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STOCHASTIC DELAY DIFFERENTIAL EQUATIONS WITH APPLICATIONS IN ECOLOGY AND EPIDEMICS

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Under the Supervision of Prof. Fathalla Ali Rihan

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Declaration of Original Work

I, Hebatallah Jamil Alsakaji, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this dissertation, entitled *"Stochastic Delay Differential Equations with Applications in Ecology and Epidemics"*, hereby, solemnly declare that this dissertation is my own original research work that has been done and prepared by me under the supervision of Prof. Fathalla Ali Rihan, in the College of Science at the UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my dissertation have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this dissertation.

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Abstract

Mathematical modeling with delay differential equations (DDEs) is widely used for analysis and predictions in various areas of life sciences, such as population dynamics, epidemiology, immunology, physiology, and neural networks. The memory or time-delays, in these models, are related to the duration of certain hidden processes like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period, and so on. In ordinary differential equations (ODEs), the unknown state and its derivatives are evaluated at the same time instant. In DDEs, however, the evolution of the system at a certain time instant depends on the past history/memory. Introduction of such time-delays in a differential model significantly improves the dynamics of the model and enriches the complexity of the system.

Moreover, natural phenomena counter an environmental noise and usually do not follow deterministic laws strictly but oscillate randomly about some average values, so that the population density never attains a fixed value with the advancement of time. Accordingly, stochastic delay differential equations (SDDEs) models play a prominent role in many application areas including biology, epidemiology and population dynamics, mostly because they can offer a more sophisticated insight through physical phenomena than their deterministic counterparts do. The SDDEs can be regarded as a generalization of stochastic differential equations (SDEs) and DDEs.

This dissertation, consists of eight Chapters, is concerned with qualitative and quantitative features of deterministic and stochastic delay differential equations with applications in ecology and epidemics. The local and global stabilities of the steady states and Hopf bifurcations with respect of interesting parameters of such models are investigated. The impact of incorporating time-delays and random noise in such class of differential equations for different types of predator-prey systems and infectious diseases is studied. Numerical simulations, using suitable and reliable numerical schemes, are provided to show the effectiveness of the obtained theoretical results.

Chapter 1 provides a brief overview about the topic and shows significance of the study. Chapter 2, is devoted to investigate the qualitative behaviours (through local and global stability of the steady states) of DDEs with predator-prey systems in case of hunting cooperation on predators. Chapter 3 deals with the dynamics of DDEs, of multiple time-delays, of two-prey one-predator system, where the growth of both preys populations subject to Allee effects, with a direct competition between the two-prey species having a common predator. A Lyapunov functional is deducted to investigate the global stability of positive interior equilibrium. Chapter 4, studies the dynamics of stochastic DDEs for predator-prey system with hunting cooperation in predators. Existence and uniqueness of global positive solution and stochastically ultimate boundedness are investigated. Some sufficient conditions for persistence and extinction, using Lyapunov functional, are obtained. Chapter 5 is devoted to investigate Stochastic DDEs of three-species predator-prey system with cooperation among prey species. Sufficient conditions of existence and uniqueness of an ergodic stationary distribution of the positive solution to the model are established, by constructing a suitable Lyapunov functional. Chapter 6 deals with stochastic epidemic SIRC model with time-delay for spread of COVID-19 among population. The basic reproduction number $\mathscr{R}_0^s$ for the stochastic model which is smaller than $\mathscr{R}_0$ of the corresponding deterministic model is deduced. Sufficient conditions that guarantee the existence of a unique ergodic stationary distribution, using the stochastic Lyapunov functional, and conditions for the extinction of the disease are obtained. In Chapter 7, some numerical schemes for SDDEs are discussed. Convergence and consistency of such schemes are investigated. Chapter 8 summaries the main finding and future directions of research.

The main findings, theoretically and numerically, show that time-delays and random noise have a significant impact in the dynamics of ecological and biological systems. They also have an important role in ecological balance and environmental stability of living organisms. A small scale of white noise can promote the survival of population; While large noises can lead to extinction of the population, this would not happen in the deterministic systems without noises. Also, white noise plays an important part in controlling the spread of the disease; When the white noise is relatively large, the infectious diseases will become extinct; Re-infection and periodic outbreaks can also occur due to the time-delay in the transmission terms.

Keywords: Allee effect, Bifurcation, Brownian motion, Epidemic models, Lyapunov functionals, Predator-prey model, Sensitivity, SIRC, Stability, Stationary distribution, Stochastic perturbations, Time-delays.

Title and Abstract (in Arabic)

المعادلات التفاضلية العشوائية ذات الذاكرة وتطبيقاتها في علوم البيئة والاوبئة

الملخص

تستخدم المعادلات التفاضلية ذات الذاكرة (Delay Differential Equations, DDEs) على نطاق واسع في النمذجة الرياضية (Mathematical Modeling) والتنبؤات في مختلف مجالات علوم الحياة، على سبيل المثال ديناميكيات السكان (Dynamical Systems)، وعلم الأوبئة (Epidemiology)، وعلم المناعة (Immunology)، وعلم وظائف الأعضاء (Physiology)، والشبكات العصبية (Neural Networks) . حيث ترتبط الذاكرة (Memory)، في النماذج الرياضية للفترات الزمنية، لتمثيل بعض العمليات الخفية مثل مراحل دورة الحياة، وكذلك الوقت اللازم بين إصابة الخلية وإنتاج فيروسات جديدة، والمدة الزمنية للفترات المعدية، والفترات المناعية. والجدير بالذكر انه في المعادلات التفاضلية العادية، يتم تقييم الحالة المتغيرات (State Variables) ومشتقاتها في الوقت الآني (Current Time) ولكن في المعادلات التفاضلية ذات الذاكرة DDEs ، يعتمد تطور النظام في الوقت الآني و على التاريخ/ الذاكرة الماضية (Memory/History) . إدخال مثل هذه التأخيرات الزمنية في نموذج رياضي تفاضلي يحسن بشكل كبير ديناميكيات النموذج ومرونتة لتمثيل الأنظمة المعقدة في علوم الحياة.

من المعلوم ان الظواهر الطبيعية قد تواجه بعض الاضطرابات البيئية العشوائية Environmental Stochastic) (Perturbations/Noise، مثل تأثير تغيرات الطقس ودرجات الحرارة، الرطوبة وإلخ إلخ، وعادة هذه الظواهر لا تتبع القوانين القطعية ولكنها تتأرجح بشكل عشوائي حول بعض القيم المتوسطة، حيث تتأرجح الكثافة السكانية و لا تصل أبدًا إلى قيمة ثابتة مع تقدم الوقت. وبناءً على ذلك، نقترح في هذه الرسالة بعض النماذج الرياضية باستخدام المعادلات التفاضلية العشوائية ذات الذاكرة (SDDEs) وذلك لتمثيل ونمذجة بعض الظواهر الطبيعية في علوم البيئة و الحياة، حيث أنها توفر درجات إضافية من الواقعية مقارنةً بنظيراتها غير العشوائي (Deterministic).يمكن اعتبار المعادلات التفاضلية العشوائية ذات الذاكرة SDDEs بمثابة تعميم المعادلات التفاضلية العشوائية (Stochastic Differential Equations, SDDEs) و المعادلات التفاضلية ذات الذاكرة. حيث تقدم هذه لأطروحة دراسة موسعة عن السمات و الخصائص النوعية والكمية (Qualitative and Quantitative Features) للمعادلات التفاضلية الحتمية والعشوائية ذات الذاكرة وتطبيقاتها المهمة و المتعددة في علوم البيئة وانتشار والأوبئة.

تتكون هذه الأطروحة من ثمانية فصول، يقدم الفصل الأول مقدمة عامة للرسالة وأهمية لهذه الدراسة. ويخصص الفصل الثاني لدراسة الخصائص النوعية لأنظمة الفريسة والمفترس (Prey-Predator Systems) في حالة الصيد مع وجود تعاون بين الحيوانات المفترسة وذلك باستخدام نماذج DDEs. يتعامل الفصل الثالث مع ديناميكيات DDEs، من الفترات الزمنية المتعددة، لأنظمة الفريسة والمفترس، حيث يكون نمو كلتا الفريستين عرضة لتأثيرات Allee، وهناك منافسة مباشرة بين الأنواع ذات الفريسة التي لديها مفترس مشترك. يقدم الفصل الرابع دراسة عن ديناميكيات المعادلات التفاضلية العشوائية ذات الذاكرة SDDEs لنظام المفترس الفريسة. حيث تم التحقق من وجود وتفرد الحل الإيجابي، بالإضافة إلى التوصل لبعض

الشروط الكافية لوجود وانقراض الفريسة والمفترس. الفصل الخامس مكرس لدراسة أنظمة الفريسة والمفترس ذات الأنواع الثلاثة مع وجود تعاون بين أنواع الفرائس وذلك باستخدام نماذج DDEs. حيث تبين أنه يمكن للضوضاء العشوائية أن تكبح انفجار الأنواع، في حالة كونها غير محدودة في النظام الحتمي. الفصل السادس، يتناول دراسة نموذج SIRC الوبائي العشوائي مع التأخيرات الزمنية وذلك لدراسة ديناميكية انتشار فيروس كورونا-٢ (COVID-19) داخل المجتمع. حيث تم استنباط الشروط اللازمة للوصول الي حالة الاستقرار وإمكانية السيطرة على انتشار المرض. يقدم الفصل السابع دراسة الحلول العددية والتقريبية للمعادلات التفاضلية العشوائية (SDDEs) والشروط اللازمة لاستقرار الحلول التقريبية، بينما يلخص الفصل الثامن ما توصلنا إليه من اهم النتائج والاتجاهات مستقبلية للبحث.

تظهر النتائج التي توصلنا إليها، نظريًا وعدديًا، أفضلية المعادلات التفاضلية العشوائية ذات الذاكرة (SDDEs) على غيرها من النماذج الرياضية الأخرى، حيث ان لها تأثير كبير في ديناميكيات الأنظمة البيئية والبيولوجية. كما أن لها دورًا مهمًا في التوازن والاستقرار البيئي للكائنات الحية. حيث ثبت ان وجود الضجيج الطفيف Small Noise)، يعزز بقاء الكائنات الحية (Persisting) ؛ اما في حين وجود الضجيج الكبير يمكن أن يؤدي إلى انقراض (Extinction) بعض الأنواع (some species) . كما وجد ان الاضطرابات العشوائية البيئية مفيدة وجانب حتمي مؤثر على ديناميكيات أي نظام بيئي، كما انه له دور فعال لقمع انفجار سكاني محتمل. و من ثم، استخدام المعادلات التفاضلية العشوائية ذات الذاكرة في نمذجة ديناميكيات السكان و ديناميكية و انتشار الأوبئة افضل من غيرها من النماذج الرياضية الأخرى التي تفتقد الذاكرة و الاضطرابات العشوائية.

مفاهيم البحث الرئيسة: تأثير Allee، التشعب، الحركة البراونية، نماذج وبائية، نماذج الفريسة والمفترس، SIRC، الاستقرار، التوزيع الثابت الاضطرابات العشوائية، التأخيرات الزمنية.

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Dedication

To my great parents, my beloved husband and my lovely kids

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List of Abbreviations

$N(0,1)$	A normally distributed random variable with zero mean and unit variance
a.s.	Almost Surely
$\mathscr{R}_0$	Basic reproduction number
$\mathscr{R}_0^s$	Basic reproductive number for stochastic models
DDEs	Delay Differential Equations
$\|.\|$	Euclidean norm: $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x \in \mathbb{R}^n$
$\mathbb{R}^n$	Euclidean space of n-dimentions
$\mathbb{E}(X)$	Expected value of X
ODEs	Ordinary Differential Equations
$\mathbb{P}(.)$	Probability measure
$\Re$	Real part
$\mathbb{Z}$	Set of all integers
$\mathbb{C}$	Set of complex numbers
$\mathscr{C}(A,B)$	Set of continuous functions mapping A to B
$\mathbb{R}_+$	Set of nonnegative real numbers
$\mathbb{N}$	Set of positive integers
$\mathscr{C}$	Short form for $\mathscr{C}([-\tau,0],\mathbb{R}^n)$, where $\tau > 0$ is a constant
SDDEs	Stochastic Delay Differential Equations
$\mathscr{T}_0^d$	Threshold parameter for deterministic models
$\mathscr{T}_0^s$	Threshold parameter for stochastic models
$W(t)$	Winner process
$\mathfrak{B}$	$\sigma-$ algebra of Borel sets in $\mathbb{R}^n$

Chapter 1: Delay Differential Equations with Population Problems

1.1 Introduction

In this chapter, some preliminaries about deterministic Delay Differential Equations (DDEs) and Stochastic Delay Differential Equations (SDDEs) are introduced. Section 2 briefly discusses the existence and uniqueness of the solutions of DDEs. Section 3 provides some concepts about the stability of DDEs. Sections 4 and 5 provide some background about the randomness, environmental noise and the existence of the solutions of SDDEs. Section 6 provides some main concepts about the stability of SDDEs. The last Section introduces the main objectives and significance of the study.

DDEs are a class of differential equations that have received a considerable attention and been shown to model many real life problems, traditionally formulated by systems of Ordinary Differential Equations (ODEs), more naturally and more accurately. Such class of DDEs are widely used for analysis and predictions of systems with memory such as population dynamics, epidemiology, immunology, physiology and neural networks [5, 20, 74, 107, 114]. In ODEs, the unknown function and its derivatives are evaluated at the same time instant. However, in a DDE the evolution of the system at a certain time instant, depends on the state of the system at an earlier time. The delay can be related to the duration of certain hidden processes like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period, and so on; See [111, 112].

In ecological systems, the individuals of the prey and predator species usually pass through various life stages during their entire life span and the involved morphology differs from one stage to another. Construction of delay differential equation models is a well known modelling strategy to take care of the stage specific activities which are responsible for significant change in the dynamics of interacting populations. In various existing literature, the biological processes like incubation, gestation, maturation, reaction time, etc., are taken care of by introducing relevant time-delay parameters to the models for predator-prey and other types of interacting populations. Incorporating time-lags (or time-delays) in biological models makes the systems much more realistic, as it can destabilize the equilibrium points and give rise to a stable limit cycle, causing oscillations to grow, and enriching the dynamics of the model. Time-delays have been considered and extremely studied by many authors in predator-prey models and biological systems; See [15, 16, 117].

Most of the studies in ecology utilize deterministic models, which of course supported the researchers with useful results for protecting species. In reality, natural phenomena counter an environmental noise and usually do not follow strictly deterministic laws but oscillate randomly about some average values, so that the population density never attains a fixed value with the advancement of time [43, 118]. Ecological systems are often subject to environmental noise, which is important factor in ecosystems, to suppress a potential population explosion [120].

A key question in population biology is understanding the conditions under which populations coexist or go extinct. Extinction is one of the most important terms in population dynamics. A species is said to be extinct when the last existing member dies. Therefore, extinction becomes a certainty when there are no surviving individuals that can reproduce and create a new generation. In ecology, extinction is often used informally to refer to local extinction, in which a species ceases to exist in the chosen area of study, but may still exist elsewhere. There are a variety of causes that can contribute directly or indirectly to the extinction of species or group of species, such as lack of food and space or toxic pollution of the entire population habitat, competition for food to better adapted competitors, predation, etc. [84]. Some examples in modelling population dynamics can be referred to [64, 84, 92, 100, 148].

In fact, stochastic perturbation factors, such as precipitation, absolute humidity, and temperature, have a significant impact on the infection force of all types of virus diseases to humans. Taking this into consideration enables a lot of authors to introduce randomness into deterministic model of biological systems to reveal the effect of environmental variability, whether it is a random noise in the system of differential equations or environmental fluctuations in parameters. Moreover, stochastic epidemic models give an extra degree of realism in comparison with their deterministic models [26, 79, 85, 134, 142, 143].

In the next Section, the existence and uniqueness of the solutions of DDEs are discussed.

1.2 Existence and Uniqueness of Solutions for DDEs

Consider the Initial Value Problem (IVP) for the system of DDEs, with multiple discrete time-delays, of the following form

$$\begin{aligned} \frac{d\mathbf{y}(t)}{dt} &= \mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau_1),\ldots,\mathbf{y}(t-\tau_m)), \quad t \geq t_0, \\ \mathbf{y}(t) &= \phi(t) \quad t \leq t_0, \end{aligned} \tag{1.1}$$

where $\mathbf{y}(t) \in \mathbb{R}^n$, $\mathbf{f}$ is a nonlinear smooth function, with respect to all of its arguments, depending on delays $\tau_i > 0$, $i = 1,\ldots,m$. Time-delay τ_i could be a constant, or variable in time $\tau_i(t)$ $(i = 1,\ldots,n)$, or even state-dependent $\tau_i = \tau_i(t,\mathbf{y}(t))$. If the right hand side of (1.1) is a function of $\mathbf{y}'(t)$, then it be called Neutral Delay Differential Equations (NDDEs). The function $\phi(t)$ is defined in $[\nu,t_0]$, where $\nu = \min_{1\leq i\leq n}\left\{\min_{t\geq t_0}(t-\tau_i)\right\}$. For simplicity, consider DDEs of the form

$$\begin{aligned} \frac{d\mathbf{y}(t)}{dt} &= \mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau(t))), \quad t \geq t_0, \\ \mathbf{y}(t) &= \phi(t) \quad t \leq t_0. \end{aligned} \tag{1.2}$$

In general, initial discontinuity $\mathbf{y}'(t_0)^+ = \mathbf{f}(t_0,\mathbf{y}(t_0),\phi(t_0-\tau))$ may differ from the value $\phi'(t_0)^-$; and its propagation from initial point t_0 along the integration interval and gives rise to subsequent discontinuity points where the solution is smoothed out more and more. On the other hand, it is well-known that every step by step numerical method for the initial value problem attains its own accuracy order provided that the solution is sufficiently smooth at each step interval $[t_n, t_{n+1}]$; More details are discussed in Appendix A.

Now, some essential results for the DDEs (1.2) will be introduced (see [17, 39]).

Theorem 1.2.1. *(Local existence) Consider the equation*

$$\begin{aligned} \frac{d\mathbf{y}(t)}{dt} &= \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau(t))) \quad t_0 \le t < t_b \\ \mathbf{y}(t_0) &= \mathbf{y}_0. \end{aligned} \tag{1.3}$$

Assume that the function $\mathbf{f}(t,u,v)$ is continuous with respect to t on $A \subseteq [t_0,t_b) \times \mathbb{R}^n \times \mathbb{R}^n$ and Lipschitz continuous with respect to u and v. Moreover, assume that the delay function $\tau(t) \ge 0$ is continuous in $[t_0,t_b)$, $\tau(t_0) = 0$ and, for some $\xi > 0$, $t - \tau(t) > t_0$ in the interval $(t_0, t_0 + \xi]$. Then the Equation (1.3) has a unique solution in $[t_0, t_0 + \delta)$ for some $\delta > 0$ and this solution continuously depends on the initial data.

To show the global existence theorem, under the same assumptions of Theorem 1.2.1, the solution can be carried on until a maximal solution defined in $[t_0, l)$, with $t_0 < l \le t_b$.

Theorem 1.2.2. *(Global existence) Under the assumptions of Theorem 1.2.1, if the unique maximal solution of (1.3) is bounded, then it exists on the entire interval $[t_0,t_b)$.*

Therefore, the following lemma is illustrated to define a bound for the solution.

Lemma 1.2.3. *Under the assumptions of Theorem 1.2.1, assume that the function $\mathbf{f}(t,u,v)$ satisfies the condition*

$$||\mathbf{f}(t,u,v)|| \le M_1(t) + M_2(t)(\|u\| + \|v\|)$$

in $[t_0,t_b) \times \mathbb{R}^n \times \mathbb{R}^n$, where $M_1(t)$ and $M_2(t)$ are continuous positive functions on $[t_0,t_b)$. Hence, the solution of (1.3) exists and is unique on the entire interval $[t_0,t_b)$.

Driver [37] proved the above results with multiple delays. Particularly, the global existence, uniqueness and continuous dependence on the initial data for the solution of the linear DDEs

$$
\begin{aligned}
\mathbf{y}^{'}(t) =& A_0\mathbf{y}(t) + \sum_{i=1}^{r} A_i(t)\mathbf{y}(t-\tau_i(t)), \quad t \geq t_0 \\
\mathbf{y}(t) =& \phi(t), \quad t \leq t_0,
\end{aligned}
$$

for any continuous functions $A_i(t)$, $i = 0, \ldots, r$, and $\phi(t)$, and for any set of continuous delays $\tau_i(t) \geq 0$.

Most DDEs don't have analytic solutions, so it is generally essential to resort to numerical methods (See Appendix A). For linear DDEs with constant delay, considering solutions of exponential form; see [35]. Now, consider a scalar linear DDE of the form

$$
\begin{aligned}
&\frac{dy}{dt} = \mu_1 y(t-\tau) + \mu_0 y(t), \quad t \geq 0, \\
&y(t) = \phi(t), \quad t \in [-\tau, 0].
\end{aligned}
\tag{1.4}
$$

Let the solution $y(t) = Ce^{\lambda t}$, where C is constant; then substituting it into Equation (1.4) gives

$$
C\lambda e^{\lambda t} = \mu_0 Ce^{\lambda t} + \mu_1 Ce^{\lambda(t-\tau)}, \tag{1.5}
$$

which can be simplify to

$$
(\lambda - \mu_0)e^{\lambda\tau} - \mu_1 = 0, \tag{1.6}
$$

Equation (1.6) is the characteristic equation and the root λ_i of (1.6) gives a solution to Equation (1.4) in the form of $Ce^{\lambda_i t}$. The following Theorem [18] illustrates the general solution of (1.6).

Theorem 1.2.4. *Assume that* $\phi(t)$ *is* $\mathscr{C}[t_0 - \tau, t_0]$, *and let* $\{\lambda_i\}$ *be a sequence zeros of*

Equation (1.6) arranged in order of decreasing real parts (or of increasing imaginary parts or absolute values). Then

$$y(t) = \sum_{i=1}^{\infty} p_i(t) e^{\lambda_i t}, \quad t \geq t_0, \quad \textit{(quasi-polynomial)}, \tag{1.7}$$

is the solution of Equation (1.4), where $p_i(t)$ is a polynomial of degree less than the multiplicity of the root $\{\lambda_i\}$.

The above approach, is not a practical method for solving DDEs. However, it provides useful information about solutions to DDEs. The following example affords the basic theory for the simplest method for solving DDEs, which is the method of steps [17].

Example 1.2.1. Assume a special case of (1.4) when $\mu_0 = 0$. Therefore, Equation (1.4) becomes

$$\begin{aligned} &\frac{dy}{dt} = \mu_1 y(t-\tau), \quad t \geq 0, \\ &y(t) = \phi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{1.8}$$

Let $\mu_1 = -1$, $\tau = 1$ and $\phi(t) = 1 + t$; In the interval [0,1], one obtains

$$y_1(t) = y(0) - \int_0^t s ds = 1 - \frac{t^2}{2},$$

in the interval [1,2], one gets

$$y_2(t) = y(1) - \int_1^t (1 - \frac{(s-1)^2}{2}) ds = \frac{(t-1)^3}{6} - t + \frac{3}{2},$$

similarly, in the interval [2,3], the solution is

$$\begin{aligned} y_3(t) &= -\frac{1}{3} - \int_2^t \Big(\frac{(t-2)^3}{6} - t + \frac{5}{2}\Big) ds \\ &= \frac{8}{3} - \frac{(t-2)^4}{24} + \frac{t^2}{2} - \frac{5}{2}t. \end{aligned}$$

Now, one can check for discontinuity in the solutions. At $t = 0$, $\phi(0) = y_1(0) = 1$, but $\phi'(0) = 1 \neq y_1'(0) = 0$. Thus, there is a discontinuity at $y'(0)$. At $t = 1$, $y_1(1) = y_2(1) = \frac{1}{2}$, $y_1'(1) = y_2'(1) = -1$, but $y_1''(1) = -1 \neq y_2''(1) = 0$. Thus, there is a discontinuity at $y''(1)$. Similarly, at $t = 2$, there is a discontinuity at $y'''(3)$.

Applying Laplace transformation [18] to (1.8) also provides some useful facts about the solutions but do not mostly acquire explicit solutions. The Laplace transform $L(y)$ of $y(t)$ is denoted by $\bar{Y}(s)$, where $\bar{Y}(s) = \int_0^\infty y(t)e^{-st}dt$.
By taking Laplace transform of $y'(t) = -y(t-1)$, one obtains

$$\begin{aligned} \int_0^\infty y'(t)e^{-st}dt &= \int_0^\infty -y(t-1)e^{-st}dt \\ s\bar{Y}(s) - 1 &= -\int_{-1}^\infty y(w)e^{-s(w+1)}dw \\ &= -e^{-s}\int_0^\infty y(w)e^{-sw}dw - e^{-s}\int_{-1}^0 y(w)e^{-sw}dw \\ &= -e^{-s}\bar{Y}(s) - e^{-s}\int_{-1}^0 (w+1)e^{-sw}dw \\ (s+e^{-s})\bar{Y}(s) &= 1 - e^{-s}[-\frac{1}{s^2} - \frac{1}{s} + \frac{e^s}{s^2}] \Rightarrow \bar{Y}(s) = \frac{1 + \frac{e^{-s}}{s} + \frac{e^{-s}-1}{s^2}}{s+e^{-s}}. \end{aligned}$$

Thus, $y(t)$ can be found by the following inverse transform

$$y(t) = L^{-1}\Big[\frac{1 + \frac{e^{-s}}{s} + \frac{e^{-s}-1}{s^2}}{s+e^{-s}}\Big].$$

Next, some stability criteria of DDEs are discussed.

1.3 Stability of Equilibria and Lyapunov Functions for DDEs

Definition 1.3.1. [12] The solution $\mathbf{y}(t)$ of (1.2) is stable if, given $\varepsilon > 0$, there exists $\Delta \equiv \Delta(\varepsilon)$ such that $\sup_{t\in[t_0-\tau,t_0]} |\mathbf{u}(t) - \phi(t)| \leq \Delta$ and $\mathbf{u}(t)$ is also a solution of (1.2), then $\delta\mathbf{y}(t) := \mathbf{u}(t) - \mathbf{y}(t)$ is uniformly bounded for $t \geq t_0$ and $\sup_{t\geq t_0} |\delta\mathbf{y}(t)| \leq \varepsilon$. The solution $\mathbf{y}(t)$ is asymptotically stable if it is stable and $|\delta\mathbf{y}(t)| \to 0$ as $t \to \infty$ for all Δ sufficiently small.

There are two standard approaches for stability theory [72, 73], the first is stability in variation (first approximation), and the second is Lyapunov theory. First, stability in variation approach are introduced, which is based on local linearization of the DDE [89]. Consider a system of DDEs with multiple constant delays

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}(t), \mathbf{y}(t-\tau_1), \ldots, \mathbf{y}(t-\tau_m)), \tag{1.9}$$

where $\mathbf{y}(t) \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ is a nonlinear smooth function depending on delays $\tau_i > 0$, $i = 1, \ldots, m$. An equilibrium, $\mathbf{y}^*(t) \equiv \mathbf{y}^*$, of (1.9) is a solution of the nonlinear algebraic system $\mathbf{f}(\mathbf{y}^*, \mathbf{y}^*, \ldots, \mathbf{y}^*)$, which is solved by Newton iteration starting from an initial guess $\mathbf{y}^*$. The local asymptotic stability of $\mathbf{y}^*$ is found out through the linearization of (1.9) around $\mathbf{y}^*$, i.e. through the following variation equation

$$\frac{d\mathbf{u}}{dt} = A_0\mathbf{u}(t) + \sum_{i=1}^{m} A_i\mathbf{u}(t-\tau_i), \tag{1.10}$$

such that A_i stands for the partial derivatives of $\mathbf{f}$ with respect to the ith variable, i.e. $A_i := \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}^i}\right|_{(\mathbf{y}^*,\mathbf{y}^*,\ldots,\mathbf{y}^*)}$, $i = 0, 1, \ldots, m$. The variational Equation (1.10) leads to the characteristic equation,

$$det\left(\lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i}\right) = 0, \tag{1.11}$$

where I is the identity matrix and $\lambda \in \mathbb{C}$. The local asymptotic stability of the equilibrium $\mathbf{y}^*$ is determined by the real parts of the characteristic roots λ, therefore, the solution is stable if $\Re(\lambda)$ is negative for all λ and it is unstable with $\Re(\lambda)$ is positive. The number of roots for Equation (1.11) could be countably infinite. However, the number of roots in any right half-plane $\Re(\lambda) > \eta$, $\eta \in \Re$, is finite such that $\Re(\lambda_j) \to -\infty$ as $j \to \infty$.

Generally, a bifurcation (or threshold point) occurs when a real characteristic root passes through zero and a Hopf bifurcation occurs when a pair of complex conjugate characteristic roots passes the imaginary axis; While a transcritical bifurcation occurs when two branches of equilibrium solutions intersect. A periodic solution $y^*(t)$ is a solution that restates itself after a finite period T i.e., $y^*(t+T) = y^*(t)$ for all $t > 0$. The local asymptotic stability of the periodic solution is determined by the time integration operator; which integrates the variational Equation (1.10) around $y^*(t)$ from time $t = 0$ over the period. This operator is also called the monodromy operator and its eigenvalues (which are independent of the starting point $t = 0$) are called Floquet Multipliers. Additionally, if $T \geq \tau$ then the oprator is compact. The periodic solution is stable if all multipliers (except the trivial one) have modulus smaller than one and it is unstable if there exists a multipliers with modulus larger than one [40].

Example 1.3.1. To study the qualitative behaviour of the linear DDE, recall Equation (1.4)

$$\frac{dy}{dt} = \mu_1 y(t-\tau) + \mu_0 y(t), \quad t \geq 0. \tag{1.12}$$

The aim is to investigate the stability around the equilibrium solution $y = 0$. Consider the exponential solution $y(t) = Ce^{\lambda t}$, where C is constant and the eigenvalue λ are solutions of the characteristic equation

$$\lambda - \mu_0 - \mu_1 e^{-\lambda\tau} = 0, \tag{1.13}$$

which is a transcendental equation. Therefore, $y = 0$ is asymptotically stable if all eigenvalues of (1.13) have negative real parts. Let $\lambda = \zeta_0 + i\zeta_1$. Substituting in the characteristic Equation (1.13) then separating the real and the imaginary parts, one obtains

$$\zeta_0 - \mu_0 - \mu_1 e^{-\zeta_0\tau}\cos\zeta_1\tau = 0, \quad \zeta_1 + \mu_1 e^{-\zeta_0\tau}\sin\zeta_1\tau = 0. \tag{1.14}$$

Notice that when $\tau = 0$, the eigenvalue of the characteristic Equation (1.13) has a negative real number if $\lambda = \mu_0 + \mu_1 < 0$. For fixed $\tau > 0$ the boundaries of the domains of the $(\mu_0, \mu_1)-$plane are formed by the line $\mu_1 = -\mu_0$ and the parametric curve $\mu_0 = \zeta_1 \cot\zeta_1\tau$, $\mu_1 = -\zeta_1/\sin\zeta_1\tau$ where $\zeta_1 \in \mathbb{R}$, for which $\zeta_0 = 0$. Therefore, the stability condition is $\mu_0 \leq -|\mu_1|$, that is independent of $\tau > 0$; see Figure 1.1, which shows the stability region of (1.12) when $\tau = 1$. For complex-valued μ_0, μ_1, the solution are stable if $|\mu_1| < -\Re(\mu_0)$.

Restricting conditions on τ such that $\Re(\lambda)$ changes from negative to positive. By the continuity, if λ changes from $\mu_0 + \mu_1$ to a certain value such that $\Re(\lambda) = \zeta_0 > 0$ as τ increases, there must be some threshold value of τ, say τ^*, at which $\Re\lambda(\tau^*) = \zeta_0(\tau^*) = 0$; In this case the characteristic Equation (1.13) must have a pair of purely imaginary roots $\pm i\zeta_1^*$, $\zeta_1^* = \zeta_1^*(\tau^*)$. Therefore, having $-\mu_0 - \mu_1\cos\zeta_1^*\tau = 0$, which implies

$$\tau_k = \frac{\cos^{-1}(-\mu_0/\mu_1)}{\sqrt{\mu_1^2 - \mu_0^2}} + \frac{2\pi k}{\sqrt{\mu_1^2 - \mu_0^2}}, k \in \mathbb{Z}. \tag{1.15}$$

Noting that $\zeta_1^* = \sqrt{\mu_1^2 - \mu_0^2} > 0$; Thus, when $\tau = \tau^* = \min\tau_k$, Equation (1.13) has a pair of purely imaginary roots. When $0 < \tau < \tau^*$, all roots of (1.13) have negative real parts then the equilibrium $y = 0$ is asymptotically stable. if $\tau > \tau^*$, then $y = 0$ is unstable. Assume $\lambda(\tau) = \zeta_0(\tau) + i\zeta_1(\tau)$ the root of Equation (1.13) satisfying $\zeta_0(\tau_k) = 0$ and

$\zeta_1(\tau_k) = \zeta_1^*$, $k = 0,1,2,\ldots$. In which the transversality condition as following

$$\frac{d\Re(\lambda)}{d\tau}\Big|_{\tau=\tau^*} = \frac{d\zeta_0}{d\tau}\Big|_{\tau=\tau^*} = (\zeta_1^*)^2 > 0, \quad j = 0,1,2,\ldots.$$

A Hopf bifurcation occurs at $y = 0$ with a period given by $T = \frac{2\pi}{\sqrt{\mu_1^2 - \mu_0^2}}$, Figure 1.2 shows a stable solution for Equation (1.12) when $\tau = 0.8 < \tau^*$; Periodic solution where a Hopf bifurcation occurs as $\tau = \tau^* = 1.209$; the solution of (1.12) becomes unstable when $\tau = 1.28 > \tau^*$. Now, one may discuss the stability of (1.12) when $\mu_0 = 0$ i.e. pure DDE,

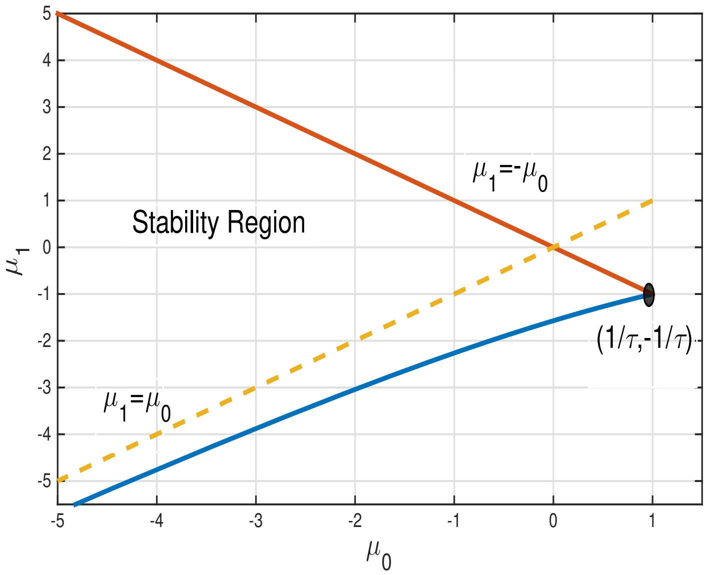

Figure 1.1: Stability region for scalar DDE $y'(t) = \mu_1 y(t-\tau) + \mu_0 y(t)$. Red line gives the real root crossing $\mu_1 = -\mu_0$, while the blue line gives the imaginary root crossing; the dash line represents the equation $\mu_1 = \mu_0$

Recall the Equation (1.8)

$$\frac{dy}{dt} = \mu_1 y(t-\tau), \tag{1.16}$$

Therefore the characteristic Equation of (1.16) is

$$\lambda - \mu_1 e^{-\lambda\tau} = 0, \tag{1.17}$$

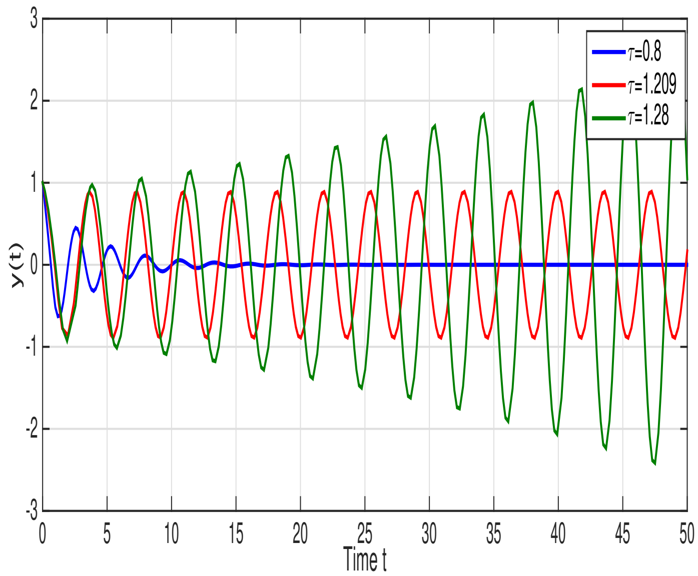

Figure 1.2: Numerical simulations for DDE $y'(t) = \mu_1 y(t-\tau) + \mu_0 y(t)$. With $\mu_1 = -2$ and $\mu_0 = -1$, where the initial function $\phi(t) = 1+t$, when $\tau = 0.8$ the steady state $y = 0$ is stable; When $\tau = \tau^* = 1.209$ a Hopf bifurcation occurs; The solution of (1.12) becomes unstable when $\tau = 1.28 > \tau^*$

First, suppose that λ is real, for $\mu_1 > 0$ the equilibrium $y = 0$ is unstable; For $\mu_1 < 0$, one can plot $z = \lambda$ and $z = \mu_1 e^{\lambda\tau}$, when $\tau = 1$; there are three cases; single intersection when $\mu_1 = \mu_1^* = -e^{-1}$ when $\lambda = -1$, for $\mu_1 \in [\mu_1^*, 0]$ there are two real negative eigenvalues, for $\mu_1 < \mu_1^*$ there are no real eigenvalues; See Figure 1.3. Assume that $\lambda = \omega_0 + i\omega_1$ is complex, in the same manner for studying the stability for linear case one may have,

$$\tau_j = \frac{\pi}{2\omega_1^*} + \frac{k\pi}{\omega_1^*}, j \in \mathbb{Z}, \tag{1.18}$$

noting that $\omega_1^* = -\mu_1 > 0$; Thus, when $\tau = \tau' = \min \tau_j$, Equation (1.17) has a pair of purely imaginary roots. When $0 < \tau < \tau'$, all roots of (1.17) have negative real parts then the equilibrium $y = 0$ is asymptotically stable; if $\tau > \tau'$, then $y = 0$ is unstable. Assume $\lambda(\tau) = \omega_0(\tau) + i\omega_1(\tau)$ the root of Equation (1.17) satisfying $\omega_0(\tau_j) = 0$ and $\omega_1(\tau_j) = \omega_1^*$,

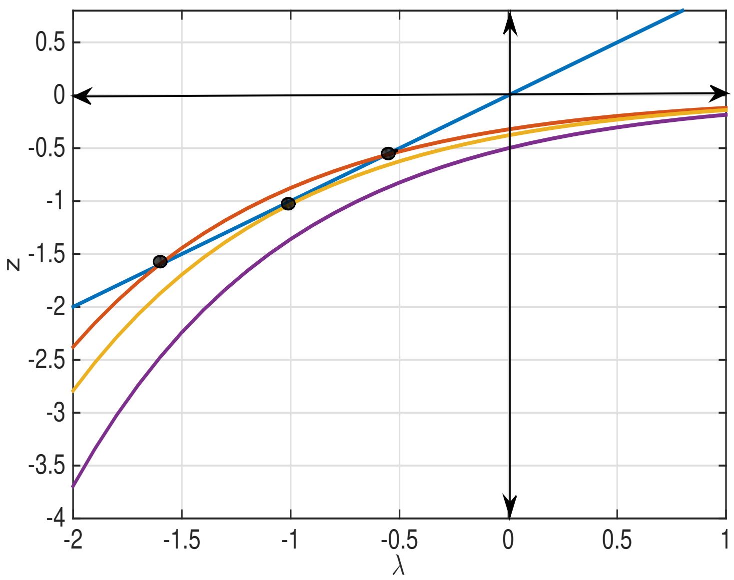

Figure 1.3: Number of real eigenvalues for the characteristic equation of DDE $y'(t) = \mu_1 y(t-\tau)$. Single intersection when $\mu_1 = \mu_1^* = -e^{-1}$ when $\lambda = -1$, for $\mu_1 \in [\mu_1^*, 0]$ there are two real negative eigenvalues, for $\mu_1 < \mu_1^*$ there are no real eigenvalues

$j = 0, 1, 2, \ldots$. The transversality condition

$$\frac{d\Re(\lambda)}{d\tau}\Big|_{\tau=\tau'} = \frac{d\omega_0}{d\tau}\Big|_{\tau=\tau'} = (\omega_1^*)^2 > 0, \quad j = 0, 1, 2, \ldots.$$

When $\tau = \frac{\pi}{2\omega_1^*}$, a Hopf bifurcation occurs at $y = 0$.

Noting from Figure 1.2 that the large time delay can induce instability and cause the solution to fluctuate when the time delay is larger than a critical value, the time delay can induce a stable limit cycle generated through the Hopf bifurcation and larger time delay can increase the amplitude of the oscillating orbits of the solution.

1.3.1 Lyapunov theory approach

Now, the author introduce the Lyapunov theory approach; by considering a more general type for DDE (1.9) with one delay. Thus, a functional delay differential system is given by

$$\begin{aligned}
\dot{\mathbf{y}}(t) &= \mathbf{f}(t, \mathbf{y}_t), \quad t \geq t_0, \\
\mathbf{y}_t(\theta) &= \mathbf{y}(t+\theta), \quad -\tau \leq \theta \leq 0, \\
\mathbf{f}(t, 0) &\equiv 0, \quad \mathbf{y}_{t_0} = \phi.
\end{aligned} \tag{1.19}$$

Let $\mathscr{C}_n : [-\tau, 0] \to \mathbb{R}^n$ be the set of continuous functions, where $\tau > 0$ is fixed. Suppose $t \in \mathbb{R}$ and $\mathbf{y} : [t-\tau, t] \to \mathbb{R}^n$ is continuous. Define $\mathbf{y}_t \in \mathscr{C}_n$ by $\mathbf{y}_t(\theta) = \mathbf{y}(t+\theta)$ for $\theta \in [-\tau, 0]$, where $\phi \in \mathscr{C}_n$, such that one may consider the existence and uniqueness of solutions, without loss of generality, the solution $\mathbf{y}_t = 0$ is an equilibrium. In this approach, the idea is to consider a classical positive definite Lyapunov function $V(t, \mathbf{y}(t))$, such that its derivative with respect to time along the trajectories of System (1.19) is negative definite. This concept is formalized in the following theorems.

Theorem 1.3.1. *[72] Let u_1, u_2 and $u_3 : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing functions such that $u_1(\theta)$ and $u_2(\theta)$ are strictly positive for all $\theta > 0$. Assume that the vector field $\mathbf{f}$ of (1.19) is bounded for bounded values of its arguments. If there exists a continuous and differentiable function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$ such that:*

i) $u_1(\|\phi(0)\|) \leq V(t, \phi) \leq u_2(\|\phi\|)$,

ii) $\dot{V}(t, \phi) \leq -u_3(\|\phi(0)\|)$ *for all trajectories of (1.19) satisfying*

$$V(t+\theta, \phi(t+\theta)) \leq V(t, \phi(t)), \quad \theta \in [-\tau, 0], \tag{1.20}$$

then the solution $\mathbf{y}_t = 0$ is uniformly stable for (1.19).

Additionally, if $u_3(\theta) > 0$ and there exists a strictly increasing function $u_4 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $u_2(\theta) > \theta$ as long with *i*) and *ii*), verifying that

$$V(t+\theta, \mathbf{y}(t+\theta)) \leq u_4(V(t, \mathbf{y}(t))), \quad \theta \in [-\tau, 0], \tag{1.21}$$

such a function V is called Lyapunov-Razumikhin function and the solution $\mathbf{y}_t = 0$ is uniformly asymptotically stable for System (1.19). Commonly, the functions u_4 are considered as $u_4 = \mathscr{N}\theta$ where $\mathscr{N}$ is a constant strictly greater than 1. The Lyapunov functions in Razumikhin approach are of the form

$$V(t) = \mathbf{y}^T P\mathbf{y}(t), \tag{1.22}$$

where P is a symmetric positive definite matrix of dimension n. Thus Equation (1.22) becomes

$$\mathbf{y}^T(t+\theta)P\mathbf{y}(t+\theta) \leq \mathscr{N}\mathbf{y}^T P\mathbf{y}(t), \quad \theta \in [-\tau, 0]. \tag{1.23}$$

Theorem 1.3.2. *[72] Let u_1, u_2 and $u_3 : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions such that $u_1(\theta)$ and $u_2(\theta)$ are strictly positive for all $\theta > 0$ and $u_1(0) = u_2(0) = 0$. Assume that the vector field $\mathbf{f}$ of (1.19) is bounded for bounded values of its arguments. If there exists a continuous and differentiable function $V : \mathbb{R} \times \mathscr{C}[-\tau, 0] \to \mathbb{R}_+$ such that $i)$ and $ii)$ of Theorem 1.3.1 satisfied then $\dot{V}(t,\phi) = \lim_{\varepsilon \to 0^+} \sup \dfrac{V(t+\varepsilon, \mathbf{y}(t+\varepsilon)) - V(t, \mathbf{y}(t))}{\varepsilon}$. Such that, the solution $\mathbf{y}_t = 0$ of (1.19) is uniformly stable. Moreover, if $u_3(\theta) > 0$, then the solution $\mathbf{y}_t = 0$ is uniformly asymptotically stable for (1.19).*

Example 1.3.2. Consider the linear delay differential equation

$$\dot{y}(t) = -\mu_1 y(t-\tau) - \mu_0 y(t), \quad t \geq t_0, \tau > 0, \tag{1.24}$$

where $\mu_0 > 0$ and μ_1 are constants. To derive stability conditions for (1.24), one may introduce a functional $V : \mathbb{R} \times \mathscr{C}[-\tau, 0] \to \mathbb{R}$ as follows

$$V(\phi) = \frac{\phi^2(0)}{2} + \frac{\mu_0}{2}\int_{-\tau}^{0} \phi^2(\theta)d\theta.$$

To find $\dot{V}$, utilizing Theorem 1.3.2, by simple computation, one obtains

$$\dot{V}(\phi) = -\frac{\mu_0}{2}\phi^2(0) - \mu_1\phi(0)\phi(-\tau) - \frac{\mu_0}{2}\phi^2(-\tau). \tag{1.25}$$

The right hand side of (1.25) is a quadratic form in $(\phi(0), \phi(-\tau))$. Thus, one may have to find parametric restrictions in which this quadratic form is positive, such that $\mu_0^2 \geq \mu_1^2$ and positive definiteness if $\mu_0^2 > \mu_1^2$; which implies region of stability as $\mu_0 \geq |\mu_1|$ and asymptotic stability as $\mu_0 > |\mu_1|$.

Next, some preliminaries for SDDEs, existence and uniqueness of the solutions and stability criteria for SDDEs are introduced.

1.4 Stochastic Delay Differential Equations

A stochastic differential equation is a differential equation whose coefficients are random numbers or random functions of the independent variables. Just as in normal differential equations, the coefficients are supposed to be given, independently of the solution that has to be found. Hence stochastic differential equations are the appropriate tool for describing systems with external noise [67]. So far deterministic delay systems have been assumed. However, there is an increasing evidence that better consistency with some phenomena can be provided if the effects of random processes in the system are taken into account [9].

Biological populations are strongly affected by the random variation in their environment. An important characteristic of environmental noise is its spectrum, which describes the variance as a sum of sinusoidal waves of different frequencies. The spectrum of frequencies in noise is particularly important to dynamics and persistence of the systems [130]. However, the Brownian motion with normally distributed errors is usually and commonly used in the continuous differential models of dynamical systems. In this dissertation white noise type is considered; In white noise, the variance is the same at all frequencies. Therefore, this is the most thoroughly studied and applied form of noise. The reason for this is that it is a simple and easily articulated model for noise. From the observation point of view, the random effect of Brownian motion is more visualized, with normally distributed errors [91, 103].

Definition 1.4.1. [56] Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space with a filtration $\{\mathscr{A}_t\}_{t\geq 0}$. A one-demential (standard) Brownian motion is a real-valued continuous $\{\mathscr{A}_t\}$-adapted process $\{W_t\}_{t\geq 0}$ satisfying the following properties

1. $W(0) = 0$ a.s. (with probability 1).
2. For $0 \leq s < t \leq T$ the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t-s$; equivalently, $W(t) - W(s) \sim \sqrt{t-s}N(0,1)$, where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.
3. For $0 \leq s < t < u < v \leq T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

Indeed, the random perturbations which are present in the real world imply that deterministic equations are often an idealization. To model the dynamics of biological delay systems under random perturbations, stochastic delay differential equations (SDDEs) are used:

$$\begin{aligned} d\mathbf{y}(t) &= \underbrace{\mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau))dt}_{(a)} + \underbrace{\mathbf{g}(t, \mathbf{y}(t), \mathbf{y}(t-\tau))dW(t)}_{(b)}, \quad t \in [0, T], \\ \mathbf{y}(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{1.26}$$

Here, $\mathbf{y}(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$, with fixed time delay, Where $\psi(t)$ is an $\mathscr{A}_t$-measurable $\mathscr{C}([-\tau, 0], \mathbb{R}^n)$-valued random variable such that $\mathbb{E}\|\psi\|^2 < \infty$; ($\mathscr{C}([-\tau, 0], \mathbb{R}^n)$ means that the Banach space of all continuous paths from $[-\tau, 0] \to \mathbb{R}^n$ equipped with the supremum norm $\|\eta\| := \sup_{s\in[-\tau,0]} |\eta(s)|$, where $\eta \in \mathscr{C}$). Term (a) is the drift term and the second term (b) is the diffusion term; let $W(t)$ be an m-dimensional Brownian motion given on the filtered probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with a filtration $(\mathscr{A}_t)$ satisfying the usual condition (it is right continuous i.e. $\mathscr{A}_t = \bigcap_{s>t} \mathscr{A}_s$ and $\mathscr{A}_t$ contains all $\mathbb{P}$-null sets). $\mathbf{f} : \mathscr{C}([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^+ \to \mathbb{R}^n$ and $\mathbf{g} : \mathscr{C}([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^+ \to \mathbb{R}^{n \times m}$ are assumed to be continuous. Such that $W(t)$ depends continuously on $t \in [0, T]$; more details and some necessary results can be found in Chapter 4.

Example 1.4.1. Consider Hutchinson equation [61]

$$\frac{dy(t)}{dt} = ry(t)\left(1 - \frac{y(t-\tau)}{K}\right). \tag{1.27}$$

Here, $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity of the population and time-delay τ was considered as hatching time. One could just add a small random perturbation σdW, which usually referred to the noise term to Equation (1.27), which becomes

$$dy(t) = \left[ry(t)\left(1 - \frac{y(t-\tau)}{K}\right)\right] dt + \sigma dW. \tag{1.28}$$

In the Equation (1.28) the noise term does not include the dependent variable y, and hence the equation is referred to as a SDDE with additive noise. However, it may be more natural to consider extension from Hutchinson equation by looking at the proportionate population change $\frac{dy(t)}{y(t)}$ and adding the stochastic term to this quantity. This gives

$$\frac{dy(t)}{y(t)} = \left[\left(1 - \frac{y(t-\tau)}{K}\right)\right] dt. \tag{1.29}$$

Therefore, Equation (1.29) becomes

$$\frac{dy(t)}{y(t)} = \left[r\left(1 - \frac{y(t-\tau)}{K}\right)\right] dt + \sigma dW. \tag{1.30}$$

Multiplying by $y(t)$ obtains the following SDDE with multiplicative noise

$$dy = \left[r\left(1 - \frac{y(t-\tau)}{K}\right) y(t) \right] dt + \sigma y(t) dW. \tag{1.31}$$

This implies the more natural procedure, therefore, equations with multiplicative noise will be only considered in this thesis. Figure 1.4 shows the effect of environmental fluctuations on a Hutchinson equation such that $r = 0.15$ and $k = 1$; Top Banners show simulation results for $\tau = 5.6$ and it indicates that the population attains its steady state value 1 regardless the external noise. Hence, it fluctuates within the interval [0.95,1.15] as $\sigma^2 = 0.01$ (top-left), and as the intensities of white noise increases to $\sigma^2 = 0.05$ it fluctuates within [0.65,1.5] (top-right). When the magnitude of time delay is increased to a threshold value $\tau = 11$ (periodic oscillations) and taking $\sigma^2 = 0.01$ the stochastic fluctuations disappears (bottom-left), and as $\sigma^2 = 0.05$ one may observe abrupt oscillation in population (bottom-right).

Remark 1.4.1. One of the important facts about the impact of the environmental noise is that it can suppress a potential population explosion [92, 93]; See Figure 1.5.

To illustrate this phenomena, consider Equation (1.8) with multiplicative noise

$$dy = \mu_1 y(t-\tau) dt + \sigma y(t) dW. \tag{1.32}$$

As $\mu_1 > 0$ the solution of (1.8) increases exponentially to infinity as $t \to \infty$. However, Figure 1.5 shows the effect of environmental fluctuations on (1.8), with $\mu_1 = 0.06$ and $\tau = 0.4$, and $\sigma^2 = 0.16$.

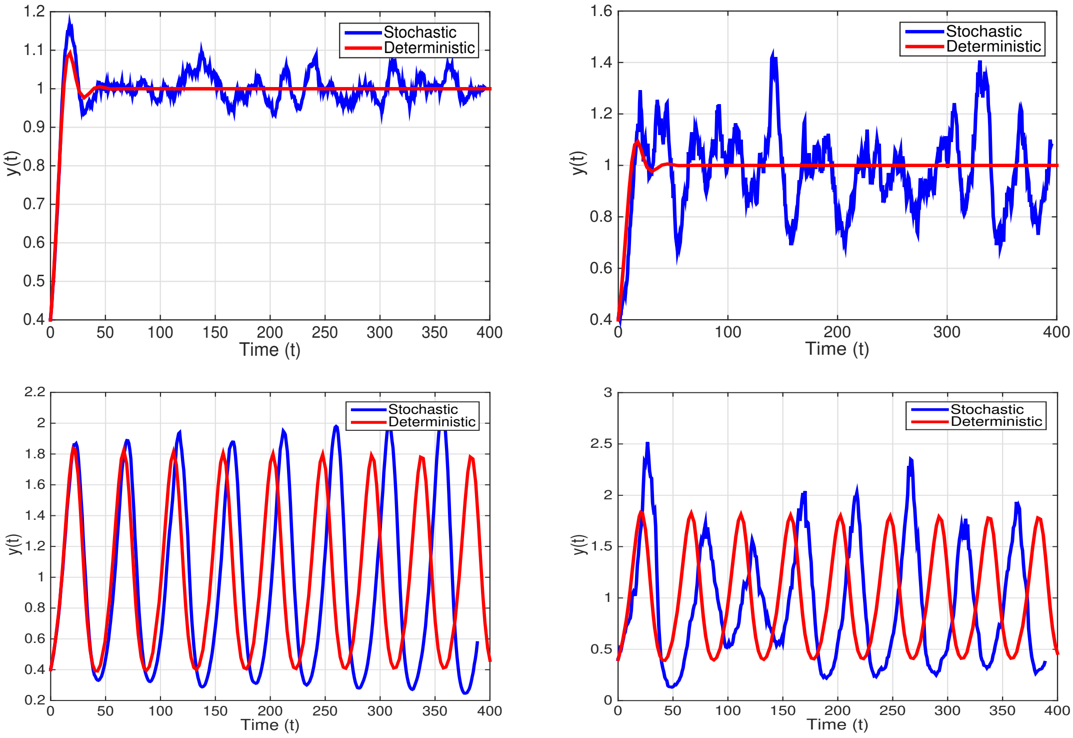

Figure 1.4: Numerical simulations of deterministic Hutchinson DDE (1.27) and its corresponding SDDE (1.31). When $r = 0.15$ and $k = 1$. Top Banners show simulation results for $\tau = 5.6$ and it indicates that the population attains its steady state value 1 regardless the external noise. Hence, it fluctuates within the interval [0.95,1.15] as $\sigma^2 = 0.01$ (top-left), and as the intensities of white noise increases to $\sigma^2 = 0.05$ it fluctuates within [0.65,1.5] (top-right). When the magnitude of time delay is increased to a threshold value $\tau = 11$ (periodic oscillations) and taking $\sigma^2 = 0.01$ the stochastic fluctuations disappears (bottom-left), and as $\sigma^2 = 0.05$ abrupt oscillation in population is observed (bottom-right)

1.5 Existence and Uniqueness of the Solutions for SDDEs

Consider $W(t)$ be a 1-dimensional Wienner process, an autonomous scalar stochastic delay differential equation of the form

$$\begin{aligned} &dy(t) = f(y(t), y(t-\tau))dt + g(y(t), y(t-\tau))dW(t), \quad t \in [0,T], \\ &y(t) = \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{1.33}$$

Equation (1.33) can be formulated as

$$y(t) = y(0) + \int_0^t f(y(s), y(s-\tau))ds + \int_0^t g(y(s), y(s-\tau))dW(s), \tag{1.34}$$

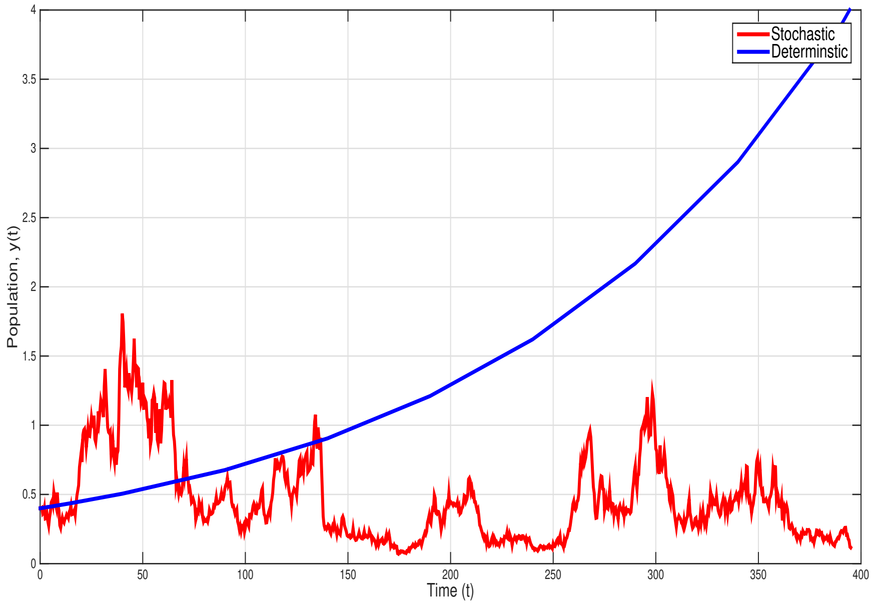

Figure 1.5: The impact of environmental Brownian noise that suppresses explosions in population dynamics. Described by $dy = \mu_1 y(t-\tau)dt + \sigma y(t)dW$ and its corresponding deterministic Equation (1.8).

for $t \in [0,T]$ and with $y(t) = \psi(t)$, for $t \in [-\tau, 0]$. The second integral in (1.34) is a stochastic integral in the Itô sense; If it is taken as Stratonovich integral the notation of the form $\int_0^t g(s,y(s)) \circ dW(s)$ is used. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\psi : [-\tau, 0] \to \mathbb{R}$. Now, one may introduce the following Theorem for Equation (1.33) [10, 23].

Theorem 1.5.1. *Problem (1.33) has a unique strong solution, provided that the uniform Lipschitz condition and a linear growth bound are satisfied for both f and g.*

Example 1.5.1. Consider the stochastic delay differential equation

$$\begin{aligned} dy(t) &= \mu_1 y(t-\tau)dt + \sigma dW(t), \quad t \geq 0, \\ y(t) &= t+1, \quad t \in [-\tau, 0]. \end{aligned} \tag{1.35}$$

Assume $\mu_1 = -1$ and $\tau = 1$; conditions of Theorem 1.5.1 can be easily verified. Thus, one may solve (1.35) by Itô's formula, in the interval [0,1], so that

$$y_1(t) = y(0) - \int_0^t s ds + \int_0^t \sigma dW(s) = 1 - \frac{t^2}{2} + \sigma W(t).$$

In the interval [1,2], one obtains

$$
\begin{aligned}
y_2(t) &= y(1) + \sigma W(1) + \int_1^t (-1 + \frac{(s-1)^2}{2} + \sigma W(s-1))ds + \int_1^t \sigma dW(s) \\
&= \frac{(t-1)^3}{6} - t + \frac{3}{2} + \int_1^t \sigma W(s-1)ds + \sigma W(t).
\end{aligned}
$$

Similarly, in the interval [2,3], the solution is

$$
\begin{aligned}
y_3(t) &= -\frac{1}{3} - \int_2^t \Big(\frac{(t-2)^3}{6} - t + \frac{5}{2}\Big)ds + \int_1^2 \sigma W(s-1)ds + \sigma W(2) \\
&\quad + \int_2^t \int_1^{s_1 - 1} \sigma W(s-1)dsds_1 + \int_2^t \sigma W(s-1)ds + \int_2^t \sigma dW(s) \\
&= \frac{8}{3} - \frac{(t-2)^4}{24} + \frac{t^2}{2} - \frac{5}{2}t + \int_1^2 \sigma W(s-1)ds + \int_2^t \int_1^{s_1-1} \sigma W(s-1)dsds_1 \\
&\quad + \int_2^t \sigma W(s-1)ds + \sigma W(t).
\end{aligned}
$$

Noting that $\int_0^t \sigma dW(s)$ is a martingale. Hence, $\mathbb{E}\Big(\int_0^t \sigma dW(s)\Big) = 0$. To find the mean function of $y(t)$, one can take the expectation of the solutions on their intervals as follows

$$
\mathbb{E}(y(t)) = \begin{cases} 1 - \frac{t^2}{2}, & t \in [0,1]; \\ \frac{(t-1)^3}{6} - t + \frac{3}{2}, & t \in [1,2]; \\ \frac{8}{3} - \frac{(t-2)^4}{24} + \frac{t^2}{2} - \frac{5}{2}t, & t \in [2,3]. \end{cases}
$$

Numerical methods for SDDEs are very under-studying and development. They must usually be used carefully from methods either for deterministic DDEs, or for Stochastic Ordinary Differential Equations (SODEs). Direct analysis of some methods for SDDEs has been considered in Chapter 7.

1.6 Stability Criteria for SDDEs

There are at least three different types of stability for SDDEs [91]. Consider the following scalar SDDE with $W(t)$ be a 1-dimensional Wienner process

$$\begin{aligned} dy(t) &= f(t,y(t),y(t-\tau))dt + g(t,y(t),y(t-\tau))dW(t), \quad t \in [0,T], \\ y(t) &= \psi(t), \quad t \in [-\tau,0]. \end{aligned} \tag{1.36}$$

Hence, Equation (1.36) can be formulated as

$$y(t) = y(0) + \int_0^t f(s,y(s),y(s-\tau))ds + \int_0^t g(s,y(s),y(s-\tau))dW(s). \tag{1.37}$$

The main ideas of pth mean stability of the trivial solution of Equation (1.37) with respect to perturbations in $\psi(.)$ (for $1 \leq p < \infty$) are discussed in the next definition, also with mean square stability when $p = 2$.

Definition 1.6.1. [11] For some $p > 0$, the trivial solution of the SDDE (1.37) is called

- Locally stable in the pth mean, if for each $\varepsilon > 0$, there exists a $\delta \geq 0$ such that $\mathbb{E}(|y(t;t_0,\psi)|^p) < \varepsilon$ whenever $t \geq t_0$ and $\mathbb{E}(\sup_{t\in[t_0-\tau,t_0]} |\psi(t)|^p) < \delta$;
- Locally asymptotically stable in the pth mean if it is stable in the pth mean and if there exists a $\delta \geq 0$ such that whenever $\mathbb{E}(\sup_{t\in[t_0-\tau,t_0]|\psi(t)|^p}) < \delta$ then $\mathbb{E}(|y(t;t_0,\psi)|^p) \to 0$ for $t \to \infty$;
- Locally exponentially stable in the pth mean if it is stable in the pth mean and if there exists a $\delta \geq 0$ such that whenever $\mathbb{E}(\sup_{t\in[t_0-\tau,t_0]|\psi(t)|^p}) < \delta$ there exists some finite constant C and a $u^* > 0$ such that $\mathbb{E}(|y(t;t_0,\psi)|^p) \leq C\mathbb{E}(\sup_{s\in[t_0-\tau,t_0]} |\psi(s)|^p) exp(-u^*(t-t_0))$ $(t_0 \leq t < \infty)$. If δ is arbitrarily large then the stability, in the above, is in each case global rather than

local.

- (Stability in probability) The trivial solution of the SDDE (1.37) is termed stochastically stable in probability, if for each $e \in (0,1)$ and $\varepsilon > 0$, there exists a $\delta \equiv \delta(e,\varepsilon) \geq 0$ such that

$$\mathbb{P}(|y(t;t_0,\psi)| \leq \varepsilon \quad \text{for all} \quad t \geq t_0) \geq 1-e,$$

whenever $t \geq t_0$ and $\sup_{t\in[t_0-\tau,t_0]} |\psi(t)|^p < \delta$ with probability 1.

Stability conditions for SDDEs can be also stated in terms of Lyapunov functionals, similar to the theorems for DDEs. Now, the Lyapunov theory approach for SDDEs is discussed; First, consider a more general type for (1.26) with one delay. Thus, an It$\hat{o}$ type SDDEs is given by

$$\begin{aligned}
&d\mathbf{y}(t) = \mathbf{f}(t,\mathbf{y}_t)dt + \mathbf{g}(t,\mathbf{y}_t)dW(t), \quad t \geq t_0, \\
&\mathbf{y}_t(\theta) = \mathbf{y}(t+\theta), \quad -\tau \leq \theta \leq 0, \\
&\mathbf{f}(t,0) \equiv 0, \quad \mathbf{y}_{t_0} = \psi.
\end{aligned} \tag{1.38}$$

Define $\mathbf{y}_t \in \mathscr{C}_n$ by $\mathbf{y}_t(\theta) = \mathbf{y}(t+\theta)$ for $\theta \in [-\tau,0]$, where $\psi \in \mathscr{C}_n$, such that the existence and uniqueness of solutions is considered, without loss of generality, the solution $\mathbf{y}_t = 0$ is an equilibrium.

Theorem 1.6.1. *[72] Suppose there is a continuous functional $V : [t_0,\infty] \times \mathscr{C}[-\tau,0] \to \mathbb{R}$ such that for any solution of (1.38), where $\mathbf{y}_t(\theta) = \mathbf{y}(t+\theta)$ as $-\tau \leq \theta \leq 0$, the following inequalities hold; such that C_i $i = 1,2,3$ are positive constants*

$$\begin{aligned}
&V(t,\mathbf{y}_t) \geq C_1|\mathbf{y}(t)|^2 \\
&\mathbb{E}V(t,\mathbf{y}_t) \leq C_2 \sup_{-\tau\leq\theta\leq 0} \mathbb{E}|\mathbf{y}(t+\theta)|^2,
\end{aligned} \tag{1.39}$$

for arbitrary $t \geq t_0$, $s \geq t$,

$$\mathbb{E}[V(s,\mathbf{y}_s) - V(t,\mathbf{y}_t)] \leq -C_3 \int_t^s \mathbb{E}|\mathbf{y}(h)|^2 dh. \tag{1.40}$$

Then the trivial solution of (1.38) is asymptotically mean-square stable.

Example 1.6.1. Consider a SDDE of the form

$$dy(t) = -\mu_1 y(t-\tau)dt + \mu_2 y(t)dW(t), \quad t > t_0, \tag{1.41}$$

where μ_1, μ_2 are positive constants. Sufficient conditions for asymptotic mean-square stability of (1.41) are:

$$0 < \mu_1\tau < 1, \qquad \mu_1(1-\mu_1\tau) > \frac{\mu_2^2}{2}.$$

To prove this, consider the functional

$$V(\psi) = \Big[\psi(0) - \mu_1 \int_{-\tau}^0 \psi(\theta)d\theta\Big]^2 + \mu_1^2 \int_{-\tau}^{\theta} ds \int_s^0 \psi^2(\theta)d\theta. \tag{1.42}$$

By Itô formula, one obtains

$$\begin{aligned}
dV(y_t) = 2\Big[y(t) - \mu_1 \int_{t-\tau}^t y(\theta)d\theta\Big](dy(t) - \mu_1 y(t)dt + \mu_1 y(t-\tau)dt) \\
+ \Big[\mu_2^2 y^2(t) + \mu_1^2 \tau y^2(t) - \mu_1^2 \int_{t-\tau}^t y^2(\theta)d\theta\Big]dt \\
= 2\Big[y(t) - \mu_1 \int_{t-\tau}^t y(\theta)d\theta\Big](\mu_2 y(t)dW(t) - \mu_1 y(t)dt) \\
+ \Big[\mu_2^2 y^2(t) + \mu_1^2 \tau y^2(t) - \mu_1^2 \int_{t-\tau}^t y^2(\theta)d\theta\Big]dt.
\end{aligned}$$

Noting that,

$$2\mu_1^2 y(t)\int_{t-\tau}^{t} y(\theta)d\theta \le \mu_1^2\Big[\tau y^2(t)+\int_{t-\tau}^{t} y^2(\theta)d\theta\Big].$$

Hence,

$$dV(y_t) \le 2\mu_2\Big[y(t)-\mu_1\int_{t-\tau}^{t} y(\theta)d\theta\Big]y(t)dW(t)-[2\mu_1(1-\mu_1\tau)-\mu_2^2]y^2(t)dt. \quad (1.43)$$

Integration of both parts of (1.43) from $s \in [t_0, t]$ to t, then taking the expectation yields

$$\mathbb{E}[V(y_t)-V(y_s)] \le -[2\mu_1(1-\mu_1\tau)-\mu_2^2]\int_{s}^{t}\mathbb{E}y^2(h)dh. \quad (1.44)$$

From inequality (1.44), one gets

$$\mathbb{E}V(y_t) \le \mathbb{E}V(y_{t_0}), \quad t \ge t_0. \quad (1.45)$$

Therefore,

$$\mathbb{E}\Big[y(t)-\mu_1\int_{t-\tau}^{t} y(\theta)d\theta\Big]^2 \le \mathbb{E}V(y_{t_0}), \quad \int_{t_0}^{\infty}\mathbb{E}y^2(s)ds < \infty. \quad (1.46)$$

Inequalities (1.46) and condition $\mu_1\tau < 1$ implies mean-square stability, since

$$\sup_{t\ge t_0}\mathbb{E}y^2(t) \le C_1 \sup_{-\tau\le\theta\le 0}\mathbb{E}\psi^2(\theta). \quad (1.47)$$

From inequalities (1.46) and since $\lim_{t\to\infty}\mathbb{E}y^2(t)=0$; this implies asymptotic mean square stability.

1.7 Main Objectives and Significance of the Research

The main objective of this research is to study the qualitative and quantitative features of deterministic and stochastic DDEs with biological systems. The study includes the following.

- Develop a class of DDEs to study and analyze modelling ecological and epidemic systems.
- Investigate the impact and role of time-delays in the dynamics of the models.
- Study the impact of Allee effect in modeling predator-prey systems and in the complexity of the model.
- Study the qualitative features of SDDEs and investigate the impact of environmental fluctuations on the dynamical behaviour of the proposed models.
- Provide and select the suitable numerical techniques for solving the resulting models of DDEs and SDDEs.

1.7.1 Research significance

Recently, there has been a worldwide movement aimed at enhancing the understanding of ecological stability. However, many significant problems are still unsolved. Most of the studies in population dynamics models utilize deterministic models. However, the natural growth of populations is always affected by stochastic perturbations which should be taken into account in the process of mathematical modelling. It is observed that small scale of white noise can promote the survival of population; while large noises can lead to extinction of the population, this would not happen in the deterministic systems without noises. Studying the existence of an ergodic stationary distribution is an interesting problem, the key difficulty is how to construct a suitable stochastic Lyapunov function and a bounded domain.

In addition, this research sheds some light on the influence of random noises that can suppress the explosion of the species, where the solutions of the undisturbed system is unbounded. Furthermore, introduction of noise in the deterministic epidemic models can modify the basic reproductive number giving rise to a new threshold quantity, in which the disease dies out more rapidly along with intensity of white noise is large.

There are many models available in the literature representing the predator-prey interactions. However, studying the impact of time delays and additive Allee effect in multi-species models is still lacking. This is significant because establishing such a model with theses properties exhibits rich dynamics behaviour such as bistability of equilibria and Hopf bifurcation. Additionally, sensitivity analysis to evaluate the uncertainty of the state variables to small changes in the Allee parameters and time delays are investigated. Throughout the thesis, examples are contributed to demonstrate the results and are augmented with Matlab numerical simulations.

This dissertation consists of 8 chapters. Chapter 2 introduces a predator-prey model with time delay and hunting cooperation on predators. Cooperative hunting parameter is assumed with a Holling type II functional response with delay. The boundedness of the system has been shown, and a local and global stability analysis of the interior equilibrium have been implemented. The critical values of delays, where the Hopf bifurcation occurs are obtained. Chapter 3 provides a system of DDEs of two-prey one-predator system, where the growth of both preys populations subject to Allee effects, and there is a direct competition between the two-prey species having a common predator. Sufficient conditions for local stability of positive interior equilibrium and existence of Hopf bifurcations in terms of threshold parameters τ_1^* and τ_2^* are obtained. A Lyapunov functional is deducted to investigate the global stability of positive interior equilibrium. Sensitivity analysis to evaluate the uncertainty of the state variables to small changes in the Allee parameters and time delays are also investigated.

Chapter 4 is devoted to investigate the dynamics of SDDEs for predator-prey system with hunting cooperation in predators. Existence and uniqueness of global positive solution and stochastically ultimate boundedness are investigated. Some sufficient conditions for persistence and extinction, using Lyapunov functional, are obtained.

Chapter 5 deals with stochastic DDEs of three-species predator-prey systems with cooperation among prey species. The proposed model takes into consideration that the intrinsic growth rate of preys and the death rate of predator are subject to environmental noise. Sufficient conditions of existence and uniqueness of an ergodic stationary distribution of the positive solution to the model have been established, by constructing suitable Lyapunov function. Sufficient criteria for extinction of the predator populations are also obtained. These conditions are expressed in terms of the threshold parameter $\mathscr{T}_0^s$ which rely strongly upon the Brownian motion.

Chapter 6 is devoted to a stochastic SIRC epidemic model for COVID-19 with time-delay. For the stochastic analysis, existence and uniqueness of positive global solutions are investigated. Some interesting sufficient conditions that guarantee the existence of unique ergodic stationary distribution for the stochastic SIRC model are also derived by using the stochastic Lyapunov function and Ito's formula. The sufficient conditions for the extinction of the disease are also obtained.

Chapter 7 is devoted to numerical solutions and suitable numerical schemes of stochastic delay differential equations.

Chapter 8 summaries the main conclusions of the research and provides some recommendations for future directions.

Next, DDEs of predator-prey system with hunting cooperation among predators is discussed.

Chapter 2: Delay Differential Equations of Predator-Prey Interactions with Hunting Cooperation in Predators

2.1 Introduction

In this chapter, the dynamics of DDEs, with two different time-delays, for a predator-prey system with hunting cooperation among predators is investigated. Section 2 introduces the model. Section 3 shows the existence of steady states and boundedness of the solutions. Section 4 studies the qualitative behaviours of the model throughout local stability of the steady states and Hopf bifurcation. The global stability, using Lyapunov functional, is investigated in Section 5. Some numerical simulations are provided in Section 6 and concluding remarks in Section 7.

Predator-Prey (PP) interaction is one of the most extensively studied issues in ecological and mathematical literature; See [34, 47, 97]. The classic predator-prey models are mostly variations of the Lotka–Volterra model, which was proposed by Lotka [88] and Volterra [132] which are a system of first order, nonlinear differential equations that describe the dynamics and interactions between two or more species of biological systems. Of course, the qualitative properties of a predator-prey system such as stability of the steady states, bifurcations analysis and oscillation of the solutions usually depend on the system parameters; See [74].

Incorporating time-delays has been considered by many authors in predator-prey models and biological systems [15, 16, 20, 105, 114, 117]. Additionally, one important component of predator-prey relationships is the functional response of predators to their prey(s)' densities. The response of predators to different prey densities depends on the feeding behavior of individual predators. In [58], Holling discussed three different types of functional responses: Holling type I (linear), type II, type III, etc. These responses are used to model the phenomena of predation, which captures the usual properties, for instance, positivity and increasing; See also [13, 45, 101, 128].

2.2 Model Formulation

Hutchinson [61], first introduced the delay in a logistic differential equation. He proposed a delay differential model for a single species of the form (Recall the equation) as follows

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t-\tau)}{K}\right), \quad \text{with} \quad x(\theta) = \phi(\theta) > 0, \theta \in [-\tau, 0], \phi(0) > 0.$$

Here, $(r > 0)$ is the intrinsic growth rate and $(K > 0)$ is the carrying capacity of the population and time-delay τ was considered as hatching time. $\phi(\theta)$ is continuous on $\theta \in [-\tau, 0]$. (This equation is referred to as the Hutchinson's equation or delayed logistic equation).

Consider a simple general two dimensional delayed model of interaction between a prey, $x(t)$, and a predator, $y(t)$, of the form

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\mathscr{G}_1(x(t-\tau_1), K) - y(t)\mathscr{F}(x(t)), \\ \frac{dy(t)}{dt} &= y(t)\mathscr{G}_2(y(t)) + \mu y(t)\mathscr{F}(x(t-\tau_2)). \end{aligned} \tag{2.1}$$

The function $\mathscr{G}_1(x(t-\tau_1), K)$ is the logistic per capita growth rate of prey, where K is the environmental carrying capacity, and and $\mathscr{G}_2(y)$ is the per capita growth rate of predator. $\mathscr{F}(x(t))$ and $\mu\mathscr{F}(x(t-\tau_2))$ are functional responses of predator for a particular prey and μ is the conversion efficiency $(0 < \mu < 1)$. Time-delay τ_1 represents the gestation period of the prey or reflects the impact of density dependent feedback mechanism [38]. Time-delay τ_2 is incorporated in the functional response of predator equation to represent the reaction time with the prey: In reality, the reproduction of predators is not immediate to the consumption of prey, as there is some discrete time lag necessary for prey gestation [105].

There exist various and extensive studies of the dynamics of the delayed PP model; See, e.g., [19, 76, 87, 140]. In [76], the authors investigated the complex dynamics of a

delayed PP system with cooperation among the prey species, they have considered time delays in the growth components for each of the species. Many studies have explored the effect of predator hunting cooperation on PP systems [3, 104, 144]. Berec [19] assumed a Holling type II functional response of the form $\mathscr{F}(x,y) = \dfrac{\sigma(y)x}{1+c(y)\sigma(y)x}$, where σ is the consumption rate of prey by their predator and c is the handling time of the predator, both σ and c are not constant quantities. Alves *et al.* [3] considered consumption rate depending on the predator density to implement predator cooperation for searching and capturing the prey. Assuming that $\alpha > 0$ be the cooperative 'hunting' parameter, with functional response of the form $\mathscr{F}(x,y) = \dfrac{(1+\alpha y)x}{1+c(1+\alpha y)x}$.
The suggested model takes the form

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t)(1-\frac{x(t-\tau_1)}{K}) - \frac{[1+\alpha y(t)]x(t)y(t)}{1+c(1+\alpha y(t))x(t)}, \\ \frac{dy(t)}{dt} &= y(t)(-\delta - ay(t)) + \frac{\mu[1+\alpha y(t)]x(t-\tau_2)y(t)}{1+c(1+\alpha y(t))x(t-\tau_2)}, \end{aligned} \tag{2.2}$$

where $\delta > 0$ is death rate of predator and $a > 0$ is an intra-specific competition rate for predators. This system is subject to initial conditions

$$\begin{aligned} &x(\theta) = \phi(\theta) \geq 0, \quad y(0) \geq 0. \\ &\quad \theta \in [-\tau, 0], \quad \tau = \max\{\tau_1, \tau_2\}, \quad \phi(0) > 0, \end{aligned} \tag{2.3}$$

ϕ is continuous bounded functions in the interval $[-\tau, 0]$. The description of the model parameters is presented in Table 2.1.

Table 2.1: One biological meaning for the parameters of Model (2.2)

Parameters	Description
r	Intrinsic growth rate
K	Environmental carrying capacity
δ	Death rate for predator
μ	Conversion efficiency
α	Cooperative hunting parameter
c	Handling time of the predator,
a	Predator intra-specific competition rate

2.3 Existence of Equilibrium Points

There are three types of equilibrium points for the deterministic System (2.2): (i) Trivial equilibrium point $\mathscr{E}_0 \equiv (0,0)$; (ii) Axial equilibrium point $\mathscr{E}_1 \equiv (K,0)$; And (iii) Interior equilibrium point $\mathscr{E}^* \equiv (x^*, y^*)$. Here,

$$x^* = \frac{\delta + ay^*}{(1+\alpha y^*)[\mu - c(\delta + ay^*)]}. \tag{2.4}$$

y^* is a positive real root of the equation

$$\eta_5 y^5 + \eta_4 y^4 + \eta_3 y^3 + \eta_2 y^2 + \eta_1 y^1 + \eta_0 = 0, \tag{2.5}$$

where

$$\begin{aligned}
&\eta_5 = Kc^2a^2, \quad \eta_4 = 2Kc^2a^2\alpha + 2\delta aKc^2 - 2K\mu ca,\\
&\eta_3 = Kc^2\delta^2 + \mu^2 K + 4\delta aKc^2\alpha + Kc^2a^2 - 2\mu Kc\delta - 4K\mu ca\alpha,\\
&\eta_2 = 2Kc^2\delta^2\alpha + 2\alpha\mu^2 K + 2\delta aKc^2 + rK\mu\alpha ca - 4\alpha\mu Kc\delta - 2K\mu ca\\
&\eta_1 = Kc^2\delta^2 + \mu^2 K + \mu ra + r\mu\alpha c\delta - 2\mu K\delta c - rK\mu^2\alpha - rK\mu a,\\
&\eta_0 = rK\mu(c\delta - \mu).
\end{aligned}$$

Equation (2.5) must have at least one positive real root if $c\delta < \mu$. Therefore, the existence of the coexisting equilibrium $\mathscr{E}^*$ assumes restrictions on the parameters so that

$$c\delta < \mu \quad and \quad y^* \leq \frac{\mu - c\delta}{ac}. \tag{2.6}$$

2.3.1 Boundedness of the solutions

One can check that the System (2.2) has a non-negative solution with a positive initial condition given in (2.3). To show the boundedness of this solution, the following

Lemma [4, 102] is introduced.

Lemma 2.3.1. *If for $t \geq 0$ and $x(0) \geq 0$ one may have $x' \leq x(a-bx)$ where $a > 0$, $b > 0$ then*

$$\lim_{t\to\infty} \sup x(t) \leq \frac{a}{b}.$$

Theorem 2.3.2. *The non-negative solution of the deterministic Model (2.2), $(x(t), y(t))$, satisfies*

$$\lim_{t\to\infty} \sup x(t) \leq Ke^{r\tau_1}, \quad \lim_{t\to\infty} \sup y(t) \leq \frac{\mu K e^{r\tau_2} - \delta}{a},$$

for $\tau_1, \tau_2 > 0$ with $\mu K e^{r\tau_2} > \delta$.

Proof. With the positive initial condition $(x(0), y(0))$, one can verify that the solution $(x(t), y(t))$ of the System (2.2) is non-negative. From the first equation of System (2.2) one may consider

$$\frac{dx(t)}{dt} \leq rx(t), \tag{2.7}$$

integrating both sides of (2.7) from $t - \tau_1$ to t one obtains

$$x(t-\tau_1) \geq x(t)e^{-r\tau_1}. \tag{2.8}$$

Using (2.8) and from the first Equation of (2.2), one gets

$$\frac{dx(t)}{dt} \leq x(t)(r - \frac{r}{K}e^{-r\tau_1}x(t)). \tag{2.9}$$

Following Lemma 2.3.1 one may have

$$\lim_{t\to\infty} \sup x(t) \leq Ke^{r\tau_1}.$$

i.e. for $\varepsilon > 0$, there exist $T_1 > 0$ such that $x(t) \leq Ke^{r\tau_1} + \varepsilon$, for all $t > T_1$. Similarly following the same computation as done for the first equation of (2.2), one obtains from the second equation of (2.2) the following

$$\lim_{t\to\infty} \sup y(t) \leq \frac{\mu K(e^{r\tau_2} + \varepsilon) - \delta}{a} = \xi, \quad \text{thus,} \tag{2.10}$$

$y(t) \leq \xi + \varepsilon$, for all $t > T_2$, conclusion of this Lemma can be achieved by letting $\varepsilon \to 0$.

2.4 Local Stability and Hopf Bifurcation

It is hard to find a closed analytical solution for the above nonlinear DDE Model (2.2), instated one can investigate their qualitative behavior by studying the stability of the steady states and Hopf bifurcation. The bifurcation analysis gives a deeper analysis about the model. It answers the query that "how does the behavior of the solutions change as parameters change".

By linearizing the system around $\mathscr{E}^* = (x^*, y^*)$, so that $x(t) = x^* + \tilde{x}(t)$, $y(t) = y^* + \tilde{y}(t)$, then one gets

$$\begin{aligned} \frac{d\tilde{x}(t)}{dt} &= a_1\tilde{x}(t) + a_2\tilde{y} + a_3\tilde{x}(t-\tau_1), \\ \frac{d\tilde{y}(t)}{dt} &= a_4\tilde{y}(t) + a_5\tilde{x}(t-\tau_2), \end{aligned} \tag{2.11}$$

where the coefficients are given by

$$\begin{aligned} &a_1 = \frac{c(1+\alpha y^*)^2 y^* x^*}{(1+c(1+\alpha y^*)x^*)^2}, \quad a_2 = -\frac{(2\alpha y^* + cx^*(1+\alpha y^*)^2 + 1)x^*}{(1+c(1+\alpha y^*)x^*)^2}, \quad a_3 = -\frac{rx^*}{K}, \\ &a_4 = -ay^*, \quad a_5 = \frac{\mu(1+\alpha y^*)y^*}{(1+c(1+\alpha y^*)x^*)^2}. \end{aligned}$$

The characteristic equation of the linearization Model (2.11) is given by

$$\lambda^2 - (a_1 + a_4)\lambda + a_1 a_4 + (a_3 a_4 - a_3\lambda)e^{-\lambda\tau_1} - a_2 a_5 e^{-\lambda\tau_2} = 0. \tag{2.12}$$

Define a threshold parameter $\mathscr{T}_0^d = \dfrac{\mu K}{\delta(1+cK)}$.

Remark 2.4.1. The extinction equilibrium $\mathscr{E}_0$ is always a saddle point, and the boundary equilibrium point $\mathscr{E}_1$ is locally asymptotically stable if $\mathscr{T}_0^d \equiv \dfrac{\mu K}{\delta(1+cK)} < 1$.

To gain insight regarding interior equilibrium $\mathscr{E}^*$, different values of time-lags τ_1 and τ_2 are considered as follows: (*i*) $\tau_1 = \tau_2 = 0$, (*ii*) $\tau_1 > 0$, $\tau_2 = 0$, (*iii*) $\tau_1 = 0$, $\tau_2 > 0$, (*iv*) $\tau_1 > 0$, $\tau_2 > 0$.

• **Case** (*i*)**:** When $\tau_1 = \tau_2 = 0$, Equation (2.12) becomes

$$\lambda^2 - (a_1 + a_3 + a_4)\lambda + a_1 a_4 + a_3 a_4 - a_2 a_5 = 0. \tag{2.13}$$

Thus all roots of (2.13) have negative real part if

$$(H_1) \qquad a_3 + a_4 < -a_1, \quad \text{and} \quad a_1 a_4 + a_3 a_4 > a_2 a_5 \qquad \text{hold.}$$

• **Case** (*ii*)**:** When $\tau_2 = 0, \tau_1 > 0$, Equation (2.12) becomes

$$\lambda^2 - (a_1 + a_4)\lambda + (a_1 a_4 - a_2 a_5) + (a_3 a_4 - a_3\lambda)e^{-\lambda\tau_1} = 0. \tag{2.14}$$

Let $\lambda = i\omega$ be root of (2.14), then it follows that

$$\begin{aligned} -\omega^2 + (a_1 a_4 - a_2 a_5) &= a_3\omega\sin\omega\tau_1 - a_3 a_4\cos\omega\tau_1 \\ -(a_1 + a_4)\omega &= a_3\omega\cos\omega\tau_1 + a_3 a_4\sin\omega\tau_1, \end{aligned} \tag{2.15}$$

which leads to

$$\omega^4 + c_1\omega^2 + c_2 = 0, \tag{2.16}$$

where $c_1 = (a_1+a_4)^2 - 2(a_1a_4 - a_2a_5) - a_3^2$ and $c_2 = (a_1a_4 - a_2a_5)^2 - (a_3a_4)^2$. Thus, Equation (2.16) has at least one positive root ω_1 if $c_2 < 0$, therefore, one may have

$$\tau_{1,j} = \frac{1}{\omega_1}\{\arccos\Big[\frac{(-\omega_1^2 + a_1a_4 - a_2a_5)a_3a_4 + (a_1+a_4)a_3\omega_1^2}{a_3^2\omega_1^2 + (a_3a_4)^2}\Big] + \frac{2j\pi}{\omega_1}\},$$
$$j = 0,1,2,\ldots. \tag{2.17}$$

Thus, $\mathscr{E}^*$ remains stable for $\tau_1 < \tau_1^{'}$, and unstable for $\tau_1 > \tau_1^{'}$ such that $\tau_1^{'} = \min\{\tau_{1,j}\}$.

• **Case** (iii)**:** For $\tau_1 = 0, \tau_2 > 0$, in the same manner, one obtains

$$\tau_{2,j} = \frac{1}{\omega_2}\{\arccos\Big[\frac{-\omega_2^2 + a_1a_4 + a_3a_4}{a_2a_5}\Big] + \frac{2j\pi}{\omega_2}\}, \quad j = 0,1,2,\ldots. \tag{2.18}$$

Therefore, $\mathscr{E}^*$ remains stable for $\tau_2 < \tau_2^{'}$, and unstable for $\tau_2 > \tau_2^{'}$ such that $\tau_2^{'} = \min\{\tau_{2,j}\}$ provided that $(a_1a_4 + a_3a_4)^2 < (a_2a_5)^2$.

• **Case** (iv)**:** When $\tau_1, \tau_2 > 0$, assuming that τ_1 is varying and τ_2 is fixed in its stable interval $\tau_2 \in [0, \tau_2')$. Assume that there exists a real number $\omega > 0$ such that $\lambda = i\omega$ is a root of the characteristic Equation (2.12), then separating real and imaginary parts, one gets

$$\begin{aligned} -\omega^2 + a_1a_4 - a_2a_5\cos\omega\tau_2 &= a_3\omega\sin\omega\tau_1 - a_3a_4\cos\omega\tau_1, \\ -(a_1+a_4)\omega + a_2a_5\sin\omega\tau_2 &= a_3\omega\cos\omega\tau_1 + a_3a_4\sin\omega\tau_1. \end{aligned} \tag{2.19}$$

Squaring and adding both sides, yields

$$\omega^4 + b_1\omega^2 + b_2\omega + b_3 = 0, \tag{2.20}$$

where,

$$b_1 = a_1^2 + a_4^2 - a_3^2 + 2a_2a_5\cos\omega\tau_2, \quad b_2 = -2(a_1+a_4)a_2a_5\sin\omega\tau_2,$$
$$b_3 = (a_1a_4)^2 - (a_3a_4)^2 + (a_2a_5)^2 - 2a_1a_2a_4a_5\cos\omega\tau_2.$$

Equation (2.20) is a peculiar equation in a complicated form, it is not easy to presume about the nature of the roots. Thus, by applying Descartes rule of signs one can say that (2.20) has at least one positive root ω_0 if

$$(H_2) \qquad (a_1a_4)^2+(a_2a_5)^2<(a_3a_4)^2+2a_1a_2a_4a_5\cos\omega\tau_2.$$

In this case, one may have

$$\begin{aligned}\tau_{1,j}=\frac{1}{\omega_0}\{\arccos\Big[&\frac{a_3\omega_0(-(a_1+a_4)\omega_0+a_2a_5\sin\omega_0\tau_2)}{(a_3\omega_0)^2+(a_3a_4)^2}\\ &-\frac{a_3a_4(-\omega_0^2+a_1a_4-a_2a_5\cos\omega_0\tau_2)}{(a_3\omega_0)^2+(a_3a_4)^2}\Big]+2j\pi\},\end{aligned} \tag{2.21}$$

where $j=0,1,2,\ldots$. Thus, $\mathscr{E}^*$ remains stable for $\tau_1<\tau_1^*$, such that
$\tau_1^*=\min\{\tau_{1,j}\}$ as in (2.21).

To check the transversality condition of Hopf bifurcation, τ_2 is fixed in its stable interval and differentiate equations (2.19) with respect to τ_1. Then substitute $\tau_1=\tau_{1,0}$ and $\omega=\omega_0$, one may have

$$\begin{aligned}A_2\Big(\frac{d(\Re\lambda)}{d\tau_1})|_{\tau_1=\tau_{1,0}}\Big)+A_1\Big(\frac{d(\omega)}{d\tau_1})|_{\tau_1=\tau_{1,0}}\Big)&=A_3\\ -A_1\Big(\frac{d(\Re\lambda)}{d\tau_1})|_{\tau_1=\tau_{1,0}}\Big)+A_2\Big(\frac{d(\omega)}{d\tau_1})|_{\tau_1=\tau_{1,0}}\Big)&=A_4,\end{aligned} \tag{2.22}$$

where

$$A_1=-2\omega_0+(-a_3-a_3a_4\tau_{1,0})\sin\omega_0\tau_{1,0}+a_2a_5\tau_2\sin\omega_0\tau_2-a_3\tau_{1,0}\omega_0\cos\omega_0\tau_{1,0},$$

$$A_2=(a_1+a_4)+(a_3+a_3a_4\tau_{1,0})\cos\omega_0\tau_{1,0}-a_3\omega_0\tau_{1,0}\sin\omega_0\tau_{1,0}-a_2a_5\tau_2\cos\omega_0\tau_2,$$

$$A_3=a_3\omega_0^2\cos\omega_0\tau_{1,0}+a_3a_4\omega_0\sin\omega_0\tau_{1,0},\;\; A_4=a_3a_4\omega_0\cos\omega_0\tau_{1,0}-a_3\omega_0^2\sin\omega_0\tau_{1,0}.$$

From (2.22), one gets

$$\Big(\frac{d(\Re\lambda)}{d\tau_1})|_{\tau_1=\tau_{1,0}}\Big)=\frac{A_2A_3-A_1A_4}{A_2^2+A_1^2}.$$

Assume that

$$(H_3) \qquad\qquad A_2A_3 > A_1A_4 \quad \text{holds,}$$

then a Hopf bifurcation occurs for $\tau_1 = \tau_{1,0}$. Therefore, for Case (*iv*), one can arrive at the following Theorem.

Theorem 2.4.1. *Suppose that $\mathscr{E}^*$ exists for System (2.2) and $(H_1)-(H_3)$ hold, such that $\tau_2 \in [0, \tau_2^{'})$, then there exists a positive threshold parameter τ_1^* such that the interior equilibrium $\mathscr{E}^*$ is locally asymptotically stable for $\tau_1 < \tau_1^*$, and unstable $\tau_1 > \tau_1^*$. Furthermore, System (2.2) undergoes a Hopf bifurcation at $\mathscr{E}^*$ where $\tau_1 = \tau_1^*$.*

If τ_1 is fixed in its stable interval and τ_2 varies, one can arrive at the following Remark.

Remark 2.4.2. If $\tau_1 \in [0, \tau_1^{'})$, there exists a threshold parameter τ_2^* such that the interior equilibrium $\mathscr{E}^*$ is locally asymptotically stable for $\tau_2 < \tau_2^*$, and unstable $\tau_2 > \tau_2^*$ where $\tau_2^* = \min\{\tau_{2,j}\}$ is given by

$$\tau_{2,j} = \frac{1}{\omega_3} \arccos\left[\frac{a_3a_4\cos\omega_3\tau_1 - \omega_3^2 - a_1a_4 - a_3\omega_3\sin\omega_3\tau_1}{a_2a_5}\right] + \frac{2j\pi}{\omega_3}, \; j = 0, 1, 2, \ldots, \tag{2.23}$$

2.5 Global Stability of the Interior Equilibrium Point

Now, the global stability of the interior equilibrium $\mathscr{E}^*$ is investigated, using Lyapunov functional.

Theorem 2.5.1. *Assume that $e_1 = 1 + c(1+\alpha y^*)x^*$, and $e_2 = 1 + c(1+\alpha y)x$. If $re_1e_2 < c(1+\alpha y^*)(1+\alpha y)(x^*y - y^*x)$, then System (2.2) is globally asymptotically stable at the interior equilibrium point.*

Proof. Assume the Lyapunov function at $\mathscr{E}^* \equiv (x^*, y^*)$ of the form

$$V(t) = \chi_1(x(t) - x^* - x^* \ln \frac{x(t)}{x^*}) + \chi_2(y(t) - y^* - y^* \ln \frac{y(t)}{y^*}), \tag{2.24}$$

where χ_1 and χ_2 are positive constants. By taking the derivative of V with respect to time t, one obtains

$$\begin{aligned}
\frac{dV(t)}{dt} &= \chi_1 \frac{x-x^*}{x}\frac{dx}{dt} + \chi_2 \frac{y-y^*}{y}\frac{dy}{dt} \\
&= \chi_1(x-x^*)\Big[r - \frac{r}{K}x(t-\tau_1) - \frac{(1+\alpha y)y}{1+c(1+\alpha y)x}\Big] \\
&\quad + \chi_2(y-y^*)\Big[-\delta - ay + \frac{\mu(1+\alpha y)x(t-\tau_2)}{1+c(1+\alpha y)x(t-\tau_2)}\Big] \\
&\leq \chi_1(x-x^*)\Big[r - \frac{r}{K}x(t-\tau_1) + x^* + \frac{(1+\alpha y^*)y^*}{1+c(1+\alpha y^*)x} - \frac{(1+\alpha y)y}{1+c(1+\alpha y)x}\Big] \\
&\quad + \chi_2(y-y^*)\Big[-a(y-y^*) + \frac{\mu(1+\alpha y)x(t-\tau_2)}{1+c(1+\alpha y)x(t-\tau_2)} - \frac{\mu(1+\alpha y)x}{1+c(1+\alpha y^*)x^*}\Big]
\end{aligned}$$

Since $e_1 = 1 + c(1+\alpha y^*)x^*$, and $e_2 = 1 + c(1+\alpha y)x$, one gets

$$\begin{aligned}
\frac{dV(t)}{dt} &\leq r\chi_1(x-x^*) - \frac{r\chi_1}{K}(x-x^*)^2 + \frac{c\chi_1(1+\alpha y^*)(1+\alpha y)(y^*x - x^*y)(x-x^*)}{e_1e_2} \\
&\quad - \frac{\chi_1(1+\alpha(y^*+y))(y-y^*)(x-x^*)}{e_1e_2} - a\chi_2(y-y^*)^2 \\
&\quad - \frac{\mu\chi_2(1+\alpha y)(y-y^*)(x-x^*)}{e_1} + \frac{\mu\chi_2(1+\alpha y)(y-y^*)(x-x^*)}{e_1e_2} \\
&\quad - \frac{\mu\chi_2\alpha c x x^*(y-y^*)^2}{e_1e_2}.
\end{aligned}$$

Based on the assumption $re_1e_2 < c(1+\alpha y^*)(1+\alpha y)(x^*y - y^*x)$ and since $e_2 > 1$, one may have

$$\begin{aligned}
\frac{dV(t)}{dt} &\leq -\frac{r\chi_1}{K}(x-x^*)^2 + \frac{[re_1e_2 + c(1+\alpha y^*)(1+\alpha y)(y^*x - x^*y)]\chi_1(x-x^*)}{e_1e_2} \\
&\quad - \frac{\chi_1(1+\alpha(y^*+y))(y-y^*)(x-x^*)}{e_1e_2} - a\chi_2(y-y^*)^2 \\
&\quad - (1 - \frac{1}{e_2})\frac{\mu\chi_2(1+\alpha y)(y-y^*)(x-x^*)}{e_2} - \frac{\mu\chi_2\alpha c x x^*(y-y^*)^2}{e_1e_2} \\
&\leq 0.
\end{aligned}$$

Hence, the proof is complete.

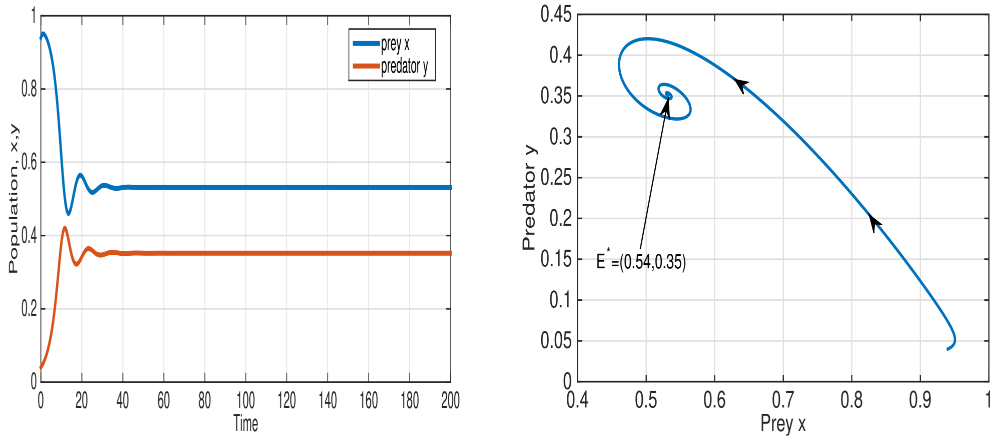

Figure 2.1: Stable population distribution (left) for Model (2.2) when $\tau_1 = \tau_2 = 0$. Figure (right) is a phase space that shows the existence of $\mathscr{E}^*$ which is locally asymptotically stable; with $a = 0.05$, $\alpha = 1.2$, $c = 0.9$, $\delta = 0.69$, $K = 1$

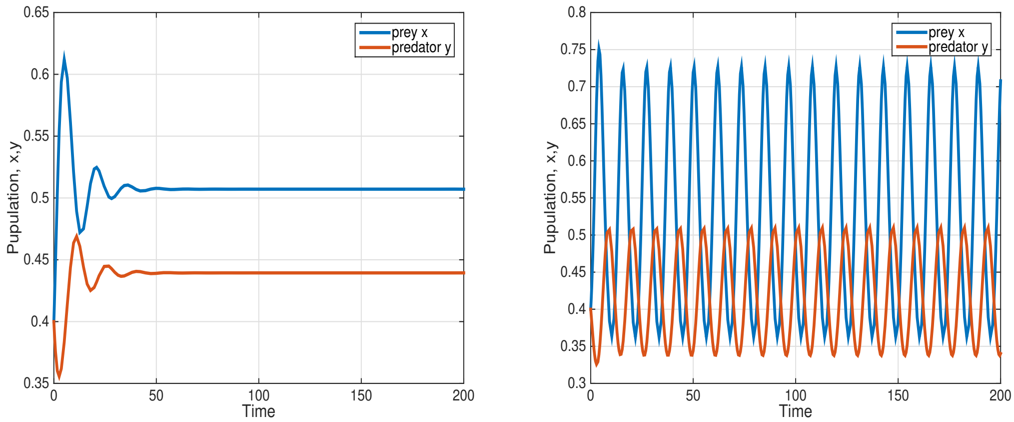

Figure 2.2: Stable population distribution (left) for Model (2.2) when $\tau_1 = 0.156 < \tau_1^*$ and $\tau_2 = 0.09 < \tau_2^*$. Hopf bifurcation periodic solution for $\tau_1^* = 1.169$ and $\tau_2 < \tau_2^* = 0.5$ (right), with the parametric values as mentioned in the text

2.6 Numerical Simulations

Some numerical simulations, leading to the approximation of the System (2.2), are performed using the Matlab DDE23 Package [126]. The parameters are taken as follows: $a = 0.05$, $\alpha = 1.6$, $c = 0.6$, $K = 1$, $\mu = 0.9$, $\delta = 0.49$, $r = 1$. Figure 2.1 shows a stable population distribution for (2.2), and $\mathscr{E}^* \equiv (0.55, 0.31)$ is locally asymptotically stable, when $\tau_1 = \tau_2 = 0$. Figure 2.2 depicts the stable population distribution for $\tau_1 = 0.156$ and $\tau_2 = 0.09$ (left). The stability behavior of the interior equilibrium $\mathscr{E}^*$ changed as τ_1 passes through critical values $\tau_1^* = 0.69$ and $\tau_2 < \tau_2^*$, where Hopf bifur-

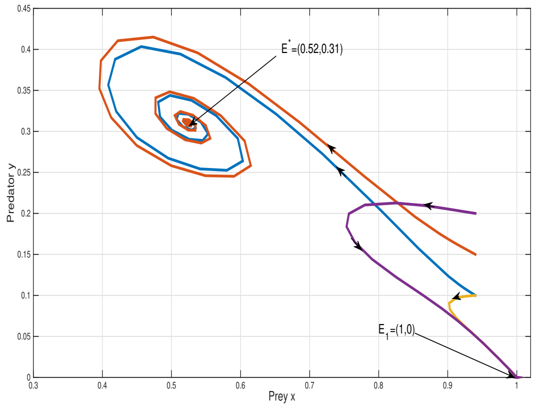

Figure 2.3: Bistability of the interior equilibrium $\mathscr{E}^*$ and $\mathscr{E}_1$ for Model (2.2). When $\tau_1 = 0.24 < \tau_1^*$ and $\tau_2 = 0.13 < \tau_2^*$; such that $0.79 < \delta < 1.1$. With $a = 0.05$, $\alpha = 1.9$, $K = 1$; $c = 0.04$, $r = 1$ and $\mu = 0.9$

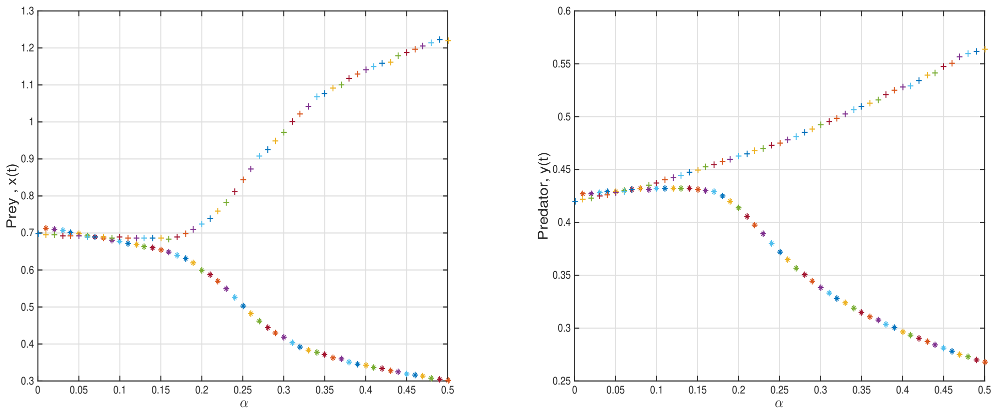

Figure 2.4: Bifurcation diagram with respect to α for System (2.2). Figure (left) shows the bifurcation diagram of the threshold parameter $\alpha = 0.1$ which is obtained numerically by maximum and minimum amplitude of prey $x(t)$. Figure (right) shows the bifurcation diagram of α with respect to the predator $y(t)$, when the other parameters are fixed as $\tau_1 = 1.7$, $\tau_2 = 0.1$, $a = 0.005$, $K = 1$, $\delta = 0.49$, $\mu = 0.9$, $c = 0.6$ and $r = 1$

cation occurs (right).

Figure 2.3 shows bistability in the presence of interior equilibrium $\mathscr{E}^*$ and the boundary $\mathscr{E}_1$. This bistability is related to coexistence of prey and predator or to the predator extinction depending on the variation of some parameters, such that $0.79 < \delta < 1.1$. Figure 2.4 shows the bifurcation diagram with respect to the hunting cooperation parameter α.

Remark 2.6.1. System (2.2) shows bistability between $\mathscr{E}^*$ and $\mathscr{E}_1$, therefore, any direction starting from the interior of $\mathbb{R}^2_+$ corresponds either to $\mathscr{E}^*$ or $\mathscr{E}_1$ based on the variation of the parameter $\delta \in [0.79, 1.1]$; (See Figure 2.3).

Remark 2.6.2. For an incremental increase of hunting cooperation parameter α, System (2.2) switches its stability from asymptotically stable to unstable limit cycle; (Figure 2.4).

2.7 Concluding Remarks

In this chapter, an ecological model which describes the combined effect of time delays and hunting cooperation in predators, on the dynamical behaviour of a predator-prey model has been proposed and studied. The boundedness of System (2.2) has been shown, and a local and global stability analysis of the interior equilibrium have been implemented. Critical values of delays, where the Hopf bifurcation occurs have also been obtained. Model (2.2) has at least one interior equilibrium under certain restrictions on the parameters defined by (2.6). The condition for the Hopf bifurcation periodic solution, by considering discrete time delay as a bifurcation parameter, is summarized in Theorem 2.4.1. Moreover, Remark 2.6.2 shows numerically that hunting cooperation acts as a bifurcation parameter for the deterministic model. The main findings, theoretically and numerically, indicate that time-delay and hunting cooperation can have a considerable impact in the dynamics of predator-prey systems. The presence of time-delays in the model improves the dynamics and enriches the complexity of the model.

In the next chapter, the author extends the analysis and propose a system of DDEs for three-species predator-prey system with Allee effect.

Chapter 3: Delay Differential Equations of Three-Species Predator-Prey Interactions with Allee Effect

3.1 Introduction

This chapter extends the analysis and studies the impact of time-delays and Allee effect on the dynamics of three-species predator-prey models. A two-prey one-predator system is considered, where the growth of both preys populations subject to Allee effects, and there exists a direct competition between the two-prey species having a common predator (see Section 2). Two discrete time-delays τ_1, τ_2 are incorporated into the predator growth equation to represent the reaction time with each prey. Local stability of the steady states, Hopf bifurcation, existence of bistability are studied in Section 3. Global stability of the interior steady state is discussed in Section 4. Sensitivity analysis to evaluate the uncertainty of the state variables to small changes in the Allee parameters and time delays is also considered in Section 5. Numerical simulations and concluding remarks are, respectively, given in Sections 6 and 7.

Allee effect and time-delays greatly increase the likelihood of local and global extinction and can produce a rich variety of dynamic effects. It is a natural question that how the introduction of Allee effect in the prey growth function changes the system dynamics of predator-prey system. However, before introducing the final model, some preliminaries about Allee effects in the predator-prey model are given briefly; See [27, 82].

3.1.1 Allee effect

Allee effect was firstly reported by the American ecologist Allee [2], when he asked *"what minimal numbers are necessary if a species is to maintain itself in nature?"* Allee, in [2], shows that the growth rate is not always positive for small densities, and it may not be decreasing as in the logistic model either. In general, Allee effect mechanisms arise from cooperation or facilitation among individuals in the species [51].

A population is said to have an Allee effect if the growth rate per capita is initially an increasing function for the low density. It can be classified into two types: strong and weak. A strong Allee effect takes place the population density is less than the specified threshold population considered, resulting in the species dying out. However, if the population density is greater than the threshold, the growth rate will remain positive [105]; While a weak Allee effect means that the per capita growth rate cannot go below zero and remains positive.

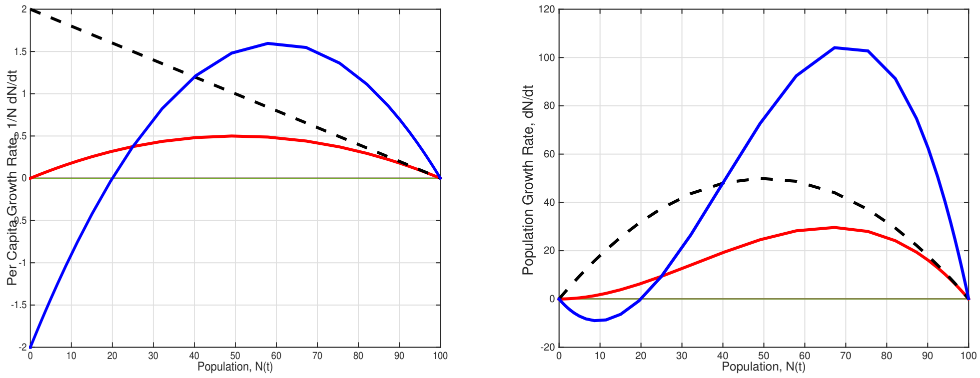

Figure 3.1: The left banner shows the per-capita growth rate $\frac{1}{N}\frac{dN}{dt}$ vs population $N(t)$. With logistic (black dashes), strong (blue curve) and weak (red curve) Allee effects. While the right banner displays the population growth rate $\frac{dN}{dt}$ vs population $N(t)$. For the strong Allee effect, the y-intercept of the per capita growth rate is less than zero at zero density, while in weak Allee effect the y-intercept cannot go below zero

Now, one may show how an Allee effect can be modelled, and how the per capita growth rate is affected with a weak Allee effect or a strong Allee effect throughout the simple examples:

$$\frac{dN}{dt} = rN^2\left(1-\frac{N}{K}\right) \text{ for a weak Allee effect,}$$
$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)\left(\frac{N}{A}-1\right) \text{ for a strong Allee effect.}$$

Figure 3.1 shows a per-capita growth rate $\frac{1}{N}\frac{dN}{dt}$ of the population with strong and weak Allee effect are represented. The straight line shows the logistic growth, and red curve displays a weak Allee effect; While the blue curve shows a strong Allee effect. The negative density dependence at low population sizes is described as a strong Allee effect, where there exists a threshold population level A, such that for $N < A$, $\frac{1}{N}\frac{dN}{dt} < 0$ (the species will die out) and for $N > A$, $\frac{1}{N}\frac{dN}{dt} > 0$, the growth will remain positive [105]. However, when the growth rate remains positive at low population densities, it is considered as a weak Allee effect.

Suppose that $N(t)$ is the size of prey population and $P(t)$ be the size of the predator population at time t, then the Lotka-Volterra model is given by the following equations:

$$\frac{dN(t)}{dt} = N(t)[\beta_1 - \gamma_1 - g_1 N(t)] - eN(t)P(t), \quad \frac{dP(t)}{dt} = P(t)[-\gamma + eN(t)], \tag{3.1}$$

with $N(0) > 0$, $P(0) > 0$. Here, β_1 is per capita maximum filtering rate and γ_1 is the death rate of the prey N(t); While the parameter g_1 denotes the strength of intra-specific competition. The predator death rate and predation rate are respectively denoted by γ and e. In the above model, it is assumed that prey population is subjected to logistic growth rate and the exponential type functional response.

For multi-species models, there are flexible ways to formulate the Allee effects. For example, due to difficulties in finding mates when prey population density becomes low, Allee effect takes place in prey species. Herein, an additive Allee effect of the form $b(N) \equiv \dfrac{N}{\alpha_1 + N}$ in the prey growth function of Model (3.1) is proposed and incorporated, which is considered as the probability of finding a mate, see [147], so that

$$\begin{aligned} \frac{dN(t)}{dt} &= N(t)[\frac{\beta_1 N(t)}{\alpha_1 + N(t)} - \gamma_1 - g_1 N(t)] - eN(t)P(t), \\ \frac{dP(t)}{dt} &= P(t)[-\gamma + eN(t)]. \end{aligned} \tag{3.2}$$

The parameter α_1 is the strength of Allee effect, $\alpha_1 = 1/R$, where R is the average area that can be searched by an individual [122]. One may notice that $b(0) = 0$, $b'(N) > 0$ if $N \in [0,\infty)$ i.e. Allee effect decreases as density increases, and $\lim_{N\to\infty} b(N) = 1$ means that Allee effect disappears at high densities. Therefore, the term $b(N)$ is considered as a weak Allee effect function in a rectangular hyperbola form, known as Michaelis-Menten-like function.

3.2 Distribution of the Model with Allee Effect

Many studies have been done on multi-species predator-prey systems, including local and global bifurcations and different types of chaos etc. (See e.g., [76, 123, 127, 129]. Sen *et al.* [123] discussed the Allee effect on two-preys' growth function, where the predator is generalized. They explained how the Allee effect can suppress the chaotic dynamics and the route to chaos in prey growth by comparing it with a model without the Allee effect. In [76], the authors studied dynamics of three species (two preys and one predator) delayed predator-prey model with cooperation among the preys against predation. The growth rate for preys is thought to be logistic. Delays are taken just in the growth components for each of the species. Takeuchi *et al.* [129] considered two-preys with logistic growth rates and an exponential functional response, where the predator survives on two-prey populations. Their results showed that the apparent chaotic behavior is a result of the periodic solution when one of the two-prey has greater competitive strength compared to the other. Song *et al.* [127] explored the dynamic behaviors of a Holling II two-prey one-predator system by introducing constant periodic releases of predators through periodically spraying a pesticide on the prey. They were then able to show that the system remains permanent under certain conditions. Herein, the author generalize

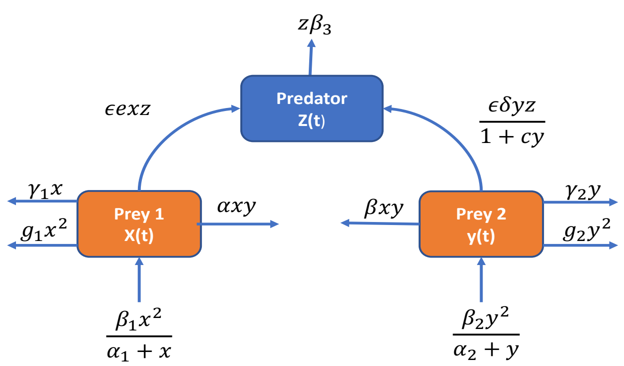

Figure 3.2: Mathematical scheme of the three-species one-predator two-prey System (3.3).

Model (3.2) to multi-species predator-prey system (two-preys one-predator). The model consists of two teams of preys with densities $x(t)$, $y(t)$, interacting with one team of predator with densities $z(t)$. Allee effects are also incorporated in the growth functions of the two-prey populations, and there exists a direct competition between the two-prey species having a common predator. The model takes the general form

$$\begin{aligned}\frac{dx(t)}{dt} =& x(t)\left[\frac{\beta_1 x(t)}{\alpha_1 + x(t)} - \gamma_1 - g_1 x(t)\right] - \alpha x(t)y(t) - ex(t)z(t)\\ \frac{dy(t)}{dt} =& y(t)\left[\frac{\beta_2 y(t)}{\alpha_2 + y(t)} - \gamma_2 - g_2 y(t)\right] - \beta x(t)y(t) - \frac{\delta y(t)z(t)}{1+cy(t)}\\ \frac{dz(t)}{dt} =& -\beta_3 z(t) + \varepsilon ex(t-\tau_1)z(t-\tau_1) + \frac{\varepsilon\delta y(t-\tau_2)z(t-\tau_2)}{1+cy(t-\tau_2)},\end{aligned} \tag{3.3}$$

with initial conditions:

$$\begin{aligned}&x(\theta) = \phi_1(\theta) > 0,\ y(\theta) = \phi_2(\theta) > 0,\ z(\theta) = \phi_3(\theta) > 0,\\ &\qquad \theta \in [-\tau, 0],\ \tau = \max\{\tau_1, \tau_2\}.\end{aligned} \tag{3.4}$$

Here, $\phi_i(\theta)$ $(i = 1,2,3)$ are smooth initial functions. The description of the model parameters is presented in Table 3.1. It is reasonable to assume that the death (predation) of preys is instantaneous when attacked by their predator but their contribution to the growth of predator population must be delayed by some time-delay. Therefore, two discrete time-delays τ_1 and τ_2 are incorporated in the reaction response functionals in the predator growth to represent the reaction time. The interaction between first species of prey and predator is assumed to be governed by Holling type I. While the interaction between the second species of prey and predator is assumed to be governed by Holling type II (cyrtoid functional) $\delta y(t)z(t)/(1+cy(t))$, response indicates that it is a hard-to-capture prey compared to the first species; See Figure 3.2. To investigate role of time-delay and Allee effect on the dynamics of the system, the author first discusses the boundedness and positivity of the solutions of the System (3.3) with the given positive initial conditions (3.4).

Table 3.1: One biological meaning for the parameters of Model (3.3)

Parameters	Description
α_1, α_2	Strength of Allee effect
β_1, β_2	Per capita maximum filtering rate of population
g_1, g_2	Strength of intra competition
γ_1, γ_2	Death rate for preys
α, β	Coefficient of competition
e, δ	Decrease rate of x(t) and y(t) due to predation by z(t)
β_3	Predator death rate
c	Magnitude of interference between the second type of prey
ε	An equal transformation rate of predator to preys $x(t)$ and $y(t)$

3.2.1 Non-negativity and boundedness of the solution

The non-negativity of the solutions indicates the existence of the population; While the boundedness explains the natural control of growth due to the restriction of resources. The author arrives at the following Lemma:

Lemma 3.2.1. *Every solution of System (3.3) corresponding to initial conditions (3.4) is defined on* $[0,\infty)$ *remains non-negative for all* $t \geq 0$*, which satisfies,*

$$\lim_{t\to\infty} \sup(x(t)+y(t)) \leq \kappa, \quad \lim_{t\to\infty} \sup z(t) \leq N,$$

where $\kappa = \min\{\beta_1, \beta_2\}$ *and* $N > 0$.

Proof. Model (3.3) can be represented in a matrix form

$$\dot{U}(t) = F(U), \tag{3.5}$$

where $U = (x,y,z)^T \in \mathbb{R}^3$, and

$$F(U) = \begin{bmatrix} F_1(U) \\ \\ F_2(U) \\ \\ F_3(U) \end{bmatrix} = \begin{bmatrix} x\left(\frac{\beta_1 x}{\alpha_1+x} - \gamma_1 - g_1 x\right) - \alpha xy - exz \\ \\ y\left(\frac{\beta_2 y}{\alpha_2+y} - \gamma_2 - g_2 y\right) - \beta xy - \frac{\delta yz}{1+cy} \\ \\ -\beta_3 z + \varepsilon ex(t-\tau_1)z(t-\tau_1) + \frac{\varepsilon\delta y(t-\tau_2)z(t-\tau_2)}{1+cy(t-\tau_2)} \end{bmatrix}.$$

Let $\mathbb{R}^3_+ = [0,\infty)^3$, since the right hand side of System (3.3) is locally Lipschitz on $\mathscr{C}$: $\mathbb{R}^{3+1}_+ \to \mathbb{R}^3$, such that $F_i(U)|_{u_i(t)=0, U\in\mathbb{R}^3_+} \geq 0$, where $u_1 = x$, $u_2 = y$ and $u_3 = z$. According to [60], the solutions of (3.5) with initial conditions (3.4) exist uniquely on the interval $[0,\xi)$, where $0 < \xi < \infty$, therefore all solutions exist on the first quadrant of the xyz-plane.

To prove the boundedness of solutions for System (3.3), first consider the case when the predator is absent, so that

$$\begin{aligned} \frac{dx}{dt} &= x(\frac{\beta_1 x}{\alpha_1 + x} - \gamma_1 - g_1 x) - \alpha xy \equiv G_1(x,y) \\ \frac{dy}{dt} &= y(\frac{\beta_2 y}{\alpha_2 + y} - \gamma_2 - g_2 y) - \beta xy \equiv G_2(x,y), \end{aligned} \tag{3.6}$$

with initial conditions $x(0) > 0$ and $y(0) > 0$, one can easily show that $G_1(x,y) \geq 0$ for $y = 0$ and $x < \frac{\beta_1 - \gamma_1}{g_1}$, such that $\beta_1 > \gamma_1$ and $G_2(x,y) \geq 0$ for $x = 0$ and $y < \frac{\beta_2 - \gamma_2}{g_2}$, where $\beta_2 > \gamma_2$. Adding the two equations of (3.6) yields

$$\begin{aligned} \frac{d}{dt}(x+y) &= x(\frac{\beta_1 x}{\alpha_1 + x} - \gamma_1 - g_1 x) + y(\frac{\beta_2 y}{\alpha_2 + y} - \gamma_2 - g_2 y) - xy(\alpha + \beta) \\ &\leq x(\beta_1 - \gamma_1 - g_1 x) + y(\beta_2 - \gamma_2 - g_2 y) \\ &\leq \beta_1 x + \beta_2 y \leq \kappa(x+y), \end{aligned} \tag{3.7}$$

where $\kappa = \min\{\beta_1, \beta_2\}$. Integrating both sides of (3.7), one gets

$$(x(t) + y(t)) \leq (x(0) + y(0))e^{-\kappa t}.$$

Since $(x(0) + y(0)) > 0$, the solutions are bounded, which clearly shows that $\lim_{t\to\infty}\sup(x(t) + y(t)) \leq \kappa$.

To extend the analysis to (3.3), consider $0 < \phi_1(\theta) + \phi_2(\theta) + \phi_3(0) < M$, $\theta \in$

$[-\tau, 0]$. Assume also that $\mathscr{H}(t) = \varepsilon x(t-\tau_1) + \varepsilon\delta y(t-\tau_2) + z$ and choose $0 < \rho < \beta_3$. By considering the derivative of $\mathscr{H}$, for $t > T + \tau$ for some fixed positive time T, one may have

$$\frac{d\mathscr{H}}{dt} + \rho\mathscr{H} \leq \varepsilon x(t-\tau_1)(\beta_1 + \rho - x(t-\tau_1)) + \varepsilon\delta y(t-\tau_2)(\beta_2 + \rho - y(t-\tau_2)) + (\rho - \beta_3)z.$$

Since x and y are nonnegative and bounded by κ,

$$\frac{d\mathscr{H}}{dt} + \rho\mathscr{H} \leq (\varepsilon + \varepsilon\delta)\kappa + (\rho - \beta_3)z \leq M.$$

Due to the non-negativity of z and the parametric condition exists for ρ, the differential inequality is bounded above, such that $\frac{d\mathscr{H}}{dt} \leq M - \rho\mathscr{H}$, i.e. there exist N where $0 < \mathscr{H}(t) < N$ for all $t > T$, which implies the boundedness of z, such that $\lim_{t\to\infty} \sup z(t) \leq N$.

3.3 Local Stability and Hopf Bifurcation

In this section, the qualitative behaviour of System (3.3) by studying the local stability of positive equilibrium points and Hopf bifurcation analysis is investigated, which provides a deeper insight into the model to address the behavioral change of solutions as a response to changes in a particular parameter. Since time-lags τ_1 and τ_2 have a significant impact in the complexity and dynamics of the model, one can consider them as bifurcation parameters.

3.3.1 Existence of equilibrium points

System (3.3) has some boundary and interior equilibrium points. However, one can only focus on the dynamic analysis of the boundary equilibrium where the first prey population is absent, and the interior equilibrium points. In order to obtain the attainable equilibrium points for the System (3.3), the zero growth isoclines of the system are given by $x\left(\frac{\beta_1 x}{\alpha_1+x}-\gamma_1-g_1x\right)-\alpha xy-exz=0$, $y\left(\frac{\beta_2 y}{\alpha_2+y}-\gamma_2-g_2y\right)-\beta xy-\frac{\delta yz}{1+cy}=0$ and $-\beta_3 z+\varepsilon exz+\frac{\varepsilon\delta yz}{1+cy}=0$, in $\mathbb{R}^3_+=\{(x,y,z)\in\mathbb{R}^3 : x,y,z\geq 0\}$. Therefore, the equilibria are the points of intersection of these zero growth isoclines regardless of the parameter values. System (3.3) has the following equilibria in $\mathbb{R}^3_+$:

(i) Trivial equilibrium $\mathscr{E}_0\equiv(0,0,0)$;

(ii) The axial equilibrium, $\mathscr{E}_1\equiv(x_1^*,0,0)$, for which the second prey and predator population are absent, where x_1^* is the root of quadratic equation $g_1x_1^2+(\gamma_1+\alpha_1g_1-\beta_1)x_1+\alpha_1\gamma_1=0$. Let $\varsigma_0=(\sqrt{\alpha_1g_1}+\sqrt{\gamma_1})^2$, the existence of $\mathscr{E}_1$ depends on the following conditions; If $\beta_1<\varsigma_0$, then $\mathscr{E}_1$ dose not exist; If $\beta_1>\varsigma_0$, then System (3.3) has two equilibria; However, if $\beta_1=\varsigma_0$, then System (3.3) has a unique equilibrium point.

In the same manner, one can show the existence of $\mathscr{E}_2\equiv(0,y_2,0)$.

(iii) In the absence of second prey population, a boundary equilibrium point $\mathscr{E}_3\equiv(x_3,0,z_3)$ exists if $\gamma_1<\varsigma_1<\varsigma_2$, such that

$$\varsigma_1=x_3\left(\frac{\beta_1\varepsilon e}{\alpha_1\varepsilon e+\beta_3}-g_1\right),\quad \varsigma_2=\frac{\beta_1\beta_3}{\alpha_1\varepsilon e+\beta_3},\tag{3.8}$$

and $x_3=\frac{\beta_3}{\varepsilon e}>0$ and $z_3=\frac{1}{e}(x_3\varsigma_2-\gamma_1)>0$.

(iv) A boundary equilibrium point $\mathscr{E}_4\equiv(0,y_4,z_4)$ exists, where the first prey population is absent, for $\gamma_2<\varsigma_3<\varsigma_4$, where

$$\begin{aligned}\varsigma_3&=\frac{\beta_2\beta_3(\varepsilon\delta-\beta_3c)-g_2\beta_2(\alpha_2(\varepsilon\delta-\beta_3c)+\beta_3)}{(\alpha_2(\varepsilon\delta-\beta_3c)+\beta_3)(\varepsilon\delta-\beta_3c)},\\ \varsigma_4&=\frac{\beta_2(\beta_3-g_2\alpha_2)}{\alpha_2(\varepsilon\delta-\beta_3c)+\beta_3},\quad\text{and}\end{aligned}\tag{3.9}$$

$$y_4=\frac{\beta_3}{\varepsilon\delta-\beta_3c}>0,\qquad z_4=\frac{1}{\delta}(1+cy_4)(\varsigma_3-\gamma_2).\tag{3.10}$$

(v) An interior equilibrium point $\mathscr{E}^* \equiv (x^*, y^*, z^*)$ exists with $\left(\frac{\beta_1 x^*}{\alpha_1 + x^*} - \gamma_1 - g_1 x^*\right) - \alpha y^* - e z^* = 0$, $\left(\frac{\beta_2 y^*}{\alpha_2 + y^*} - \gamma_2 - g_2 y^*\right) - \beta x^* - \frac{\delta z^*}{1 + c y^*} = 0$, $-\beta_3 + \varepsilon e x^* + \frac{\varepsilon \delta y^*}{1 + c y^*} = 0$. Such that $x^* = \frac{1}{\varepsilon e}(\beta_3 - \frac{\varepsilon \delta y^*}{1 + c y^*}) > 0$, $z^* = \frac{1}{e}\left(\frac{\beta_1(\beta_3(1 + c y^*) - \varepsilon \delta y^*)}{(1 + c y^*)(\varepsilon e + \beta_3)} + g_1(\beta_3 - \frac{\varepsilon \delta y^*}{1 + c y^*}) - \gamma_1 - \alpha y^*\right) > 0$, where y^* is the root(s) the following equation

$$G(y) = \sigma_1 y^4 + \sigma_2 y^3 + \sigma_3 y^2 + \sigma_4 y + \sigma_5 = 0. \tag{3.11}$$

The coefficients σ_i , $i = 1, \ldots, 5$ are defined by

$$\sigma_1 = \alpha_2 c^2, \quad \sigma_2 = c(2\alpha_2 + \alpha_2 c g_2 + \frac{\beta \beta_2 c}{\varepsilon e} - \frac{\beta \delta}{e} - \delta \alpha + c \gamma_2),$$

$$\sigma_3 = \frac{\beta_1 \delta^2 \varepsilon - c \delta \beta_1 \beta_2}{\varepsilon e + \beta_3} + \frac{\alpha_2 \beta \delta \varepsilon c - \beta \beta_2 \alpha_2 c^2 - \beta \delta \varepsilon}{\varepsilon e} - \delta^2 \varepsilon g_1 + c \beta_2$$
$$+ c^2 \beta_2 + c g_1 \delta \beta_3 + c \delta \gamma_1 + \alpha_2 - \delta \alpha,$$

$$\sigma_4 = \frac{\beta \beta_2 - \alpha_2 \beta \delta \varepsilon}{\varepsilon e} + \frac{\delta \beta_1 \beta_2 + c \delta \beta_1 \beta_3 \alpha_2 + \beta_1 \alpha_2 \delta^2}{\varepsilon e + \beta_3} - g_1 \alpha_2 \delta^2 \varepsilon$$
$$+ c \alpha_2 \delta \beta_3 g_1 + c \alpha_2 \delta \gamma_1 + c \alpha_2 \gamma_2 - \beta_2 - \beta_2 c + \alpha_2 g_2 - \delta \beta_2 g_1 - \delta \gamma_1 + \gamma_2,$$
$$\sigma_5 = \frac{\delta \beta_1 \beta_3 \alpha_2}{\varepsilon e + \beta_3} - \frac{\alpha_2 \beta_2 \beta}{\varepsilon e} - \alpha_2 \delta \beta_3 g_1 - \alpha_2 \delta \gamma_1 + \alpha_2 \gamma_2.$$

The nature of the roots for (3.11) is determined by noting the sign of its discriminant [44]. Therefore, a sufficient condition that guarantees that (3.11) has at least one positive root is $\sigma_5 < 0$, which leads to $\dfrac{\delta \beta_1 \beta_3 \alpha_2}{\varepsilon e + \beta_3} + \alpha_2 \gamma_2 < \dfrac{\alpha_2 \beta_2 \beta}{\varepsilon e} + \alpha_2 \delta \beta_3 g_1 + \alpha_2 \delta \gamma_1$. Thus, System (3.3) can have at most four interior equilibria in the presence of the Allee effect. However, in the absence of Allee effect the author arrives at the following Remark.

Remark 3.3.1. In the absence of the Allee effect ($\alpha_1 = \alpha_2 = 0$), the interior equilibria for System (3.3) are reduced to at most three interior equilibria. Consequently, Allee effect can generate or eradicate interior equilibria. It may stabilize or destabilize the system.

3.3.2 Existence of bistability

The phenomenon of bistability has been recognized experimentally in some biological situations, but much more commonly in theoretical models, such as the dynamics of animal populations [46]. The coexistence between two stable attractors can be determined by increasing or decreasing the value of some control parameters. Therefore, the system pursues one branch of equilibrium points when increasing a control parameter until a threshold limit point is reached at which the system jumps to another branch of stable equilibrium points. Bistability occurs when the system can converge to two different equilibrium points, depending on the variation of the initial conditions in the same parametric region; Or the system is able to evolve into either one of two equilibrium points by increasing or decreasing the level of one of the system's parameters.

The underlying Model (3.3) displays bistability of two interior equilibrium, which is based on the variation of the coefficient of competition α; See Figure 3.8. Both equilibria are locally asymptotically stable.

3.3.3 Stability and bifurcation analysis of the equilibria $\mathscr{E}_4$ and $\mathscr{E}^*$

The author focuses on studying local stability and bifurcation conditions for System (3.3), by analyzing the characteristic equations of the linearized system at $\mathscr{E}_4$ and $\mathscr{E}^*$. The Jcobian matrix at $\mathscr{E}_4$ is given by:

$$J_{\mathscr{E}_4} = \begin{pmatrix} C_1 & 0 & 0 \\ C_2 & C_3 & C_4 \\ D_1 & D_2 e^{-\lambda\tau_2} & C_5 + D_3 e^{-\lambda\tau_2} \end{pmatrix},$$

where

$$\begin{aligned} C_1 &= -\gamma_1 - \alpha y_4 - e z_4, \quad C_2 = -\beta y_4, \quad C_3 = \frac{\beta_2 y_4}{(\alpha_2 + y_4)}\left(1 + \frac{\alpha_2}{(\alpha_2 + y_4)}\right) \\ &\quad - 2 g_2 y_4 - \gamma_2 - \frac{\delta z_4}{(1 + c y_4)^2}, \quad C_4 = -\frac{\delta y_4}{1 + c y_4}, \quad C_5 = -\beta_3, \\ D_1 &= \varepsilon e z_4, \quad D_2 = \frac{\varepsilon \delta z_4}{(1 + c y_4)^2}, \quad D_3 = \frac{\varepsilon \delta y_4}{1 + c y_4}. \end{aligned}$$

The characteristic equation for System (3.3) at $\mathscr{E}_4$ can be written as:

$$C(\lambda)+D(\lambda)e^{-\lambda\tau_2}=0, \tag{3.12}$$

such that

$$C(\lambda)=\lambda^3+\psi_0\lambda^2+\psi_1\lambda+\psi_2,\ \ D(\lambda)=\phi_0\lambda^2+\phi_1\lambda+\phi_2,$$

where

$$\psi_0=-(C_1+C_3+C_5);\quad \psi_1=C_1C_3+C_3C_5+C_1C_5;\quad \psi_2=-C_1C_3C_5,$$
$$\phi_0=-D_3;\quad \phi_1=C_1D_3+C_3D_3-C_4D_2;\quad \phi_2=C_1C_4D_2-C_1C_3D_3.$$

When $\tau_2=0$, Equation (3.12) becomes:

$$\lambda^3+(\psi_0+\phi_0)\lambda^2+(\psi_1+\phi_1)\lambda+\psi_2+\phi_2=0. \tag{3.13}$$

Based on Routh-Hurwitz Criteria, all roots of (3.13) are negative if $\psi_0+\phi_0>0$, $\psi_2+\phi_2>0$ and $(\psi_0+\phi_0)(\psi_1+\phi_1)>(\psi_2+\phi_2)$. Thus, $\mathscr{E}_4$ is locally asymptotically stable when $\tau_2=0$.

If $\tau_2\neq 0$, assume $\lambda=i\omega,\omega>0$, then (3.12) becomes:

$$\begin{aligned}-\psi_0\omega^2+\psi_2&=(\phi_0\omega^2-\phi_2)\cos\omega\tau_2-\phi_1\omega\sin\omega\tau_2,\\ -\omega^3+\phi_1\omega&=(\phi_2-\phi_0\omega^2)\sin\omega\tau_2-\phi_1\omega\cos\omega\tau_2,\end{aligned} \tag{3.14}$$

squaring and adding both sides, one gets:

$$\omega^6+q_2\omega^4+q_1\omega^2+q_0=0, \tag{3.15}$$

where

$$q_2 = \psi_0^2 - 2\psi_1 - \phi_0^2, \;\; q_1 = \psi_1^2 - 2\psi_0\psi_2 + \phi_0\phi_2 - \phi_1^2, \;\; q_0 = \psi_2^2 - \phi_2^2.$$

The equilibrium point $\mathscr{E}_4$ is locally asymptotically stable, by Descartes rule of signs, if Equation (3.15) has at least one positive root $\hat{\omega}$, if $\psi_1^2 + \phi_0\phi_2 > 2\psi_0\psi_2 + \phi_1^2$ and $\psi_2^2 < \phi_2^2$. From Equation (3.14), one obtains

$$\tau_{2,k} = \frac{1}{\hat{\omega}} \arccos\left[\frac{(\psi_2 - \psi_0\hat{\omega}^2)(\phi_0\hat{\omega}^2 - \phi_2) + \phi_1\psi_1\hat{\omega}^2 - \phi_1\hat{\omega}^4}{(\phi_2 - \phi_0\hat{\omega}^2)^2 - (\phi_1\hat{\omega})^2}\right] + \frac{2k\pi}{\hat{\omega}}, \tag{3.16}$$

where $k = 0, 1, 2, \ldots$. By differentiating (3.12) with respect to τ_2 such that $\omega = \hat{\omega}$ and $\tau_2 = \tau_{2,k}$, the transversality condition can be obtained in this form

$$\Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{\hat{E}_1\hat{E}_4 - \hat{E}_2\hat{E}_3}{\hat{E}_2\hat{E}_4}. \tag{3.17}$$

Here,

$$\begin{aligned}
\hat{E}_1 &= [\psi_1 - 3\hat{\omega}^2](\psi_1\hat{\omega}^2 - \hat{\omega}^4) + 2\psi_0\hat{\omega}[\psi_2\hat{\omega} - \psi_0\hat{\omega}^3], \\
\hat{E}_2 &= (\hat{\omega}^4 - \psi_1\hat{\omega}^2)^2 + (\psi_2\hat{\omega} - \psi_0\hat{\omega}^3)^2, \\
\hat{E}_3 &= \phi_1^2\hat{\omega}^2 + 2(\phi_0\hat{\omega}^3 - \phi_2\hat{\omega})\phi_0\hat{\omega}, \\
\hat{E}_4 &= (\phi_1\hat{\omega}^2)^2 + (\phi_2\hat{\omega} - \phi_0\hat{\omega}^3)^2.
\end{aligned}$$

Then a Hopf bifurcation occurs for $\hat{\tau}_2 = \min\{\tau_{2,k}\}$, if $\Re(\frac{d\lambda}{d\tau_2})^{-1} > 0$. Therefore, based on the above analysis, the author arrives to the following result.

Theorem 3.3.1. *The boundary equilibrium $\mathscr{E}_4$ remains stable for $\tau_2 < \hat{\tau}_2$, and unstable for $\tau_2 > \hat{\tau}_2$, where $\hat{\tau}_2 = \min\{\tau_{2,k}\}$ defined by (3.16). Moreover, System (3.3) undergoes a Hopf bifurcation at $\mathscr{E}_4$ when $\tau_2 = \hat{\tau}_2$.*

Herein, some numerical results and simulations for System (3.3) are provided, by using DDE-BIFTOOL [40, 131] Matlab packages, and choosing suitable values of parameters. To investigate the stability of $\mathscr{E}_4$, one can show how the real part of (3.12) changes as τ_2 varies, and fix parameters $\varsigma_0 = 1.106$ and consider τ_2 as a bifurcation parameter and varying it from 0.2 to 16 (See Figure 3.3). However, from this figure alone it is not clear which real parts correspond to real roots respectively complex pairs of roots. In Figure 3.4 (left) taking $\tau_2 = 4.6$, shows that the eigenvalues of (3.12) have negative real part, the eigenvalues representing by circles seem to be similar to zero, but indeed, they are a pair of pure imaginary eigenvalues with real part a little bit less than zero. However, Figure 3.4 (right) shows that there is a pair of pure imaginary eigenvalues where the occurrence of Hopf bifurcation is possible at $\hat{\omega} = \pm 2.4$.

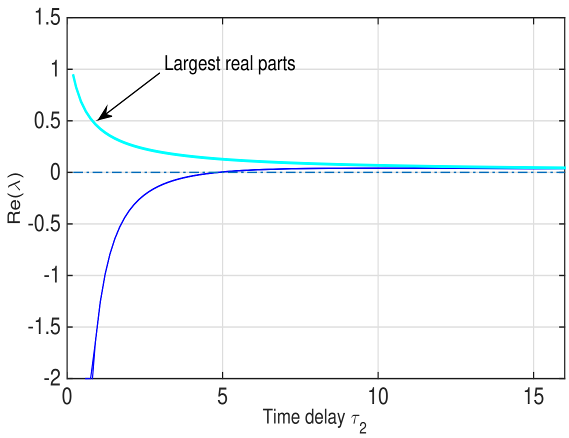

Figure 3.3: Real parts of the approximated and corrected roots of the characteristic Equation (3.12). Which shows variation of real part of the eigenvalues as the bifurcation parameter τ_2 is varied at $\mathscr{E}_4$.

Now, the stability of the interior equilibrium $\mathscr{E}^* \equiv (x^*, y^*, z^*)$ is discussed in detail, at which the Jcobian matrix is

$$J_{\mathscr{E}^*} = \begin{bmatrix} F_1 & F_2 & F_3 \\ F_4 & F_5 & F_6 \\ I_1 e^{-\lambda\tau_1} & I_2 e^{-\lambda\tau_2} & F_7 + I_3 e^{-\lambda\tau_1} + I_4 e^{-\lambda\tau_2} \end{bmatrix}.$$

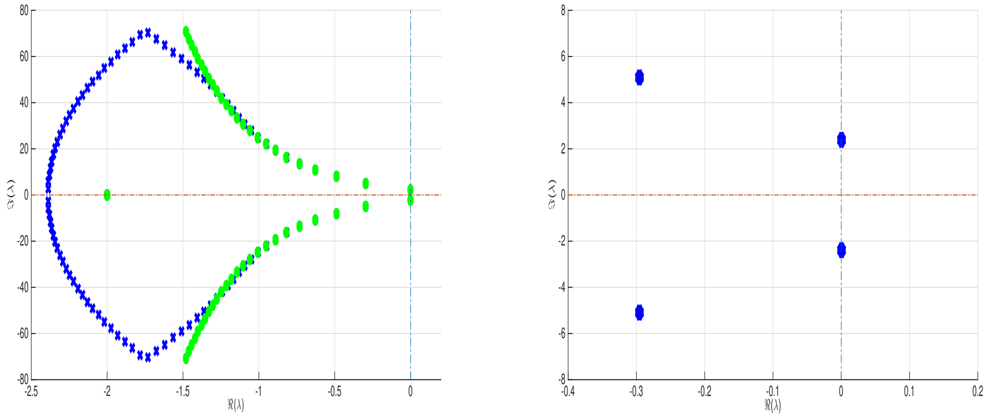

Figure 3.4: The eigenvalues of the characteristic Equation (3.12) at $\mathscr{E}_4$. Left banner shows eigenvalues of the characteristic Equation (3.12) at $\mathscr{E}_4$ (approximated (+) and corrected (•)), with the same parameters as in Figure 3.3. According to the scaling, right banner illustrates that there is a pair of pure imaginary eigenvalues which is consistent with the theoretical results that showed (3.12) has a pair of pure imaginary eigenvalues where $\hat{\omega} = \pm 2.4$

Here,

$$F_1 = \frac{\beta_1 x^*}{(\alpha_1 + x^*)}\big(1 + \frac{\alpha_1}{(\alpha_1 + x^*)}\big) - 2g_1 x^* - \gamma_1 - \alpha y^* - ez^* < 0, \quad F_2 = -\alpha x^*, \quad F_3 = -ex^*,$$
$$F_4 = -\beta y^*, \quad F_5 = \frac{\beta_2 y^*}{(\alpha_2 + y^*)}\big(1 + \frac{\alpha_2}{(\alpha_2 + y^*)}\big) - 2g_2 y^* - \gamma_2 - \beta x^* - \frac{\delta z^*}{(1 + cy^*)^2} < 0,$$
$$F_6 = -\frac{\delta y^*}{1 + cy^*}, F_7 = -\beta_3, \quad I_1 = \varepsilon e z^*, \quad I_2 = \frac{\varepsilon \delta z^*}{(1 + cy^*)^2}, \quad I_3 = \varepsilon e x^*, \quad I_4 = \frac{\varepsilon \delta y^*}{1 + cy^*}.$$

The characteristic equation for the interior point $\mathscr{E}^* \equiv (x^*, y^*, z^*)$ is then given by

$$A(\lambda) + B(\lambda)e^{-\lambda\tau_1} + C(\lambda)e^{-\lambda\tau_2} = 0. \tag{3.18}$$

Here,

$$A(\lambda) = \lambda^3 + R_1\lambda^2 + R_2\lambda + R_3, \ B(\lambda) = N_1\lambda^2 + N_2\lambda + N_3, \ C(\lambda) = M_1\lambda^2 + M_2\lambda + M_3,$$

such that

$$R_1 = -F_1 - F_5 - F_7, \quad R_2 = F_1F_5 + F_1F_7 + F_5F_7 - F_2F_4, \quad R_3 = F_2F_4F_7 - F_1F_5F_7,$$
$$N_1 = -I_3, \quad N_2 = (F_1 + F_5)I_3 - F_3I_1, \quad N_3 = F_2F_4I_3 + F_3F_5I_1 - F_2F_6I_1 - F_1F_5I_3,$$

$$M_1 = -I_4, \quad M_2 = (F_1 + F_5)I_4 - F_6 I_2, \quad M_3 = F_2F_4I_4 + F_1F_6I_2 - F_3F_4I_2 - F_1F_5I_4.$$

To gain insight regarding interior equilibrium $\mathscr{E}^*$, the author discusses the stability of interior steady states and Hopf bifurcation conditions of the threshold parameters τ_1 and τ_2 by considering the following different cases:

• **Case 1:** When $\tau_1 = \tau_2 = 0$, Equation (3.18) becomes

$$\lambda^3 + (R_1 + N_1 + M_1)\lambda^2 + (R_2 + N_2 + M_2)\lambda + (R_3 + N_3 + M_3) = 0. \tag{3.19}$$

Therefore, the interior equilibrium $\mathscr{E}^*$ is locally asymptotically stable if

(H_1) $R_1 + N_1 + M_1 > 0, \quad R_3 + N_3 + M_3 > 0 \quad \& \quad (R_1 + N_1 + M_1)(R_2 + N_2 + M_2) > R_3 + N_3 + M_3$ hold. Thus, based on Routh-Hurwitz Criteria, all the roots of (3.19) have negative real parts.

• **Case 2:** For $\tau_1 = 0, \tau_2 > 0$, then Equation (3.18) becomes

$$\lambda^3 + (R_1 + N_1)\lambda^2 + (R_2 + N_2)\lambda + (R_3 + N_3) + (M_1\lambda^2 + M_2\lambda + M_3)e^{-\lambda\tau_2} = 0. \tag{3.20}$$

For some values of ($\tau_2 > 0$), there exists a real number ω such that $\lambda = i\omega$ is a root of (3.20), then one gets

$$\begin{aligned} -(R_1 + M_1)\omega^2 + (R_3 + N_3) &= (M_1\omega^2 - M_3)\cos\omega\tau_2 - M_2\omega\sin\omega\tau_2 \\ -\omega^3 + (R_2 + N_2)\omega &= (M_3 - M_1\omega^2)\sin\omega\tau_2 - M_2\omega\cos\omega\tau_2. \end{aligned} \tag{3.21}$$

Squaring and adding both of the equations, yields

$$\omega^6 + a_1\omega^4 + a_2\omega^2 + a_3 = 0, \tag{3.22}$$

where

$$a_1 = (R_1 + M_1)^2 - 2(R_2 + N_2) - M_1^2,$$

$$a_2 = (R_2 + N_2)^2 - 2(R_1 + M_1)(R_3 + N_3) + 2M_1M_3 - M_2^2,$$

$$a_3 = (R_3 + N_3)^2 - M_3^2.$$

By Descartes' rule of signs, Equation (3.22) has at least one positive root ω_1 if

(H_2) $\quad R_1^2 + 2R_1M_1 > 2(R_2 + N_2) \quad \& \quad (R_3 + N_3)^2 < M_3^2$ hold.

Eliminating $\sin\omega_1\tau_2$ from (3.21), yields

$$\begin{aligned} \tau_{2,j} = \frac{1}{\omega_1} \arccos\Big[&\frac{((R_3+N_3)-(R_1+N_1)\omega_1^2)(M_1\omega_1^2-M_3)}{(M_3-M_1\omega_1^2)^2-(M_2\omega_1)^2} \\ &+ \frac{M_2(R_2+N_2)\omega_1^2 - M_2\omega_1^4}{(M_3-M_1\omega_1^2)^2-(M_2\omega_1)^2}\Big] + \frac{2j\pi}{\omega_1}, \end{aligned} \tag{3.23}$$

where $j = 0, 1, 2, \ldots$. By differentiating (3.20) with respect to τ_2 such that $\omega = \omega_1$ and $\tau_2 = \tau_{2,j}$, the transversality condition can be obtained in this form

$$\Re\Big(\frac{d\lambda}{d\tau_2}\Big)^{-1} = \frac{A_1A_4 - A_2A_3}{A_2A_4}. \tag{3.24}$$

Here,

$$A_1 = [(R_2+N_2) - 3\omega_1^2]((R_2+N_2)\omega_1^2 - \omega_1^4) + 2(R_1+N_1)\omega_1[(R_3+N_3)\omega_1 - (R_1+N_1)\omega_1^3],$$

$$A_2 = (\omega_1^4 - (R_2+N_2)\omega_1^2)^2 + ((R_3+N_3)\omega_1 - (R_1+N_1)\omega_1^3)^2,$$

$$A_3 = M_2^2\omega_1^2 + 2(M_1\omega_1^3 - M_3\omega_1)M_1\omega_1,$$

$$A_4 = (M_2\omega_1^2)^2 + (M_3\omega_1 - M_1\omega_1^3)^2.$$

Then a Hopf bifurcation occurs for τ_2 if $\Re(\frac{d\lambda}{d\tau_2})^{-1} > 0$; i.e. $A_1A_4 > A_2A_3$. The authors arrives at the following Theorem:

Theorem 3.3.2. *Let (H_1)-(H_2) hold, where $\tau_1 = 0$, then there exists $\tau_2 > 0$ such that $\mathscr{E}^*$ remains stable for $\tau_2 < \tau_2'$, and unstable for $\tau_2 > \tau_2'$, where $\tau_2' = \min\{\tau_{2,j}\}$ defined by (3.23). Moreover, System (3.3) undergoes a Hopf bifurcation at $\mathscr{E}^*$ when $\tau_2 = \tau_2'$.*

• **Case 3:** When $\tau_2 = 0, \tau_1 > 0$, in the same manner of the pervious case, one can arrive to the following Theorem

Theorem 3.3.3. *For System (3.3), with $\tau_2 = 0$, there exists a positive number τ_1, such that the equilibrium point $\mathscr{E}^*$ is locally asymptotically stable for $\tau_1 < \tau_1^{'}$, and unstable for $\tau_1 > \tau_1^{'}$, where $\tau_1^{'} = \min\{\tau_{1,j}\}$. Furthermore, Hopf bifurcation occurs at $\tau_1 = \tau_1^{'}$.*

$$\begin{aligned}\tau_{1,j} = \frac{1}{\omega_2} \arccos \big[&\frac{((R_3+M_3)-(R_1+M_1)\omega_2^2)(N_1\omega_2^2-N_3)}{(N_1\omega_2^2-N_3)^2+(N_2\omega_2)^2} \\ &- \frac{N_2(R_2+M_2)\omega_2^2+N_2\omega_2^4}{(N_1\omega_2^2-N_3)^2+(N_2\omega_2)^2}\big] + \frac{2j\pi}{\omega_2},\end{aligned} \tag{3.25}$$

where $j = 0, 1, 2, \ldots$.

• **Case 4:** When $\tau_1 > 0$ & $\tau_2 > 0$, assuming that τ_1 is a variable parameter and τ_2 as fixed on its stable interval. Let $\lambda = i\omega$ as a root of (3.18); Separating real and imaginary parts, implies

$$\begin{aligned}-\omega^3 + R_2\omega + (M_1\omega^2 - M_3)\sin\omega\tau_2 + &M_2\omega\cos\omega\tau_2 \\ &= (N_3 - N_1\omega^2)\sin\omega\tau_1 - N_2\omega\cos\omega\tau_1,\end{aligned} \tag{3.26}$$

$$\begin{aligned}-R_1\omega^2 + R_3 + (M_3 - M_1\omega^2)\cos\omega\tau_2 + &M_2\omega\sin\omega\tau_2 \\ &= (N_1\omega^2 - N_3)\cos\omega\tau_1 - N_2\omega\sin\omega\tau_1.\end{aligned} \tag{3.27}$$

Thus, by eliminating the trigonometric functions ($\sin\omega\tau_1$ and $\cos\omega\tau_1$) from (3.26) and (3.27), yields

$$\omega^6 + \xi_4\omega^5 + \xi_3\omega^4 + \xi_2\omega^3 + \xi_1\omega^2 + \xi_0 = 0, \tag{3.28}$$

where

$$\xi_4 = -2M_1\sin\omega\tau_2, \;\; \xi_3 = R_1 + M_1^2 - 2R_2 - N_1^2 - 2M_2\cos\omega\tau_2,$$

$$\xi_2 = 2(M_1R_2 + M_3)\sin\omega\tau_2 - 2M_3R_1\cos\omega\tau_2, \;\; \xi_1 = -2M_3R_2\sin\omega\tau_2,$$

$$\xi_0 = R_3^2 + M_3^2 - N_3^2 + (2M_3R_3 + R_1M_1)\cos\omega\tau_2.$$

Equation (3.28) is a preternatural equation in a complicated form, it is quite difficult to predict the nature of its roots. Thus, by applying Descartes' rule of signs one can say that (3.28) has at least one positive root ω_0 if (H_3) $\quad \xi_4 > 0 \quad$ since $\quad M_1 < 0 \quad \& \quad \xi_0 < 0$; therefore, one obtains

$$\tau_{1,j} = \frac{1}{\omega_0} \arccos \Big[\frac{AD+CB}{A^2+C^2}\Big] + \frac{2j\pi}{\omega_0}, \; j = 0,1,2,\dots. \tag{3.29}$$

Here,

$$A = N_1\omega_0^2 - N_3, \quad B = -\omega_0^3 + R_2\omega_0 + (M_3 - M_1\omega_0^2)\sin\omega_0\tau_2 + \cos\omega_0\tau_2,$$

$$C = N_2\omega_0, \quad D = -R_1\omega_0^2 + R_3 + (M_1\omega_0^2 - M_3)\cos\omega_0\tau_2 + M_2\omega_0\sin\omega_0\tau_2.$$

To study the Hopf bifurcation analysis, by fixing τ_2 in its stable interval and differentiate Equations (3.26) and (3.27) with respect to τ_1. Then substitute $\tau_1 = \tau_{1,0}$ and $\omega = \omega_0$, one gets

$$\begin{aligned} Q_2\Big(\frac{d(\Re\lambda)}{d\tau_1}\Big)|_{\tau_1=\tau_{1,0}}\Big) + Q_1\Big(\frac{d(\omega)}{d\tau_1}\Big)|_{\tau_1=\tau_{1,0}}\Big) &= Q_3 \\ -Q_1\Big(\frac{d(\Re\lambda)}{d\tau_1}\Big)|_{\tau_1=\tau_{1,0}}\Big) + Q_2\Big(\frac{d(\omega)}{d\tau_1}\Big)|_{\tau_1=\tau_{1,0}}\Big) &= Q_4, \quad \text{where} \end{aligned} \tag{3.30}$$

$$\begin{aligned} Q_1 &= -3\omega_0^2 + R_2 + (2N_1\omega_0 - N_2\omega_0\tau_{1,0})\sin\omega_0\tau_{1,0} + (N_2 + N_1\tau_1\omega_0^2 - N_3\tau_{1,0})\cos\omega_0\tau_{1,0} \\ &\quad + (2\omega_0M_1 - M_2\tau_2\omega_0)\sin\omega_0\tau_2 + (M_1\tau_2\omega_0^2 - M_3\tau_2 + M_2)\cos\omega_0\tau_2, \\ Q_2 &= -2R_1\omega_0 + (N_1\omega_0^2\tau_{1,0} - N_3\tau_{1,0} + N_2)\sin\omega_0\tau_{1,0} + (N_2\omega_0\tau_1 - 2N_1\omega_0)\cos\omega_0\tau_{1,0} \\ &\quad + (M_2 + M_1\omega_0^2\tau_2 - M_3\tau_2)\sin\omega_0\tau_2 + (M_2\omega_0\tau_2 - 2M_1\omega_0)\cos\omega_0\tau_2, \\ Q_3 &= N_2\omega_0^2\sin\omega_0\tau_{1,0} + (N_3\omega_0 - N_1\omega_0^3)\cos\omega_0\tau_{1,0}, \\ Q_4 &= N_2\omega_0^2\cos\omega_0\tau_{1,0} + (N_1\omega_0^3 - N_3\omega_0)\sin\omega_0\tau_{1,0}. \end{aligned}$$

From (3.30), one obtains

$$\Big(\frac{d(\Re\lambda)}{d\tau_1}\Big)|_{\tau_1=\tau_{1,0}}\Big) = \frac{Q_2Q_3 - Q_1Q_4}{Q_2^2 + Q_1^2}. \tag{3.31}$$

As $Q_2Q_3 > Q_1Q_4$, then Hopf bifurcation occurs for $\tau_1 = \tau_{1,0}$. Therefore, the author arrives at the following Theorem:

Theorem 3.3.4. *If $\mathscr{E}^*$ exists, such that (H_1) and (H_3) hold, with $\tau_2 \in (0, \tau_2^{'})$, then there exists a positive threshold parameter τ_1^* such that the interior equilibrium $\mathscr{E}^*$ is locally asymptotically stable for $\tau_1 < \tau_1^*$, and unstable $\tau_1 > \tau_1^*$, where $\tau_1^* = \min\{\tau_{1,j}\}$ as in (3.29). Additionally, System (3.3) undergoes Hopf bifurcation at $\mathscr{E}^*$ when $\tau_1 = \tau_1^*$.*

Remark 3.3.2. Similarly, for $\tau_1 \in (0, \tau_1^{'})$, there exists a threshold parameter τ_2^* such that the interior equilibrium $\mathscr{E}^*$ is locally asymptotically stable for $\tau_2 < \tau_2^*$, and unstable $\tau_2 > \tau_2^*$. Also, Hopf bifurcation occurs for System (3.3) as $\tau_2 = \tau_2^*$; Where $\tau_2^* = \min\{\tau_{2,j}\}$ is given by

$$\tau_{2,j} = \frac{1}{\omega_3} \arccos\Big[\frac{A_1 D_1 + C_1 B_1}{A_1^2 + C_1^2}\Big] + \frac{2j\pi}{\omega_3}, \ j = 0, 1, 2, \ldots, \quad \text{with} \tag{3.32}$$

$$A_1 = M_1 \omega_3^2 - M_3, \quad B_1 = \omega_3^3 - R_2 \omega_3 + (N_3 - N_1 \omega_3^2) \sin \omega_3 \tau_1 - N_2 \omega_3 \cos \omega_3 \tau_1,$$

$$C_1 = M_2 \omega_3, \quad D_1 = -R_1 \omega_3^2 + R_3 + \cos \omega_3 \tau_1 + N_2 \omega_3 \sin \omega_3 \tau_1.$$

The proofs are obtained in the same manner of the above analysis.

3.4 Global Stability of Interior Steady State $\mathscr{E}^*$

In this Section, the global stability of System (3.3) around interior steady state $\mathscr{E}^* \equiv (x^*, y^*, z^*)$ is studied.

Theorem 3.4.1. *If $\beta_1 \alpha_1 < g_1(\alpha_1 + x^*)(\alpha_1 + x)$ and $\beta_2 \alpha_2 (1 + cy^*)(1 + cy) + \delta c z^* (\alpha_2 + y^*)(\alpha_2 + y) < g_2(\alpha_2 + y^*)(\alpha_2 + y)(1 + cy^*)(1 + cy)$, then System (3.3) is globally asymptotically stable at the interior equilibrium point $\mathscr{E}^*$.*

Proof. By suggesting the following Lyapunov function at $\mathscr{E}^* \equiv (x^*, y^*, z^*)$ of the form

$$V(t) = \rho_1\big(x(t) - x^* - x^* \ln\frac{x(t)}{x^*}\big) + \rho_2\big(y(t) - y^* - y^* \ln\frac{y(t)}{y^*}\big) + \rho_3\big(z(t) - z^* - z^* \ln\frac{z(t)}{z^*}\big)$$

where ρ_1, ρ_2, ρ_3 are nonnegative constants. Take derivative V with respect to time t, yielding to

$$\begin{aligned}
\dot{V}(t) &= \rho_1 \frac{x-x^*}{x}\dot{x}(t) + \rho_2 \frac{y-y^*}{y}\dot{y}(t) + \rho_3 \frac{z-z^*}{z}\dot{z}(t) \\
&= \rho_1(x-x^*)\big(\frac{\beta_1 x}{\alpha_1 + x} - \gamma_1 - g_1 x - \alpha y - ez\big) \\
&\quad + \rho_2(y-y^*)\big(\frac{\beta_2 y}{\alpha_2 + y} - \gamma_2 - g_2 y - \beta x - \frac{\delta yz}{y(1+cy)}\big) \\
&\quad + \rho_3(z-z^*)\big(-\beta_3 + \frac{\varepsilon e x(t-\tau_1) z(t-\tau_1)}{z} + \frac{\varepsilon\delta y(t-\tau_2) z(t-\tau_2)}{z(1+cy(t-\tau_2))}\big)
\end{aligned}$$

$$\begin{aligned}
&\le \rho_1(x-x^*)\big(\frac{\beta_1 x}{\alpha_1 + x} - \frac{\beta_1 x^*}{\alpha_1 + x^*} - g_1(x-x^*) - \alpha(y-y^*) - e(z-z^*)\big) \\
&\quad + \rho_2(y-y^*)\big(\frac{\beta_2 y}{\alpha_2 + y} - \frac{\beta_2 y^*}{\alpha_2 + y^*} - g_2(y-y^*) - \beta(x-x^*) + \frac{\delta y^* z^*}{y^*(1+cy^*)} - \frac{\delta yz}{y(1+cy)}\big) \\
&\quad + \rho_3(z-z^*)\big(\frac{\varepsilon e x(t-\tau_1) z(t-\tau_1)}{z} + \frac{\varepsilon\delta y(t-\tau_2) z(t-\tau_2)}{z(1+cy(t-\tau_2))} - (\varepsilon e x^* + \frac{\varepsilon\delta y^*}{1+cy^*})\big) \\
&\le -\rho_1 g_1(x-x^*)^2 - \rho_2 g_2(y-y^*)^2 - (\rho_1\alpha + \rho_2\beta)(x-x^*)(y-y^*) \\
&\quad + (\varepsilon e\rho_3 - e\rho_1)(x-x^*)(z-z^*) + \rho_1(x-x^*)\big(\frac{\beta_1 x}{\alpha_1 + x} - \frac{\beta_1 x^*}{\alpha_1 + x^*}\big) \\
&\quad + \rho_2(y-y^*)\big(\frac{\beta_2 y}{\alpha_2 + y} - \frac{\beta_2 y^*}{\alpha_2 + y^*}\big) + \rho_2(y-y^*)\big(\frac{\delta y^* z^*}{y^*(1+cy^*)} - \frac{\delta yz}{y(1+cy)}\big) \\
&\quad + \rho_3(z-z^*)\big(\frac{\varepsilon\delta y}{1+cy} - \frac{\varepsilon\delta y^*}{1+cy^*}\big) \\
&\le -\rho_1 g_1(x-x^*)^2 - \rho_2 g_2(y-y^*)^2 - (\rho_1\alpha + \rho_2\beta)(x-x^*)(y-y^*) \\
&\quad + (\varepsilon e\rho_3 - e\rho_1)(x-x^*)(z-z^*) + \beta_1\rho_1(x-x^*)^2\big(\frac{\alpha_1}{(\alpha_1 + x^*)(\alpha_1 + x)}\big) \\
&\quad + \beta_2\rho_2(y-y^*)^2\big(\frac{\alpha_2}{(\alpha_2 + y^*)(\alpha_2 + y)}\big) + \delta\rho_2(y-y^*)\big(\frac{-(z-z^*)}{1+cy} + \frac{cz^*(y-y^*)}{(1+cy^*)(1+cy)}\big) \\
&\quad + \varepsilon\delta\rho_3(y-y^*)(z-z^*)\big(\frac{1}{1+cy} - \frac{cy}{(1+cy^*)(1+cy)}\big).
\end{aligned}$$

Thus, based on the assumptions: $\beta_1\alpha_1 < g_1(\alpha_1+x^*)(\alpha_1+x)$, $\beta_2\alpha_2(1+cy^*)(1+cy)+\delta cz^*(\alpha_2+y^*)(\alpha_2+y) < g_2(\alpha_2+y^*)(\alpha_2+y)(1+cy^*)(1+cy)$, $\varepsilon\rho_3 < \max\{\rho_1,\rho_2\}$, one can get

$$\begin{aligned}\dot{V}(t) \leq & \Big(\frac{\rho_1\alpha_1\beta_1}{(\alpha_1+x^*)(\alpha_1+x)} - \rho_1 g_1\Big)(x-x^*)^2 + \Big(\frac{\varepsilon\delta\rho_3-\delta\rho_2}{1+cy}\Big)(y-y^*)(z-z^*)\\ &+\Big(\frac{\delta\rho_2 cz^*}{(1+cy^*)(1+cy)} + \frac{\rho_2\alpha_2\beta_2}{(\alpha_2+y^*)(\alpha_2+y)} - \rho_2 g_2\Big)(y-y^*)^2\\ &+(\varepsilon e\rho_3-\rho_1 e)(x-x^*)(z-z^*) - (\rho_1\alpha+\rho_2\beta)(x-x^*)(y-y^*)\\ &-\frac{\varepsilon\delta\rho_3 cy}{(1+cy^*)(1+cy)}(z-z^*)(y-y^*) \leq 0.\end{aligned}$$

Hence the proof is complete.

3.5 Sensitivity Analysis

Sensitivity analysis of a particular model is the most important tool for investigating the quantitative (or qualitative) influence of perturbing the parameters on the model. The objective of a sensitivity analysis is to determine systematically the effect of uncertain parameters on system solutions and the effect of the noisy data on the accuracy to which parameters may be determined [110]. Assume that System (3.3) is represented by a state variable $\mathbf{y}(t,\mathbf{p}) \in \mathbb{R}^d$ $(d=3)$, for $t \in [t_0, t_b]$ which is the unique non-negative solution of the IVP

$$\begin{aligned}&\mathbf{y}'(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{y}(t,\mathbf{p}),\mathbf{y}(t-\tau_1,\mathbf{p}),\mathbf{y}(t-\tau_2,\mathbf{p});\mathbf{p}), \quad t_0 \leq t \leq t_b,\\ &\mathbf{y}(t,\mathbf{p}) = \phi(t,\mathbf{p}), \quad t \leq t_0.\end{aligned} \tag{3.33}$$

The right-hand side of Equation (3.33) depends on the constant vector of parameters $\mathbf{p}$, which includes the initial values and time-lags. $\mathbf{f}$ is considered precisely if $\mathbf{p}$ is specified and generally it is continuously differentiable with respect to the arguments in bio-mathematical systems; While the initial functions $\phi(t,\mathbf{p})$ are piecewise continuous with possible jump discontinuities at a finite number of points [8].

It is quite common for a model to exhibit high sensitivity to small variations in some parameters; While showing robustness to variation in other parameters. There are different ways to find the sensitivity functions of DDEs [110]. Nevertheless, the so called "direct approach" is utilized to find sensitivity functions of Model (3.3).

By taking all the parameters appearing in Model (3.33) to be constants, then sensitivity analysis, in this case, may just entail finding the partial derivatives of the solution with respect to each parameter p_l. One can denote by $S(t)$ the matrix $S(t;\mathbf{p})$ of the sensitivity functions

$$S(t) = S(t;\mathbf{p}) := \Big[\frac{\partial y^i(t;\mathbf{p})}{\partial p_j}\Big]_{j=1,2,\ldots,L}^{i=1,2,\ldots,d} \tag{3.34}$$

In Model (3.3), $d = 3$ and $L = 20$. Introducing the notation $\left\{\frac{\partial}{\partial \mathbf{p}}\right\}^T$, the matrix of local sensitivity functions takes the form

$$S(t,\mathbf{p})(t) = \left\{\frac{\partial}{\partial \mathbf{p}}\right\}^T y(t,\mathbf{p}) \in \mathbb{R}^{d\times L}. \quad \text{Its } i^{th} \text{ column is} \tag{3.35}$$

$$S_i(t,\mathbf{p}) = \Big[\frac{\partial y_i(t,\mathbf{p})}{\partial p_1}, \frac{\partial y_i(t,\mathbf{p})}{\partial p_2}, \ldots, \frac{\partial y_i(t,\mathbf{p})}{\partial p_L}\Big]. \tag{3.36}$$

Therefore, $S_i(t,\mathbf{p})$ is a vector whose components denote the sensitivity of the solution $y_i(t,\mathbf{p})$ of the model to small variations in the parameters p_j, $j = 1,2,\ldots,L$. Applying $\frac{\partial}{\partial \mathbf{p}}$ to Equation (3.33) yields the variational equation

$$S'(t,\mathbf{p}) = \mathbf{J}(t)S(t,\mathbf{p}) + \mathbf{J}_{\tau_1}(t)S(t-\tau_1,\mathbf{p}) + \mathbf{J}_{\tau_2}(t)S(t-\tau_2,\mathbf{p}) + \mathbf{B}(t), \quad t \geq 0, \tag{3.37}$$

$$\begin{aligned}
\mathbf{J}(t) &= \frac{\partial}{\partial \mathbf{y}(t)}\mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau_1),\mathbf{y}(t-\tau_2);\mathbf{p}),\\
\mathbf{J}_{\tau_1}(t) &= \frac{\partial}{\partial \mathbf{y}(t-\tau_1)}\mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau_1),\mathbf{y}(t-\tau_2);\mathbf{p}),\\
\mathbf{J}_{\tau_2}(t) &= \frac{\partial}{\partial \mathbf{y}(t-\tau_2)}\mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau_1),\mathbf{y}(t-\tau_2);\mathbf{p}),\\
\mathbf{B}(t) &= \frac{\partial}{\partial \mathbf{p}}\mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau_1),\mathbf{y}(t-\tau_2);\mathbf{p}).
\end{aligned}$$

The sensitivity functions are directly obtained by solving the $d \times L$ sensitivity equations (3.37) together with the original System (3.33). However, this is a challenging problem when d and L are large, and when the model equations are stiff.

Now, consider Model (3.3), with the vector of parameters **p**=[α_1, α_2,β_1, β_2,g_1, g_2,γ_1, γ_2,α, β,e, δ,β_3,c,ε, τ_1,τ_2, x_0,y_0,z_0]. The sensitivity functions with respect to the parameters p_i $(i = 1,2,\ldots,20)$ are denoted by

$$S_{x_{p_i}}(t) := \frac{\partial}{\partial p_i}x(t), \quad S_{y_{p_i}}(t) := \frac{\partial}{\partial p_i}y(t), \quad S_{z_{p_i}}(t) := \frac{\partial}{\partial p_i}z(t). \tag{3.38}$$

3.5.1 Sensitivity to severity of Allee effect

Here, the sensitivity of model solution of (3.3) is studied, with respect to the parameters α_1 and α_2 (strength Allee effect). Hence, sensitivity functions due to small perturbations in Allee parameter α_1 are given by system of DDEs

$$\begin{aligned}
S'_{x\alpha_1}(t) &= S_{x\alpha_1}(t)[\frac{\beta_1 x(t)}{\alpha_1 + x(t)} - \gamma_1 - 2g_1x(t) - \alpha y(t) - ez(t)] - \alpha S_{y\alpha_1}(t)x(t) \\
&\quad - eS_{z\alpha_1}(t)x(t) + \beta_1 x(t)(\frac{\alpha_1 S_{x\alpha_1}(t) - x(t)}{(\alpha_1 + x(t))^2}), \\
S'_{y\alpha_1}(t) &= S_{y\alpha_1}(t)[\frac{\beta_2 y(t)}{\alpha_2 + y(t)} - \gamma_2 - 2g_2y(t) - \beta x(t)] + y(t)[-\beta S_{x\alpha_1}(t) \\
&\quad + \frac{\alpha_2\beta_2 S_{y\alpha_1}(t)}{(\alpha_2 + y(t))^2}] - \delta[\frac{S_{y\alpha_1}(t)z(t)}{(1+cy(t))^2} + \frac{S_{z\alpha_1}(t)y(t)}{1+cy(t)}], \\
S'_{z\alpha_1}(t) &= -\beta_3 S_{z\alpha_1}(t) + \varepsilon e[S_{x\alpha_1}(t-\tau_1)z(t-\tau_1) + S_{z\alpha_1}(t-\tau_1)x(t-\tau_1)] \\
&\quad + \varepsilon\delta[\frac{S_{y\alpha_1}(t-\tau_2)z(t-\tau_2)}{(1+cy(t-\tau_2))^2} + \frac{S_{z\alpha_1}(t-\tau_2)y(t-\tau_2)}{1+cy(t-\tau_2)}].
\end{aligned} \tag{3.39}$$

To estimate the sensitivity functions $S_{x_{\alpha_1}}(t)$, $S_{y_{\alpha_1}}(t)$ and $S_{z_{\alpha_1}}(t)$, then one may have to solve the system of sensitivity equations (3.39) together with the original System (3.3).

Similarly, the sensitivity functions due to small changes in Allee coefficient α_2

satisfy the system of DDEs

$$
\begin{aligned}
S'_{x\alpha_2}(t) &= S_{x\alpha_2}(t)[\frac{\beta_1 x(t)}{\alpha_1 + x(t)} - \gamma_1 - 2g_1 x(t) - \alpha y(t) - ez(t)] \\
&\quad + \beta_1 x(t)(\frac{\alpha_1 S_{x\alpha_2}(t)}{(\alpha_1 + x(t))^2}) - \alpha S_{y\alpha_2}(t)x(t) - eS_{z\alpha_2}(t)x(t), \\
S'_{y\alpha_2}(t) &= S_{y\alpha_2}(t)[\frac{\beta_2 y(t)}{\alpha_2 + y(t)} - \gamma_2 - 2g_2 y(t) - \beta x(t)] - \beta S_{x\alpha_2}(t)y(t) \\
&\quad + \beta_2 y(t)[\frac{\alpha_2 S_{y\alpha 2}(t) - y(t)}{(\alpha_2 + y(t))^2}] - \delta[\frac{S_{y\alpha_2}(t)z(t)}{(1+cy(t))^2} + \frac{S_{z\alpha_2}(t)y(t)}{1+cy(t)}], \\
S'_{z\alpha_2}(t) &= -\beta_3 S_{z\alpha_2}(t) + \varepsilon e[S_{x\alpha_2}(t-\tau_1)z(t-\tau_1) + S_{z\alpha_2}(t-\tau_1)x(t-\tau_1)] \\
&\quad + \varepsilon\delta[\frac{S_{y\alpha_2}(t-\tau_2)z(t-\tau_2)}{(1+cy(t-\tau_2))^2} + \frac{S_{z\alpha_2}(t-\tau_2)y(t-\tau_2)}{1+cy(t-\tau_2)}].
\end{aligned}
\tag{3.40}
$$

After-that, one may solve (3.40) along with (3.3) to evaluate $S_{x_{\alpha_2}}(t)$, $S_{y_{\alpha_2}}(t)$ and $S_{z_{\alpha_3}}(t)$; See Figure 3.10.

3.5.2 Sensitivity to time-delays

The sensitivity functions due to small changes in the time-lag parameters τ_1 and τ_2 are obtained by solving the neutral delay differential equations (NDDEs)

$$
\begin{aligned}
S'_{x\tau_1}(t) &= S_{x\tau_1}(t)[\frac{\beta_1 x(t)}{\alpha_1 + x(t)} - \gamma_1 - 2g_1 x(t) - \alpha y(t) - ez(t)] \\
&\quad + \beta_1 x(t)(\frac{\alpha_1 S_{x\tau_1}(t)}{(\alpha_1 + x(t))^2}) - \alpha S_{y\tau_1}(t)x(t) - eS_{z\tau_1}(t)x(t), \\
S'_{y\tau_1}(t) &= S_{y\tau_1}(t)[\frac{\beta_2 y(t)}{\alpha_2 + y(t)} - \gamma_2 - 2g_2 y(t) - \beta x(t)] + y(t)[-\beta S_{x\tau_1}(t) \\
&\quad + \frac{\alpha_2 \beta_2 S_{y\tau_1}(t)}{(\alpha_2 + y(t))^2}] - \delta[\frac{S_{y\tau_1}(t)z(t)}{(1+cy(t))^2} + \frac{S_{z\tau_1}(t)y(t)}{1+cy(t)}], \\
S'_{z\tau_1}(t) &= -\beta_3 S_{z\tau_1}(t) + \varepsilon e[S_{x\tau_1}(t-\tau_1)z(t-\tau_1) + (S_{z\tau_1}(t-\tau_1) \\
&\quad - z'(t-\tau_1))x(t-\tau_1)] + \varepsilon\delta[\frac{S_{y\tau_1}(t-\tau_2)z(t-\tau_2)}{(1+cy(t-\tau_2))^2} \\
&\quad + \frac{S_{z\tau_1}(t-\tau_2)y(t-\tau_2)}{1+cy(t-\tau_2)}].
\end{aligned}
\tag{3.41}
$$

Noting that it is possible to obtain similar sensitivity equations with respect to the time delay τ_2. One can estimate the relative sensitivity functions,

$$\frac{\partial \mathbf{y}/\mathbf{y}}{\partial \mathbf{p}/\mathbf{p}} = \frac{\text{relative change in} \quad \mathbf{y}}{\text{relative change in} \quad \mathbf{p}};$$

which are useful to compare different parameters. The higher the relative sensitivity, the more important the input parameter in the model.

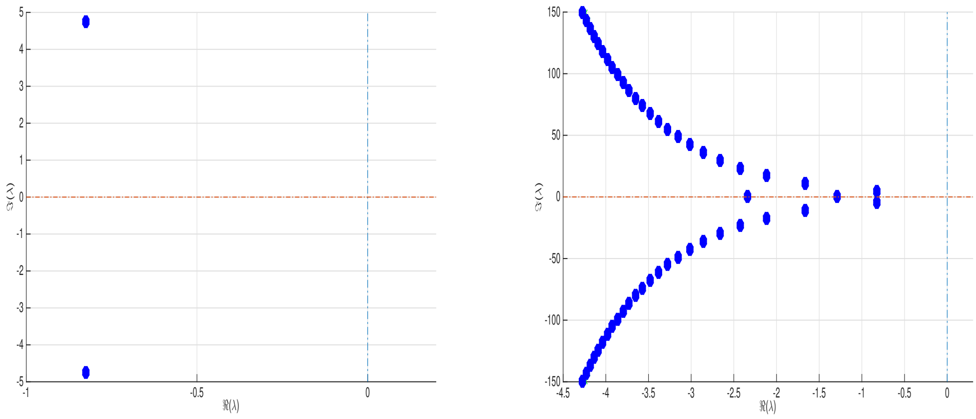

Figure 3.5: The roots of the characteristic equation of System (3.3) with negative real parts at the stable steady state $\mathscr{E}^*$. Real parts computed up to $\Re(\lambda) \geq -1$ (left), $\Re(\lambda) \geq -5$ (right). Parameter values are given in the text

3.6 Numerical Simulations

Some numerical simulations of System (3.3) are carried out, using Matlab package DDE23 and DDE-BIFTOOL, to confirm the theoretical results. The author first investigates the behavior of model around $\mathscr{E}^*$ with parameter values:

$$\begin{aligned} &\alpha = 0.9, \alpha_1 = 0.001, \alpha_2 = 0.001, \beta = 1.35, \gamma_2 = 1, \gamma_1 = 1, \\ &\beta_1 = 2, \beta_2 = 2, \beta_3 = 1, \varepsilon = 0.5, e = 5, \delta = 1. \end{aligned} \tag{3.42}$$

In Figure 3.5, the roots of (3.18) were computed, by setting minimal real part to a more negative value (the roots are computed up to $\Re(\lambda) \geq -\frac{1}{\tau}$) (left), where $\tau = \max\{\tau_1, \tau_2\}$, and recompute stability up to $\Re(\lambda) \geq -5$ (right).

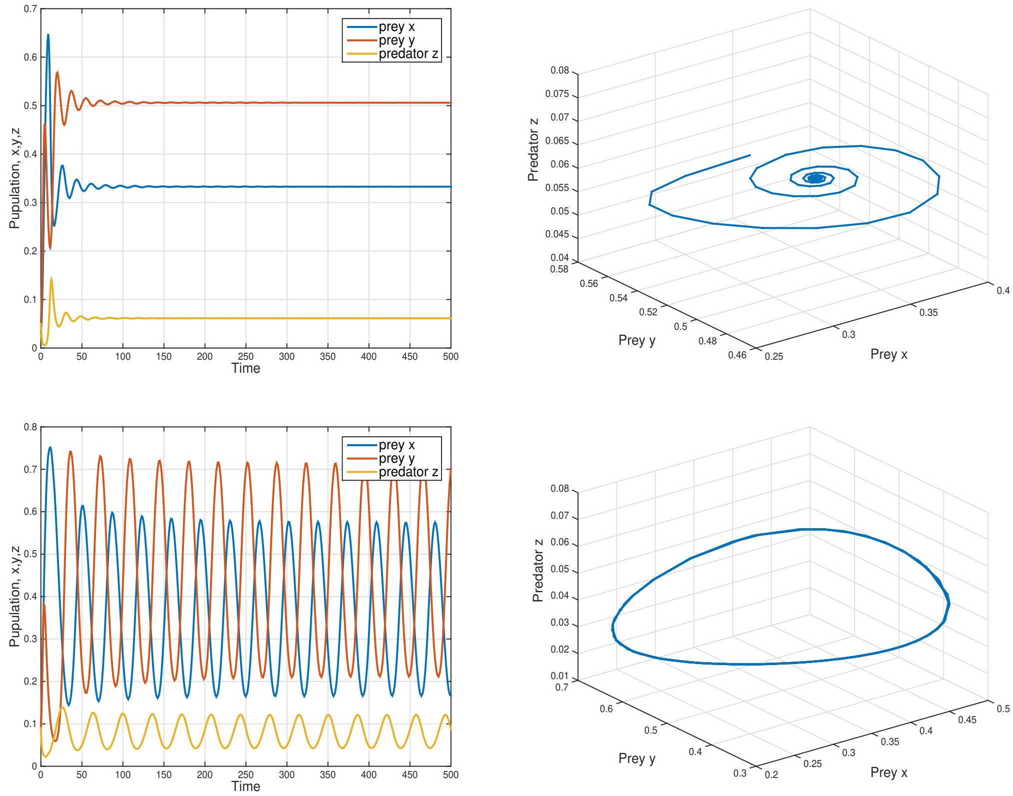

Figure 3.6: Numerical simulations of System (3.3) around the steady state $\mathscr{E}^*$. Top Banners show that $\mathscr{E}^*$ is asymptotically stable when $\tau_1 = 3.54 < \tau_1^*$ and $\tau_2 \in (0, \tau_2^*)$; Below Banners display a Hopf bifurcation when $\tau_1 = \tau_1^* = 4.34$ and $\tau_2 < \tau_2^* = 5.34$; the other parameter values are given in (3.42).

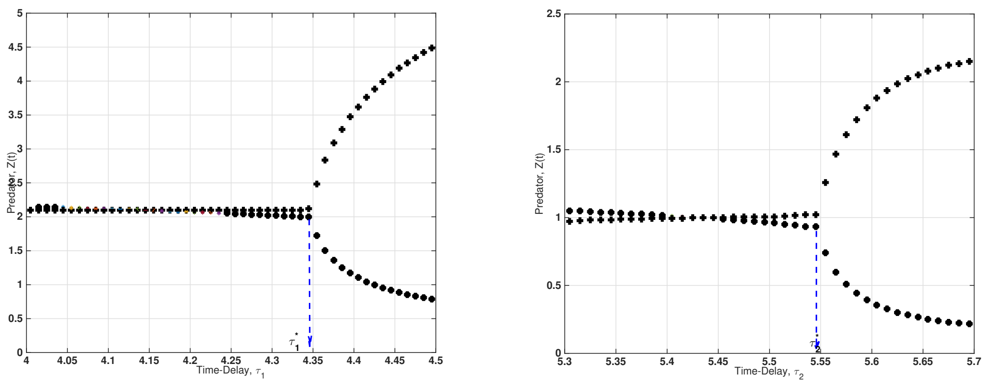

Figure 3.7: The Hopf bifurcation diagrams of τ_1 and τ_2 for System (3.3); which are obtained numerically by maximum and minimum amplitude of $z(t)$. The left banner displays the threshold parameter $\tau_1^* = 4.34$ with $\tau_2 < \tau_2^*$; While right banner shows that the threshold parameter $\tau_2^* = 5.54$ with $\tau_1 < \tau_1^*$

Figure 3.6 shows the numerical simulations of the delayed System (3.3) around the steady state $\mathscr{E}^*$. The interior steady state $\mathscr{E}^*$ is asymptotically stable when $\tau_1 < \tau_1^*$ and $\tau_2 \in (0, \tau_2^*)$; The model undergoes a Hopf bifurcation when $\tau_1 = \tau_1^* = 4.34$ and $\tau_2 < \tau_2^* =$

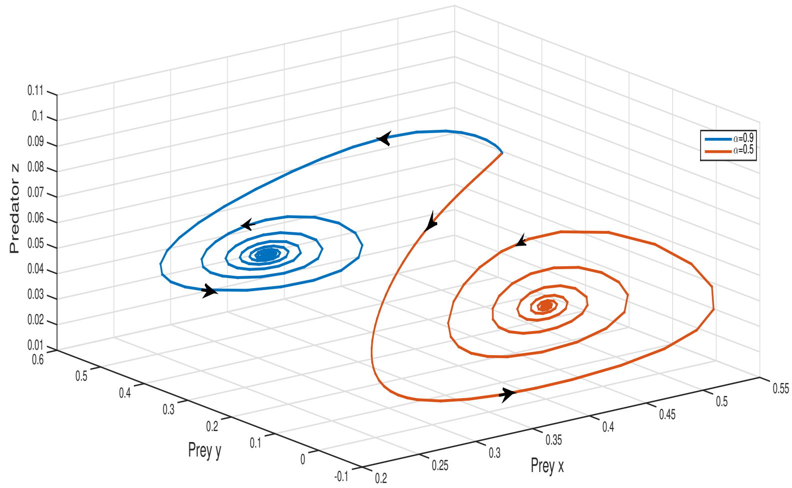

Figure 3.8: Bistability of two interior equilibria for the delayed System (3.3); with $\alpha = 0.9$ and $\alpha = 0.5$. Both equilibria are locally asymptotically stable, other parameter values are given in (3.42)

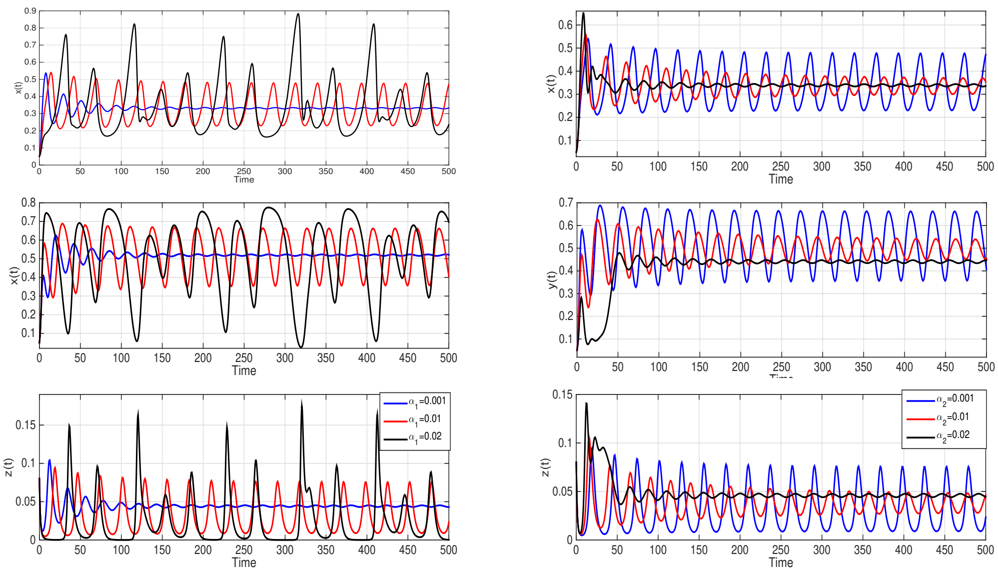

Figure 3.9: The sensitivity of the dynamics of the System (3.3) due to small changes in the severity of Allee effect α_1 and α_2. The left banners show the numerical simulations with different values of α_1 ($0.001 \leq \alpha_1 \leq 0.02$) and fixed value of $\alpha_2 = 0.001$; While right banners display the simulations with different values of α_2 ($0.01 \leq \alpha_2 \leq 0.02$) and fixed value of $\alpha_1 = 0.01$. The phase portrait gets stretched over time as α_1 reduced; While low values of α_2 increases the oscillations over time. The presence of Allee effect in the model enriches the dynamics of the system

5.34. Figure 3.7 displays the Hopf bifurcation diagrams of τ_1 and τ_2 which are obtained numerically by maximum and minimum amplitude of $z(t)$. The left banner displays the threshold parameter $\tau_1^* = 4.34$ with $\tau_2 < \tau_2^*$; While right banner shows that the threshold parameter $\tau_2^* = 5.54$ with $\tau_1 < \tau_1^*$.

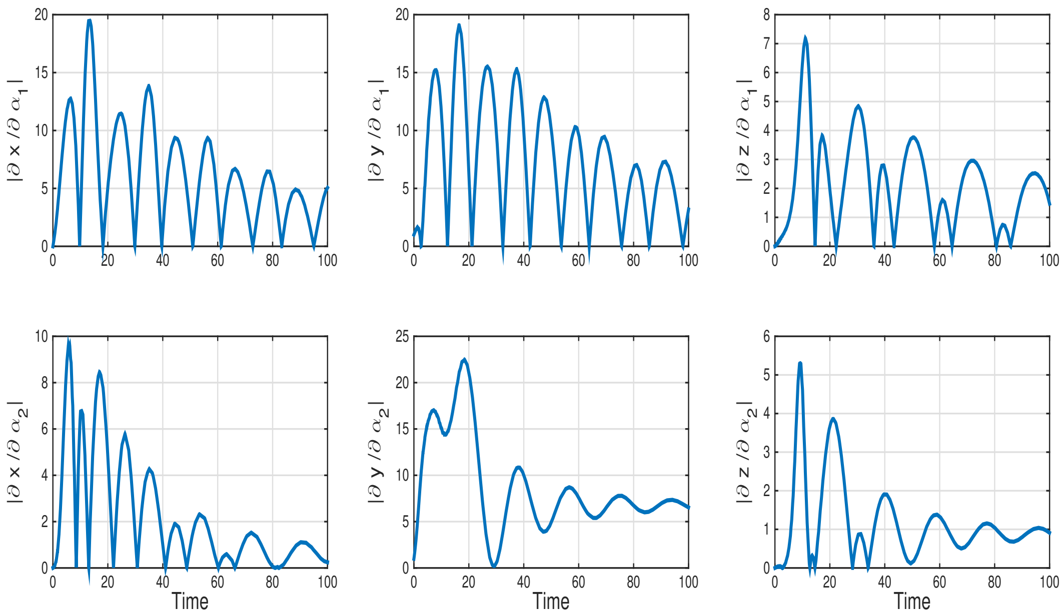

Figure 3.10: Sensitivity functions of model solution of System (3.3) with respect to Allee parameters α_1 and α_2. Top banners show the sensitivity functions for $x(t)$, $y(t)$ and $z(t)$ with respect to small changes in Allee parameter α_1. However, the bottom banners display the sensitivity with respect to α_2. They show that the model is very sensitive to the small perturbations of Allee parameters in early time intervals and the sensitivity decreases by time. The two parameters α_1 and α_2 are significant in the model, and cause high impact in early stages of interactions

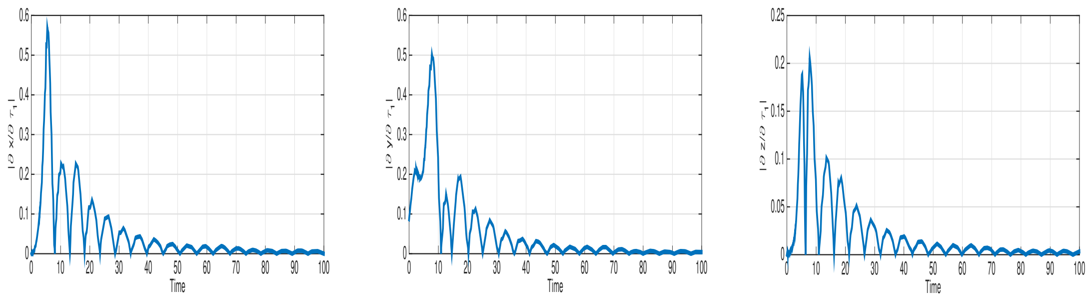

Figure 3.11: Sensitivity functions of model solution of System (3.3) with respect to time delay τ_1; which show the sensitivity functions for $x(t)$, $y(t)$ and $z(t)$ with respect to small changes in τ_1

Figure 3.8 displays a bistability of two interior equilibrium points, for the DDEs Model (3.3), when parameter α varies from $\alpha = 0.5$ to $\alpha = 0.9$. If the interior equilibria exists, any trajectory starting from the interior of $\mathbb{R}^3_+$ converges to one of the interior equilibria.

Figure 3.9 shows the sensitivity of the dynamics of the System (3.3) due to small changes in the severity of Allee effect α_1 and α_2. The left banners show the numerical simulations with different values of α_1 ($0.001 \leq \alpha_1 \leq 0.02$) and fixed value of $\alpha_2 =$

0.001; While right banners display the simulations with different values of α_2 ($0.01 \leq \alpha_2 \leq 0.02$) and fixed value of $\alpha_1 = 0.01$. The phase portrait gets stretched over time as α_1 reduced; While low values of α_2 increases the oscillations over time. The presence of Allee effect in the model enriches the dynamics of the system; While Figure 3.10 exhibits the absolute values of sensitivity functions: $|\partial x(t)/\partial \alpha_{1,2}|$, $|\partial y(t)/\partial \alpha_{1,2}|$ and $|\partial z(t)/\partial \alpha_{1,2}|$ to evaluate the sensitivity of the state variables due to a small perturbations in α_1 and α_2. The oscillation behaviour indicates that the species population is very sensitive to small changes in the parameter. It is clear that α_1 and α_2 are important in the model and have a significant impact in the dynamics, specially in the early stages of time. However, the sensitivity to these parameters decreases with time. Figure 3.11 shows that the parameter τ_1 has a significant effect in the model at the first subintervals and this sensitivity decreases by time.

3.7 Concluding Remarks

In this chapter, the author established two-prey one-predator model with time-delays and a weak Allee effect in the preys' growth functions, where there is a direct competition between prey populations. Non-negativity and boundedness of the solutions have been investigated. Some new sufficient conditions for local and global asymptotic stability of interior steady states have been deduced. In addition, Hopf bifurcation with respect to time-delays threshold parameters τ_1^* and τ_2^* have been studied. The model undergoes a Hopf bifurcation when time-delays pass through its critical values. The sensitivity of model solutions to small perturbations in the severity of Allee effect α_1 and α_2 and time delays was investigated. The obtained results confirm that Allee effect has a significant impact in the dynamics in the early stages of interaction. Introduction of time-delay and Allee effects, in the model, improves the stability results, and enrich the dynamics of the system, keep the populations densities in balance, and makes the model closer to reality.

It is known that the previous deterministic DDEs models are sometimes stable with a cyclic behaviors in the common period for the sizes of species populations. However, in practice, stochastic variations will occur in the values of x and y, which may produce a qualitatively different behavior. Thus, in next chapters, the author extends the analysis and study impact of environmental noise, by using stochastic DDEs with biological systems.

Chapter 4: Stochastic DDEs of Predator-Prey System with Hunting Cooperation in Predators

4.1 Introduction

Stochastic differential models provide an additional degree of realism compared to their corresponding deterministic counterparts because of the randomness and stochasticity of real life. This work extends the analysis and studies the dynamics of a stochastic delay differential model for predator-prey system with hunting cooperation in predators; Existence and uniqueness of global positive solution are discussed in Section 3. Stochastically ultimate boundedness and almost surely asymptotic properties are investigated in Sections 4 and 5. Some sufficient conditions for persistence and extinction, using Lyapunov functional, are obtained in Section 6. Illustrative examples and numerical simulations, using Milstein's scheme, are carried out to validate the analytical findings; See Section 7. The concluding remarks are given in Section 8.

Many studies have explored the effect of predator hunting cooperation on Predator-Prey (PP) systems [3, 104, 144]. Deterministic models such as (2.2) may be inadequate for capturing the exact variability in nature. Then, stochastic models are required for an accurate approximation of the dynamics of such interactions. The random fluctuations result in changing some degree of parameters in the deterministic environment. Many authors have studied stochastic population models and revealed the effects of environmental noises on the dynamics of population models (see [31, 32, 54, 141]). In [108], the authors studied the effect of environmental fluctuations of a delayed Harrison-type PP model, they analyzed the impact of the combination of delay and noise in the dynamical behavior of the model. In [14], the authors studied the effect of environmental fluctuations on a competitive model for two phytoplankton species where one species liberate toxic substances by considering a discrete time delay parameter in the growth equations of both species.

Before starting the analysis, one can provide some necessary results which will be used in this and next chapters.

4.2 Preliminaries

The mathematical model for a random quantity is a "random variable". Herein, the author recall some concepts from general probability theory. Consider

$$\mathbb{R}^n_+ = \{y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : y_i > 0, 1 \le i \le n\}.$$

Definition 4.2.1. [138] If Ω is a given set, then a $\sigma-$algebra $\mathscr{A}$ on Ω is a family $\mathscr{A}$ of subset Ω with the following properties:

(i) $\phi \in \mathscr{A}$;

(ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$, where $A^c = \Omega \backslash A$ is the complement of A in Ω;

(iii) $A_1, A_2, \dots \in \mathscr{A} \Rightarrow A := \cup_{i=1}^{\infty} A_i \in \mathscr{A}$.

Thus, the pair $(\Omega, \mathscr{A})$ is a measurable space. If $\mathfrak{C}$ is a family of subsets of Ω, there is a smallest σ-algebra $\sigma(\mathfrak{C})$ on Ω which contains $\mathfrak{C}$. Hence, $\sigma(\mathfrak{C})$ is the σ-algebra generated by $\mathfrak{C}$. Assume $\Omega = \mathbb{R}^n$ and $\mathfrak{C}$ is the family of all open sets in $\mathbb{R}^n$, then $\mathfrak{B}^n = \sigma(\mathfrak{C})$ is the Borel σ-algebra and the elements of $\mathfrak{B}^n$ are the Borel sets. A real valued-function $y : \Omega \to \mathbb{R}$ is $\mathscr{A}$-measurable if

$$\{\omega : y(\omega) \le c\} \in \mathscr{A} \quad \text{for all} \quad c \in \mathbb{R}.$$

An $\mathbb{R}^n$-valued function $\mathbf{y}(\omega) = (y_1(\omega), y_2(\omega), \dots, y_n(\omega))^T$ is $\mathscr{A}$-measurable if y_i is $\mathscr{A}$-measurable for all $i = 1, \dots, n$. Additionally, a $n \times m$-matrix-valued function $\mathbf{y}(\omega) = (y_{ij}(\omega))$ is $\mathscr{A}$-measurable if y_{ij} is $\mathscr{A}$-measurable for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathscr{A})$ is a function $\mathbb{P} : \mathscr{A} \to [0, 1]$ such that:

(a) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$;

(b) if $A_1, A_2, \cdots \in \mathscr{A}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint, then $\mathbb{P}\Big(\cup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

The triple $(\Omega, \mathscr{A}, \mathbb{P})$ is called a probability space [92]. If y is a real-valued random variable and is integrable with respect to the probability measure $\mathbb{P}$, the expectation of y with respect to $\mathbb{P}$ is

$$\mathbb{E}(y) = \int_{\Omega} y(\omega) d\mathbb{P}(\omega).$$

The variance of y is

$$Var(y) = \mathbb{E}(y - \mathbb{E}(y))^2.$$

The pth moment of y is denoted as $\mathbb{E}|y|^p$ $(p > 0)$. A statement $\mathscr{S}$ about outcomes is said to be true almost surely (a.s.), or with probability 1, if

$$A := \{\omega : \mathscr{S}(\omega) \quad \text{is true}\} \in \mathscr{A} \quad \text{and} \quad \mathbb{P}(A) = 1.$$

4.2.1 Stochastic processes

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. A filtration is a family $\{\mathscr{A}_t\}_{t \geq 0}$ of increasing sub-σ-algebras of $\mathscr{A}$ (i.e. $\mathscr{A}_t \subset \mathscr{A}_s \subset \mathscr{A}$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $\mathscr{A}_t = \cap_{s>t} \mathscr{A}_s$ for all $t \geq 0$. Considering the probability space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathscr{A}_0$ contains all $\mathbb{P}$-null sets. Additionally, one can define $\mathscr{A}_\infty = \sigma(\cup_{t \geq 0} \mathscr{A}_t)$.

In general, a stochastic process is a family $\{\mathbf{y}_t\}_{t \in I}$ of $\mathbb{R}^n$-valued random variables with parameter set I which could be ($\mathbb{R}_+ = [0, \infty)$, an interval $[a, b]$, the non-negative integers or subsets of $\mathbb{R}^n$), and state space $\mathbb{R}^n$. For a fixed $t \in I$, a random variable $\Omega \ni \omega \to \mathbf{y}_t(\omega) \in \mathbb{R}^n$ is considered. Wheres for a fixed $\omega \in \Omega$, a function

$I \ni t \to \mathbf{y}_t(\omega) \in \mathbb{R}^n$ is assumed, which is called a sample path of the process, also one can denote $\mathbf{y}_t(\omega)$ by $\mathbf{y}(t,\omega)$ and the stochastic process can be considered as a function of two components (t,ω) from $I \times \Omega$ to $\mathbb{R}^n$; a stochastic process is often written as $\{\mathbf{y}_t\}$, $\mathbf{y}_t$ or $\mathbf{y}(t)$. $\{\mathscr{A}_t\}$ is said to be adapted if for every t, $\mathbf{y}_t$ is $\mathscr{A}_t$-measurable; and it is said to be measurable if the stochastic process considered as a function of two components (t,ω) from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^n$ is $\mathfrak{B}(\mathbb{R}_+) \times \mathscr{A}$-measurable.

Now, one may define a random variable $\tau : \Omega \to [0,\infty)$, which is called an $\{\mathscr{A}_t\}$-stopping time if $\{\omega : \tau(\omega) \leq t\} \in \mathscr{A}_t$ for any $t \geq 0$. An $\mathbb{R}^n$-valued $\{\mathscr{A}_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ is a martingale with respect to $\{\mathscr{A}_t\}$ if $\mathbb{E}(M_t|\mathscr{A}_s) = M_s$ a.s. for all $0 \leq s < t < \infty$. A stochastic process $\mathbf{y} = \{\mathbf{y}_t\}_{t\geq 0}$ is called square integrable if $\mathbb{E}|\mathbf{y}_t|^2 < \infty$ for every $t \geq 0$. If $M = \{M_t\}_{t\geq 0}$ is a real-valued square-integrable continuous martingale, then there exists a unique continuous integrable adapted increasing process $\{\langle M, M_t\rangle_t\}$ (quadratic variation of M) such that $\{M_t^2 - \langle M, M\rangle_t\}$ is a continuous martingale vanishing at $t = 0$. A right continuous adapted process $M = \{M_t\}_{t>0}$ is a local martingale if there exists a nondecreasing sequence $\{\tau_k\}_{k\geq 1}$ of stopping times with $\tau_k \to \infty$ a.s. such that every $\{M_{\tau_k \wedge t} - M_0\}_{t\geq 0}$ is a martingale.

Lemma 4.2.1. *(Strong Law of Large Numbers) [81]. Let $M = \{M_t\}_{t\geq 0}$ be a real valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \quad \Rightarrow \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s.,$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

In Chapter 1 the properties of Brownian motions were discussed, now one can introduce the stochastic integrals and Itô formula.

4.2.2 Itô formula

Consider the following indicator function

$$I_{[t_i,t_{i+1}]}(t) = \begin{cases} 1 & \text{when} \quad t \in [t_i, t_{i+1}]; \\ 0 & \text{otherwise.} \end{cases}$$

The stochastic integral $\int_0^t f(s)dW_s$ with an $m-$dimensional Brownian motion $\{W_t\}$ for a class of $n \times m$-matrix-valued stochastic processes $\{f(t)\}$ is defined; Unfortunately, $W(t)$ does not have a derivative and so one cannot write the integral as a Riemann integral [103]. Let $t_i^* \in [t_i, t_{i+1}]$ then one can approximate $f(t)$ by $\sum_i f(t_i^*).I_{[t_i,t_{i+1}]}(t)$. Therefore, one can define $\int_0^t f(s)dW_s$ as the limit $\sum_i f(t_i^*)[W_{t_{i+1}} - W_{t_i}]$ as $n \to \infty$.

Noting that one may consider $t_i^* = t_i$ then the Itô integral have been defined. However, if $t_i^* = \frac{t_i + t_{i+1}}{2}$, this gives the Stratonovich integral. For example the stochastic integral $\int_0^t WdW$ by Itô approach is $\frac{1}{2}(W^2(t) - t)$; While with the Stratonovich definition yields $\frac{1}{2}W^2(t)$. The Itô integral is a martingale and the Stratonovich provides the results expected from ordinary calculus; the difference between these two integrals comes from the lack of smoothness of $W(t)$; which can be illustrated by Itô stochastic chain rule formula [70, 103].

Definition 4.2.2. ([52]) The transition probability function $\mathbb{P}(s, \mathbf{y}, t, \mathscr{A})$ is said to be time-homogeneous if the function $\mathbb{P}(s, \mathbf{y}, t+s, \mathscr{A})$ is independent of s, where $0 \leq s \leq t, \mathbf{y} \in \mathbb{R}^n$ and $\mathscr{A} \in \mathfrak{B}$, $\mathfrak{B}$ denotes the $\sigma-$ algebra of Borel sets in $\mathbb{R}^n$.

Assume $\mathbf{y}(t)$ is a regular time homogeneous Markov process in $\mathscr{C}([-\tau, 0]; \mathbb{R}_+^n)$ and satisfies the following stochastic delay differential equations (SDDEs)

$$d\mathbf{y}(t) = \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau))dt + \sum_{r=1}^{n} \mathbf{g}_r(t, \mathbf{y}(t))dW_r(t) \quad \text{for} \quad t \geq -\tau, \tau \geq 0 \tag{4.1}$$

with the initial value $\mathbf{y}(t) = \mathbf{y}_0 \in \mathscr{C}([-\tau, 0]; \mathbb{R}_+^n)$. The diffusion matrix of the process $\mathbf{y}(t)$

is defined as follows:

$$A(y) = \sum_{r=1}^{k} (g_r^i)^T (t, y(t)) g_r^j (t, y(t)) = (a_{ij}(y)).$$

Define a $\mathscr{C}^{2,1}(\mathscr{C}([-\tau,0];\mathbb{R}_+^n) \times [-\tau,\infty);\mathbb{R}_+)$ the family of all nonnegative functions $V(t,\mathbf{y}_t)$ such that it is continuously twice differentiable in $\mathbf{y}$ and once in t. The differentiable operator $\mathscr{L}$ of (4.1) is defined by [91]

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mathbf{f}_i(t,\mathbf{y}(t),\mathbf{y}(t-\tau))\frac{\partial}{\partial \mathbf{y}_i} + \frac{1}{2}\sum_{i,j=1}^{n} [\mathbf{g}^T(t,\mathbf{y}(t))\mathbf{g}(t,\mathbf{y}(t))]_{ij}\frac{\partial^2}{\partial \mathbf{y}_i \partial \mathbf{y}_j}. \quad (4.2)$$

If $\mathscr{L}$ acts on a functional $V(t,\mathbf{y}_t) \in \mathscr{C}^{2,1}(\mathscr{C}([-\tau,0];\mathbb{R}_+^n) \times [\tau,\infty);\mathbb{R}_+)$, then

$$\mathscr{L}V(t,\mathbf{y}_t) = V_t(t,\mathbf{y}_t) + V_{\mathbf{y}}(t,\mathbf{y}_t)\mathbf{f}(t,\mathbf{y}(t),\mathbf{y}(t-\tau)) + \frac{1}{2} trace[\mathbf{g}^T(t,\mathbf{y}(t))V_{\mathbf{yy}}(t,\mathbf{y}_t)\mathbf{g}(t,\mathbf{y}(t))],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_{\mathbf{y}} = (\frac{\partial V}{\partial y_1}, \ldots, \frac{\partial V}{\partial y_n})$, $V_{\mathbf{yy}} = (\frac{\partial^2 V}{\partial y_i \partial y_j})_{n\times n}$. According to It$\hat{o}$ formula, if $\mathbf{y}(t) \in \mathscr{C}([-\tau,0];\mathbb{R}_+^n)$, then

$$dV(t,\mathbf{y}_t) = \mathscr{L}V(t,\mathbf{y}_t)dt + V_{\mathbf{y}}(t,\mathbf{y}_t)\mathbf{g}(t,\mathbf{y}(t))dW(t).$$

4.3 SDDEs for Predator-Prey System

In this chapter, the author considers a stochastic version of a predator-prey System (2.2), where white noise is incorporated into the growth equations of both prey and predator, so that

$$\begin{aligned} dx(t) &= \Big[rx(t)(1 - \frac{x(t-\tau_1)}{K}) - \frac{[1+\alpha y(t)]x(t)y(t)}{1+c(1+\alpha y(t))x(t)}\Big]dt + \sigma_1 x(t)dW_1(t), \\ dy(t) &= \Big[-\delta y(t) - ay^2(t) + \frac{\mu[1+\alpha y(t)]x(t-\tau_2)y(t)}{1+c(1+\alpha y(t))x(t-\tau_2)}\Big]dt + \sigma_2 y(t)dW_2(t). \end{aligned} \quad (4.3)$$

$W_1(t), W_2(t)$ are standard independent Wiener processes defined on a complete probability space $(\Omega, \mathscr{A}, \{\mathscr{A}\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{A}\}_{t\geq 0}$ satisfying the usual conditions; and σ_1, σ_2 are the positive intensities of white noises. Assuming that $\kappa \in [-\tau, 0]$, $\tau = \max\{\tau_1, \tau_2\}$, i.e. $(x_0, y_0) \in \mathscr{C}([-\tau, 0], \mathbb{R}^2_+)$ with $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$, if $(x,y) \in \mathbb{R}^2$, its norm is denoted by $|(x,y)| = \sqrt{x^2+y^2}$. The initial value of System (4.3) becomes

$$(x(\kappa), y(\kappa)) = \{(x(\kappa), y(\kappa)) : -\tau \leq \kappa \leq 0\} \in \mathscr{C}([-\tau, 0]; \mathbb{R}^2_+). \tag{4.4}$$

Now, the existence and uniqueness of positive solutions is investigated.

4.3.1 Existence and uniqueness of positive solution

In order to prove that the model of SDDEs (4.3) has a unique global solution (i.e. no explosion in a finite-time) for any given initial condition, the coefficients of the System (4.3) are generally required to satisfy the linear growth condition and local Lipschitz condition [23, 92]. Although, the response function $f(x,y) = \frac{(1+\alpha y)x}{1+c(1+\alpha y)x}$ is nonlinear, coefficients of (4.3) don not satisfy the linear growth condition. Thus, to show that Model (4.3) has a global positive solution, let firstly prove that the model has a positive local solution by making the change of variables. Then, one can prove that this solution will also not explode to infinity at any finite time, by using a suitable stochastic Lyapunov functional.

Theorem 4.3.1. *Let the coefficients of the System (4.3) be locally Lipschitz continuous, then for any given initial data (4.4) there is a unique positive solution $(x(t), y(t))$ of System (4.3) on $t \geq -\tau$, and the solution will remain in $\mathbb{R}^2_+$ with probability one.*

Proof. Let $n(t) = \ln x(t)$, $p(t) = \ln y(t)$, one may have the system

$$\begin{aligned} dn(t) &= \Big(r - \frac{r}{K}e^{n(t-\tau_1)} - \frac{(1+\alpha e^{p(t)})e^{n(t)}}{1+c(1+\alpha e^{p(t)})e^{n(t)}} - \frac{\sigma_1^2}{2}\Big)dt + \sigma_1 dW_1(t) \\ dp(t) &= \Big(\frac{\mu(1+\alpha e^{p(t)})e^{n(t-\tau_2)}}{1+c(1+\alpha e^{p(t)})e^{n(t-\tau_2)}} - \delta - ae^{p(t)} - \frac{\sigma_2^2}{2}\Big)dt + \sigma_2 dW_2(t), \end{aligned} \tag{4.5}$$

for any initial values $n(\kappa)=\ln x(\kappa)$, $p(\kappa)=\ln y(\kappa)$, $\kappa\in[-\tau,0]$. It is easy to show that all the coefficients of (4.5) satisfy the local Lipschitz condition, therefore, there is a unique local solution $(n(t),p(t))$ on $[-\tau,\tau_e)$, where τ_e is explosion time. By Ito's formula, one can see that $x(t)=e^{n(t)}$, $y(t)=e^{p(t)}$, therefore, there is a unique local positive solution of (4.3) for any given initial value $(x_0,y_0)\in\mathbb{R}^2_+$.

To show this solution is global, one may need to show $\tau_e=\infty$ a.s. (almost surely). Let $l_0>0$ be sufficiently large so that $(x(t),y(t))=\{(\phi_1(t),\phi_2(t)):-\tau\le t\le 0\}\in\mathscr{C}([-\tau,0];\mathbb{R}^2_+)$ all lie within the interval $[\frac{1}{l_0},l_0]$. Now for each integer $l\ge l_0$, define the stopping time $\tau_l=\inf\{t\in[-\tau,\tau_e):x(t)\notin(\frac{1}{l},l),y(t)\notin(\frac{1}{l},l)\}$, let $\inf\phi=\infty$. τ_l is increasing with l and let $\tau_\infty=\lim\limits_{l\to\infty}\tau_l$, then $\tau_\infty\le\tau_e$ and by showing $\tau_\infty=\infty$ a.s., now the idea is to conclude that $\tau_e=\infty$ a.s. If this assertion is erroneous, then there exists a pair of constants $T>0$ and $\varepsilon\in(0,1)$ such that $\mathbb{P}\{\tau_\infty\le T\}>\varepsilon$. Therefore, there is an integer $l_1\ge l_0$

$$\mathbb{P}\{\tau_l\le T\}>\varepsilon,\quad\text{for all}\quad l\ge l_1. \tag{4.6}$$

Define a $\mathscr{C}^2$-function $V(x,y):\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ by

$$V(x,y)=(x-\log x-1)+(y-\log y-1)+\frac{r}{K}\int_t^{t+\tau_1}x(s-\tau_1)ds.$$

Clearly, this function is non-negative for all $x,y\ge 0$. Let $l\ge l_0$ and $T>0$ be arbitrary. For $0\le t\le\tau_l\wedge T$, by It$\hat{o}$'s formula for V, one gets

$$\begin{aligned}dV=&\Big[(x-1)[r-\frac{r}{K}x(t-\tau_1)-\frac{(1+\alpha y)y}{1+c(1+\alpha y)x}]+(y-1)[\frac{\mu(1+\alpha y)x(t-\tau_2)}{1+c(1+\alpha y)x(t-\tau_2)}\\&-\delta-ay]+\frac{\sigma_1^2+\sigma_2^2}{2}+\frac{r}{K}x-\frac{r}{K}x(t-\tau_1)\Big]dt+\sigma_1(x-1)dW_1(t)+\sigma_2(y-1)dW_2(t),\end{aligned}$$

$$\begin{aligned} dV &\le \Big[-ay^2+(1+\mu+a-\delta)y+r\frac{K+1}{K}x+(\delta-r)+\frac{\sigma_1^2+\sigma_2^2}{2}\Big]dt \\ &\quad +\sigma_1(x-1)dW_1(t)+\sigma_2(y-1)dW_2(t), \\ &\le \gamma dt+\sigma_1(x-1)dW_1(t)+\sigma_2(y-1)dW_2(t), \end{aligned}$$

where γ is a positive number. Therefore

$$\begin{aligned} \int_{\tau_l\wedge T-\tau}^{\tau_l\wedge T} dV(x,y) \le \int_{\tau_l\wedge T-\tau}^{\tau_l\wedge T} \gamma dt + \int_{\tau_l\wedge T-\tau}^{\tau_l\wedge T} \sigma_1(x-1)dW_1(t) \\ + \int_{\tau_l\wedge T-\tau}^{\tau_l\wedge T} \sigma_2(y-1)dW_2(t). \end{aligned} \tag{4.7}$$

Taking expectation of both sides implies

$$\mathbb{E}[V(x(t_l\wedge T),y(t_l\wedge T))] \le V(x(0),y(0))+\gamma T. \tag{4.8}$$

Set $\Omega_l=\{\tau_l\le T\}$ for $l\ge l_1$ and by the virtue of (4.6), one obtains $\mathbb{P}(\Omega_l)\ge\varepsilon$. For every $\eta\in\Omega_l$, $x(\tau_l,\eta)$ and $y(\tau_l,\eta)$ equal either to l or $\frac{1}{l}$, Consequently, $V(x(\tau_l,\eta),y(\tau_l,\eta))$ is no less than either $l-\log l-1$ or $\frac{1}{l}+\log l-1$. Therefore, one can get

$$V(x(\tau_l,\eta),y(\tau_l,\eta)) \ge \min\{l-\log l-1, \frac{1}{l}+\log l-1\}.$$

It follows from (4.8) that

$$V(x(0),y(0))+\gamma T \ge \mathbb{E}[1_{\Omega_l}(\eta)V(x(\tau_l),y(\tau_l))] \ge \varepsilon[l-\log l-1]\wedge[\frac{1}{l}+\log l-1],$$

where 1_{Ω_l} is the indicator function of Ω_l. Letting $l\to\infty$ leads to a contradiction that $\infty > V(x(0),y(0))+\gamma T=\infty$. Therefore, one gets $\tau_\infty=\infty$ a.s.

4.4 Stochastically Ultimate Boundedness

After discussion existence and uniqueness of positive solution of SDDEs (4.3), one can show that the positive solution does not explode to infinity in a finite time.

Definition 4.4.1. [91] The solution of the SDDE Model (4.3) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there is a positive constant $\varphi = \varphi(\varepsilon)$, such that for any initial value (4.4), the solution of Model (4.3) has the property that $\lim_{t\to\infty} \sup \mathbb{P}\{|(x(t),y(t))| > \varphi\} < \varepsilon$.

The system is said to be stochastically ultimately bounded if the following Theorem is satisfied.

Theorem 4.4.1. *For any $\theta \in (0,1)$ and $\mu\theta(\mu K e^{\tau_2} - \delta) > a$, there is a positive constant $N = N(\theta)$, which is independent of the initial value (4.4), such that the solution Model (4.3) has the following properties*

$$\lim_{t\to\infty} \sup \mathbb{E}[|(x(t),y(t))|^{\theta}] \leq N. \tag{4.9}$$

Then System (4.3) is stochastically ultimately bounded.

Proof. To prove (4.9), define

$$V(x,y) = x^{\theta} + y^{\theta}, \quad (x,y) \in \mathbb{R}^2_+. \tag{4.10}$$

Applying Itô's formula, gives

$$\begin{aligned}\mathscr{L}V(x,y) = \theta x^{\theta}\big[r - \frac{r}{K}x(t-\tau_1) - \frac{(1+\alpha y)y}{1+c(1+\alpha y)x}\big] + \frac{\sigma_1^2}{2}\theta(\theta-1)x^{\theta} \\ + \theta y^{\theta}[\frac{\mu(1+\alpha y)x(t-\tau_2)}{1+c(1+\alpha y)x(t-\tau_2)} - \delta - ay] + \frac{\sigma_2^2}{2}\theta(\theta-1)y^{\theta},\end{aligned}$$

$$
\begin{aligned}
\mathscr{L}V(x,y) &\leq \theta x^{\theta}(r - \frac{r}{K}x(t-\tau_1)) - \frac{\sigma_1^2}{2}\theta(1-\theta)x^{\theta} + \frac{\mu\theta y^{\theta}(1+\alpha y)x(t-\tau_2)}{1+c(1+\alpha y)x(t-\tau_2)} \\
&\quad - \frac{\sigma_2^2}{2}\theta(1-\theta)y^{\theta} - \theta a y^{\theta+1} \\
&\leq r\theta x^{\theta} - \frac{\sigma_1^2}{2}\theta(1-\theta)x^{\theta} - \frac{\sigma_2^2}{2}\theta(1-\theta)y^{\theta} \\
&\quad + \mu\theta(\frac{\mu K e^{\tau_2} - \delta}{a})|x(t-\tau_2)|^2 \\
&= H(x,y) - V(x,y) - e^{\tau_2}|x(t)|^2 + \mu\theta(\frac{\mu K e^{\tau_2} - \delta}{a})|x(t-\tau_2)|^2, \quad \text{where,}
\end{aligned}
$$

$$
H(x,y) = (r\theta + 1)x^{\theta} + y^{\theta} - \frac{\sigma_1^2}{2}(1-\theta)x^{\theta} - \frac{\sigma_2^2}{2}\theta(1-\theta)y^{\theta} + e^{\tau_2}|x(t)|^2 \leq N_0,
$$

for $(x,y) \in \mathbb{R}_+^2$. Note that $H(x,y)$ is bounded in $\mathbb{R}_+^2$. Hence, one gets

$$
\mathscr{L}V(x,y) \leq N_0 - V(x,y) - e^{\tau_2}|x(t)|^2 + \frac{\mu\theta(\mu K e^{\tau_2} - \delta)}{a}|x(t-\tau_2)|^2.
$$

Thus, one obtains

$$
\begin{aligned}
dV(x,y) &= \mathscr{L}V(x,y)dt + \sigma_1\theta x^{\theta}dW_1(t) + \sigma_2\theta y^{\theta}dW_2(t) \\
&\leq (N_0 - V(x,y) - e^{\tau_2}|x(t)|^2 + \frac{\mu\theta(\mu K e^{\tau_2} - \delta)}{a}|x(t-\tau_2)|^2)dt \\
&\quad + \sigma_1\theta x^{\theta}dW_1(t) + \sigma_2\theta y^{\theta}dW_2(t).
\end{aligned}
$$

Again, using It$\hat{o}$'s formula, one may have

$$
\begin{aligned}
d(e^t V(x,y)) &= e^t V(x,y)dt + e^t dV(x,y) \\
&\leq e^t[N_0 - e^{\tau_2}|x(t)|^2 + \frac{\mu\theta(\mu K e^{\tau_2} - \delta)}{a}|x(t-\tau_2)|^2]dt \\
&\quad + e^t\sigma_1\theta x^{\theta}dW_1(t) + e^t\sigma_2\theta y^{\theta}dW_2(t).
\end{aligned}
$$

If $\mu\theta(\mu K e^{\tau_2}-\delta) > a$, then

$$\begin{aligned}
e^t\mathbb{E}[V(x,y)] &\le V(x(0),y(0)) + N_0 e^t - \mathbb{E}[\int_0^t e^{s+\tau_2}|x(s)|^2 ds] \\
&\quad + \mu\theta\frac{\mu K e^{\tau_2}-\delta}{a}\mathbb{E}[\int_0^t e^s|x(s-\tau_2)|^2 ds] \\
&= V(x(0),y(0)) + N_0 e^t - \mathbb{E}[\int_0^t e^{s+\tau_2}|x(s)|^2 ds] \\
&\quad + \frac{\mu^2\theta K}{a}\mathbb{E}[\int_{-\tau_2}^{t-\tau_2} e^{s+\tau_2}|x(s)|^2 ds] - \frac{\delta\mu\theta}{a}\mathbb{E}[\int_0^t e^s|x(s-\tau_2)|^2 ds] \\
&\le V(x(0),y(0)) + N_0 e^t + \frac{\mu^2\theta K}{a}\mathbb{E}[\int_{-\tau_2}^{0} e^{s+\tau_2}|x(s)|^2 ds],
\end{aligned}$$

which implies that $\lim\limits_{t\to\infty}\sup\mathbb{E}[V(x(t),y(t))] \le N_0$. Therefore, one gets

$$\lim_{t\to\infty}\sup\mathbb{E}[|x(t),y(t)|^\theta] \le \sqrt{2^\theta}\lim_{t\to\infty}\sup\mathbb{E}[V(x(t),y(t))] \le \sqrt{2^\theta}N_0 = N(\theta). \tag{4.11}$$

Since $\lim\limits_{t\to\infty}\sup\mathbb{E}[|x(t),y(t)|^\theta] \le N$, then for any $\varepsilon > 0$ let $\varphi = N^2/\varepsilon^2$. By Chebyshev inequality, $\mathbb{P}\{|(x(t),y(t))| > \varphi\} \le \dfrac{\mathbb{E}[(\sqrt{|(x(t),y(t))|})]}{\sqrt{\varphi}}$, one obtains $\lim\limits_{t\to\infty}\sup\mathbb{P}\{|(x(t),y(t))| > \varphi\} \le \dfrac{N}{\sqrt{\varphi}} := \varepsilon$, which implies

$$\lim_{t\to\infty}\sup\mathbb{P}\{|(x(t),y(t))| \le \varphi\} \ge 1-\varepsilon.$$

Thus, Model (4.3) is stochastically ultimately bounded.

4.5 Almost Surely Asymptotic Properties

When the model is subject to stochastic noises, it is valuable and interesting to examine whether the stochastic model preserve some stability properties for the deterministic model. For simplicity, one can introduce the following notations.

- $M_i(t) = \int_0^t \frac{\sigma_i(x(t),y(t))_i}{V(x(s),y(s))}dW(s), \quad i = 1,2$. Such that $M_i(t)$ is a real-valued continuous local martingale vanishing at $t = 0$ and its quadratic form is given by

$\langle M_i(t), M_i(t)\rangle = \int_0^t \frac{\sigma_i^2(x(t),y(t))_i^2}{V(x(s),y(s))} ds;\ i = 1,2.$

- Let $\varepsilon \in (0,1)$ and $\zeta > 0$, by the exponential martingale inequality [92], for each $n \geq 1$,

$$\mathbb{P}\{\sup_{0\leq t\leq n}(M_i(t) - \frac{\varepsilon}{2}\langle M_i(t), M_i(t)\rangle) > \frac{\zeta \ln n}{\varepsilon}\} \leq n^{-\zeta}.$$

Since $\Sigma_{n=1}^{\infty} n^{-\zeta}$ is convergent, By using Borel-Cantelli Lemma, there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for $\rho \in \Omega_0$ there exists an integer $n_0 = n_0(\rho)$ and choosing $\zeta = 2$, one may have

$$M_i(t) \leq \frac{\varepsilon}{2}\langle M_i(t), M_i(t)\rangle + \frac{2}{\varepsilon}\ln n, \quad \text{for all} \quad 0 \leq t \leq n \wedge n \geq n_0(\rho). \tag{4.12}$$

Theorem 4.5.1. *For any given initial value (4.4), such that the solution of Model (4.3) has the property that*

$$\limsup_{t\to\infty} t^{-1} \ln|(x(t),y(t))| \leq D_2 \quad a.s,$$

where $D_2 = 1 + \delta + a + \frac{\hat{D}^2}{2\hat{\sigma}^2}$, *such that* $\hat{\sigma} = \min\{\sigma_1 a_1, \sigma_2 a_2\}$ *and* $\hat{D} = \max\{\alpha, \mu\}$.

Proof. Define $V(x(t),y(t)) = x(t) + y(t)$, using Itô's formula it gives that

$$\begin{aligned}
\ln V(x(t),y(t)) - \ln V(x(0),y(0)) = &\int_0^t \Big(\frac{x}{(x(s)+y(s))}(1 - \frac{r}{K}x(s-\tau_1) \\
&- \frac{(1+\alpha y)y}{1+c(1+\alpha y)x}) - \frac{\sigma_1^2 x^2}{2(x(s)+y(s))^2}\Big)ds \\
&+ \int_0^t \Big(\frac{y}{(x(s)+y(s))}(\frac{\mu(1+\alpha y)x(s-\tau_2)}{1+c(1+\alpha y)x(s-\tau_2)} - \delta - ay) \\
&- \frac{\sigma_2^2 y^2}{2(x(s)+y(s))^2}\Big)ds + M_1(t) + M_2(t).
\end{aligned} \tag{4.13}$$

Substituting Inequality (4.12) into (4.13) one obtains

$$\begin{aligned}\ln\frac{V(x(t),y(t))}{V(x(0),y(0))} \le \int_0^t \Big(&\frac{x}{(x(s)+y(s))}(1-\frac{r}{K}x(s-\tau_1)-\frac{(1+\alpha y)y}{1+c(1+\alpha y)x})\\ &-\frac{(1-\varepsilon)\sigma_1^2x^2}{2(x(s)+y(s))^2}\Big)ds\\ +\int_0^t \Big(&\frac{y}{(x(s)+y(s))}(\frac{\mu(1+\alpha y)x(s-\tau_2)}{1+c(1+\alpha y)x(s-\tau_2)}-\delta-ay)\\ &-\frac{(1-\varepsilon)\sigma_2^2y^2}{2(x(s)+y(s))^2}\Big)ds\\ &+\frac{4}{\varepsilon}\ln n,\end{aligned} \tag{4.14}$$

assume that $a_1,a_2\in(0,1)$ are positive constants. Thus,

$$\begin{aligned}\ln\frac{V(x(t),y(t))}{V(x(0),y(0))} \le& \int_0^t (1+\alpha y+\mu x(s-\tau_2)+\delta+a-\frac{(1-\varepsilon)\hat{\sigma}^2|(x(s),y(s))|^2}{2})ds\\ &+\frac{4}{\varepsilon}\ln n,\end{aligned} \tag{4.15}$$

where, $\hat{\sigma}=\min\{\sigma_1a_1,\sigma_2a_2\}$ for all $0\le t\le n$, $n\ge n_0(\rho)$ and $\rho\in\Omega_0$. From (4.15), one gets

$$\begin{aligned}&\ln V(x(t),y(t))+\frac{(1-2\varepsilon)\hat{\sigma}^2}{4}\int_0^t|(x(s),y(s))|^2ds\\ &\qquad\le D_1+\int_0^t(1+\alpha y+\mu x+\delta+a-\frac{\hat{\sigma}^2|(x(s),y(s))|^2}{2})ds+\frac{4}{\varepsilon}\ln n,\end{aligned} \tag{4.16}$$

such that $D_1=\ln V(x(0),y(0))+\mu\int_{-\tau_2}^0 x(s)ds$. Due to

$$1+\alpha y+\mu x+\delta+a-\frac{\hat{\sigma}^2|(x(t),y(t))|^2}{2}\le 1+\delta+a+\frac{\hat{D}^2}{2\hat{\sigma}^2}=D_2,$$

where $\hat{D}=\max\{\alpha,\mu\}$, therefore, if $\rho\in\Omega_0$,

$$\ln V(x(t),y(t))+\frac{(1-2\varepsilon)\hat{\sigma}^2}{4}\int_0^t|(x(s),y(s))|^2ds\le D_1+D_2t+\frac{4}{\varepsilon}\ln n,$$

for all $0 \le t \le n$, $n \ge n_0(\rho)$. Hence, for all $\rho \in \Omega_0$, if $n-1 \le t \le n$, $n \ge n_0(\rho)$, as $\varepsilon \to 0$ it gives that

$$\limsup_{t\to\infty}(t^{-1}\ln V(x(t),y(t)) + t^{-1}\frac{(1-2\varepsilon)\hat{\sigma}^2}{4}\int_0^t |(x(s),y(s))|^2 ds) \le D_2 \quad a.s,$$

using $V(x(t),y(t)) \ge \frac{|(x(t),y(t))|}{\sqrt{2}}$, implies $\limsup_{t\to\infty}(t^{-1}\ln|(x(t),y(t))|) \le D_2 \quad a.s.$

4.6 Persistence and Extinction of the Solution

Herein, sufficient conditions for persistence and extinction are provided, using Lyapunov functionals.

4.6.1 Persistence

Under certain restrictions on the parameters values with small intensities of white noise, the conditions under which persistence of the system SDDEs (4.3) occurs are investigated. Let first define persistence in the mean of a dynamical system.

Definition 4.6.1. The species $y(t)$ is said to be persistence (See [84].) in the mean if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds > 0, \quad a.s.$$

In order to show the persistence, the author go through the following Lemma.

Lemma 4.6.1. *[63]. Let $y(t) \in \mathscr{C}[[0,\infty)\times\Omega,(0,\infty)]$. If there exist positive constants λ_0, λ such that*

$$\ln y(t) \ge \lambda t - \lambda_0 \int_0^t y(s)ds + F(t) \quad a.s.,$$

for all $t \geq 0$, where $F \in \mathscr{C}[[0,\infty)\times\Omega,\mathbb{R}]$ and $\lim_{t\to\infty}\frac{F(t)}{t} = 0$ a.s., then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \geq \frac{\lambda}{\lambda_0} \quad a.s.$$

Define a threshold parameter $\mathscr{T}_0^s$ as follows.

$$\mathscr{T}_0^s = \frac{\mu K}{\tilde{\delta}(1+c)} > 0, \quad \text{where} \quad \tilde{\delta} = \delta + \frac{\sigma_2^2}{2}. \tag{4.17}$$

Theorem 4.6.2. *Let $(x(t), y(t))$ be the solution the SDDEs (4.3) with initial conditions (4.4). Assume that $2r > \sigma_1^2$, then the System (4.3) will be persistence if $\mathscr{T}_0^s > 1$. So that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds > 0, \quad a.s.$$

Proof. Using of Itô's formula to the first equation of System (4.3), yields

$$d(\ln x(t) - \frac{r}{K}\int_t^{t+\tau_1} x(s-\tau_1)ds) \leq ((r-\frac{\sigma_1^2}{2}) - \frac{r}{K}x(t))dt + \sigma_1 dW_1(t). \tag{4.18}$$

Integrating of Inequality (4.18) from 0 to t results in

$$\frac{\ln x(t) - \frac{r}{K}\int_t^{t+\tau_1} x(s-\tau_1)ds}{t} - \frac{\ln x(0) - \frac{r}{K}\int_0^{\tau_1} x(s-\tau_1)ds}{t} \leq (r-\frac{\sigma_1^2}{2}) - \frac{r}{K}\langle x(t)\rangle + \frac{\sigma_1 W_1(t)}{t}.$$

Thus,

$$\langle x(t)\rangle \leq \frac{K}{r}(r-\frac{\sigma_1^2}{2}) + \gamma_1(t), \quad \text{where,} \tag{4.19}$$

$$\gamma_1(t) = \frac{K}{r}\Big[\frac{\sigma_1 W_1(t)}{t} - \frac{\ln x(t) - \frac{r}{K}\int_t^{t+\tau_1} x(s-\tau_1)ds}{t} + \frac{\ln x(0) - \frac{r}{K}\int_0^{\tau_1} x(s-\tau_1)ds}{t}\Big],$$

it follows from Lemma 4.2.1 that

$$\lim_{t\to\infty}\frac{W_1(t)}{t}=0 \quad a.s.$$

Noting that

$$\frac{\int_t^{t+\tau_1}x(s-\tau_1)ds}{t}=\frac{1}{t}\int_{t-\tau_1}^{t}x(s)ds=\frac{1}{t}\Big[\int_0^t x(s)ds-\int_0^{t-\tau_1}x(s)ds\Big].$$

Therefore, $\lim_{t\to\infty}\int_t^{t+\tau_1}\frac{x(s-\tau_1)ds}{t}=0$. Moreover, $\lim_{t\to\infty}\int_0^{\tau_1}\frac{x(s-\tau_1)ds}{t}=\lim_{t\to\infty}\frac{\int_{-\tau_1}^0\phi_1(t)dt}{t}=0$. Thus, one obtains

$$\lim_{t\to\infty}\gamma_1=0 \quad a.s. \tag{4.20}$$

By It$\hat{o}$'s formula, one gets

$$\begin{aligned}
&d(\ln x(t)-\frac{r}{K}\int_t^{t+\tau_1}x(s-\tau_1)ds)\\
&=\Big[r(1-\frac{x(t-\tau_1)}{K})-\frac{(1+\alpha y)y(t)}{1+c(1+\alpha y)x(t)}-\frac{r}{K}x(t)+\frac{r}{K}x(t-\tau_1)-\frac{\sigma_1^2}{2}\Big]dt+\sigma_1 dW_1(t)\\
&\geq\Big[r-\frac{r}{K}x(t)-2(1+\alpha)y(t)-\frac{\sigma_1^2}{2}\Big]dt+\sigma_1 dW_1(t),
\end{aligned}$$

so one may have

$$\begin{aligned}
&\frac{\ln x(t)-\frac{r}{K}\int_t^{t+\tau_1}x(s-\tau_1)ds}{t}-\frac{\ln x(0)-\frac{r}{K}\int_0^{\tau_1}x(s-\tau_1)ds}{t}\\
&\geq(r-\frac{\sigma_1^2}{2})-2(1+\alpha)\langle y\rangle-\frac{r}{K}\langle x\rangle+\frac{\sigma_1 W_1(t)}{t}.
\end{aligned}$$

Therefore,

$$\langle x(t)\rangle\geq-\frac{2K(1+\alpha)}{r}\langle y(t)\rangle+\frac{K}{r}(r-\frac{\sigma_1^2}{2})+\gamma_1(t). \tag{4.21}$$

Let

$$V = \ln y(t) + \int_t^{t+\tau_2} \Big[\frac{\mu x(s-\tau_2)}{1+cx(s-\tau_2)}\Big] ds, \tag{4.22}$$

utilizing It$\hat{o}$ formula, one obtains

$$\begin{aligned} dV = [&\frac{\mu(1+\alpha y)x(t-\tau_2)}{1+c(1+\alpha y)x(t-\tau_2)} - \delta - ay - \frac{1}{2}\sigma_2^2 + \frac{\mu x(t)}{1+cx(t)} \\ &- \frac{\mu x(t-\tau_2)}{1+cx(t-\tau_2)}]dt + \sigma_2 dW_2(t), \end{aligned} \tag{4.23}$$

According to (4.23), one may have

$$dV \geq [\frac{\mu x(t)}{1+cx(t)} - \delta - ay - \frac{1}{2}\sigma_2^2]dt + \sigma_2 dW_2(t), \tag{4.24}$$

- Case (1) When $x(t) \leq 1$, then

$$dV \geq [\frac{\mu x(t)}{1+c} - \delta - ay - \frac{1}{2}\sigma_2^2]dt + \sigma_2 dW_2(t). \tag{4.25}$$

$$\frac{V(t)-V(0)}{t} \geq \frac{\mu}{1+c}\langle x(t)\rangle - a\langle y(t)\rangle - [\delta + \frac{1}{2}\sigma_2^2] + \frac{\sigma_2}{t}W_2(t). \tag{4.26}$$

Substituting (4.21) into (4.26), one obtains

$$\begin{aligned} \frac{V(t)-V(0)}{t} \geq \frac{\mu}{1+c}&\Big[\frac{-2K(1+\alpha)}{r}\langle y(t)\rangle + K(1-\frac{\sigma_1^2}{2r}) + \gamma_1(t)\Big] \\ &- a\langle y(t)\rangle - \tilde{\delta} + \frac{\sigma_2}{t}W_2(t). \quad \text{Therefore,} \end{aligned} \tag{4.27}$$

$$\begin{aligned} \ln y(t) \geq &\Big[\frac{\mu K}{(1+c)}(1-\frac{\sigma_1^2}{2r}) - \tilde{\delta}\Big]t - \Big[\frac{2\mu K(1+\alpha)}{r(1+c)} + a\Big]\langle y(t)\rangle t \\ &+ \frac{\mu}{1+c}\gamma_1(t)t + \sigma_2 W_2(t) + \ln y(0) + \int_0^{\tau_2} \frac{\mu x(s-\tau_2)}{1+cx(s-\tau_2)}ds \\ &- \int_t^{t+\tau_2} \frac{\mu x(t-\tau_2)}{1+cx(t-\tau_2)}ds \end{aligned} \tag{4.28}$$

$$\ln y(t) \geq \lambda t - \lambda_0 \int_0^t y(s)ds + F(t), \tag{4.29}$$

where

$$\lambda = \frac{\mu K}{(1+c)}(1-\frac{\sigma_1^2}{2r}) - \tilde{\delta}, \quad \lambda_0 = \frac{2\mu K(1+\alpha)}{r(1+c)} + a,$$
$$F(t) = \frac{\mu}{1+c}\gamma_1(t)t + \sigma_2 W_2(t) + \ln y(0) + \int_0^{\tau_2} \frac{\mu x(s-\tau_2)}{1+cx(s-\tau_2)}ds - \int_t^{t+\tau_2} \frac{\mu x(t-\tau_2)}{1+cx(t-\tau_2)}ds.$$

Since $2r > \sigma_1^2$ and $\mathscr{T}_0^s > 1$, this implies that $\lambda > 0$. This together with (4.20) and Lemma 4.6.1, one obtains

$$\liminf_{t\to\infty} \langle y(t) \rangle \geq \frac{r(1+c)}{2\mu K(1+\alpha) + ar(1+c)} \Big(\frac{\mu K(2r-\sigma_1^2)}{2r(1+c)} - \tilde{\delta} \Big) > 0.$$

- Case (2) When $x(t) > 1$, from (4.24), one can get

$$dV \geq [\frac{\mu}{1+c} - \delta - ay - \frac{1}{2}\sigma_2^2]dt + \sigma_2 dW_2(t), \tag{4.30}$$

$$\frac{V(t)-V(0)}{t} \geq \frac{\mu}{1+c} - a\langle y(t) \rangle - \tilde{\delta} + \frac{\sigma_2}{t} W_2(t). \tag{4.31}$$

Since $\mathscr{T}_0^s > 1$, and following a similar proof to Case (1), one can obtain

$$\liminf_{t\to\infty} \langle y(t) \rangle \geq \frac{1}{a}\Big(\frac{\mu}{1+c} - \tilde{\delta} \Big) > 0.$$

This completes the proof.

4.6.2 Extinction

Extinction is one of the most important term in population dynamics. A species is said to be extinct if there is no existing member in the habitat. Although, under some conditions, the solution to the original deterministic DDEs (2.2) may be persistent. However, the solution to the associated SDDEs will become extinct with probability one. This reveals the important fact that the environmental noise may make the population extinct. Now, the conditions under which extinction of predator population occurs are established.

Definition 4.6.2. [92] The species $y(t)$ is said to go to extinction with probability one if $\lim_{t\to\infty} y(t) = 0$ a.s.

Theorem 4.6.3. *Let $a > \mu\alpha$. If $\mathscr{T}_0^s < 1$, then the solution $(x(t), y(t))$ of Model (4.3), for any given initial value (4.4), satisfies*

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} < 0 \quad a.s., \tag{4.32}$$

which means $\lim_{t\to\infty} y(t) = 0$ exponentially a.s. In other words, the predators die out with probability one. In addition,

$$\lim_{t\to\infty} \langle x(t) \rangle = K(1 - \frac{\sigma_1^2}{2r}).$$

Proof. According to (4.23), one obtains

$$\begin{aligned} dV &\leq [\frac{\mu(1+\alpha y)x(t-\tau_2)}{1+cx(t-\tau_2)} - \delta - ay - \frac{1}{2}\sigma_2^2 + \frac{\mu x(t)}{1+cx(t)} \\ &\quad - \frac{\mu x(t-\tau_2)}{1+cx(t-\tau_2)}]dt + \sigma_2 dW_2(t) \\ &\leq [\frac{\mu x(t)}{1+cx(t)} - \delta - (a-\mu\alpha)y(t) - \frac{1}{2}\sigma_2^2]dt + \sigma_2 dW_2(t), \end{aligned} \tag{4.33}$$

since $(a > \mu\alpha)$, then

$$dV \leq [\frac{\mu x(t)}{1+cx(t)} - \delta - \frac{1}{2}\sigma_2^2]dt + \sigma_2 dW_2(t). \tag{4.34}$$

Thus, one may have two cases. **Case I**. When $x(t) \leq 1$, according to (4.34), one can get

$$dV \leq [\frac{\mu}{1+c} - \tilde{\delta}]dt + \sigma_2 dW_2(t), \tag{4.35}$$

therefore, one gets

$$\frac{V(t)-V(0)}{t} \leq \frac{\mu}{1+c} - \tilde{\delta} + \frac{\sigma_2}{t}W_2(t). \quad \text{So,} \tag{4.36}$$

$$\frac{\ln y(t)}{t} \leq \frac{\mu}{1+c} - \tilde{\delta} + \chi_1(t), \quad \text{where} \tag{4.37}$$

$$\chi_1(t) = \frac{\sigma_2}{t}W_2(t) + \frac{\ln y(0)}{t} + \frac{1}{t}\int_0^{\tau_2} \frac{\mu x(s-\tau_2)}{1+cx(t-\tau_2)}ds - \frac{1}{t}\int_t^{t+\tau_2} \frac{\mu x(s-\tau_2)}{1+cx(s-\tau_2)}ds. \tag{4.38}$$

In view of the strong law of large numbers of Brownian motion, one can easily obtain that $\lim_{t\to\infty}\chi_1(t) = 0 \quad a.s.$ Thus, it follows from (4.37) and since $\mathscr{T}_0^s < 1$

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq \frac{\mu}{1+c} - \tilde{\delta} < 0. \quad a.s. \tag{4.39}$$

Case II. When $x(t) > 1$, by (4.34), one may have

$$dV \leq [\frac{\mu x(t)}{1+c} - \tilde{\delta}]dt + \sigma_2 dW_2(t), \quad \text{then,} \tag{4.40}$$

$$\frac{V(t)-V(0)}{t} \leq \frac{\mu K}{1+c} - \frac{\mu}{1+c}\gamma_1(t) - \tilde{\delta} + \frac{\sigma_2}{t}W_2(t). \tag{4.41}$$

$$\text{Therefore,} \quad \frac{\ln y(t)}{t} \leq \frac{\mu K}{1+c} - \tilde{\delta} + \chi_2(t), \quad \text{where} \tag{4.42}$$

$$\begin{aligned}\chi_2(t) = {} & \frac{\sigma_2}{t}W_2(t) + \frac{\ln y(0)}{t} + \frac{1}{t}\int_0^{\tau_2} \frac{\mu x(s-\tau_2)}{1+cx(s-\tau_2)}ds - \frac{1}{t}\int_t^{t+\tau_2} (\frac{\mu x(s-\tau_2)}{1+cx(s-\tau_2)}ds \\ & - \frac{\mu}{1+c}\gamma_1(t).\end{aligned} \tag{4.43}$$

In view of the strong law of large numbers of Brownian motion, one can easily obtain that $\lim_{t\to\infty}\chi_2(t) = 0 \quad a.s.$ Therefore,

$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq \frac{\mu K}{1+c} - \tilde{\delta} < 0. \quad a.s. \tag{4.44}$$

Which implies that, $y(t)$ tends to zero exponentially with probability one,

$$\lim_{t\to\infty} y(t) = 0 \quad a.s. \tag{4.45}$$

By taking the limit both sides of (4.19) and (4.21) at the same time, one can get

$$\lim_{t\to\infty} \langle x(t)\rangle = K(1-\frac{\sigma_1^2}{2r}).$$

This completes the proof.

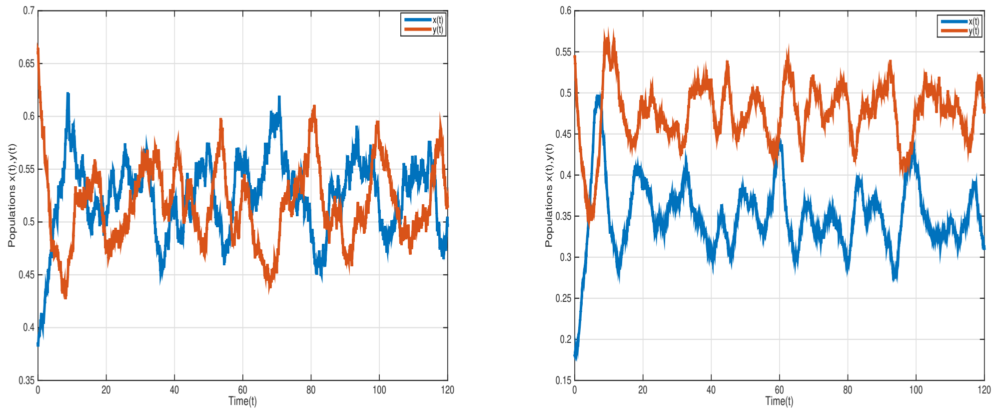

Figure 4.1: Numerical simulations of System (4.3) which displays the persistence of the system, when $\mathscr{T}_0^s > 1$. With $\alpha = 0.12$, $a = 0.08$, $c = 0.3$, $r = 1$, $\mu = 0.9$, $\delta = 0.39$, $K = 1$, $\sigma_1 = \sigma_2 = 0.001$ and $\tau_1 = \tau_2 = 0.1$. However, the right banner illustrates that the predator population dominates the prey population as time goes when α is increased to $\alpha = 1.2$

4.7 Numerical Simulations

In this Section, the author attempts to validate the mathematical results obtained in the previous sections. Milstein's scheme with a strong order of convergence one, discussed in [71] is utilized. The corresponding discretization system to SDDEs (4.3) is

$$\begin{aligned}
x_{n+1} &= x_n + hx_n[r(1-\frac{x_{n-m_1}}{K}) - \frac{(1+\alpha y_n)y_n}{1+c(1+\alpha y_n)x_n}] + \sigma_1 x_n \xi_{1,n} + \frac{\sigma_1^2}{2} x_n[(\xi_{1,n}(h)^{\frac{1}{2}})^2 - h] \\
y_{n+1} &= y_n + hy_n[\frac{\mu(1+\alpha y_n)x_{n-m_2}}{1+c(1+\alpha y_n)x_{n-m_2}} - \delta - ay_n] + \sigma_2 y_n \xi_{2,n} + \frac{\sigma_2^2}{2} y_n[(\xi_{2,n}(h)^{\frac{1}{2}})^2 - h].
\end{aligned}$$

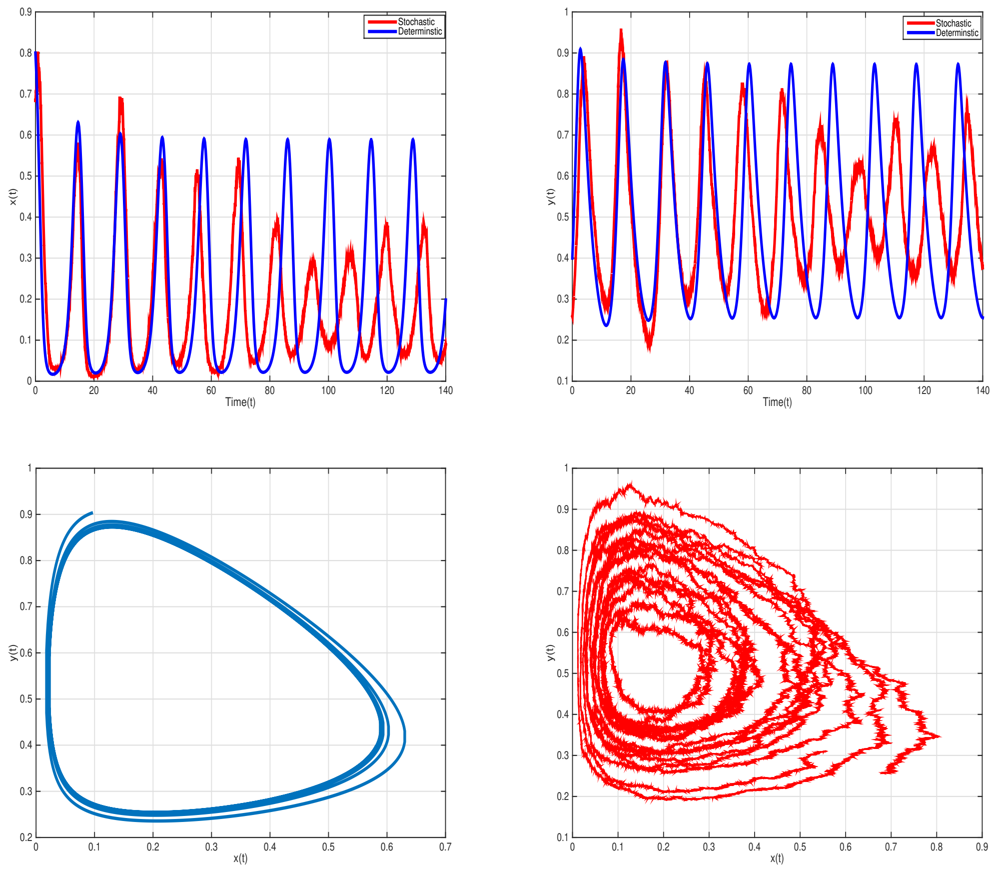

Figure 4.2: Numerical simulations of the solutions for System (4.3) and the corresponding deterministic System (2.2), when $\mathscr{T}_0^s > 1$. With $\tau_1 = \tau_1^* = 0.8$ and $\tau_2 = 0.1 < \tau_2^*$, $a = 0.08$, $\delta = 0.19$, $\alpha = 1.6$, $c = 0.6$, $K = 1$, $\mu = 0.9$, $r = 1$, while the intensities of Brownian motions are relatively small $\sigma_1 = 0.004$ and $\sigma_2 = 0.0001$. Top (left) displays a periodic solution of the deterministic model for the prey population; while in the stochastic model a damped periodic solution is observed. Top (right) shows a periodic solution of the deterministic model for predator population with a damped periodic solution in the stochastic model. Bottom (left) is a phase space that shows the existence of a limit cycle around $\mathscr{E}^*$. Bottom (right) is a numerical simulation that shows the damped periodic oscillation around $\mathscr{E}^*$ in the stochastic model.

Here, $\xi_{1,n}$ and $\xi_{2,n}$ are mutually independent $N(0,1)$ random variables, m_1, m_2 are integers such that the time delays can be expressed in terms of the step-size as $\tau_1 = m_1 h$ & $\tau_2 = m_2 h$. Some numerical simulations of the stochastic Model (4.3) and its corresponding deterministic Model (2.2) are provided.

Taking the parameter values $\alpha = 0.12$, $a = 0.08$, $c = 0.3$, $r = 1$, $\mu = 0.9$, $\delta = 0.39$, $K = 1$, $\sigma_1 = \sigma_2 = 0.001$, and $\tau_1 = \tau_2 = 0.1$. Figure 4.1 shows persistence of System (4.3) with initial values $(0.4, 0.8)$, such that $\mathscr{T}_0^s = \frac{\mu K}{\tilde{\delta}(1+c)} = 1.78 > 1$. If the hunting cooperative parameter α is increased as $\alpha = 1.2$ keeping all other parameters the same,

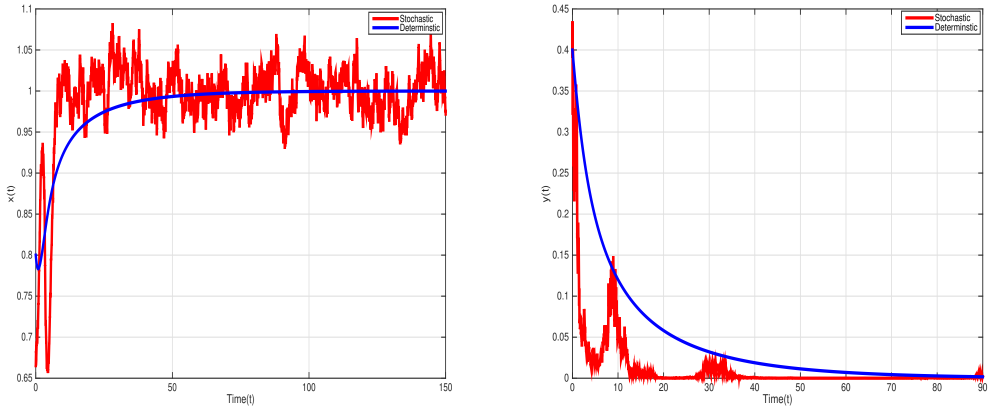

Figure 4.3: Numerical simulations of the solutions for System (4.3) and the corresponding deterministic System (2.2), when $\mathcal{T}_0^s < 1$. With $a = 0.19$, $\mu = 0.8$, $\alpha = 0.1$, $c = 0.8$, $K = 1$, $\delta = 0.59$, $r = 1$, $\sigma_1 = 0.001$, $\sigma_2 = 0.023$ and $\tau_1 = \tau_2 = 0.1$. In the left banner the population of prey varies around the deterministic steady state value. In the right banner predator population goes to extinct at $t = 70$ for deterministic system; While extinction occurs at $t = 20$ with stochastic model.

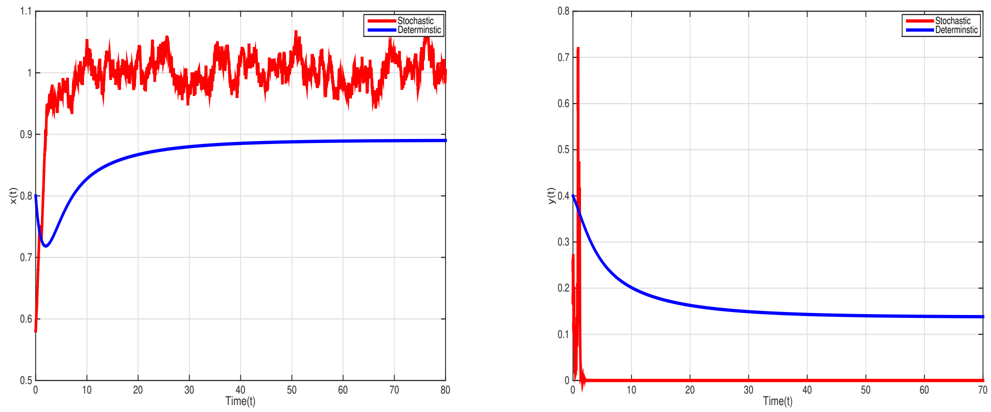

Figure 4.4: Numerical simulations of the solutions for System (4.3) and the corresponding deterministic System (2.2), when $\mathcal{T}_0^s < 1$ with $\tau_1 = \tau_2 = 0.1$, $\sigma_1 = 0.001$ and $\sigma_2 = 0.1$. With $a = 0.19$, $\mu = 0.6$, $\alpha = 0.1$, $c = 0.3$, $K = 1$, $\delta = 0.4$, $r = 1$. In the left banner, the increasing of the intensities of white noise promote the prey population densities. in the right banner the large scale of white noises may lead to no surviving predator individuals that can reproduce and create a new generation; While in the deterministic one the predator individuals are still survival.

one can observe that the predator dominates the prey population; See Figure 4.1 (right). The population densities vary around the deterministic steady state values.

Figure 4.2 shows the impact of small white noise in dynamics of the system. The Figure displays a periodic solution of the deterministic system when $\tau_1 = \tau_1^* = 0.8$ and $\tau_2 = 0.1 < \tau_2^*$, $a = 0.08$, $\delta = 0.19$, $\alpha = 1.6$, $c = 0.6$, $K = 1$, $\mu = 0.9$, $r = 1$. However, with small noises $\sigma_1 = 0.004$ and $\sigma_2 = 0.0001$, where $\mathcal{T}_0^s = 2.96 > 1$, one can observe

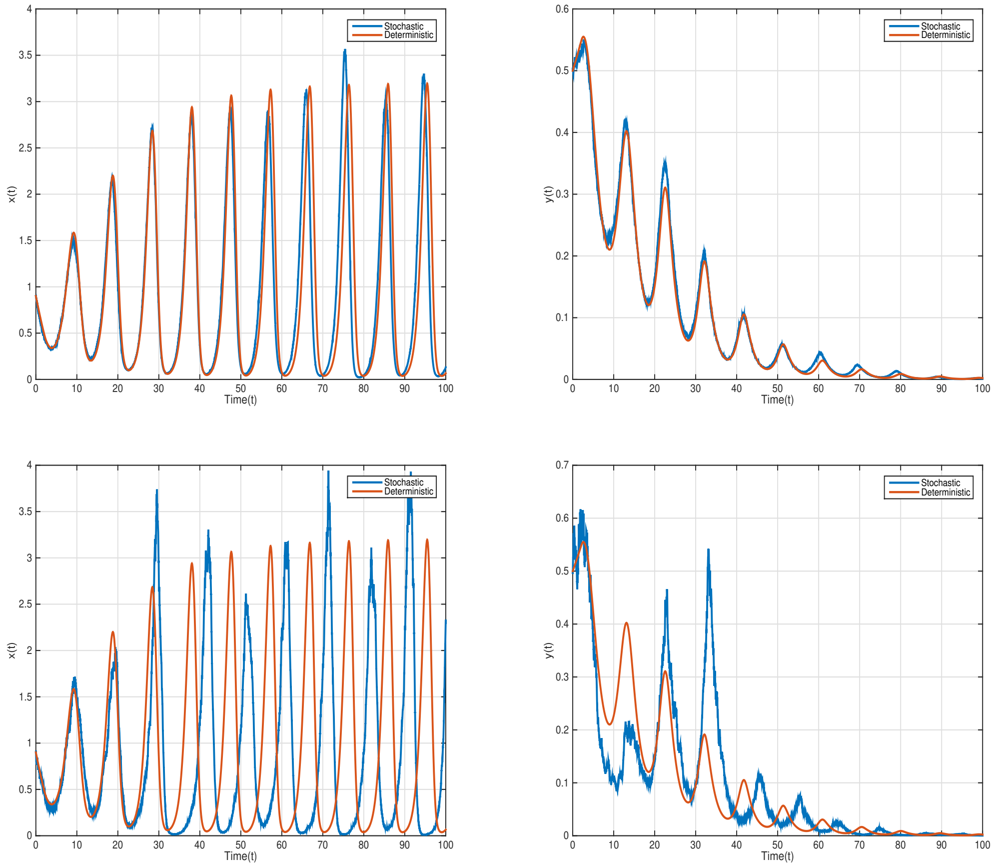

Figure 4.5: Numerical simulations of the solutions for System (4.3) and the corresponding deterministic System (2.2) with time delay $\tau_1 > \tau_1^* = 2.1$. An unstable solution for prey population (left-top and bottom); while the large oscillation leads to the extinction of predator both in the deterministic and stochastic models (right-top and bottom)

that the periodic solution is damped in both population densities. If the intensity of White noises increases then the predator goes to extinct, as $\mathscr{T}_0^s < 1$.

Figure 4.3 shows that the population of prey varies around the deterministic steady state value (left), and predator population goes to extinction at $t = 70$ with deterministic model; while with white noise at $t = 20$ (right). In this simulation, the initial value $(x(0), y(0)) = (0.8, 0.4)$ and parameter values $a = 0.19$, $\mu = 0.8$, $\alpha = 0.1$, $c = 0.8$, $K = 1$, $\delta = 0.59$, $r = 1$, $\sigma_1 = 0.001$, $\sigma_2 = 0.023$, $\tau_1 = \tau_2 = 0.01$ are chosen. Then $\mathscr{T}_0^s = 0.75 < 1$. According to Theorem 4.6.3, the solution of (4.3) obeys $\limsup_{t\to\infty} \dfrac{\ln y(t)}{t} < 0$ a.s., that is y(t) tends to zero exponentially with probability one. In the other hand, for the deterministic Model (2.2), the condition of $\mathscr{T}_0^d = \dfrac{\mu K}{\delta(1+cK)} = 0.7 < 1$ is satisfied, so the boundary equilibrium point $\mathscr{E}_1 \equiv (1, 0)$, is a stable point.

Figure 4.4 shows that the environmental noise plays an important role in extinction of predator population. When the intensity of Brownian motion $W_2(t)$ is increased to $\sigma_2 = 0.1$, $c = 0.3$, $\mu = 0.6$ and $\delta = 0.4$ with other parameters as of Figure 4.3, the extinction occurs in predator population, when $\mathscr{T}_0^s = 0.92 < 1$. This means $y(t)$ of System (4.3) will go to extinction with probability one. However, with the same parameters, the deterministic Model (2.2), has an interior stable equilibrium $\mathscr{E}^* = (0.82, 0.22)$. Therefore, the population $y(t)$ becomes extinct exponentially with probability one when white noise increases.

Figure 4.5 shows fluctuation in population densities of the prey and predator when time delay $\tau_1 = 2.1$ cases large oscillation for deterministic Model (2.2) as well as stochastic Model (4.3). Let $\sigma_1 = \sigma_2 = 0.001$ (top) one can see that the stochastic fluctuation disappears in both prey predator species and they behave as if there is no external noise. On the other hand, let $\sigma_1 = 0.004, \sigma_2 = 0.005$ one can see (bottom) that with the increase of noise intensities, the amplitude also slightly increased. Moreover, the predator population go to extinct with large value of time delay.

Remark 4.7.1. Extinction of predator population is possibly occur when the intensity of white noise is large, such that $\mathscr{T}_0^s < 1$. This would not happen in the deterministic System (2.2) without noises (See Figure 4.4). If the predators' death rate is large, extinction of the predators can also occur in the deterministic Model (2.2); While a small noise in the stochastic model, the extinction of predator population occurs faster than the deterministic model; (See Figure 4.3).

4.8 Concluding Remarks

In this chapter, the dynamics of SDDEs for predator-prey system with hunting cooperation in predators was studied. Considering the environmental noise, the existence and uniqueness of global positive solution and the stochastically ultimate boundedness of the system were established. The effect of environmental noises on persistence and possible extinction of prey and predator populations have been shown. The obtained analytical results with supportive numerical simulations, have been verified using Milstein's scheme. Conditions under which persistence of the system occurs have been established, when $\mathscr{T}_0^s > 1$ have been deduced; While with $\mathscr{T}_0^s < 1$, extinction of predator occurs. It can also be observed that the extinction of the predator population occurs more rapidly for the stochastic System (4.3) when the intensity of white noise increases, see Figures 4.3-4.4. It has also been shown numerically that the predator population dominates the prey population as cooperative hunting parameter increases (See Figure 4.1). The main findings, theoretically and numerically, are all represented in terms of the system parameters and the intensity of randomly fluctuating driving forces. This indicates that time-delay and white noise have a considerable impact on the dynamics and presence of prey predator populations.

In the next chapter, the author extends the stochastic analysis to a three-species model consisting of two-prey one-predator model with time-delays and cooperation among prey species.

Chapter 5: Stochastic DDEs of Three-Species Predator-Prey Systems with Cooperation among Prey Species

5.1 Introduction

In this chapter, a stochastic delay differential model for three species (two preys, one predator) predator-prey system with cooperation among prey species against predator is proposed. Section 2 investigates the existence and uniqueness of global positive solution of the SDDEs model. Section 3 establishes sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the positive solutions to the model. Section 4 shows the extinction of the predator populations under certain parametric restrictions. Some numerical simulations and discussions are carried out, in Section 5, to illustrate the theoretical results. Finally, concluding remarks are presented in Section 6.

A typical cooperative Lotka-Volterra system for two teams of preys with densities $x(t)$, $y(t)$, interacting with one team of predator with densities z(t), can be written in the following form

$$
\begin{aligned}
\frac{dx(t)}{dt} &= x(t)[r_1(1-x(t)) - \alpha_1 z(t) + \beta y(t)z(t)] \\
\frac{dy(t)}{dt} &= y(t)[r_2(1-y(t)) - \alpha_2 z(t) + \beta x(t)z(t)] \\
\frac{dz(t)}{dt} &= z(t)[-\delta - \alpha_3 z + a_1 x(t) + a_2 y(t)].
\end{aligned} \tag{5.1}
$$

The coefficients α_1 and α_2 are the rate of predation, and β is the rate of cooperation for the preys $x(t)$ and $y(t)$, respectively. The preys are chosen such that they can support each other's existence and there is no competition among them, mutualism can be established among the preys against predation. Table 5.1 displays the biological meaning of the model parameters.

Incorporating time-lags in biological models makes the systems much more realistic. Therefore, In [76], the authors studied the following system

$$\begin{aligned}
\frac{dx(t)}{dt} &= r_1x(t)(1-\frac{x(t-\tau_1)}{k_1}) - \alpha_1x(t)z(t) + \beta x(t)y(t)z(t) \\
\frac{dy(t)}{dt} &= r_2y(t)(1-\frac{y(t-\tau_2)}{k_2}) - \alpha_2y(t)z(t) + \beta x(t)y(t)z(t) \\
\frac{dz(t)}{dt} &= -\delta z(t) - \alpha_3z^2 + a_1x(t-\tau_3)z(t) + a_2y(t-\tau_3)z(t).
\end{aligned} \tag{5.2}$$

It is assumed that the rate of cooperation β is not as much as the rate of predation, (i.e. $x < \frac{\alpha_2}{\beta}$ and $y < \frac{\alpha_1}{\beta}$); Which follows that System (5.2) has permanence and its positive equilibrium is locally asymptotically stable for all τ_1, τ_2 and τ_3, under certain parametric restrictions. However, when the rate of cooperation is greater than the rate of predation the number of species becomes unbounded. It is known that deterministic models, such

Table 5.1: One biological meaning for the parameters of Model (5.3)

Parameters	Description
r_1, r_2	Intrinsic growth rate for x and y
k_1 , k_2	Carrying capacity for x and y
α_1, α_2	The rate of predation to decrease the preys growth rate
β	The rate of cooperation for the preys x and y
δ	Predator death rate
α_3	The rate of intra-species competition within the predators
a_1, a_2	An equal transformation rate of predator to preys x and y.

as (5.2), are stable with a cyclic behaviour in the common period for the sizes of species populations. However, in practice, stochastic variations will occur in the values of x, y and z, which may produce a qualitatively different behaviour. These variations may lead to an extinction of the predator as a result of a possible extinction of the prey. Deterministic models may be inadequate for capturing the exact variability in nature. Then, stochastic models are required for an accurate approximation of the dynamics of such interactions. The random fluctuations result in changing some degree of parameters in the deterministic environment. Moreover, the natural growth of populations is always affected by environmental stochastic perturbations which is an inevitable aspect of dynamics of any ecosystems, to suppress a potential population explosion [43, 113, 118, 120]. In Chapter 4 the dynamics of a stochastic delay differential model for predator-prey system with hunting cooperation in predators was studied [113]. Sufficient conditions for persistence and extinction of predator population have been investigated. Liu *et al.* [86] studied the impact of random noise in the dynamics of predator-prey model with herd behaviour. They established sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the positive solutions to the model. In [145], the authors studied the effect of environmental fluctuations on a predator-prey model with stage structure for predator population and ratio dependent functional response.

As a matter of fact, the random fluctuations result in changing some degree of parameters in the deterministic environment [86, 145]. Motivation to what have been mentioned above, it is interesting and important to study, in this chapter, the impact of stochastic perturbations on the dynamics of three-species predator-prey Model (5.2). Assuming that the intrinsic growth rate of preys and the death rate of predator are subject to environmental noise. Suppose that r_1, r_2 and $-\delta$ are stochastically perturbed with

$$r_1 \to r_1 + \sigma_1 \dot{W}_1 \quad r_2 \to r_2 + \sigma_2 \dot{W}_2 \quad -\delta \to -\delta + \sigma_3 \dot{W}_3,$$

where σ_1^2, σ_2^2 and σ_3^2 abide the intensities of the white noise, W_1, W_2, and W_3 denote the independent standard Brownian motions. Thus, the stochastic version of a predator-prey Model (5.2) can be written in the form

$$\begin{aligned} dx(t) &= [r_1 x(t)(1 - \frac{x(t-\tau_1)}{k_1}) - \alpha_1 x(t)z(t) + \beta x(t)y(t)z(t)]dt + \sigma_1 x(t) dW_1(t) \\ dx(t) &= [r_2 y(t)(1 - \frac{y(t-\tau_2)}{k_2}) - \alpha_2 y(t)z(t) + \beta x(t)y(t)z(t)]dt + \sigma_2 y(t) dW_2(t) \\ dz(t) &= [-\delta z(t) - \alpha_3 z^2 + a_1 x(t-\tau_3)z(t) + a_2 y(t-\tau_3)z(t)]dt + \sigma_3 z(t) dW_3(t). \end{aligned} \tag{5.3}$$

Here,

$$x(\kappa) = \phi_1(\kappa),\ y(\kappa) = \phi_2(\kappa),\ z(\kappa) = \phi_3(\kappa),\ \kappa \in [-\tau, 0],\ \tau = \max\{\tau_1, \tau_2, \tau_3\}, \tag{5.4}$$

$\phi_i(0) > 0$ and $\phi_i(\kappa)$, $(i = 1,2,3)$, are nonnegative continuous initial functions on $[-\tau, 0]$. i.e. $(x_0, y_0, z_0) = (\phi_1, \phi_2, \phi_3)^T \in \mathscr{C}([-\tau, 0], \mathbb{R}^3_+)$ with $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 : x > 0, y >$

$0, z > 0\}$, if $(x, y, z) \in \mathbb{R}^3$, its norm is denoted by $|(x, y, z)| = \sqrt{x^2 + y^2 + z^2}$.

Studying the existence of an ergodic stationary distribution of the *stochastic delay differential equations* (SDDEs) (5.3) for the three-species predator-prey system is an interesting problem. In the current study, a suitable stochastic Lyapunov function and a bounded domain of $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_i > 0, i = 1, 2, 3\}$ are established.

5.2 Existence and Uniqueness of Positive Solution

In order to prove that the SDDEs (5.3) has a unique global solution (i.e. no explosion in a finite-time) for any given initial condition, the coefficients of the System (5.3) are generally required to satisfy the linear growth condition and local Lipschitz condition [23, 92].

Theorem 5.2.1. *If the coefficients of System (5.3) are locally Lipschitz continuous, then for any given initial condition (5.4) there is a unique positive solution $(x(t), y(t), z(t))$ of System (5.3) on $t \geq -\tau$, and the solution will remain in $\mathbb{R}^3_+$ with probability one, if $\beta x < \alpha_2$, $\beta y < \alpha_1$.*

Proof. Since all the coefficients of System (5.3) are Lipschitz continuous, therefore, there is a unique local solution $(x(t), y(t), z(t))$ on $[-\tau, \tau_e)$, where τ_e is an explosion time. To show this solution is global, one may need to show $\tau_e = \infty$ a.s. (almost surely). Let $l_0 > 0$ be sufficiently large so that $(x(t), y(t), z(t)) = \{(\phi_1(t), \phi_2(t), \phi_3(t)) : -\tau \leq t \leq 0\} \in \mathscr{C}([-\tau, 0]; \mathbb{R}^3_+)$ all lie within the interval $[\frac{1}{l_0}, l_0]$. Now for each integer $l \geq l_0$, define the stopping time $\tau_l = \inf\{t \in [-\tau, \tau_e) : x(t) \notin (\frac{1}{l}, l), y(t) \notin (\frac{1}{l}, l)\}$, let $\inf \phi = \infty$. τ_l is increasing with l and let $\tau_\infty = \lim_{l \to \infty} \tau_l$, then $\tau_\infty \leq \tau_e$ and by showing $\tau_\infty = \infty$ a.s., the aim is to conclude that $\tau_e = \infty$ a.s. If this assertion is erroneous, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Therefore, there is an integer

$l_1 \geq l_0$ such that

$$\mathbb{P}\{\tau_l \leq T\} > \varepsilon, \quad \text{for all} \quad l \geq l_1. \tag{5.5}$$

Define a $\mathscr{C}^2$-function $V(x,y,z) : \mathbb{R}^3_+ \to \mathbb{R}_+$ by $V(x,y,z) = V_1 + V_2 + V_3$. Where, $V_1 = (x - \log x - 1) + \frac{r_1}{k_1}\int_t^{t+\tau_1} x(t-\tau_1)ds$, $V_2 = (y - \log y - 1) + \frac{r_2}{k_2}\int_t^{t+\tau_2} y(t-\tau_2)ds$ and $V_3 = d_0(z - \log z - 1) + \frac{d_0}{2}\int_t^{t+\tau_3}(a_1 x^2(t-\tau_3) + a_2 y^2(t-\tau_3))ds$, such that $d_0 = \frac{\alpha_1 a_2 + \alpha_2 a_1}{a_1 a_2}$. Clearly, this function is non-negative for all $x,y,z \geq 0$. Let $l \geq l_0$ and $T > 0$ be arbitrary. For $0 \leq t \leq \tau_l \wedge T$, by Itô's formula for V, one gets

$$dV(x,y,z) = \mathscr{L}V(x,y,z) + \sigma_1(x-1)dW_1(t) + \sigma_2(y-1)dW_2(t) + d_0\sigma_3(z-1)dW_3(t).$$

$$\begin{aligned} \mathscr{L}V_1 &= r_1 x - \frac{r_1}{K_1}xx(t-\tau_1) + xz(\beta y - \alpha_1) - r_1 + \alpha_1 z - \beta zy + \frac{r_1}{K_1}x + \frac{\sigma_1^2}{2} \\ &\leq (r_1 + \frac{r_1}{K_1})x - r_1 + \frac{\sigma_1^2}{2}. \end{aligned} \tag{5.6}$$

$$\begin{aligned} \mathscr{L}V_2 &= r_2 y - \frac{r_2}{K_2}yy(t-\tau_1) + yz(\beta x - \alpha_2) - r_2 + \alpha_2 z - \beta zx + \frac{r_2}{K_2}y + \frac{\sigma_2^2}{2} \\ &\leq (r_2 + \frac{r_2}{K_2})y - r_2 + \frac{\sigma_2^2}{2}. \end{aligned} \tag{5.7}$$

$$\begin{aligned} \mathscr{L}V_3 &= -d_0\delta z - d_0\alpha_3 z^2 + d_0 a_1 x(t-\tau_3)z + d_0 a_2 y(t-\tau_3)z + d_0 z + d_0\alpha_3 z \\ &\quad - d_0 a_1 x(t-\tau_3) - d_0 a_2 y(t-\tau_3) + \frac{d_0 a_1}{2}x^2 + \frac{d_0 a_2}{2}y^2 - \frac{d_0 a_1}{2}x^2(t-\tau_3) \\ &\quad - \frac{d_0 a_2}{2}y^2(t-\tau_3) + \frac{d_0\sigma_3^2}{2} \\ &\leq (d_0\alpha_3 - d_0)z + (a_1 + a_2 - \alpha_3)d_0 z^2 + d_0\delta + \frac{d_0 a_1}{2}x^2 + \frac{d_0 a_2}{2}y^2 + \frac{d_0\sigma_3^2}{2}. \end{aligned} \tag{5.8}$$

$$\begin{aligned} \mathscr{L}V &= \mathscr{L}V_1 + \mathscr{L}V_2 + \mathscr{L}V_3 \\ &\leq d_0\delta - r_1 - r_2 + r_1(\frac{K_1+1}{K_1})x + \frac{d_0 a_1}{2}x^2 + r_2(\frac{K_2+1}{K_2})y + \frac{d_0 a_2}{2}y^2 \\ &\quad + d_0(\alpha_3 - \delta)z + d_0(a_1 + a_2 - \alpha_3)z^2 + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{d_0\sigma_3^2}{2} \end{aligned}$$

$$
\begin{aligned}
\mathscr{L}V \leq & \sup_{x\in\mathbb{R}_+}\left\{r_1\Big(\frac{K_1+1}{K_1}\Big)x+\frac{d_0a_1}{2}x^2\right\}+\sup_{y\in\mathbb{R}_+}\left\{r_2\Big(\frac{K_2+1}{K_2}\Big)y+\frac{d_0a_2}{2}y^2\right\}\\
& +\sup_{z\in\mathbb{R}_+}\left\{d_0(\alpha_3-\delta)z+d_0(a_1+a_2-\alpha_3)z^2\right\}+d_0\delta\\
& -r_1-r_2+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{d_0\sigma_3^2}{2}\leq K,
\end{aligned}
$$

where K is a positive constant. It follows that $\mathscr{L}V$ is bounded. Hence,

$$
dV \leq Kdt+\sigma_1(x-1)dW_1(t)+\sigma_2(y-1)dW_2(t)+d_0\sigma_3(z-1)dW_3(t). \tag{5.9}
$$

Integrating (5.9) from 0 to $\tau_l\wedge T=\min\{\tau_l,T\}$ and then taking the expectation on both sides, one may have

$$
\mathbb{E}[V(x(\tau_l\wedge T),y(\tau_l\wedge T),z(\tau_l\wedge T))]\leq \mathbb{E}[V(x(0),y(0),z(0))]+KT. \tag{5.10}
$$

Let $\Omega_l=\{\tau_l\leq T\}$, for $l\geq l_1$ and in view of (5.5), one obtains $\mathbb{P}(\Omega_l)\geq\varepsilon$. Such that, for every $\omega\in\Omega_l$, there is at least one of $x(\tau_l,\omega)$,$y(\tau_l,\omega)$, or $z(\tau_l,\omega)$ equaling either to l or $\frac{1}{l}$ and then, one obtains

$$
V(x(\tau_l\wedge T),y(\tau_l\wedge T),z(\tau_l\wedge T))\geq (l-1-\ln l)\wedge\Big(\frac{1}{l}-1-\ln\frac{1}{l}\Big). \tag{5.11}
$$

According to (5.10), one gets

$$
\begin{aligned}
\mathbb{E}[V(x(0),y(0),z(0))]+KT &\geq \mathbb{E}[1_{\Omega_l(\omega)}V(x(\tau_l,\omega),y(\tau_l,\omega),z(\tau_l,\omega))]\\
&\geq \varepsilon(l-1-\ln l)\wedge\Big(\frac{1}{l}-1-\ln\frac{1}{l}\Big),
\end{aligned} \tag{5.12}
$$

where 1_{Ω_l} is the indicator function of Ω_l. Letting $l \to \infty$ yields

$$\infty > \mathbb{E}[V(x(0), y(0), z(0))] + KT = \infty, \tag{5.13}$$

which leads to the contradiction, so it is a must to have $\tau_\infty = \infty$ a.s.

5.3 Existence of Ergodic Stationary Distribution

In this section, a suitable stochastic Lyapunov function is constructed to study existence of a unique ergodic stationary distribution of the positive solutions to System (5.3). First, assume $\mathbf{x}(t)$ is a regular time-homogenous Markov process in $\mathbb{R}^d$, illustrated by the SDDEs

$$d\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{x}(t-\tau))dt + \sum_{r=1}^{d} \mathbf{g}_r(t, \mathbf{x}(t))dW_r(t). \tag{5.14}$$

The diffusion matrix of the process $\mathbf{x}(t)$ is

$$\Lambda(x) = (\lambda_{ij}(x)), \quad \lambda_{ij}(x) = \sum_{r=1}^{d} \mathbf{g}_r^i(x)\mathbf{g}_r^j(x).$$

Lemma 5.3.1. *[52]. The Markov process $\mathbf{x}(t)$ has a unique ergodic stationary distribution $\pi(.)$ if there exist a bounded domain $\mathscr{U} \subset \mathbb{R}^d$ with regular boundary Γ and*

(i): there is a positive number $\mathscr{M}$ such that $\sum_{i,j=1}^{d} \lambda_{ij}(x)\xi_i\xi_j \geq \mathscr{M}|\xi|^2, x \in \mathscr{U}, \xi \in \mathbb{R}^d$.

(ii): there exists a nonnegative $\mathscr{C}^2$-function V such that $\mathscr{L}V$ is negative for any $\mathbb{R}^d \setminus \mathscr{U}$.

A threshold parameter $\mathscr{T}_0^s$ of SDDEs (5.3) is defined as follows

$$\mathscr{T}_0^s = \frac{r_1\rho_1 + r_2\rho_2}{(\rho_1\frac{\sigma_1^2}{2} + \rho_2\frac{\sigma_2^2}{2} + \rho_3\frac{\sigma_3^2}{2})}, \tag{5.15}$$

where $\rho_1 = \frac{k_1}{(k_1+1)}$, $\rho_2 = \frac{k_2}{(k_2+1)}$, and $\rho_3 = \max\{r_1, r_2\}$.

Theorem 5.3.2. *If $r_1 > \frac{\sigma_1^2}{2}$, $r_2 > \frac{\sigma_2^2}{2}$ and $\mathscr{T}_0^s > 1$, then for any initial conditions (5.4), the system of SDDEs (5.3) admits a stationary distribution $\pi(.)$, and the solution of the system is ergodic.*

Proof. In order to prove Theorem 5.3.2, it is enough to validate conditions (i) and (ii) of Lemma 5.3.1. To prove condition (i); the diffusion matrix of System (5.3) is given by

$$\Lambda(x,y,z) = \begin{pmatrix} \sigma_1^2 x^2 & 0 & 0 \\ 0 & \sigma_2^2 y^2 & 0 \\ 0 & 0 & \sigma_3^2 z^2 \end{pmatrix}.$$

Let $\mathscr{U}$ be any bounded domain in $\mathbb{R}_+^3$, then there exists a positive constant

$$\mathscr{M}_0 = \min_{(x,y,z)\in\bar{\mathscr{U}}_\sigma} \{\sigma_1^2 x^2, \sigma_2^2 y^2, \sigma_3^2 z^2\},$$

such that

$$\sum_{i,j=1}^{3} \lambda_{ij}(x,y,z)\xi_i\xi_j = \sigma_1^2 x^2 \xi_1^2 + \sigma_2^2 y^2 \xi_2^2 + \sigma_3^2 z^2 \xi_3^2 \geq \mathscr{M}_0 |\xi|^2,$$

for any $(x,y,z) \in \bar{\mathscr{U}}_\sigma, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}_+^3$. Thus, condition (i) of Lemma 5.3.1 is satisfied. Then one needs to prove condition (ii) of Lemma 5.3.1. By System (5.3),

$$\begin{aligned} &\mathscr{L}(\rho_1[x-1-lnx] + \frac{\rho_1 r_1}{K_1}\int_t^{t+\tau_1} x(s-\tau_1)ds) \leq \rho_1 r_1 x - r_1\rho_1 + \frac{\rho_1 r_1}{K_1}x + \rho_1\frac{\sigma_1^2}{2}, \\ &\mathscr{L}(\rho_2[y-1-lny] + \frac{\rho_2 r_2}{K_2}\int_t^{t+\tau_2} y(s-\tau_2)ds) \leq \rho_2 r_2 y - r_2\rho_2 + \frac{\rho_2 r_2}{K_2}y + \rho_2\frac{\sigma_2^2}{2}, \end{aligned} \tag{5.16}$$

$$\mathscr{L}(-\rho_3 \ln z) \leq \rho_3\delta + \rho_3\alpha_3 z + \rho_3\frac{\sigma_3^2}{2}. \tag{5.17}$$

Then, assume $V_1 : \mathbb{R}^3_+ \to \mathbb{R}$ as follows

$$V_1(x,y,z) = \rho_1(x-1-\ln x) + \rho_2(y-1-\ln y) - \rho_3 \ln z + \frac{\rho_1 r_1}{k_1}\int_t^{t+\tau_1} x(t-\tau_1)ds \\ + \frac{\rho_2 r_2}{k_2}\int_t^{t+\tau_2} y(t-\tau_2)ds,$$

thus, according to (5.16) and (5.17), one gets

$$\begin{aligned} \mathscr{L}V_1 &\leq -r_1\rho_1 - r_2\rho_2 + (\rho_1\frac{\sigma_1^2}{2} + \rho_2\frac{\sigma_2^2}{2} + \rho_3\frac{\sigma_3^2}{2}) + r_1x + r_2y + \alpha_3\rho_3 z, \\ &= -\mu + r_1x + r_2y + \alpha_3\rho_3 z. \end{aligned} \tag{5.18}$$

Here, $\mu = r_1\rho_1 + r_2\rho_2 - (\rho_1\frac{\sigma_1^2}{2} + \rho_2\frac{\sigma_2^2}{2} + \rho_3\frac{\sigma_3^2}{2}) > 0$, since $\mathscr{T}_0^s > 1$.

Now, a $\mathscr{C}^2$-function $\tilde{V} : \mathbb{R}^3_+ \to \mathbb{R}$ is defined as follows

$$\begin{aligned} \tilde{V}(x,y,z) = {} & \mathscr{Q}V_1(x,y,z) + z^{-\theta} + x - y + z + \frac{r_1}{K_1}\int_t^{t+\tau_2} y^2(s-\tau_2)ds \\ & + \int_t^{t+\tau_3}(a_1x^2(s-\tau_3) + a_2y^2(s-\tau_3))ds, \end{aligned} \tag{5.19}$$

where $0 < \theta < 1$, is sufficiently small constant satisfying $(\varphi - \delta > \frac{\theta+1}{2}\sigma_3^2)$, where $\varphi = \inf_{(x,y)\in\mathbb{R}^2_+}\{a_1x(t-\tau_3) + a_2y(t-\tau_3)\}$, and $\mathscr{Q} = \frac{2}{\mu}\max\{2, \sup_{(x,y,z)\in\mathbb{R}^3_+}\{-\theta z^{-\theta}(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2) + \alpha_3\theta z^{1-\theta} + \frac{3r_1}{2}x - \frac{r_2}{2}y - \frac{\delta}{2}z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\}\}$. Note that $V(x,y,z)$ is not only continuous, but also tends to ∞ as (x,y,z) approaches the boundary of $\mathbb{R}^3_+$ and as $\|(x,y,z)\| \to \infty$, where $\|.\|$ denotes the Euclidean norm of a point in $\mathbb{R}^3_+$. Therefore, it must be a lower bounded and achieve this lower bound at a point (x_0,y_0,z_0) in the interior of $\mathbb{R}^3_+$.

Define a $\mathscr{C}^2$-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+ \cup \{0\}$ of the form

$$\begin{aligned}
V(x,y,z) &= \tilde{V}(x,y,z) - \tilde{V}(x_0,y_0,z_0) \\
&= \mathscr{Q}V_1(x,y,z) + z^{-\theta} + x - y + z + \int_t^{t+\tau_3} (a_1 x^2(s-\tau_3) + a_2 y^2(s-\tau_3))ds \\
&\quad + \frac{r_1}{K_1}\int_t^{t+\tau_2} y^2(s-\tau_2)ds - \tilde{V}(x_0,y_0,z_0) \\
&= \mathscr{Q}V_1(x,y,z) + V_2(z) + V_3(x,y,z).
\end{aligned} \tag{5.20}$$

Such that $V_2(z) = z^{-\theta}$, $V_3(x,y,z) = x - y + z + \frac{r_1}{K_1}\int_t^{t+\tau_2} y^2(s-\tau_2)ds + \int_t^{t+\tau_3}(a_1 x^2(s-\tau_3) + a_2 y^2(s-\tau_3))ds - \tilde{V}(x_0,y_0,z_0)$. By It$\hat{o}$ formula to $V_2(z)$, one obtains

$$\begin{aligned}
\mathscr{L}V_2 &= -\theta z^{-\theta-1}\big(-\delta z - \alpha_3 z^2 + a_1 x(t-\tau_3)z + a_2 y(t-\tau_2)z\big) + \frac{\theta(\theta+1)}{2}\sigma_3^2 z^{-\theta} \\
&\leq -\theta z^{-\theta-1}\big((\varphi-\delta)z - \alpha_3 z^2\big) + \frac{\theta(\theta+1)}{2}\sigma_3^2 z^{-\theta} \\
&\leq -\theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta}.
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
\mathscr{L}V_3 &\leq r_1 x - r_2 y - \delta z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 \\
&\quad + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2.
\end{aligned} \tag{5.22}$$

Thus, in view of (5.18), (5.21) and (5.22), one gets

$$\begin{aligned}
\mathscr{L}V(x,y,z) &\leq \mathscr{Q}(-\mu + r_1 x + r_2 y + \alpha_3\rho_3 z) - \theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta} \\
&\quad + r_1 x - r_2 y - \delta z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 \\
&\quad + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2.
\end{aligned} \tag{5.23}$$

To create a compact subset $\mathscr{U}_\varepsilon$ such that condition (ii) of Lemma 5.3.1 holds. Define a bounded closed set as follows

$$\mathscr{U}_\varepsilon = \Big\{(x,y,z) \in \mathbb{R}^3_+ : \varepsilon \leq x \leq \frac{1}{\varepsilon}, \varepsilon \leq y \leq \frac{1}{\varepsilon}, \varepsilon \leq z \leq \frac{1}{\varepsilon}\Big\}, \tag{5.24}$$

where $0 < \varepsilon < 1$ is a sufficiently small constant. In the set $\mathbb{R}^3_+ \setminus \mathscr{U}_\varepsilon$, one can choose ε sufficiently small such that the following conditions hold

$$\varepsilon \leq \min\{\frac{\mu}{4r_1}, \frac{1}{2\mathscr{Q}}, \frac{\delta}{2\mathscr{Q}\alpha_3\rho_3}\}, \tag{5.25a}$$

$$\varepsilon \leq \min\{\frac{1}{2\mathscr{Q}}, \frac{\delta}{2\mathscr{Q}\alpha_3\rho_3}, \frac{\mu}{4r_2}\}, \tag{5.25b}$$

$$\varepsilon \leq \min\{\frac{1}{2\mathscr{Q}}, \frac{\mu}{4\mathscr{Q}\alpha_3\rho_3}\}, \tag{5.25c}$$

$$-\frac{r_1}{2\varepsilon} + \mathscr{F} \leq -1, \tag{5.25d}$$

$$-\frac{r_2}{2\varepsilon} + \mathscr{F} \leq -1, \tag{5.25e}$$

$$-\frac{\delta}{2\varepsilon} + \mathscr{F} \leq -1, \tag{5.25f}$$

where

$$\begin{aligned}\mathscr{F} = \sup_{(x,y,z)\in\mathbb{R}^3_+} &\{\mathscr{Q}(r_1x + r_2y + \rho_3\alpha_3z) - \theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta} \\ &+ \frac{3r_1}{2}x - \frac{r_2}{2}y - \frac{\delta}{2}z + (a_1 - \alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 \\ &+ (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\}.\end{aligned} \tag{5.26}$$

One can divide $\mathbb{R}^3_+ \setminus \mathscr{U}_\varepsilon$ into six subdomains:

$\mathscr{U}_\varepsilon^1 = \{(x,y,z) \in \mathbb{R}^3_+ : x \leq \varepsilon\}$, $\mathscr{U}_\varepsilon^2 = \{(x,y,z) \in \mathbb{R}^3_+ : y \leq \varepsilon\}$,

$\mathscr{U}_\varepsilon^3 = \{(x,y,z) \in \mathbb{R}^3_+ : z \leq \varepsilon\}$, $\mathscr{U}_\varepsilon^4 = \{(x,y,z) \in \mathbb{R}^3_+ : x \geq \frac{1}{\varepsilon}\}$,

$\mathscr{U}_\varepsilon^5 = \{(x,y,z) \in \mathbb{R}^3_+ : y \geq \frac{1}{\varepsilon}\}$, $\mathscr{U}_\varepsilon^6 = \{(x,y,z) \in \mathbb{R}^3_+ : z \geq \frac{1}{\varepsilon}\}$. Clearly, $\mathbb{R}^3_+ \setminus \mathscr{U}_\varepsilon = \mathscr{U}_\varepsilon^1 \cup \mathscr{U}_\varepsilon^2 \cup \mathscr{U}_\varepsilon^3 \cup \mathscr{U}_\varepsilon^4 \cup \mathscr{U}_\varepsilon^5 \cup \mathscr{U}_\varepsilon^6$. Next, one may show that $\mathscr{L}V(x,y,z) \leq -1$ for any $(x,y,z) \in \mathbb{R}^3_+ \setminus \mathscr{U}_\varepsilon$, which is equivalent to proving it on the above six domains:

Case I. For any $(x,y,z) \in \mathscr{U}_\varepsilon^1$, noting that $x \le x(1+y+z) \le \varepsilon(1+y+z)$, one gets

$$\begin{aligned}
\mathscr{L}V(x,y,z) \le & -\frac{\mathscr{Q}}{4}\mu + \Big[-\frac{\mathscr{Q}}{4}\mu + \mathscr{Q}r_1\varepsilon\Big] - \frac{r_1}{2}x + [\mathscr{Q}r_2\varepsilon - \frac{r_2}{2}]y + [\mathscr{Q}\alpha_3\rho_3\varepsilon - \frac{\delta}{2}]z \\
& + \Big[-\frac{\mathscr{Q}}{2}\mu - \theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta} + \frac{3r_1}{2}x - \frac{r_2}{2}y - \frac{\delta}{2}z \\
& + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\Big] \\
\le & -\frac{\mathscr{Q}}{4}\mu + \Big[-\frac{\mathscr{Q}}{4}\mu + \mathscr{Q}r_1\varepsilon\Big] - \frac{r_1}{2}x + [\mathscr{Q}r_2\varepsilon - \frac{r_2}{2}]y + [\mathscr{Q}\alpha_3\rho_3\varepsilon - \frac{\delta}{2}]z \\
& + \Big[-\frac{\mathscr{Q}}{2}\mu + \sup_{(x,y,z)\in\mathbb{R}_+^3}\Big\{-\theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta} + \frac{3r_1}{2}x \\
& - \frac{r_2}{2}y - \frac{\delta}{2}z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 \\
& + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\Big\}\Big].
\end{aligned}$$

Since $\mathscr{Q} = \frac{2}{\mu}\max\{2, \sup_{(x,y,z)\in\mathbb{R}_+^3}\big\{-\theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta} + \frac{3r_1}{2}x - \frac{r_2}{2}y - \frac{\delta}{2}z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\big\}\}$, one obtains $\frac{\mathscr{Q}}{4}\mu \ge 1$. Therefore, from (5.25a), one gets

$$\mathscr{L}V(x,y,z) \le -\frac{\mathscr{Q}}{4}\mu - \frac{r_1}{2}x \le -\frac{\mathscr{Q}}{4}\mu \le -1 \tag{5.27}$$

Consequently, $\mathscr{L}V(x,y,z) \le -1$ for any $(x,y,z) \in \mathscr{U}_\varepsilon^1$.

Case II. For any $(x,y,z) \in \mathscr{U}_\varepsilon^2$, since $y \le y(1+x+z) \le \varepsilon(1+x+z)$, one gets

$$\begin{aligned}
\mathscr{L}V(x,y) \le & -\frac{\mathscr{Q}}{4}\mu + \Big[-\frac{\mathscr{Q}}{4}\mu + \mathscr{Q}r_2\varepsilon\Big] - \frac{r_2}{2}y + [\mathscr{Q}r_1\varepsilon - \frac{r_1}{2}]x + [\mathscr{Q}\alpha_3\rho_3\varepsilon - \frac{\delta}{2}]z \\
& + \Big[-\frac{\mathscr{Q}}{2}\mu + \sup_{(x,y,z)\in\mathbb{R}_+^3}\Big\{-\theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) + \alpha_3\theta z^{1-\theta} \\
& + \frac{3r_1}{2}x - \frac{r_2}{2}y - \frac{\delta}{2}z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 \\
& + (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\Big\}\Big].
\end{aligned} \tag{5.28}$$

It follows from (5.25b) that

$$\mathscr{L}V(x,y,z) \leq -\frac{\mathscr{Q}}{4}\mu - \frac{r_2}{2}y \leq -\frac{\mathscr{Q}}{4}\mu \leq -1. \quad \text{For any} \quad (x,y,z) \in \mathscr{U}_\varepsilon^2. \tag{5.29}$$

Case III. For any $(x,y,z) \in \mathscr{U}_\varepsilon^3$, such that $z \leq z(1+x+y) \leq \varepsilon(1+x+y)$, one obtains

$$\begin{aligned}\mathscr{L}V(x,y,z) \leq &-\frac{\mathscr{Q}}{4}\mu + \Big[-\frac{\mathscr{Q}}{4}\mu + \mathscr{Q}\alpha_3\rho_3\varepsilon\Big] - \frac{\delta}{2}z + [\mathscr{Q}r_1\varepsilon - \frac{r_1}{2}]x \\ &+ [\mathscr{Q}r_2\varepsilon - \frac{r_2}{2}]y + \Big[-\frac{\mathscr{Q}}{2}\mu + \sup_{(x,y,z)\in\mathbb{R}_+^3}\Big\{-\theta z^{-\theta}\big(\varphi - \delta - \frac{\theta+1}{2}\sigma_3^2\big) \\ &+ \alpha_3\theta z^{1-\theta} + \frac{3r_1}{2}x - \frac{r_2}{2}y - \frac{\delta}{2}z + (a_1-\alpha_1)x^2 + (\frac{r_2}{K_2} + \alpha_2 + a_2)y^2 \\ &+ (\alpha_2 + a_1 + a_2 - \alpha_1 - \alpha_3)z^2\Big\}\Big].\end{aligned} \tag{5.30}$$

In view of (5.25c), one gets

$$\mathscr{L}V(x,y,z) \leq -\frac{\mathscr{Q}}{4}\mu - \frac{\delta}{2}z \leq -\frac{\mathscr{Q}}{4}\mu \leq -1. \tag{5.31}$$

Therefore, $\mathscr{L}V(x,y,z) \leq -1$ for any $(x,y,z) \in \mathscr{U}_\varepsilon^3$.

Case IV. For any $(x,y,z) \in \mathscr{U}_\varepsilon^4$, one may have

$$\mathscr{L}V(x,y,z) \leq -\frac{r_1}{2}x + \mathscr{F} \leq -\frac{r_1}{2\varepsilon} + \mathscr{F} \leq -1, \tag{5.32}$$

which follows from (5.25d). Thus, $\mathscr{L}V(x,y,z) \leq -1$ for any $(x,y,z) \in \mathscr{U}_\varepsilon^4$.

Case V. For any $(x,y,z) \in \mathscr{U}_\varepsilon^5$, one obtains

$$\mathscr{L}V(x,y,z) \leq -\frac{r_2}{2}y + \mathscr{F} \leq -\frac{r_2}{2\varepsilon} + \mathscr{F} \leq -1, \tag{5.33}$$

which follows from (5.25e). Thus, $\mathscr{L}V(x,y,z) \leq -1$ for any $(x,y,z) \in \mathscr{U}_\varepsilon^5$.

Case VI. For any $(x,y,z) \in \mathscr{U}_\varepsilon^6$, one may have

$$\mathscr{L}V(x,y,z) \leq -\frac{\delta}{2}z + \mathscr{F} \leq -\frac{\delta}{2\varepsilon} + \mathscr{F} \leq -1, \tag{5.34}$$

which follows from (5.25f). Thus, $\mathscr{L}V(x,y,z) \leq -1$ for any $(x,y,z) \in \mathscr{U}_\varepsilon^6$.

Hence, from (5.27)-(5.34), one can obtain that for a sufficiently small ε,

$$\mathscr{L}V(x,y,z) \leq -1 \quad \text{for any} \quad (x,y,z) \in \mathbb{R}_+^3 \setminus \mathscr{U}_\varepsilon.$$

By Lemma 5.3.1, the solution of System (5.3) is ergodic and has a unique stationary distribution $\pi(.)$.

5.4 Extinction

In this section, some sufficient conditions for the extinction of predator populations are investigated, that is, when the prey populations survival and the predator population goes to extinct. It has been shown that a strong intensity of noise can be a cause for extinction of the prey species, which will also drive predator population to extinct.

Theorem 5.4.1. *Let $(x(t),y(t),z(t))$ be the solution the SDDEs (5.3) with initial conditions (5.4).*

(a) *If $r_1 > \frac{\sigma_1^2}{2}$, $r_2 > \frac{\sigma_2^2}{2}$ and $\mathscr{T}_0^s < 1$ then the predator population will die out that is to say*

$$\lim_{t\to} z(t) = 0 \quad a.s. \tag{5.35}$$

(b) *If* $r_1 < \frac{\sigma_1^2}{2}$ *and* $r_2 < \frac{\sigma_2^2}{2}$, *then the prey and predator populations will die out, such that*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s., \quad \lim_{t\to\infty} y(t) = 0 \quad a.s., \quad \lim_{t\to\infty} z(t) = 0 \quad a.s. \tag{5.36}$$

Proof. Using of Itô's formula to the first equation of System (5.3), yields

$$\begin{aligned} d(\ln x(t) - \frac{r_1}{K_1}\int_t^{t+\tau_1} x(s-\tau_1)ds) &= ((r_1 - \frac{\sigma_1^2}{2}) - \frac{r_1}{K_1}x(t) \\ &\quad - z(t)[\alpha_1 - \beta y(t)])dt + \sigma_1 dW_1(t). \\ &\leq ((r_1 - \frac{\sigma_1^2}{2}) - \frac{r_1}{K_1}x(t))dt + \sigma_1 dW_1(t). \end{aligned} \tag{5.37}$$

Integrating of inequality (5.37) from 0 to t results in

$$\frac{\ln x(t) - \frac{r_1}{K_1}\int_t^{t+\tau_1} x(s-\tau_1)ds}{t} - \frac{\ln x(0) - \frac{r_1}{K_1}\int_0^{\tau_1} x(s-\tau_1)ds}{t} \leq (r_1 - \frac{\sigma_1^2}{2}) - \frac{r_1}{K_1}\langle x(t)\rangle + \frac{\sigma_1 W_1(t)}{t}.$$

Thus,

$$\langle x(t)\rangle \leq \frac{K_1}{r_1}(r_1 - \frac{\sigma_1^2}{2}) + \zeta_1(t), \quad \text{where,} \tag{5.38}$$

$$\zeta_1(t) = \frac{K_1}{r_1}\Big[\frac{\sigma_1 W_1(t)}{t} - \frac{\ln x(t) - \frac{r_1}{K_1}\int_t^{t+\tau_1} x(s-\tau_1)ds}{t} + \frac{\ln x(0) - \frac{r_1}{K_1}\int_0^{\tau_1} x(s-\tau_1)ds}{t}\Big].$$

It follows from [81] that $\lim_{t\to\infty} \frac{W_1(t)}{t} = 0$ a.s. Note that

$$\frac{\int_t^{t+\tau_1} x(s-\tau_1)ds}{t} = \frac{1}{t}\int_{t-\tau_1}^{t} x(s)ds = \frac{1}{t}\Big[\int_0^t x(s)ds - \int_0^{t-\tau_1} x(s)ds\Big].$$

Therefore, $\lim_{t\to\infty} \int_t^{t+\tau_1} \frac{x(s-\tau_1)ds}{t} = 0.$

Moreover, $\lim_{t\to\infty}\int_0^{\tau_1}\frac{x(s-\tau_1)ds}{t}=\lim_{t\to\infty}\frac{\int_{-\tau_1}^0\phi_1(t)dt}{t}=0$. Thus, one obtains

$$\lim_{t\to\infty}\zeta_1=0\quad a.s. \tag{5.39}$$

By It$\hat{o}$'s formula, one gets

$$d(\ln y(t)-\frac{r_2}{K_2}\int_t^{t+\tau_2}x(s-\tau_2)ds)\le((r_2-\frac{\sigma_2^2}{2})-\frac{r_2}{K_2}y(t))dt+\sigma_2 dW_2(t). \tag{5.40}$$

Similarly, it follows

$$\langle y(t)\rangle\le\frac{K_2}{r_2}(r_2-\frac{\sigma_2^2}{2})+\zeta_2(t),\quad \text{where} \tag{5.41}$$

$$\zeta_2(t)=\frac{K_2}{r_2}\Big[\frac{\sigma_2 W_2(t)}{t}-\frac{\ln y(t)-\frac{r_2}{K_2}\int_t^{t+\tau_1}y(s-\tau_2)ds}{t}+\frac{\ln y(0)-\frac{r_2}{K_2}\int_0^{\tau_2}y(s-\tau_2)ds}{t}\Big],$$

Therefore, in the same manner, one can obtain, $\lim_{t\to\infty}\zeta_2=0$ a.s. Let

$$V=\ln z(t)+\int_t^{t+\tau_3}[a_1x(s-\tau_3)+a_3y(s-\tau_3)]ds, \tag{5.42}$$

utilizing It$\hat{o}$ formula, one obtains

$$dV=[-\delta-\alpha_3 z(t)+a_1x(t)+a_2y(t)-\frac{1}{2}\sigma_3^2]dt+\sigma_3 dW_3(t). \tag{5.43}$$

Therefore, one may have

$$\frac{V(t)-V(0)}{t}\le a_1\langle x(t)\rangle+a_2\langle y(t)\rangle-[\delta+\frac{1}{2}\sigma_3^2]+\frac{\sigma_3}{t}W_3(t), \tag{5.44}$$

So, from (5.38) and (5.41), one gets

$$\frac{\ln z(t)}{t} \leq \frac{a_1K_1}{r_1}(r_1 - \frac{\sigma_1^2}{2}) + \frac{a_2K_2}{r_2}(r_2 - \frac{\sigma_2^2}{2}) - (\delta + \frac{1}{2}\sigma_3^2) + \zeta_1(t) + \zeta_2(t) + \zeta_3(t), \quad (5.45)$$

such that, one may have

$$\begin{aligned} \zeta_3(t) = & \frac{\ln z(0) + \int_0^{\tau_3}[a_1x(s-\tau_3) + a_2y(s-\tau_3)]ds}{t} \\ & - \frac{\int_t^{t+\tau_3}[a_1x(s-\tau_3) + a_2y(s-\tau_3)]ds}{t} + \frac{\sigma_3 W_3(t)}{t}. \end{aligned} \quad (5.46)$$

In view of the strong law of large numbers of Brownian motion, one can easily obtain $\lim_{t\to\infty} \zeta_3(t) = 0 \quad a.s.$ Therefore, by taking the superior limit on both sides of (5.45), one may have

$$\limsup_{t\to\infty} \frac{\ln z(t)}{t} \leq \frac{a_1K_1}{r_1}(r_1 - \frac{\sigma_1^2}{2}) + \frac{a_2K_2}{r_2}(r_2 - \frac{\sigma_2^2}{2}) - (\delta + \frac{1}{2}\sigma_3^2), \quad (5.47)$$

To prove (a), having the conditions $r_1 > \frac{\sigma_1^2}{2}$, $r_2 > \frac{\sigma_2^2}{2}$ and $\mathscr{T}_0^s < 1$, yields

$$\limsup_{t\to\infty} \frac{\ln z(t)}{t} \leq a_1K_1 + a_2K_2 - \min\{\frac{a_1K_1}{2r_1}, \frac{a_2K_2}{2r_2}, \frac{1}{2}\}(\sigma_1^2 + \sigma_2^2 + \sigma_3^3) - \delta < 0. \quad (5.48)$$

It implies that $\lim_{t\to\infty} z(t) = 0$ a.s.

To prove (b), applying It$\hat{o}$'s formula to the first equation of System (5.3), yields

$$\begin{aligned} d(\ln x(t)) & \leq ((r_1 - \frac{\sigma_1^2}{2}) - z(t)[\alpha_1 - \beta y(t)])dt + \sigma_1 dW_1(t), \\ & \leq (r_1 - \frac{\sigma_1^2}{2})dt + \sigma_1 dW_1(t). \end{aligned} \quad (5.49)$$

Integrating (5.49) from 0 to t, one obtains

$$\frac{\ln x(t)}{t} - \frac{\ln x(0)}{t} \le (r_1 - \frac{\sigma_1^2}{2}) + \frac{\sigma_1 W_1(t)}{t}. \tag{5.50}$$

Noting that $\lim\limits_{t\to\infty} \frac{W_1(t)}{t} = 0$, a.s., and since $r_1 < \frac{\sigma_1^2}{2}$, therefore, by taking the superior limit of both sides of (5.50), one obtains

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le r_1 - \frac{\sigma_1^2}{2} < 0, \tag{5.51}$$

which implies $\lim\limits_{t\to\infty} x(t) = 0$ a.s.

Similarly, one can show that $\limsup\limits_{t\to\infty} \frac{\ln y(t)}{t} \le r_2 - \frac{\sigma_2^2}{2} < 0$, results in $\lim_{t\to\infty} y(t) = 0$ a.s. That is to say both preys $x(t)$ and $y(t)$ will die out with probability one. From (5.47), and considering $r_1 < \frac{\sigma_1^2}{2}$ and $r_2 < \frac{\sigma_2^2}{2}$; one can obtain that $\lim\limits_{t\to\infty} z(t) = 0$ a.s.

Lemma 5.4.2. *In the absence of time-delays i.e.* $\tau_1 = \tau_2 = \tau_3 = 0$. *If* $\chi := a_1 \int_0^\infty x\pi(x)dx + a_2 \int_0^\infty y\pi(y)dy - \delta - \frac{\sigma_3^2}{2} < 0$, $r_1 > \frac{\sigma_1^2}{2}$ *and* $r_2 > \frac{\sigma_2^2}{2}$, *then the solution* $(x(t), y(t), z(t))$ *of System (5.3) with any initial value* $(x(0), y(0), z(0)) \in \mathbb{R}^3_+$ *satisfies* $\lim\limits_{t\to\infty} z(t) = 0 \quad$ *a.s, such that the distributions of* $x(t)$ *and* $y(t)$ *converge weakly a.s., to the measures which have the following densities respectively*

$$\begin{aligned} \pi(x) &= G_1 \sigma_1^{-2} x^{-2+\frac{2r_1}{\sigma_1^2}} e^{-\frac{2r_1}{K_1\sigma_1^2}x}, x \in (0,\infty); \\ \pi(y) &= G_2 \sigma_2^{-2} y^{-2+\frac{2r_2}{\sigma_2^2}} e^{-\frac{2r_2}{K_2\sigma_2^2}y}, y \in (0,\infty), \end{aligned} \tag{5.52}$$

where $G_1 = [\sigma_1^{-2} (\frac{K_1\sigma_1^2}{2r_1})^{\frac{2r_1}{\sigma_1^2}-1} \Gamma(\frac{2r_1}{\sigma_1^2} - 1)]^{-1}$ *and* $G_2 = [\sigma_2^{-2} (\frac{K_2\sigma_2^2}{2r_2})^{\frac{2r_2}{\sigma_2^2}-1} \Gamma(\frac{2r_2}{\sigma_2^2} - 1)]^{-1}$, *are constants such that* $\int_0^\infty \pi(x)dx = 1$ *and* $\int_0^\infty \pi(y)dy = 1$.

Proof. Since the solution of System (5.3) is positive for any initial value $(x(0), y(0), z(0)) \in$

$\mathbb{R}^3_+$, so one gets

$$\begin{aligned} dx(t) &= \big(r_1x(t)(1-\frac{x(t)}{K_1}) - x(t)z(t)[\alpha_1 - \beta y(t)]\big)dt + \sigma_1 x(t)dW_1(t) \\ &\leq \big(r_1x(t)(1-\frac{x(t)}{K_1})\big)dt + \sigma_1 x(t)dW_1(t). \end{aligned} \tag{5.53}$$

Consider the following supplementary logistic equation with random perturbation

$$dX(t) = \big[r_1X(t)(1-\frac{X}{K_1})\big]dt + \sigma_1 X(t)dW_1(t), \tag{5.54}$$

with initial values $X(0) = x(0) > 0$.

Let $g_1(x) = r_1x(t)\big(1-\frac{x(t)}{K_1}\big)$, $\nu_1(x) = \sigma_1 x(t)$, $x \in (0,\infty)$, such that

$$\int \frac{g_1(s)}{\nu_1^2(s)} = \int (\frac{r_1}{\sigma_1^2 s} - \frac{r_1}{K_1\sigma_1^2})ds = \frac{r_1}{\sigma_1^2}\ln s - \frac{r_1}{K_1\sigma_1^2}s + G_1.$$

Therefore, $e^{\int \frac{g_1(s)}{\nu_1^2(s)}} = e^{G_1} s^{\frac{r_1}{\sigma_1^2}} e^{-\frac{r_1}{K_1\sigma_1^2}s}$.

One can verify that

$$\int_0^\infty \frac{1}{\nu_1^2(x)} e^{\int_1^x \frac{2g_1(s)}{\nu_1^2(s)}ds} dx = \frac{e^{\frac{2r_1}{K_1\sigma_1^2}}}{\sigma_1^2} \int_0^\infty x^{-2} x^{\frac{2r_1}{\sigma_1^2}} e^{-\frac{2r_1}{K_1\sigma_1^2}x} dx < \infty. \tag{5.55}$$

From (5.55), one can say that System (5.54) has the ergodic property and the invariant density is $\pi(x) = G_1\sigma_1^{-2} x^{-2+\frac{2r_1}{\sigma_1^2}} e^{-\frac{2r_1}{K_1\sigma_1^2}x}, x \in (0,\infty)$. Therefore, from ergodic Theorem it follows that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \int_0^\infty x\pi(x)dx \quad a.s. \tag{5.56}$$

Let $X(t)$ be the solution of SDE (5.54) with the initial value $X(0) = x(0) > 0$. Therefore, one obtains

$$x(t) \leq X(t), \quad \forall t \geq 0 \quad a.s. \tag{5.57}$$

For the second prey $y(t)$, one can obtain

$$\begin{aligned} dy(t) &= \big(r_2 y(t)(1-\frac{y(t)}{K_2}) - y(t)z(t)[\alpha_2 - \beta x(t)]\big)dt + \sigma_2 y(t)dW_2(t) \\ &\leq \big(r_2 y(t)(1-\frac{y(t)}{K_2})\big)dt + \sigma_2 y(t)dW_2(t). \end{aligned} \tag{5.58}$$

Assume the following logistic equation with noise

$$dY(t) = \big[r_2 Y(t)(1-\frac{Y}{K_2})\big]dt + \sigma_2 Y(t)dW_2(t), \tag{5.59}$$

with initial value $Y(0) = y(0) > 0$. Setting $g_2(y) = r_2 y(t)\big(1-\frac{y(t)}{K_2}\big)$, $\nu_2(y) = \sigma_2 y(t)$, $y \in (0,\infty)$, one gets $\int \frac{g_2(s)}{\nu_2^2(s)} = \int (\frac{r_2}{\sigma_2^2 s} - \frac{r_2}{K_2\sigma_2^2})ds = \frac{r_2}{\sigma_2^2}\ln s - \frac{r_2}{K_2\sigma_2^2}s + G_2$.
Thus, $e^{\int \frac{g_2(s)}{\nu_2^2(s)}} = e^{G_2} s^{\frac{r_2}{\sigma_2^2}} e^{-\frac{r_2}{K_2\sigma_2^2}s}$. Hence, one may have

$$\int_0^\infty \frac{1}{\nu_2^2(y)} e^{\int_1^y \frac{2g_2(s)}{\nu_2^2(s)}ds} dy = \frac{e^{\frac{2r_2}{K_2\sigma_2^2}}}{\sigma_2^2}\int_0^\infty y^{-2} y^{\frac{2r_2}{\sigma_2^2}} e^{-\frac{2r_2}{K_2\sigma_2^2}y} dy < \infty. \tag{5.60}$$

According to (5.60), one can conclude that System (5.59) has the ergodic property and the invariant density is $\pi(y) = G_2\sigma_2^{-2} y^{-2+\frac{2r_2}{\sigma_2^2}} e^{-\frac{2r_2}{K_2\sigma_2^2}y}, y \in (0,\infty)$. Therefore, it follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds = \int_0^\infty y\pi(y)dy \quad a.s. \tag{5.61}$$

Let $Y(t)$ be the solution of SDE (5.59) with the initial value $Y(0) = Y(0) > 0$, then one obtains

$$y(t) \leq Y(t), \quad \forall t \geq 0 \quad a.s. \tag{5.62}$$

By It$\hat{o}$ formula to the third equation of System (5.3), one can derive that

$$d(\ln z(t)) = \big(-\delta - \alpha_3 z(t) + a_1 x(t) + a_2 y(t) - \frac{\sigma_3^2}{2}\big)dt + \sigma_3 dW_3(t), \tag{5.63}$$

Integrating both sides of (5.63) from 0 to t, yields

$$\begin{aligned}\ln z(t) - \ln z(0) &= (-\delta - \frac{\sigma_3^2}{2})t + a_1 \int_0^t x(s)ds + a_2 \int_0^t y(s)ds \\ &\quad - \alpha_3 \int_0^t z(s)ds + \sigma_3 W_3(t) \\ &\leq (-\delta - \frac{\sigma_3^2}{2})t + a_1 \int_0^t x(s)ds + a_2 \int_0^t y(s)ds + \sigma_3 W_3(t) \\ &\leq (-\delta - \frac{\sigma_3^2}{2})t + a_1 \int_0^t X(s)ds + a_2 \int_0^t Y(s)ds + \sigma_3 W_3(t),\end{aligned} \tag{5.64}$$

where in the last Inequality (5.57) and (5.62) have been used. Take the limit superior on both sides of (5.64), together with (5.56) and (5.61), since $\lim_{t\to\infty} \frac{W_3(t)}{t} = 0$ a.s, one gets

$$\limsup_{t\to\infty} \frac{\ln z(t)}{t} \leq -\delta - \frac{\sigma_3^2}{2} + a_1 \int_0^\infty x\pi(x)dx + a_2 \int_0^\infty y\pi(y)dy := \chi < 0 \quad a.s. \tag{5.65}$$

Therefore, $\lim\limits_{t\to\infty} z(t) = 0 \quad a.s.$

5.5 Numerical Simulations

In this Section, some numerical simulations to validate the obtained theoretical results are provided. Milstein's higher order scheme with a strong order of convergence one, discussed in [56, 71], to solve SDDEs (5.3) is provided. The corresponding discretization

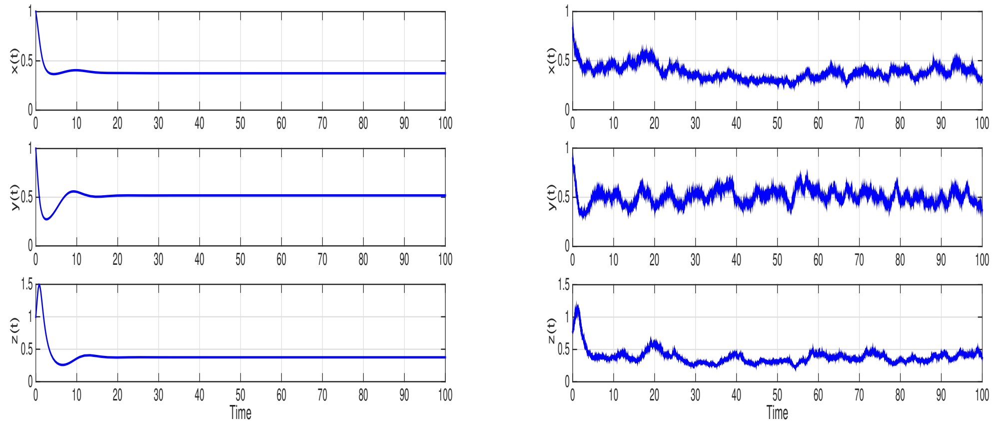

Figure 5.1: Numerical simulations of deterministic DDEs (5.2) (left) and SDDEs (5.3) (right), when $\tau_1 = 1.25$, $\tau_2 = 0.6$ and $\tau_3 = 0.5$; with noise intensities $\sigma_1^2 = 0.08$, $\sigma_2^2 = 0.1$, $\sigma_3^2 = 0.06$. With parameter values: $r_1 = 0.2$, $r_2 = 0.6$, $K_1 = 0.7$, $K_2 = 0.8$, $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\beta = 0.1$, $\delta = 0.8$, $a_1 = 1$, $a_2 = 1.4$. For $\mathscr{T}_0^s > 1$, the stochastic model has a unique ergodic stationary distribution $\pi(.)$ of System (5.3)

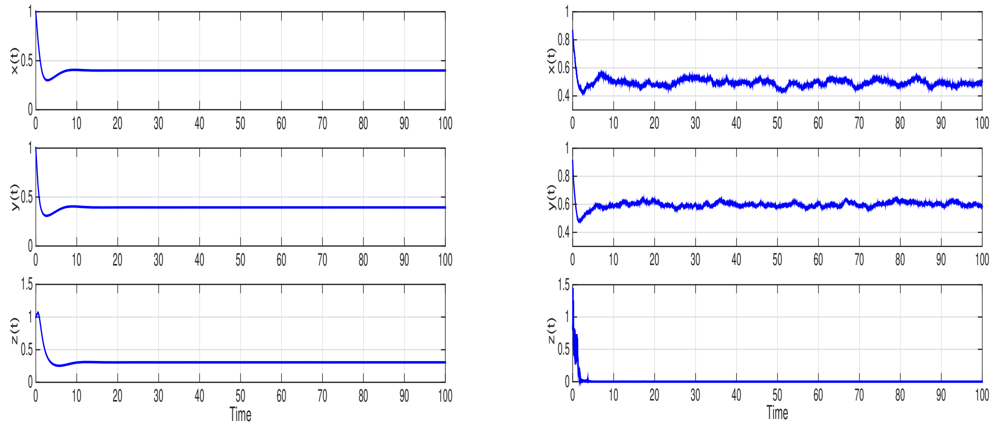

Figure 5.2: Numerical simulations of the solutions for System (5.3) (right) and the corresponding undisturbed System (5.2) (left), with $\tau_1 = 1.25$, $\tau_2 = 0.6$ and $\tau_3 = 0.5$, one can clearly see that the predator goes to extinct; under the noise intensities $\sigma_1^2 = 0.03$, $\sigma_2^2 = 0.02$ and $\sigma_3^2 = 1.4$. When $\mathscr{T}_0^s < 1$

system is then

$$\begin{aligned}
x_{n+1} &= x_n + hx_n[r_1(1 - \frac{x_{n-m_1}}{k_1}) - \alpha_1 z_n + \beta y_n z_n] + \sigma_1 x_n \xi_{1,n}\sqrt{h} + \frac{\sigma_1^2}{2} x_n[\xi_{1,n}^2 - 1]h \\
y_{n+1} &= y_n + hy_n[r_2(1 - \frac{y_{n-m_2}}{k_2}) - \alpha_2 z_n + \beta x_n z_n] + \sigma_2 y_n \xi_{2,n}\sqrt{h} + \frac{\sigma_2^2}{2} y_n[\xi_{2,n}^2 - 1]h \\
z_{n+1} &= z_n + hz_n[\delta + \alpha_3 z_n + a_1 x_{n-m_3} + a_2 y_{n-m_3}] + \sigma_3 z_n \xi_{3,n}\sqrt{h} + \frac{\sigma_3^2}{2} z_n[\xi_{3,n}^2 - 1]h.
\end{aligned} \tag{5.66}$$

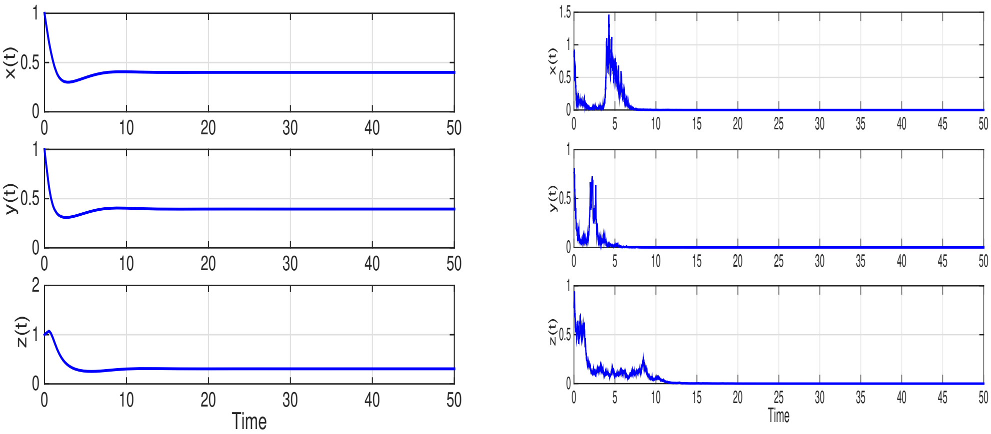

Figure 5.3: Numerical simulations of the solutions for System (5.3) (right) and the corresponding undisturbed System (5.2) (left), with $\tau_1 = 1.25$, $\tau_2 = 0.6$ and $\tau_3 = 0.5$, one can clearly see that all the species go to extinct . Under the noise intensities $\sigma_1^2 = 1.2$, $\sigma_2^2 = 1.2$ and $\sigma_3^2 = 0.5$. When $r_1 < \frac{\sigma_1^2}{2}$, $r_2 < \frac{\sigma_2^2}{2}$ and $\mathscr{T}_0^s < 1$

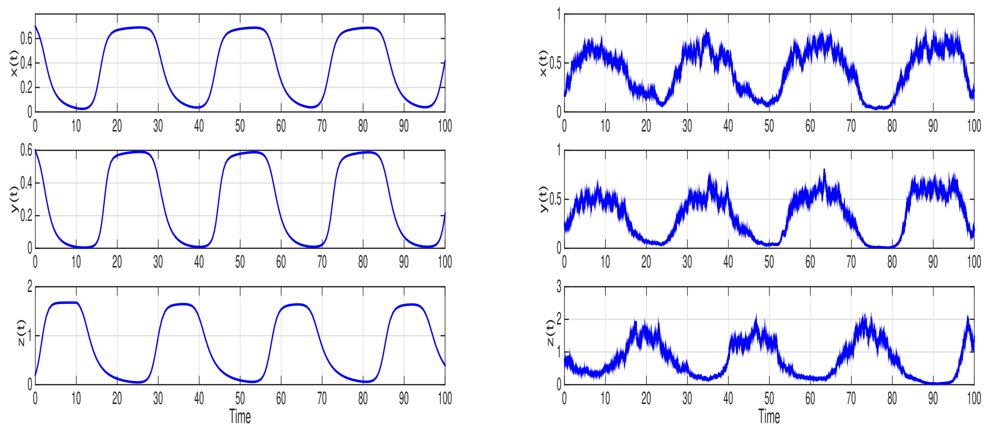

Figure 5.4: Numerical simulations of the solutions for System (5.3) (right) and the corresponding undisturbed System (5.2) (left), with $\tau_1 = 10$, $\tau_2 = 0.1$ and $\tau_3 = 0.1$; under the noise intensities $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.2$ and $\sigma_3^2 = 0.2$. Clearly, the number of large oscillations in prey and predator species (right) is more or less the same in comparison to its undisturbed counterpart

Here, $\xi_{1,n}$, $\xi_{3,n}$ and $\xi_{2,n}$ are mutually independent $N(0,1)$ random variables, m_1, m_2, m_3 are integers such that the time-delays can be expressed in terms of the step-size as $\tau_1 = m_1 h$, $\tau_2 = m_2 h$ and $\tau_3 = m_3 h$.

Example 5.5.1. Given $\sigma_1^2 = 0.08$, $\sigma_2^2 = 0.1$, $\sigma_3^2 = 0.06$, $\tau_1 = 1.25$, $\tau_2 = 0.6$, $\tau_3 = 0.5$ and parameter values: $r_1 = 0.2$, $r_2 = 0.6$, $K_1 = 0.7$, $K_2 = 0.8$, $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$, $\beta = 0.1$, $\delta = 0.8$, $a_1 = 1$, $a_2 = 1.4$. Direct calculation leads to $\mathscr{T}_0^s = \dfrac{r_1\rho_1 + r_2\rho_2}{(\rho_1\frac{\sigma_1^2}{2} + \rho_2\frac{\sigma_2^2}{2} + \rho_3\frac{\sigma_3^2}{2})} = 5.9 > 1$, $(r_1 = 0.2 > 0.04 = \frac{\sigma_1^2}{2})$ and $(r_2 = 0.6 > 0.05 = \frac{\sigma_2^2}{2})$.

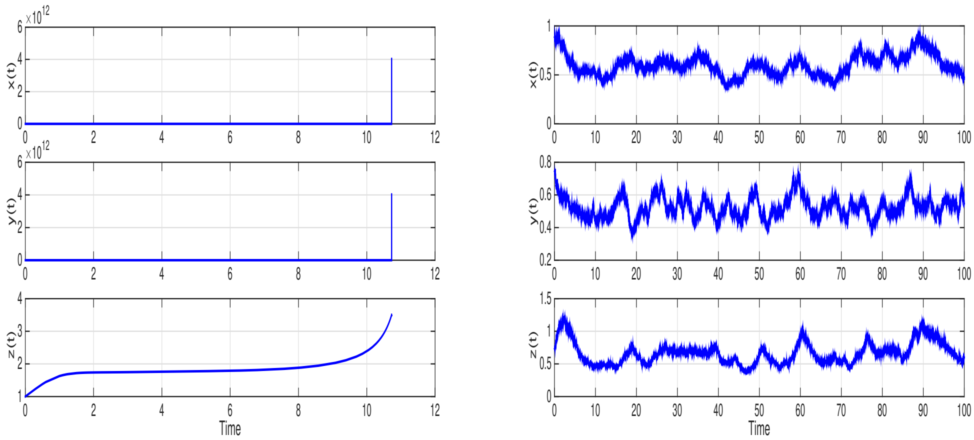

Figure 5.5: The effect of white noise to prevent the explosion of the population, when $\beta = 0.5$. The explosion may occur when the cooperation parameter β is large, with the same parameter values of Figure 5.1, there is an explosion of population with deterministic model (left); While the noise prevent such explosion of the population (right)

Thus, the conditions of Theorem 5.3.2 hold. In view of Theorem 5.3.2, Figure 5.1 shows that there is a unique ergodic stationary distribution $\pi(.)$ of System (5.3).

Example 5.5.2. Choosing $\sigma_1^2 = 0.03$, $\sigma_2^2 = 0.02$, $\sigma_3^2 = 1.4$, $\tau_1 = 1.25$,$\tau_2 = 0.6$, $\tau_3 = 0.5$ and parameter values: $r_1 = 0.4$, $r_2 = 0.5$, $K_1 = 0.5$, $K_2 = 0.6$, $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\beta = 0.1$, $\delta = 0.4$, $\alpha_3 = 1.8$, $a_1 = 1$, $a_2 = 1.4$. By a simple calculation, $(r_1 = 0.4 > 0.015 = \frac{\sigma_1^2}{2})$, $(r_2 = 0.5 > 0.01 = \frac{\sigma_2^2}{2})$, and $\mathcal{T}_0^s = 0.89 < 1$. In view of Theorem 5.4.1 (a), extinction of predator can occur. The predator populations dies out exponentially with probability one; See Figure 5.2.

Example 5.5.3. With $\sigma_1^2 = 1.2$, $\sigma_2^2 = 1.2$, $\sigma_3^2 = 0.5$, $\tau_1 = 1.25$, $\tau_2 = 0.6$, $\tau_3 = 0.5$ and parameter values: $r_1 = 0.4$, $r_2 = 0.5$, $K_1 = 0.5$, $K_2 = 0.6$, $\alpha_1 = 0.3$, $\alpha_2 = 0.6$, $\beta = 0.1$, $\delta = 0.4$, $\alpha_3 = 1.8$, $a_1 = 1$, $a_2 = 1.4$. By a simple calculation, $r_1 = 0.4 < 0.6 = \frac{\sigma_1^2}{2}$, $r_2 = 0.5 < 0.6 = \frac{\sigma_2^2}{2}$, and $\mathcal{T}_0^s = 0.394 < 1$. In view of Theorem 5.4.1 (b), one can see that the preys $x(t)$, $y(t)$ and the predator $z(t)$ populations all die out exponentially with probability one; see Figure 5.3 which shows that a strong intensity of noise can be a cause for extinction of the prey species that will then teed to predator population to extinct.

Example 5.5.4. Figure 5.4 shows the periodicity of the solutions of deterministic and stochastic models when $\tau_1 = 10$, $\tau_2 = 0.1$, $\tau_3 = 0.1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.2$, $\sigma_3^2 = 0.2$, and other parameter values $r_1 = 2$, $r_2 = 2$, $K_1 = 0.6$, $K_2 = 0.6$, $\alpha_1 = 1.3$, $\alpha_2 = 1.5$, $\delta = 0.8$, $\alpha_3 = 1.6$, $a_1 = 1$, $a_2 = 1.4$.

The author arrives at the following Remarks.

Remark 5.5.1. As $\mathscr{T}_0^s > 1$, the stationary distribution indicates that all the species can be exist in a long period of time, provided that the intensities of white noise are adequately small. On the other hand, when $\mathscr{T}_0^s < 1$, the tendency for extinction increases, which never happens in the undisturbed system, without intensity of environmental perturbations; See Figures 5.2 and 5.3.

Remark 5.5.2. Environmental Brownian noise suppresses the explosion of the population (see Figure 5.5). Combination of time-delays and white noise enriches the dynamics of the model and increases the complexity of the system, which rationally meets with the reality.

5.6 Concluding Remarks

In this chapter, a stochastic delay differential model for the dynamics of two-preys one-predator system, with cooperation among the prey species against predator was proposed. By constructing a suitable stochastic Lyapunov function, sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the positive solutions to the model have been established. Sufficient conditions for extinction of the predator population in two cases have been deduced, that is, the first case is the prey populations survival and the predator population extinction; the second case is all the preys and predator populations extinction. A threshold parameter $\mathscr{T}_0^s$ was also established. The solutions of SDDEs (5.3) fluctuate in the vicinity of the positive equilibrium of the corresponding undisturbed system when $\mathscr{T}_0^s > 1$, which can be considered as weak stability. Whereas, the predator dies out if $\mathscr{T}_0^s < 1$. It has been seen, from the numerical simulations, that random noises can suppress the explosion of the species, where the solutions of the undisturbed system is unbounded. The combination time-delays and white noise have a great impact on the dynamics, complexity and permanence of prey and predator populations. Existence of the ergodic stationary distribution of the positive solutions to the proposed model is a very important issue for the population system and affects the survival of the species in the environment.

In the next chapter, the author introduces a stochastic SIRC epidemic model for COVID-19. In which the impact of stochastic perturbation factors in the model are investigated.

Chapter 6: Stochastic SIRC Epidemic Model with Time-Delay for COVID-19

6.1 Introduction

Environmental factors, such as humidity, precipitation, and temperature, have significant impacts on the spread of the new strain coronavirus COVID-19 to humans. In this chapter, a stochastic epidemic SIRC model, with cross-immune class and time-delay in transmission terms, for the spread of COVID-19 is used. The model is analyzed in which the existence and uniqueness of positive global solution are proved. The basic reproduction number $\mathscr{R}_0^s$ for the stochastic model which is smaller than $\mathscr{R}_0$ of the corresponding deterministic model is deduced. Sufficient conditions that guarantee the existence of a unique ergodic stationary distribution, using the stochastic Lyapunov function, and conditions for the extinction of the disease are obtained. A stochastic SIRC model with time delay is provided in Section 2. Section 3 studies the existence and uniqueness of global positive solution for stochastic delayed SIRC model. In Sections 4 and 5, a stationary distribution and extinction analysis of the underlying model are investigated. Some virtual numerical examples are present, in Section 6. Finally, concluding remarks are given in Section 7.

The ongoing pandemic Coronavirus Disease (COVID-19) becomes a worldwide emergency. This infectious disease is spreading fast, endangering large number of people health, and thus needs immediate actions and intensive studies to control the disease in communities [41]. COVID-19 is the seventh member of the coronavirus (CoV) family, such as MERS-CoV and SARS-CoV [48]. Although SARS-CoV was more deadly, it was much less infectious than COVID-19. There have been no outbreaks of SARS anywhere in the world since 2003. The symptoms of COVID-19 infection include cough, fever, tiredness, diarrhea, and shortness of breath. Mostly in severe cases, COVID-19 causes pneumonia and death [136]. The primary studies show that the incubation period of COVID-19 is between 3–14 days or longer [137]. Additionally, the average of basic reproduction number $\mathscr{R}_0$ for COVID-19 is about 2–2.8. The disease may still be infectious in the latent infection period. Studies to date suggest that the virus is very serious and spreads fast from person to person through close contact and respiratory droplets rather than through the air [137]. Table 6.1 shows the incubation period of several common infectious diseases.

Mathematical modeling of the infectious diseases has an important role in the epidemiological aspect of disease control [29]. Several epidemic models, with various characteristics, have been described and investigated in the literature. Most of these models are based on Susceptible-Infected-Removed (SIR) model. Casagrandi *et al.* [30] introduced SIRC model to describe the dynamical behaviors of Influenza A, by inserting a new compartment, namely Cross-Immuney (C) component[1] of people who have been recovered after being infected by different strains of the same viral subtype in previous years. The component C describes an intermediate state between the susceptible S and the recovered R one. Rihan *et al.* [115] investigated the qualitative behaviours of fractional-order SIRC model for Salmonella bacterial infection. Recently in [68], the authors provided a deterministic SEIR epidemic model of fractional-order to describe the dynamics of COVID-19. In other descriptions, quarantine state (Q) may be include in the presence of subjects, such as SIRQ models [55].

In fact, stochastic perturbation factors, such as precipitation, absolute humidity, and temperature, have a significant impact on the infection force of all types of virus diseases to humans. Taking this into consideration enables to present randomness into deterministic biological models to expose the environmental variability effect, whether it is a environmental fluctuations in parameters or random noise in the differential systems [77, 85, 96, 134, 145]. Moreover, stochastic models give an extra degree of freedom and realism in comparison with their corresponding deterministic models. Stochastic population dynamics perturbed by white noise (or Brownian motion) has been studied extensively by many authors [7, 94, 95]. It has been investigated in [93] that a envi-

[1] Cross-immunity (or cross-reactivity) is a major evolutionary force that affects pathogen diversity (i.e. it drives viruses and microbes to be as distinct as possible from one another in order to avoid immunity detection, memory recognition and clearance).

ronmental Brownian noise can suppress explosions in population dynamics. Yuan *et al.* [142] discussed the results of stochastic viral infection, immune response dynamics and analyzed the human immuno-deficiency virus infection. In [62], the author investigated the existence results of ergodic distribution for stochastic hepatitis B virus model based on Lyapunov function. In [63], the authors explore the dynamics of SIR epidemic model with environmental fluctuations. Additionally, they calculated a threshold parameter to demonstrate the persistence and extinction of the disease. Recently, Lakshmi *et al.* [78] identified some environmental factors such as geographic location of the countries, the upcoming climate, atmospheric temperature, humidity, sociobiological factors, etc., that influence the global spread of the COVID-19.

Up-to date studies, it has been reported that there are many COVID-19 carriers who are not suffering the disease. This may be due to cross-immunity of other virus survivors, people who have been recovered from the virus, such as other stains of coronavirus, H1N1, or influenza *A*. It has been reported in [48] that "*SARS-CoV-2 immunity has some degree of cross-reactive coronavirus immunity in a fraction of the human population, and this fraction of population has influence susceptibility to COVID-19 disease*". Accordingly, in the present chapter, an SIRC epidemic model of cross-immune class for the dynamics of transmission COVID-19 among groups is investigated. Time-delay is included in the transmission terms to represent the incubation period of the virus (the time between infection and symptom onset). White noise type of perturbations is also incorporated to reveal the effect of environmental fluctuations and variability in parameters. Based on existing literatures, this is the first work dealing with the persistence and extinction of a stochastic epidemic model for COVID-19 infection. The impact of small and large values of white noise in the ’persistence’ and ’extinction’ of the disease are investigated. The existence results of stationary distribution and extinction of the disease are also derived, using a novel combination of stochastic Lyapunov functional.

Table 6.1: Incubation period of several common infectious diseases

Disease	Range	Ref.
COVID-19	3–14 days	[137]
Cholera	0.5–4.5 days	[6]
Common cold	1–3 days	[80]
Ebola	1–21 days	[135]
HIV	2–3 weeks to months or longer	[66]
Influenza	1–3 days	[42]
MERS	2–14 days	[1]
SARS	1–10 days	[124]

6.2 Stochastic SIRC Epidemic Model

For the spread of COVID-19 disease in humans, the population is classified into four categories: $S(t), I(t), R(t)$, and $C(t)$ are the proportion of susceptible, infected, recovered and cross-immune ones at time t respectively. Let $N(t) = S(t) + I(t) + R(t) + C(t)$ be the total population. At this stage, SIRC model efficiently describes the mechanism for the spreading of the COVID-19 virus. The classical SIRC model [30, 65] takes the form

$$
\begin{aligned}
\dot{S}(t) &= \eta(1 - S(t)) - \xi S(t)I(t-\tau) + \beta C(t), \\
\dot{I}(t) &= \xi S(t)I(t-\tau) + \sigma\xi C(t)I(t) - (\eta + \alpha)I(t), \\
\dot{R}(t) &= (1-\sigma)\xi C(t)I(t) + \alpha I(t) - (\eta + \gamma)R(t), \\
\dot{C}(t) &= \gamma R(t) - \xi C(t)I(t) - (\eta + \beta)C(t).
\end{aligned}
\tag{6.1}
$$

A discrete time-delay τ is incorporated into the SIRC model, to represent the incubation period, which is about 3-14 days [137]. All the parameters appearing in the model are nonnegative see Table 6.2. In the absence of cross-immunity i.e. $(1 - \sigma = 0)$, the SIRC model curtails to the SIRS model, since the two individuals S and C become immunologically indistinguishable. Figure 6.1 shows the scheme of SIRC model.

Time-delay $\tau > 0$ is incorporated in the transmission terms to represent the incubation period of the viral infection, the time between infection and symptom onset. The current studies show that the average/median of incubation period of early confirmed cases of COVID-19 is about 5.5 days, which is similar to SARS-CoV. Presence of time-delay in the model may cause periodic solutions many times for different time-delay values τ

Table 6.2: Description of the model parameters

Parameters	Description
η	Mortality rate in every compartment and is assumed equal to the rate of newborn in the population [30]
β	Rate at which the cross-immune population becomes susceptible again
ξ	Contact/transmission rate
σ	The average reinfection probability of a cross-immune individual
α	Recovery rate of the infected population
γ	Rate at which the recovered population becomes the cross-immune population and moves from total to partial immunity

[109]. The Model (6.1) has a disease-free equilibrium $\mathscr{E}_0 = [1,0,0,0]$, and an endemic

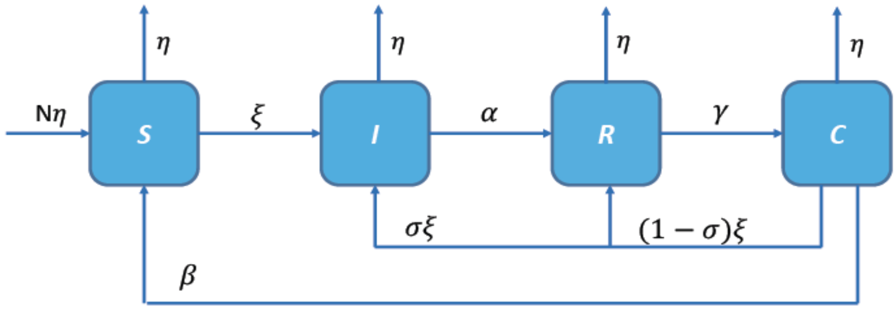

Figure 6.1: Scheme of SIRC Model (6.1), assuming that the total population $N = 1$.

equilibrium $\mathscr{E}_+ = [S^*, I^*, R^*, C^*]$, where

$$
\begin{aligned}
S^* &= \frac{\eta+\alpha}{\xi} - \frac{\beta\gamma\alpha I^*}{[(\eta+\gamma)-(1-\sigma)\gamma]\xi I^* + (\eta+\beta)(\eta+\gamma)},\\
R^* &= \frac{\alpha I^*(\xi I^* + \eta + \beta)}{[(\eta+\gamma)-(1-\sigma)\gamma]\xi I^* + (\eta+\beta)(\eta+\gamma)},\\
C^* &= \frac{\gamma\alpha I^*}{[(\eta+\gamma)-(1-\sigma)\gamma]\xi I^* + (\eta+\beta)(\eta+\gamma)},
\end{aligned}
$$

and I^* is a root of quadratic equation $pI^2 + qI + r = 0$, where

$$
\begin{aligned}
p &= \eta\xi(\eta+\alpha+\sigma\gamma),\\
q &= \eta\xi\left[\alpha(2\eta+\gamma+\beta) + (\eta+\beta)(\eta+\gamma) + (\eta+\sigma\gamma)(\eta-\xi)\right],\\
r &= \eta(\eta+\beta)(\eta+\gamma)(\eta+\alpha)(1-\mathscr{R}_0).
\end{aligned}
$$

Here $\mathscr{R}_0 = \dfrac{\xi}{\eta+\alpha}$ is known as the *basic reproduction number* of the deterministic model.

In fact, there is increasing indication that superior consistency with some phenomena can be contributed if the effects of environmental noises in the system are taken into account [113]. The epidemic Model (6.1) assumes that the observed dynamics are driven exclusively by internal deterministic cases. Ignoring environmental variability in the modeling may affect the dynamics of the model and transmission of the disease. Accordingly, there is need to extend the deterministic systems described by differential equations into *Stochastic Differential Equations* (SDEs), where related parameters are modeled as suitable stochastic processes, added to the driving system equations.

From the mathematical and biological point of view, there are some assumptions to incorporate stochastic perturbations into the epidemiological model, such as Markov chain process, parameter perturbations, white noise type, etc. Here, white noise type perturbation is incorporated into Model (6.1), which is proportional to S, I, R, C classes, so that

$$\begin{aligned}
dS(t) =&[\eta(1-S(t)) - \xi S(t)I(t-\tau) + \beta C(t)]dt + \nu_1 S(t)dW_1(t),\\
dI(t) =&[\xi S(t)I(t-\tau) + \sigma\xi C(t)I(t) - (\eta+\alpha)I(t)]dt + \nu_2 I(t)dW_2(t),\\
dR(t) =&[(1-\sigma)\xi C(t)I(t) + \alpha I(t) - (\eta+\gamma)R(t)]dt + \nu_3 R(t)dW_3(t),\\
dC(t) =&[\gamma R(t) - \xi C(t)I(t) - (\eta+\beta)C(t)]dt + \nu_4 C(t)dW_4(t),
\end{aligned} \tag{6.2}$$

where $W_1(t), W_2(t), W_3(t)$, and $W_4(t)$ stand for the independent Brownian motions. $\nu_1^2, \nu_2^2, \nu_3^2$, and ν_4^2 represent the intensity of the environmental white noises, $\nu_i > 0$ $(i = 1,2,3,4)$, subject to the following initial conditions

$$\begin{aligned}
&S(\theta) = \phi_1(\theta), \quad I(\theta) = \phi_2(\theta),\\
&R(\theta) = \phi_3(\theta), \quad C(\theta) = \phi_4(\theta), \quad \theta \in [-\tau, 0]\\
&\phi_i(\theta) \in \mathscr{C}, \quad i = 1,2,3,4,
\end{aligned} \tag{6.3}$$

such that $\mathscr{C}$ is the family of Lebesgue integrable functions from $[-\tau, 0]$ into $\mathbb{R}_+^4$.

6.3 Existence and Uniqueness of the Positive Solution

To investigate the dynamical characteristics of SDDEs (6.2), the first consideration is to verify System(6.2) has a unique global positive solution. As the coefficients of System(6.2) satisfy the local Lipschitz condition together with the linear growth condition [23]; Consequently, there exists a unique local solution. Now, one needs to prove that the solution is positive and global, using Lyapunov analysis method [91].

Theorem 6.3.1. *System (6.2) has a unique positive solution $(S(t), I(t), R(t), C(t))$ on $t \geq -\tau$, and the solution will remain in $\mathbb{R}^4_+$ for the given initial condition (6.3) with probability one.*

Proof. For any initial value (6.3), as the coefficients of System(6.2) satisfy the local Lipschitz condition, so System(6.2) has a unique local solution $(S(t), I(t), R(t), C(t))$ on $t \in [-\tau, \tau_e)$, a.s., where τ_e represents the explosion time [91].

The purpose is to show that this solution is global i.e. $\tau_e = \infty$, a.s. Assume $n_0 \geq 1$ be sufficiently large such that $S(\theta), I(\theta), R(\theta)$ and $C(\theta)$ $(\theta \in [-\tau, 0])$ are lying in the interval $\left[\frac{1}{n_0}, n_0\right]$. For each $n \geq n_0, n \in \mathbb{N}$, define the stopping time

$$\tau_n = \inf\left\{t \in [-\tau, \tau_e) : \min\{S(t), I(t), R(t), C(t)\} \leq \frac{1}{n} \quad or \quad \max\{S(t), I(t), R(t), C(t)\} \geq n\right\},$$

fixing $\inf \phi = \infty$ (ϕ be the empty set). Apparently, τ_n is increasing as $n \to \infty$. Assume $\tau_\infty = \lim_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ a.s. Therefore, one needs to show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(S(t), I(t), R(t), C(t)) \in \mathbb{R}^4_+$ a.s. for all $t \geq -\tau$. If it is erroneous, there is a pair $\varepsilon \in (0, 1)$ and $\tilde{T} > 0$ such that $P\{\tau_\infty \leq \tilde{T}\} > \varepsilon$. Then, there is an integer $n_1 \geq n_0$

such that

$$P\{\tau_n \le \tilde{T}\} \ge \varepsilon, \forall n \ge n_1. \tag{6.4}$$

Define a $\mathscr{C}^2$ function $\mathscr{V} : \mathbb{R}_+^4 \to \mathbb{R}_+$ as

$$\mathscr{V}(S,I,R,C) = (S - \kappa - \kappa\frac{\ln S}{\kappa}) + (I - 1 - \ln I) + (R - 1 - \ln R) + (C - 1 - \ln C) + \int_t^{t+\tau} \kappa\xi I(s-\tau)ds,$$

where $\kappa > 0$ is a constant to be determined. By Ito's formula, one can obtain

$$d\mathscr{V} = \mathscr{L}\mathscr{V}dt + \nu_1(S-\kappa)dW_1(t) + \nu_2(I-1)dW_2(t) + \nu_3(R-1)dW_3(t) + \nu_4(C-1)dW_4(t),$$

where

$$\begin{aligned}
\mathscr{L}\mathscr{V} =& (1-\frac{\kappa}{S})(\eta - \eta S - \xi SI(t-\tau) + \beta C) + (1-\frac{1}{I})(\xi SI(t-\tau) + \sigma\xi CI - (\eta+\alpha)I) \\
& + (1-\frac{1}{R})(\xi CI - \sigma\xi CI + \alpha I - \eta R - \gamma R) + (1-\frac{1}{C})(\gamma R - \xi CI \\
& - (\eta+\beta)C) + \frac{\kappa\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2}{2} + \kappa\xi I(t) - \kappa\xi I(t-\tau), \\
& \le 4\eta + \kappa\eta + \alpha + \beta + \gamma - \eta C - \eta R + (\xi(1+\kappa) - \alpha)I - \eta I - \eta S + \\
& \frac{\kappa\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2}{2}.
\end{aligned}$$

Let $\kappa = \frac{\alpha-\xi}{\xi}$, then one may have

$$\begin{aligned}
\mathscr{L}\mathscr{V} &\le 4\eta + \kappa\eta + \alpha + \beta + \gamma + \frac{\kappa\nu_1^2 + \nu_2^2 + \nu_3^2 + \nu_4^2}{2} \\
&\le \mathscr{M},
\end{aligned} \tag{6.5}$$

where $\mathscr{M} > 0$ is a constant which is independent of $S(t)$, $I(t)$, $R(t)$ and $C(t)$. Therefore,

$$\begin{aligned} d\mathscr{V}(S,I,R,C) \leq \mathscr{M}dt + \nu_1(S-\kappa)dW_1(t) + \nu_2(I-1)dW_2(t) + \nu_3(R-1)dW_3(t) \\ + \nu_4(C-1)dW_4(t). \end{aligned} \tag{6.6}$$

Integrating (6.6) from 0 to $\tau_n \wedge \tilde{T} = \min\{\tau_n, \tilde{T}\}$ and then taking the expectation E on both sides, one gets

$$\begin{aligned} & E[\mathscr{V}(S(\tau_n \wedge \tilde{T}), I(\tau_n \wedge \tilde{T}), R(\tau_n \wedge \tilde{T}), C(\tau_n \wedge \tilde{T}))] \\ & \qquad \leq E[\mathscr{V}(S(0), I(0), R(0), C(0))] + \mathscr{M}\tilde{T}. \end{aligned} \tag{6.7}$$

Let $\Omega_n = \{\tau_n \leq \tilde{T}\}$, for $n \geq n_1$ and in view of (6.4), one obtains $P(\Omega_n) \geq \varepsilon$. Such that, for every $\omega \in \Omega_n$, there is at least one of $S(\tau_n, \omega)$,$I(\tau_n, \omega)$, $R(\tau_n, \omega)$ or $C(\tau_n, \omega)$ equaling either to n or $\frac{1}{n}$ and then, one obtains

$$\mathscr{V}(S(\tau_n \wedge \tilde{T}), I(\tau_n \wedge \tilde{T}), R(\tau_n \wedge \tilde{T}), C(\tau_n \wedge \tilde{T})) \geq (n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln \frac{1}{n}). \tag{6.8}$$

According to (6.7), one gets

$$\begin{aligned} & E\mathscr{V}(S(0), I(0), R(0), C(0)) + \mathscr{M}\tilde{T} \geq E[1_{\Omega_n(\omega)}\mathscr{V}(S(\tau_n, \omega), \\ & \qquad I(\tau_n, \omega), R(\tau_n, \omega)), C(\tau_n, \omega)]\varepsilon(n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln \frac{1}{n}), \end{aligned} \tag{6.9}$$

where 1_{Ω_n} represents the indicator function of Ω_n. Letting $n \to \infty$ yields

$$\infty > E\mathscr{V}(S(0), I(0), R(0), C(0)) + \mathscr{M}\tilde{T} = \infty, \tag{6.10}$$

which leads to a contradiction, it can be conclude that $\tau_\infty = \infty$ a.s.

6.4 Existence of Ergodic Stationary Distribution

In this section, the existence of a unique ergodic stationary distribution of the positive solutions to System(6.2) is discussed.

Define the Reproduction Number of the stochastic model as

$$\mathscr{R}_0^s = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\eta}\hat{\alpha}\hat{\gamma}\hat{\beta}}, \tag{6.11}$$

where $\hat{\eta} = \eta + \frac{v_1^2}{2}$, $\hat{\alpha} = \eta + \alpha + \frac{v_2^2}{2}$, $\hat{\gamma} = \eta + \gamma + \frac{v_3^2}{2}$, and $\hat{\beta} = \eta + \beta + \frac{v_4^2}{2}$.

Theorem 6.4.1. *Assume that* $\mathscr{R}_0^s > 1$, *and* $\eta - \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2} > 0$, *then for any initial value* $(S(0), I(0), R(0), C(0)) \in \mathbb{R}_+^4$, *System*$(6.2)$ *has a unique ergodic stationary distribution* $\pi(\cdot)$.

Proof. First, one needs to validate conditions (i) and (ii) of Lemma 5.3.1. To prove condition (i), the diffusion matrix of Model (6.2) is described as

$$\Lambda = \begin{pmatrix} v_1^2 S^2 & 0 & 0 & 0 \\ 0 & v_2^2 I^2 & 0 & 0 \\ 0 & 0 & v_3^2 R^2 & 0 \\ 0 & 0 & 0 & v_4^2 C^2 \end{pmatrix}.$$

Then, the matrix Λ is positive definite for any compact subset of $\mathbb{R}_+^4$, then condition (i) of Lemma 5.3.1 is satisfied. Next, condition (ii) is proved. To this end, define $\mathscr{C}^2$-function $\mathscr{V} : \mathbb{R}_+^4 \to \mathbb{R}$ as follows

$$\begin{aligned} \mathscr{V}(S,I,R,C) = Q\Big(&-\ln S - c_1 \ln I - c_2 \ln R - c_3 \ln C + \xi \int_t^{t+\tau} I(s-\tau)ds\Big) \\ &-\ln S + \xi \int_t^{t+\tau} I(s-\tau)ds - \ln R - \ln C + \frac{1}{\rho+1}(S+I+R+C)^{\rho+1}, \end{aligned}$$

$$\mathscr{V}(S,I,R,C) = Q\mathscr{V}_1 + \mathscr{V}_2 + \mathscr{V}_3 + \mathscr{V}_4 + \mathscr{V}_5,$$

where $c_1 = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\alpha}^2\hat{\gamma}\hat{\beta}}$, $c_2 = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\alpha}\hat{\gamma}^2\hat{\beta}}$, and $c_3 = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\alpha}\hat{\gamma}\hat{\beta}^2}$. Noting that $\mathscr{V}(S,I,R,C)$ is not only continuous, but also tends to $+\infty$ as (S,I,R,C) approaches to the boundary of $\mathbb{R}^4_+$ and $\|(S,I,R,C)\| \to \infty$. Therefore, $\mathscr{V}$ must have a minimum point $(S(0),I(0),R(0),C(0))$ in the interior of $\mathbb{R}^4_+$. Define a $\mathscr{C}^2-$function $\tilde{V} : \mathbb{R}^4_+ \to \mathbb{R}_+$ as

$$\begin{aligned}\widetilde{V}(S,I,R,C) &= Q\Big(-\ln S - c_1\ln I - c_2\ln R - c_3\ln C + \xi\int_t^{t+\tau} I(s-\tau)ds\Big)\\ &\quad -\ln S + \xi\int_t^{t+\tau} I(s-\tau)ds - \ln R - \ln C + \frac{1}{\rho+1}(S+I+R+C)^{\rho+1}\\ &\quad -\mathscr{V}(S(0),I(0),R(0),C(0)),\\ &:= Q\mathscr{V}_1 + \mathscr{V}_2 + \mathscr{V}_3 + \mathscr{V}_4 + \mathscr{V}_5 - \mathscr{V}(S(0),I(0),R(0),C(0))\end{aligned} \tag{6.12}$$

where $(S,I,R,C) \in (\frac{1}{n},n)\times(\frac{1}{n},n)\times(\frac{1}{n},n)\times(\frac{1}{n},n)$ and $n > 1$ is a sufficiently large integer, $\mathscr{V}_1 = -\ln S - c_1 lnI - c_2\ln R - c_3\ln C + \xi\int_t^{t+\tau} I(s-\tau)ds$, $\mathscr{V}_2 = -\ln S + \xi\int_t^{t+\tau} I(s-\tau)ds$, $\mathscr{V}_3 = -\ln R$, $\mathscr{V}_4 = -\ln C$ and $\mathscr{V}_5 = \frac{1}{\rho+1}(S+I+R+C)^{\rho+1}$. $\rho > 1$ is a constant satisfying

$$\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2) > 0,$$

and $Q > 0$ is a sufficiently large number satisfying the following condition

$$\begin{aligned}&-Q\mu + w \leq -2, \quad \text{where}\\ &\mu = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\alpha}\hat{\gamma}\hat{\beta}} - (\eta + \frac{v_1^2}{2}) > 0, \quad \text{since} \quad \mathscr{R}_0^s > 1,\end{aligned} \tag{6.13}$$

$$\begin{aligned}w = \sup_{(S,I,R,C)\in\mathbb{R}^4_+} &\Big\{-\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]I^{\rho+1} + 3\eta + \gamma\\ &+\beta + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2}\Big\}.\end{aligned} \tag{6.14}$$

$$A = \sup_{(S,I,R,C)\in\mathbb{R}^4_+} \Big\{ \eta(S+I+R+C)^{\rho} - \frac{1}{2}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)] \times (S+I+R+C)^{\rho+1} \Big\} < \infty. \tag{6.15}$$

Applying It$\hat{o}$'s formula to $\mathscr{V}_1$, one obtains

$$\begin{aligned}
\mathscr{L}\mathscr{V}_1 &= -\frac{\eta}{S} + \eta + \xi I - \frac{\beta C}{S} - \frac{c_1 \xi S I(t-\tau)}{I} - c_1 \sigma \xi C + c_1(\eta+\alpha) \\
&\quad - \frac{c_2(1-\sigma)\xi C I}{R} - \frac{c_2 \alpha I}{R} + c_2(\eta+\gamma) - \frac{c_3 \gamma R}{C} + c_3 \xi I + c_3(\eta+\beta) + \frac{v_1^2}{2} \\
&\quad + \frac{c_1 v_2^2}{2} + \frac{c_2 v_3^2}{2} + \frac{c_3 v_4^2}{2} \\
&\le -4\sqrt[4]{\eta\gamma\xi^2(1-\sigma)c_1c_2c_3} + \eta + \frac{v_1^2}{2} + c_1(\eta+\alpha+\frac{v_2^2}{2}) + c_2(\eta+\gamma+\frac{v_3^2}{2}) \\
&\quad + c_3(\eta+\beta+\frac{v_4^2}{2}) + \xi(1+c_3)I \\
&\le -\frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\alpha}\hat{\gamma}\hat{\beta}} + \eta + \frac{v_1^2}{2} + \xi(1+c_3)I = -\mu + \xi(1+c_3)I,
\end{aligned} \tag{6.16}$$

Similarly, one can get

$$\mathscr{L}\mathscr{V}_2 = -\frac{\eta}{S} + \eta + \xi I - \frac{\beta C}{S} + \frac{v_1^2}{2}, \tag{6.17}$$

$$\mathscr{L}\mathscr{V}_3 = -\frac{(1-\sigma)\xi C I}{R} - \frac{\alpha I}{R} + \eta + \gamma + \frac{v_3^2}{2}, \tag{6.18}$$

$$\mathscr{L}\mathscr{V}_4 = -\frac{\gamma R}{C} + \xi I + \eta + \beta + \frac{v_4^2}{2}, \tag{6.19}$$

$$\begin{aligned}
\mathscr{L}\mathscr{V}_5 &= (S+I+R+C)^{\rho}[\eta - \eta(S+I+R+C)] + \frac{\rho}{2}(S+I+R+C)^{\rho-1} \\
&\quad \times [v_1^2 S^2 + v_2^2 I^2 + v_3^2 R^2 + v_4^2 C^2], \\
&\le (S+I+R+C)^{\rho}[\eta - \eta(S+I+R+C)] + \frac{\rho}{2}(S+I+R+C)^{\rho+1}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2), \\
&\le \eta(S+I+R+C)^{\rho} - (S+I+R+C)^{\rho+1}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)],
\end{aligned}$$

$$
\begin{aligned}
\mathscr{L}\mathscr{V}_5 &\leq A - \frac{1}{2}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)](S+I+R+C)^{\rho+1} \\
&\leq A - \frac{1}{2}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)](S^{\rho+1}+I^{\rho+1}+R^{\rho+1}+C^{\rho+1}),
\end{aligned} \tag{6.20}
$$

where A is defined by (6.15). From the Equations (6.16)-(6.20), one gets

$$
\begin{aligned}
\mathscr{L}\tilde{V} \leq & -Q\mu + Q\xi(1+c_3)I - \frac{1}{2}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)](S^{\rho+1}+I^{\rho+1}+R^{\rho+1}+C^{\rho+1}) \\
& -\frac{\eta}{S} + 3\eta - \frac{\beta C}{S} + \frac{v_1^2}{2} - \frac{\alpha I}{R} + \gamma + \frac{v_3^2}{2} - \frac{\gamma R}{C} + 2\xi I + A + \beta + \frac{v_4^2}{2}, \\
\leq & -Q\mu + Q\xi(1+c_3)I - \frac{1}{4}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)](S^{\rho+1}+I^{\rho+1}+R^{\rho+1}+C^{\rho+1}) \\
& -\frac{\eta}{S} - \frac{1}{4}[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)]I^{\rho+1} + 3\eta - \frac{\beta C}{S} + \frac{v_1^2}{2} - \frac{\alpha I}{R} + \gamma \\
& + \frac{v_3^2}{2} - \frac{\gamma R}{C} + 2\xi I + A + \beta + \frac{v_4^2}{2}.
\end{aligned}
$$

For $\varepsilon > 0$, define a bounded closed set

$$
\mathscr{D} = \left\{(S,I,R,C) \in \mathbb{R}_+^4 : \varepsilon \leq S \leq \frac{1}{\varepsilon}, \varepsilon \leq I \leq \frac{1}{\varepsilon}, \varepsilon^2 \leq R \leq \frac{1}{\varepsilon^2}, \varepsilon^3 \leq C \leq \frac{1}{\varepsilon^3}\right\}.
$$

In the set $\mathbb{R}_+^4 \setminus \mathscr{D}$, choose ε satisfies the following conditions

$$
-\frac{\eta}{\varepsilon} + H \leq -1, \tag{6.21}
$$

$$
-Q\mu + Q\xi(1+c_3)\varepsilon + w \leq -1, \tag{6.22}
$$

$$
-\frac{\alpha}{\varepsilon} + H \leq -1, \tag{6.23}
$$

$$
-\frac{\gamma}{\varepsilon} + H \leq -1, \tag{6.24}
$$

$$
-\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]\frac{1}{\varepsilon^{\rho+1}} + H \leq -1, \tag{6.25}
$$

$$
-\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]\frac{1}{\varepsilon^{2(\rho+1)}} + H \leq -1, \tag{6.26}
$$

$$-\frac{1}{4}\Big[\eta-\frac{\rho}{2}(v_1^2\vee v_2^2\vee v_3^2\vee v_4^2)\Big]\frac{1}{\varepsilon^{3(\rho+1)}}+H\leq -1,\quad \text{where} \tag{6.27}$$

$$H=\sup_{(S,I,R,C)\in\mathbb{R}_+^4}\Big\{Q(c_3+1)\xi I-\frac{1}{4}\Big[\eta-\frac{\rho}{2}(v_1^2\vee v_2^2\vee v_3^2\vee v_4^2)\Big]I^{\rho+1}+3\eta+\gamma+\beta+2\xi I$$
$$+A+\frac{v_1^2}{2}+\frac{v_3^2}{2}+\frac{v_4^2}{2}\Big\}.$$

One needs to show that $\mathscr{L}\tilde{V}\leq -1$ for any $(S,I,R,C)\in\mathbb{R}_+^4\setminus\mathscr{D}$, and $\mathbb{R}_+^4\setminus\mathscr{D}=\bigcup_{i=1}^8\mathscr{D}_i$, where

$$\mathscr{D}_1=\{(S,I,R,C)\in\mathbb{R}_+^4;0<S<\varepsilon\},\ \ \mathscr{D}_2=\{(S,I,R,C)\in\mathbb{R}_+^4;0<I<\varepsilon\},$$

$$\mathscr{D}_3=\{(S,I,R,C)\in\mathbb{R}_+^4;0<R<\varepsilon^2,I\geq\varepsilon\},$$

$$\mathscr{D}_4=\{(S,I,R,C)\in\mathbb{R}_+^4;0<C<\varepsilon^3,R\geq\varepsilon^2\},$$

$$\mathscr{D}_5=\{(S,I,R,C)\in\mathbb{R}_+^4;S>\frac{1}{\varepsilon}\},\ \ \mathscr{D}_6=\{(S,I,R,C)\in\mathbb{R}_+^4;I>\frac{1}{\varepsilon}\},$$

$$\mathscr{D}_7=\{(S,I,R,C)\in\mathbb{R}_+^4;R>\frac{1}{\varepsilon^2}\},\ \ \mathscr{D}_8=\{(S,I,R,C)\in\mathbb{R}_+^4;C>\frac{1}{\varepsilon^3}\}.$$

Case 1. For any $(S,I,R,C)\in\mathscr{D}_1$, one obtains

$$\begin{aligned}\mathscr{L}\tilde{V}&\leq-\frac{\eta}{S}+Q(c_3+1)\xi I-\frac{1}{4}\Big[\eta-\frac{\rho}{2}(v_1^2\vee v_2^2\vee v_3^2\vee v_4^2)\Big]I^{\rho+1}+3\eta+\gamma+\beta\\&\quad+2\xi I+A+\frac{v_1^2}{2}+\frac{v_3^2}{2}+\frac{v_4^2}{2}\\&\leq-\frac{\eta}{S}+H,\\&\leq-\frac{\eta}{\varepsilon}+H\leq-1,\end{aligned}$$

which is obtained from (6.21). Therefore, $\mathscr{L}\mathscr{V}\leq -1$ for any $(S,I,R,C)\in D_1$.

Case 2. For any $(S,I,R,C) \in \mathscr{D}_2$, one gets

$$\begin{aligned}
\mathscr{L}\tilde{V} &\leq -Q\mu + Q\xi(1+c_3)I - \frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big] I^{\rho+1} + 3\eta \\
&\quad + \gamma + \beta + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2}, \\
&\leq -Q\mu + Q\xi(1+c_3)I + w \\
&\leq -Q\mu + Q\xi(1+c_3)\varepsilon + w < -1,
\end{aligned}$$

which follows from (6.22) and (6.13). Thus, $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_2$.

Case 3. For any $(S,I,R,C) \in \mathscr{D}_3$, one can get

$$\begin{aligned}
\mathscr{L}\tilde{V} &\leq -\frac{\alpha I}{R} + Q(c_3+1)\xi I - \frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big] I^{\rho+1} + 3\eta + \gamma + \beta \\
&\quad + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2} \\
&\leq -\frac{\alpha\varepsilon}{\varepsilon^2} + H \leq -1,
\end{aligned}$$

which follows from (6.23). Consequently, $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_3$.

Case 4. For any $(S,I,R,C) \in \mathscr{D}_4$, it yields

$$\begin{aligned}
\mathscr{L}\tilde{V} &\leq -\frac{\gamma R}{C} + Q(c_3+1)\xi I - \frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big] I^{\rho+1} + 3\eta + \gamma + \beta \\
&\quad + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2} \\
&\leq -\frac{\gamma\varepsilon^2}{\varepsilon^3} + H \leq -1,
\end{aligned}$$

which is obtained from (6.24). Thus, $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_4$.

Case 5. If $(S,I,R,C) \in \mathscr{D}_5$, one may have

$$\begin{aligned}\mathscr{L}\tilde{V} \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]S^{\rho+1} + Q(c_3+1)\xi I \\ & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]I^{\rho+1} + 3\eta \\ & +\gamma + \beta + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2} \\ \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]\frac{1}{\varepsilon^{\rho+1}} + H \leq -1,\end{aligned}$$

which is obtained from (6.25). Therefore, one obtains $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_5$.

Case 6 If $(S,I,R,C) \in \mathscr{D}_6$, one gets

$$\begin{aligned}\mathscr{L}\tilde{V} \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]I^{\rho+1} + Q(c_3+1)\xi I \\ & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]I^{\rho+1} + 3\eta \\ & +\gamma + \beta + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2} \\ \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]\frac{1}{\varepsilon^{\rho+1}} + H \leq -1,\end{aligned}$$

which is obtained from (6.25). Hence, $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_6$.

Case 7. If $(S,I,R,C) \in \mathscr{D}_7$, then

$$\begin{aligned}\mathscr{L}\tilde{V} \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]R^{\rho+1} + Q(c_3+1)\xi I \\ & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]I^{\rho+1} + 3\eta \\ & +\gamma + \beta + 2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2} \\ \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]\frac{1}{\varepsilon^{2\rho+2}} + H \leq -1,\end{aligned}$$

which is obtained from (6.26). Hence, $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_7$.

Case 8. If $(S,I,R,C) \in \mathscr{D}_8$, one can see that

$$\begin{aligned}\mathscr{L}\tilde{V} \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]C^{\rho+1} + Q(c_3+1)\xi I \\ & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]I^{\rho+1} + 3\eta \\ & +\gamma+\beta+2\xi I + A + \frac{v_1^2}{2} + \frac{v_3^2}{2} + \frac{v_4^2}{2} \\ \leq & -\frac{1}{4}\Big[\eta - \frac{\rho}{2}(v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2)\Big]\frac{1}{\varepsilon^{3\rho+3}} + H \leq -1,\end{aligned}$$

which is obtained from (6.27). Therefore, $\mathscr{L}\mathscr{V} \leq -1$ for any $(S,I,R,C) \in D_8$.

Clearly, condition (ii) of Lemma 5.3.1 holds. Therefore, the System(6.2) identifies a unique stationary distribution $\pi(.)$.

6.5 Extinction

In order to show the extinction of the disease, one may go through the following Lemma.

Lemma 6.5.1. *(See Lemmas 2.1 and 2.2 in [146]) Let $(S(t),I(t),R(t),C(t))$ be the solution of (6.2) with any $(S(0),I(0),R(0),C(0)) \in \mathbb{R}^4_+$, then*

$$\lim_{t\to\infty}\frac{S(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{I(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{R(t)}{t} = 0, \quad \lim_{t\to\infty}\frac{C(t)}{t} = 0, \quad a.s.$$

Furthermore, if $\eta > \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2}$, then

$\lim_{t\to\infty}\frac{\int_0^t S(s)dW_1(s)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t I(s)dW_2(s)}{t} = 0, \quad \lim_{t\to\infty}\frac{\int_0^t R(s)dW_3(s)}{t} = 0,$

$\lim_{t\to\infty}\frac{\int_0^t C(s)dW_4(s)}{t} = 0, \quad a.s.$

Theorem 6.5.2. *If $\mathscr{R}_0^s < 1$ and $\eta > \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2}$ then the solution of (6.2) satisfies the following* $\limsup_{t\to\infty}\frac{1}{t}\ln(\alpha(I(t)+C(t))+(\eta+\alpha)R(t)) \leq \xi - \frac{1}{2(\alpha)^2}\Big\{\alpha^2\frac{v_2^2}{2} \wedge (\eta(\eta+\alpha+\gamma)+(\eta+\alpha)^2\frac{v_3^2}{2}) \wedge \alpha^2(\eta+\beta+\frac{v_4^2}{2})\Big\} < 0$ *and* $\lim_{t\to\infty}\langle S\rangle = 1 \quad a.s.$

Proof. Define $U(t) = \alpha(I(t)+C(t)) + (\eta+\alpha)R(t)$, taking Ito's formula, one can get

$$
\begin{aligned}
d\ln U(t) = & \Big\{ \frac{1}{\alpha(I+C)+(\eta+\alpha)R} \Big[\alpha\xi SI(t-\tau) - \alpha(\eta+\beta)C - (\eta^2+\eta\alpha+\eta\gamma)R\Big] \\
& - \frac{\Big[\alpha^2 v_2^2 I^2 + (\eta+\alpha)^2 v_3^2 R^2 + \alpha^2 v_4^2 C^2\Big]}{2(\alpha(I+C)+(\eta+\alpha)R)^2} \Big\} dt + \frac{\alpha v_2 I}{\alpha(I+C)+(\eta+\alpha)R} dW_2(t) \\
& + \frac{(\eta+\alpha)v_3 R}{\alpha(I+C)+(\eta+\alpha)R} dW_3(t) + \frac{\alpha v_4 C}{\alpha(I+C)+(\eta+\alpha)R} dW_4(t), \\
\leq & \xi S dt - \frac{1}{(\alpha(I+C)+(\eta+\alpha)R)^2} \Big\{ \alpha^2 \frac{v_2^2}{2} I^2 + \alpha^2(\eta+\beta+\frac{v_4^2}{2})C^2 \\
& + (\eta(\eta+\alpha+\gamma) + (\eta+\alpha)^2 \frac{v_3^2}{2})R^2 \Big\} dt + \frac{\alpha v_2 I}{\alpha(I+C)+(\eta+\alpha)R} dW_2(t) \\
& + \frac{(\eta+\alpha)v_3 R}{\alpha(I+C)+(\eta+\alpha)R} dW_3(t) + \frac{\alpha v_4 C}{\alpha(I+C)+(\eta+\alpha)R} dW_4(t), \\
\leq & \xi S dt - \frac{1}{2(\alpha)^2} \Big\{ \alpha^2 \frac{v_2^2}{2} \wedge (\eta(\eta+\alpha+\gamma) + (\eta+\alpha)^2 \frac{v_3^2}{2}) \wedge \alpha^2(\eta+\beta+\frac{v_4^2}{2}) \Big\} dt \\
& + \frac{\alpha v_2 I}{\alpha(I+C)+(\eta+\alpha)R} dW_2(t) + \frac{(\eta+\alpha)v_3 R}{\alpha(I+C)+(\eta+\alpha)R} dW_3(t) \\
& + \frac{\alpha v_4 C}{\alpha(I+C)+(\eta+\alpha)R} dW_4(t). \qquad (6.28)
\end{aligned}
$$

From Model (6.2), one gets

$$
\begin{aligned}
d(S(t)+I(t)+R(t)+C(t)) = & \Big[\eta - \eta(S(t)+I(t)+R(t)+C(t))\Big] dt + v_1 S(t) dW_1(t) \\
& + v_2 I(t) dW_2(t) + v_3 R(t) dW_3(t) + v_4 C(t) dW_4(t). \qquad (6.29)
\end{aligned}
$$

Taking integration of (6.29) from 0 to t, one obtains

$$\langle S(t)+I(t)+R(t)+C(t) \rangle = 1 + \psi_1(t), \quad \text{where} \qquad (6.30)$$

$$\begin{aligned}\psi_1(t) = \frac{1}{\eta}\Big[&\frac{1}{t}(S(0)+I(0)+R(0)+C(0)) - \frac{1}{t}(S(t)+I(t)+R(t)+C(t))\\ &+\frac{\nu_1\int_0^t S(s)dW_1(s)}{t} + \frac{\nu_2\int_0^t I(s)dW_2(s)}{t} + \frac{\nu_3\int_0^t R(s)dW_3(s)}{t}\\ &+\frac{\nu_4\int_0^t C(s)dW_4(s)}{t}\Big].\end{aligned} \tag{6.31}$$

By Lemmas 4.2.1 and 6.5.1, one can easily obtain that $\lim_{t\to\infty}\psi_1(t)=0$ a.s. Therefore, by taking the superior limit on both sides of (6.30), one may have

$$\lim_{t\to\infty}\sup\langle S(t)+I(t)+R(t)+C(t)\rangle \quad = 1 \quad a.s. \tag{6.32}$$

Integrating (6.28) from 0 to t, one obtains

$$\begin{aligned}\frac{\ln U(t)}{t} \leq \xi - \frac{1}{2(\alpha)^2}\Big\{&\alpha^2\frac{\nu_2^2}{2}\wedge(\eta(\eta+\alpha+\gamma)\\ &+(\eta+\alpha)^2\frac{\nu_3^2}{2})\wedge\alpha^2(\eta+\beta+\frac{\nu_4^2}{2})\Big\}+\psi_2(t), \quad \text{where}\end{aligned} \tag{6.33}$$

$$\begin{aligned}\psi_2(t) = \frac{\ln U(0)}{t} &+ \frac{\alpha\nu_2}{t}\int_0^t\Big(\frac{I(s)}{\alpha(I(s)+C(s))+(\eta+\alpha)R(s)}dW_2(s)\Big)\\ &+\frac{(\eta+\alpha)\nu_3}{t}\int_0^t\Big(\frac{R(s)}{\alpha(I(s)+C(s))+(\eta+\alpha)R(s)}dW_3(s)\Big)\\ &+\frac{\alpha\nu_4}{t}\int_0^t\Big(\frac{C(s)}{\alpha(I(s)+C(s))+(\eta+\alpha)R(s)}dW_4(s)\Big).\end{aligned}$$

In the same manner; by Lemmas 4.2.1 and 6.5.1, one gets $\lim_{t\to\infty}\psi_2(t)=0$ a.s. Since $\mathscr{R}_0^s<1$, therefore, by taking the superior limit of both sides of (6.33), one obtains

$$\begin{aligned}\lim_{t\to\infty}\sup\frac{\ln U(t)}{t} \leq \xi - \frac{1}{2(\alpha)^2}\Big\{&\alpha^2\frac{\nu_2^2}{2}\wedge(\eta(\eta+\alpha+\gamma)\\ &+(\eta+\alpha)^2\frac{\nu_3^2}{2})\wedge\alpha^2(\eta+\beta+\frac{\nu_4^2}{2})\Big\}<0,\end{aligned} \tag{6.34}$$

which implies that $\lim_{t\to\infty}I(t)=0, \lim_{t\to\infty}R(t)=0, \lim_{t\to\infty}C(t)=0.$ a.s., which confirms that the disease I can die out with probability one. It is easy, by using (6.32) and (6.34), to show that $\lim_{t\to\infty}\langle S\rangle = 1 \quad a.s.$

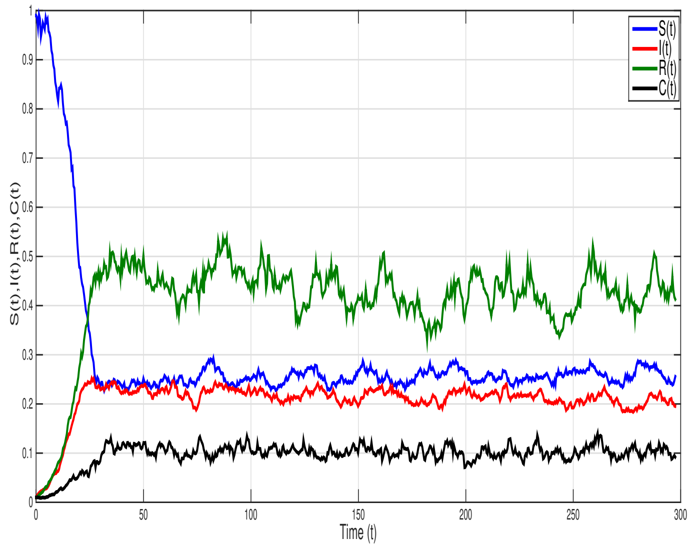

Figure 6.2: Numerical simulations of stochastic Model (6.2), when $\mathscr{R}_0^s = 1.3 > 1$. With $\eta = 0.09, \xi = 1.3, \beta = 0.05, \sigma = 0.9, \alpha = 0.36, \gamma = 0.1; \tau = 1$ and white noises $\nu_1 = 0.1, \nu_2 = 0.09, \nu_3 = 0.09, \nu_4 = 0.07$. The model has a unique ergodic stationary distribution and the infection is persistent

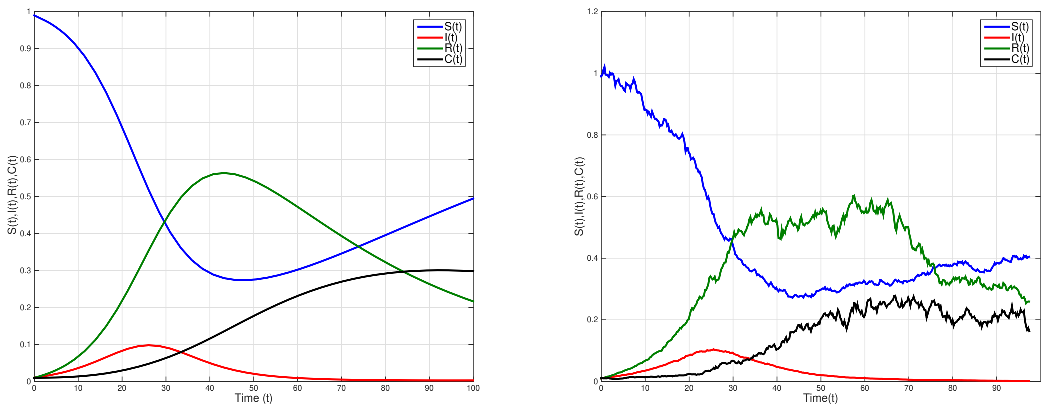

Figure 6.3: Time domain behaviors of solutions of SDDEs Model (6.2) (right) and the corresponding deterministic Model (6.1) (left), when $\mathscr{R}_0^s = 0.38 < 1$. With $\eta = 0.0005, \xi = 0.6, \beta = 0.01, \sigma = 0.12, \alpha = 0.3, \gamma = 0.02; \tau = 1.4$ and white noises $\nu_1 = \nu_2 = 0.02, \nu_3 = 0.01, \nu_4 = 0.02$. The infection dies out with probability one

6.6 Numerical Simulations and Discussions

Numerical simulations are given to validate the theoretical results, through Euler-Maruyama method for SDDEs, reported in [22, 91], to numerically solve SDDEs (6.2).

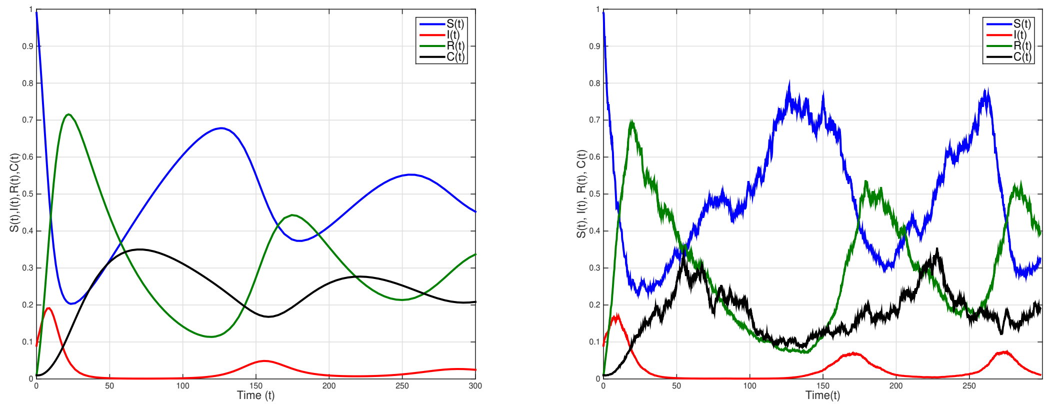

Figure 6.4: Time domain behaviors of SDDEs Model (6.2) (right) and corresponding deterministic Model (6.1) (left), the Figure shows a periodic outbreak due to the time-delay τ. When $\mathscr{R}_0^s = 0.38 < 1$, with $\eta = 0.0005, \xi = 0.6, \beta = 0.01, \sigma = 0.12, \alpha = 0.3, \gamma = 0.02; \tau = 2.5$ and white noises $v_1 = 0.02, v_2 = 0.2, v_3 = 0.02, v_4 = 0.2$

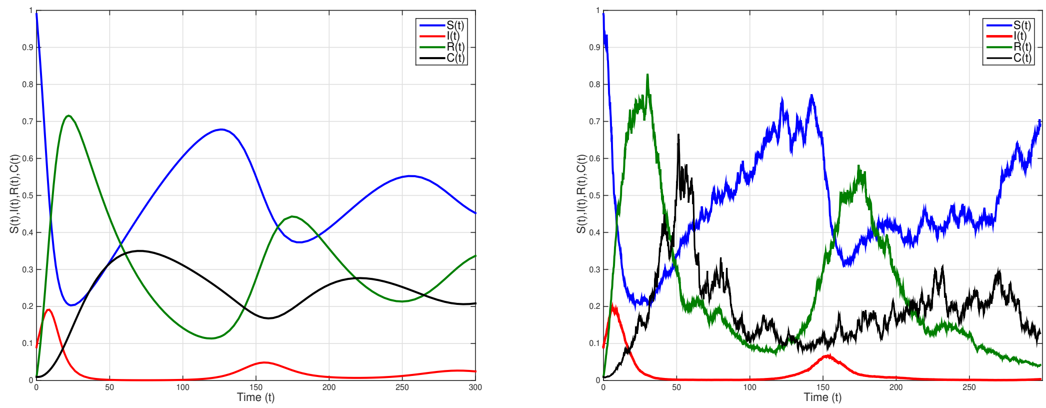

Figure 6.5: Simulations of stochastic Model (6.2) (right) and the corresponding deterministic Model (6.1) (left), when $\mathscr{R}_0^s = 0.38 < 1$. With $\eta = 0.0005, \xi = 0.6, \beta = 0.01, \sigma = 0.12, \alpha = 0.3, \gamma = 0.02; \tau = 2.5$ and white noises $v_1 = 0.2, v_2 = 0.2, v_3 = 0.1, v_4 = 0.2$. The deterministic model shows a periodic outbreak due to the time-delay τ. The infection dies out with time when white noise is large

The discretization transformation takes the form

$$\begin{aligned}
S_{j+1} =& S_j + [\eta(1-S_j) - \xi S_j I_{j-m} + \beta C_j]\Delta t + v_1 S_j \sqrt{\Delta t}\zeta_{1,j}, \\
I_{j+1} =& I_j + [\xi S_j I_{j-m} + \sigma \xi C_j I_j - (\eta+\alpha) I_j]\Delta t + v_2 I_j \sqrt{\Delta t}\zeta_{2,j}, \\
R_{j+1} =& R_j + [(1-\sigma)\xi C_j I_j + \alpha I_j - (\eta+\gamma) R_j]\Delta t + v_3 R_j \sqrt{\Delta t}\zeta_{3,j}, \\
C_{j+1} =& C_j + [\gamma R_j - \xi C_j I_j - (\eta+\beta) C_j]\Delta t + v_4 C_j \sqrt{\Delta t}\zeta_{4,j}.
\end{aligned} \tag{6.35}$$

The independent Gaussian random variables denoted as $\zeta_{i,j}, (i = 1,2,3,4)$, which follow the distribution $N(0,1)$, the time delay defines as $\tau = m\Delta t$, m is an integer and the step

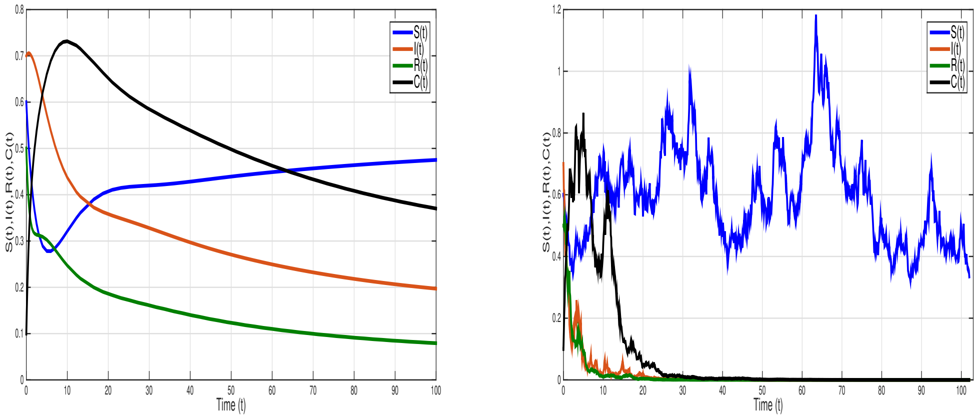

Figure 6.6: Time domain behaviors of SDDEs Model (6.2) (right) and corresponding deterministic Model (6.1) (left), where $\tau = 1$, when $\mathscr{R}_0 = 1.78 > 1$. The infection persists in the deterministic model; when $\mathscr{R}_0^s = 0.75 < 1$, the infection dies out in the stochastic model. With parameter values $\eta = 0.02, \xi = 0.5, \beta = 0.1, \sigma = 0.2, \alpha = 0.26, \gamma = 1$, and white noises $\nu_1 = 0.13,\ \nu_2 = 0.54,\ \nu_3 = 0.26,\ \nu_4 = 0.75$

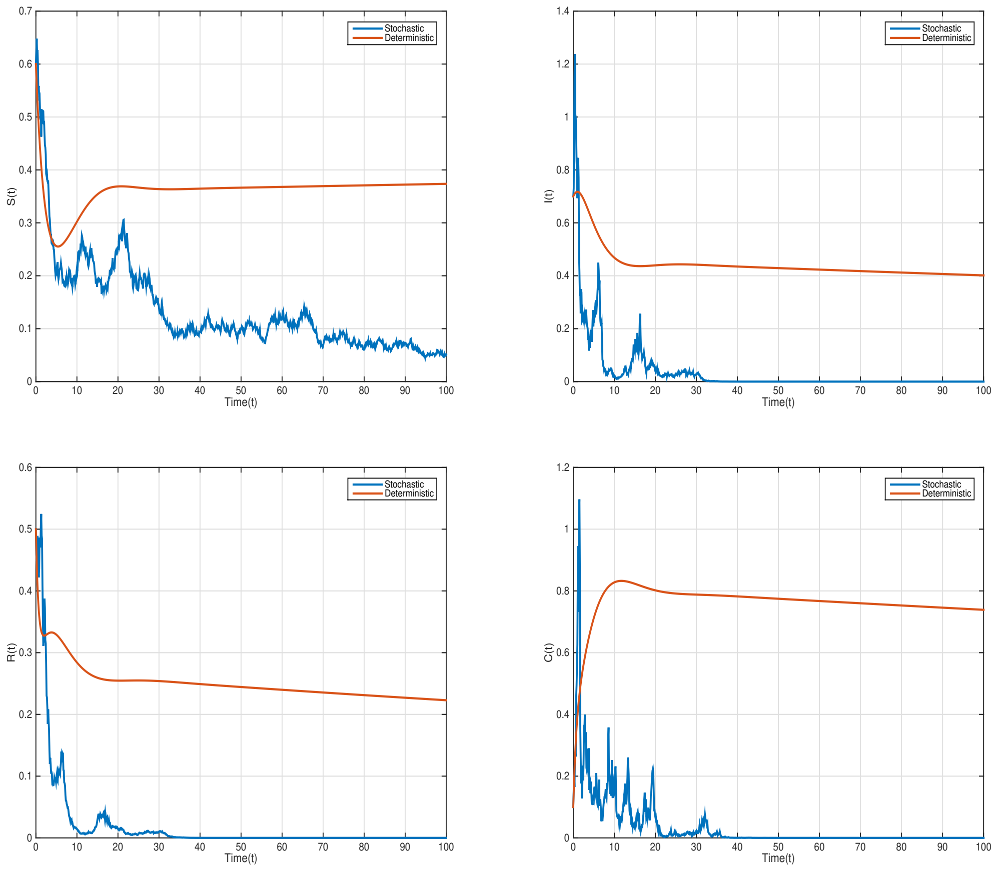

Figure 6.7: Time response of solutions for Model (6.2) (right) and corresponding deterministic Model (6.1) (left), when $\tau = 0$. Such that $\mathscr{R}_0 = 1.78 > 1$, the infection persists in the deterministic model; when $\mathscr{R}_0^s = 0.75 < 1$, the infection dies out in the stochastic model. With parameter values $\eta = 0.02, \xi = 0.5, \beta = 0.1, \sigma = 0.2, \alpha = 0.26, \gamma = 1$, and white noises $\nu_1 = 0.13,\ \nu_2 = 0.54,\ \nu_3 = 0.26,\ \nu_4 = 0.75$.

size Δt. Let $v_i > 0, (i = 1, 2, 3, 4)$ be the white noise values.

Example 6.6.1. Consider Model (6.2), with white noise values: $v_1 = 0.1,\ v_2 = 0.09,\ v_3 = 0.09,\ v_4 = 0.07$, and parameter values: $\eta = 0.09, \xi = 1.3, \beta = 0.05, \sigma = 0.9, \gamma = 0.1, \alpha = 0.36, \tau = 1.2$. Simple calculation leads to $\mathscr{R}_0^s = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\eta}\hat{\alpha}\hat{\gamma}\hat{\beta}} = 1.3 > 1$, and $\eta - \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2} = 0.087 > 0$. Therefore, the conditions of Theorem 6.4.1 hold. Based on Theorem 6.4.1, there is a unique ergodic stationary distribution $\pi(.)$ of Model (6.2). Thus, the disease I is persistent; See Figure 6.2.

Example 6.6.2. Given the Model (6.2), with parameters values: $\eta = 0.0005; \xi = 0.6; \beta = 0.01; \sigma = 0.12; \alpha = 0.3; \gamma = 0.02, \tau = 1.4$,and white noises: $v_1 = 0.02, v_2 = 0.02, v_3 = 0.01, v_4 = 0.2$. One obtains $\mathscr{R}_0^s = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\eta}\hat{\alpha}\hat{\gamma}\hat{\beta}} = 0.38 < 1$, and $\eta - \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2} = -0.0195 < 0$. In this case, the conditions of Theorem 6.4.1 are not satisfied. From Figure 6.3, one can clearly find that the disease goes to extinction. In Figure 6.4 time-delay is increased to $\tau = 2.5$, with white noises $v_1 = 0.01, v_2 = 0.2, v_3 = 0.02, v_4 = 0.03$, other parameter values are the same as in Figure 6.3. Therefore $\mathscr{R}_0^s < 1$, and $\eta - \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2} = -0.0445 < 0$. The conditions of Theorem 6.4.1 are not satisfied. Figure 6.5 shows a periodic outbreak due to the time-delay τ. However, the infection dies out with time with bigger white noise.

Example 6.6.3. To further explain the impact of time-delay and white noises on System(6.2), choose $\tau = 2.5$ and parameter values: $\eta = 0.0005; \xi = 0.6; \beta = 0.01; \sigma = 0.12; \alpha = 0.3; \gamma = 0.02$, and white noises $v_1 = 0.2, v_2 = 0.2, v_3 = 0.1, v_4 = 0.3$. Such that, $\mathscr{R}_0^s = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\eta}\hat{\alpha}\hat{\gamma}\hat{\beta}} = 0.38 < 1$, and $\eta - \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2} = -0.045 < 0$. Thus, the conditions of Theorem 6.4.1 are not satisfied. Figure 6.5 shows a periodic outbreak due to the time-delay τ, when the white noise increased the periodicity of the outbreak decreased. The infection dies out with time as white noise increases.

Example 6.6.4. In order to show the impact of random perturbation. With $\tau = 1$, and by increasing the white noise values to $v_1 = 0.13,\ v_2 = 0.54,\ v_3 = 0.26,\ v_4 = 0.75$, with parameter values $\eta = 0.02; \xi = 0.5; \beta = 0.1; \sigma = 0.2; \alpha = 0.26; \gamma = 1$. Thus, $\mathscr{R}_0^s = \frac{\eta\gamma\xi^2(1-\sigma)}{\hat{\eta}\hat{\alpha}\hat{\gamma}\hat{\beta}} = 0.75 < 1 < 1.78 = \frac{\xi}{\alpha+\eta} = \mathscr{R}_0$, and $\eta - \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2} = 0.0115 > 0$. Therefore, the conditions of Theorem 6.5.2 hold, and disease dies out exponentially with probability one. However, the disease persists with deterministic model; See Figure 6.6.

Example 6.6.5. Consider the same parameter values of Example 6.6.4, but with time-delay $\tau = 0$. Thus, according to Theorem 6.5.2 the disease dies out exponentially with probability one; See Figure 6.7. Therefore, the smaller values of white noise ensure the existence of unique stationary distribution, which gives the persistence of the disease; While larger values of white noise can lead to disease extinction.

Remark 6.6.1. Given the deterministic SIRC Model (6.1), if the basic reproduction number $\mathscr{R}_0 = \dfrac{\xi}{\alpha + \eta} < 1$, then the disease-free equilibrium point is globally asymptotically stable; Whereas, if $\mathscr{R}_0 > 1$, the unique endemic equilibrium point is globally asymptotically stable. Repeated outbreaks of the infection can occur due to the time-delay in the transmission terms. In the stochastic SIRC Model (6.2), if $\mathscr{R}_0^s = \dfrac{\eta\gamma\xi^2(1-\sigma)}{\hat{\eta}\hat{\alpha}\hat{\gamma}\hat{\beta}} < 1 < \mathscr{R}_0$, and $\eta > \frac{v_1^2 \vee v_2^2 \vee v_3^2 \vee v_4^2}{2}$ the stochastic Model (6.2) has disease extinction with probability one, and for $\mathscr{R}_0^s > 1$, the stochastic Model (6.2) has a unique ergodic stationary distribution. See Figures 6.6 and 6.7.

6.7 Conclusion

In this chapter, a stochastic SIRC epidemic model with time-delay for the new strain coronavirus COVID-19 has been provided. The stochastic components, due to environmental variability, are incorporated in the model as Gaussian white noise. Some sufficient conditions for persistence and extinction in the mean of the disease have been established. The model has a unique stationary distribution which is ergodic if the intensity of white noise is small. Introduction of noise in the deterministic SIRC model modifies the basic reproductive number $\mathscr{R}_0$ giving rise to a new threshold quantity $\mathscr{R}_0^s$. It has been proved that the disease dies out if $\mathscr{R}_0^s < 1 < \mathscr{R}_0$. On the other hand, if $\mathscr{R}_0^s > 1$ and $\mathscr{R}_0 > 1$, the disease persists with both models, but with different behaviors. In other words, extinction of the infection possibly occurs when $\mathscr{R}_0^s < 1 < \mathscr{R}_0$, along with intensity of white noise is large. This would not happen in the deterministic models. The potential of using stochastic SIRC model for COVID-19 is to consider the environmental fluctuation that all affects the spread of the virus. Periodicity of the outbreaks is possible due to the presence of time-delay (memory) in the transmission terms.

The author believes that the stochastic SIRC model is an attempt to understand epidemiological characteristics of COVID-19. The model provides new insights into epidemiological situations when the environmental noise (perturbations) and cross-immunity are considered in the COVID-19 epidemic models. The combination of white noise and time-delay, in the epidemic model, has a considerable impact on the persistence and extinction of the infection and enriches the dynamics of the model. This work can be extended to include control variables for a vaccination, treatment and/or quarantine actions. More sophisticated model is also required to investigate the dynamics of COVID-19 with immune system in cells level [119].

In the next chapter, some sufficient numerical schemes for stochastic delay differential equations is introduced.

Chapter 7: Numerical Schemes for Stochastic Delay Differential Equations

7.1 Introduction

Stochastic delay differential equations (SDDEs) are very important in ecology, epidemiology and many of other fields. This chapter introduces some numerical approaches for the derivation of discrete time approximations for solutions of SDDEs. The proposed schemes converge in a strong sense. Section 2 provides some required preliminaries. Section 3 introduces a numerical scheme for an autonomous SDDE and investigate local and global errors; convergence and consistency of the scheme. Section 4 discusses strong discrete time approximations of solutions of non-autonomous SDDEs, including Euler and Taylor schemes and implicit schemes. The mean square stability of Milstein scheme is discussed in Section 5. Concluding remarks are given in Section 6.

SDDEs are considered as generalization of both deterministic delay differential equations (DDEs) and stochastic ordinary differential equations (SODEs). Some basic consents about stochastic differential equations are discussed in [103, 121, 138]. The fundamental theory of existence and uniqueness of the solution of SDDEs has been studied by Mao [90] and Mohammed [99]. Some stability properties of numerical schemes of SDDEs are also studied in [52, 73, 91]. In the literature, some numerical schemes for SDDEs have been investigated, such as Euler-type schemes [10, 75], drift-implicit Euler scheme [59, 83], Milstein schemes [24, 57], split-step schemes [50, 139], and additionally multi-step schemes [25]. The extension of numerical approaches for SODEs to SDDEs is non-trivial, particularly since the time-delays may induce instabilities in the basic SDDEs; while its corresponding SODEs are stable [59]. In addition, the presence of time-delays influences on the convergence order and computational complexity of the numerical schemes [28]. In general, there is no analytical closed-form solution of the models considered in this dissertation, and usually numerical techniques are required to investigate the models quantitatively.

7.2 Preliminaries

Consider the d-dimensional SDDEs with r-dimensional standard Wiener processes given on the filtered probability space $(\Omega, \mathscr{A}, \mathscr{A}_{t_0}, \mathbb{P})$. Therefore, one may have equations of the form

$$\begin{aligned} d\mathbf{y}(t) &= \underbrace{\mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau))}_{\text{drift coefficient}} dt + \underbrace{\sum_{j=1}^{r} \mathbf{g}_j(t, \mathbf{y}(t), \mathbf{y}(t-\tau))\, d\mathbf{W}_j(t)}_{\text{diffusion coefficient}}, \quad t \in [0, T], \\ \mathbf{y}(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7.1}$$

With one fixed delay τ, where $\psi(t)$ is an $\mathscr{A}_{t_0}$-measurable $\mathscr{C}([-\tau, 0], \mathbb{R}^d)$-valued random variable. The drift coefficient $\mathbf{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient $\mathbf{g}_j : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $j = 1, 2, \ldots, r$, are given d-dimensional. Equation (7.1) can be formulated as

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{f}(s, \mathbf{y}(s), \mathbf{y}(s-\tau)) ds + \sum_{j=1}^{r} \int_0^t \mathbf{g}(s, \mathbf{y}(s), \mathbf{y}(s-\tau)) d\mathbf{W}_j(s), \tag{7.2}$$

for $t \in [0, T]$ and with $\mathbf{y}(t) = \psi(t)$, for $t \in [-\tau, 0]$.

Definition 7.2.1 (Strong solution)**.** A d-dimensional stochastic process $\mathbf{y} = \{\mathbf{y}(t) : [-\tau, T]\}$ is called a strong solution of (7.1) if it has the following properties:

- $\{\mathbf{y}(\mathbf{t})\}$ is measurable, sample continuous process and $(\mathscr{A}_t)_{0 \le t \le T}$-adapted;
- Equations (7.1) and (7.2) hold for every $t \in [0, T]$ almost surly.

Definition 7.2.2 (Path-wise unique solution)**.** Let the set $\mathscr{X}$ denotes some class of stochastic processes that solve (7.1). If any two processes $y^{(i)} = \{y^{(i)}(t), t \in [-\tau, T]\}, i = 1, 2$ from $\mathscr{X}$ with the same initial functions have the same path on $[0, T]$, almost surely, that is

$$\mathbb{P}(\sup_{0 \le t \le T} |y^{(1)}(t) - y^{(2)}(t)| > 0) = 0, \tag{7.3}$$

then the solution of (7.1) is path-wise unique within $\mathscr{X}$.

Herein, the Lipschitz condition (L_1) and Growth condition (L_2) are formulated to guarantee the existence of a unique solution of (7.1). Assuming that $|.|$ denotes the Eucilidian norm, one may have

(L_1) *Lipschitz condition*: There exists a constant $K \in (0,\infty)$, such that

$$|\mathbf{f}(t,x_1,y_1)-\mathbf{f}(t,x_2,y_2)|+|\mathbf{g}_1(t,x_1,y_1)-\mathbf{g}_1(t,x_2,y_2)+\cdots+$$
$$|\mathbf{g}_r(t,x_1,y_1)-\mathbf{g}_r(t,x_2,y_2)| \leq K(|x_2-x_1|+|y_2-y_1|),$$

for $t \in [0,T]$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}^d$.

(L_2) Growth condition: There exists a constant $G \in (0,\infty)$, such that

$$|\mathbf{f}(t,x,y)|^2+|\mathbf{g}_1(t,x,y)|^2+\cdots+|\mathbf{g}_r(t,x,y)|^2 \leq G(1+|x|^2+|y|^2),$$

for $t \in [0,T]$ and $x,y \in \mathbb{R}^d$.

Let $\mathscr{C}=\mathscr{C}([-\tau,0],\mathbb{R}^d)$ be the Banach space of all $d-$dimensional continuous functions η on $[-\tau,0]$ equipped with the sup-norm $\|\eta\|_{\mathscr{C}}=\sup_{s\in[-\tau,0]}|\eta(s)|$. For every function $\xi|[-\tau,T]\to\mathbb{R}^d$ and every $t\in[0,T]$, so that

$$\xi_t=\{a_t(s):=\xi(t+s), s\in[-\tau,0]\},$$

is a function defined on $[-\tau,0]$, which is the segment of ξ at t. In the same manner, the segment-valued function $t\to\xi_t$ for $t\in[0,T]$ is obtained. Additionally, denoting $\mathscr{L}_2(\Omega,\mathscr{C},\mathscr{A}_0)$, the set of $\mathbb{R}^d$-valued continuous processes $\eta=\{\eta(s),s\in[-\tau,0]\}$ with $\eta(s)$ being $\mathscr{A}_0$-measurable for all $s\in[-\tau,0]$ and

$$\mathbb{E}\|\eta\|^2_{\mathscr{C}}=\mathbb{E}\sup_{s\in[-\tau,0]}|\eta(s)|^2<\infty. \tag{7.4}$$

Noting that the initial function ψ can be considered as a square integrable $\mathscr{C} = \mathscr{C}([-\tau,0],\mathbb{R}^d)$-valued random variable on $(\Omega,\mathscr{A}_0,\mathbb{P})$. Hence, the above assumptions lead to the following theorem.

Theorem 7.2.1. *[75] Assume that (L_1) and (L_2) hold for both* $\mathbf{f}$ *and* $\mathbf{g}$*, where* ψ *be in* $\mathscr{L}_2(\Omega,\mathscr{C},\mathscr{A}_0)$*. Then with initial segment* ψ*, the SDDE (7.1) has a path-wise unique strong solution* $\mathbf{y} = \{\mathbf{y}(t), t \in [-\tau,T]\}$ *in* $\mathscr{L}_2(\Omega,\mathscr{C},\mathscr{A}_0)$*. Moreover,* $\mathbb{E}\Big(\sup_{t\in[-\tau,t]} |\mathbf{y}(t)|^2\Big) < \infty$*, and for each* $t \in [0,T]$*, the segment* $\mathbf{y}_t = \{\mathbf{y}(t+s), s \in [-\tau,0]\}$ *is a* $\mathscr{C}([-\tau,0],\mathbb{R}^d)$*-valued process having continuous paths. Additionally, if* $\mathbb{E}\|\psi\|_{\mathscr{C}}^{2k} < \infty$ *for some* $k \geq 1$*, then*

$$\mathbb{E}\|\mathbf{y}_t\|_{\mathscr{C}}^{2k} = \mathbb{E}\Big(\sup_{s\in[-\tau,0]} |\mathbf{y}(t+s)|^{2k}\Big) < \infty \tag{7.5}$$

and

$$\mathbb{E}\|\mathbf{y}_t\|_{\mathscr{C}}^{2k} \leq C_k[1+\mathbb{E}\|\psi\|_{\mathscr{C}}^{2k}], \tag{7.6}$$

for all $t \in [0,T]$ *and some positive constant* C_k*.*

For the proof of the above Theorem one can refer to [99]. Next numerical schemes for autonomous and non-autonomous SDDEs are provided.

7.3 Numerical Scheme for an Autonomous SDDEs

For simplicity, SDDEs (7.1) is reduced to a scalar autonomous stochastic delay differential equation of the form

$$\begin{aligned} dy(t) &= f(y(t),y(t-\tau))dt + g(y(t),y(t-\tau))dW(t), \quad t \in [0,T], \\ y(t) &= \psi(t), \quad t \in [-\tau,0]. \end{aligned} \tag{7.7}$$

Equation (7.7) can be formulated as

$$y(t) = y(0) + \int_0^t f(y(s),y(s-\tau))ds + \int_0^t g(y(s),y(s-\tau))dW(s), \tag{7.8}$$

for $t \in [0,T]$ and with $y(t) = \psi(t)$, for $t \in [-\tau, 0]$. The second integral in (7.8) is a stochastic integral in the It$\hat{o}$ sense. For problem (7.7), mesh points are defined with a uniform step on the interval $[0,T]$, so that $\Delta t = h = T/N$, $t_n = nh$, where $n = 0,\ldots,N$. Also for the given h it is assumed that there is a corresponding integer m, where the time-delay can be expressed in terms of the step-size as $\tau = mh$. For all indices $n-m \leq 0$ define $\tilde{y}_{n-m} := \psi(t_n - \tau)$, otherwise

$$\tilde{y}_{n+1} = \tilde{y}_n + \phi(h, \tilde{y}_n, \tilde{y}_{n-m}, I_\phi), \quad n = 0,\ldots,N-1. \tag{7.9}$$

The increment function $\phi(h, \tilde{y}_n, \tilde{y}_{n-m}, I_\phi) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ includes a finite number of multiple It$\hat{o}$-integrals (see [69, 98]) of the form

$$I_{(j_1,\ldots,j_l),h} = \int_t^{t+h} \int_t^{s_l} \cdots \int_t^{s_2} dW^{j1}(s_1) \ldots dW^{j_{l-1}}, (s_{l-1}) dW^{j_l}(s_l)$$

where $j_i \in \{0,1\}$ and $dW^0(t) = dt$, and with $t = t_n$ for (7.9), one can denote I_ϕ the collection of It$\hat{o}$-integrals required to compute the increment function ϕ.

Assumptions on the increment function ϕ of (7.9):

Suppose that V_1, V_2, V_3 are positive constants, such that for all $\kappa, \kappa', \omega, \omega' \in \mathbb{R}$, one may have

$$\begin{aligned} \left|\mathbb{E}\Big(\phi(h,\kappa,\omega,I_\phi) - \phi(h,\kappa',\omega',I_\phi)\Big)\right| &\leq V_1 h(|\kappa-\kappa'| + |\omega-\omega'|), \\ \mathbb{E}\Big(|\phi(h,\kappa,\omega,I_\phi) - \phi(h,\kappa',\omega',I_\phi)|^2\Big) &\leq V_2 h\Big(|\kappa-\kappa'|^2 + |\omega-\omega'|^2\Big), \quad \text{and} \end{aligned} \tag{7.10}$$

$$\mathbb{E}\Big(|\phi(h,\kappa,\omega,I_\phi)|^2\Big) \leq V_3 h\Big(1 + |\kappa|^2 + |\omega|^2\Big). \tag{7.11}$$

Lemma 7.3.1. *[10] If the increment function ϕ in Equation (7.9) satisfies condition (7.11), then $\mathbb{E}|\tilde{y}_n|^2 < \infty$ for all $n \leq N$.*

Let $y(t_{n+1})$ be the exact solution of (7.7) at the mesh point t_{n+1}, $\tilde{y}_{n+1}$ is the value of the approximate solution given by (7.9) and $\tilde{y}(t_{n+1})$ is the solution of (7.9) after just one step, so that

$$\tilde{y}(t_{n+1}) = y(t_n) + \phi(h, y(t_n), y(t_n - \tau), I_\phi). \tag{7.12}$$

7.3.1 Local and global errors

Definition 7.3.1. The local error that occurs in one step of the approximation $\{\tilde{y}_n\}$ is the sequence of random variables

$$\delta_{n+1} = y(t_{n+1}) - \tilde{y}(t_{n+1}), \quad n = 0, \ldots, N-1. \tag{7.13}$$

However, the global error is the amount of error that occurs in the use of a numerical approximation to solve a problem, which is the sequence of random variables

$$\varepsilon_n := y(t_n) - \tilde{y}_n, \quad n = 1, \ldots, N. \tag{7.14}$$

Noting that ε_n is $\mathscr{A}_{t_n}$-measurable since both $y(t_n)$ and $\tilde{y}_n$ are $\mathscr{A}_{t_n}$-measurable random variable, such that $\left(\mathbb{E}|\varepsilon_n|^2\right)^{1/2}$ is the $\mathscr{L}^2$-norm of (7.14).

7.3.2 Convergence and consistency

Definition 7.3.2. Assume that, $\delta_{n+1} = y(t_{n+1}) - \tilde{y}(t_{n+1}), \quad n = 0, \ldots, N-1$, then method (7.9) is said to be consistent with order p_1 in the mean and with order p_2 in the mean square with

$$p_2 \geq \frac{1}{2} \quad \text{and} \quad p_1 \geq p_2 + \frac{1}{2}, \tag{7.15}$$

if the estimates

$$\max_{0\le n\le N-1} |\mathbb{E}(\delta_{n+1})| \le Ch^{p_1} \quad \text{as} \quad h\to 0, \quad \text{and} \tag{7.16}$$

$$\max_{0\le n\le N-1} \left(|\mathbb{E}(\delta_{n+1})|^2\right)^{1/2} \le Ch^{p_2} \quad \text{as} \quad h\to 0, \tag{7.17}$$

hold, where the constant C does not depend on h, but may depend on T, and on the initial data.

Theorem 7.3.2. *[10] Assume that the conditions of Theorem 7.2.1 are satisfied. Such that the method defined by Equation (7.9) is consistent with order p_1 in the mean and order p_2 in the mean square sense, where p_1, p_2 fulfilling (7.15), and the increment function ϕ on Equation (7.9) satisfies the estimates (7.10). Then the approximation (7.9) for Equation (7.7) is convergent in $\mathscr{L}^2$ (as $h\to 0$ with $\tau/h\in\mathbb{N}$) with order $p = p_2 - 1/2$. That is, convergence is in the mean square sense, such that*

$$\max_{0\le n\le N-1} \left(|\mathbb{E}(\delta_{n+1})|^2\right)^{1/2} \le Ch^{p} \quad \text{as} \quad h\to 0, \tag{7.18}$$

Theorem 7.3.3. *[10] If the increment function ϕ of the approximation (7.9) satisfies the estimates (7.10), then the one-step method (7.9) is zero stable in the quadratic mean-square sense.*

Next, the analysis to non-autonomous system of SDDEs (7.1) is extended.

7.4 Numerical Schemes for Non-autonomous SDDEs

There are some specific discrete time approximations for (7.1). The simplest scheme which is defined by stochastic difference equation is represented by Euler approximation as the following

$$\tilde{\mathbf{y}}_{n+1} = \tilde{\mathbf{y}}_n + \mathbf{f}(t_n,\tilde{\mathbf{y}}_n,\tilde{\mathbf{y}}_{n-m})h + \sum_{j=1}^{r} \mathbf{g}_j(t_n,\tilde{\mathbf{y}}_n,\tilde{\mathbf{y}}_{n-m})\Delta W_n^j, \tag{7.19}$$

where $\tilde{\mathbf{y}} = \{\tilde{\mathbf{y}}(t), t \in [-\tau, T]\}$ is right continuous with left hands limits, a discrete time approximation with step-size h, such that for each $n = 0, 1, \ldots, N-1$, the random variable $\tilde{\mathbf{y}}(t_n)$ is $\mathscr{A}_{t_n}$-measurable and $\tilde{\mathbf{y}}(t_{n+1})$ can be expressed as a function of $\tilde{\mathbf{y}}(t_{-m})$, $\tilde{\mathbf{y}}(t_{-m+1})$,$\ldots$, $\tilde{\mathbf{y}}(t_n)$, discretization time t_n and a finite number of $\mathscr{A}_{t_{n+1}}$-measurable random variable. With $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$, for $n = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, r$. By more general assumptions, one can check that Euler approximation is strongly converges with order $1/2$ [75].

7.4.1 Taylor approximation

For stochastic differential equations, it is common that by application of the Wagner-Platen stochastic Taylor expansion [70], one can construct discrete time approximations that converge with a given order of strong convergence, which involve in each time step certain multiple integrals. For the general multi-dimensional case $d, r = 1, 2, \ldots$ the order-one strong Taylor approximation has the form

$$\begin{aligned}
\tilde{\mathbf{y}}_{n+1} &= \tilde{\mathbf{y}}_n + \mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})h + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j \\
&+ \sum_{j_1, j_2=1}^{r} \sum_{i=1}^{d} g_{i,j_1}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \frac{\partial}{\partial \tilde{y}_n^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\
&\times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2) dW^{j_2}(s_1) \\
&+ \sum_{j_1, j_2=1}^{r} \sum_{i_1=1}^{d} g_{i,j_1}(t_{n-m}, \tilde{y}_{n-m}, \tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\
&\times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2 - \tau) dW^{j_2}(s_1),
\end{aligned} \tag{7.20}$$

for $n = 0, 1, \ldots, N-1$, $i = 1, 2, \ldots, d$. One can check that approximation (7.20) converges under suitable assumptions with strong order one [75]. In the one-dimensional case when $\tau = 0$ scheme (7.20) coincides with the well-known Milstien Scheme for SDEs. However, the time delay in (7.20) generates an extra term which describes a double Wiener integral that integrates an earlier segment of the Wiener path with respect to the actual Wiener path.

7.4.2 Implicit strong approximations

In practice, explicit schemes have smaller computational costs but with lower accuracy compared to implicit methods. It is sometimes recommended to use implicit schemes to have numerically stable approximate solutions for SDDEs, as in the case of stiff problem[1]. For the general multi-dimensional case (7.1), the family of implicit Euler approximations are as the following

$$\begin{aligned}\tilde{\mathbf{y}}_{n+1} &= \tilde{\mathbf{y}}_n + [\theta \mathbf{f}(t_{n+1}, \tilde{\mathbf{y}}_{n+1}, \tilde{\mathbf{y}}_{n-m+1}) + (1-\theta)\mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})]h \\ &\quad + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j,\end{aligned} \tag{7.21}$$

for $n = 0, 1, \ldots, N-1$, such that $\theta \in [0,1]$ stands for the degree of implicitness. If $\theta = 0$, one may have the explicit Euler approximation (7.19). For $\theta = 1$, one obtains the fully implicit Euler approximation. The approximation (7.21) converge with strong order $1/2$ [83]. In the same manner one can establish an order-one strong implicit Taylor approximation with

$$\begin{aligned}\tilde{\mathbf{y}}_{n+1} &= \tilde{\mathbf{y}}_n + [\theta \mathbf{f}(t_{n+1}, \tilde{\mathbf{y}}_{n+1}, \tilde{\mathbf{y}}_{n-m+1}) + (1-\theta)\mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})]h \\ &\quad + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j + \sum_{j_1,j_2=1}^{r} \sum_{i=1}^{d} g_{i,j_1}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \frac{\partial}{\partial \tilde{y}_n^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\ &\quad \times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2) dW^{j_2}(s_1) \\ &\quad + \sum_{j_1,j_2=1}^{r} \sum_{i=1}^{d} g_{i,j_1}(t_{n-m}, \tilde{y}_{n-m}, \tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\ &\quad \times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2 - \tau) dW^{j_2}(s_1).\end{aligned} \tag{7.22}$$

Next, some details about the mean square stability of Milstein method will be given, since this scheme has been used in the numerical simulations for SDDEs models through the

[1] A stiff problem is defined as that in which the global accuracy of the numerical solution is determined by stability rather than local error and implicit methods are more appropriate for it.

dissertation.

7.5 Milstien Scheme for SDDEs

In this section, Milstein scheme is introduced for SDDE and one can show that the numerical method is mean square stable under suitable conditions.

Given the one-dimensional version of (7.1), $r = d = 1$, of the following form

$$\begin{aligned} dy(t) &= f(t,y(t),y(t-\tau))dt + g(t,y(t),y(t-\tau))dW, \quad t \in [0,T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7.23}$$

Order one strong Taylor approximation for (7.23) in the one-dimensional case, is defined by

$$\begin{aligned} \tilde{y}_{n+1} = \tilde{y}_n &+ f(t_n,\tilde{y}_n,\tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} ds_1 + g(t_n,\tilde{y}_n,\tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} dW(s_1) \\ &+ g(t_n,\tilde{y}_n,\tilde{y}_{n-m}) \frac{\partial}{\partial \tilde{y}_n} g(t_n,\tilde{y}_n,\tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW(s_2)dW(s_1) \\ &+ g(t_{n-m},\tilde{y}_{n-m},\tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}} g(t_n,\tilde{y}_n,\tilde{y}_{n-m}) \\ &\times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW(s_2-\tau)dW(s_1). \end{aligned} \tag{7.24}$$

Once the Taylor approximation is considered, Milstein scheme can be constructed for (7.23).

$$\begin{aligned} \tilde{y}_{n+1} = \tilde{y}_n &+ f(t_n,\tilde{y}_n,\tilde{y}_{n-m})h + g(t_n,\tilde{y}_n,\tilde{y}_{n-m})\Delta W_n \\ &+ \frac{1}{2} g(t_n,\tilde{y}_n,\tilde{y}_{n-m}) g'(t_n,\tilde{y}_n,\tilde{y}_{n-m})[(\Delta W_n)^2 - h] \\ &+ g(t_{n-m},\tilde{y}_{n-m},\tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}} g(t_n,\tilde{y}_n,\tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW(s_2-\tau)dW(s_1), \end{aligned} \tag{7.25}$$

7.5.1 Convergence and mean square stability of Milstein scheme

Consider the linear scalar SDDE of the form

$$\begin{aligned} dy(t) &= [\rho_0 y(t) + \rho_1 y(t-\tau)]dt + [\rho_2 y(t) + \rho_3 y(t-\tau)]dW(t), \quad t \in [0,T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7.26}$$

Where $\rho_0, \rho_1, \rho_2, \rho_3 \in \mathbb{R}$, $W(t)$ is an one-dimensional standard Wiener process, and $\psi(t)$ is continuous and bounded function with $\mathbb{E}[\|\psi\|^2] < \infty$, where $\|\psi\| = \sup_{-\tau \le t \le 0} |\psi(t)|$.

Theorem 7.5.1. *([75]) Suppose that*

$$\rho_0 < -|\rho_1| - \frac{(|\rho_2| + |\rho_3|)^2}{2}, \tag{7.27}$$

then the solution of (7.26) satisfies $\lim_{t\to\infty} \mathbb{E}[|y(t)|^2] = 0$*, i.e. the solution is mean square stable.*

Using order 1 strong Taylor approximation formula to the linear one delay System (7.26), one gets

$$\begin{aligned} y_{n+1} = & y_n + (\rho_0 y_n + \rho_1 y_{n-m})h + (\rho_2 y_n + \rho_3 y_{n-m})\Delta W_n \\ & + \rho_3(\rho_2 y_{n-m} + \rho_3 y_{n-2m})I_1 + \rho_2(\rho_2 y_n + \rho_3 y_{n-m})I_2, \end{aligned} \tag{7.28}$$

where y_n is an approximation to $y(t_n)$, such that

$$I_1 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW(t-\tau)dW(s), \quad I_2 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW(t)dW(s).$$

The convergence order of (7.28) can obtained by Theorem 10.2 in [75], since the coefficients of (7.28) are satisfy Lipschitz condition and growth condition. Thus, Milstein scheme (7.28) is strongly convergent of order 1.

Theorem 7.5.2. *[133] The Milstein scheme (7.28) is mean square stable, if condition*

(7.27) satisfied.

Proof. By reorganizing the terms of (7.28), one gets

$$\begin{aligned} y_{n+1} =&(1+\rho_0 h+\rho_2\Delta W_n)y_n+(\rho_1 h+\rho_3\Delta W_n)y_{n-m}+\rho_3(\rho_2 y_{n-m}+\rho_3 y_{n-2m})I_1 \\ &+\rho_2(\rho_2 y_n+\rho_3 y_{n-m})I_2. \end{aligned} \tag{7.29}$$

Squaring both sides of (7.29), then it follows from $2ab \le a^2+b^2$ ($\forall a,b\in\mathbb{R}$), one may have

$$\begin{aligned} y_{n+1}^2 \le&(1+\rho_1 h+\rho_2\Delta W_n)^2 y_n^2+(\rho_1 h+\rho_3\Delta W_n)^2 y_{n-m}^2 \\ &+\rho_2^2[(\rho_2^2+|\rho_2\rho_3|)y_n^2+(\rho_3^2+|\rho_2\rho_3|)y_{n-m}^2]I_2^2 \\ &+\rho_3^2[(\rho_2^2+|\rho_2\rho_3|)y_{n-m}^2+(\rho_3^2+|\rho_2\rho_3|)y_{n-2m}^2]I_1^2+|1+\rho_0 h||\rho_1|h(y_n^2+y_{n-m}^2) \\ &+|\rho_2\rho_3|\Delta W_n^2(y_n^2+y_{n-m}^2)+2[(1+\rho_0 h)\rho_3+\rho_1\rho_2 h]\Delta W_n y_n y_{n-m} \\ &+2\rho_2\rho_3(\rho_2 y_n+\rho_3 y_{n-m})(\rho_2 y_{n-m}+\rho_3 y_{n-2m})I_1 I_2 \\ &+2\rho_2(1+\rho_0 h+\rho_2\Delta_n)(\rho_2 y_n+\rho_3 y_{n-m})y_n I_2 \\ &+2\rho_3(1+\rho_0 h+\rho_2\Delta_n)(\rho_2 y_{n-m}+\rho_3 y_{n-2m})y_n I_1 \\ &+2\rho_2(\rho_1 h+\rho_3\Delta W_n)(\rho_2 y_n+\rho_3 y_{n-m})y_{n-m}I_2 \\ &+2\rho_3(\rho_1 h+\rho_3\Delta W_n)(\rho_2 y_{n-m}+\rho_3 y_{n-2m})y_{n-m}I_1 \end{aligned} \tag{7.30}$$

Assume that $x_n=\mathbb{E}[y_n^2]$, then take expectation for both sides of (7.30), yields

$$x_{n+1} \le A_1 x_n+A_2 x_{n-m}+A_3 x_{n-2m}, \quad \text{where} \tag{7.31}$$

$$A_1=(1+\rho_0 h)^2+\rho_2^2 h+|1+\rho_0 h||\rho_1|h+|\rho_2\rho_3|h+\frac{h^2}{2}\rho_2^2(\rho_2^2+|\rho_2\rho_3|),$$

$$A_2 = \rho_1^2 h^2 + \rho_3^2 h + |1+\rho_0 h||\rho_1|h + |\rho_2\rho_3|h + \frac{h^2}{2}\rho_2^2(\rho_3^2 + |\rho_2\rho_3|)$$
$$+ \frac{h^2}{2}\rho_3^2(\rho_2^2 + |\rho_2\rho_3|), \quad A_3 = \frac{h^2}{2}\rho_3^2(\rho_2^2 + |\rho_2\rho_3|).$$

Such that the following inequality holds

$$\begin{aligned}(1+\rho_0 h)^2 + \rho_1^2 h^2 + (\rho_2^2 + \rho_3^2 + 2|\rho_2\rho_3|)h + 2|1+\rho_0 h||\rho_1|h \\ + \frac{h^2}{2}(\rho_2^2+\rho_3^2)(|\rho_2|+|\rho_3|)^2 < 1.\end{aligned} \tag{7.32}$$

Consider

$$\begin{aligned} h_1 =& \frac{-[2\rho_0 + 2|\rho_1| + (|\rho_2|+|\rho_3|)^2]}{(|\rho_0|+|\rho_1|)^2 + \frac{1}{2}(\rho_2^2+\rho_3^2)(|\rho_2|+|\rho_3|)^2} > 0, \\ h_2 =& \min\{\frac{1}{|\rho_0|}, \frac{-[2\rho_0 + 2|\rho_1| + (|\rho_2|+|\rho_3|)^2]}{(|\rho_0|+|\rho_1|)^2 + \frac{1}{2}(\rho_2^2+\rho_3^2)(|\rho_2|+|\rho_3|)^2}\} > 0, \end{aligned} \tag{7.33}$$

- If $h \in (0, h_1)$, Inequality (7.32) holds;
- If $h \in (0, h_2)$, then $1 + \rho_0 h > 0$ (wider range of stable stepsize values) and Inequality (7.32) holds.

Let $h_0 = \max\{h_1, h_2\}$; Thus, Milstein scheme is MS-stable, whenever $h \in (0, h_0)$.

A Matlab program to produce the numerical results, using Milstien scheme is provided in Appendix B.

7.6 Concluding Remarks

In this chapter, some numerical schemes for SDDEs were briefly discussed. Convergence and consistency of such schemes were investigated. The mean square stability of Milstein scheme had been discussed and the obtained result shows that the method preserves the stability property of a class of linear scalar SDDE. In this dissertation, the above discussed Milstein scheme for solving different examples and models of SDDEs had been discussed. A Matlab program for an example is displayed in Appendix B.

Chapter 8: Summary and Concluding Remarks

8.1 Introduction

In the present thesis the qualitative behaviour of deterministic and stochastic delay differential equations with ecology and epidemics have been investigated. The main novelty is to investigate the impact of time-delays in the models and effects of perturbations to equations caused by random changes/noise in the system. Time-delays and random noise have significant impact in the predator-prey systems and infectious diseases.

Chapter 2 provided a system of DDEs for predator-prey system with hunting cooperation. Local and global asymptotic stabilities of the steady states, Hopf bifurcations of interesting parameter τ have been investigated. The combination of time-delay and hunting cooperation have a considerable impact in the ecosystem. Chapter 3 introduced a system of DDEs for a three species predator-prey system (two-prey one-predator) with time-delays and an additive Allee effect in the prey's growth functions, where there is a direct competition between prey populations. Local stability of the system has been analyzed in detail. Verifiable sufficient conditions which guarantee the global stability around the interior equilibrium using Lyapunov function, have been discussed. Sensitivity of the model solution with respect to Allee parameters and time delays have been evaluated, using the so-called "direct approach".

Chapter 4 studied the dynamics of SDDEs for predator-prey system with hunting cooperation in predators. Relevant properties of the corresponding stochastic delayed predator-prey model have been illustrated and revealed the effect of environmental noise on the model. Under certain conditions the stochastic model will remain to have a positive stable solution which gives a result of the robustness of the solution. The effect of environmental noises on persistence and possible extinction of prey and predator populations have been investigated.

Chapter 5 extended the analysis and investigated a system of SDDEs of two-preys one-predator system, with cooperation among the prey species against predator. The basic features of the model in presence of multiplicative noise terms were discussed, in order to understand the dynamics along with the environmental driving forces. Sharp criteria for the existence of a unique ergodic stationary distribution, of the positive solution of the model, under certain parametric restrictions have been analyzed. Sufficient conditions for extinction of the predator population in two cases have been deduced. The first case is the prey populations survival and the predator population extinction; the second case is all the preys and predator populations die out.

Chapter 6 provided a stochastic SIRC epidemic model with time-delay for the new strain coronavirus COVID-19. The stochastic components, due to environmental variability, are incorporated in the model as Gaussian white noise. Some sufficient conditions for persistence and extinction in the mean of the disease were established. The model has a unique stationary distribution which is ergodic if the intensity of white noise is small. Introduction of noise in the deterministic SIRC model modifies the basic reproductive number $\mathscr{R}_0$ giving rise to a new threshold quantity $\mathscr{R}_0^s$.

Chapter 7 discussed some numerical schemes for stochastic delay differential equations.

8.2 Concluding Remarks and Findings

By using a variety of analytical methods for studying the qualitative features of *deterministic* and *stochastic* delay differential equations systems, the following results have been seen:

- Time-delay (time-lag) parameters play an important role in the dynamics of predator-prey systems, and improve the complexity of the models.
- Combination of time-delays and Allee effect enriches the the dynamics of the system and can lead to bistability of equilibria.
- The model is very sensitive to the small perturbations of Allee parameters in early time intervals and the sensitivity decreases by time.

- Possible Hopf bifurcations around the equilibrium points have been studied in detail. In particular, the threshold parameters where the Hopf bifurcation occurs are deduced.
- The random noises can suppress the explosion of the species, where the solutions of the deterministic system is unbounded. Furthermore, introduction of noise in the deterministic epidemic models can modifies the basic reproductive number giving rise to a new threshold quantity,
- Extinction of predator population is possibly occur when the intensity of white noise is relatively large.
- In the case of the existence of a unique equilibrium point, the stability of the equilibrium point and oscillations could exist globally.
- Small scale of environmental fluctuations can promote the survival of species; while large noises can lead to extinction of the species, this would not happen in the deterministic systems without noises.
- Long-term behaviour of the systems has been studied, and conditions for persistence have been derived.
- Verifiable criteria were developed, which guarantee the existence of a unique ergodic stationary distribution of the positive solutions to stochastic models, using a novel multiple Lyapunov functions.
- White noise plays an important role in controlling the spread of the disease; The large environmental noises may help to bring about extinction of diseases. When the white noise is relatively large, the infectious diseases will become extinct; Reinfection and periodic outbreaks can occur due to the time-delay in the transmission terms.

8.3 Software Used

The numerical experiments and numerical simulations of deterministic DDEs have been carried out using DDE23 [126], which is based on explicit Runge-Kutta scheme; See Appendix A. The numerical simulations of bifurcation diagrams have been carried out by using the Matlab software DDE-BIFTOOLS [40].

For the numerical simulations of SDDEs, Euler Maruyame scheme of order of convergence 1/2 in the mean square sense was utilized. Also, Milstein's scheme for SDDEs of order one in the mean square sense had been used; See Appendix B.

8.4 Future Directions

In the next project, the proposed Models (3.3) and (5.2) can be further extended, to investigate the effect of the combination of Monod-Haldane and Holling type II functional response, of a two competing prey and one predator system; During predation both teams of prey help each other and the rate of predation on both teams are different; Time-delays can be considered due to reaction time of the predations. Including control variables are also possible.

There are still some interesting topics deserve further investigation, such as introducing the color noise or the telegraph noise, for example continuous-time Markov chain, into Models (4.3) and (5.3), since the dynamics of population may suffer sudden-environmental changes which can be modelled by a continuous-time Markov chain. Therefore, the sufficient conditions for ergodicity are supposed to be expressed in terms of model parameters, the intensities of Brownian motion along with the distribution of Markov chain.

For the stochastic SIRC model, discussed in Chapter 6, it is possible to extend this work and include control variables for a vaccination, treatment and/or quarantine actions. More sophisticated model is also required to investigate the dynamics of COVID-19 with immune system in cells level [119].

Models with variable and state dependent time-lags deserve further study and investigation. Furthermore, development of ordinary delay differential equations, and stochastic delay differential equations to include the spatial state variables will be also observed in the future work.

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2. F. A. Rihan and H. J. Alsakaji. Persistence and Extinction for Stochastic Delay Differential Model of Prey Predator System with Hunting Cooperation in Predators. *Advances in Difference Equations*, 124 (1):1-22, 2020. https://doi.org/10.1186/s13662-020-02579-z
3. F. A. Rihan, H. J. Alsakaji and C. Rajivganthi. Stability and Hopf Bifurcation of Three-Species Prey-Predator System with Time Delays and Allee Effect. *Complexity*, 2020. https://doi.org/10.1155/2020/7306412
4. F.A. Rihan, Q. AL-Mdallal, H. J. Alsakaji and A. Hashish. A Fractional-Order Epidemic Model with Time-Delay and Nonlinear Incidence Rate. *Chaos, Soliton & Fractals*, 126:97-105, 2019. https://doi.org/10.1016/j.chaos.2019.05.039
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6. C. Rajivganthi, F. A. Rihan and H. J. Alsakaji. A Fractional-Order Predator-Prey Model with Beddington–DeAngelis Functional Response and Time-Delay. *The Journal of Analysis*, 27 (2), 525-538, Springer, 2018. https://doi.org/10.1007/s41478-018-0092-7
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8. H. J. Alsakaji, F. A. Rihan, Dynamics of A Delayed Predator Prey System with Stochastic Fluctuation, June 30-July 6. 2019, CMMSE 2019, Costa Ballena, Spain. (In press) To appear in CME-Willey.
9. H. J. Alsakaji, F. A. Rihan, Stochastic Delay Differential Equations of Prey Predator System: Analytic and numeric, April. 2019, IACMC 2019 proceedings, Springer, Jordan. http://iacmc.zu.edu.jo/eng/images/stories/paper2020.pdf
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Appendices

Appendix A: Software for Solving DDEs

Consider a scalar DDE with multiple delays of the form

$$y^{'}(t) = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_j)), \tag{8.1}$$

with constant delays τ_j such that $\tau = \min(\tau_1, \tau_2, \ldots, \tau_j) > 0$. The equation is to hold on $a \le t \le b$, which requires the history $y(t) = \phi(t)$ to be given for $t \le a$. Runge-Kutta scheme is very common standard way to solve the ODE problem $y^{'} = f(t,y)$ on $[a,b]$ with given $y(a)$, which can be extended to solve DDEs. Actually, $dde23$ is closely related to the ODE solver $ode23$ [125] which use the BS(2,3) triple [21]. Assume that an approximation y_n to $y(t)$ at t_n and wish to compute an approximation at $t_{n+1} = t_n + h_n$. For $i = 1, \ldots, s$, the stages $f_{ni} = f(t_{ni}, y_{ni})$ are defined in terms of $t_{ni} = t_n + c_i h_n$, such that $y_{ni} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} f_{nj}$, one may advance a step by defining $y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i f_{ni}$. For briefness one can write this in terms of the increment function $\Phi(t_n, y_n) = \sum_{i=1}^{s} b_i f_{ni}$. The solution satisfies this formula with a local truncation error lte_n,

$$y(t_{n+1}) = y(t_n) + h_n \Phi(t_n, y(t_n)) + lte_n.$$

For smooth f and $y(t)$ the local truncation error is $\mathscr{O}(h_n^{p+1})$. The triple includes anther formula,

$$y_{n+1}^{*} = y_n + h_n \sum_{i=1}^{s} b_i^{*} f_{ni} = y_n + h_n \Phi^{*}(t_n, y_n).$$

The solution satisfies this formula with a local truncation error lte_n^{*} is $\mathscr{O}(h_n^{p})$. This second formula is considered only for choosing the stepsize. The third formula is

$$y_{n+\rho} = y_n + h_n \sum_{i=1}^{s} b_i(\rho) f_{ni}.$$

The coefficients $b_i(\rho)$ are polynomials in ρ, which shows a polynomial approximation to $y(t_n + \rho h_n)$ for $0 \le \rho \le 1$. One may assume that this formula obtains the value y_n as $\rho = 0$ and y_{n+1} as $\rho = 1$. Therefore, the third formula is considered as a continuous extension of the first. As a special case when $\rho = 1$, for such a triples one can assume that the formula is used to advance the integration of the continuous extension as follows

$$y_{n+\rho} = y_n + h_n \Phi(t_n, y_n, \rho),$$

where the local truncation error of the continuous extension is defined by

$$y(t_n+\rho h_n)=y(t_n)+h_n\Phi(t_n,y(t_n),\rho)+lte_n(\rho).$$

Consider that for smooth f and $y(t)$, there is a constant R_1 such that $|lte_n(\rho)|\leq R_1 h_n^{p+1}$ for $0\leq\rho\leq 1$. To utilize an explicit Runge-Kutta triple to solve the DDE (8.1), one may need a plan to deal with the history terms $y(t_{ni}-\tau_j)$ that present in

$$f_{ni}=f(t_{ni},y_{ni},y(t_{ni}-\tau),\ldots,y(t_{ni}-\tau_k)). \tag{8.2}$$

Two criteria should be distinguished; $h_n\leq\tau$ and $h_n>\tau_j$ for some j.

- For $h_n\leq\tau$ assume that an approximation $\phi(t)$ to $y(t)$ for all $t\leq t_n$ is considered, then all $t_{ni}-\tau_j\leq t_n$ and

  $$f_{ni}=f(t_{ni},y_{ni},\phi(t_{ni}-\tau),\ldots,\phi(t_{ni}-\tau_k)).$$

  After taking the step to t_{n+1}, one may use the continuous extension to define $\phi(t)$ on $[t_n,t_{n+1}]$ as $\phi(t_n+\rho h_n)=y_{n+\rho}$, then one can take another step, which is enough for proving convergence as the maximum stepsize tends to zero.

- When $h_n>\tau_j$, the initial function $\phi(t)$ evaluated in the span of the current step where the formulas are defined implicitly. Such that one can evaluate the formulas with simple iteration. By reaching t_n, one may have define $\phi(t)$ for $t\leq t_n$. Thus, its definition to $(t_n,t_n+h_n]$ as long with the resulting function $\phi^{(0)}(t)$ can be extended. Usually, the stage of simple iteration begins with the approximate solution $\phi^{(m)}(t)$, where the next iteration is computed with the explicit formula

  $$\phi^{(m+1)}(t_n+\rho h_n)=y_n+h_n\Phi(t_n,y_n,\rho:\phi^{(m)}(t)).$$

  In dde23 $\phi^{(0)}(t)$ is assumed to be constant y_0 for the first step. Since the solution is not smooth at a, one can not attempt to predict more accurately then. Thus, one may do not know a suitable stepsize to the scale of the problem. Actually, reducing the stepsize as needed to obtain convergence of simple iteration is a useful technique for finding an initial stepsize that is on scale. Therefore, after the first step, the continuous extension of the preceding step as $\phi^{(0)}(t)$ for the current step is used. This prediction has a good order of accuracy, following Dormand [36] approach for treating the local truncation error of the BS(2,3) triples, the accuracy quantitatively can be determined using

  $$A^4(\rho)=\frac{\rho^2}{288}(1728\rho^4-7536\rho^3+13132\rho^2-11148\rho+3969)^{1/2}. \tag{8.3}$$

Convergence

Theorem A.1: [126] Suppose an explicit Runge-Kutta triple is used to solve (8.1), such that the meshes $\{t_n\}$ include all discontinuities of low orders and that the maximum stepsize H satisfies (8.6) and (8.8). If f satisfies a Lipschitz condition in its dependent variables and is sufficiently smooth in all its variables, then there exist a constant C such that for $a \leq t \leq b$,

$$\|y(t) - \phi(t)\| \leq CH^p. \tag{8.4}$$

For simplicity, one assumes that there is only one delay and take the Lipschitz condition in the form

$$\|f(t,\tilde{y},\tilde{z}) - f(t,y,z)\| \leq L\max(\|\tilde{y}-y\|, \|\tilde{z}-z\|).$$

The Runge-Kutta formulas utilized to DDEs involve a history term in which one may write in the increment function as $\theta(t)$. Therefore, the following two lemmas about how the increment functions depend on their arguments.

Lemma A.1: [126] There is a constant L such that for $0 \leq \rho \leq 1$.

$$\|\Phi(t_n,\tilde{y}_n,\rho;\theta(t)) - \Phi(t_n,y_n,\rho,\theta(t))\| \leq L\|\tilde{y}_n - y_n\|. \tag{8.5}$$

Lemma A.2: Let Δ be a bound on $\|\Lambda(t) - \theta(t)\|$ for all $t \leq t_{n+1}$. If the maximum stepsize H is small enough that

$$HL\max_i\Big(\sum_{j=1}^{i-1}|a_{ij}|\Big) \leq 1, \tag{8.6}$$

then there is a constant Γ such that

$$\|\Phi(t_n,\tilde{y}_n,\rho;\Lambda(t)) - \Phi(t_n,y_n,\rho,\theta(t))\| \leq \Gamma\Delta. \tag{8.7}$$

Additionally,

$$H\Gamma = H\Gamma\sum_{i=1}^{s}\max(|b_i(\rho)|) \leq \frac{1}{2}. \tag{8.8}$$

The proof of the of Lemma A.2 is discussed in [125].

Error Estimation

dde23 estimates the local truncation error of lower order formula, lte_n^*, by making use of local extrapolation, with $est = y_{n+1} - y_{n+1}^*$. The local truncation error

$$lte*_n = (y(t_{n+1}) - y(t_n)) - h_n\Phi^*(t_n, y(t_n); y(t)),$$

it follows from the definition of lte_n that

$$lte_n^* = h_n\Phi(t_n, y(t_n); y(t)) - h_n\Phi^*(t_n, y(t_n); y(t)) + \mathscr{O}(h_n^{p+1}).$$

Since $h_n\Phi(t_n, y(t_n); y(t)) = h_n\Phi(t_n, y_n; \phi(t)) + \mathscr{O}(h_n H^p)$, from Lemma A.1 one obtains

$$\|h_n\Phi(t_n, y(t_n); y(t)) - h_n\Phi(t_n, y_n; y(t))\| \leq h_n L\|y(t_n)y_n\| \leq h_n\mathscr{L}CH^p,$$

therefore, from (8.7)

$$\|h_n\Phi(t_n, y_n; y(t)) - h_n\Phi(t_n, y_n; \phi(t))\| \leq h_n\Gamma CH^p.$$

Since $h_n\Phi(t_n, y_n; \phi(t)) = y_{n+1} - y_n$, one obtains

$$\begin{aligned} lte_n^* &= h_n\Phi(t_n, y_n; \phi(t)) - h_n\Phi^*(t_n, y_n; \phi(t)) + \mathscr{O}(h_n H^p) \\ &= (y_{n+1} - y_n) - (y_{n+1}^* - y_n) + \mathscr{O}(h_n H^p) \\ &= est + \mathscr{O}(h_n H^p), \end{aligned}$$

which confirms the estimate of the local truncation error, hence, it can be expanded to

$$lte_n^* = \Phi^*(t_n, y(t_n); y(t))h_n^p + \mathscr{O}(h_n^{p+1}).$$

If the stepsize h_n is rejected from (t_n, y_n), the local truncation error of another attempt of size h can be predicted as $est(h/h_n)^p \approx \Phi^*(t_n, y(t_n); y(t))h^p$. In the same manner the prediction applies to a stepsize h from (t_{n+1}, y_{n+1}) as $\Phi^*(t_{n+1}, y(t_{n+1}); y(t)) \approx \Phi^*(t_n, y(t_n); y(t))$, since the change in each arguments is $\mathscr{O}(h_n)$.

Several packages and software are available for the numerical integration and/or the study of bifurcations in delay differential equations. Here is a short list for available software:

- **DDE23** (Shampine and Thompson [126]) simulates retarded differential equations with several fixed discrete delays.
- **Archi** (Paul [106]) simulates a large class of functional differential equations.

- **RADAR5** (Guglielmi and Hairer [49]) simulates stiff problems, including differential-algebraic and neutral delay equations with constant or state-dependent (eventually vanishing) delay.
- **DKLAG6** (Thompson [33]) simulates retarded and neutral differential equations with state dependent delays.
- **MIDDE** (Rihan *et al.* [116]) simulates stiff and non-stiff delay differential equations and Volterra delay integro-differential equations, using mono-implicit RK methods.
- **BIFDD** (Hassard [53]) (Fortran 77) normal form analysis of Hopf bifurcations of differential equations with several fixed discrete delays.
- **DDE-BIFTOOL** (Engelborghs [40]) (MatLab) allows computation and stability analysis of steady state solutions, their fold and Hopf bifurcations and periodic solutions of differential equations with several fixed discrete delays.

Appendix B: Sample of Matlab Program for SDDEs Using Milstein Scheme

Here, a sample of Matlab program for solving SDDE is provided, Figures 4.1–4.4.

```
%This program to solve SDDE model and its deterministic model
%dx=[rx(t)(1-x(t-\tau_1)/k)-(1+alpha y(t))y(t)x(t)/(1+c(1+alpha
%y(t))x(t))]dt+sigma_1 x(t)dB_1(t)
%dy=[-delta y(t)-ay^2(t)+ mu(1+alpha y(t))x(t-\tau_2)/(1+c(1+alpha
%y(t))x(t-\tau_2))]dt+sigma_2y(t) dB_2(t)
clc;
x1=cell(1,1);
x2=cell(1,1);
x3=cell(1,1);
x4=cell(1,1);
x1{1}(1)=sin(-0.3*rand)+.8;%intial data for x(t) SDDE
x2{1}(1)=sin(-0.3*rand)+0.4;%initial data for y(t) SDDE
x3{1}(1)=.8;%initial condition for x(t) DDE
x4{1}(1)=.4;%initial condition for y(t) DDE
h=0.001;
tend=100;
tau1=.1;
tau2=.1;
alpha=.1;%hunting cooperative parameter
a=0.19;%intra-competition between predator
c=0.8;%the handling time of the predator
delta=0.59;%death rate of predator
mu=0.8;%conversion efficiency
r=1;%growth rate of prey
K=1;%environmental carrying capacity
sig1=0.001;%white noise for x(t)
sig2=0.023;%white noise for y(t)
t=0;
k=1;
while t<=tend
```

```
x1{1}(k+1)=x1{1}(k)+h*(r*x1{1}(k)*(1-x1{1}(max(1,k-tau1/h))/K)-(1+alpha*x2{1}(k))*x1{1}(k)*x2{1}(k)/(1+c*(1+alpha*x2{1}(k))*x1{1}(k)))+...
    sig1*x1{1}(k)*randn+sig1^2*0.5*x1{1}(k)*(randn^2-h);
x2{1}(k+1)=x2{1}(k)+h*(mu*(1+alpha*x2{1}(k))*x1{1}(max(1,k-tau2/h))*x2{1}(k)/(1+c*(1+alpha*x2{1}(k))*x1{1}(max(1,k-tau2/h)))-delta*x2{1}(k)-a*x2{1}(k).^2)+...
    sig2*x2{1}(k)*randn+sig2^2*0.5*x2{1}(k)*(randn^2-h);
x3{1}(k+1)=x3{1}(k)+h*(r*x3{1}(k)*(1-x3{1}(max(1,k-tau1/h))/K)-(1+alpha*x4{1}(k))*x3{1}(k)*x4{1}(k)/(1+c*(1+alpha*x4{1}(k))*x3{1}(k)));
x4{1}(k+1)=x4{1}(k)+h*(mu*(1+alpha*x4{1}(k))*x3{1}(max(1,k-tau2/h))*x4{1}(k)/(1+c*(1+alpha*x4{1}(k))*x3{1}(max(1,k-tau2/h)))-delta*x4{1}(k)-a*x4{1}(k).^2);
    k=k+1;
t=t+h;
end
figure(1)
tt=(0:h:tend);
plot(tt,x1{1},tt,x3{1},'LineWidth',3)
xlabel('Time(t)');
ylabel('x(t)');
legend('Stochastic','Determinstic');
grid on;
figure(2)
tt=(0:h:tend);
plot(tt,x2{1},tt,x4{1},'LineWidth',3)
xlabel('Time(t)');
ylabel('y(t)');
legend('Stochastic','Determinstic');
grid on;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%PERSISTENCE%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
x1{1}(1)=sin(-0.3*rand)+.4;%intial data for x(t) SDDE
x2{1}(1)=sin(-0.3*rand)+0.8;%initial data for y(t) SDDE
h=0.001;
tend=200;
tau1=.1;
tau2=.1;
```

```
alpha=.12; a=.08; c=0.3; delta=0.39; mu=0.9; r=1;K=1;
sig1=0.001;%white noise for x(t)
sig2=0.001;%white noise for y(t)
t=0;
k=1;
while t<=tend
x1{1}(k+1)=x1{1}(k)+h*(r*x1{1}(k)*(1-x1{1}(max(1,k-tau1/h))
    /K)-(1+alpha*x2{1}(k))*x1{1}(k)*x2{1}(k)/(1+c*(1+alpha*
    x2{1}(k))*x1{1}(k)))+...
     sig1*x1{1}(k)*randn+sig1^2*0.5*x1{1}(k)*(randn^2-h);
x2{1}(k+1)=x2{1}(k)+h*(mu*(1+alpha*x2{1}(k))*x1{1}(max(1,k-
    tau2/h))*x2{1}(k)/(1+c*(1+alpha*x2{1}(k))*x1{1}(max(1,k-
    tau2/h)))-delta*x2{1}(k)-a*x2{1}(k).^2)+...
     sig2*x2{1}(k)*randn+sig2^2*0.5*x2{1}(k)*(randn^2-h);
     k=k+1;
t=t+h;
end
figure(4)
tt=(0:h:tend);
plot(tt,x1{1},tt,x2{1},'LineWidth',3)
xlabel('Time(t)');
ylabel('Populations x(t),y(t)');
legend('x(t)','y(t)');
grid on;
x1{1}(1)=sin(-0.3*rand)+.4;%intial data for x(t) SDDE
x2{1}(1)=sin(-0.3*rand)+0.8;%initial data for y(t) SDDE
h=0.001;
tend=200;
tau1=.1;
tau2=.1;
alpha=1.2; a=.08; c=0.3; delta=0.39; mu=0.9; r=1;K=1;
sig1=0.001;%white noise for x(t)
sig2=0.001;%white noise for y(t)
t=0;
k=1;
while t<=tend
x1{1}(k+1)=x1{1}(k)+h*(r*x1{1}(k)*(1-x1{1}(max(1,k-tau1/h))
    /K)-(1+alpha*x2{1}(k))*x1{1}(k)*x2{1}(k)/(1+c*(1+alpha*
    x2{1}(k))*x1{1}(k)))+...
```

```
    sig1*x1{1}(k)*randn+sig1^2*0.5*x1{1}(k)*(randn^2-h);
x2{1}(k+1)=x2{1}(k)+h*(mu*(1+alpha*x2{1}(k))*x1{1}(max(1,k-
   tau2/h))*x2{1}(k)/(1+c*(1+alpha*x2{1}(k))*x1{1}(max(1,k-
   tau2/h)))-delta*x2{1}(k)-a*x2{1}(k).^2)+...
    sig2*x2{1}(k)*randn+sig2^2*0.5*x2{1}(k)*(randn^2-h);
    k=k+1;
t=t+h;
end
figure(5)
tt=(0:h:tend);
plot(tt,x1{1},tt,x2{1},'LineWidth',3)
xlabel('Time(t)');
ylabel('Populations x(t),y(t)');
legend('x(t)','y(t)');
grid on;
```