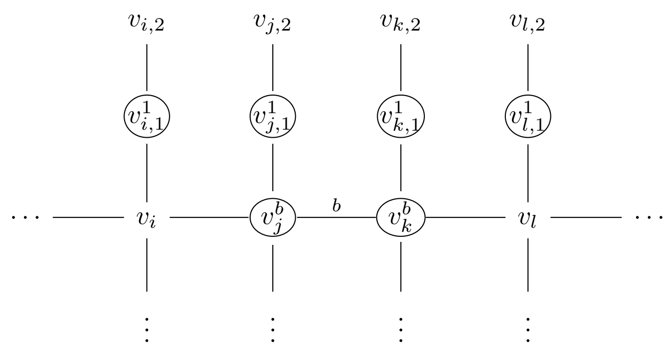
We discuss the necessary conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed for the case $r = 2$.

Theorem 3.3.2. *Let $(\Sigma_2 G)_\lambda$ be a weighted 2-path suspension of G_ω. If $I_{2,f}\big((\Sigma_2 G)_\lambda\big)$ is unmixed, then the weight function λ satisfies the constraints in Proposition 3.2.1.*

Proof. By Lemma 3.3.1 and its proof, it is enough to show that if the constraints on 3-paths or 3-cycles are not satisfied, then there exists an f-weighted 2-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_2 G)_\lambda$ such that $|V''| = d + 1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$. Without loss of generality, we assume that the weight function λ satisfies constraints in Lemma 3.3.1.

(a) Let $v_iv_jv_kv_l$ be a 3-path in G_ω such that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) < \lambda(v_jv_k) =: b$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_jv_k) = b$.

(1) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_l)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^b, v_{k,1}^1, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

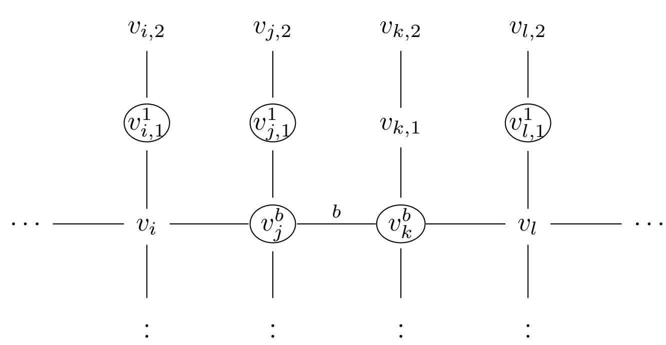
is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, and

$$v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_{j,1}v_jv_i \in \mathfrak{P}_{j,1},\ v_jv_kv_l \in \mathfrak{P}_{j,0},\ v_{k,1}v_kv_l \in \mathfrak{P}_{k,1},$$

$$v_iv_jv_k \in \mathfrak{P}_{k,0},\ v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1},\ v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

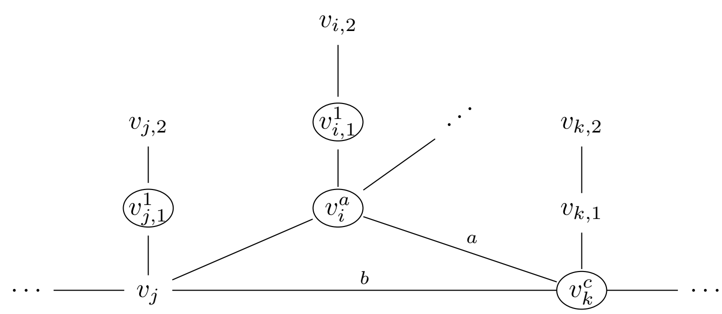
for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(2) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_l)) \geq b$.



We have that $\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^b, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$ is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, and $v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1}$, $v_{j,1}v_jv_i \in \mathfrak{P}_{j,1}$, $v_jv_kv_l \in \mathfrak{P}_{j,0}$, $v_iv_jv_k \in \mathfrak{P}_{k,0}$, $v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1}$, $v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(b) Let $v_iv_jv_kv_i$ be a 3-cycle in G with $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_k) =: a$. Suppose that we have $f(\lambda(v_kv_i), \lambda(v_kv_j)) < \lambda(v_kv_i) = a$ and $f(\lambda(v_kv_i), \lambda(v_kv_j)) < \lambda(v_kv_j) =: b$. So we have that $f(\lambda(v_kv_i), \lambda(v_kv_j)) < \min\{a, b\} =: c$.



Then it is straightforward to show that

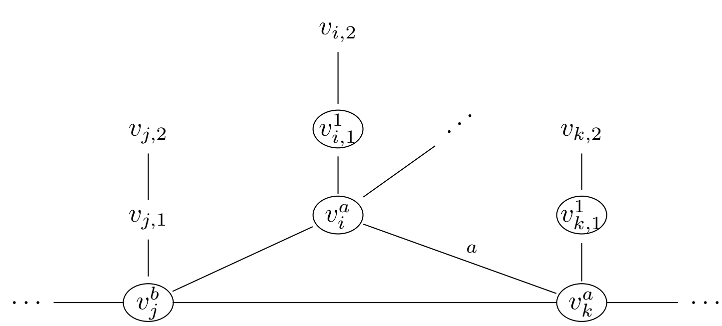
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,1}^1, v_i^a, v_k^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, and for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$,

$$v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1},\ v_{i,1}v_iv_j \in \mathfrak{P}_{i,1},\ v_iv_kv_j \in \mathfrak{P}_{i,0},\ v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,0},\ v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(c) Let $v_iv_jv_kv_i$ be a 3-cycle in G with $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_k) =: a$. Suppose that we have $f(\lambda(v_kv_i), \lambda(v_kv_j)) < \lambda(v_kv_i) = a$ and $b := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.

(1) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) < a$.



It is straightforward to show that

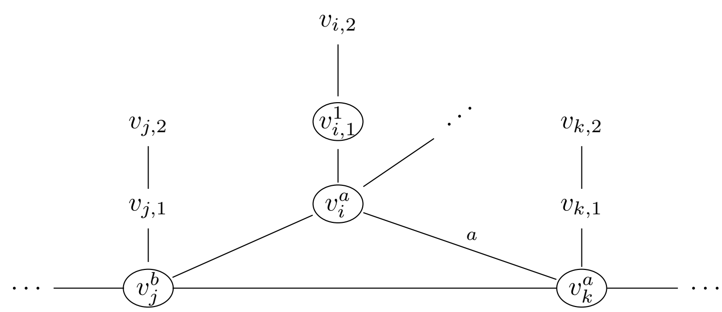
$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,1}^1, v_i^a, v_k^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, and

$$v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_j \in \mathfrak{P}_{i,1},\ v_iv_kv_j \in \mathfrak{P}_{i,0},\ v_jv_iv_k \in \mathfrak{P}_{k,0},\ v_{k,1}v_kv_j \in \mathfrak{P}_{k,1},\ v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(2) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) \geq a$.



It is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,1}^1, v_i^a, v_k^a\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\right\}$$

is an f-weighted 2-path vertex cover of $(\Sigma_2 G)_\lambda$, and

$$v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_j \in \mathfrak{P}_{i,1},\ v_iv_kv_j \in \mathfrak{P}_{i,0},\ v_jv_iv_k \in \mathfrak{P}_{k,0},\ v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$. □

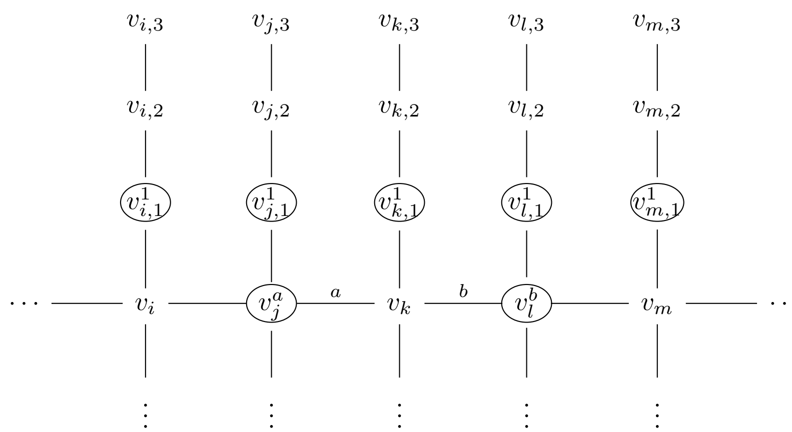
We discuss the necessary conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed for the case $r = 3$.

Theorem 3.3.3. *Let $(\Sigma_3 G)_\lambda$ be a weighted 3-path suspension of G_ω. If $I_{3,f}\big((\Sigma_3 G)_\lambda\big)$ is unmixed, then the weight function λ satisfies the constraints in Proposition 3.2.43.*

Proof. By Lemma 3.3.1 and its proof, it is enough to show that if the constraints on 4-paths or 3-cycles or 4-cycles are not satisfied, then there exists an f-weighted 3-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_3 G)_\lambda$ such that $|V''| = d + 1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$. Without loss of generality, we assume that the weight function λ satisfies the constraints in Lemma 3.3.1.

(a) Let $v_iv_jv_kv_lv_m$ be a 4-path in G such that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) < \lambda(v_jv_k) =: a$. Suppose that $f(\lambda(v_iv_j), \lambda(v_jv_k)) < \lambda(v_jv_k) = a$ and $f(\lambda(v_kv_l), \lambda(v_lv_m)) < \lambda(v_kv_l) =: b$.

(1) Assume that $f(\lambda(v_{l,1}v_l), \lambda(v_l v_m)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_{l,1}^1, v_l^b, v_{m,1}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

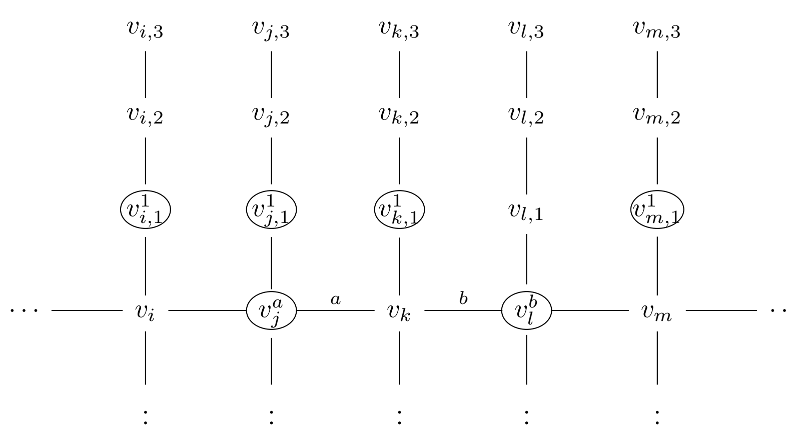
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_{j,2}v_{j,1}v_j v_i \in \mathfrak{P}_{j,1},\ v_j v_k v_l v_m \in \mathfrak{P}_{j,0},\ v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1},$$

$$v_{l,2}v_{l,1}v_l v_m \in \mathfrak{P}_{l,1},\ v_i v_j v_k v_l \in \mathfrak{P}_{l,0},\ v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, m\}$.

(2) Assume that $f(\lambda(v_{l,1}v_l), \lambda(v_l v_m)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_l^b, v_{m,1}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

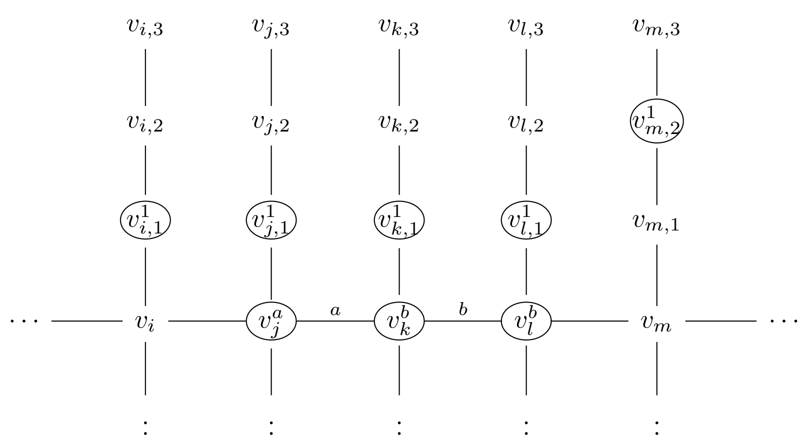
$$v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,1},\ v_jv_kv_lv_m \in \mathfrak{P}_{j,0},\ v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1},$$

$$v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}$.

(b) Let $v_iv_jv_kv_lv_m$ be a 4-path in G such that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) < \lambda(v_kv_l) =: b$. Suppose that $f(\lambda(v_jv_k), \lambda(v_jv_i)) < \lambda(v_jv_k) =: a$ and $f(\lambda(v_kv_l), \lambda(v_lv_m)) < \lambda(v_kv_l) = b$.

(1) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) < a$ and $f(\lambda(v_{l,1}v_l), \lambda(v_lv_m)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_k^b, v_{l,1}^1, v_l^b, v_{m,2}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

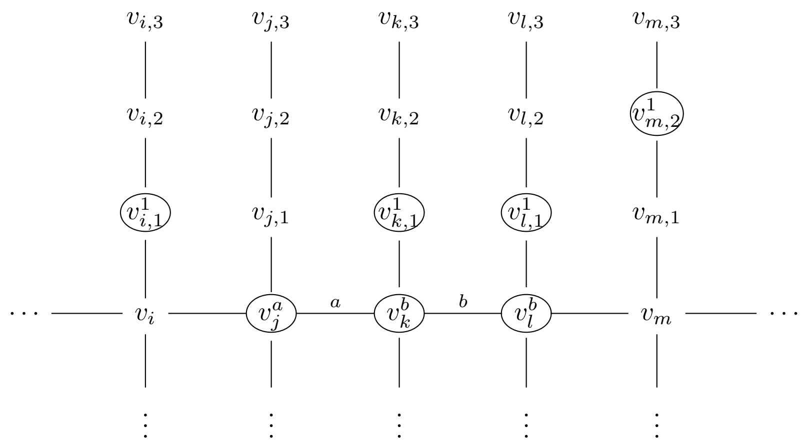
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,1},\ v_jv_kv_lv_m \in \mathfrak{P}_{j,0},\ v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1},\ v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0},$$

$$v_{l,2}v_{l,1}v_lv_m \in \mathfrak{P}_{l,1},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}$.

(2) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) \geq a$ and $f(\lambda(v_{l,1}v_l), \lambda(v_lv_m)) < b$.



Then it is straightforward to show that

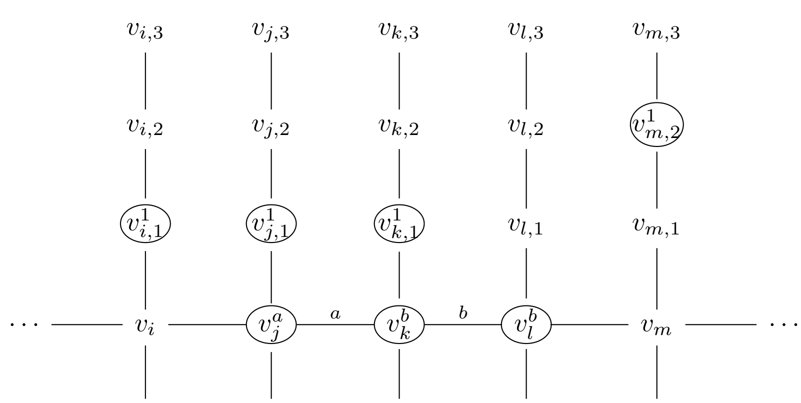
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_j^a, v_{k,1}^1, v_k^b, v_{l,1}^1, v_l^b, v_{m,2}^1\right\} \sqcup \left\{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}$,

$$v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_jv_kv_lv_m \in \mathfrak{P}_{j,0},\ v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1},\ v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0},$$

$$v_{l,2}v_{l,1}v_lv_m \in \mathfrak{P}_{l,1},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(3) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) < a$ and $f(\lambda(v_{l,1}v_l), \lambda(v_lv_m)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{i,1}^1, v_{j,1}^1, v_j^a, v_{k,1}^1, v_k^b, v_l^b, v_{m,2}^1\} \sqcup \{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\}$$

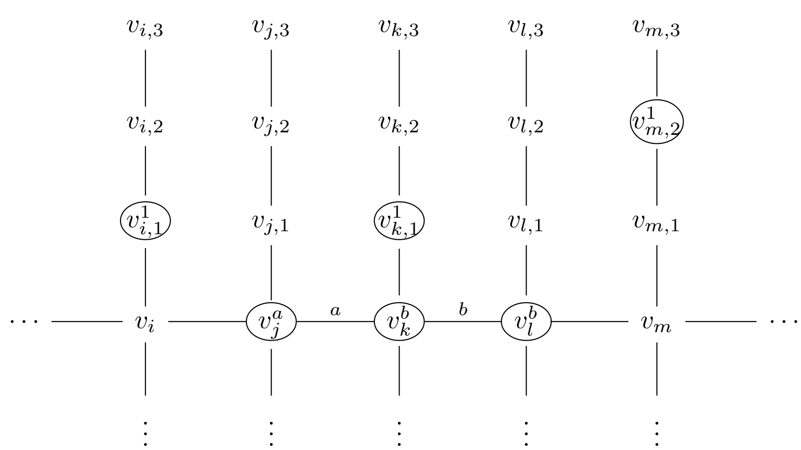
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,1},\ v_jv_kv_lv_m \in \mathfrak{P}_{j,0},\ v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1},$$

$$v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}$.

(4) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) \geq a$ and $f(\lambda(v_{l,1}v_l), \lambda(v_lv_m)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{i,1}^1, v_j^a, v_{k,1}^1, v_k^b, v_l^b, v_{m,2}^1\} \sqcup \{v_n^1 \mid n \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}\}$$

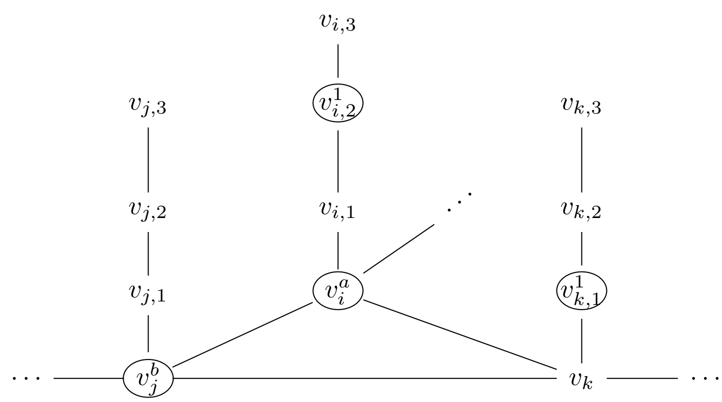
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,3}v_{i,2}v_{i,1}v_i \in \mathfrak{P}_{i,1},\ v_jv_kv_lv_m \in \mathfrak{P}_{j,0},\ v_{k,1}v_kv_jv_i \in \mathfrak{P}_{k,1},\ v_kv_lv_mv_{m,1} \in \mathfrak{P}_{k,0},$$

$$v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{m,3}v_{m,2}v_{m,1}v_m \in \mathfrak{P}_{m,2},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l, m\}$.

(c) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}, \lambda(v_iv_j))) < f(\lambda(v_iv_{i,1}), \lambda(v_iv_k)) =: a$. Suppose that $b := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

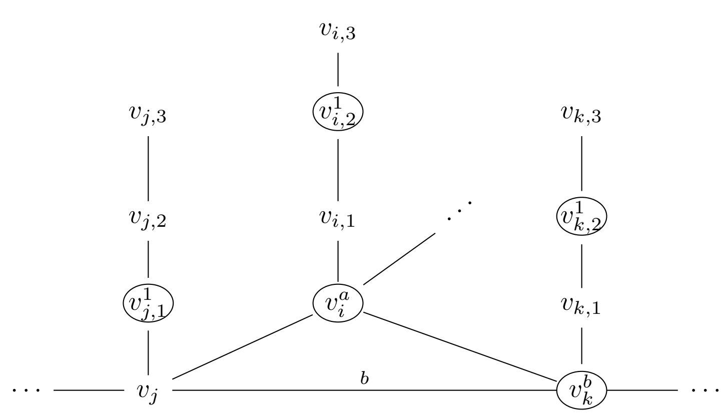
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,1}v_iv_jv_k \in \mathfrak{P}_{j,0},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_{i,1}v_iv_kv_j \in \mathfrak{P}_{i,0},\ v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$.

(d) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}, \lambda(v_iv_j))) < f(\lambda(v_iv_{i,1}), \lambda(v_iv_k)) =: a$. Suppose that $f(\lambda(v_kv_i), \lambda(v_kv_j)) < \lambda(v_kv_j) =: b$.

(1) Assume that $\lambda(v_iv_k) < a$ and $f(\lambda(v_{k,1}v_k), \lambda(v_kv_i)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_{k,2}^1, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

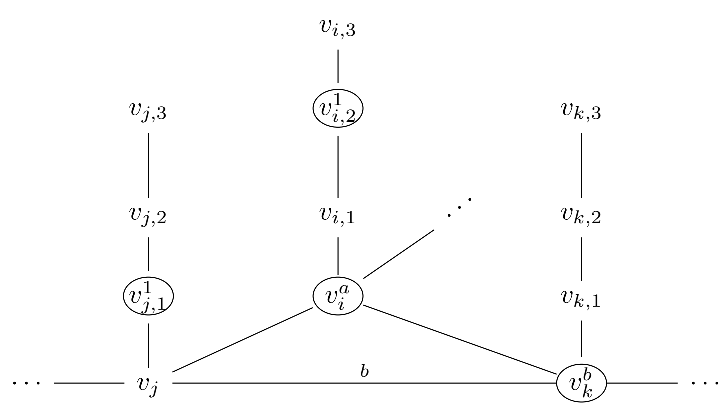
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_{i,1}v_iv_kv_j \in \mathfrak{P}_{i,0},$$

$$v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{k,2},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$.

(2) Assume that $\lambda(v_iv_k) \geq a$ or $f(\lambda(v_{k,1}v_k), \lambda(v_kv_i)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

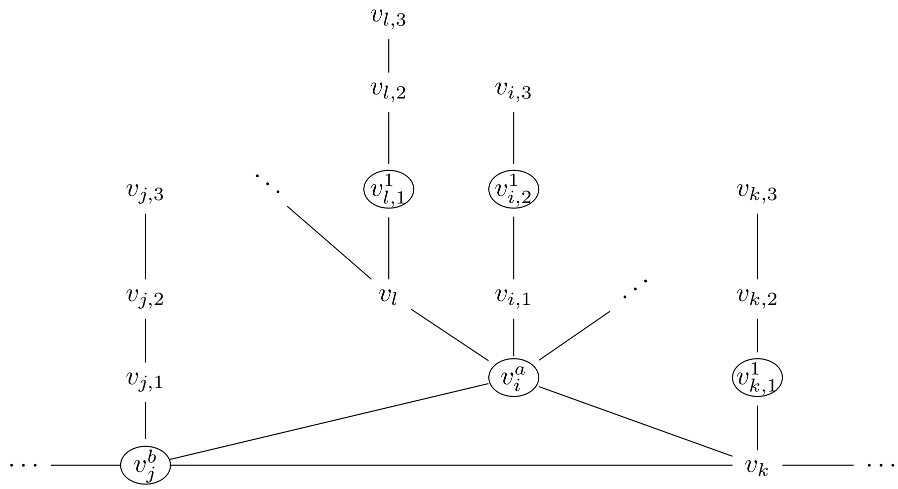
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_{i,1}v_iv_kv_j \in \mathfrak{P}_{i,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$.

(e) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < f(\lambda(v_iv_l), \lambda(v_iv_k)) =: a$ for some

$v_iv_l \in E(G)$ with $j \neq l \neq k$. Suppose that $b := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.



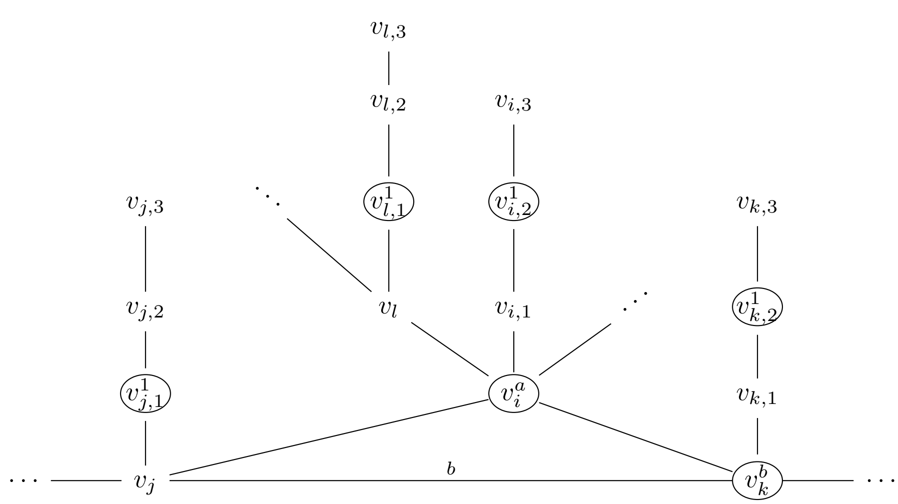
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$ and that $v_{i,1}v_iv_jv_k \in \mathfrak{P}_{j,0}$, $v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2}$, $v_lv_iv_kv_j \in \mathfrak{P}_{i,0}$, $v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1}$, and $v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \setminus \{i, j, k, l\}$.

(f) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < f(\lambda(v_iv_l), \lambda(v_iv_k)) =: a$ for some $v_iv_l \in E(G)$ with $j \neq l \neq k$. Suppose that $f(\lambda(v_kv_i), \lambda(v_kv_j)) < \lambda(v_kv_j) =: b$.

(1) Assume that $\lambda(v_iv_k) < a$ and $f(\lambda(v_{k,1}v_k), \lambda(v_kv_i)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_{k,1}^1, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \setminus \{i, j, k, l\}\right\}$$

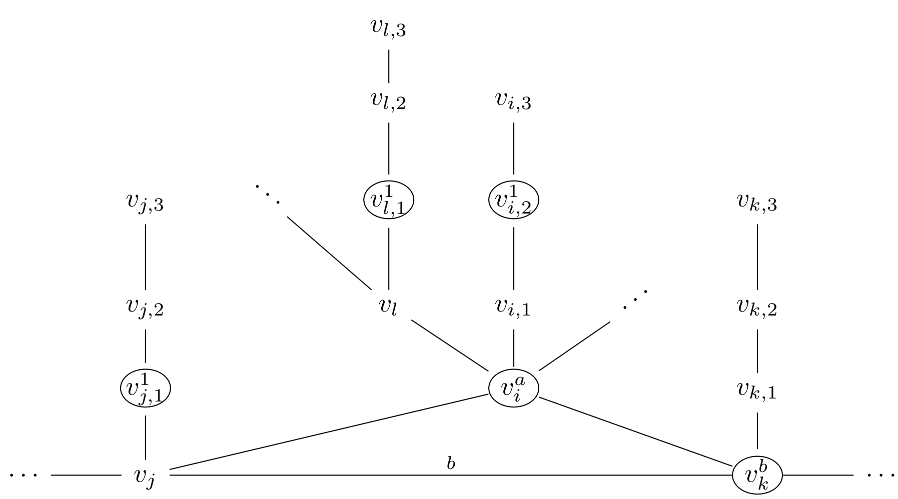
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_lv_iv_kv_j \in \mathfrak{P}_{i,0},$$

$$v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{k,2},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \setminus \{i, j, k, l\}$.

(2) Assume that $\lambda(v_iv_k) \geq a$ or $f(\lambda(v_{k,1}v_k), \lambda(v_kv_i)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,1}^1, v_{i,2}^1, v_i^a, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \setminus \{i, j, k, l\}\right\}$$

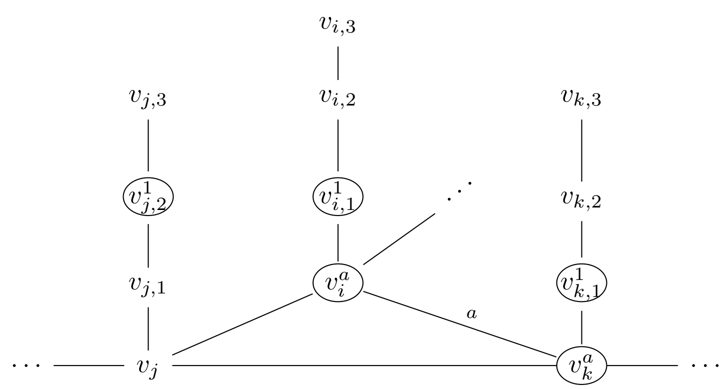
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,1}v_jv_iv_{i,1} \in \mathfrak{P}_{j,1},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_lv_iv_kv_j \in \mathfrak{P}_{i,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \setminus \{i, j, k, l\}$.

(g) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}, \lambda(v_iv_j))) < \lambda(v_iv_k) =: a$. Suppose that we have $f(\lambda(v_k v_i), \lambda(v_k v_j)) < \lambda(v_k v_i) = a$.

(1) Assume that $f(\lambda(v_{k,1} v_k), \lambda(v_k v_j)) < a$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{j,2}^1, v_{i,1}^1, v_i^a, v_{k,1}^1, v_k^a\} \sqcup \{v_m^1 \mid m \in \{1, \dots, d\} \smallsetminus \{i, j, k\}\}$$

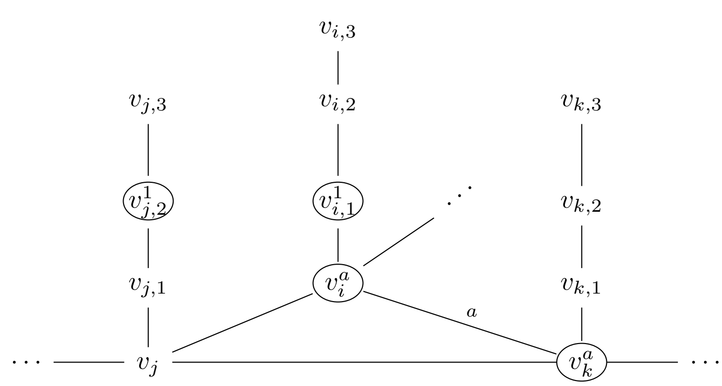
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,2} v_{j,1} v_j v_i \in \mathfrak{P}_{j,2},\ v_{i,1} v_i v_j v_{j,1} \in \mathfrak{P}_{i,1},\ v_i v_k v_j v_{j,1} \in \mathfrak{P}_{i,0},$$

$$v_{j,1} v_j v_k v_{k,1} \in \mathfrak{P}_{k,1},\ v_{j,1} v_j v_i v_k \in \mathfrak{P}_{k,0},\ v_{t,3} v_{t,2} v_{t,1} v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k\}$.

(2) Assume that $f(\lambda(v_{k,1} v_k), \lambda(v_k v_j)) \geq a$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{j,2}^1, v_{i,1}^1, v_i^a, v_k^a\} \sqcup \{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

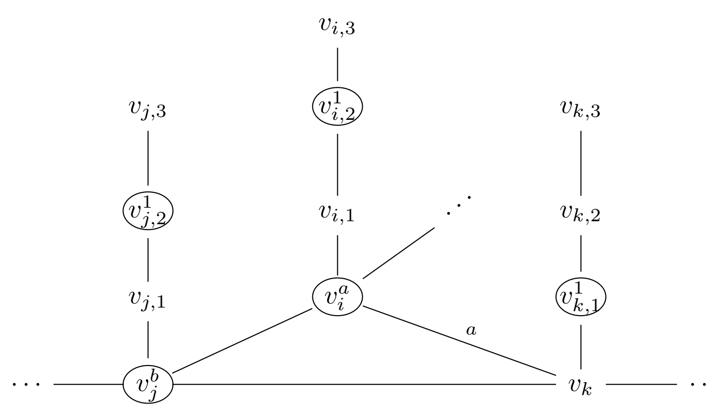
$$v_{j,2}v_{j,1}v_jv_i \in \mathfrak{P}_{j,2},\ v_{i,1}v_iv_jv_{j,1} \in \mathfrak{P}_{i,1},\ v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0},\ v_{j,1}v_jv_iv_k \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(h) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}, \lambda(v_iv_j))) < \lambda(v_iv_k) =: a$ and there exists $v_jv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_k$. Suppose

$$b := f(\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2})) > \max\{\lambda(v_jv_i), f(\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2}))\}.$$

(1) Assume that $l_1 = j$, then $l_2 = 1$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{j,2}^1, v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\} \sqcup \{v_m^1 \mid m \in \{1, \ldots, d\} \setminus \{i, j, k\}\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

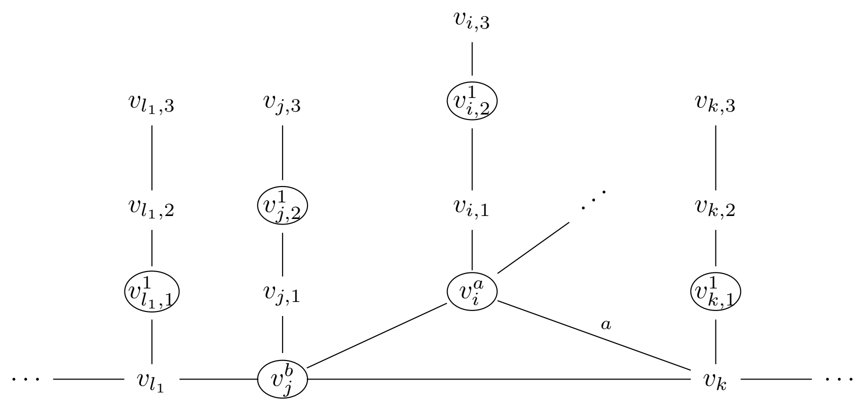
$$v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2},\ v_{j,1}v_jv_iv_k \in \mathfrak{P}_{j,0},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},$$

$$v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0},\ v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.

(2) Assume that $l_1 \neq j$, then $l_2 = 0$.

(i) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_j v_k)) < b$.



Then it is straightforward to show that

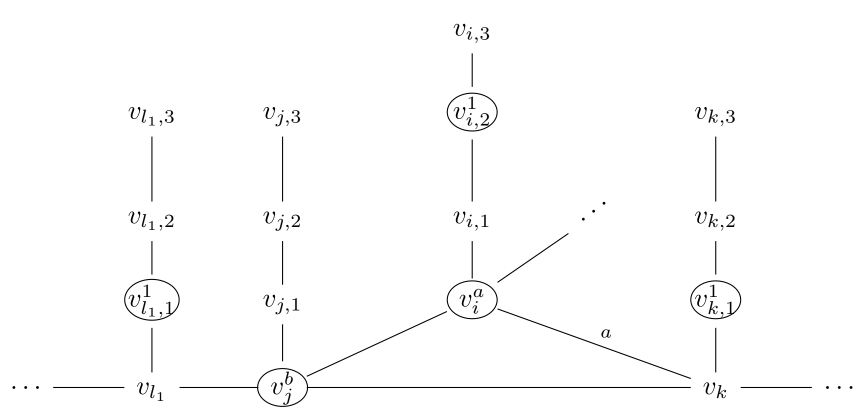
$$\mathfrak{P} := (V'', \delta'') := \left\{v^1_{l_1,1}, v^1_{j,2}, v^b_j, v^1_{i,2}, v^a_i, v^1_{k,1}\right\} \sqcup \left\{v^1_m \mid m \in \{1, \ldots, d\} \setminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,0},\ v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2},\ v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},$$
$$v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0},\ v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \setminus \{i, j, k, l\}$.

(ii) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_j v_k)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{l_1,1}^1, v_j^b, v_{i,2}^1, v_i^a, v_{k,1}^1\} \sqcup \{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l_1\}\}$$

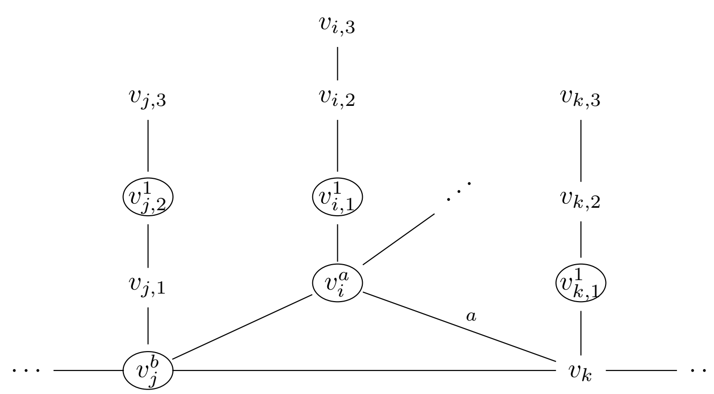
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$,

$$v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,0},\ v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0},\ v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},$$
$$v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0},\ v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(i) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}, \lambda(v_iv_j))) < \lambda(v_iv_k) =: a$ and there exists $v_jv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_k$. Suppose that

$$b := f(\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2})) > \max\{f(\lambda(v_jv_i), \lambda(v_jv_k)), f(\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2}))\}.$$

(1) Assume that $l_1 = j$, then $l_2 = 1$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v_{j,2}^1, v_j^b, v_{i,1}^1, v_i^a, v_{k,1}^1\} \sqcup \{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\}$$

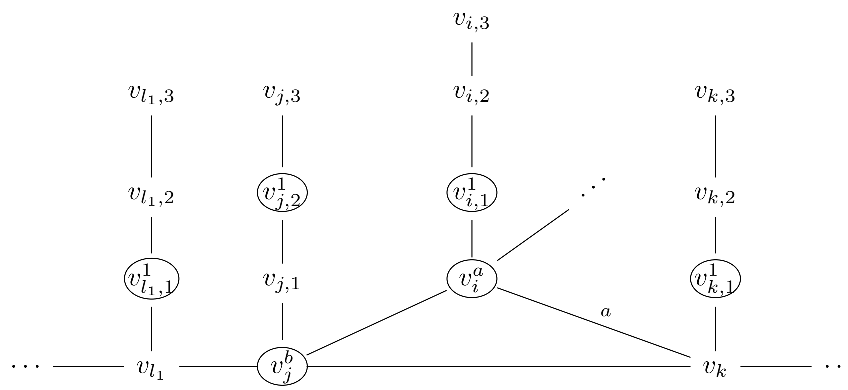
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2},\ v_{j,1}v_jv_iv_k \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},$$
$$v_iv_kv_jv_{j,1} \in \mathfrak{P}_{i,0},\ v_{j,1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k\}$.

(2) Assume that $l_1 \neq j$, then $l_2 = 0$.

(i) Assume that $f(\lambda(v_j v_{j,1}), \lambda(v_j v_k)) < b$.



Then it is straightforward to show that

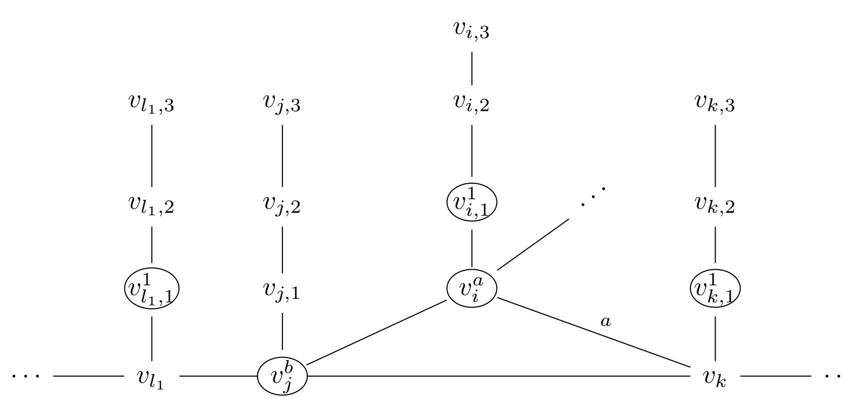
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{l_1,1}^1, v_{j,2}^1, v_j^b, v_{i,1}^1, v_i^a, v_{k,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \smallsetminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1},\ v_{j,2}v_{j,1}v_j v_k \in \mathfrak{P}_{j,2},\ v_{l_1}v_j v_i v_k \in \mathfrak{P}_{j,0},\ v_{i,1}v_i v_j v_k \in \mathfrak{P}_{i,1},$$
$$v_i v_k v_j v_{l_1} \in \mathfrak{P}_{i,0},\ v_{l_1}v_j v_k v_{k,1} \in \mathfrak{P}_{k,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l_1\}$.

(ii) Assume that $f(\lambda(v_j v_{j,1}), \lambda(v_j v_k)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v^1_{l_1,1}, v^b_j, v^1_{i,1}, v^a_i, v^1_{k,1}\} \sqcup \{v^1_m \mid m \in \{1, \dots, d\} \setminus \{i, j, k, l_1\}\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1},\ v_{l_1}v_jv_iv_k \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},$$

$$v_iv_kv_jv_{l_1} \in \mathfrak{P}_{i,0},\ v_{l_1}v_jv_kv_{k,1} \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \setminus \{i, j, k, l_1\}$.

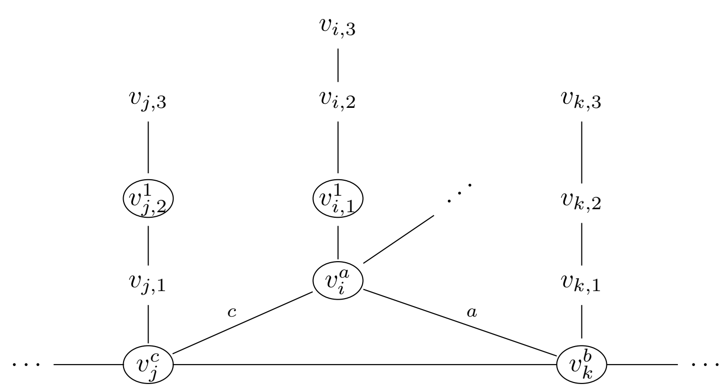
(j) Let $v_iv_jv_k$ be a 3-cycle in G such that $f(\lambda(v_iv_{i,1}, \lambda(v_iv_j))) < \lambda(v_iv_k) =: a$ and there exists $v_kv_{l_1,l_2} \in E\big((\Sigma_3 G)_\lambda\big)$ with $v_i \neq v_{l_1,l_2} \neq v_j$. Suppose that

$$b := f(\lambda(v_kv_j), \lambda(v_kv_{l_1,l_2})) > \max\{\lambda(v_kv_j), f(\lambda(v_kv_i), \lambda(v_kv_{l_1,l_2}))\}$$

and $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i) =: c$.

(1) Assume that $l_1 = k$, then $l_2 = 1$.

(i) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_k)) < c$.



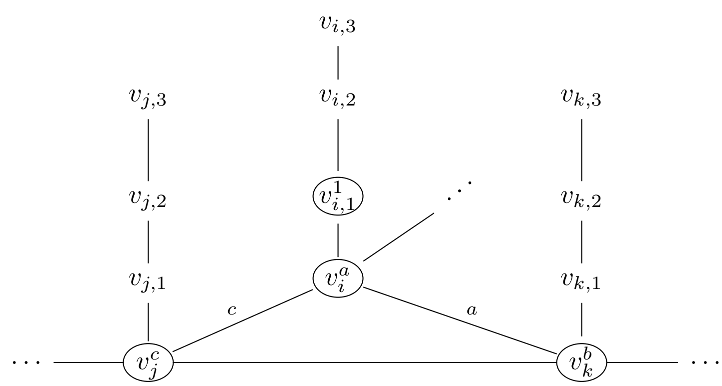
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \{v^1_{j,2}, v^c_j, v^1_{i,1}, v^a_i, v^b_k\} \sqcup \{v^1_m \mid m \in \{1, \dots, d\} \setminus \{i, j, k\}\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$,

$$v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2},\ v_jv_iv_kv_{k,1} \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},$$
$$v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_{k,1} \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(ii) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_k)) \geq c$.



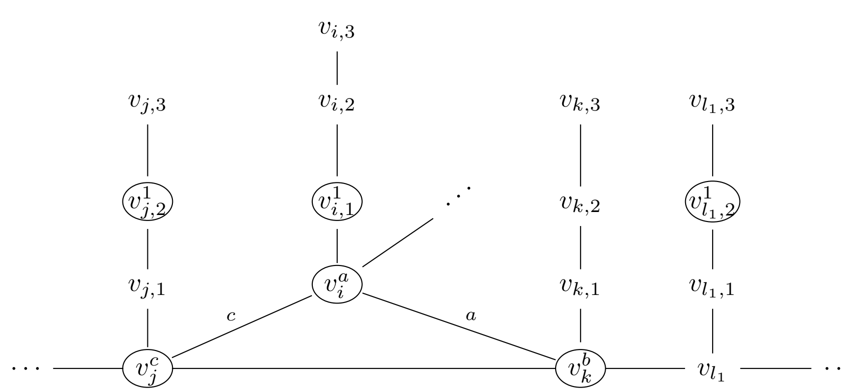
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^c, v_{i,1}^1, v_i^a, v_k^b\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, such that one has $v_jv_iv_kv_{k,1} \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_kv_{k,1}v_{k,2} \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_{k,1} \in \mathfrak{P}_{k,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$ for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}$.

(2) Assume that $l_1 \neq k$, then $l_2 = 0$.

(i) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_k)) < c$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{j,2}^1, v_j^c, v_{i,1}^1, v_i^a, v_k^b, v_{l_1,2}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \smallsetminus \{i, j, k, l_1\}\right\}$$

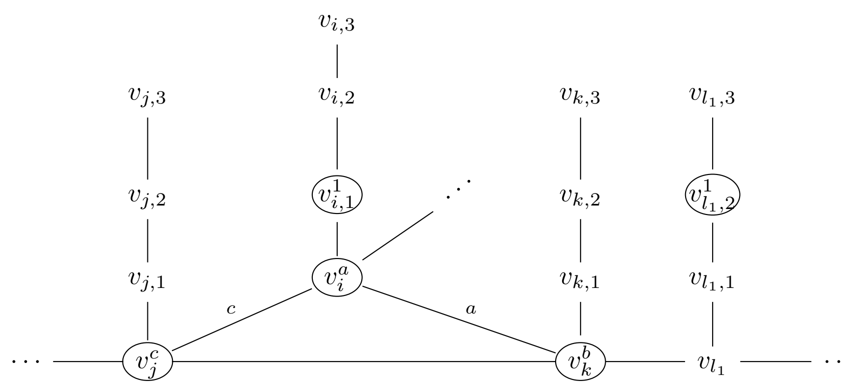
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,2},\ v_jv_iv_kv_{l_1} \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_kv_{l_1,0}v_{l_1,1} \in \mathfrak{P}_{i,0},$$

$$v_iv_jv_kv_{l_1} \in \mathfrak{P}_{k,0},\ v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,2},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l_1\}$.

(ii) Assume that $f(\lambda(v_{j,1}v_j), \lambda(v_jv_k)) \geq c$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_j^c, v_{i,1}^1, v_i^a, v_k^b, v_{l_1,2}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \dots, d\} \smallsetminus \{i, j, k, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_jv_iv_kv_{l_1} \in \mathfrak{P}_{j,0},\ v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_kv_{l_1,0}v_{l_1,1} \in \mathfrak{P}_{i,0},$$

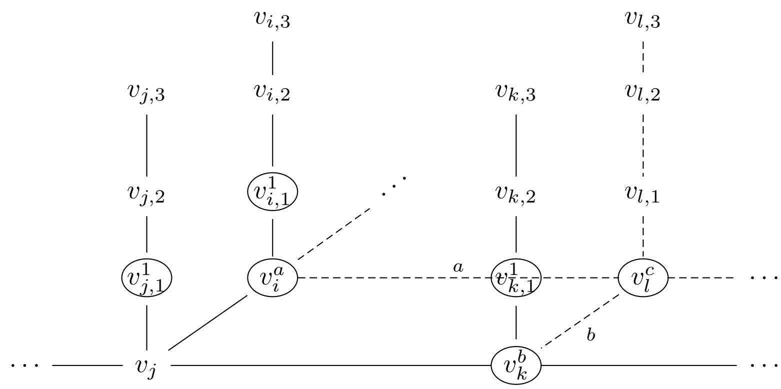
$$v_iv_jv_kv_{l_1} \in \mathfrak{P}_{k,0},\ v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,2},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \dots, d\} \smallsetminus \{i, j, k, l_1\}$.

(k) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have

$f(\lambda(v_kv_j),\lambda(v_kv_l)) < \lambda(v_kv_l) =: b$ and $f(\lambda(v_lv_i),\lambda(v_lv_k)) < \min\{\lambda(v_lv_i),\lambda(v_lv_k)\} =: c$.

(1) Assume that $f(\lambda(v_{k,1}v_k),\lambda(v_kv_j)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'',\delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}^1, v_{k,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}\right\}$$

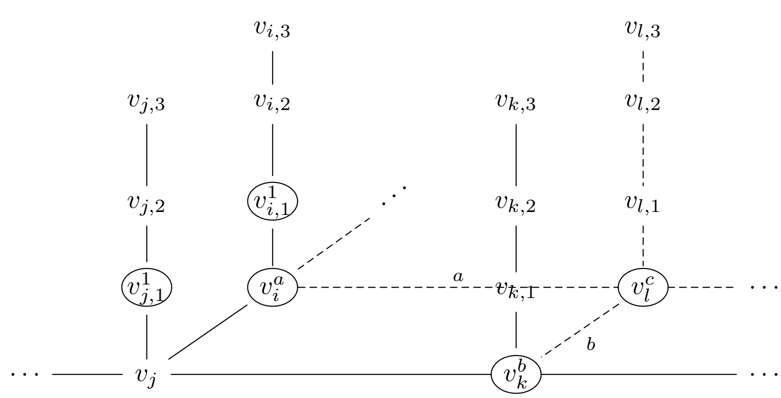
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1},\ v_iv_jv_kv_{k,1} \in \mathfrak{P}_{k,1},$$

$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(2) Assume that $f(\lambda(v_{k,1}v_k),\lambda(v_kv_j)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

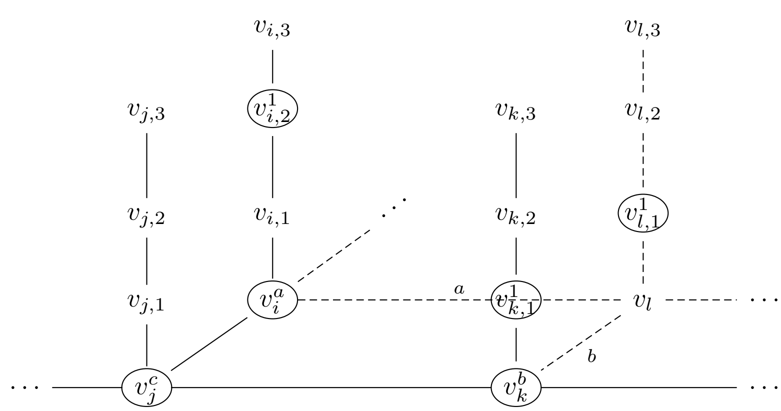
$$v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1},$$

$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0}\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(l) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$ and $c := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$.

(1) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) < b$.



Then it is straightforward to show that

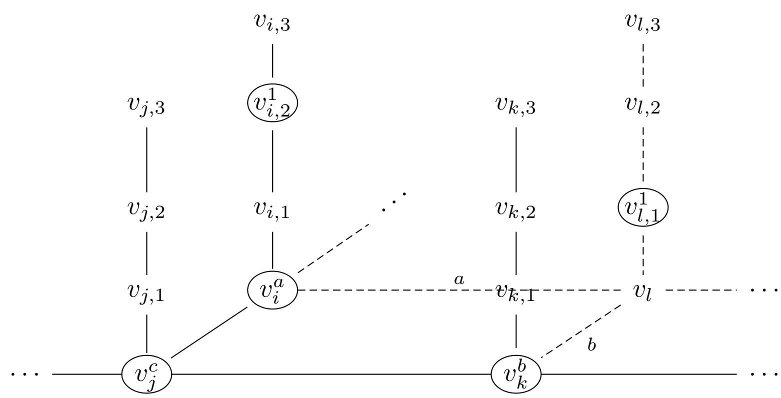
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,2}^1, v_i^a, v_j^c, v_{k,1}^1, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$,

$$v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{j,0},\ v_{k,2}v_{k,1}v_kv_j \in \mathfrak{P}_{k,1},$$

$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(2) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,2}^1, v_i^a, v_j^c, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i,j,k,l\}\right\}$$

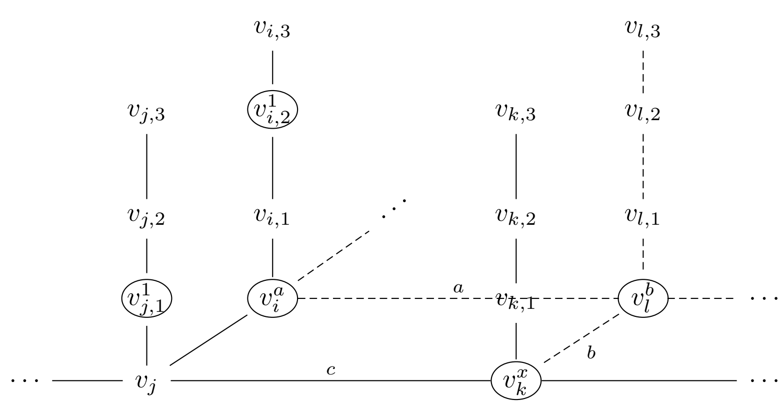
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{j,0},$$

$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i,j,k,l\}$.

(m) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_k) = b$, and $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_j) =: c$. Let $x := \min\{b, c\}$.



Then it is straightforward to show that

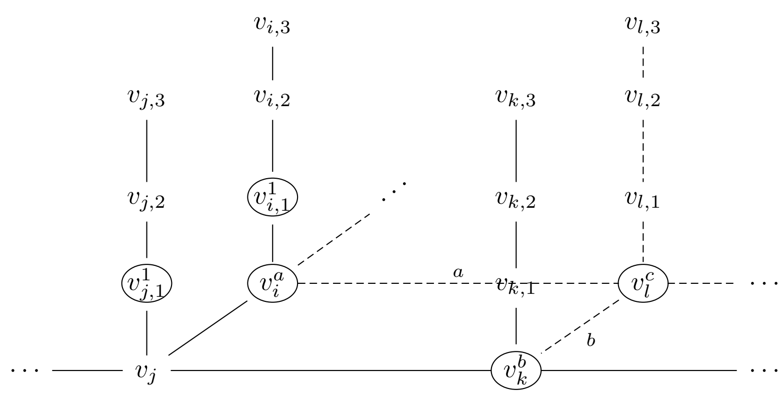
$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,2}^1, v_i^a, v_{j,1}, v_{k,1}^1, v_k^x, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,2},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1},$$
$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(n) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_k) = b$, and $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_i) = a$. Let $c := \min\{a, b\}$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}, v_{k,1}^1, v_k^x, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$,

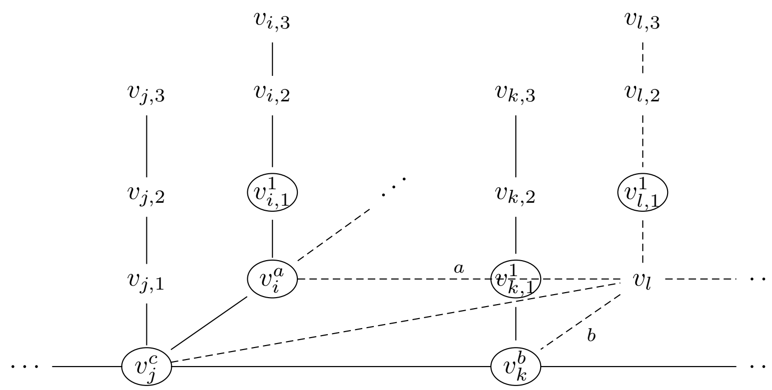
$$v_{i,2}v_{i,1}v_iv_j \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,1},$$
$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(o) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have

$f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $v_jv_l \in E(G)$, and

$$c := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\big\{\lambda(v_jv_k), f(\lambda(v_jv_i), \lambda(v_jv_l))\big\}.$$

(1) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) < b$.



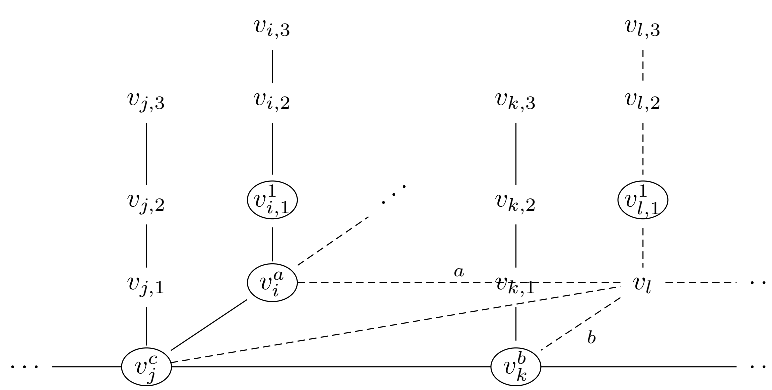
Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \big\{v_{i,1}^1, v_i^a, v_j^c, v_{k,1}^1, v_k^b, v_{l,1}^1\big\} \sqcup \big\{v_m^1 \mid m \in \{1, \dots, d\} \setminus \{i, j, k, l\}\big\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \dots, d\} \setminus \{i, j, k, l\}$,

$$v_{i,1}v_iv_jv_l \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{j,0},\ v_{k,2}v_{k,1}v_kv_j \in \mathfrak{P}_{k,1},$$
$$v_iv_jv_lv_k \in \mathfrak{P}_{k,0},\ v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(2) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^c, v_k^b, v_{l,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

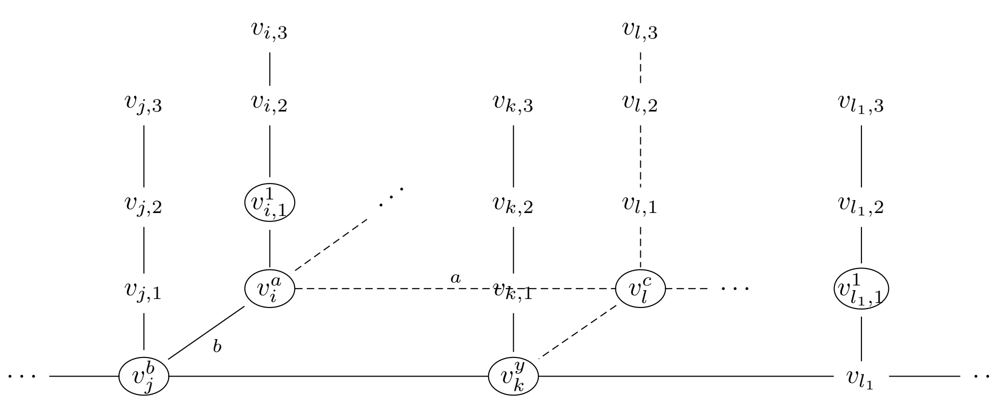
$$v_{i,1}v_iv_jv_l \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{j,0},$$
$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_{l,3}v_{l,2}v_{l,1}v_l \in \mathfrak{P}_{l,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(p) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l)$, $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_i) = a$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \min\{\lambda(v_lv_i), \lambda(v_lv_k)\} =: c$, and there exists $v_kv_{l_1,l_2} \in E((\Sigma_r G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f(\lambda(v_kv_j), \lambda(v_kv_l)) > \max\left\{\lambda(v_kv_j), f(\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2}))\right\},$$
$$y := f(\lambda(v_kv_j), \lambda(v_kv_{l_1,l_2})) > \max\left\{\lambda(v_kv_j), f(\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2}))\right\}.$$

Then $l_1 \neq k$, and so $l_2 = 0$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^b, v_k^y, v_{l,1}^1, v_l^a\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, l_1\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l, l_1\}$,

$$v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_{l_1} \in \mathfrak{P}_{i,0},\ v_{j,3}v_{j,2}v_{j,1}v_j \in \mathfrak{P}_{j,0},\ v_iv_jv_kv_{l_1} \in \mathfrak{P}_{k,0},$$
$$v_lv_iv_jv_k \in \mathfrak{P}_{l,0},\ v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}.$$

(q) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l)$, $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_i) = a$, $f(\lambda(v_kv_j), \lambda(v_kv_l)) > \lambda(v_kv_l)$, and there exists $v_kv_{l_1,l_2} \in E((\Sigma_r G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f(\lambda(v_kv_j), \lambda(v_kv_l)) > \max\bigl\{\lambda(v_kv_j), f(\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2}))\bigr\},$$
$$y := f(\lambda(v_kv_j), \lambda(v_kv_{l_1,l_2})) > \max\bigl\{\lambda(v_kv_j), f(\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2}))\bigr\}.$$

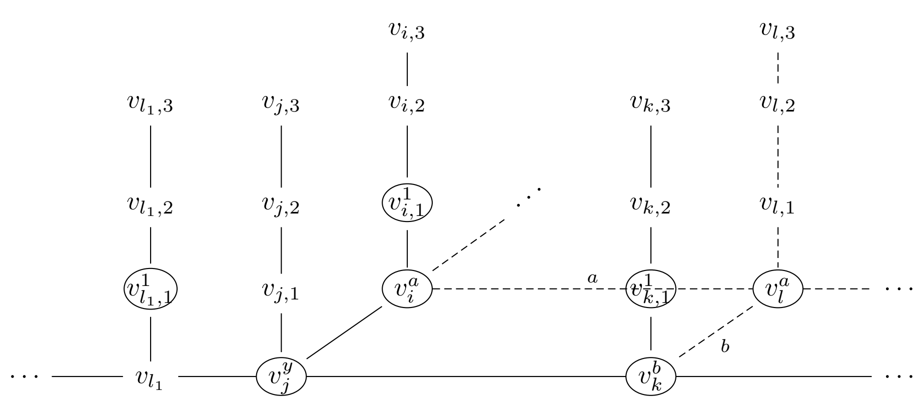
Then we have that $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) < f(\lambda(v_kv_j), \lambda(v_kv_l))$, a contradiction.

(r) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_i) = a$, and we have that there exists $v_jv_{l_1,l_2} \in E((\Sigma_r G)_\lambda)$ with $v_i \neq v_{l_1,l_2} \neq v_k$ such that

$$c := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\bigl\{\lambda(v_jv_k), f(\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2}))\bigr\},$$
$$x := f(\lambda(v_jv_k), \lambda(v_jv_{l_1,l_2})) > \max\bigl\{\lambda(v_jv_k), f(\lambda(v_jv_i), \lambda(v_jv_{l_1,l_2}))\bigr\}.$$

Then $l_1 \neq k$, and so $l_2 = 0$. Let $y := \min\{x, c\}$.

(1) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^y, v_{k,1}^1, v_k^b, v_l^a, v_{l_1,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\dots,d\} \smallsetminus \{i,j,k,l,l_1\}\right\}$$

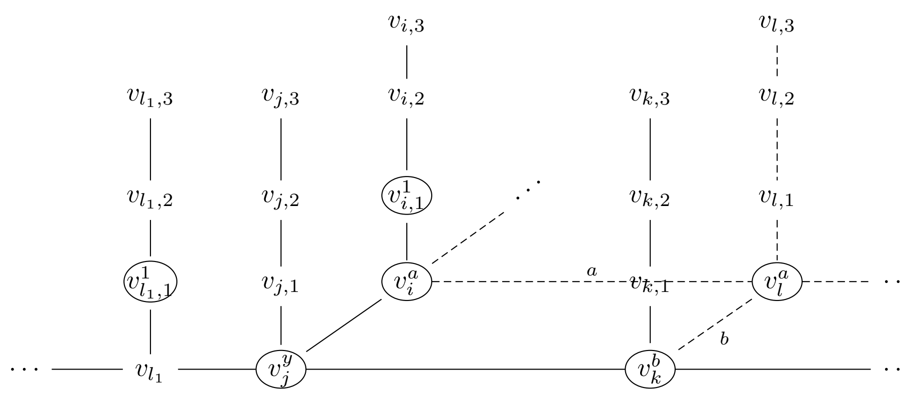
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,1}v_iv_jv_{l_1} \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{j,0},\ v_{k,2}v_{k,1}v_kv_j \in \mathfrak{P}_{k,1},$$

$$v_jv_iv_lv_k \in \mathfrak{P}_{k,0},\ v_lv_iv_jv_{l_1} \in \mathfrak{P}_{l,0},\ v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(2) Assume that $f(\lambda(v_{k,1}v_k), \lambda(v_kv_j)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^y, v_k^b, v_l^a, v_{l_1,1}^1\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\dots,d\} \smallsetminus \{i,j,k,l,l_1\}\right\}$$

is an f-weighted 3-path vertex cover, and

$$v_{i,1}v_iv_jv_{l_1} \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_iv_jv_kv_l \in \mathfrak{P}_{j,0},\ v_jv_iv_lv_k \in \mathfrak{P}_{k,0},,$$

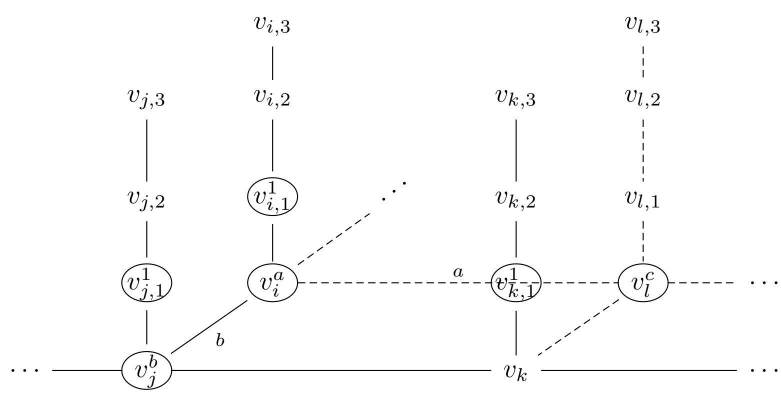
$$v_lv_iv_jv_{l_1} \in \mathfrak{P}_{l,0},\ v_{l_1,3}v_{l_1,2}v_{l_1,1}v_{l_1} \in \mathfrak{P}_{l_1,1},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1,\dots,d\} \smallsetminus \{i,j,k,l\}$.

(s) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have

$f(\lambda(v_jv_i),\lambda(v_jv_k)) < \lambda(v_jv_i) =: b$ and $f(\lambda(v_lv_i),\lambda(v_lv_k)) < \min\{\lambda(v_lv_i),\lambda(v_lv_k)\} =: c$.

(1) Assume that $f(\lambda(v_{j,1}v_j),\lambda(v_jv_k)) < b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'',\delta'') := \left\{v_{i,1}^1, v_i^a, v_{j,1}^1, v_j^b, v_{k,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l\}\right\}$$

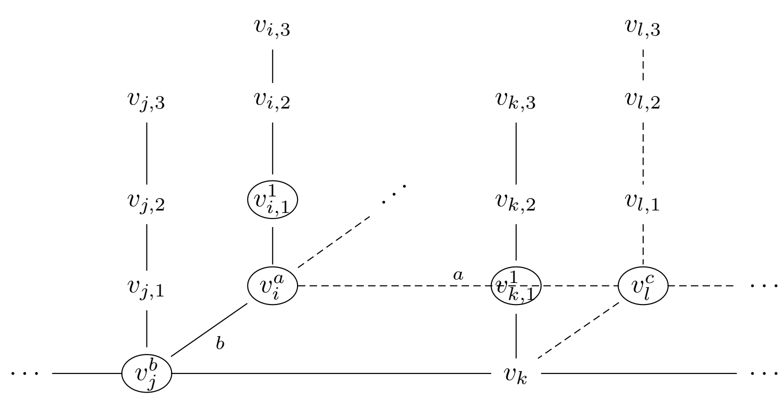
is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$, and

$$v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_{j,2}v_{j,1}v_jv_k \in \mathfrak{P}_{j,1},\ v_jv_iv_lv_k \in \mathfrak{P}_{j,0},$$

$$v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1,\ldots,d\} \smallsetminus \{i,j,k,l\}$.

(2) Assume that $f(\lambda(v_{j,1}v_j),\lambda(v_jv_k)) \geq b$.



Then it is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \left\{v_{i,1}^1, v_i^a, v_j^b, v_{k,1}^1, v_k^b, v_l^c\right\} \sqcup \left\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}\right\}$$

is an f-weighted 3-path vertex cover of $(\Sigma_3 G)_\lambda$,

$$v_{i,1}v_iv_jv_k \in \mathfrak{P}_{i,1},\ v_iv_lv_kv_j \in \mathfrak{P}_{i,0},\ v_jv_iv_lv_k \in \mathfrak{P}_{j,0},$$

$$v_{k,3}v_{k,2}v_{k,1}v_k \in \mathfrak{P}_{k,1},\ v_iv_jv_kv_l \in \mathfrak{P}_{l,0},\ v_{t,3}v_{t,2}v_{t,1}v_t \in \mathfrak{P}_{t,0}$$

for any $t \in \{1, \ldots, d\} \smallsetminus \{i, j, k, l\}$.

(t) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i) =: b$ and $f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\{\lambda(v_jv_i), \lambda(v_jv_k)\} \geq b$, a contradiction.

(u) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i)$, $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_k) = b$, and $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_j) =: c$. Then this case is similar to Case (m).

(v) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i)$, $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_k) = b$, and $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_i) = a$. Then this case is similar to Case (n).

(w) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i)$, $f(\lambda(v_kv_j), \lambda(v_kv_l)) < \lambda(v_kv_l) =: b$, $v_jv_l \in E(G)$, and

$$c := f(\lambda(v_jv_i), \lambda(v_jv_k)) > \max\left\{\lambda(v_jv_k), f(\lambda(v_jv_i), \lambda(v_jv_l))\right\}.$$

Then this case is similar to Case (o).

(x) Let $v_iv_jv_kv_l$ be a 4-cycle such that $f(\lambda(v_{i,1}v_i), \lambda(v_iv_j)) < \lambda(v_iv_l) =: a$. Suppose that we have $f(\lambda(v_jv_i), \lambda(v_jv_k)) < \lambda(v_jv_i) =: b$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \lambda(v_lv_i) =: c$, $f(\lambda(v_lv_i), \lambda(v_lv_k)) < \min\{\lambda(v_lv_i), \lambda(v_lv_k)\}$, and there exists $v_kv_{l_1,l_2} \in E((\Sigma_r G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f(\lambda(v_kv_j), \lambda(v_kv_l)) > \max\left\{\lambda(v_kv_j), f(\lambda(v_kv_l), \lambda(v_kv_{l_1,l_2}))\right\},$$

$$y := f(\lambda(v_k v_j), \lambda(v_k v_{l_1,l_2})) > \max\big\{\lambda(v_k v_j), f(\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2}))\big\}.$$

Then this case is similar to Case (p).

(y) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f(\lambda(v_{i,1} v_i), \lambda(v_i v_j)) < \lambda(v_i v_l) =: a$. Suppose that we have $f(\lambda(v_j v_i), \lambda(v_j v_k)) < \lambda(v_j v_i) =: b$, $f(\lambda(v_l v_i), \lambda(v_l v_k)) < \lambda(v_l v_i) =: c$, $f(\lambda(v_k v_j), \lambda(v_k v_l)) > \lambda(v_k v_l)$, and there exists $v_k v_{l_1,l_2} \in E((\Sigma_r G)_\lambda)$ with $v_j \neq v_{l_1,l_2} \neq v_l$ such that

$$x := f(\lambda(v_k v_j), \lambda(v_k v_l)) > \max\big\{\lambda(v_k v_j), f(\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2}))\big\},$$
$$y := f(\lambda(v_k v_j), \lambda(v_k v_{l_1,l_2})) > \max\big\{\lambda(v_k v_j), f(\lambda(v_k v_l), \lambda(v_k v_{l_1,l_2}))\big\}.$$

Then this case is also similar to Case (p).

(z) Let $v_i v_j v_k v_l$ be a 4-cycle such that $f(\lambda(v_{i,1} v_i), \lambda(v_i v_j)) < \lambda(v_i v_l) =: a$. Suppose that we have $f(\lambda(v_j v_i), \lambda(v_j v_k)) < \lambda(v_j v_i)$, $f(\lambda(v_k v_j), \lambda(v_k v_l)) < \lambda(v_k v_l) =: b$, $f(\lambda(v_l v_i), \lambda(v_l v_k)) < \lambda(v_l v_i) = a$, and there exists $v_j v_{l_1,l_2} \in E((\Sigma_r G)_\lambda)$ with $v_i \neq v_{l_1,l_2} \neq v_k$ such that

$$c := f(\lambda(v_j v_i), \lambda(v_j v_k)) > \max\big\{\lambda(v_j v_k), f(\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2}))\big\},$$
$$x := f(\lambda(v_j v_k), \lambda(v_j v_{l_1,l_2})) > \max\big\{\lambda(v_j v_k), f(\lambda(v_j v_i), \lambda(v_j v_{l_1,l_2}))\big\}.$$

Then this case is similar to Case (r). □

We discuss the necessary conditions for $I_{r,f}((\Sigma_r G)_\lambda)$ to be unmixed when $r \geq 4$.

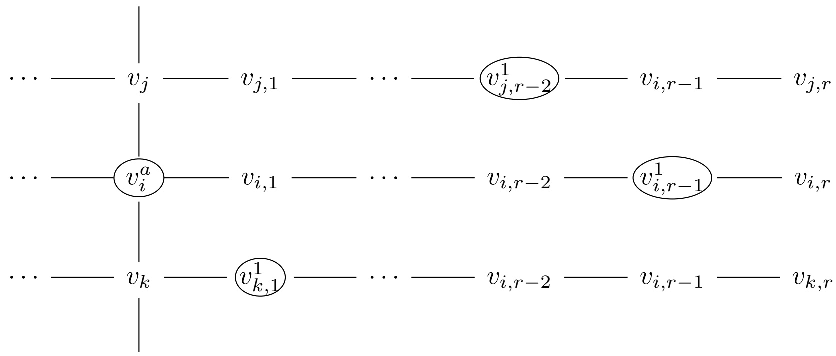
Theorem 3.3.4. *Assume that $r \geq 4$. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. If $I_{r,f}\big((\Sigma_r G)_\lambda\big)$ is unmixed, then the weight function λ satisfies the constraints in Proposition 3.2.44.*

Proof. By Lemma 3.3.1 and its proof, it is enough to show that if $a := f(\lambda(v_{i,1} v_i), \lambda(v_i v_j)) > f(\lambda(v_{i,1} v_i), \lambda(v_i v_k)) =: b$ for a 2-path $v_j v_i v_k$ in G_ω, then there exists an f-weighted r-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_r G)_\lambda$ such that $|V''| = d + 1$ and $\mathfrak{P}_{i,j} \neq \emptyset$ for each $v_{i,j} \in V''$. It is straightforward to show that

$$\mathfrak{P} := (V'', \delta'') := \big\{v_{j,r-2}^1, v_i^a, v_{i,r-1}^1, v_{k,1}^1\big\} \sqcup \big\{v_m^1 \mid m \in \{1, \ldots, d\} \smallsetminus \{i, j, k\}\big\}$$

is an f-weighted r-path vertex cover of $(\Sigma_r G)_\lambda$, $v_{j,r} \cdots v_{j,1} v_j \in \mathfrak{P}_{j,r-2}$, $v_{i,r-1} \cdots v_{i,1} v_i v_k \in \mathfrak{P}_{i,r-1}$, $v_{i,r-2} \cdots v_{i,1} v_i v_j v_{j,1} \in \mathfrak{P}_{i,0}$, $v_{k,r} \cdots v_{k,1} v_k \in \mathfrak{P}_{k,1}$ and to show that $v_{t,r} \cdots v_{t,1} v_t \in \mathfrak{P}_{t,0}$ for any

$t \in \{1, \ldots, d\} \setminus \{i, j, k\}$.



□

3.4 Sufficient Conditions for Cohen-Macaulayness

In this section, we prove the sufficient conditions for which the f-weighted r-path ideal of a weighted r-path suspension is Cohen-Macaulay for all $r \geq 2$.

Theorem 3.4.1. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that the weight function λ satisfies the constraints in Propositions 3.2.1, 3.2.43, or 3.2.44 when $r = 2$ or 3 or $r \geq 4$, respectively. Then $I := I_{r,f}\big((\Sigma_r G)_\lambda\big)$ is Cohen-Macaulay, $\{X_{i,j} - X_{i,j-1} \mid i = 1, \ldots, d, j = 1, \ldots, r\}$ is a homogeneous regular sequence for R'/I, and*

$$\frac{R'}{I + (X_{i,j} - X_{i,j-1} \mid i = 1, \ldots, d, j = 1, \ldots, r)R'} \cong \frac{R}{IR}.$$

Proof. For $k = 1, \ldots, (r-1)d$, let $i_k = \lfloor \frac{k+d-1}{d} \rfloor$, $j_k = k + (1 - i_k)d$ and

$$\underline{n}_k = (\underbrace{r - i_k + 1, \ldots, r - i_k + 1}_{j_k \text{ times}}, \underbrace{r - i_k + 2, \ldots, r - i_k + 2}_{d - j_k \text{ times}}) \in \mathbb{N}^d.$$

For $k = 1, \ldots, (r-1)d$, define a polynomial ring R_k by

$$R_k = A \begin{bmatrix} 0, & \cdots & 0, & X_{j_k+1, r-i_k+1}, & \cdots & X_{d, r-i_k+1} \\ X_{1, r-i_k}, & \cdots & X_{j_k, r-i_k}, & X_{j_k+1, r-i_k}, & \cdots & X_{d, r-i_k}, \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{1,0}, & \cdots & X_{j_k,0}, & X_{j_k+1,0}, & \cdots & X_{d,0} \end{bmatrix}.$$

The polynomial ring R_k has $j_k(r-i_k+1)+(d-j_k)(r-i_k+2)$ variables. Then for $k=1,\dots,(r-1)d$, $p_{\underline{n}_k}(I)R_k$ is the monomial ideal of R_k obtained from I by setting $X_{a,b}=X_{a,r-i_k}$ for $a=1,\dots,j_k$ and $b=r-i_k+1,\dots,r$ and setting $X_{a,b}=X_{a,r-i_k+1}$ for $a=j_k+1,\dots,d$ and $b=r-i_k+2,\dots,r$. Note that

$$\frac{R_1}{p_{\underline{n}_1}(I)} \cong \frac{R'}{I+(X_{1,r}-X_{1,r-1})},$$

and for $k=2,\dots,(r-1)d$ we have inductively

$$\begin{aligned}
\frac{R_k}{p_{\underline{n}_k}(I)} &\cong \frac{R_{k-1}}{p_{\underline{n}_{k-1}(I)}+(X_{j_k,r-i_k+1}-X_{j_k,r-i_k})} \cong \frac{R_{k-1}/p_{\underline{n}_{k-1}}(I)}{(X_{j_k,r-i_k+1}-X_{j_k,r-i_k})}\\
&\cong \frac{R_{k-2}/(p_{\underline{n}_{k-2}}(I)+(X_{j_{k-1},r-i_{k-1}+1}-X_{j_{k-1},r-i_{k-1}}))}{(X_{j_k,r-i_k+1}-X_{j_k,r-i_k})}\\
&\cong \frac{R_{k-2}}{p_{\underline{n}_{k-2}}(I)+(X_{j_{k-1},r-i_{k-1}+1}-X_{j_{k-1},r-i_{k-1}},X_{j_k,r-i_k+1}-X_{j_k,r-i_k})}\\
&\cong \cdots\\
&\cong \frac{R_1}{p_{\underline{n}_1}(I)+(X_{j_l,r-i_l+1}-X_{j_l,r-i_l}\mid l=2,\dots,k)}\\
&\cong \frac{R'}{I+(X_{j_l,r-i_l+1}-X_{j_l,r-i_l}\mid l=1,\dots,k)},
\end{aligned}$$

since $j_1=1$, $r-i_1+1=r-1+1=r$ and $r-i_1=r-1$. Hence

$$\begin{aligned}
\frac{R}{IR} = \frac{R_{(r-1)d}}{p_{\underline{n}_{(r-1)d}}(I)} &\cong \frac{R'}{I+(X_{j_l,r-j_l+1}-X_{j_l,r-j_l}\mid l=1,\dots,(r-1)d)}\\
&= \frac{R'}{I+(X_{i,j}-X_{i,j-1}\mid i=1,\dots,d,\ j=2,\dots,r)}.
\end{aligned}$$

Let $k\in\{1,\dots,(r-1)d\}$. We set $R_0:=R'$ and $\underline{n}_0:=(r+1,\dots,r+1)$. Then by Propositions 3.2.1, 3.2.43 and 3.2.44, for any (V'',δ'') such that $P(V'',\delta'')$ occurs in an irredundant m-irreducible decomposition of $p_{\underline{n}_{k-1}}(I)$, we have that there exists a unique $v_{i,i_\alpha}\in V''$ for $i=1,\dots,d$. So in $R_{k-1}/p_{\underline{n}_{k-1}}(I)$, the associated primes of $0=p_{\underline{n}_{k-1}}(I)/p_{\underline{n}_{k-1}}(I)$ are of the form $\overline{(X_{1,\beta_1},\dots,X_{d,\beta_d})R_{k-1}}$. So we have that $X_{j_k,r-i_k+1}-X_{j_k,r-i_k}\in \mathrm{NZD}_R\big(R_{k-1}/p_{\underline{n}_{k-1}}(I)\big)$. So we have that $X_{j_k,r-i_k+1}-X_{j_k,r-i_k}$ is $R_{k-1}/p_{\underline{n}_{k-1}}(I)$-regular. Thus, by the definition of the R'/I-regular sequence, we have

$$\{X_{i,j}-X_{i,j-1}\mid i=1,\dots,d,j=2,\dots,r\}=\{X_{j_l,r-j_l+1}-X_{j_l,r-j_l}\mid l=1,\dots,(r-1)d\}$$

is a homogeneous regular sequence for R'/I. Since R/IR is Artinian, it is Cohen-Macaulay. So by Fact 2.5.5, we have that R'/I is Cohen-Macaulay. $\square$

3.5 Main Results

The main results of this chapter are in Theorems 3.5.5 and 3.5.6.

Corollary 3.5.1. Let $f = \max$. Then the constraints for λ in Propositions 3.2.1, 3.2.43, and 3.2.44 become

$$\lambda(v_iv_j) \leq \min\{\lambda(v_iv_{i,1}), \lambda(v_jv_{j,1})\},\ \forall\ v_iv_j \in E(G).$$

Corollary 3.5.2. Let $f = \text{lcm}$. Then the constraints for λ in Propositions 3.2.1, 3.2.43, and 3.2.44 become

$$\lambda(v_iv_j) \mid \lambda(v_iv_{i,1}) \text{ and } \lambda(v_iv_j) \mid \lambda(v_jv_{j,1}),\ \forall\ v_iv_j \in E(G).$$

Corollary 3.5.3. Let $f = \min$. Then the constraints for λ in Proposition 3.2.1, 3.2.43, and 3.2.44 become

$$\lambda(v_iv_j) \leq \min\{\lambda(v_iv_{i,1}), \lambda(v_jv_{j,1})\},\ \forall\ v_iv_j \in E(G),$$

$$\lambda(v_{i,k}, v_{i,k+1}) \leq \lambda(v_{i,k+1}v_{i,k+2}),\ \forall\ i = 1, \dots, d \text{ and } k = 0, \dots, r-2,$$

and

$$\begin{cases} \lambda(v_iv_j) \geq \lambda(v_jv_k) \text{ or } \lambda(v_kv_l) \geq \lambda(v_jv_k) \text{ for all 3-paths } v_iv_jv_kv_l \text{ in } G, & \text{if } r = 2, \\ \lambda(v_iv_j) \geq \lambda(v_jv_k) \text{ or } \lambda(v_lv_m) \geq \lambda(v_kv_l) \text{ for all 4-paths } v_iv_jv_kv_lv_m \text{ in } G, \text{ and} & \\ \text{the weights on edges satisfy } a = b \geq c \text{ for all 3-cycles in } G, & \text{if } r = 3, \\ \text{all edges in } G \text{ have the same weight} & \text{if } r \geq 4. \end{cases}$$

Proof. We first show the equivalence for weight constraints on 4-paths in G when $r = 3$. Let $v_iv_jv_kv_lv_m$ be a 4-path in G. On one hand, let $\lambda(v_iv_j) \geq \lambda(v_jv_k)$ or $\lambda(v_lv_m) \geq \lambda(v_kv_l)$, then by Notation 3.2.2(a), we have that $\min(\lambda(v_iv_j), \lambda(v_jv_k)) = \lambda(v_jv_k) \geq \lambda(v_jv_k)$ or $\min(\lambda(v_kv_l), \lambda(v_lv_m)) = \lambda(v_kv_l) \geq \lambda(v_kv_l)$, so Notation 3.2.2(d) holds. On the other hand, without loss of generality, assume that $\lambda(v_iv_j) < \lambda(v_jv_k)$, then $\min\{\lambda(v_{j,1}v_j), \lambda(v_jv_i)\} = \lambda(v_jv_i) < \lambda(v_jv_k)$ by Notation 3.2.2(a) and $\min(\lambda(v_iv_j), \lambda(v_jv_k)) < \lambda(v_jv_k)$, so $\min(\lambda(v_kv_l), \lambda(v_lv_m)) \geq \lambda(v_kv_l)$ by Notation 3.2.2(d). Hence $\lambda(v_lv_m) \geq \lambda(v_kv_l)$.

We then show the equivalence for weight constraints on 3-cycles in G when $r = 3$. Let $v_iv_jv_kv_i$ be a 3-cycle in G. On one hand, let the weights on edges of the 3-cycle $v_iv_jv_kv_i$ satisfy $a = b \geq c$, without loss of generality, assume that $\lambda(v_iv_j) \leq \lambda(v_jv_k) = \lambda(v_kv_i)$, then we have that

$$\min(\lambda(v_jv_i), \lambda(v_jv_k)) = \lambda(v_jv_i) < \lambda(v_jv_k) = \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$$

and $\min(\lambda(v_kv_i), \lambda(v_kv_j)) = \lambda(v_kv_j) \geq \lambda(v_kv_j)$, and $\min(\lambda(v_kv_i), \lambda(v_kv_j)) = \lambda(v_kv_j) \geq \lambda(v_kv_i)$, so we proved Notations 3.2.2(e)(1) and (e)(2)(i), it is straightforward to show that Notations (e)(2)(ii) and (e)(2)(iii) hold. On the other hand, if $\lambda(v_iv_j) = \lambda(v_jv_k) = \lambda(v_kv_i)$, then we are done, so without loss of generality, assume $\lambda(v_iv_j) < \lambda(v_iv_k)$, then $\min(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < \min(\lambda(v_iv_{i,1}), \lambda(v_iv_k))$ by Notation 3.2.2(a), so $\min(\lambda(v_kv_i), \lambda(v_kv_j)) \geq \lambda(v_kv_j)$ by Notation 3.2.2(e)(1), hence $\lambda(v_kv_i) \geq \lambda(v_kv_j)$, similarly, we have $\min(\lambda(v_iv_{i,1}), \lambda(v_iv_j)) < \lambda(v_iv_k)$, so $\min(\lambda(v_kv_i), \lambda(v_kv_j)) \geq \lambda(v_kv_i)$ by Notation 3.2.2(e)(2), so $\lambda(v_kv_j) \geq \lambda(v_kv_i)$, hence $\lambda(v_kv_i) \geq \lambda(v_kv_j) \geq \lambda(v_kv_i)$ and so $\lambda(v_kv_i) = \lambda(v_kv_j) > \lambda(v_iv_j)$.

We show there is no weight constraint on any 4-cycles in G. It suffices to show that Notation 3.2.2(f)(2) holds automatically provided that $f = \min$. It is straightforward to show that Notations (f)(2)(i), (f)(2)(ii), (f)(2)(iv) and (f)(2)(v) holds automatically. But Notation 3.2.2(2)(iii) is equivalent to either that $\lambda(v_kv_j) \geq \lambda(v_kv_l)$, that $\lambda(v_lv_i) \geq \lambda(v_lv_k)$, or that $(\lambda(v_kv_l) \geq \lambda(v_kv_j)$ and $\lambda(v_lv_k) \geq \lambda(v_lv_i))$, which is equivalent to $\lambda(v_kv_l) \leq \max\{\lambda(v_kv_j), \lambda(v_lv_i)\}$ or $\lambda(v_kv_l) \geq \max\{\lambda(v_kv_j), \lambda(v_lv_i)\}$, but this holds automatically.

It is straightforward to show the equivalence for $r = 2$ and $r \geq 4$. □

Corollary 3.5.4. Let $f = \gcd$. Then the constraints for λ in Notation 3.2.2 become

$$\lambda(v_iv_j) \mid \lambda(v_iv_{i,1}) \text{ and } \lambda(v_iv_j) \mid \lambda(v_jv_{j,1}),\ \forall\ v_iv_j \in E(G),$$

$$\lambda(v_{i,k}, v_{i,k+1}) \mid \lambda(v_{i,k+1}v_{i,k+2}),\ \forall\ i = 1, \ldots, d \text{ and } k = 0, \ldots, r-2,$$

and

(a) if $r = 2$, then $\lambda(v_jv_k) \mid \lambda(v_iv_j)$ or $\lambda(v_jv_k) \mid \lambda(v_kv_l)$ for all 3-paths $v_iv_jv_kv_l$ in G,

(b) if $r = 3$, then

(1) for all 4-paths $v_iv_jv_kv_lv_m$ in G: if $\lambda(v_jv_i) < \lambda(v_jv_k)$, then $\lambda(v_kv_l) \mid \lambda(v_lv_m)$,

(2) for all 3-cycles $v_iv_jv_kv_i$ in G: if $\lambda(v_iv_j) < \lambda(v_iv_k)$, then $\lambda(v_kv_i) = \lambda(v_kv_j)$ and

$$\gcd(\lambda(v_jv_i), \lambda(v_jv_{n,0})) \leq \max\big\{\gcd(\lambda(v_jv_i), \lambda(v_jv_k)), \gcd(\lambda(v_jv_k), \lambda(v_jv_{n,0}))\big\}$$
$$\forall\ v_jv_{n,0} \in E(G) \text{ with } i \neq n \neq k,$$

(3) for all 4-cycles $v_iv_jv_kv_lv_i$: if $\lambda(v_iv_j) < \lambda(v_iv_l)$, then

$$\begin{cases} \text{either } (\lambda(v_kv_l) \mid \lambda(v_kv_j) \text{ and } \lambda(v_jv_i) \mid \lambda(v_jv_k)), \\ \lambda(v_lv_i) \mid \lambda(v_lv_k), \\ \text{or } \lambda(v_lv_k) \mid \lambda(v_lv_i), \end{cases}$$

and

$$\begin{cases} \text{either } \lambda(v_kv_l) \mid \lambda(v_kv_j), \\ \lambda(v_lv_k) \mid \lambda(v_lv_i), \\ \text{or } (\lambda(v_kv_j) \mid \lambda(v_kv_l) \text{ and } \lambda(v_lv_i) \mid \lambda(v_lv_k)), \end{cases}$$

(c) if $r \geq 4$, then all edges in G have the same weight.

Proof. We first show the equivalence for weight constraints on 4-paths in G when $r = 3$. Let $v_iv_jv_kv_lv_m$ be a 4-path in G. On one hand, let $\lambda(v_iv_j) < \lambda(v_jv_k)$ and assume that $\lambda(v_kv_l) \mid \lambda(v_lv_m)$, then $\gcd(\lambda(v_kv_l), \lambda(v_lv_m)) = \lambda(v_kv_l) \geq \lambda(v_kv_l)$, so Notation 3.2.2(d) holds. On the other hand, assume that $\lambda(v_iv_j) < \lambda(v_jv_k)$, then $\gcd(\lambda(v_{j,1}v_j), \lambda(v_jv_i)) = \lambda(v_jv_i) < \lambda(v_jv_k)$ by Notation 3.2.2(a) and $\gcd(\lambda(v_iv_j), \lambda(v_jv_k)) < \lambda(v_jv_k)$, so $\gcd(\lambda(v_kv_l), \lambda(v_lv_m)) \geq \lambda(v_kv_l)$ by Notation 3.2.2(d), hence $\lambda(v_kv_l) \mid \lambda(v_lv_m)$.

We then show the equivalence for weight constraints on 3-cycles in G when $r = 3$. Let $v_iv_jv_kv_i$ be a 3-cycle in G. On one hand, let $\lambda(v_iv_j) < \lambda(v_iv_k)$, assume that $\lambda(v_kv_i) = \lambda(v_kv_j)$ and that

$$\gcd(\lambda(v_jv_i), \lambda(v_jv_{n,0})) \leq \max\big\{\gcd(\lambda(v_jv_i), \lambda(v_jv_k)), \gcd(\lambda(v_jv_k), \lambda(v_jv_{n,0}))\big\}$$
$$\forall\ v_jv_{n,0} \in E(G) \text{ with } i \neq n \neq k,$$

then $\gcd(\lambda(v_jv_i), \lambda(v_jv_k)) \leq \max\{\lambda(v_jv_i), \lambda(v_jv_k)\}$ and $\gcd(\lambda(v_kv_i), \lambda(v_kv_j)) = \lambda(v_kv_j) \geq \lambda(v_kv_j)$, and $\gcd(\lambda(v_kv_i), \lambda(v_kv_j)) = \lambda(v_kv_i) \geq \lambda(v_kv_i)$, so Notations 3.2.2(e)(1) and (e)(2)(i) was proved. Hence it is straightforward to show that Notations 3.2.2(e)(2)(ii) and (e)(2)(iii) holds automatically.

On the other hand, then it is straightforward to show that we can deduce Notation 3.2.2(b)(2) in the corollary from Notation 3.2.2(e). □

Theorem 3.5.5. *Assume that H_λ is obtained by pruning a sequence of r-pathless leaves from G_ω and that H_λ is an r-path suspension of a weighted graph Γ_μ. Then the following conditions are equivalent:*

(i) $I_{r,f}(G_\omega)$ is Cohen-Macaulay;

(ii) $I_{r,f}(G_\omega)$ is unmixed; and

(iii) the weight function λ satisfies the constraints in Propositions 3.2.1, 3.2.43, or 3.2.44 when $r = 2$ or 3 or $r \geq 4$, respectively, where we rename the vertices of H_λ such that $V(\Gamma_\mu) = \{v_i \mid i = 1, \ldots, d\}$,

$$V(H_\lambda) = \{v_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\} \text{ with } v_{i,0} = v_i, \ \forall\ i = 1, \ldots, d,$$

and $\{v_{i,0}v_{i,1} \cdots v_{i,r}\}_{i=1}^{d}$ are all the d r-whiskers.

Proof. (i)$\Longrightarrow$(ii) follows from Fact 2.7.8.

(ii)$\Longrightarrow$(iii) Assume that $I_{r,f}(G_\omega)$ is unmixed. By Lemma 3.1.11(b), $I_{r,f}(H_\lambda)$ is also unmixed. Then Statement (iii) follows from Theorem 3.3.2, 3.3.3 and 3.3.4.

(iii)$\Longrightarrow$(i) Assume condition (iii) holds. Then Theorem 3.4.1 implies that $I_{r,f}(H_\lambda)$ is Cohen-Macaulay. So Lemma 3.1.11(c) implies that $I_{r,f}(G_\omega)$ is as well. □

Because of the following fact and Theorem 3.4.1, the main result of this chapter gives a formula to compute $\mathrm{r}_R\big(R/I_{r,f}(G_\omega)\big)$ for all trees such that $R/I_{r,f}(G_\omega)$ is Cohen-Macaulay.

Theorem 3.5.6. *Assume that G_ω is a weighted tree. Then the following conditions are equivalent:*

(i) $I_{r,f}(G_\omega)$ is Cohen-Macaulay;

(ii) $I_{r,f}(G_\omega)$ is unmixed; and

(iii) there exists a weighted tree Γ_μ and an r-path suspension H_λ of Γ_μ such that H_λ is obtained by pruning a sequence of r-pathless leaves from G_ω, the weight function λ satisfies the constraints in Propositions 3.2.1, 3.2.43, or 3.2.44 when $r = 2$ or 3 or $r \geq 4$, respectively, where we rename the

vertices of H_λ *such that we have that* $V(\Gamma_\mu) = \{v_i \mid i = 1, \ldots, d\}$,

$$V(H_\lambda) = \{v_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\} \text{ with } v_{i,0} = v_i, \ \forall\ i = 1, \ldots, d,$$

and $\{v_{i,0}v_{i,1} \cdots v_{i,r}\}_{i=1}^{d}$ *are all the* d r*-whiskers.*

Proof. (iii)$\Longrightarrow$(i)$\Longrightarrow$(ii) follows from Proposition 3.5.5.

(ii)$\Longrightarrow$(iii) Assume that $I_{r,f}$ is unmixed. Since G is finite, we prune a sequence of r-pathless leaves from G_ω to obtain a weighted graph H_λ that has no r-pathless leaves. Lemma 3.1.11(b) implies that $I_{r,f}(H_\lambda)$ is unmixed. So we have that $I_r(H_\lambda)$ is unmixed by Lemma 3.1.8. Hence H is an r-path suspension of a tree Γ by [2, Theorem 3.8 and Remark 3.9]. Finally, Proposition 3.5.5 implies the weight conditions on $E(H_\lambda)$. □

Chapter 4

Cohen-Macaulay Type of Weighted r-Path Ideals

Let $\mathbb{K}$ be a field, $d \geq 2$, $R = \mathbb{K}[X_1, \ldots, X_d]$ and $\mathfrak{m} = (X_1, \ldots, X_d)R$. Let $G = (V, E)$ be a (finite simple) graph with vertex set $V = \{v_1, \ldots, v_d\}$ and edge set E. Let $r \geq 1$ be a positive integer and $R' = \mathbb{K}\big[\{X_{i,j} \mid i = 1, \ldots, d, j = 0, \ldots, r\}\big]$.

In this chapter, we compute the type of the rings $R/I_r(G_\omega)$ when they are Cohen-Macaulay. This main result is in Theorem 4.2.25.

4.1 Background

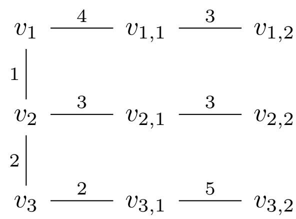
Definition 4.1.1. The *weighted r-path ideal* associated to G_ω is the ideal $I_r(G_\omega) := I_{r,\max}(G_\omega) \subseteq R$ that is generated by the max-weighted paths in G of length r:

$$I_r(G_\omega) = \left(X_{i_1}^{e_{i_1}} \ldots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \ldots v_{i_{r+1}} \text{ is a path in } G \text{ with } e_{i_1} = \omega(v_{i_1} v_{i_2}), \\ e_{i_j} = \max(\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})) \text{ for } 1 < j \leq r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r} v_{i_{r+1}}) \end{array} \right) R.$$

Remark. (a) $I_r(G_1) = I_r(G)$, where $1 : E \to \mathbb{N}$ is the constant function $1(e) = 1$.

(b) $I_1(G_\omega) = I(G_\omega)$.

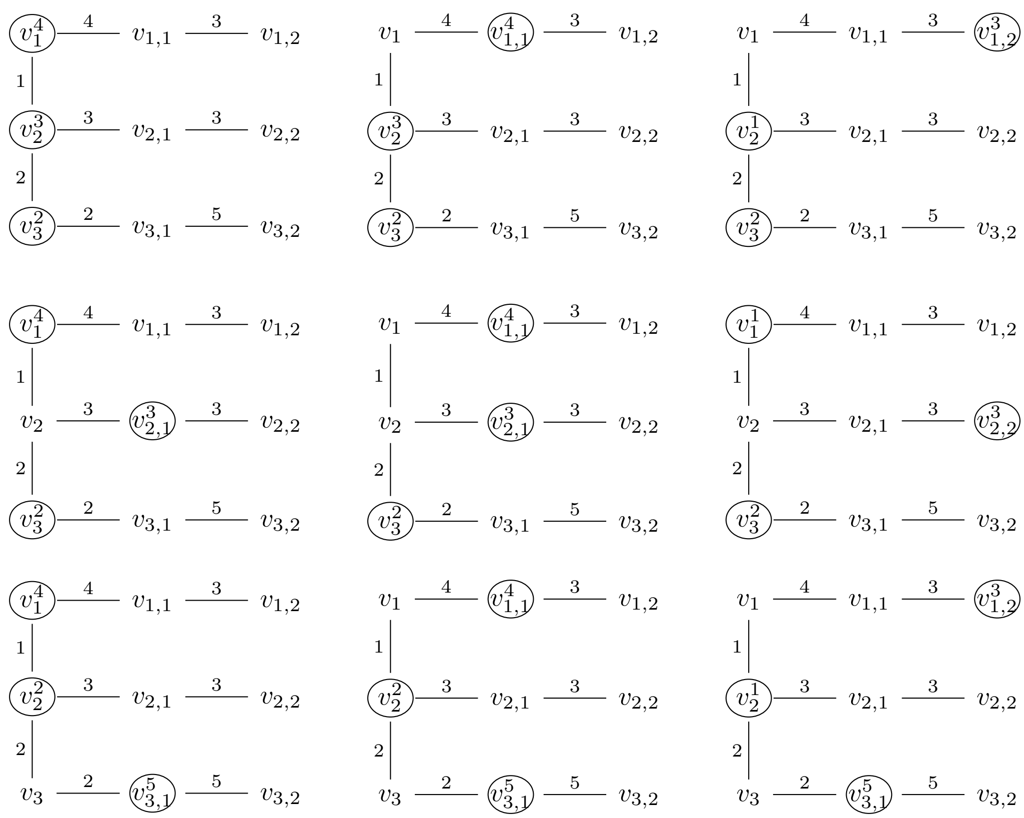
Example 4.1.2. Consider the following weighted graph $(\Sigma_2 P_2)_\lambda$ from Example 3.1.17.

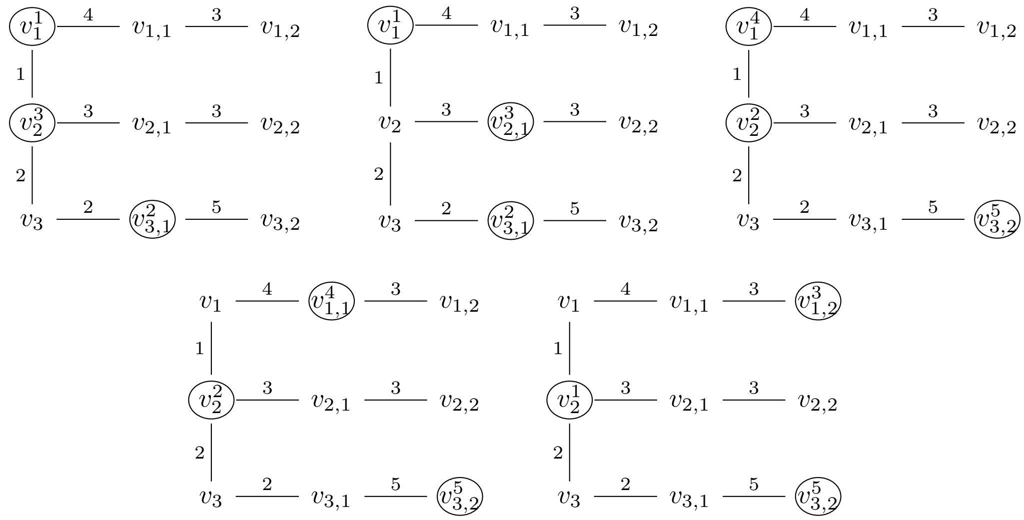


Then the weighted 2-path ideal of $(\Sigma_2 P_2)_\lambda$ is

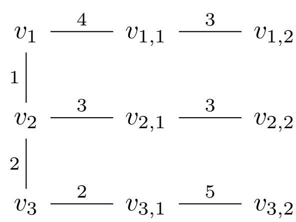
$$I_2\big((\Sigma_2 P_2)_\lambda\big) = \big(X_{1,2}^3 X_{1,1}^4 X_1^4, X_{1,1}^4 X_1^4 X_2, X_1 X_2^3 X_{2,1}^3, X_1 X_2^2 X_3^2, X_{2,2}^3 X_{2,1}^3 X_2^3,$$
$$X_{2,1}^3 X_2^3 X_3^2, X_2^2 X_3^2 X_{3,1}^2, X_{3,2}^5 X_{3,1}^5 X_3^2\big) R'.$$

Example 4.1.3. The minimal weighted 2-path vertex covers of $(\Sigma_2 P_2)_\lambda$ from Example 3.1.17 are displayed in the following sketches. In each diagram, all of the vertices encompassed by a circle form a weighted 2-path vertex cover of $(\Sigma_2 P_2)_\lambda$.





Example 4.1.4. Consider the following graph $(\Sigma_2 P_2)_\lambda$ from Example 3.1.17.



By Fact 3.1.7 and Example 4.1.3, the irredundant m-irreducible decomposition of $I_2\big((\Sigma_2 P_2)_\lambda\big)$ is

$$\begin{aligned} I_2\big((\Sigma_2 P_2)_\lambda\big) = {}& \big(X_1^4, X_2^3, X_3^2\big)R' \cap \big(X_{1,1}^4, X_2^3, X_3^2\big)R' \cap \big(X_{1,2}^3, X_2, X_3^2\big)R' \cap \big(X_1^4, X_{2,1}^3, X_3^2\big)R' \\ & \cap \big(X_{1,1}^4, X_{2,1}^3, X_3^2\big)R' \cap \big(X_1, X_{2,2}^3, X_3^2\big)R' \cap \big(X_1^4, X_2^2, X_{3,1}^5\big)R' \cap \big(X_{1,1}^4, X_2^2, X_{3,1}^5\big)R' \\ & \cap \big(X_{1,2}^3, X_2, X_{3,1}^5\big)R' \cap \big(X_1, X_2^3, X_{3,1}^2\big)R' \cap \big(X_1, X_{2,1}^3, X_{3,1}^2\big)R' \cap \big(X_1^4, X_2^2, X_{3,2}^5\big)R' \\ & \cap \big(X_{1,1}^4, X_2^2, X_{3,2}^5\big)R' \cap \big(X_{1,2}^3, X_2, X_{3,2}^5\big)R'. \end{aligned}$$

Definition 4.1.5. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω. Consider the ideal $\mathfrak{m}^{[\underline{a}(\lambda)]} = \big(X_1^{a_1}, \ldots, X_d^{a_d}\big)R$, where for $i = 1, \ldots, d$, $a_i = \sum_{k=0}^{r} e_{i,k}$ with

$$e_{i,k} = \begin{cases} \lambda(v_i v_{i,1}) & \text{if } k = 0, \\ \max\{\lambda(v_{i,k-1}v_{i,k}), \lambda(v_{i,k}v_{i,k+1})\} & \text{if } k = 1, \ldots, r-1, \\ \lambda(v_{i,r-1}v_{i,r}) & \text{if } k = r. \end{cases}$$

In words, $\mathfrak{m}^{\underline{a}(\lambda)}$ is the monomial ideal of R obtained from the monomial ideal $(g_1, \ldots, g_d)R'$ by setting $\mathfrak{m}^{\underline{a}(\lambda)} = \big(p(g_1), \ldots, p(g_d)\big)R$, where g_i is the corresponding generator in $I_r\big((\Sigma_r G)_\lambda\big)$ of the r-whisker $v_i v_{i,1} \ldots v_{i,r}$ from $(\Sigma_r G)_\lambda$ for $i = 1, \ldots, d$.

Example 4.1.6. In Example 4.1.4, $\mathfrak{m}^{[\underline{a}(\lambda)]} = \big(X_1^{a_1}, X_2^{a_2}, X_3^{a_3}\big)R$ with

$$
\begin{aligned}
a_1 &= \sum_{k=0}^{2} e_{1,k} = \lambda(v_1 v_{1,1}) + \max\{\lambda(v_1 v_{1,1}), \lambda(v_{1,1} v_{1,2})\} + \lambda(v_{1,1} v_{1,2}) = 4 + 4 + 3 = 11,\\
a_2 &= \sum_{k=0}^{2} e_{2,k} = \lambda(v_2 v_{2,1}) + \max\{\lambda(v_2 v_{2,1}), \lambda(v_{2,1} v_{2,2})\} + \lambda(v_{2,1} v_{2,2}) = 3 + 3 + 3 = 9,\\
a_3 &= \sum_{k=0}^{2} e_{3,k} = \lambda(v_3 v_{3,1}) + \max\{\lambda(v_3 v_{3,1}), \lambda(v_{3,1} v_{3,2})\} + \lambda(v_{3,1} v_{3,2}) = 2 + 5 + 5 = 12.
\end{aligned}
$$

Fact 4.1.7. It is straightforward to show that

$$I_r\big((\Sigma_r G)_\lambda\big)R = I_r\big((\Sigma_{r-1} G)_{\lambda'}\big)R + \mathfrak{m}^{[\underline{a}(\lambda)]}, \text{ where } \lambda' = \lambda|_{\Sigma_{r-1} G}.$$

Because of the following fact, the main result of this chapter gives a formula to compute $\mathrm{r}_R\big(R/I_r(G_\omega)\big)$ for all trees such that $R/I_r(G_\omega)$ is Cohen-Macaulay.

Fact 4.1.8. [7, Proposition 3.7 and Theorem 3.11] Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_i v_j) \leq \min\{\lambda(v_i, v_{i,1}), \lambda(v_j, v_{j,1})\}$ for all edges $v_i v_j \in E$.

(a) $R'/I_r\big((\Sigma_r G)_\lambda\big)$ is Cohen-Macaulay.

(b) If $\Gamma_{\lambda'}$ is a weighted tree and $R/I_r(\Gamma_{\lambda'})$ is Cohen-Macaulay, then there exists a weighted tree $H_{\omega'}$ such that $(\Sigma_r H)_{\lambda''}$ is obtained by pruning a sequence of r-pathless leaves from $\Gamma_{\lambda'}$ with $\lambda'' = \lambda'|_{\Sigma_r H}$ and the weight function λ' satisfies the above condition.

4.2 Type

Definition 4.2.1. Let $(\Sigma_{r-1} G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω. We define $q : V\big((\Sigma_{r-1} G)_\lambda\big) \to V(G)$ as $q(v_{i,j}) = v_i$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_{r-1} G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. Then

$$q(V'') = \{v_i \mid \exists\ v_{i,j} \in V''\}.$$

Set

$$\mathrm{WCA}_i(\mathfrak{P}) = \{v_{i,j} \in V'' \mid \delta''(v_{i,j}) \leq \lambda(v_{i,j}v) \text{ for some edge } v_{i,j}v \text{ in } (\Sigma_{r-1}G)_\lambda\},\ \forall\, i = 1, \ldots, d,$$

and

$$h_{i,k} = \max\bigl\{\lambda(v_{i,k}v) \mid v_{i,k}v \in E\bigl((\Sigma_{r-1}G)_\lambda\bigr)\bigr\},\ \forall\, i = 1, \ldots, d,\ k = 0, \ldots, r-1.$$

Define

$$\begin{aligned}\gamma_{(V'',\delta'')} : q(V'') &\longrightarrow \mathbb{N} \sqcup \{\infty\}\\ v_i &\longmapsto \begin{cases} \min\Bigl\{\delta''(v_{i,j}) + \sum_{k=0}^{j-1} h_{i,k} \Bigm| v_{i,j} \in \mathrm{WCA}_i(\mathfrak{P})\Bigr\} & \text{if } \mathrm{WCA}_i(\mathfrak{P}) \neq \emptyset,\\ \infty & \text{otherwise.}\end{cases}\end{aligned}$$

Proposition 4.2.2. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\bigl((\Sigma_{r-1}G)_\lambda\bigr)$ and $\delta'' : V'' \to \mathbb{N}$. Assume that $\mathrm{WCA}_i(\mathfrak{P}) \neq \emptyset$ for some $i \in \{1, \ldots, d\}$. If $v_{i,j_1}, v_{i,j_2} \in \mathrm{WCA}_i(\mathfrak{P})$ with $j_1 < j_2$, then

$$\delta''(v_{i,j_1}) + \sum_{k=0}^{j_1-1} h_{i,k} < \delta''(v_{i,j_2}) + \sum_{k=0}^{j_2-1} h_{i,k}.$$

So we have that

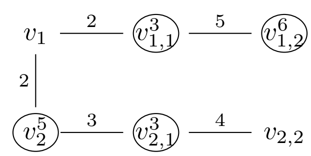
$$\gamma_{(V'',\delta'')}(v_i) = \delta''(v_{i,j_0}) + \sum_{k=0}^{j_0-1} h_{i,k},\ \text{ where } j_0 := \min\{j \mid v_{i,j} \in \mathrm{WCA}_i(\mathfrak{P})\}.$$

Proof. Suppose that $\delta''(v_{i,j_1}) + \sum_{k=0}^{j_1-1} h_{i,k} \geq \delta''(v_{i,j_2}) + \sum_{k=0}^{j_2-1} h_{i,k}$. Then we have that $\delta''(v_{i,j_1}) \geq \delta''(v_{i,j_2}) + \sum_{k=j_1}^{j_2-1} h_{i,k}$. So we have that

$$h_{i,j_1} < \delta''(v_{i,j_2}) + h_{i,j_1} \leq \delta''(v_{i,j_2}) + \sum_{k=j_1}^{j_2-1} h_{i,k} \leq \delta''(v_{i,j_1}),\ \text{ i.e., } h_{i,j_1} < \delta''(v_{i,j_1}).$$

Hence we get that $\delta''(v_{i,j_1}) > h_{i,j_1} = \{\lambda(v_{i,j_1}v) \mid v_{i,j_1}v \in E((\Sigma_{r-1}G)_\lambda)\}$, contradicting the definition of $\mathrm{WCA}_i(\mathfrak{P})$ and that $v_{i,j_1} \in \mathrm{WCA}_i(\mathfrak{P})$. $\square$

Example 4.2.3. A weighted 2-path suspension $(\Sigma_2 P_1)_\lambda$ of $G_\omega := (P_1)_\omega = \bigl(\ v_1 \xrightarrow{\ 2\ } v_2\ \bigr)$ with a weighted 3-path vertex cover $\mathfrak{P} := (V'', \delta'')$ of $(\Sigma_2 P_1)_\lambda$ is given in the following sketch:



Since $I_3\big((\Sigma_2 P_1)_\lambda\big) = \big(X_{1,2}^5 X_{1,1}^5 X_1^2 X_2^2, X_{1,1}^2 X_1^2 X_2^3 X_{2,1}^3, X_1^2 X_2^3 X_{2,1}^4 X_{2,2}^4\big)R'$, we have

$$I_3\big((\Sigma_2 P_1)_\lambda\big)R = \big(X_1^{12} X_2^2, X_1^4 X_2^6, X_1^2 X_2^{11}\big)R.$$

Note that $q(V'') = \{v_1, v_2\}$. Since $\delta''(v_{1,1}) = 3 < 5 = \lambda(v_{1,1}v_{1,2})$ and $\delta''(v_{1,2}) = 6 > 5 = \lambda(v_{1,1}v_{1,2})$, we have that $\mathrm{WCA}_1(\mathfrak{P}) = \{v_{1,1}\}$. Similarly, we have that $\mathrm{WCA}_2(\mathfrak{P}) = \{v_{2,1}\}$, and so

$$\gamma_{(V'',\delta'')}(v_1) = \delta''(v_{1,1}) + \sum_{k=0}^{1-1} h_{1,k} = \delta''(v_{1,1}) + \max\{\lambda(v_1v_2), \lambda(v_1v_{1,1})\} = 3 + \max\{2, 2\} = 5,$$
$$\gamma_{(V'',\delta'')}(v_2) = \delta''(v_{2,1}) + \sum_{k=0}^{1-1} h_{2,k} = \delta''(v_{2,1}) + \max\{\lambda(v_1v_2), \lambda(v_2v_{2,1})\} = 3 + \max\{2, 3\} = 6.$$

Therefore, $P\big(V'', \gamma_{(V'',\delta'')}\big) = \big(X_1^5, X_2^6\big)R \supseteq \big(X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11}\big)R = I_3\big((\Sigma_2 P_1)_\lambda\big)R$.

The following theorem is a key for decomposing $I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R$ with $\lambda' = \lambda|_{\Sigma_{r-1}G}$ and hence $I_r\big((\Sigma_r G)_\lambda\big)R$. The proof is somewhat technical. The reader may wish to follow the argument with Example 4.2.3 as a motivating example.

Theorem 4.2.4. *Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E$. Let $\mathfrak{P} := (V'', \delta'')$ be such that $V'' \subseteq V\big((\Sigma_{r-1}G)_\lambda\big)$ and $\delta'' : V'' \to \mathbb{N}$. Then $I_r\big((\Sigma_{r-1}G)_\lambda\big)R \subseteq P\big(q(V''), \gamma_{(V'',\delta'')}\big)$ if and only if (V'', δ'') is a weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$.*

Proof. $\Longrightarrow$ Assume that $I_r\big((\Sigma_{r-1}G)_\lambda\big)R \subseteq P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Let $P_r := v_{p_1,q_1} \cdots v_{p_{r+1},q_{r+1}}$ be an r-path in $(\Sigma_{r-1}G)_\lambda$. Set

$$e_{p_k,q_k} = \begin{cases} \lambda(v_{p_1,q_1}v_{p_2,q_2}) & \text{if } k = 1, \\ \max\{\lambda(v_{p_{k-1},q_{k-1}}v_{p_k,q_k}), \lambda(v_{p_k,q_k}v_{p_{k+1},q_{k+1}})\} & \text{if } k = 2, \dots, r, \\ \lambda(v_{p_r,q_r}v_{p_{r+1},q_{r+1}}) & \text{if } k = r+1. \end{cases}$$

Then $X_{p_1}^{e_{p_1,q_1}} \cdots X_{p_{r+1}}^{e_{p_{r+1},q_{r+1}}} \in [\![I_r((\Sigma_{r-1}G)_\lambda)R]\!] \subseteq [\![P(q(V''), \gamma_{(V'',\delta'')})]\!]$. So we have that

$$X_{i_0}^{\gamma_{(V'',\delta'')}(v_{i_0})} \mid X_{p_1}^{e_{p_1,q_1}} \cdots X_{p_{r+1}}^{e_{p_{r+1},q_{r+1}}} \text{ for some } v_{i_0} \in q(V'').$$

Hence we have that $v_{i_0} = v_{p_l}$ for some $l \in \{1, \ldots, r+1\}$ and

$$\min_{v_{i_0,j} \in \mathrm{WCA}_{i_0}(\mathfrak{P})} \left\{ \delta''(v_{i_0,j}) + \sum_{k=0}^{j-1} h_{i_0,k} \right\} = \gamma_{(V'',\delta'')}(v_{i_0}) \leq \sum_{k=0}^{r+1} \mathbb{1}_k e_{p_k,q_k}, \text{ where } \mathbb{1}_k = \begin{cases} 1 & \text{if } p_k = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

So we have that $v_{p_l} = v_{i_0} \in q(V'')$. Since P_r is an r-path in $\Sigma_{r-1}G$, we have that P_r is of the following form.

$$\begin{array}{ccccccc}
v_{p_{1+q_1},0} & & & & & & \\
\| & & & & & & \\
v_{p_1,0} & \text{———} & v_{p_1,1} & \text{———} & \cdots & \text{———} & v_{p_1,q_1} \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
v_{p_{r+1},0} & \text{———} & v_{p_{r+1},1} & \text{———} & \cdots & \text{———} & v_{p_{r+1},q_{r+1}} \\
\| & & & & & & \\
v_{p_{1+r-q_{r+1}},0} & & & & & &
\end{array}$$

where q_1 or q_{r+1} may be 0. Let $M_0 := \max_{1 \leq k \leq r+1}\{q_k \mid i_0 = p_k\}$. Then we have that

$$M_0 = \begin{cases} q_1 & \text{if } i_0 = p_1, \\ q_{r+1} & \text{if } i_0 = p_{r+1}. \end{cases}$$

Since $\gamma_{(V'',\delta'')}(v_i) < \infty$, we have that $\mathrm{WCA}_{i_0}(\mathfrak{P}) \neq \emptyset$. Set $j_0 := \min\{j \mid v_{i_0,j} \in \mathrm{WCA}_{i_0}(\mathfrak{P})\}$. Then by Proposition 4.2.2, we have that

$$\delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} h_{i_0,k} = \min_{v_{i_0,j} \in V''} \left\{ \delta''(v_{i_0,j}) + \sum_{k=0}^{j-1} h_{i_0,k} \right\} \leq \sum_{k=0}^{r+1} \mathbb{1}_k \cdot e_{p_k,q_k} = \sum_{k=0}^{M_0} e_{i_0,k}. \tag{4.2.4.1}$$

Suppose that $j_0 > M_0$. Then since $e_{i_0,k} \leq h_{i_0,k}$ for $k = 0, \ldots, M_0$, by Inequality (4.2.4.1), we have that

$$\delta''(v_{i_0,j_0}) + \sum_{k=0}^{M_0} e_{i_0,k} \leq \delta''(v_{i_0,j_0}) + \sum_{k=0}^{M_0} h_{i_0,k} \leq \delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} h_{i_0,k} \leq \sum_{k=0}^{M_0} e_{i_0,k}, \text{ i.e., } \delta''(v_{i_0,j_0}) \leq 0,$$

contradicting $\delta''(v_{i_0,j_0}) \geq 1$ by the definition of δ''. So $j_0 \leq M_0$ and there must exist a sub-path of P_r of the form

$$v_{i_0,0} \text{ ------ } v_{i_0,1} \text{ ------ } \cdots \text{ ------ } v_{i_0,M_0}.$$

Since $0 \leq j_0 \leq M_0$, there exists a vertex in this path of the form $v_{i_0,j_0} = v_{p_k,q_k}$ for some k in $\{1,\ldots,r+1\}$. So $v_{p_k,q_k} = v_{i_0,j_0} \in \mathrm{WCA}_i(\mathfrak{P}) \subseteq V''$.

(a) Assume that $0 = j_0 < M_0$. Since $\lambda(v_iv_j) \leq \min\{\lambda(v_i, v_{i,1}), \lambda(v_j, v_{j,1})\}$ for all edges $v_iv_j \in E$ and $M_0 \geq 1$, we have that $e_{i_0,0} = \lambda(v_{i_0,0}v_{i_0,1}) = h_{i_0,0}$. Since $v_{i_0,j_0} \in \mathrm{WCA}_{i_0}(\mathfrak{P})$, we have that $\delta''(v_{i_0,0}) \leq h_{i_0,0} = e_{i_0,0}$.

(b) Assume that $0 < j_0 < M_0$. Since $v_{i_0,j_0} \in \mathrm{WCA}_{i_0}(\mathfrak{P})$, we have that v_{i_0,j_0} weighted-covers the edge $v_{i_0,j_0-1}v_{i_0,j_0}$ or $v_{i_0,j_0}v_{i_0,j_0+1}$, i.e., $\delta''(v_{i_0,j_0}) \leq \max\{\lambda(v_{i_0,j_0-1}v_{i_0,j_0}), \lambda(v_{i_0,j_0}v_{i_0,j_0+1})\} = e_{i_0,j_0}$.

(c) Assume that $j_0 = M_0$. Since $e_{i_0,k} \leq h_{i_0,k}$ for $k = 0,\ldots,j_0-1$, by Inequality (4.2.4.1), we have

$$\delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} e_{i_0,k} \leq \delta''(v_{i_0,j_0}) + \sum_{k=0}^{j_0-1} h_{i_0,k} \leq \sum_{k=0}^{M_0} e_{i_0,k} = \sum_{k=0}^{j_0} e_{i_0,k}, \text{ i.e., } \delta''(v_{i_0,j_0}) \leq e_{i_0,j_0}.$$

So v_{i_0,j_0} weighted-covers P_r. Thus, V'' is a weighted r-path vertex cover of $\Sigma_{r-1}G$.

$\Longleftarrow$ Assume that (V'', δ'') is a weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$. We need to show that every monomial generator of $I_r\big((\Sigma_{r-1}G)_\lambda\big)R$ is in $P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. We let $\underline{X}^{\underline{b}} := X_{i_1}^{e_{i_1,j_1}} \ldots X_{i_{r+1}}^{e_{i_{r+1},j_{r+1}}}$ be such a generator corresponding to an r-path $P_r := v_{i_1,j_1} \cdots v_{i_{r+1},j_{r+1}}$ in $(\Sigma_{r-1}G)_\lambda$. We need to show that $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. Note that $X_{i_1,j_1}^{e_{i_1,j_1}} \cdots X_{i_{r+1},j_{r+1}}^{e_{i_{r+1},j_{r+1}}}$ is of the following form. We replace each vertex in P_r with the corresponding variable and its exponent.

$$\begin{array}{ccccccc}
X_{i_{1+j_1},0}^{e_{i_{1+j_1},0}} & & & & & & \\
\| & & & & & & \\
X_{i_1,0}^{e_{i_1,0}} & \text{------} & X_{i_1,1}^{e_{i_1,1}} & \text{------} & \cdots & \text{------} & X_{i_1,j_1}^{e_{i_1,j_1}} \\
| & & & & & & \\
\vdots & & & & & & \\
| & & & & & & \\
X_{i_{r+1},0}^{e_{i_{r+1},0}} & \text{------} & X_{i_{r+1},1}^{e_{i_{r+1},1}} & \text{------} & \cdots & \text{------} & X_{i_{r+1},j_{r+1}}^{e_{i_{r+1},j_{r+1}}} \\
\| & & & & & & \\
X_{i_{1+r-j_{r+1}},0}^{e_{i_{1+r-j_{r+1}},0}} & & & & & &
\end{array}$$

where j_1 or j_{r+1} may be 0. Since P_r is an r-path in $(\Sigma_{r-1}G)_\lambda$ and (V'', δ'') is a weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, we have that v_{i_l,j_l} weighted-covers the r-path P_r for some $l \in \{1, \ldots, r+1\}$. So $v_{i_l,j_l} \in \mathrm{WCA}_{i_l}(\mathfrak{P})$ and then

$$\gamma_{(V'',\delta'')}(v_{i_l}) = \min_{v_{i_l,t} \in \mathrm{WCA}_{i_l}(\mathfrak{P})} \left\{ \delta''(v_{i_l,t}) + \sum_{k=0}^{t-1} h_{i_l,k} \right\} \leq \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k}.$$

Let $M_0 := \max_{1 \leq k \leq r+1}\{j_k \mid i_l = i_k\}$. Then $j_l \leq M_0$. Since v_{i_l,j_l} weighted-covers the r-path P_r, $\delta''(v_{i_l,j_l}) \leq e_{i_l,j_l}$. So

$$\delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq e_{i_l,j_l} + \sum_{k=0}^{j_l-1} e_{i_l,k} = \sum_{k=0}^{j_l} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l.k}, \text{ i.e., } \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k}.$$

(a) Assume that $j_l = 0$. Then

$$\gamma_{(V'',\delta'')}(v_{i_l}) \leq \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k} = \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k} = \sum_{k=0}^{r+1} \mathbb{1}_{l,k} e_{i_k,j_k} = b_{i_l},$$

where $\mathbb{1}_{l,k} = \begin{cases} 1 & \text{if } i_k = i_l \\ 0 & \text{otherwise} \end{cases}, \ \forall\ k = 1, \ldots, r+1.$

(b) Assume that $j_l > 0$. Then $M_0 \geq 1$. Since $\lambda(v_i v_j) \leq \min\{\lambda(v_i, v_{i,1}), \lambda(v_j, v_{j,1})\}$ for all edges $v_i v_j \in E$, we have that $e_{i_0,0} = \lambda(v_{i_0,0} v_{i_0,1}) = h_{i_0,0}$. Also, since $e_{i_0,k} = h_{i_0,k}$ for $k = 1, \ldots, j_l - 1$, we have that

$$\gamma_{(V'',\delta'')}(v_{i_l}) \leq \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} h_{i_l,k} = \delta''(v_{i_l,j_l}) + \sum_{k=0}^{j_l-1} e_{i_l,k} \leq \sum_{k=0}^{M_0} e_{i_l,k} = \sum_{k=0}^{r+1} \mathbb{1}_{l,k} e_{i_k,j_k} = b_{i_l}.$$

So $X_{i_l}^{\gamma_{(V'',\delta'')}(v_{i_l})} \mid \underline{X}^{\underline{b}}$. Thus, $\underline{X}^{\underline{b}} \in P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. □

Proposition 4.2.5. Let $(\Sigma_{r-1}G)_\lambda$ be a weighted $(r-1)$-path suspension of G_ω such that $\lambda(v_i v_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_i v_j) \leq \lambda(v_j, v_{j,1})$ for all $v_i v_j \in E$. The monomial ideal $I_r\big((\Sigma_{r-1}G)_\lambda\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V''), \gamma_{(V'',\delta'')}\big)$ with $V'' \subseteq V(\Sigma_{r-1}G)$ and $\delta'' : V'' \to \mathbb{N}$.

Proof. Fact 2.2.38 gives a decomposition of $I_r\big((\Sigma_{r-1}G)_\lambda\big)R$. Let $J := \big(X_{b_1}^{\beta_{b_1}}, \ldots, X_{b_s}^{\beta_{b_s}}\big)R$ occur in the decomposition. Without loss of generality, assume that $b_1, \ldots, b_s \in \mathbb{N}$ are distinct, and let

$k \in \{1, \ldots, s\}$. By Fact 2.2.38, there exists a generator $p(X_{i_1,j_1} \ldots X_{i_{r+1},j_{r+1}})$ with $v_{i_1,j_1} \ldots v_{i_{r+1},j_{r+1}}$ an r-path in $(\Sigma_{r-1}G)_\lambda$ such that for some $c(k) \in \{1, \ldots, r+1\}$, we have that $i_{c(k)} = b_k$ and

$$\beta_{b_k} = \begin{cases} e_{b_k,0} & \text{if } M_k = 0, \\ \lambda(v_{b_k,M_k} v_{b_k,M_k-1}) + \sum_{l=0}^{M_k-1} h_{b_k,l} & \text{if } M_k \geq 1, \end{cases}$$

where $M_k := \max_{1 \leq n \leq r+1}\{j_n \mid b_k = i_n\} \leq r-1$ and

$$e_{i_m,j_m} = \begin{cases} \lambda(v_{i_1,j_1} v_{i_2,j_2}) & \text{if } m = 1, \\ \max\{\lambda(v_{i_{m-1},j_{m-1}} v_{i_m,j_m}), \lambda(v_{i_m,j_m} v_{i_{m+1},j_{m+1}})\} & \text{if } m = 2, \ldots, r, \\ \lambda(v_{i_r,j_r} v_{i_{r+1},j_{r+1}}) & \text{if } m = r+1. \end{cases}$$

We repeat the process for each $k \in \{1, \ldots, s\}$ and set $V'' = \{v_{b_1,M_1}, \ldots, v_{b_s,M_s}\}$. Then $q(V'') = \{v_{b_1}, \ldots, v_{b_s}\}$. Define

$$\begin{aligned} \delta'' : V'' &\longrightarrow \mathbb{N} \\ v_{b_k,M_k} &\longmapsto \begin{cases} \lambda(v_{b_k,M_k} v_{b_k,M_k-1}) & \text{if } M_k \geq 1 \\ \beta_{b_k} (= e_{b_k,0}) & \text{if } M_k = 0 \end{cases}, \ \forall\, k = 1, \ldots, s. \end{aligned}$$

We claim $J = P\big(q(V''), \gamma_{(V'',\delta'')}\big)$. It is enough to show $\gamma_{(V'',\delta'')}(v_{b_k}) = \beta_{b_k}$ for $k = 1, \ldots, s$. Let $\mathfrak{P} := (V'', \delta'')$. Since $|V''| = |q(V'')|$, we have $|\mathrm{WCA}_k(\mathfrak{P})| \leq 1$ for $k = 1, \ldots, s$. There are two cases:

(a) Assume that $M_k \geq 1$. Since $v_{b_k,M_k} \in V''$ and $\delta''(v_{b_k,M_k}) = \lambda(v_{b_k,M_k} v_{b_k,M_k-1})$, we have that $v_{b_k,M_k} \in \mathrm{WCA}_{b_k}(\mathfrak{P})$. Therefore, we have that $\mathrm{WCA}_{b_k}(\mathfrak{P}) = \{v_{b_k,M_k}\}$, and hence

$$\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,M_k}) + \sum_{l=0}^{M_k-1} h_{b_k,l} = \lambda(v_{b_k,M_k} v_{b_k,M_k-1}) + \sum_{l=0}^{M_k-1} h_{b_k,l} = \beta_{b_k}.$$

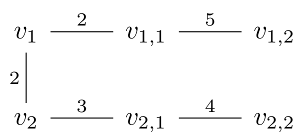
(b) Assume that $M_k = 0$. Then $j_{c(k)} \in \{0, \ldots, M_k\} = \{0\}$ and so $j_{c(k)} = 0$. Then

$$e_{i_{c(k)},j_{c(k)}} = \begin{cases} \lambda(v_{i_1,j_1} v_{i_2,j_2}) & \text{if } c(k) = 1, \\ \lambda(v_{i_r,j_r} v_{i_{r+1},j_{r+1}}) & \text{if } c(k) = r+1, \\ \max\big\{\lambda(v_{i_{c(k)-1},j_{c(k)-1}} v_{i_{c(k)},j_{c(k)}}), \lambda(v_{i_{c(k)},j_{c(k)}} v_{i_{c(k)+1},j_{c(k)+1}})\big\} & \text{if } 2 \leq c(k) \leq r-1 \end{cases}$$

$\delta''(v_{i_{c(k)},j_{c(k)}}) = \delta''(v_{b_k,0}) = \beta_{b_k} = e_{b_k,0} = e_{i_{c(k)},j_{c(k)}}$, and $v_{i_{c(k)},M_k} = v_{b_k,0} = v_{b_k,M_k} \in V''$. So $v_{b_k,0} \in \mathrm{WCA}_{b_k}(\mathfrak{P})$ and thus $\mathrm{WCA}_{b_k}(\mathfrak{P}) = \{v_{b_k,0}\}$. Hence

$$\gamma_{(V'',\delta'')}(v_{b_k}) = \delta''(v_{b_k,0}) + \sum_{l=0}^{0-1} h_{b_k,l} = \delta''(v_{b_k,0}) = \beta_{b_k}. \qquad \square$$

Example 4.2.6. Consider the following graph $(\Sigma_2 P_1)_\lambda$ from Example 4.2.3.



By Example 4.2.3, $I_3\big((\Sigma_2 P_1)\big)R = (X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11})R$. By Fact 2.2.38,

$$\begin{aligned} I_3\big((\Sigma_2P_1)\big)R &= \big(X_1^{12}, X_1^4, X_1^2\big)R \cap \big(X_1^{12}, X_1^4, X_2^{11}\big)R \cap \big(X_1^{12}, X_2^6, X_1^2\big)R \cap \big(X_1^{12}, X_2^6, X_2^{11}\big)R \\ &\quad \cap \big(X_2^2, X_1^4, X_1^2\big)R \cap \big(X_2^2, X_1^4, X_2^{11}\big)R \cap \big(X_2^2, X_2^6, X_1^2\big)R \cap \big(X_2^2, X_2^6, X_2^{11}\big)R \\ &= \big(X_1^2\big)R \cap \big(X_1^4, X_2^{11}\big)R \cap \big(X_1^2, X_2^6\big)R \cap \big(X_1^{12}, X_2^6\big)R \\ &\quad \cap \big(X_1^2, X_2^2\big)R \cap \big(X_1^4, X_2^2\big)R \cap \big(X_1^2, X_2^2\big)R \cap \big(X_2^2\big)R. \end{aligned}$$

Let $J_1 = (X_1^2)R$. Then $b_1 = 1$ and $\beta_{b_1} = \beta_1 = 2$. Consider the generator $X_{1,0}^{e_{1,0}}X_{2,0}^{e_{2,0}}X_{2,1}^{e_{2,1}}X_{2,2}^{e_{2,2}} := X_1^2X_2^3X_{2,1}^4X_{2,2}^4$ of $I_3\big((\Sigma_2P_1)_\lambda\big)$. Then $M_1 := 0$ and $\beta_1 = e_{1,0} = 2$. Let $V'' = \{v_{1,0}\}$ and $\delta'' : V'' \longrightarrow \mathbb{N}$ be given by $v_1 \longmapsto e_{1,0} = 2$. Since $q(V'') = \{v_1\}$, $\gamma_{(V'',\delta'')}(v_1) = \delta''(v_{1,0}) = 2$. So we have that

$$P\big(q(V''), \gamma_{(V'',\delta'')}\big) = P\big(\{v_1^2\}\big) = \big(X_1^2\big)R = J_1.$$

Let $J_2 = (X_1^4, X_2^{11})R$. Then $b_1 = 1, b_2 = 2$, and $\beta_{b_1} = \beta_1 = 4$ and $\beta_{b_2} = \beta_2 = 11$. Consider the generator $X_{1,1}^{e_{1,1}}X_{1,0}^{e_{1,0}}X_{2,0}^{e_{2,0}}X_{2,1}^{e_{2,1}} := X_{11}^2X_1^2X_2^3X_{2,1}^3$ of $I_3\big((\Sigma_2P_1)_\lambda\big)$. Then $M_1 := 1$ and $\beta_1 = \lambda(v_{1,1}v_{1,0}) + h_{1,0} = 2 + 2 = 4$. Consider the generator $X_{1,0}^{e_{1,0}}X_{2,0}^{e_{2,0}}X_{2,1}^{e_{2,1}}X_{2,2}^{e_{2,2}} := X_1^2X_2^3X_{2,1}^4X_{2,2}^4$ of $I_3\big((\Sigma_2P_1)_\lambda\big)$. Then $M_2 := 2$ and $\beta_2 = \lambda(v_{2,2}v_{2,1}) + h_{2,0} + h_{2,1} = 4+3+4 = 11$. Let $V'' = \{v_{1,1}, v_{2,2}\}$ and $\delta'' : V'' \longrightarrow \mathbb{N}$ be given by $v_1 \longmapsto \lambda(v_{1,1}v_{1,0}) = 2$ and $v_{2,2} \longmapsto \lambda(v_{2,2}v_{2,1}) = 4$. Then $q(V'') = \{v_1, v_2\}$ and so $\gamma_{(V'',\delta'')}(v_1) = \delta''(v_{1,1}) + h_{1,0} = 2 + 2$ and

$$\gamma_{(V'',\delta'')}(v_2) = \delta''(v_{2,2}) + h_{2,0} + h_{2,1} = 4 + 3 + 4 = 11.$$

So we have that

$$P\big(q(V''), \gamma(V'', \delta'')\big) = P\big(\{v_1^4, v_2^{11}\}\big) = \big(X_1^4, X_2^{11}\big)R = J_2.$$

The next result is our first decomposition needed for computing $\mathrm{r}_{R'}\big(R'/I_r((\Sigma_r G)_\lambda)\big)$.

Theorem 4.2.7. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_iv_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_jv_{j,1})$ for all edges $v_iv_j \in E$. We have*

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big), \text{ where } \lambda' = \lambda|_{\Sigma_{r-1}G},$$

and

$$I_r\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]}.$$

Proof. Note that $I_r\big((\Sigma_r G)_\lambda\big)R = I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R + \mathfrak{m}^{[\underline{a}(\lambda)]}$ by Fact 4.1.7. Then it is enough to show that, by[9, Theorem 7.5.3],

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

By Proposition 4.2.5, the monomial ideal $I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R$ can be written as a finite intersection of m-irreducible ideals of the form $P\big(q(V'') := \{v_{i_1}, \ldots, v_{i_t}\}, \gamma_{(V'',\delta'')}\big)$ with $V'' \subseteq V(\Sigma_{r-1}G)$ and $\delta'' : V'' \to \mathbb{N}$. Then by Theorem 4.2.4,

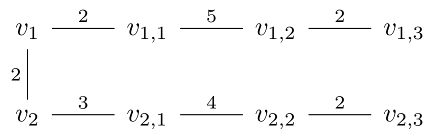
$$\begin{aligned} I_r((\Sigma_{r-1}G)_{\lambda'})R &\subseteq \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) \\ &\subseteq \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'} \text{ in the decomp. of } I_r((\Sigma_{r-1}G)_{\lambda'})R} P\big(q(V''), \gamma_{(V'',\delta'')}\big) \\ &= I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R. \end{aligned}$$

So we have that

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big). \qquad \square$$

Example 4.2.8. Consider the following weighted 3-path suspension $(\Sigma_3 P_1)_\lambda$ of $G_\omega := (P_1)_\omega =$

$\big(v_1 \overset{2}{\text{———}} v_2 \big)$.



Let $\lambda' = \lambda|_{\Sigma_2 P_1}$. Since $I_3\big((\Sigma_2 P_1)_{\lambda'}\big) = (X_{1,2}^5 X_{1,1}^5 X_1^2 X_2^2, X_{1,1}^2 X_1^2 X_2^3 X_{2,1}^3, X_1^2 X_2^3 X_{2,1}^4 X_{2,2}^4)$, by Theorem 4.2.7, we have two infinite intersections:

$$I_3\big((\Sigma_2 P_1)_{\lambda'}\big)R = \big(X_1^{12} X_2^2, X_1^4 X_2^6, X_1^2 X_2^{11}\big)R = \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_2 P_1)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big),$$

and

$$\begin{aligned} I_3\big((\Sigma_3 P_1)_{\lambda'}\big)R &= \big(X_1^{12} X_2^2, X_1^4 X_2^6, X_1^2 X_2^{11}\big)R + (X_1^{14}, X_2^{13})R \\ &= \bigcap_{(V'',\delta'') \text{ w. } r\text{-path v. cover of } (\Sigma_2 P_1)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big). \end{aligned}$$

The next result is key for our second decomposition result, Corollary 4.2.11.

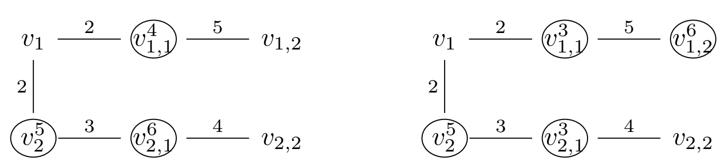
Lemma 4.2.9. Let $\mathfrak{p} := (V_1'', \delta_1''), \mathfrak{P} := (V_2'', \delta_2'')$ be such that $V_1'', V_2'' \subseteq V\big((\Sigma_{r-1} G)_\lambda\big)$ and $\delta_1'', \delta_2'' : V'' \to \mathbb{N}$. If $(V_1'', \delta_1'') \leq (V_2'', \delta_2'')$, then $P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big) \subseteq P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big)$.

Proof. Let $X_i^{\gamma_{(V_1'',\delta_1'')}(v_i)}$ be a generator of $P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big)$. Then $V_1'' \subseteq V_2''$ implies we have that $X_i^{\gamma_{(V_2'',\delta_2'')}(v_i)} \in P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big)$ and

$$\begin{aligned} \gamma_{(V_1'',\delta_1'')}(v_i) &= \min\Big\{ \delta_1''(v_{i_j,t}) + \sum_{k=0}^{t-1} h_{i_j,k} \;\Big|\; v_{i_j,t} \in \mathrm{WCA}_{i_j}(\mathfrak{p}) \Big\} \\ &\geq \min\Big\{ \delta_2''(v_{i_j,t}) + \sum_{k=0}^{t-1} h_{i_j,k} \;\Big|\; v_{i_j,t} \in \mathrm{WCA}_{i_j}(\mathfrak{P}) \Big\} = \gamma_{(V_2'',\delta_2'')}(v_i). \end{aligned}$$

It follows that $X_i^{\gamma_{(V_2'',\delta_2'')}(v_i)} \mid X_i^{\gamma_{(V_1'',\delta_1'')}(v_i)}$, and hence $P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big) \subseteq P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big)$. $\square$

Example 4.2.10. Consider the following two pairs of sets $\mathfrak{p} := (V_1'', \delta_1'') := \{v_{1,1}^4, v_2^5, v_{2,1}^6\}$ and $\mathfrak{P} := (V_2'', \delta_2'') := \{v_{1,1}^3, v_{1,2}^6, v_2^5, v_{2,1}^3\}$ of $(\Sigma_2 P_1)_\lambda$ from Example 4.2.3.



Since $V_1'' \subseteq V_2''$ and $\delta_1'' \geq \delta_2''|_{V_1''}$, we have that $(V_1'', \delta_1'') \leq (V_2'', \delta_2'')$. Similar to Example 4.2.3, we have that $\mathrm{WCA}_1(\mathfrak{p}) = \{v_{1,1}\}$ and $\mathrm{WCA}_2(\mathfrak{p}) = \emptyset$. Therefore, $\gamma_{(V_1'', \delta_1'')}(v_2) = \infty$ and

$$\gamma_{(V_1'', \delta_1'')}(v_1) = \delta_1''(v_{1,1}) + \sum_{k=0}^{1-1} h_{1,k} = \delta_1''(v_{1,1}) + \max\{\lambda(v_1 v_2), \lambda(v_1 v_{1,1})\} = 4 + \max\{2, 2\} = 5.$$

Also, since $q(V_1'') = \{v_1, v_2\}$, we have that $P\big(q(V_1''), \gamma_{(V_1'', \delta_1'')}\big) = (X_1^5, X_2^\infty)R = (X_1^5)R$. Then from Example 4.2.3, we have that

$$P\big(q(V_2''), \gamma_{(V_2'', \delta_2'')}\big) = (X_1^5, X_2^6)R \supseteq (X_1^5)R = P\big(q(V_1''), \gamma_{(V_1'', \delta_1'')}\big).$$

Here is our second decomposition result for computing $\mathrm{r}_{R'}\big(R'/I_r((\Sigma_r G)_\lambda)\big)$.

Corollary 4.2.11. Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_i v_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_i v_j) \leq \lambda(v_j, v_{j,1})$ for all edges $v_i v_j \in E$. We have

$$I_r\big((\Sigma_{r-1} G)_{\lambda'}\big)R = \bigcap_{(V'', \delta'') \text{ min. w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big), \text{ where } \lambda' = \lambda|_{\Sigma_{r-1}G},$$

and

$$I_r\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'', \delta'') \text{ min. w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]}.$$

Proof. By Fact 4.1.7 and [9, Theorem 7.5.3], it is enough to prove that

$$I_r\big((\Sigma_{r-1} G)_{\lambda'}\big)R = \bigcap_{(V'', \delta'') \text{ min. w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big).$$

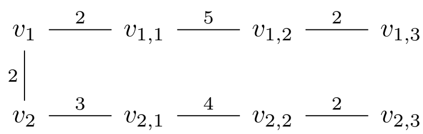
By Theorem 4.2.7, it is enough to show that

$$\begin{aligned} &\bigcap_{(V'', \delta'') \text{ weighted. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big) \\ = &\bigcap_{(V'', \delta'') \text{ min. weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'', \delta'')}\big). \end{aligned}$$

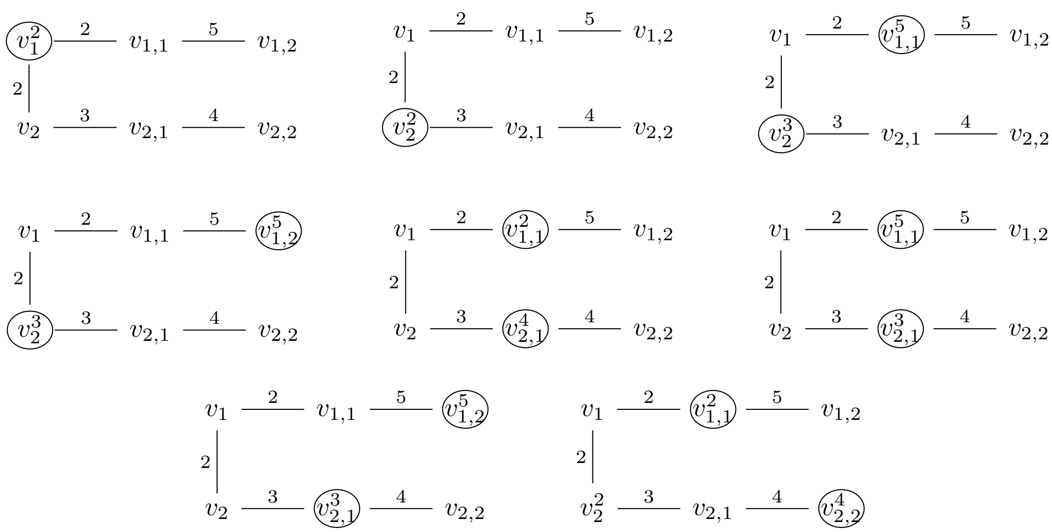
$\subseteq$ follows because every minimal weighted r-path vertex cover is a weighted r-path vertex cover.

$\supseteq$ follows from Fact 3.1.6 and Lemma 4.2.9. $\square$

Example 4.2.12. Consider the following weighted 3-path suspension $(\Sigma_3 P_1)_\lambda$ of $G_\omega := (P_1)_\omega = \left(v_1 \overset{2}{\text{———}} v_2 \right)$.



We depict the minimal weighted 3-path vertex covers of $(\Sigma_2 P_1)_{\lambda'}$ with $\lambda' = \lambda|_{\Sigma_2 P_1}$ in the following sketches:



Since $I_3\big((\Sigma_2 P_1)_{\lambda'}\big) = (X_{1,2}^5 X_{1,1}^5 X_1^2 X_2^2, X_{1,1}^2 X_1^2 X_2^3 X_{2,1}^3, X_1^2 X_2^3 X_{2,1}^4 X_{2,2}^4)$, by Corollary 4.2.11, we have that

$$I_3\big((\Sigma_2 P_1)_{\lambda'}\big)R = \big(X_1^{12}X_2^2, X_1^4 X_2^6, X_1^2 X_2^{11}\big)R = \big(X_1^2\big)R \cap \big(X_2^2\big)R \cap \big(X_1^7, X_2^3\big)R \cap \big(X_1^{12}, X_2^3\big)R$$
$$\cap \big(X_1^4, X_2^7\big) \cap \big(X_1^7, X_2^6\big) \cap \big(X_1^{12}, X_2^6\big)R \cap \big(X_1^4, X_2^{11}\big)R.$$

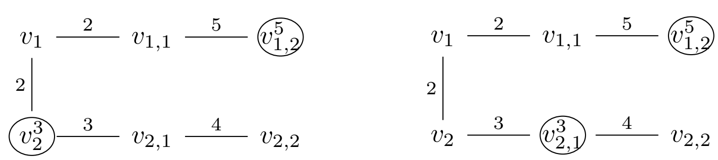
This decomposition is redundant. Thus, the decomposition in Corollary 4.2.11 may be redundant.

In light of the preceding example, we define another order from which we can produce an irredundant decomposition. Lemma 4.2.21 is the key for understanding how this ordering helps with irredundancy.

Definition 4.2.13. Given minimal weighted r-path vertex covers $(V_1'', \delta_1''), (V_2'', \delta_2'')$ of $(\Sigma_{r-1} G)_\lambda$, we write $(V_1'', \delta_1'') \leq_{\wp} (V_2'', \delta_2'')$ if $q(V_1'') \subseteq q(V_2'')$ and $\gamma_{(V_1'', \delta_1'')} \geq \gamma_{(V_2'', \delta_2'')}|_{q(V_1'')}$. A minimal weighted

r-path vertex cover (V'', δ'') is $\wp$-*minimal* if there is not another minimal weighted r-path vertex cover (V''', δ''') such that $(V'', \delta'') <_{\wp} (V''', \delta''')$.

Example 4.2.14. Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,2}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_{2,1}^3\}$ of $(\Sigma_2 P_1)_\lambda$ from Example 4.2.12.



Then $q(V_1'') = \{v_1, v_2\} = q(V_2'')$. Since

$$\gamma_{(V_1'',\delta_1'')}(v_1) = \delta_1''(v_{1,2}) + h_{1,1} + h_{1,0} = 5 + 5 + 2 = \delta_2''(v_{1,2}) + h_{1,1} + h_{1,0} = \gamma_{(V_2'',\delta_2'')}(v_1),$$

and $\gamma_{(V_1'',\delta_1'')}(v_2) = \delta_1''(v_2) = 3 < 3 + 3 = \delta_2''(v_{2,1}) + h_{2,0} = \gamma_{(V_2'',\delta_2'')}(v_2)$, we have that $\gamma_{(V_1'',\delta_1'')} < \gamma_{(V_2'',\delta_2'')}$. So $(V_1'', \delta_1'') >_{\wp} (V_2'', \delta_2'')$. Hence (V_1'', δ_1'') is not $\wp$-minimal.

Lemma 4.2.15. Let $\mathfrak{p} := (W', \delta')$ and $\mathfrak{P} := (W'', \delta'')$ be two minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_{\wp} (W', \delta')$, then $|(W'', \delta'')| = |(W', \delta')|$ and $q(W'') = q(W')$.

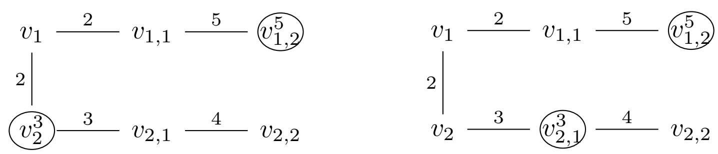
Proof. Since (W', δ') is a minimal weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, for a distinct pair $v_{i_1,j_1}, v_{i_2,j_2} \in W'$, we have that $i_1 \neq i_2$. Also, since $q(W'') \subseteq q(W')$, $|W''| = |q(W'')| \leq |q(W')| = |W'|$. Suppose that $|W''| < |W'|$. Then there exists $v_{i,j} \in W'$ such that $v_i \notin q(W'')$. Since (W', δ') is a minimal weighted r-path vertex cover of $(\Sigma_{r-1}G)_\lambda$, there is an r-path P_r in $(\Sigma_{r-1}G)_\lambda$ that can only be weighted-covered by $v_{i,j}$. By assumption, P_r can be weighted-covered by some $v_{k,l} \in W''$, so $v_k \in q(W'')$. Also, since $v_i \notin q(W'')$, we have that $k \neq i$. Let $\alpha = \min\{b \mid v_{k,b} \in \mathrm{WCA}_k(\mathfrak{p})\}$ and $\beta = \min\{b \mid v_{k,b} \in \mathrm{WCA}_k(\mathfrak{P})\}$. So we have that $\alpha, \beta \leq l$. Since $\gamma_{(W'',\delta'')} \geq \gamma_{(W',\delta')}$, we have that $\alpha \leq \beta$ similar to the proof of Proposition 4.2.2. If $\alpha < l$, then it is straightforward to show that P_r can also be weighted-covered by $v_{k,\alpha} \in W'$, a contradiction. Assume that $\alpha = l$. Then $\alpha = \beta = l$ and so $v_{k,\beta} \in W''$ weighted-cover P_r. Since

$$\delta''(v_{k,\alpha}) + \sum_{b=0}^{\alpha-1} h_{k,b} = \gamma_{(W'',\delta'')}(v_k) \geq \gamma_{(W',\delta')}(v_k) = \delta'(v_{k,\alpha}) + \sum_{b=0}^{\alpha-1} h_{k,b},$$

we have that $\delta'(v_{k,\alpha}) \leq \delta''(v_{k,\alpha})$. So P_r can also be weighted-covered by $v_{k,\alpha} \in W'$, a contradiction.

Hence $|W''| = |W'|$ and thus $|q(W'')| = |q(W')|$. Since $q(W'') \subseteq q(W')$, we have that $q(W'') = q(W')$. $\square$

Example 4.2.16. Consider the minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,2}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_{2,1}^3\}$ of $(\Sigma_2 P_1)_\lambda$ from Example 4.2.18(a).



By Example 4.2.18(a), $(V_1'', \delta_1'') <_\wp (V_2'', \delta_2'')$. Then $|(V_1'', \delta_1'')| = |\{v_{1,2}, v_2\}| = 2 = |\{v_{1,2}, v_{2,1}\}| = |(V_2'', \delta_2'')|$ and $q(V_1'') = \{v_1, v_2\} = q(V_2'')$.

The following theorem can be used as an algorithm to find the set of $\wp$-minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$ from the set of minimal weighted r-path vertex covers.

Theorem 4.2.17. *Let $\mathfrak{p} := (V_1'', \delta_1''), \mathfrak{P} := (V_2'', \delta_2'')$ be two minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$. Then $(V_1'', \delta_1'') \leq_\wp (V_2'', \delta_2'')$ if and only if $q(V_1'') = q(V_2'')$ and for any $v_{i_l} \in q(V_1'')$: $j_{1,l} > j_{2,l}$ or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l, j_{1,l}}) \geq \delta_2''(v_{i_l, j_{2,l}})$ with $j_{1,l} := \{j \mid v_{i_l, j} \in V_1''\}$ and $j_{2,l} = \{j \mid v_{i_l, j} \in V_2''\}$.*

Proof. By Lemma 4.2.15, $(V_1'', \delta_1'') \leq_\wp (V_2'', \delta_2'')$ if and only if $q(V_1'') = q(V_2'')$ and $\gamma_{(V_1'', \delta_1'')}|_{q(V_1'')} \geq \gamma_{(V_2'', \delta_2'')}|_{q(V_1'')}$ if and only if $q(V_1'') = q(V_2'')$ and for any $v_{i_l} \in q(V_1'')$, $\gamma_{(V_1'', \delta_1'')}(v_{i_l}) \geq \gamma_{(V_2'', \delta_2'')}(v_{i_l})$ if and only if $q(V_1'') = q(V_2'')$ and for any $v_{i_l} \in q(V_1'')$, $\delta_1''(v_{i_l, j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k} \geq \delta_2''(v_{i_l, j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}$ by Proposition 4.2.2. We claim that for $v_{i_l} \in q(V_1'')$, $\delta_1''(v_{i_l, j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k} \geq \delta_2''(v_{i_l, j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}$ if and only if $j_{1,l} > j_{2,l}$, or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l, j_{1,l}}) \geq \delta_2''(v_{i_l, j_{2,l}})$.

$\Longleftarrow$ Assume that $j_{1,l} > j_{2,l}$, or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l, j_{1,l}}) \geq \delta_2''(v_{i_l, j_{2,l}})$. Then

$$\alpha := \left(\delta_1''(v_{i_l, j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k}\right) - \left(\delta_2''(v_{i_l, j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}\right) = \delta_1''(v_{i_l, j_{1,l}}) - \delta_2''(v_{i_l, j_{2,l}}) + \sum_{k=j_{2,l}}^{j_{1,l}-1} h_{i_l,k}.$$

To prove our statement, it is equivalent to show that $\alpha \geq 0$.

(a) If $j_{1,l} > j_{2,l}$, then $\alpha \geq \delta_1''(v_{i_l, j_{1,l}}) - \delta_2''(v_{i_l, j_{2,l}}) + h_{i_l, j_{2,l}} > h_{i_l, j_{2,l}} - \delta_2''(v_{i_l, j_{2,l}}) \geq 0$.

(b) If $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l, j_{1,l}}) \geq \delta_2''(v_{i_l, j_{2,l}})$, then $\alpha = \delta_1''(v_{i_l, j_{1,l}}) - \delta_2''(v_{i_l, j_{2,l}}) \geq 0$.

$\Longrightarrow$ Suppose that $j_{1,l} < j_{2,l}$, or $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) < \delta_2''(v_{i_l,j_{2,l}})$. Then

$$\alpha := \left(\delta_1''(v_{i_l,j_{1,l}}) + \sum_{k=0}^{j_{1,l}-1} h_{i_l,k}\right) - \left(\delta_2''(v_{i_l,j_{2,l}}) + \sum_{k=0}^{j_{2,l}-1} h_{i_l,k}\right) = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) - \sum_{k=j_{1,l}}^{j_{2,l}-1} h_{i_l,k}.$$

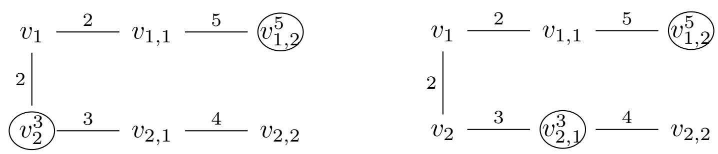
To prove our statement, it is equivalent to show that $\alpha < 0$.

(a) If $j_{1,l} = j_{2,l}$ and $\delta_1''(v_{i_l,j_{1,l}}) < \delta_2''(v_{i_l,j_{2,l}})$, then $\alpha = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) < 0$.

(b) If $j_{1,l} < j_{2,l}$, since $v_{i_l,j_{1,l}} \in V_1''$ and V_1'' is a minimal weighted r-path vertex cover, we have $\delta_1''(v_{i_l}, j_{1,l}) \leq h_{i_l,j_{1,l}}$, so $\alpha = \delta_1''(v_{i_l,j_{1,l}}) - \delta_2''(v_{i_l,j_{2,l}}) - \sum_{k=j_{1,l}}^{j_{2,l}-1} h_{i_l,k} < \delta_1''(v_{i_l,j_{1,l}}) - h_{i_l,j_{1,l}} \leq 0$. $\square$

Example 4.2.18. We have the following examples:

(a) Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,2}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_{2,1}^3\}$ of $(\Sigma_2 P_1)_\lambda$ from Example 4.2.14.



Then $q(V_1'') = \{v_{i_1} := v_1, v_{i_2} := v_2\} = q(V_2'')$. Note that

$$j_{1,1} = \min\{j \mid v_{i_1,j} \in V_1''\} = \min\{j \mid v_{1,j} \in V_1''\} = 2,$$

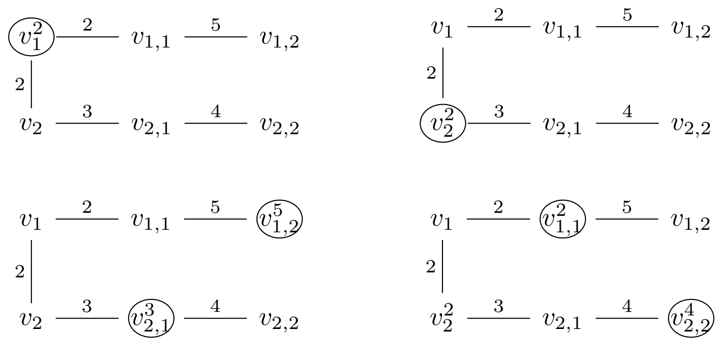
$$j_{1,2} = \min\{j \mid v_{i_2,j} \in V_1''\} = \min\{j \mid v_{2,j} \in V_1''\} = 0,$$

$$j_{2,1} = \min\{j \mid v_{i_1,j} \in V_2''\} = \min\{j \mid v_{2,j} \in V_2''\} = 2,$$

$$j_{2,2} = \min\{j \mid v_{i_2,j} \in V_2''\} = \min\{j \mid v_{2,j} \in V_2''\} = 1.$$

Since $j_{1,1} = 2 = j_{2,1}$ and $\delta_1''(v_{1,j_{1,1}}) = \delta_1''(v_{1,2}) = 5 = \delta_2''(v_{1,2}) = \delta_2''(v_{1,j_{2,1}})$, and $j_{1,2} = 0 < 1 = j_{2,2}$, we have that $(V_1'', \delta_1'') <_{\wp} (V_2'', \delta_2'')$ by Theorem 4.2.17.

(b) Consider all the minimal weighted 3-path vertex covers of $(\Sigma_2 P_1)_{\lambda'}$ from Example 4.2.12. Applying Theorem 4.2.17 repeatedly, we get all the $\wp$-minimal weighted 3-path vertex covers depicted in the following.

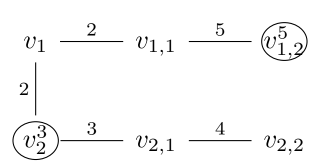


The next two results are key for our third and final decomposition result.

Proposition 4.2.19. For every minimal weighted r-path vertex cover $\mathfrak{p} := (W', \delta')$ of $(\Sigma_{r-1}G)_\lambda$, there is a $\wp$-minimal weighted r-path vertex cover (W'', δ'') of $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_\wp (W', \delta')$.

Proof. If (W', δ') is itself a $\wp$-minimal weighted r-path vertex cover for $(\Sigma_{r-1}G)_\lambda$, then we are done. If (W', δ') is not $\wp$-minimal, then by Lemma 4.2.15, the size of $q(W')$ cannot be decreased, so for some $v_i \in q(W')$ the function $\gamma_{(W',\delta')}(v_i) = \delta'(v_{i,j_0}) + \sum_{k=0}^{j_0-1} h_{i,k}$ with $j_0 := \{j \mid v_{i,j} \in \mathrm{WCA}_i(\mathfrak{p})\}$ from Proposition 4.2.2 can be increased, which is done by increasing j_0 and assigning an appropriate value to $\delta'(v_{i,j_0})$ since (W', δ') is minimal. We increase $\gamma_{(W',\delta')}(v_i)$ for each $v_i \in q(W')$ such that any further increase would cause the set not to be a weighted r-path vertex cover. This process terminates in finitely many steps because $j_0 \leq r$. Denote the new set (W'', δ''). Then (W'', δ'') is minimal since the size of W'' cannot be decreased by Lemma 4.2.15 and δ'' cannot be increased. Thus, by construction, (W'', δ'') is a $\wp$-minimal weighted r-path vertex cover for $(\Sigma_{r-1}G)_\lambda$ such that $(W'', \delta'') \leq_\wp (W', \delta')$. $\square$

Example 4.2.20. Consider the following minimal weighted 3-path vertex cover $\mathfrak{p} := (V_1'', \delta_1'') := \{v_{1,2}^5, v_2^3\}$ of $(\Sigma_2 P_1)_\lambda$ from Example 4.2.18(a).

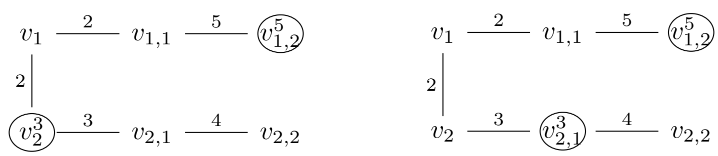


Note that $\gamma_{(V_1'',\delta_1'')}(v_1)$ cannot be increased. Assume that $v_{2,1} \in V''$. Then setting $\delta''(v_{2,1}) = 3$, we have that $\mathfrak{p}' := (V_1''', \delta_1''') = \{v_{1,2}^5, v_{2,1}^3\}$ is a minimal weighted 3-path vertex cover by Example 4.2.18(a). However, since $v_{1,2} \in V''$, we have that $v_{2,2}$ cannot be used to replace $v_{2,1}$ in V_1''' to generate a minimal 3-path vertex cover, otherwise, the 3-path $v_{1,1}v_1v_2v_{2,1}$ will be left uncovered. Thus, (V_1''', δ_1''') is $\wp$-minimal and $(V_1''', \delta_1''') <_{\wp} (V_1'', \delta_1'')$.

Lemma 4.2.21. Let $(V_1', \delta_1'), (V_2', \delta_2')$ be two minimal weighted r-path vertex covers of $(\Sigma_{r-1}G)_\lambda$. Then $(V_1', \delta_1') \leq_{\wp} (V_2', \delta_2')$ if and only if $P\big(q(V_1'), \gamma_{(V_1',\delta_1')}\big) \subseteq P\big(q(V_2'), \gamma_{(V_2',\delta_2')}\big)$.

Proof. $(V_1', \delta_1') \leq_{\wp} (V_2', \delta_2')$ if and only if $q(V_1') \subseteq q(V_2')$ and $\gamma_{(V_1',\delta_1')}|_{q(V_1')} \geq \gamma_{(V_2',\delta_2')}|_{q(V_1')}$ if and only if $P\big(q(V_1'), \gamma_{(V_1',\delta_1')}\big) \subseteq P\big(q(V_2'), \gamma_{(V_2',\delta_2')}\big)$. ☐

Example 4.2.22. Consider the following two minimal weighted 3-path vertex covers $(V_1'', \delta_1'') := \{v_{1,2}^5, v_2^3\}$ and $(V_2'', \delta_2'') := \{v_{1,2}^5, v_{2,1}^3\}$ of $(\Sigma_2 P_1)_\lambda$ from Example 4.2.18(a).



Then $(V_2'', \delta_2'') <_{\wp} (V_1'', \delta_1'')$ by Example 4.2.18(a). Note also that

$$P\big(q(V_2''), \gamma_{(V_2'',\delta_2'')}\big) = \big(X_1^{12}, X_2^6\big)R \subseteq \big(X_1^{12}, X_2^3\big)R = P\big(q(V_1''), \gamma_{(V_1'',\delta_1'')}\big).$$

Next, we present our third and final decomposition result which will yield the type computation in Theorem 4.2.25.

Theorem 4.2.23. *Given a weighted r-path suspension of G_ω $(\Sigma_r G)_\lambda$ such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_j, v_{j,1})$ for all edges $v_iv_j \in E$, we have an irredundant parametric decomposition*

$$I_r\big((\Sigma_r G)_\lambda\big)R = \bigcap_{(V'',\delta'')\ \wp\text{-min. w. } r\text{-path v. c. of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big) + \mathfrak{m}^{[\underline{a}(\lambda)]},\ \ \lambda' = \lambda|_{\Sigma_{r-1}G}.$$

Proof. By Fact 4.1.7 and [9, Theorem 7.5.3], to verify this result, it is enough to show that we have an irredundant decomposition

$$I_r\big((\Sigma_{r-1}G)_{\lambda'}\big)R = \bigcap_{(V'',\delta'')\ \wp\text{-min. w. } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

Lemma 4.2.21 shows that this intersection is irredundant. So by Corollary 4.2.11, it is enough to show that

$$\bigcap_{(V'',\delta'') \text{ min. weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big)$$
$$= \bigcap_{(V'',\delta'') \ p\text{-min. weighted } r\text{-path v. cover of } (\Sigma_{r-1}G)_{\lambda'}} P\big(q(V''), \gamma_{(V'',\delta'')}\big).$$

$\subseteq$ follows as every p-minimal weighted r-path vertex cover is a minimal weighted r-path vertex cover.

$\supseteq$ follows from Proposition 4.2.19 and Lemma 4.2.21. □

Example 4.2.24. Consider the graph $(\Sigma_3 P_1)_\lambda$ from Example 4.2.12. Then by Theorem 4.2.23 and Example 4.2.18(b), we have an irredundant parametric decomposition

$$\begin{aligned} I_3\big((\Sigma_3 P_1)_\lambda\big) &= \big(X_1^{12}X_2^2, X_1^4X_2^6, X_1^2X_2^{11}\big)R + \mathfrak{m}^{[\underline{a}(\lambda)]} \\ &= \big[\big(X_1^2\big)R \cap \big(X_2^2\big)R \cap \big(X_1^{12}, X_2^6\big)R \cap \big(X_1^4, X_2^{11}\big)R\big] + \big(X_1^{14}, X_2^{13}\big)R. \end{aligned}$$

The next theorem is the fourth main result of this thesis.

Theorem 4.2.25. *Let $(\Sigma_r G)_\lambda$ be a weighted r-path suspension of G_ω such that $\lambda(v_iv_j) \leq \lambda(v_i, v_{i,1})$ and $\lambda(v_iv_j) \leq \lambda(v_j, v_{j,1})$ for all edges $v_iv_j \in E$.*

$$\mathrm{r}_{R'}\left(\frac{R'}{I_r((\Sigma_r G)_\lambda)}\right) = \sharp\,\{p\text{-minimal weighted } r\text{-path vertex covers of } (\Sigma_{r-1}G)_{\lambda'}\},\ \lambda' = \lambda|_{\Sigma_{r-1}G}.$$

Proof. We compute

$$\begin{aligned} \mathrm{r}_{R'}\left(\frac{R'}{I_r((\Sigma_r G)_\lambda)}\right) &= \mathrm{r}_{R'}\left(\frac{R'}{I_r((\Sigma_r G)_\lambda) + (X_i - X_{i,k} \mid 1 \leq i \leq d, 1 \leq k \leq r)R'}\right) \\ &= \mathrm{r}_R\left(\frac{R}{I_r((\Sigma_r G)_\lambda)R}\right) \\ &= \sharp\,\{\text{ideals in an irredundant parametric decomposition of } I_r\big((\Sigma_r G)_\lambda\big)R\} \\ &= \sharp\,\{p\text{-minimal weighted } r\text{-path vertex covers of } (\Sigma_{r-1}G)_{\lambda'}\}, \end{aligned}$$

where the first equality is from Facts 4.1.8(a) and 2.7.6, and Theorem 3.4.1, the second equality is from Theorem 3.4.1, the third equality is from Fact 2.7.7 since $\dim\big(\frac{R}{I_r((\Sigma_r G)_\lambda)R}\big) = 0$, and the last equality is from Fact 4.2.23. $\square$

Remark. Because of Fact 4.1.8, we use Theorem 4.2.25 to compute $\mathrm{r}_R\big(R/I_r(G_\omega)\big)$ for all weighted trees G_ω such that $I_r(G_\omega)$ is Cohen-Macaulay.

Example 4.2.26. Consider Example 4.2.24. Then by Theorem 4.2.25, we have that

$$\mathrm{r}_{R'}\big(R'/I_3(\Sigma_3 P_1)_\lambda\big) = 4.$$

We observe that the smallest number of vertices for one of the 3-path vertex covers of $(\Sigma_3 P_1)_\lambda$ is 2. Then by Facts 3.1.7 and 2.2.28, $\dim\big(R'/I_3((\Sigma_3 P_1)_\lambda)\big) = 8 - 2 = 6$. Since $R'/I_3((\Sigma_3 P_1)_\lambda)$ is Cohen-Macaulay by Fact 4.1.8(a), $\operatorname{depth}\big(R'/I_3((\Sigma_3 P_1)_\lambda)\big) = \dim\big(R'/I_3((\Sigma_3 P_1)_\lambda)\big) = 6$. Hence

$$\operatorname{Ext}^6_{R'}\big(\mathbb{K}, R'/I_3((\Sigma_3 P_1)_\lambda)\big) \cong \mathbb{K}^4.$$

Chapter 5

Future work

5.1 Generalized Weighted Simplicial Complex

The Stanley-Reisner correspondence uses simplicial complexes to study square-free monomial ideals. In order to use similar techniques to study certain non-square-free monomial ideals, in the future, we will define a weighted version of the notion and then define a weighted version of Stanley-Reisner ideals. As in the classical setting, we'll see whether these ideals yield an irredundant irreducible decomposition. In terms of the decomposition, we will define the Alexander dual of a weighted simplicial complex and dual of any monomial ideal, to see whether Alexander duality commutes with the weighted Stanley-Reisner correspondence, and see how it is related to the dual defined by E. Miller [8].

5.2 More classifications

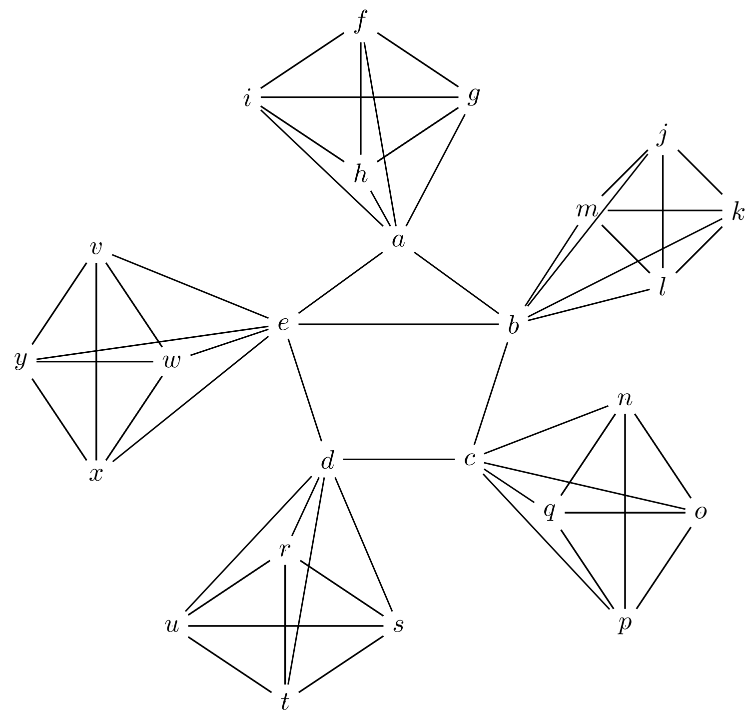
We focus on classifying the edge-weighted graphs whose f-weighted r-path ideal is Cohen-Macaulay over a field $\mathbb{K}$. As for the unweighted case, we cannot expect a general classification theorem. We've completed the classification for weighted r-suspensions. In the future, we plan to use combinatorial analysis to classify all weighed K_n-coronas with $n \geq 2$ and weighted chordal graphs such that their f-weighted r-path ideals are Cohen-Macaulay.

Definition 5.2.1. Let $n \geq 1$. A graph G is called a K_n*-corona* if there is a subgraph H of G such that each vertex of H is affixed a distinct completed graph K_n. An edge-weighted graph G_ω is called a *weighted K_n-corona* if the underlying graph G is K_n-corona.

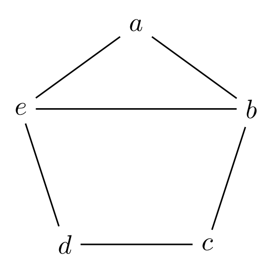
Definition 5.2.2. A graph G is called *chordal* if every cycle of length > 3 has a chord. An edge-weighted graph G_ω is called a *weighted chord graph* if the underlying graph G is chordal.

The examples for the K_n-corona and chordal are in the following:

Example 5.2.3. Let G be the following graph.

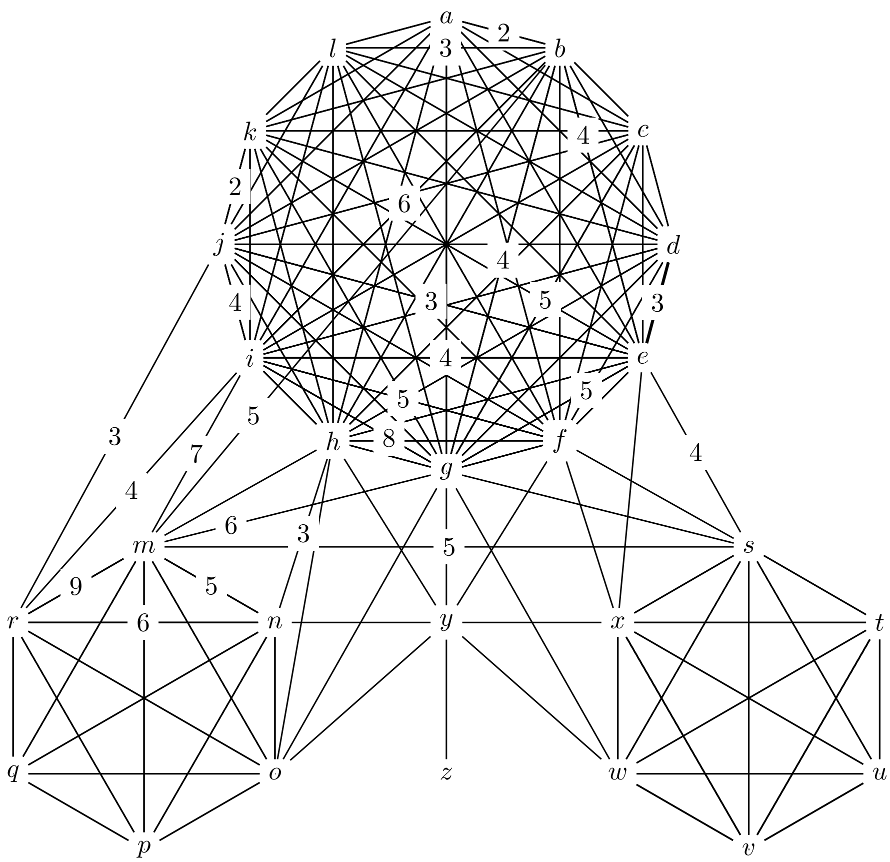


Then G is K_4-corona, since we have a subgraph H of G



such that each vertex a, b, c, d, e of H is affixed to a distinct complete graph K_4. Note that G is not chordal since the 4-cycle b —— c —— d —— e —— b in G doesn't have a chord.

Example 5.2.4. For the following weighted chordal graph G_ω, we draw part of the weights of G. For the unweighted edges, one can put any reasonable weights on them to define $\omega : E \to \mathbb{N}$.



We can show that the f-weighted r-path ideal $I_{r,f}(G_\omega)$ of G_ω cannot be Cohen-Macaulay when $r = 1$. By definition, $I_{1,f}(G_\omega)$ is the same the weighted edge ideal of G_ω [10, Definition 3.1].

Conjecture 5.2.5. For weighted K_n-coronas with $n \geq 2$, there is a classification result of Cohen-Macaulay f-weighted r-path ideals.

Conjecture 5.2.6. For weighted chordal graphs, there is a classification result of Cohen-Macaulay f-weighted r-path ideals as in [6].

Conjecture 5.2.7. For all weighted r-path suspensions, weighted K_n-coronas and weighted chordal graphs and any function f, when the f-weighted r-path ideals are Cohen-Macaulay, we can compute their type combinatorially.

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