2-2021

A NUMERICAL METHOD FOR SOLVING FUZZY INITIAL VALUE PROBLEMS

Safa Emad Al-Refai

United Arab Emirates University

College of Science

Department of Mathematical Sciences

A NUMERICAL METHOD FOR SOLVING FUZZY INITIAL VALUE PROBLEMS

Safa Emad Al-Refai

This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Prof. Muhammed I. Syam

February 2021

Declaration of Original Work

I, Safa Emad Al-Refai , the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"A Numerical Method for Solving Fuzzy Initial Value Problems"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Muhammed I. Syam, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

Student's Signature ________________ Date 9/3/2021

Advisory Committee

1) Advisor: Muhammed I. Syam

Title: Professor

Department of Mathematical Sciences

College of Science

2) Co-advisor: Mohammed Al-Refai

Title: Professor

Department of Mathematical Sciences

College of Science

Yarmouk University, Jordan

Approval of the Master Thesis

This Master Thesis is approved by the following Examining Committee Members:

1) Advisor (Committee Chair): Muhammed I. Syam

Title: Professor

Department of Mathematical Sciences

College of Science

Signature ____________________ Date 9/3/2021

2) Member: Fathalla Rihan

Title: Professor

Department of Mathematical Sciences

College of Science

Signature ____________________ Date 9/3/2021

3) Member (External Examiner): Marwan Alquran

Title: Professor

Department of Mathematics and Statistics

Institution: Jordan University of Science and Technology

Signature ____________________ Date 9/3/2021

This Master Thesis is accepted by:

Dean of the College of Science: Professor Maamar Ben Kraouda

Signature Maamar Benkraouda Date 18/03/2021

Dean of the College of Graduate Studies: Professor Ali Al-Marzouqi

Signature Ali Hassan Date 18/3/2021

Copy ________ of ________

Abstract

In this thesis, the optimized one-step methods based on the hybrid block method (HBM) are derived for solving first and second-order fuzzy initial value problems. The off-step points are chosen to minimize the local truncation error of the proposed methods. Several theoretical properties of the proposed methods, such as stability, convergence, and consistency are investigated. Moreover, the regions of absolute stability of the proposed methods are plotted. Numerical results indicate that the proposed methods have order three and they are stable and convergent. In addition, several numerical examples are presented to show the efficiency and accuracy of the proposed methods. Results are compared with the existing ones in the literature. Even though the one off-step point is used, the results of the proposed methods are better than the ones obtained by other methods with a less computational cost.

Keywords: Fuzzy initial value problems, Convergence, Stability, Consistency.

Title and Abstract (in Arabic)

طريقة عددية لحل المسائل الضبابية ذات القيم البدائية

الملخص

في هذه الأطروحة، سيتم اشتقاق طرق الخطوة الواحدة المثلى لحل المسائل الضبابية ذات القيم البدائية من الرتبة الأولى والثانية. يتم اختيار نقاط خارج الخطوة من اجل التقليل الامثل لخطأ الاقتطاع المحلي للطرق المقترحة. تمت دراسة العديد من الخصائص النظرية للطرق المقترحة مثل الثبات والتقارب والاتساق. علاوة على ذلك، يتم رسم منطقة الاستقرار المطلق للطرق المقترحة. أظهرت النتائج أن الطرق المقترحة لها ترتيب ثلاثي وأنها مستقرة ومتقاربة. بالإضافة إلى ذلك، تم تقديم العديد من الأمثلة العددية لإظهار فعالية ودقة الطرق المقترحة. تم مقارنة النتائج مع النتائج الموجودة في الأبحاث الاخرى و لاحضنا انه حتى اباستخدام نقطة واحدة خارج الخطوة، نحصل على نتائج شبيهة و احياناً أفضل من الطرق الأخرى بتكلفة حسابية أقل.

مفاهيم البحث الرئيسة: المسائل الضبابية ذات القيم البدائية، التقارب، الاستقرار، الثبات.

Acknowledgements

Throughout the writing of this thesis, I have received a great deal of support and assistance. This journey would not have been possible without the support of my professors, family, and friends. I would like to take this opportunity to thank them.

I would first like to thank my supervisor, Professor Muhammed I Syam, who was an inexhaustible source of support. Thank you for your continuous help, hard-work, and the motivational words that encouraged and supported me to complete this research work with love and sincerity. Working with you was a great experience for me; your insightful feedback pushed me to sharpen my thinking and brought my work to a higher level. Besides, I am grateful for your guidance which is delivered for me to uphold my experience and research skills, and open my horizon to pave my way for better understanding. I can never thank you enough for all that you have done for me. It's my pleasure and pride to be your student.

I would like to extend special thanks to my co-supervisor, Professor Mohammed Al-Refai, who never felt tired of affording help and sharing valuable information. I am very grateful to work with you. Then, I would like to thank the internal examiner, Prof. Fathalla Rihan, and the external examiner, Prof. Marwan Alquran, for their constructive remarks and for reviewing my thesis.

I would like to thank the Faculty Members of Mathematics Department, especially the head of the department, Dr. Adama Diene, the master's coordinator, Dr. Ahmad Al Rwashdeh, Prof. Mohammed Alhammadi, Dr. Mohamed Hajji, Dr. Youssef El Khatib and all my teachers for their encouragement, even those who supported me with just one word in this journey. Thank you all for your assistance and input in getting me acquainted with the research skills and my master's thesis.

My special thanks go to my parents, who are the main reason for reaching this achievement. They never felt tired of encouraging me in all of my pursuits and inspiring me to follow my dreams. Thank you for supporting me from all aspects; emotionally and financially. Thank you for the strength and faith that you have given me, to walk on the path of success. I can never thank you enough for everything you have done for me. Also, many thanks to my twin sister Marwa, my sister Taqwa, my brothers Mohammed and Ali and all my relatives, who always encouraged me to complete this journey in peace.

Last but not least, I would like to thank all my friends, particularly, Rasha, Asma, Sarah, Ragad, Huda, Bayan, Amena, Sondos, and Namarig, who were my second family.

Dedication

To my beloved parents and teachers

Table of Contents

List of Tables

List of Figures

Chapter 1: Fuzzy Logic and Fuzzy Sets

1.1 Introduction

The term fuzzy means things that are not very clear or vague. In real life, a situation might come across us, and it can't be decided whether this statement is true or false. At that time, fuzzy logic offers very valuable flexibility for reasoning. Also, it considers the uncertainties of any situation. The fuzzy logic algorithm helps to solve a problem after considering all available data. Then, it takes the best possible decision for the given input. This logic imitates the human way of decision-making, which considers all the possibilities between digital values true and false.

Traditional boolean logic deals with two values which are 0 (false, No) and 1 (True, yes). In the 19th century, a system of algebra and set theory was created by George Boole. This system could deal mathematically with such two-valued logic, mapping true and false to 1 and 0, respectively. Then, in the 20th century, a three-valued logic which is true, possible, and false, was proposed by Jan Lukasiewicz which was not widely accepted. After that, the interest in fuzzy logic notion was starting when Zadeh noted that conventional computer logic was not capable of manipulating data representing subjective or unclear human ideas. He created fuzzy logic to allow computers to determine the distinctions among data with shades of gray, similar to the process of human reasoning. Moreover, Zadeh published a paper. This paper initially did not receive special attention in Western countries. However, over time it began to gain enough supporters, which led to the expansion of the theories of this paper. Thus, the paper gained more attention and began to spread in many countries, such as Japan, South Korea, China, and India. Europe and the States also have been combined gradually into this new area of fuzzy logic, which spread widely and used in various scientific fields [1]. Recently, fuzzy logic has become attractive to many researchers due to its applications in various fields. Fuzzy logic has been applied in various fields, such as computer sciences, information sciences, mathematics, engineering, economics, business, and finance. Fuzzy logic and fuzzy set are powerful mathematical tools in the mathematical modeling of uncertain systems in industry, nature, and humanity. For more details about the history of fuzzy logic and its applications, see [1, 2, 3].

Fuzzy differential equations have several real-life applications in many interesting areas, such as physics, control theory, economics, population models, and ecology [4, 5, 6]. These applications have attracted researchers to investigate such problems. One of these interesting problems is the fuzzy initial value problems (FIVPs). These problems consist of fuzzy differential equations (FDEs) with fuzzy initial conditions. The FIVPs are often incomplete or ambiguous. For instance, the initial conditions or the coefficients of the fuzzy differential equations may not be known accurately. In this situation, FDEs appear as a natural way to model dynamical systems under uncertain possibilities. To solve these equations, the derivative is defined by one of three different approaches, see [7]. The first approach is based on the Hukuhara derivative which was given by Puri-Ralescu in 1983. The second approach is known as Zadeh's extension principle and the last approach is strongly generalized differentiability which was presented by Bede and Gal in 2005. In this study, the Hukuhara derivative will be chosen in order to define the differential equations.

Finding the exact solutions of first and second-order fuzzy initial value problems is a hard task, and it is sometimes not possible. For this reason, researchers were interested to find numerical solutions by using different methods, such as the decomposition method [8, 9], homotopy analysis method [10, 11], Runge-Kutta method [12, 13, 14, 15], block method [16], fuzzy Laplace transform method [17], Lagrange's multiplier method [18], and characterization Theorem [19]. The proposed methods will be investigated to find numerical solutions for these problems. These methods depend on the one-step hybrid block method. In these methods, the local truncation errors are tried to be optimized in order to find the best choice of the step-point. The main advantage of the proposed methods is that they are self-starter which means there is no need to use other methods to generate more initial starting conditions.

Numerical methods for solving initial value problems are divided into two main categories. The first category is the one-step methods, which depend only on one initial value and its derivative to determine other approximations to the exact solution. For example, Euler, Runge-Kutta, and Taylor methods are one–step methods. The second category is the multistep methods, which depend on more than one initial value and their derivative to determine other approximations for the solution. There are several examples of multistep methods such as Adams-Bashforth and Adams-Moulton methods. The one-step hybrid block method is a method that belongs to the one-step family since it depends on one initial value only. However, this method has off-step points, which makes it have the same properties as multistep methods. Thus, this method is a mixture between one-step and multistep methods.

The purpose of this research work is to find numerical solutions of the fuzzy initial value problems of first and second-order using the proposed methods which are giving a high order of accuracy and convergent to the exact solutions even whenever it is impossible to find the exact solutions in the closed-form. To achieve this desired aim, the one-step hybrid block method will be applied in the initial value problems. Then, the proposed methods will be extended to solve the fuzzy type of these problems using some properties of fuzzy operation. In addition, convergence and stability results of the proposed methods will be studied. Also, several examples to illustrate the efficiency and accuracy of the proposed methods are exhibited and the numerical results are compared with the existing ones in the literature.

1.2 Direct Method for Solving Fuzzy Initial Value Problems

In this section, the idea of the Direct method to solve fuzzy IVPs will be presented. The difficulties of using this method will be explained. To explain the idea of this method, consider the following fuzzy initial value problem (FIVP) of the form

$$y^{(k)} = f(t, y', y'', \ldots, y^{(k-1)}), \quad t_0 < t < T \tag{1.1}$$

subject to

$$y(t_0) = y_0,\ \ y'(t_0) = y'_0,\ \ \ldots,\ \ y(t_0)^{(k-1)} = y_0^{(k-1)}, \tag{1.2}$$

where $y_0, \ldots, y_0^{(k-1)}$ are fuzzy numbers and $f, y^k, \ldots, y$ are fuzzy functions on $[t_0, T] \times F_R$. Since the function and initial conditions are fuzzy, the α-level sets operations are applied to obtain the components of the problem are as follows

$$(y(t))_\alpha = (y_{1,\alpha}(t), y_{2,\alpha}(t)),\ \ (y'(t))_\alpha = (y'_{1,\alpha}(t), y'_{2,\alpha}(t)), \ldots,\ \ (y^k(t))_\alpha = (y^k_{1,\alpha}(t), y^k_{2,\alpha}(t)),$$

$$(y(t_0))_\alpha = (y_{0,1}, y_{0,2}),\ \ y'(t_0)_\alpha = (y'_{0,1}, y'_{0,2}),\ \ \ldots,\ \ (y(t_0)^{(k-1)})_\alpha = (y_{0,1}^{(k-1)}, y_{0,2}^{(k-1)}),$$

$$(f(t,y,y',\ldots,y^{k-1}))_\alpha = \Big(f_1\Big(t, (y_{1,\alpha}(t), y_{2,\alpha}(t)), (y'_{1,\alpha}(t), y'_{2,\alpha}(t)), \ldots, \Big(y^{k-1}_{1,\alpha}(t), y^{k-1}_{2,\alpha}(t)\Big)\Big),$$

$$f_2\Big(t, (y_{1,\alpha}(t), y_{2,\alpha}(t)), (y'_{1,\alpha}(t), y'_{2,\alpha}(t)), \ldots, \Big(y^{k-1}_{1,\alpha}(t), y^{k-1}_{2,\alpha}(t)\Big)\Big)\Big),$$

$$f_1 = \min\{f(t,w,w',\ldots,w^{k-1}) : w \in (y_{1,\alpha}(t), y_{2,\alpha}(t)), w' \in (y'_{1,\alpha}(t), y'_{2,\alpha}(t)), \ldots,$$
$$w^{k-1} \in (y^{k-1}_{1,\alpha}(t), y^{k-1}_{2,\alpha}(t))\},$$

$$f_2 = \max\{f(t,w,w',\ldots,w^{k-1}) : w \in (y_{1,\alpha}(t), y_{2,\alpha}(t)), w' \in (y'_{1,\alpha}(t), y'_{2,\alpha}(t)), \ldots,$$
$$w^{k-1} \in (y^{k-1}_{1,\alpha}(t), y^{k-1}_{2,\alpha}(t))\},$$

The above min-max problems are tried to solve directly. Also, the final answer should be checked. If the solution satisfies the following conditions $\frac{\partial y_1(t,\alpha)}{\partial \alpha} \geq 0$, $\frac{\partial y_2(t,\alpha)}{\partial \alpha} \leq 0$, and $y_1(t,\alpha) \leq y_2(t,\alpha)$. Then it is a fuzzy solution of the fuzzy initial value problem.

Now, the direct method will be applied to solve the FIVP of second order.

Example 1.2.1. [9] Consider the following linear second order fuzzy initial value problem

$$\hat{\sigma}\, y''(x) = \hat{\gamma}, \quad y(0) = \hat{\theta}, \; y'(0) = \hat{\beta}, \tag{1.3}$$

where α-level set are $\hat{\sigma}_\alpha = [1, 2-\alpha]$, $\hat{\gamma}_\alpha = [\alpha+1, 3-\alpha]$, $\hat{\theta}_\alpha = [\alpha-1, 1-\alpha]$ and, $\hat{\beta}_\alpha = [\alpha, 2-\alpha]$. Let $y(x) = [y_1(x,\alpha), y_2(x,\alpha)]$ be the fuzzy solution and $y'(x) = [y_1'(x,\alpha), y_2'(x,\alpha)]$, $y''(x) = [y_1''(x,\alpha), y_2''(x,\alpha)]$.

By implementing α-level sets, the problem 1.3 becomes

$$[1, 2-\alpha]\; [y_1''(x,\alpha), y_2''(x,\alpha)] = [\alpha+1, 3-\alpha],$$

$$[y_1(0,\alpha), y_2(0,\alpha)] = [\alpha-1, 1-\alpha], \;\; [y_1'(0,\alpha), y_2'(0,\alpha)] = [\alpha, 2-\alpha].$$

Then, the min-max problem becomes,

$$\min\{y_1''(x,\alpha), y_2''(x,\alpha), (2-\alpha)y_1''(x,\alpha), (2-\alpha)y_2''(x,\alpha)\} = \alpha+1,$$

$$y_1(0,\alpha) = \alpha-1, \; y_1'(0,\alpha) = \alpha,$$

$$\max\{y_1''(x,\alpha), y_2''(x,\alpha), (2-\alpha)y_1''(x,\alpha), (2-\alpha)y_2''(x,\alpha)\} = 3-\alpha,$$

$$y_2(0,\alpha) = 1-\alpha, \; y_2'(0,\alpha) = 2-\alpha.$$

Since $y_1''(x,\alpha) \le y_2''(x,\alpha)$ and $2-\alpha \ge 1, \;\; \forall \alpha \in [0,1]$, then,

$$y_1''(x,\alpha) = \alpha+1, \;\; y_1(0,\alpha) = \alpha-1, \; y_1'(0,\alpha) = \alpha,$$

$$(2-\alpha)y_2''(x,\alpha) = 3-\alpha, \;\; y_2(0,\alpha) = 1-\alpha, \; y_2'(0,\alpha) = 2-\alpha. \tag{1.4}$$

Thus, the initial value problems in System 1.4 can be solved directly by applying the integration two times, then

$$y_1(x,\alpha) = \frac{\alpha+1}{2}x^2 + c_1x + c_2,$$
$$y_2(x,\alpha) = \frac{3-\alpha}{2(2-\alpha)}x^2 + c_3x + c_4.$$

Applying initial conditions in System 1.4 , then the solution will be

$$y_1(x,\alpha) = \frac{\alpha+1}{2}x^2 + \alpha x + \alpha - 1,$$
$$y_2(x,\alpha) = \frac{3-\alpha}{2(2-\alpha)}x^2 + (2-\alpha)x + 1 - \alpha.$$

The last step, the solution should be checked if its satisfied the following conditions

1. $\frac{\partial y_1}{\partial \alpha} = \frac{1}{2}x^2 + x + 1 = \frac{1}{2}[(x+1)^2 + 1] \geq 0.$

2. $\frac{\partial y_2}{\partial \alpha} = \frac{-(2-\alpha)+(3-\alpha)}{2(2-\alpha)^2}x^2 - x - 1 = \frac{1}{2(2-\alpha)^2}x^2 - x - 1 \leq 0.$

3. $y_1(x,\alpha) \leq y_2(x,\alpha).$

Condition 1 holds $\forall\, x \geq 0$, Condition 2 holds $\forall x \in \left[1-\sqrt{\frac{3}{2}}, 1+\sqrt{\frac{3}{2}}\right]$ and, Condition 3 holds in interval $\left[1-\sqrt{\frac{3}{2}}, 1+\sqrt{\frac{3}{2}}\right]$. Thus the solution is fuzzy solution in interval $\left[1-\sqrt{\frac{3}{2}}, 1+\sqrt{\frac{3}{2}}\right]$.

In this example, the solution can be found easily using the direct method. However, the solution to such kind of problems might be difficult to solve directly, and obtaining exact solutions is not always possible by using the direct method. Thus, proposed methods are created to solve this kind of problems. It will be presented in Chapter 3 to solve the FIVPs.

Chapter 2: Preliminaries

In this chapter, the research preliminaries will be introduced such as fuzzy numbers, and differentiation of fuzzy functions.

2.1 Fuzzy Numbers

In this section, some definition and theorems of fuzzy numbers are introduced. The definitions and theorems are referenced from the Lee book [20].

Definition 2.1.1. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two intervals. The, addition and subtraction of A and B are defined as

$$A + B = [a_1 + b_1, a_2 + b_2]$$

and

$$A - B = [a_1 - b_2, a_2 - b_1].$$

Definition 2.1.2. Let $\mathbb{R}$ be the set of real numbers and $\hat{a} : \mathbb{R} \to [0,1]$ be a fuzzy set. Then, $\hat{a}$ is said to be a fuzzy number if it satisfies the followings

a. $\hat{a}$ is normal, that is, there exists $c \in \mathbb{R}$ such that $\hat{a}(c) = 1$.

b. $\hat{a}$ is fuzzy convex, that is $\hat{a}(\lambda u + (1-\lambda)v) \geq \min\{\hat{a}(u), \hat{a}(v)\}$ for $u, v \in \mathbb{R}, \lambda \in [0,1]$.

c. $\hat{a}$ is piecewise continuous.

d. $\hat{a}$ is defined in real number.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F_{\mathbb{R}}$. For any $\alpha \in (0,1]$, α-level set $\hat{a}_\alpha$ of any $\hat{a} \in F_{\mathbb{R}}$ is defined by

$$\hat{a}_\alpha = \{x \in \mathbb{R} : \hat{a}(x) \geq \alpha\}.$$

The 0-level set $\hat{a}_0$ is defined by the closure of $\{x \in \mathbb{R} : \hat{a}(x) > 0\}$. Then $\hat{a}_\alpha$ is convex subset of $\mathbb{R}$ and it is written as $\hat{a}_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$. One can see that

1. $\hat{a}_\alpha \subseteq \hat{a}_c$ if $0 < c \leq \alpha \leq 1$.

2. If the sequence $\{\alpha_n\}$ is an increasing sequence in $(0,1]$ converges to α, then

$$\underset{n\to\infty}{Lim} \hat{a}_{\alpha_n} = \hat{a}_\alpha.$$

3. For any $\alpha \in (0,1]$, $-\infty < \underline{\hat{a}}_\alpha \leq \overline{\hat{a}}_\alpha < \infty$.

Some special fuzzy numbers can be defined for $\alpha \in [0,1]$ such as triangle fuzzy number.

Example 2.1.1. Let $\hat{a} : \mathbb{R} \to [0,1]$ be defined by

$$\hat{a}(x) = \frac{1}{1+x^2}. \tag{2.1}$$

Graph of $\hat{a}$ is given by Figure 2.1.

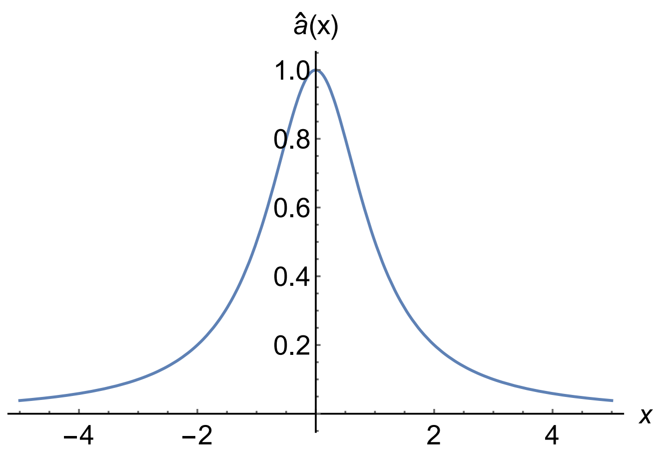


Figure 2.1: The graph of Equation (2.1)

Then, $\hat{a}$ is normal since $\hat{a}(0) = 1$. Moreover, for any $u, v \in \mathbb{R}, \lambda \in [0,1], \lambda u + (1-\lambda)v$ is between u and v. Indeed, $\hat{a}(\lambda u + (1-\lambda)v)$is between $\hat{a}(u)$ and $\hat{a}(v)$. Thus,

$$\hat{a}(\lambda u + (1-\lambda)v) \geq \min\{\hat{a}(u), \hat{a}(v)\}.$$

Hence, $\hat{a}$ is fuzzy convex. For any $\alpha \in (0,1]$, $\{x \in \mathbb{R} : \hat{a}(x) \geq \alpha\} = \left[-\sqrt{\frac{1-\alpha}{\alpha}}, \sqrt{\frac{1-\alpha}{\alpha}}\right]$ is subset of $\mathbb{R}$. Therefore, $\hat{a}$ is fuzzy number.

Definition 2.1.3. Let $\hat{a}, \hat{b} \in F_{\mathbb{R}}$ with $\hat{a}_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$ and $\hat{b}_\alpha = [\underline{b}_\alpha, \overline{b}_\alpha]$. Then, the addition, the subtraction, and the scalar multiplication are defined by

$$(\hat{a} \oplus \hat{b})_\alpha = [\underline{a}_\alpha + \underline{b}_\alpha, \overline{a}_\alpha + \overline{b}_\alpha],$$

$$(\hat{a} \ominus \hat{b})_\alpha = [\underline{a}_\alpha - \overline{b}_\alpha, \overline{a}_\alpha - \underline{b}_\alpha],$$

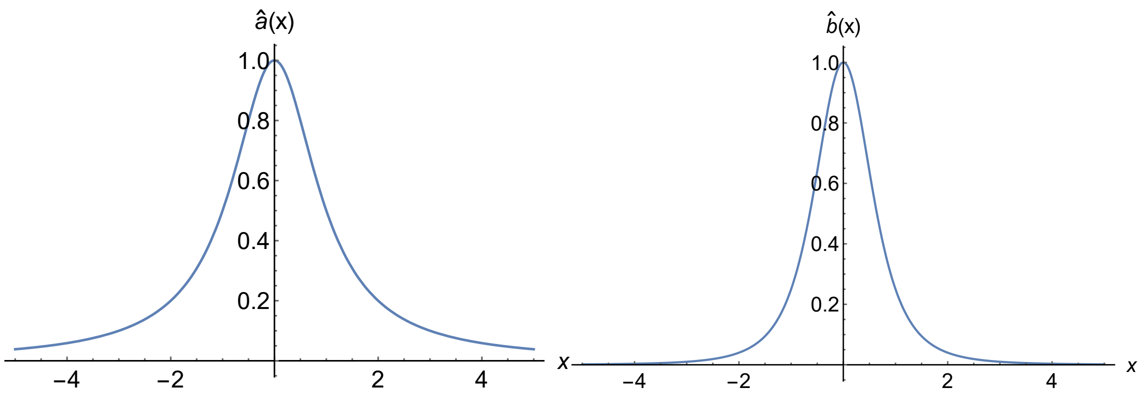
$$(\lambda \otimes \hat{a})_\alpha = \begin{cases} [\lambda \underline{a}_\alpha, \lambda \overline{a}_\alpha], & \lambda \geq 0 \\ [\lambda \overline{a}_\alpha, \lambda \underline{a}_\alpha], & \lambda < 0, \end{cases}$$

for $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$.

Example 2.1.2. Let $\hat{a}, \hat{b} : \mathbb{R} \to [0,1]$ be defined by

$$\hat{a}(x) = \frac{1}{1+x^2}, \qquad \hat{b}(x) = \frac{1}{(1+x^2)^2}.$$

Their graphs are given by Figures (2.2a) and (2.2b).



(a) graph of $\hat{a}(x)$ (b) graph of $\hat{b}(x)$

Figure 2.2: The graphs of $\hat{a}(x)$ and $\hat{b}(x)$ in Example 2.1.2

Then,

$$\hat{a}_{0.5} = [-1, 1] \quad and \quad \hat{b}_{0.5} = \left[-\sqrt{\sqrt{2}-1}, \sqrt{\sqrt{2}-1}\right].$$

Also,

$$(\hat{a} \oplus \hat{b})_{0.5} = \left[-1-\sqrt{\sqrt{2}-1}, 1+\sqrt{\sqrt{2}-1}\right],$$

$$(\hat{a} \ominus \hat{b})_{0.5} = \left[-1-\sqrt{\sqrt{2}-1}, 1+\sqrt{\sqrt{2}-1}\right],$$

$$(2 \odot \hat{a})_{0.5} = (-2 \odot \hat{a})_{0.5} = [-2, 2].$$

Next, two important fuzzy numbers will be defined as follows

Definition 2.1.4. The trapezoidal fuzzy number $\hat{a}$ is defined by $[a_1 \;\; a_2 \;\; a_3 \;\; a_4]$ where

$$\hat{a}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \le x \le a_4 \\ 0, & x > a_4 \end{cases}$$

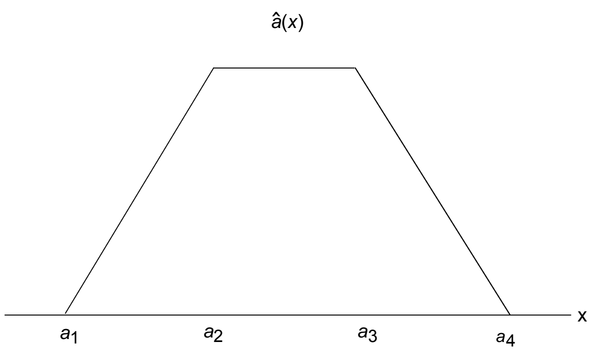


Figure 2.3: Trapezoidal fuzzy number

and the graph of trapezoidal fuzzy number $\hat{a}$ is given in the Figure 2.3.

Definition 2.1.5. The triangle fuzzy number $\hat{a}$ is defined by $[a_1 \;\; a_2 \;\; a_3]$ where

$$\hat{a}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}$$

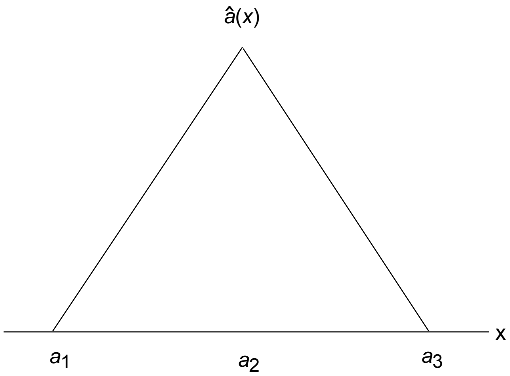


Figure 2.4: Triangle fuzzy number

and the graph of triangle fuzzy number $\hat{a}$ is given in the Figure 2.4.

Particular examples for fuzzy numbers are given in the next example.

Example 2.1.3. The trapezoidal fuzzy number $\hat{a} = [-2 \;\; -1 \;\; 1 \;\; 2]$ is given by the following Figure 2.5.

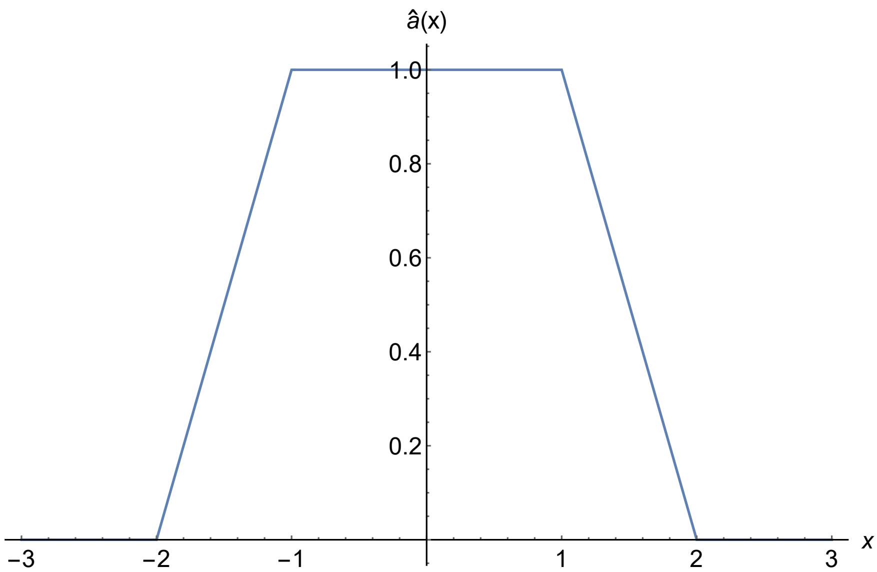


Figure 2.5: The graph of $\hat{a} = [-2 \;\; -1 \;\; 1 \;\; 2]$

while the triangle fuzzy number $\hat{a} = [-2 \ \ 0 \ \ 2]$ is given by the following Figure 2.6.

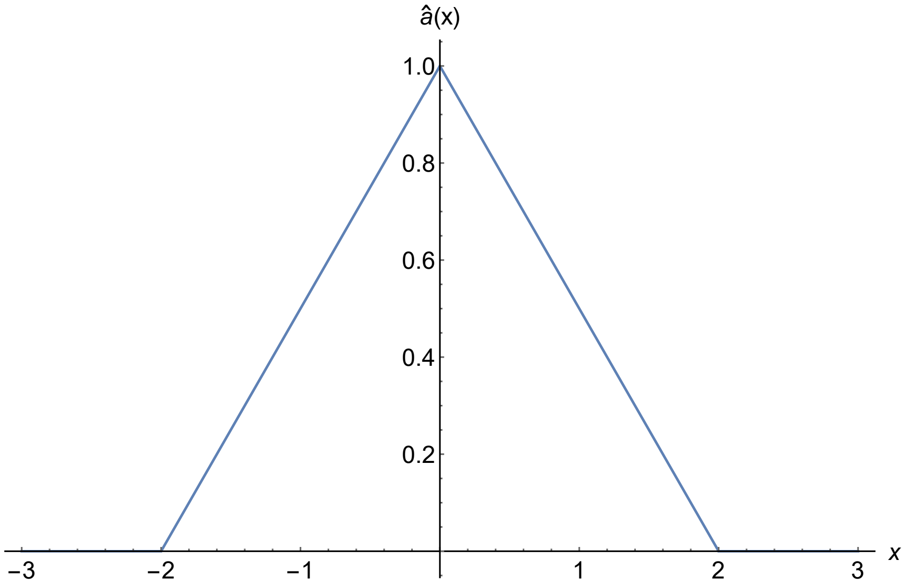


Figure 2.6: The graph of $\hat{a} = [-2 \ \ 0 \ \ 2]$

A metric on the set of all fuzzy numbers is defined in the following manner.

Definition 2.1.6. Let $A, B \subseteq \mathbb{R}^n$. The Hausdorff matric d_H is defined by

$$d_H(A,B) = \max\left\{\sup_{a\in A} \ \inf_{b\in B} ||a-b||, \sup_{b\in B} \ \inf_{a\in A} ||a-b||\right\}.$$

Then, the metric d_F on $F_{\mathbb{R}}$ is defined by

$$d_F(\hat{a},\hat{b}) = \sup_{0\leq\alpha\leq 1} \left\{d_H(\hat{a}_\alpha, \hat{b}_\alpha)\right\},$$

for all $\hat{a}$, $\hat{b} \in F_{\mathbb{R}}$. Since $\hat{a}_\alpha$ *and* $\hat{b}_\alpha$ are compact intervals in $\mathbb{R}$,

$$d_F(\hat{a},\hat{b}) = \sup_{0\leq\alpha\leq 1} max\left\{|\underline{a}_\alpha - \underline{b}_\alpha|, |\overline{a}_\alpha - \overline{b}_\alpha|\right\}.$$

To explain the previous definition, the following example is given.

Example 2.1.4. Let $A = [-1,1]$ and $B = [0,2]$. Then,

$$d_H(A,B) = \max\left\{\sup_{a\in A}\ \inf_{b\in B} ||x-y||, \sup_{b\in B}\ \inf_{a\in A} ||x-y||\right\} = max\{1,1\} = 1.$$

Let,

$$\hat{a}(x) = \begin{cases} 0, & x < -1 \\ x+1, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases} \quad and \quad \hat{b}(x) = \begin{cases} 0, & x < -2 \\ \frac{x+2}{2}, & -2 \le x \le 0 \\ \frac{2-x}{2}, & 0 \le x \le 2 \\ 0, & x > 2 \end{cases}.$$

Then, for $\alpha \in [0,1]$, $\hat{a}_\alpha = [\alpha-1, 1-\alpha]$ *and* $\hat{b}_\alpha = [2\alpha-2, 2-2\alpha]$. Thus,

$$d_F(\hat{a},\hat{b}) = \sup_{0\le\alpha\le 1} max\{|\alpha-1-(2\alpha-2)|, |1-\alpha-(2-2\alpha)|\}$$

$$= \sup_{0\le\alpha\le 1} max\{|1-\alpha|, |\alpha-1|\} = 1.$$

Theorem 2.1.1. *Let* $(R_{\mathbb{R}}, d_F)$ *be a complete metric space. For all* $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ $\in F_R$, *and* $\gamma \in \mathbb{R}$, *it holds that*

1. $d_F(\hat{a}\oplus\hat{c}, \hat{b}\oplus\hat{c}) = d_F(\hat{a}\oplus\hat{b})$.

2. $d_F(\lambda\odot\hat{a}, \lambda\odot\hat{c}) = |\lambda|\ d_F(\hat{a},\hat{c})$.

3. $d_F(\hat{a}\oplus\hat{b}, \hat{c}\oplus\hat{d}) \le d_F(\hat{a}\oplus\hat{c}) + d_F(\hat{b}\oplus\hat{d})$.

Proof. Simple calculations imply that

1.

$$d_F(\hat{a}\oplus\hat{c},\hat{b}\oplus\hat{c}) = \sup_{0\leq\alpha\leq1} max\left\{|(\underline{a}_\alpha+\underline{c}_\alpha)-(\underline{b}_\alpha+\underline{c}_\alpha)|,|(\overline{a}_\alpha+\overline{c}_\alpha)-(\overline{b}_\alpha+\overline{c}_\alpha)|\right\}$$

$$= max\left\{|\underline{a}_\alpha-\underline{b}_\alpha|,|\overline{a}_\alpha-\overline{b}_\alpha|\right\} = d_F(\hat{a}\oplus\hat{b}).$$

2.

$$d_F(\lambda\odot\hat{a},\lambda\odot\hat{c}) = \sup_{0\leq\alpha\leq1} max\{|\lambda\underline{a}_\alpha-\lambda\underline{c}_\alpha|,|\lambda\overline{a}_\alpha-\lambda\overline{c}_\alpha|\}$$

$$= |\lambda|\sup_{0\leq\alpha\leq1} max\{|\underline{a}_\alpha-\underline{c}_\alpha|,|\overline{a}_\alpha-\overline{c}_\alpha|\} = |\lambda|d_F(\hat{a},\hat{c}).$$

3. The triangle inequality implies that

$$|(\underline{a}_\alpha+\underline{b}_\alpha)-(\underline{c}_\alpha+\underline{d}_\alpha)| \leq |\underline{a}_\alpha-\underline{c}_\alpha|+|\underline{b}_\alpha-\underline{d}_\alpha|$$

and

$$|(\overline{a}_\alpha+\overline{b}_\alpha)-(\overline{c}_\alpha+\overline{d}_\alpha)| \leq |\overline{a}_\alpha-\overline{c}_\alpha|+|\overline{b}_\alpha-\overline{d}_\alpha)|.$$

Thus,

$$max\left\{|(\underline{a}_\alpha+\underline{b}_\alpha)-(\underline{c}_\alpha+\underline{d}_\alpha)|,|(\overline{a}_\alpha+\overline{b}_\alpha)-(\overline{c}_\alpha+\overline{d}_\alpha)|\right\} \leq$$

$$max\{|\underline{a}_\alpha-\underline{c}_\alpha|,|\overline{a}_\alpha-\overline{c}_\alpha|\}+max\left\{|\underline{b}_\alpha-\underline{d}_\alpha|,|\overline{b}_\alpha-\overline{d}_\alpha)|\right\}.$$

Hence, $d_F(\hat{a}\oplus\hat{b},\hat{c}\oplus\hat{d}) \leq d_F(\hat{a}\oplus\hat{c})+d_F(\hat{b}\oplus\hat{d})$.

More proprieties of fuzzy numbers are given in the next theorem.

Theorem 2.1.2. *Let $F_{\mathbb{R}}$ be the set of all fuzzy numbers, then*

1. $\hat{0}(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}$ *, and, $\hat{0} \in F_{\mathbb{R}}$ is identity element with respect to $\oplus$.*

2. *None of $\hat{a} \in F_{\mathbb{R}} - \mathbb{R}$ has inverse in $F_{\mathbb{R}}$ with respect to $\oplus$.*

3. *For any $x,\ y \geq 0$ or $x,\ y \leq 0$ and any $\hat{a} \in F_{\mathbb{R}}$, $(x+y) \odot \hat{a} = x \odot \hat{a} \oplus y \odot \hat{a}$. The result is not true in general.*

4. *For any $\lambda \in \mathbb{R}$ and any $\hat{a},\ \hat{b} \in F_{\mathbb{R}}$, $\lambda \odot (\hat{a} \oplus \hat{b}) = \lambda \odot \hat{a} \oplus \lambda \odot \hat{b}$.*

5. *For any $\lambda,\ \mu \in \mathbb{R}$ and any $\hat{a} \in F_{\mathbb{R}}$, $\lambda \odot (\mu \oplus \hat{a}) = (\lambda\mu) \odot \hat{a}$.*

Proof.

1. Let $\hat{a} \in F_{\mathbb{R}}$. Then, for $\alpha \in [0,1]$,

$$(\hat{a} \oplus \hat{0})_\alpha = [\underline{a}_\alpha + 0, \overline{a} + 0] = [\underline{a}_\alpha, \overline{a}_\alpha] = \hat{a}_\alpha$$

and

$$(\hat{0} \oplus \hat{a})_\alpha = [0 + \underline{a}_\alpha, 0 + \overline{a}] = [\underline{a}_\alpha, \overline{a}_\alpha] = \hat{a}_\alpha.$$

Thus, $\hat{a} \oplus \hat{0} = \hat{0} \oplus \hat{a} = \hat{a}_\alpha$.

2. Let $\hat{a} \in F_{\mathbb{R}} - \mathbb{R}$ and $\hat{b} \in F_{\mathbb{R}}$ such that

$$(\hat{a} \oplus \hat{b})_\alpha = [\underline{a}_\alpha + \underline{b}_\alpha, \overline{a}_\alpha + \overline{b}_\alpha] = [0,0]$$

Then, $\underline{b}_\alpha = -\underline{a}_\alpha$ and $\overline{b}_\alpha = -\overline{a}_\alpha$. Since $\underline{a}_\alpha \leq \overline{a}_\alpha$ and $\underline{b}_\alpha \leq \overline{b}_\alpha$, $\underline{a}_\alpha = \overline{a}_\alpha$ for $\alpha \in [0,1]$. Hence, $\hat{a} \in \mathbb{R}$ which is a contradiction. Thus, for all $\hat{a} \in F_{\mathbb{R}} - \mathbb{R}$ it has no inverse in $F_{\mathbb{R}}$, with respect to $\oplus$.

3. For any $x,\ y \geq 0$ and any $\hat{a} \in F_{\mathbb{R}}$,

$$((x+y)\odot\hat{a})_\alpha = [(x+y)\odot\underline{a}_\alpha, (x+y)\odot\overline{a}_\alpha] = [x\underline{a}_\alpha, x\overline{a}_\alpha] \oplus [y\underline{a}_\alpha, y\overline{a}_\alpha] = (x\odot\hat{a}\oplus y\odot\hat{b})_\alpha.$$

Also, for any $x,\ y \leq 0$ and any $\hat{a} \in F_{\mathbb{R}}$, it holds that

$$((x+y)\odot\hat{a})_\alpha = [(x+y)\odot\overline{a}_\alpha, (x+y)\odot\underline{a}_\alpha] = [x\overline{a}_\alpha, x\underline{a}_\alpha] \oplus [y\overline{a}_\alpha, y\underline{a}_\alpha] = (x\odot\hat{a}\oplus y\odot\hat{b})_\alpha$$

for any $\alpha \in [0,1]$. In general, the result is not true. Let $x = 1$, $y = -2$, and $\hat{a} : \mathbb{R} \to [0,1]$ be defined by

$$\hat{a}(x) = \frac{1}{1+x^2}.$$

Then,

$$((x+y)\odot\hat{a})_{0.5} = [-1,1]$$

and

$$(x\odot\hat{a}\oplus y\odot\hat{a})_{0.5} = [-1,1]\oplus[-2,2] = [-3,3].$$

4. For any $\lambda \geq 0$ and any $\hat{a},\ \hat{b} \in F_{\mathbb{R}}$,

$$(\lambda\odot(\hat{a}\oplus\hat{b}))_\alpha = [\lambda(\underline{a}_\alpha+\underline{b}_\alpha), \lambda(\overline{a}_\alpha+\overline{b}_\alpha)] = [\lambda\underline{a}_\alpha, \lambda\overline{a}_\alpha] \oplus [\lambda\underline{b}_\alpha, \lambda\overline{b}_\alpha]$$

$$= (\lambda\odot[\underline{a}_\alpha, \overline{a}_\alpha]) \oplus (\lambda\odot[\underline{b}_\alpha, \overline{b}_\alpha]) = (\lambda\odot\hat{a}\oplus\lambda\odot\hat{b})_\alpha,$$

and for any $\lambda \leq 0$ and any $\hat{a},\ \hat{b} \in F_{\mathbb{R}}$, it holds that

$$(\lambda\odot(\hat{a}\oplus\hat{b}))_\alpha = [\lambda(\overline{a}_\alpha+\overline{b}_\alpha), \lambda(\underline{a}_\alpha+\underline{b}_\alpha)] = [\lambda\overline{a}_\alpha, \lambda\underline{a}_\alpha] \oplus [\lambda\overline{b}_\alpha, \lambda\underline{b}_\alpha]$$

$$= (\lambda\odot[\underline{a}_\alpha, \overline{a}_\alpha]) \oplus (\lambda\odot[\underline{b}_\alpha, \overline{b}_\alpha]) = (\lambda\odot\hat{a}\oplus\lambda\odot\hat{b})_\alpha,$$

for any $\alpha \in [0,1]$.

5. For any $\lambda,\ \mu \in \mathbb{R}$ and any $\hat{a} \in F_{\mathbb{R}}$,

$$\lambda \odot (\mu \oplus \hat{a})_{\alpha} = \begin{cases} \lambda \odot [\mu \underline{a}_{\alpha}, \mu \overline{a}_{\alpha}], & \mu \geq 0 \\ \lambda \odot [\mu \overline{a}_{\alpha}, \mu \underline{a}_{\alpha}], & \mu \leq 0 \end{cases}$$

$$= \begin{cases} [\lambda \mu \underline{a}_{\alpha}, \lambda \mu \overline{a}_{\alpha}], & \mu \geq 0, \lambda \geq 0 \\ [\lambda \mu \overline{a}_{\alpha}, \lambda \mu \underline{a}_{\alpha}], & \mu \geq 0, \lambda < 0 \\ [\lambda \mu \overline{a}_{\alpha}, \lambda \mu \underline{a}_{\alpha}], & \mu < 0, \lambda \geq 0 \\ [\lambda \mu \underline{a}_{\alpha}, \lambda \mu \overline{a}_{\alpha}], & \mu < 0, \lambda < 0 \end{cases}$$

$$= (\lambda \mu \odot \hat{a})_{\alpha}$$

for any $\alpha \in [0,1]$.

2.2 Differentiation of Fuzzy Functions

In this section, differentiation of fuzzy functions will be presented using different approaches such as the Hukuhara differentiation and the gH differentiation. Besides, some related properties and results will be given.

Definition 2.2.1. Let V be a real vector space and $F_{\mathbb{R}}$ be the set of fuzzy numbers. Then, a function $\hat{h}: V \to F_{\mathbb{R}}$ is called fuzzy-valued function on V. For any $\alpha \in [0,1]$, $\hat{h}(x)$ can be written as $[f_{\alpha}(x), g_{\alpha}(x)]$ for all $x \in V$. The functions $f_{\alpha}(x)$ and $g_{\alpha}(x)$ are called α-level functions of the fuzzy-valued function $\hat{h}$.

Example 2.2.1. Let $\hat{h} : V \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{h}(x) = \hat{a} \odot x,$$

where $\hat{a}$ is a fuzzy number. Then, for any $\alpha \in [0,1]$,

$$\hat{h}(x) = \begin{cases} [x\underline{a}_\alpha, x\overline{a}_\alpha], & x \geq 0 \\ [x\overline{a}_\alpha, x\underline{a}_\alpha], & x < 0 \end{cases}.$$

Definition 2.2.2. Let $\hat{a}$ and $\hat{b}$ be two fuzzy numbers. If there exist number $\hat{c}$ such that $\hat{c} \oplus \hat{b} = \hat{a}$. Then, $\hat{c}$ is called Hukuhara difference of $\hat{a}$ and $\hat{b}$ and it is denoted by $\hat{a} \ominus_H \hat{b}$.

Example 2.2.2. Let $\hat{a} = (-2 \;\; 0 \;\; 2)$ and $\hat{b} = (-1 \;\; 3 \;\; 7)$ be two triangular fuzzy numbers. Then,

$$\hat{a} = \hat{c} \oplus \hat{b},$$

where $\hat{c} = (-1 \quad -3 \quad -5)$. Then,

$$\hat{c} = \hat{a} \ominus_H \hat{b}.$$

It is worth to mention that the following two important properties should satisfy which are

1. $\hat{0} = \hat{a} \ominus_H \hat{a}$.
2. $((\hat{a} \oplus \hat{b}) \ominus_H \hat{b})_\alpha = \hat{a}_\alpha \quad$ for all $\alpha \in [0,1]$.

In Example 2.2.2, the two properties are hold.

Definition 2.2.3. [7] Let A be a subset of $\mathbb{R}$. A fuzzy function $\hat{f} : A \to F_{\mathbb{R}}$ is said to be H-differntiable at $x_0 \in A$ if and only if there exists a fuzzy number $D\hat{f}(x_0)$ such that the following limits (with respect to metric d_F) hold true

$$D\hat{f}(x_0) = \lim_{h \to 0^+} \frac{1}{h} \odot (\hat{f}(x_0 + h) \ominus_H \hat{f}(x_0))$$

and

$$D\hat{f}(x_0) = \lim_{h \to 0^+} \frac{1}{h} \odot (\hat{f}(x_0) \ominus_H \hat{f}(x_0 - h)).$$

In this case, $D\hat{f}(x_0) = \hat{f}'(x_0)$ is called H-derivative of $\hat{f}$ at x_0 . If $\hat{f}$ is H-differntiable $\forall x \in A$, then, $\hat{f}$ is H-differntiable over A.

Example 2.2.3. Let $\hat{f} : A \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{f}(x) = \hat{a} \odot x,$$

where $\hat{a}$ is a fuzzy number . For $x > 0$,

$$(\hat{f}(x+h) \ominus_H \hat{f}(x))_\alpha = ((\hat{a} \odot (x+h)) \ominus_H (\hat{a} \odot x))_\alpha$$

$$= [(x+h)\underline{a}_\alpha, (x+h)\overline{a}_\alpha] \ominus_H [x\underline{a}_\alpha, x\overline{a}_\alpha] = [h\underline{a}_\alpha, h\overline{a}_\alpha].$$

Thus, for $h > 0$,

$$\frac{1}{h} \odot (\hat{f}(x_0 + h) \ominus_H \hat{f}(x_0)))_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha] = \hat{a}_\alpha$$

which implies that

$$\lim_{h \to 0^+} \frac{1}{h} \odot (\hat{f}(x+h) \ominus_H \hat{f}(x)) = \hat{a}.$$

Similarly, for small $h > 0$, and $x - h > 0$,

$$(\hat{f}(x) \ominus_H \hat{f}(x-h))_\alpha = ((\hat{a} \odot x) \ominus_H (\hat{a} \odot (h-x)))_\alpha$$

$$= [x\underline{a}_\alpha, x\overline{a}_\alpha] \ominus_H [(h-x)\underline{a}_\alpha, (h-x)\overline{a}_\alpha] = [h\underline{a}_\alpha, h\overline{a}_\alpha].$$

Thus,

$$\frac{1}{h} \odot (\hat{f}(x) \ominus_H \hat{f}(x-h)))_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha] = \hat{a}_\alpha$$

which implies that

$$\lim_{h \to 0^+} \frac{1}{h} \odot (\hat{f}(x) \ominus_H \hat{f}(x-h)) = \hat{a}.$$

Thus, $D\hat{f}(x) = \hat{a}$. For $x < 0$, $x + h < 0$ for small $h > 0$. Thus,

$$(\hat{f}(x+h) \ominus_H \hat{f}(x))_\alpha = ((\hat{a} \odot (x+h)) \ominus_H (\hat{a} \odot x))_\alpha$$

$$= [(x+h)\overline{a}_\alpha, (x+h)\underline{a}_\alpha] \ominus_H [x\overline{a}_\alpha, x\underline{a}_\alpha] = [h\overline{a}_\alpha, h\underline{a}_\alpha].$$

However, $h\overline{a} \nleq h\underline{a}$ for $\alpha \in [0,1]$. Thus, Hukuhara difference does not exist which means that $\hat{f}(x)$ is not H-differntiable. When $x = 0$,

$$(\hat{f}(0) \ominus_H \hat{f}(0-h))_\alpha = ((\hat{a} \odot 0) \ominus_H (\hat{a} \odot (0-h)))_\alpha = [h\overline{a}_\alpha, h\underline{a}_\alpha].$$

Thus,

$$(\frac{1}{h} \odot (\hat{f}(0) \ominus_H \hat{f}(0-h)))_\alpha = [\overline{a}_\alpha, \underline{a}_\alpha]$$

which implies that

$$\lim_{h \to 0^+} (\frac{1}{h} \odot (\hat{f}(0) \ominus_H \hat{f}(0-h)) = [\overline{a}, \underline{a}].$$

Also,

$$(\hat{f}(0+h) \ominus_H \hat{f}(0))_\alpha = ((\hat{a} \odot h) \ominus_H (\hat{a} \odot 0))_\alpha = [h\underline{a}_\alpha, h\overline{a}_\alpha].$$

Thus,

$$(\frac{1}{h} \odot (\hat{f}(0+h) \ominus_H \hat{f}(0)))_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$$

which implies that

$$\lim_{h \to 0^+} (\frac{1}{h} \odot (\hat{f}(0+h) \ominus_H \hat{f}(0)) = \hat{a}.$$

Thus, $\hat{f}(x)$ is not H-differntiable at $x = 0$. Therefore, $\hat{f}(x)$ is H-differntiable when $x > 0$ and $D\hat{f}(x) = \hat{a}$.

Theorem 2.2.1. *[21] Let $\hat{f} : I \to F_{\mathbb{R}}$ be a fuzzy function defined by*

$$\hat{f}(x) = \hat{a} \odot g(x),$$

where $\hat{a}$ is a fuzzy number and $I = (b,c) \subseteq \mathbb{R}$. Let $g : I \to \mathbb{R}^+$ be differentiable function at $x_0 \in I$. If $g'(x_0) > 0$, then,

1. *Hukuhara differences in Definition 2.2.3 of $\hat{f}$ exist at x_0.*
2. *$\hat{f}$ is H-differntiable at x_0.*
3. *$\hat{f}'(x) = \hat{a} \odot \hat{g}'(x)$.*

Example 2.2.4. a) Let $\hat{f} : \mathbb{R} \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{f}(x) = \hat{a} \odot x,$$

where $\hat{a}$ is a fuzzy number and $g(x) = x$. Then, $g'(x) = 1$. Hence, $g(x)$ and $g'(x)$ are positive when $x > 0$. Thus $\hat{f}$ is H-differntiable on $(0,\infty)$ and $\hat{f}'(x) = \hat{a}$. However, last theorem can not be used when $x \leq 0$ since $g(x) \leq 0$.

b) Let $\hat{f} : \mathbb{R} \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{f}(x) = \hat{a},$$

where $\hat{a}$ is a fuzzy number and $g(x) = 1$. Then, $g'(x) = 0$. Hence, Theorem 2.2.1 can not be used. However, using the definition, it can be seen that

$$D\hat{f}(x) = \lim_{h \to 0^+} \frac{1}{h} \odot (\hat{f}(x+h) \ominus_H \hat{f}(x)) = \hat{0},$$

and

$$D\hat{f}(x) = \lim_{h \to 0^+} \frac{1}{h} \odot (\hat{f}(x) \ominus_H \hat{f}(x-h)) = \hat{0}.$$

Thus, $\hat{f}$ is H-differntiable on $(-\infty,\infty)$ and $\hat{f}'(x) = 0$.

c) Let $\hat{f} : \mathbb{R}^+ \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{f}(x) = \hat{a} \odot x^2,$$

where $\hat{a}$ is a fuzzy number and $g(x) = x^2$. Then, $g'(x) = 2x$. Hence, $g(x)$ and $g'(x)$ are positive when $x > 0$. Thus $\hat{f}$ is H-differntiable on $(0,\infty)$ and $\hat{f}'(x) = \hat{a} \odot 2x$.

d) Let $\hat{f} : (0,\infty) \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{f}(x) = \hat{a} \odot sinh(x),$$

where $\hat{a}$ is a fuzzy number and $g(x) = sinh(x)$. Then, $g'(x) = cosh(x)$. Hence, $g(x)$

and $g'(x)$ are positive when $x > 0$. Thus $\hat{f}$ is H-differntiable on $(0,\infty)$ and $\hat{f}'(x) = \hat{a} \odot cosh(x)$. Moreover,

$$\hat{g}^{(n)}(x) = \begin{cases} cosh(x), & n \text{ is odd} \\ sinh(x), & n \text{ is even} \end{cases}$$

and so $\hat{f}(x)$ is n-time H-differntiable on $(0,\infty)$ and

$$D^n \hat{f}(x) = \hat{f}^{(n)}(x) = \begin{cases} \hat{a} \odot cosh(x), & n \text{ is odd} \\ \hat{a} \odot sinh(x), & n \text{ is even.} \end{cases}$$

Theorem 2.2.2. *Let $\hat{h} : I \to F_{\mathbb{R}}$ be H-differntiable at x_0 with derivative $h'(x_0)$ and $\hat{h} = [f_\alpha(x), g_\alpha(x)]$ where $I \subset \mathbb{R}$ and $x_0 \in I$. Then, $(\hat{h}'(x_0))_\alpha = [f'_\alpha(x_0), g'_\alpha(x_0)]$ and $f'_\alpha(x_0)$, $g'_\alpha(x_0)$ are differentable at x_0 for all $\alpha \in [0,1]$.*

Definition 2.2.4. [7] Given two fuzzy numbers $\hat{a}$, $\hat{b} \in F_{\mathbb{R}}$, the gH-difference is the fuzzy number $\hat{c}$, if exists, such that

$$\hat{a} \ominus_{gH} \hat{b} = \hat{c} \; if \; either \; \hat{a} = \hat{b} + \hat{c} \; or \; \hat{b} = \hat{a} - \hat{c}.$$

Thus,

$$(\hat{a} \ominus_{gH} \hat{b})_\alpha = [min\{\underline{a}_\alpha - \underline{b}_\alpha, \overline{a}_\alpha - \overline{b}_\alpha\}, max\{\underline{a}_\alpha - \underline{b}_\alpha, \overline{a}_\alpha - \overline{b}_\alpha\}],$$

and if H-difference exists, then $\hat{a} \ominus_{gH} \hat{b} = \hat{a} \ominus_H \hat{b}$. Hence, $\hat{a} \ominus_{gH} \hat{b} = \hat{c}$ exists if either

1. $\underline{c}_\alpha = \underline{a}_\alpha - \underline{b}_\alpha$ and $\overline{c}_\alpha = \overline{a}_\alpha - \overline{b}_\alpha$ with $\underline{c}_\alpha$ is increasing and $\overline{c}_\alpha$ is decreasing with $\underline{c}_\alpha \leq \overline{c}_\alpha$ for all $\alpha \in [0,1]$, or

2. $\underline{c}_\alpha = \overline{a}_\alpha - \overline{b}_\alpha$ and $\overline{c}_\alpha = \underline{a}_\alpha - \underline{b}_\alpha$ with $\underline{c}_\alpha$ is increasing and $\overline{c}_\alpha$ is decreasing with $\underline{c}_\alpha \leq \overline{c}_\alpha$ for all $\alpha \in [0,1]$.

Example 2.2.5. a) Let $\hat{a} = (-1 \ \ 0 \ \ 1)$ and $\hat{b} = (3 \ \ 4 \ \ 5)$ be two triangle fuzzy numbers. Then,

$$\underline{(\hat{a} \ominus_{gH} \hat{b})}_\alpha = min\{\alpha - 1 - (\alpha + 3), 1 - \alpha - (5 - \alpha)\} = -4$$

and

$$\overline{(\hat{a} \ominus_{gH} \hat{b})}_\alpha = max\{\alpha - 1 - (\alpha + 3), 1 - \alpha - (5 - \alpha)\} = -4.$$

Thus, conditions (1) and (2) in Definition 2.2.4 are hold. Hence, $(\hat{a} \ominus_{gH} \hat{b})_\alpha$ exists.

b) Let $\hat{a} = (0 \ \ 2 \ \ 5)$ and $\hat{b} = (0 \ \ 1 \ \ 2 \ \ 3)$ be triangle and trapezoidal fuzzy numbers. Then, using Condition (1) in Definition 2.2.4 when $\alpha = 1$,

$$\underline{(\hat{a} \ominus_{gH} \hat{b})}_1 = 2 - 1 = 1$$

and

$$\overline{(\hat{a} \ominus_{gH} \hat{b})}_1 = 2 - 2 = 0.$$

Then, $\underline{(\hat{a} \ominus_{gH} \hat{b})}_1 \nleq \overline{(\hat{a} \ominus_{gH} \hat{b})}_1$. Also, using Condition (2) in Definition 2.2.4 when $\alpha = 0$,

$$\underline{(\hat{a} \ominus_{gH} \hat{b})}_0 = 5 - 3 = 2,$$

and

$$\overline{(\hat{a} \ominus_{gH} \hat{b})}_0 = 0 - 0 = 0.$$

Then, $\underline{(\hat{a}\ominus_{gH}\hat{b})}_0 \nleq \overline{(\hat{a}\ominus_{gH}\hat{b})}_0$. Condition (1) and (2) in Definition 2.2.4 do not hold. Thus, $(\hat{a}\ominus_{gH}\hat{b})_\alpha$ does not exists.

Definition 2.2.5. Let A be an interval of $\mathbb{R}$. Let $x_0,\ x_0+h\in A$. A fuzzy function $\hat{f}:A\to F_{\mathbb{R}}$ is said to be gH-differntiable at x_0 if and only if there exists a fuzzy number $\hat{f}'_{gH}(x_0)$ such that (with respect to metric d_F)

$$\hat{f}'_{gH}(x_0)=\lim_{h\to 0}\frac{1}{h}(\hat{f}(x_0+h)\ominus_{gH}\hat{f}(x_0)).$$

In this case, $D\hat{f}_{gH}(x_0)=\hat{f}'_{gH}(x_0)$ is called gH-derivative of $\hat{f}$ at x_0. If $\hat{f}$ is gH-differntiable at all $x\in A$, $\hat{f}$ then, is gH-differntiable over A.

Example 2.2.6. a) Let $\hat{f}:\mathbb{R}\to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{f}(x)=\hat{a}\odot x,$$

where $\hat{a}$ is a fuzzy number. Then, for $h>0$,

$$(\hat{f}(x+h)\ominus_{gH}\hat{f}(x))_\alpha=[min\{\underline{f(x+h)}_\alpha-\underline{f(x)}_\alpha,\overline{f(x+h)}_\alpha-\overline{f(x)}_\alpha\},max\{\underline{f(x+h)}_\alpha$$

$$-\underline{f(x)}_\alpha,\overline{f(x+h)}_\alpha-\overline{f(x)}_\alpha\}]=[\underline{a}_\alpha h,\overline{a}_\alpha h],$$

and

$$(\hat{f}(x)\ominus_{gH}\hat{f}(x-h))_\alpha=[min\{\underline{f(x)}_\alpha-\underline{f(x-h)}_\alpha,\overline{f(x)}_\alpha-\overline{f(x-h)}_\alpha\},max\{\underline{f(x)}_\alpha$$

$$-\underline{f(x-h)}_\alpha,\overline{f(x)}_\alpha-\overline{f(x-h)}_\alpha\}]=[\underline{a}_\alpha h,\overline{a}_\alpha h],$$

for all $\alpha \in [0,1]$. Then,

$$\hat{f}'_{gH}(x) = \lim_{h\to 0} \frac{1}{h}(\hat{f}(x+h) \ominus_{gH} \hat{f}(x)) = [\underline{a}, \overline{a}] = \hat{a}.$$

Thus, $\hat{f}$ is gH-differntiable on $(-\infty,\infty)$ and $\hat{f}'_{gH}(x) = \hat{a}$.

b) Let $\hat{h} : \mathbb{R} \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{h}(x) = \hat{a},$$

where $\hat{a}$ is a fuzzy number. Then $h(x) = [f(x), g(x)] = [\underline{a}, \overline{a}]$ are differentiable, and using the same argument as in part (a), $\hat{h}$ is gH-differntiable on $(-\infty,\infty)$ and $\hat{h}'_{gH}(x) = \hat{0}$.

c) Let $\hat{h} : \mathbb{R} \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{h}(x) = \hat{a} \odot x^2,$$

where $\hat{a}$ is a fuzzy number. Then $f(x) = \underline{a}x^2$ and $g(x) = \overline{a}x^2$ are differentiable, and using the same argument as in part (a), $\hat{h}$ is gH-differntiable on $(-\infty,\infty)$ and $\hat{h}'_{gH}(x) = \hat{a} \odot 2x$.

d) Let $\hat{h} : \mathbb{R} \to F_{\mathbb{R}}$ be a fuzzy function defined by

$$\hat{h}(x) = \hat{a} \odot \sinh(x),$$

where $\hat{a}$ is a fuzzy number. Then $f(x) = \underline{a}\sinh(x)$ and $g(x) = \overline{a}\sinh(x)$ are differentiable, and using the same argument as in part (a), $\hat{h}$ is gH-differntiable on $(-\infty,\infty)$

and $\hat{h}'_{gH}(x) = \hat{a} \odot \cosh(x)$.

Theorem 2.2.3. *If $\hat{f}$, $\hat{g} : A \to F_{\mathbb{R}}$ are H-differentiable at $x_0 \in A \subseteq \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $\hat{f} \oplus \hat{g}$ and $\gamma \odot \hat{f}$ are H-differentiable at x_0 and,*

$$(\hat{f} \oplus \hat{g})'(x_0) = \hat{f}'(x_0) \oplus \hat{g}'(x_0), \quad (\gamma \odot \hat{f})'(x_0) = \gamma \odot \hat{f}'(x_0).$$

Also, $\hat{f} \in C^n(A, F_{\mathbb{R}})$ if $(\hat{f}^{(i)}(x))_\alpha = [(\underline{f}(x))^{(i)}_\alpha, (\overline{f}(x))^{(i)}_\alpha]$ for $i = 0, 1, ..., n$ and $\alpha \in [0,1]$.

Example 2.2.7. Let the fuzzy function $\hat{f}(x) = \hat{a}_n \odot x^n \oplus \hat{a}_{n-1} \odot x^{n-1} \oplus ... \oplus \hat{a}_1 \odot x, \;\; n > 0$. Then, $\hat{f}$ are H-differentiable on $(0, \infty)$ and

$$\hat{f}'(x) = \hat{a}_n \odot nx^{n-1} \oplus \hat{a}_{n-1} \odot (n-1)x^{n-2} \oplus ... \oplus \hat{a}_1 \odot 1.$$

Theorem 2.2.4. *Let $\hat{a} \in F_{\mathbb{R}}$, $g : I^n \to \mathbb{R}^+$ and $I = (b,c) \subset \mathbb{R}^+$ be differentiable at $x_0 \in I^n$. Let $\hat{f} : I^n \to F_{\mathbb{R}}$ be defined by $\hat{f}(x) = \hat{a} \odot g(x)$. If $\dfrac{\partial g(x_0)}{\partial x_i} > 0$, for $i = 1, 2, ..., n$, then the partial derivative exists at x_0 and $\dfrac{\partial \hat{f}(x_0)}{\partial x_i} = \hat{a} \odot \dfrac{\partial g(x_0)}{\partial x_i}$ for $i = 1, 2, ..., n$.*

Example 2.2.8. Consider the fuzzy function $\hat{f}(x) = \hat{a} \odot e^{3x+2y}$.Then $g(x,y) = e^{3x+2y} > 0$. Since

$$\frac{\partial g}{\partial x} = 3\, e^{3x+2y} > 0, \;\; \frac{\partial g}{\partial y} = 2\, e^{3x+2y} > 0, \;\; \frac{\partial^2 g}{\partial x^2} = 9\, e^{3x+2y} > 0,$$

$$\frac{\partial^2 g}{\partial y^2} = 4\, e^{3x+2y} > 0, \;\; \frac{\partial^2 g}{\partial x \partial y} = 6\, e^{3x+2y} > 0,$$

then,

$$\frac{\partial \hat{f}}{\partial x} = \hat{a} \odot 3\, e^{3x+2y} > 0, \;\; \frac{\partial \hat{f}}{\partial y} = \hat{a} \odot 2\, e^{3x+2y} > 0, \;\; \frac{\partial^2 \hat{f}}{\partial x^2} = \hat{a} \odot 9\, e^{3x+2y} > 0,$$

$$\frac{\partial^2 \hat{f}}{\partial y^2} = \hat{a} \odot 4\, e^{3x+2y} > 0, \quad \frac{\partial^2 \hat{f}}{\partial x \partial y} = \hat{a} \odot 6\, e^{3x+2y} > 0.$$

It is easy to see that $\hat{f} \in C^{\infty}(\mathbb{R}^2, F_{\mathbb{R}})$.

Chapter 3: One-Step Hybrid Block Method with One Off-Step Point for Solving Initial Value Problems (IVPs)

In this chapter, the one-step hybrid block methods with one off-step point will be derived for first and second-order initial value problems. Some theoretical results will be presented. In addition, some numerical results will be given to show the efficiency of those methods.

3.1 Hybrid Block Method

In this section, the idea of the implicit hybrid one step method will be presented. Some related definitions are given. To explain the idea of these methods, consider the following initial value problem (IVP) of the form

$$y^{(k)} = f(t, y', y'', \ldots, y^{(k-1)}), \quad t_0 < t < T \tag{3.1}$$

subject to

$$y(t_0) = y_0,\ y'(t_0) = y'_0,\ \ldots,\ y(t_0)^{(k-1)} = y_0^{(k-1)}, \tag{3.2}$$

where $y_0, \ldots, y_0^{(k-1)}$ are real constants and f is smooth function on $[t_0, T] \times \mathbb{R}^k$. There are several methods to solve IVP (3.1-3.2) such as Euler, Taylor, Runge-kutta, Adams-Bashforth, and Adam-Moulton methods. The one-step methods such as Euler, Taylor, and Runge-kutta methods are suitable only for first order IVP since they have low order of accuracy. If Taylor or Runge-kutta methods are used to solve higher order IVP, then large function evaluations per step are needed. Therefore, solving Problem (3.1-3.2) by one step method is required to rewrite the problem into a system of first order IVPs which make the dimension of the problem and its scale are high. Thus, the approach will be costly with low accuracy. On the other side, Adams-Bashforth and Moulton methods do not need to rewrite Problem (3.1-3.2) in a system of first order IVPs. In addition, higher order accuracy will be produced by these methods. However, two main disadvantages for these methods, they are not efficient in terms of function evaluations and not self starter. To overcome these disadvantage the continuous implicit hybrid one step methods are used. Let $\{t_0, t_1, \ldots, t_m\}$ be a uniform partition of $[t_0, T]$ with $t_j = t_0 + j\,h, \quad j = 0, 1, \ldots, m$ and $h = \frac{T - t_0}{m}$. For $n \in \{0, 1, \ldots, m-1\}$, let $v_1, v_2, \ldots, v_L \in (0,1)$ be real numbers with

$$v_1 < v_2 < \cdots < v_L$$

and $t_{n+v_i} = t_n + v_i\,h, \quad i = 1, 2, \ldots, L$. The points $\{t_{n+v_i} : i = 1, 2, \ldots, L\}$ are called off-step points. The definition of one-step hybrid methods with L off-step points is given in the following definition.

Definition 3.1.1. [22] Let k be the order of IVP. A one-step hybrid formula with L off-step points $\{t_{n+v_i} : i = 1, 2, \ldots, L\}$ is given by

$$y_{n+1} + \sum_{i=0}^{k-1} a_i\, h^i\, y_n^{(i)} = h^k \left[\sum_{i=0}^{1} b_i\, f_{n+i} + \sum_{i=1}^{L} b_{n+v_i}\, f_{n+v_i} \right], \tag{3.3}$$

where a_0 and b_0 are non-zeros, $y_{n+i} \approx y(t_n + i\,h)$, and $f_{n+v_i} \approx f(t_{n+v_i}, y_{n+v_i})$.

The order of formula 3.3 can be found by the following definition.

Definition 3.1.2. [22] Let

$$\mathscr{L}\left[y\left(t_n\right); h\right] = \sum_{s=0}^{\infty} \left[\frac{y^{(s)}\left(t_n\right)\, h^s}{s!} + \sum_{i=0}^{k-1} \frac{a_i\, y^{(s+i)}\left(t_n\right)\, h^{s+i}}{s!} - \sum_{i=0}^{1} \frac{b_i\, y^{(s+k)}\left(t_n\right)\, i^s\, h^{2k+s}}{s!} \right.$$

$$\left. - \sum_{i=1}^{L} \frac{b_{n+v_i} + f_{n+v_i}\, y^{(s+k)}\left(t_n\right)\, \left(v_i\right)^s\, h^{2k+s}}{s!} \right] = c_0 y_n + c_1 y_n' + c_2 y_n'' + \ldots.$$

If $c_0 = c_1 = c_2 = \cdots = c_{p+k-1} = 0$ and $c_{p+k} \neq 0$, then the order of the method is p and the error constant is c_{p+k}.

Rewrite System 3.3 in matrix form as

$$A_0\, Y_m = A_1\, y_m + A_2\, F_m,$$

where

$$Y_m = \begin{pmatrix} y_{n+v_1} \\ \vdots \\ y_{n+v_L} \\ y_{n+1} \\ h\, y'_{n+v_1} \\ \vdots \\ h\, y'_{n+v_L} \\ h\, y'_{n+1} \\ \vdots \\ h^{k-1}\, y^{k-1}_{n+v_1} \\ \vdots \\ h^{k-1}\, y^{k-1}_{n+v_L} \\ h^{k-1}\, y^{k-1}_{n+1} \end{pmatrix}, \quad y_m = \begin{pmatrix} y_{n-v_1} \\ \vdots \\ y_{n-v_L} \\ y_n \\ h\, y'_{n-v_1} \\ \vdots \\ h\, y'_{n-v_L} \\ h\, y'_n \\ \vdots \\ h^{k-1}\, y^{(k-1)}_{n-v_1} \\ \vdots \\ h^{k-1}\, y^{(k-1)}_{n-v_L} \\ h^{k-1}\, y^{(k-1)}_n \end{pmatrix}, \quad F_m = \begin{pmatrix} h^k\, f_n \\ h^k\, f_{n+v_1} \\ \vdots \\ h^k\, f_{n+v_L} \\ h^k\, f_{n+1} \end{pmatrix}.$$

Then, following to Fatunla's approach [23] the characteristic equation is given by

$$\det(\mu\, A_0 - A_1) = 0. \tag{3.4}$$

If all roots of Equation 3.4 have modules less than or equal 1 and the algebraic multiplicity of each nonzero root of Equation 3.4 is less than or equal k the order of the IVP, then the method is called zero stable. If the order of the method is greater than or equal 1, then it is called consistent. If it is zero stable and consistent, the method is convergent.

To find the region of absolute stability, the following test problem is considered

$$y' = \lambda\, y, \ \ where \ \ \lambda < 0.$$

Then,

$$y^{(s)} = \lambda^s\, y, \quad s \in \{1, 2, \ldots, k\}. \tag{3.5}$$

Substitute (3.5) in (3.3) to get

$$Y_{n+1} = M(\alpha)\, Y_n, \quad \alpha = \lambda\, h.$$

The eigenvalue of $M(\alpha)$ are zeros except the last eigenvalue is $\lambda_n(\alpha)$.
Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \lambda_n(z)$. Then the region of absolute stability is

$$R = \{z \in \mathbb{C} : |f(z)| < 1\}.$$

If $\{z \in \mathbb{C} : Re(z) < 0\} \subset R$, then the method is called A-stable.

3.2 First Order Initial Value Problems

In this section, a numerical method based on the one-step hybrid block method with one off-step point (HBM1), t_{n+k} where $0 < k < 1$, will be used to solve the following initial value problem (IVP)

$$y'(t) = f(t, y(t)), \ \ t \geq 0 \tag{3.6}$$

$$y(t_0) = y_0. \tag{3.7}$$

To derive HBM1, assume that $t_n = nh$, where h is the stepsize. The solution of the IVP (3.6-3.7) will be approximated by a polynomial of degree 3 as follows

$$y(t) \simeq \sum_{j=0}^{3} c_j t^j, \tag{3.8}$$

and its derivative by

$$y'(t) \simeq \sum_{j=1}^{3} j c_j t^{j-1}. \tag{3.9}$$

By interpolating Equation (3.8) at the point t_n and collocating Equation (3.9) at the points t_n, $t_{n+k} = t_n + kh$, and $t_{n+1} = t_n + h$, the following system become

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 \\ 0 & 1 & 2t_n & 3t_n^2 \\ 0 & 1 & 2t_{n+k} & 3t_{n+k}^2 \\ 0 & 1 & 2t_{n+1} & 3t_{n+1}^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} y_n \\ f_n \\ f_{n+k} \\ f_{n+1} \end{pmatrix}, \tag{3.10}$$

where $y_{n+j} \approx y(t_{n+j})$ and $f_{n+j} \approx y'(t_{n+j})$, $j = 0, k, 1$. Then, the System (3.10) is solved after substituting $t = t_n + wh$, to get

$$y(w) \approx y_n + h(\alpha_0 f_n + \alpha_k f_{n+k} + \alpha_1 f_{n+1}), \tag{3.11}$$

where α_0, α_k, and α_1 are functions of w. Evaluating the approximation of $y(w)$ at $w = k$ and 1 yields

$$y_{n+j} = y_n + h(\alpha_{0,j} f_n + \alpha_{k,j} f_{n+k} + \alpha_{1,j} f_{n+1}), \tag{3.12}$$

where $j = k, 1$ and

$$\begin{aligned} \alpha_{0,1} &= \frac{3k-1}{6k}, \alpha_{k,1} = \frac{1}{6k-6k^2}, \alpha_{1,1} = \frac{3k-2}{6(k-1)}, \\ \alpha_{0,k} &= \frac{k(3-k)}{6}, \alpha_{k,k} = \frac{k(2k-3)}{6(k-1)}, \alpha_{1,k} = \frac{k^3}{6(k-1)}. \end{aligned}$$

To maximize the order of the implicit block method (3.12), the local truncation errors in the formula for y_{n+1} is optimized as follow,

$$\mathscr{L}\left(y\left(t_{n+1}\right);h\right)=\frac{2k-1}{72}h^{4}y^{(4)}\left(t_{n}\right)+\frac{h^{5}\ \left(5k^{2}+5k-4\right)}{720}y^{(5)}\left(t_{n}\right)+\mathscr{O}\left(h^{6}\right). \tag{3.13}$$

To maximize the order of formula (3.12) where $j=1$, the following equation for k is solved, where $0<k<1$,

$$\frac{2k-1}{72}=0\ . \tag{3.14}$$

Hence,

$$k=\frac{1}{2}, \tag{3.15}$$

and the local truncation errors is given by

$$\mathscr{L}\left[y\left(t_{n+1}\right);h\right]=-\frac{h^{5}y^{(5)}\left(t_{n}\right)}{2880}+\mathscr{O}\left(h^{6}\right)=-3.4722\times10^{-4}h^{5}y^{(5)}\left(t_{n}\right)+\mathscr{O}\left(h^{6}\right).$$

Therefore, the HBM1 is given by

$$\begin{aligned} y_{n+1}&=y_{n}+\frac{h}{6}(f_{n}+4f_{n+\frac{1}{2}}+f_{n+1}),\\ y_{n+\frac{1}{2}}&=y_{n}+\frac{h}{24}(5f_{n}+8f_{n+\frac{1}{2}}-f_{n+1}). \end{aligned} \tag{3.16}$$

Now, some theoretical results on the one-step hybrid block method with one off-step point are presented. These results include consistency, stability, and convergence results. Let us write the System (3.16) as follows

$$\begin{aligned} y_{n+\frac{1}{2}}&=y_{n}+\frac{h}{24}(5f_{n}+8f_{n+\frac{1}{2}}-f_{n+1}),\\ y_{n+1}&=y_{n}+\frac{h}{6}(f_{n}+4f_{n+\frac{1}{2}}+f_{n+1}). \end{aligned} \tag{3.17}$$

First, rewrite the HBM1 in Equation (3.17) in the form

$$A_1\,Y_m = A_0\,y_m + A_2\,F_m, \tag{3.18}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\; A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\; A_2 = \begin{pmatrix} \frac{5}{24} & \frac{8}{24} & \frac{-1}{24} \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{pmatrix},$$

$$y_m = \begin{pmatrix} y_{n-\frac{1}{2}} \\ y_n \end{pmatrix}, Y_m = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \end{pmatrix}, F_m = \begin{pmatrix} h\,f_n \\ h\,f_{n+\frac{1}{2}} \\ h\,f_{n+1} \end{pmatrix}.$$

Thus, the characteristic equation of the HBM1 in Equation (3.21) is given by

$$\rho(z) = \det(zA_1 - A_0) = z(z-1) = 0, \tag{3.19}$$

which implies that $z_1 = 0$ and $z_2 = 1$. Then, the multiplicity of the nonzero root of the characteristic equation is 1, which does not exceed the order of the differential equation. Hence, the method is zero stable.

The local truncation error of the System (3.16) is

$$\begin{aligned} \mathscr{L}[y(t_n);h] &= \left(\mathscr{L}\left[y\left(t_{n+\frac{1}{2}}\right);h\right], \mathscr{L}[y(t_{n+1});h]\right) \\ &= \sum_{i=0}^{\infty} \alpha_i h^i y^{(i)}(t_n) = \alpha_4 h^4 y^{(4)}(t_n) + \sum_{i=5}^{\infty} \alpha_i h^i y^{(i)}(t_n), \end{aligned} \tag{3.20}$$

where $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_4 = (\frac{1}{384}, 0)^T$. Thus, System (3.16) has order $(3,3)^T$. For simplicity, the order is denoted by 3. Since the order of the System (3.16) is $3 \geq 1$, then it is consistent. The consistency and the zero stability of the System (3.16) imply that it is convergent [22, 24]. To find the region of absolute stability, consider the following test problem $y' = \lambda y$ where $\lambda < 0$, then

$$y' = f(t,y) = \lambda y.$$

Substitute f in the following matrix form

$$A_1\, Y_{n+1} = A_0\, Y_n + h\, \left(B_0\, F_n + B_1\, F_{n+1}\right), \tag{3.21}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B_0 = \begin{pmatrix} 0 & \frac{5}{24} \\ 0 & \frac{1}{6} \end{pmatrix}, B_1 = \begin{pmatrix} \frac{8}{24} & -\frac{1}{24} \\ \frac{4}{6} & \frac{1}{6} \end{pmatrix},$$
$$Y_n = \begin{pmatrix} y_{n-\frac{1}{2}} \\ y_n \end{pmatrix}, Y_{n+1} = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \end{pmatrix}, F_n = \begin{pmatrix} f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}, F_{n+1} = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix},$$

to get

$$Y_{n+1} = M(z)\, Y_n,\ z = \lambda h, \tag{3.22}$$

where the matrix $M(z)$ is given by

$$M(z) = \left(A_1 - zB_1\right)^{-1} \left(A_0 + zB_0\right). \tag{3.23}$$

The matrix $M(z)$ has eigenvalues $\left\{0, -\frac{2(z-3)(z+6)}{3\left(z^2-6z+12\right)}\right\}$. Consider $R(z) : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $R(z) = -\frac{2(z-3)(z+6)}{3\left(z^2-6z+12\right)}$. The region of absolute stability S is defined as $S = \{z \in \mathbb{C} : |R(z)| < 1\}$. The Region of absolute stability of the method is presented in Figure 3.1. The stability region contains the entire left half complex plane and thus, the method is A-stable.

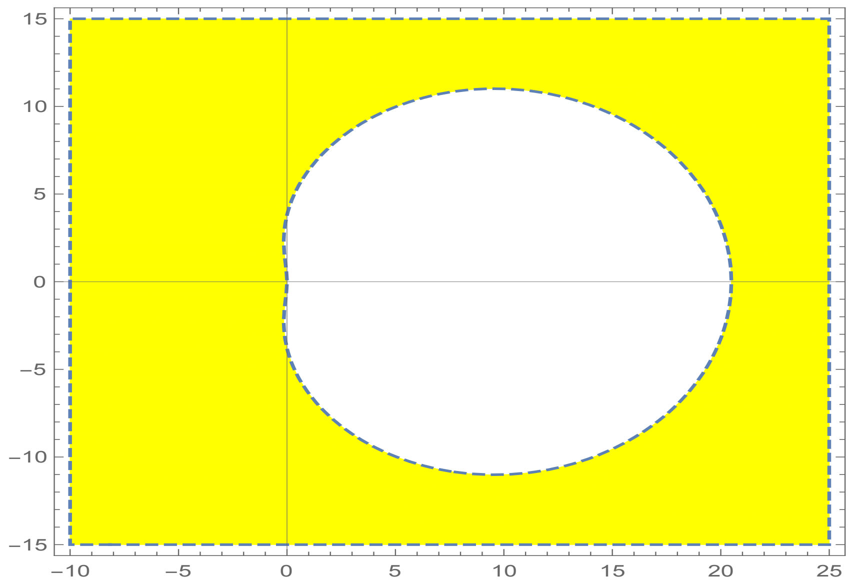


Figure 3.1: The region of absolute stability

3.3 Second Order Initial Value Problems

In this section, a numerical method based on the one-step hybrid block method with one off-step point (HBM1), t_{n+k} with $0 < k < 1$, is presented to solve the following differential equation of the form

$$y''(t) = f(t, y(t), y'(t)), \ \ t \geq 0, \tag{3.24}$$

$$y(t_0) = y_0, \tag{3.25}$$

$$y'(t_0) = y_1. \tag{3.26}$$

To derive HBM1, assume that $t_n = nh$ where h is the stepsize. The solution of Problem (3.24-3.26) is approximated by a polynomial of degree 4 as follows,

$$y(t) \simeq \sum_{j=0}^{4} a_j t^j, \tag{3.27}$$

and its first derivative by

$$y'(t) \simeq \sum_{j=1}^{4} j a_j t^{j-1}, \tag{3.28}$$

and its second derivative by

$$y''(t) \simeq \sum_{j=2}^{4} j(j-1)\,a_j\,t^{j-2}. \tag{3.29}$$

Interpolating Equations (3.27-3.28) at the point t_n and collocating Equation (3.29) at the points t_n, $t_{n+k} = t_n + kh$, and $t_{n+1} = t_n + h$ to get the following system

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 \\ 0 & 0 & 2 & 6t_{n+k} & 12t_{n+k}^2 \\ 0 & 0 & 2 & 6t_{n+1} & 12t_{n+1}^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_n \\ y'_n \\ f_n \\ f_{n+k} \\ f_{n+1} \end{pmatrix}, \tag{3.30}$$

where $y_{n+j} \approx y(t_{n+j})$, $y'_{n+j} \approx y'(t_{n+j})$, and $f_{n+j} \approx y''(t_{n+j})$, $j = 0, k, 1$. Solving System (3.30) after substituting $t = t_n + wh$ to get

$$y(w) \approx y_n + h\,\alpha\,y'_n + h^2\,(\alpha_0 f_n + \alpha_k f_{n+k} + \alpha_1 f_{n+1}) \tag{3.31}$$

$$y'(w) \approx \beta\,y'_n + h\,(\beta_0 f_n + \beta_k f_{n+k} + \beta_1 f_{n+1}) \tag{3.32}$$

where α, α_0, α_k, α_1, β, β_0, β_k, and β_1 are functions of w. Then, evaluate the approximation of $y(w)$ and $y'(w)$ at $w = k$ and 1, to get

$$y_{n+j} = y_n + h\,\alpha_j\,y'_n + h^2\,(\alpha_{0,j} f_n + \alpha_{k,j} f_{n+k} + \alpha_{1,j} f_{n+1})$$

$$y'_{n+j} = \beta_j\,y'_n + h\,(\beta_{0,j} f_n + \beta_{k,j} f_{n+k} + \beta_{1,j} f_{n+1}) \tag{3.33}$$

where $j = k, 1$ and

$$\alpha_1 = 1,\ \alpha_{0,1} = \frac{1}{3} - \frac{1}{12k},\ \alpha_{k,1} = \frac{1}{12k - 12k^2},\ \alpha_{1,1} = \frac{1}{12}\left(2 + \frac{1}{k-1}\right),$$
$$\alpha_k = k,\ \alpha_{0,k} = -\frac{1}{12}(k-4)k^2,\ \alpha_{k,k} = \frac{(k-2)k^2}{12(k-1)},\ \alpha_{1,k} = \frac{k^4}{12(k-1)},$$
$$\beta_1 = 1,\ \beta_{0,1} = \frac{1}{2} - \frac{1}{6k},\ \beta_{k,1} = \frac{1}{6k - 6k^2},\ \beta_{1,1} = \frac{1}{6}\left(3 + \frac{1}{k-1}\right),$$
$$\beta_k = 1,\ \beta_{0,k} = -\frac{1}{6}(k-3)k,\ \beta_{k,k} = \frac{(2k-3)k}{6(k-1)},\ \beta_{1,k} = \frac{k^3}{6(k-1)}.$$

To maximize the order of the implicit block method (3.33) when $j = 1$, the local truncation errors in the formula for y_{n+1}, is optimized as follow,

$$\mathscr{L}(y(t_{n+1});h) = \frac{5k-2}{360}h^5 y^{(5)}(t_n) + \frac{(5k^2+5k-3)}{1440}h^6 y^{(6)}(t_n) + \mathscr{O}\left(h^7\right).$$

To maximize the order, the following equation for k where $0 < k < 1$ is solved

$$\frac{5k-2}{360} = 0\ . \tag{3.34}$$

Hence,

$$k = \frac{2}{5}, \tag{3.35}$$

and the local truncation errors for y_{n+1}, y_{n+k}, y'_{n+1}, and y'_{n+k} are

$$\mathscr{L}[y(t_{n+1});h] = -\frac{h^6 y^{(6)}(t_n)}{7200} + \mathscr{O}\left(h^7\right) = -1.38889 \times 10^{-4} h^6 y^{(6)}(t_n) + \mathscr{O}\left(h^7\right),$$

$$\mathscr{L}[y(t_{n+k});h] = \frac{14h^5 y^{(5)}(t_n)}{46875} + \mathscr{O}\left(h^6\right) = 2.98667 \times 10^{-4} h^5 h^5 y^{(5)}(t_n) + \mathscr{O}\left(h^6\right),$$

$$\mathscr{L}\left[y'(t_{n+1});h\right] = -\frac{h^5 y^{(5)}(t_n)}{360} + \mathscr{O}\left(h^6\right) = -2.7777 \times 10^{-3} h^5 y^{(5)}(t_n) + \mathscr{O}\left(h^6\right),$$

$$\mathscr{L}\left[y'(t_{n+k});h\right] = \frac{8h^5 y^{(5)}(t_n)}{5625} + \mathscr{O}\left(h^6\right) = 1.42222 \times 10^{-3} h^5 y^{(5)}(t_n) + \mathscr{O}\left(h^6\right).$$

Then the order of HBM1 is $(3,3,3,3)$ and the error constant is $\left(0, \frac{14}{46875}, \frac{-1}{360}, \frac{8}{5625}\right)$. Therefore, the HBM1 is given as follows

$$
\begin{aligned}
y_{n+1} &= y_n + h\, y'_n + \frac{h^2}{72}(9f_n + 25f_{n+\frac{2}{5}} + 2f_{n+1}),\\
y_{n+\frac{2}{5}} &= y_n + \frac{2h}{5}\, y'_n + \frac{2h^2}{1125}(27f_n + 20f_{n+\frac{2}{5}} - 2f_{n+1}),\\
y'_{n+1} &= y'_n + \frac{h}{36}(3f_n + 25f_{n+\frac{2}{5}} + 8f_{n+1}),\\
y'_{n+\frac{2}{5}} &= y'_n + \frac{h}{225}(39f_n + 55f_{n+\frac{2}{5}} - 4f_{n+1}).
\end{aligned} \tag{3.36}
$$

Now, the main properties of the proposed method such as consistency, stability, and convergence will be studied. Let us write System (3.36) in the form

$$
\begin{aligned}
y_{n+\frac{2}{5}} &= y_n + \frac{2h}{5}\, y'_n + \frac{2h^2}{1125}(27f_n + 20f_{n+\frac{2}{5}} - 2f_{n+1}),\\
y_{n+1} &= y_n + h\, y'_n + \frac{h^2}{72}(9f_n + 25f_{n+\frac{2}{5}} + 2f_{n+1}),\\
hy'_{n+\frac{2}{5}} &= hy'_n + \frac{h^2}{225}(39f_n + 55f_{n+\frac{2}{5}} - 4f_{n+1}),\\
hy'_{n+1} &= hy'_n + \frac{h^2}{36}(3f_n + 25f_{n+\frac{2}{5}} + 8f_{n+1}).
\end{aligned} \tag{3.37}
$$

Then, the System (3.37) can be rewritten in the matrix form as

$$
A_0 Y_m = A_1 y_m + A_2 F_m, \tag{3.38}
$$

where

$$
Y_m = \begin{pmatrix} y_{n+\frac{2}{5}} \\ y_{n+1} \\ h\, y'_{n+\frac{2}{5}} \\ h\, y'_{n+1} \end{pmatrix}, \quad y_m = \begin{pmatrix} y_{n-\frac{2}{5}} \\ y_n \\ h\, y'_{n-\frac{2}{5}} \\ h\, y'_n \end{pmatrix}, \quad F_m = \begin{pmatrix} h^2\, f_n \\ h^2\, f_{n+\frac{2}{5}} \\ h^2\, f_{n+1} \end{pmatrix},
$$

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{54}{1125} & \frac{40}{1125} & \frac{-4}{1125} \\ \frac{9}{72} & \frac{25}{72} & \frac{2}{72} \\ \frac{39}{225} & \frac{55}{225} & \frac{-4}{225} \\ \frac{3}{36} & \frac{25}{36} & \frac{8}{36} \end{pmatrix}.$$

Following to Fatunla's approach [23], the characteristic equation of HBM1 is

$$det(\mu A_0 - A_1) = det \begin{pmatrix} \mu & -1 & 0 & -\frac{2}{5} \\ 0 & \mu-1 & 0 & 1 \\ 0 & 0 & \mu & -1 \\ 0 & 0 & 0 & \mu-1 \end{pmatrix} = \mu^2(\mu-1)^2 = 0,$$

which implies that $\mu_1 = \mu_2 = 0$ and $\mu_3 = \mu_4 = 1$. Then, the multiplicity of the nonzero roots of the characteristic equation is 2 which does not exceed the order of the differential equation. Hence, it is zero stable.

The local truncation error of the System (3.36) is

$$\begin{aligned} \mathscr{L}[y(t_n);h] &= \left(\mathscr{L}\left[y\left(t_{n+\frac{2}{5}}\right);h\right], \mathscr{L}[y(t_{n+1});h], \mathscr{L}\left[y'\left(t_{n+\frac{2}{5}}\right);h\right], \mathscr{L}\left[y'(t_{n+1});h\right]\right) \\ &= \sum_{i=0}^{\infty} \gamma_i h^i y^{(i)}(t_n) = \gamma_5 h^5 y^{(5)}(t_n) + \sum_{i=6}^{\infty} \gamma_i h^i y^{(i)}(t_n), \end{aligned} \tag{3.39}$$

where $\gamma_0 = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$ and $\gamma_5 = (0, \frac{14}{46875}, \frac{8}{5625}, \frac{1}{360})^T$. Thus, System (3.36) has order $(3,3,3,3)^T$. For simplicity, the order is denoted by 3. Since the order of System (3.36) is $3 \geq 1$, then it is consistent. The consistency and the zero stability of System (3.36) imply that it is convergent [22, 24]. To find the region of absolute stability, consider the following test problem $y' = \lambda y$ where $\lambda < 0$, then

$$y'' = f(t,y,y') = \lambda^2 y. \tag{3.40}$$

Substitute f in the following matrix form

$$B_0 Y_{n+1} = B_1 Y_n + hB_2 Y'_{n+1} + h^2(C_0 F_n + C_1 F_{n+1}), \tag{3.41}$$

where

$$Y_{n+1} = \begin{pmatrix} y_{n+\frac{2}{5}} \\ y_{n+1} \end{pmatrix}, \ Y_n = \begin{pmatrix} y_{n-\frac{2}{5}} \\ y_n \end{pmatrix}, \ Y'_n = \begin{pmatrix} y'_{n-\frac{2}{5}} \\ y'_n \end{pmatrix}, \ F_n = \begin{pmatrix} f_{n-\frac{2}{5}} \\ f_n \end{pmatrix}, \ F_{n+1} = \begin{pmatrix} f_{n+\frac{2}{5}} \\ f_{n+1} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}, \ C_0 = \begin{pmatrix} 0 & \frac{54}{1125} \\ 0 & \frac{9}{72} \end{pmatrix}, \ C_1 = \begin{pmatrix} \frac{40}{1125} & \frac{-4}{1125} \\ 0\frac{25}{72} & \frac{2}{72} \end{pmatrix},$$

to get

$$B_0 Y_{n+1} = B_1 Y_n + hB_2 \lambda Y_n + h^2(C_0 \lambda^2 Y_n + C_1 \lambda^2 Y_{n+1}). \tag{3.42}$$

Let $S = \lambda h$, then

$$Y_{n+1} = M(S)\, Y_n,$$

where $M(S) = (B_0 - S^2 C1)^{-1}(B_1 + SB_2 + S^2 C_0)$. The eigenvalue of $M(S)$ are

$$(0, \frac{900 + 900S + 393S^2 + 93S^3 + 11S^4}{900 - 57S^2 + 2S^4}).$$

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(S) = \frac{900+900S+393S^2+93S^3+11S^4}{900-57S^2+2S^4}$ where $S = \lambda h$. The region of absolute stability will be all $S \in \mathbb{C}$ such that $|f(S)| < 1$. This region is given in Figure 3.2 and the interval of stability is $(-4.08611, 0)$.

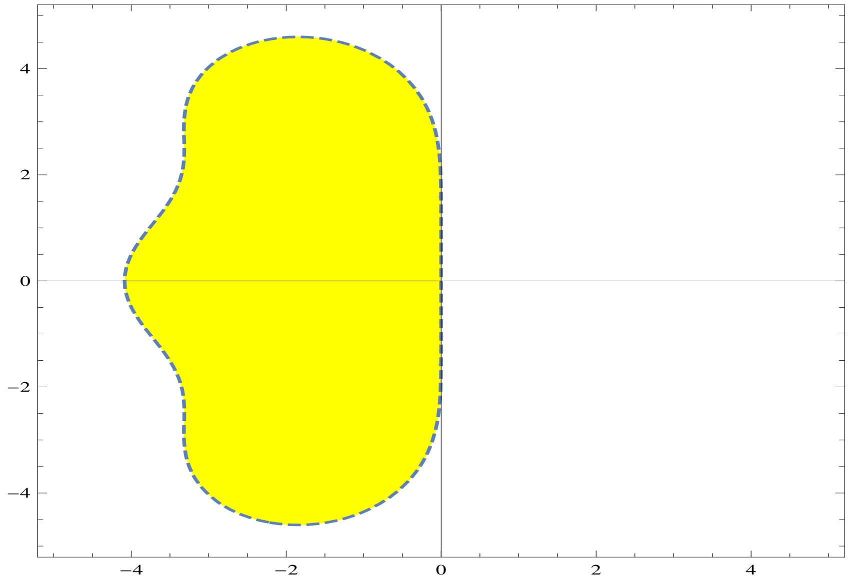


Figure 3.2: Region of absolute stability

3.4 Numerical Results of IVPs

In this section, numerical examples will be presented to show the efficiency of the proposed methods to solve first and second-order IVPs, respectively.

Example 3.4.1. Consider the following linear first order initial value problem

$$y'(x) = y(x), \quad y(0) = 1,$$

the value of h is chosen to be 0.01 and the exact solution is given by

$$y(x) = e^x.$$

The absolute errors using the proposed method of first order on $[0,1]$ are presented in Table 3.1.

Table 3.1: The absolute errors in Example 3.4.1 for $h = 0.01$.

x	Er for y_n
0	0
0.1	1.53499×10^{-12}
0.2	3.39262×10^{-12}
0.3	5.62417×10^{-12}
0.4	8.28759×10^{-12}
0.5	1.14493×10^{-11}
0.6	1.51834×10^{-11}
0.7	1.95768×10^{-11}
0.8	2.47273×10^{-11}
0.9	3.07434×10^{-11}
1	3.77529×10^{-11}

Example 3.4.2. Consider the following linear first order initial value problem

$$y'(x) = y(x) + 1, \quad y(0) = 0,$$

the value of h is chosen to be 0.01 and the exact solution is given by

$$y(x) = e^x - 1.$$

The absolute errors using the proposed method of first order on $[0, 1]$ are presented in Table 3.2.

Table 3.2: The absolute errors in Example 3.4.2 for $h = 0.01$.

x	Er for y_n
0	0
0.1	1.53505×10^{-12}
0.2	3.3927×10^{-12}
0.3	5.62428×10^{-12}
0.4	8.28754×10^{-12}
0.5	1.14487×10^{-11}
0.6	1.51833×10^{-11}
0.7	1.9577×10^{-11}
0.8	2.47271×10^{-11}
0.9	3.07432×10^{-11}
1	3.77507×10^{-11}

Example 3.4.3. Consider the nonlinear first order initial value problem

$$y'(x) = y(x)^2 + 1, \quad y(0) = 0,$$

the value of h is chosen to be 0.01 and the exact solution is given by

$$y(x) = \tan(x).$$

In Table 3.3, the absolute errors obtained by the current method of first order are compared with ones obtained in [25]. One can see that the results are better than the ones in [25], even when the method is used only with one off-step point.

Table 3.3: The absolute errors in Example 3.4.3 for $h = 0.01$.

x	Er for y_n	Er for y_n in [25]
0	0	0
0.1	5.5639×10^{-12}	—
0.2	1.11475×10^{-11}	1×10^{-5}
0.3	1.65997×10^{-11}	1×10^{-5}
0.4	2.12868×10^{-11}	2×10^{-5}
0.5	2.3369×10^{-11}	2×10^{-5}
0.6	1.79635×10^{-11}	3×10^{-5}
0.7	7.81752×10^{-12}	4×10^{-5}
0.8	8.9329×10^{-11}	5×10^{-5}
0.9	3.30909×10^{-10}	8×10^{-5}
1	1.07632×10^{-9}	1.1×10^{-4}

Example 3.4.4. Consider the following second order linear initial value problem

$$y'' = -y(x), \quad y(0) = y'(0) = 1,$$

the value of h is chosen to be 0.1 and the exact solution is given by

$$y(x) = \cos(x) + \sin(x).$$

Using HBM1,

$$\begin{aligned}
y_{n+1} &= y_n + h\, y'_n - \frac{h^2}{72}(9\, y_n + 25\, y_{n+\frac{2}{5}} + 2\, y_{n+1}), \\
y_{n+\frac{2}{5}} &= y_n + \frac{2h}{5}\, y'_n - \frac{2h^2}{1125}(27\, y_n + 20\, y_{n+\frac{2}{5}} - 2\, y_{n+1}), \\
y'_{n+1} &= y'_n - \frac{h}{36}(3\, y_n + 25\, y_{n+\frac{2}{5}} + 8\, y_{n+1}), \\
y'_{n+\frac{2}{5}} &= y'_n - \frac{h}{225}(39\, y_n + 55\, y_{n+\frac{2}{5}} - 4\, y_{n+1}).
\end{aligned}$$

The above system can be written in the matrix form as

$$Y_m = A^{-1}B\, y_n + A^{-1}C\, y_n',$$

where

$$A = \begin{pmatrix} 1+\frac{h^2}{36} & \frac{25h^2}{72} & 0 & 0 \\ -\frac{4h^2}{1125} & 1+\frac{8h^2}{225} & 0 & 0 \\ \frac{2h}{9} & \frac{25h}{36} & 1 & 0 \\ -\frac{4h}{225} & \frac{11h}{45} & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1-\frac{h^2}{8} \\ 1-\frac{6h^2}{125} \\ -\frac{h}{12} \\ -\frac{13h}{75} \end{pmatrix},\ C = \begin{pmatrix} h \\ \frac{2h}{5} \\ 1 \\ 1 \end{pmatrix},\ Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+\frac{2}{5}} \\ y'_{n+1} \\ y'_{n+\frac{2}{5}} \end{pmatrix}.$$

In Table 3.4, the absolute errors obtained by the current method of second order are compared with ones obtained in [26]. One can see that the similar results are got using one off-step point with low computation cost comparing with [26].

Table 3.4: The absolute errors in Example 3.4.4 for $h = 0.1$.

x	Er for y_n	Er for y_n in [26]
0	0	0
0.1	1.35006×10^{-10}	6.92×10^{-9}
0.2	2.57539×10^{-8}	1.76×10^{-8}
0.3	7.43478×10^{-8}	1.62×10^{-8}
0.4	1.41872×10^{-7}	4.73×10^{-8}
0.5	2.24165×10^{-7}	1.20×10^{-7}
0.6	3.16758×10^{-7}	1.87×10^{-7}
0.7	4.14951×10^{-7}	3.07×10^{-7}
0.8	5.13904×10^{-7}	4.19×10^{-7}
0.9	6.08717×10^{-7}	5.79×10^{-7}
1	6.94524×10^{-7}	7.27×10^{-7}

Example 3.4.5. Consider the following second order nonlinear initial value problem

$$y'' = x\,(y'(x))^2, \quad y(0) = 1, \quad y'(0) = \frac{1}{2},$$

the value of h is chosen to be $\frac{1}{320}$ and the exact solution is given by

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{x+2}{2-x}\right)$$

The absolute errors using the proposed method of second order on $[0,1]$ are presented in Table 3.5.

Table 3.5: The absolute errors in Example 3.4.5 for $h = \frac{1}{320}$.

x	Er for y_n
0	0
$\frac{1}{320}$	2.22045×10^{-16}
$\frac{2}{320}$	2.22045×10^{-16}
$\frac{3}{320}$	1.55431×10^{-15}
$\frac{4}{320}$	3.33067×10^{-15}
$\frac{5}{320}$	5.77316×10^{-15}
$\frac{6}{320}$	8.65974×10^{-15}
$\frac{7}{320}$	1.26565×10^{-14}
$\frac{8}{320}$	1.68754×10^{-14}
$\frac{9}{320}$	2.17604×10^{-14}
$\frac{10}{320}$	2.70894×10^{-14}

Chapter 4: First and Second Order Fuzzy Initial Value Problems

In this chapter, the methods were derived in Chapter 3 will be used to solve the first and second-order fuzzy initial value problems. New theorems will be introduced to solve such problems. Several examples will be given to show the accuracy and efficiency of these methods.

4.1 First Order Fuzzy Initial Value Problems

In this section, a proposed method of first order will be implemented to solve the first order fuzzy initial value problem. In addition, some theoretical results will be presented.

Consider the following fuzzy initial value problem

$$y'(t) = f(t,y), \;\; t \geq 0, \tag{4.1}$$

$$y(0) = y_0. \tag{4.2}$$

Denotes the $\alpha-$level of the solution $y(t),\ y_0$, and the function $f(t,y)$ by

$$y(t,\alpha) = [y_1(t,\alpha), y_2(t,\alpha)]\,,$$

$$y(0,\alpha) = [y_{1,0}, y_{2,0}\,]\,,$$

$$f(t,y,\alpha) = [f_1(t,y(t,\alpha)), f_2(t,y(t,\alpha))]\,.$$

Following the technique described in Chapter 3, the fuzzy HBM1 is given by

$$
\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(f_{1,n,\alpha} + 4f_{1,n+\frac{1}{2},\alpha} + f_{1,n+1,\alpha}) \\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5f_{1,n,\alpha} + 8f_{1,n+\frac{1}{2},\alpha} - f_{1,n+1,\alpha}) \\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(f_{2,n,\alpha} + 4f_{2,n+\frac{1}{2},\alpha} + f_{2,n+1,\alpha}) \\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5f_{2,n,\alpha} + 8f_{2,n+\frac{1}{2},\alpha} - f_{2,n+1,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
f_{1,n,\alpha} &= \min\{f(t_n, w) : w \in [y_{1,n,\alpha}, y_{2,n,\alpha}]\}, \\
f_{2,n,\alpha} &= \max\{f(t_n, w) : w \in [y_{1,n,\alpha}, y_{2,n,\alpha}]\}, \\
f_{1,n+\frac{1}{2},\alpha} &= \min\{f(t_{n+\frac{1}{2}}, w) : w \in \left[y_{1,n+\frac{1}{2},\alpha}, y_{2,n+\frac{1}{2},\alpha}\right]\}, \\
f_{2,n+\frac{1}{2},\alpha} &= \max\{f(t_{n+\frac{1}{2}}, w) : w \in \left[y_{1,n+\frac{1}{2},\alpha}, y_{2,n+\frac{1}{2},\alpha}\right]\}, \\
f_{1,n+1,\alpha} &= \min\{f(t_{n+1}, w) : w \in [y_{1,n+1,\alpha}, y_{2,n+1,\alpha}]\}, \\
f_{2,n+1,\alpha} &= \min\{f(t_{n+1}, w) : w \in [y_{1,n+1,\alpha}, y_{2,n+1,\alpha}]\}.
\end{aligned}
\tag{4.3}
$$

In the next theorem, the HBM1 is studied when $f(t, y)$ is monotonic function of y.

Theorem 4.1.1. *If $f(t, y)$ is increasing on y, then the fuzzy HBM1 becomes*

$$
\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(f(t_n, y_{1,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{1,n+1,\alpha})) \\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5f(t_n, y_{1,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{1,n+1,\alpha})) \\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(f(t_n, y_{2,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{2,n+1,\alpha})) \\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5f(t_n, y_{2,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{2,n+1,\alpha})),
\end{aligned}
$$

while if $f(t,y)$ is decreasing on y, then the fuzzy HBM1 becomes

$$
\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(f(t_n, y_{2,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{2,n+1,\alpha})) \\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5f(t_n, y_{2,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{2,n+1,\alpha})) \\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(f(t_n, y_{1,n,\alpha}) + 4f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) + f(t_{n+1}, y_{1,n+1,\alpha})) \\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5f(t_n, y_{1,n,\alpha}) + 8f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}) - f(t_{n+1}, y_{1,n+1,\alpha})).
\end{aligned}
$$

Proof. If $f(t,y)$ is increasing on y, it follows from (4.3) that

$$
\begin{aligned}
f_{1,n,\alpha} &= f(t_n, y_{1,n,\alpha}),\ f_{2,n,\alpha} = f(t_n, y_{2,n,\alpha}), \\
f_{1,n+\frac{1}{2},\alpha} &= f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}),\ f_{2,n+\frac{1}{2},\alpha} = f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}), \\
f_{1,n+1,\alpha} &= f(t_{n+1}, y_{1,n+1,\alpha}),\ f_{2,n+1,\alpha} = f(t_{n+1}, y_{2,n+1,\alpha}),
\end{aligned} \tag{4.4}
$$

and if $f(t,y)$ is decreasing on y, it follows from (4.3) that

$$
\begin{aligned}
f_{1,n,\alpha} &= f(t_n, y_{2,n,\alpha}),\ f_{2,n,\alpha} = f(t_n, y_{1,n,\alpha}), \\
f_{1,n+\frac{1}{2},\alpha} &= f(t_{n+\frac{1}{2}}, y_{2,n+\frac{1}{2},\alpha}),\ f_{2,n+\frac{1}{2},\alpha} = f(t_{n+\frac{1}{2}}, y_{1,n+\frac{1}{2},\alpha}), \\
f_{1,n+1,\alpha} &= f(t_{n+1}, y_{2,n+1,\alpha}),\ f_{2,n+1,\alpha} = f(t_{n+1}, y_{1,n+1,\alpha}),
\end{aligned} \tag{4.5}
$$

which completes the proof.

In next theorem, the case when $f(t,y)$ is linear function of y is studied.

Theorem 4.1.2. *Let $b = [b_1, b_2]$ be a fuzzy number, $a \in \mathbb{R}$ and $f(t, y) = a\,y + b$.*

1. If $a \geq 0$, then

$$y_{1,n+1,\alpha} = \varphi^{n+1} y_{1,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k,$$

and

$$y_{2,n+1,\alpha} = \varphi^{n+1} y_{2,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k.$$

2. If $a < 0$, then

$$y_{1,n+1,\alpha} = \theta_1 y_{1,n,\alpha} + \gamma_1 b_1 + \theta_2 y_{2,n,\alpha} + \gamma_2 b_2,$$

and

$$y_{2,n+1,\alpha} = \theta_2 y_{1,n,\alpha} + \gamma_2 b_1 + \theta_1 y_{2,n,\alpha} + \gamma_1 b_2.$$

Proof. 1) Let $a \geq 0$. Then, $f(t_{n+j}, y_{i,n+j,\alpha}) = b_i + a\,y_{i,n+j,\alpha}$, where $i = 1, 2$ and $j = 0, \frac{1}{2}, 1$. Thus,

$$\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}(b_1 + a\,y_{1,n,\alpha} + 4(b_1 + a\,y_{1,n+\frac{1}{2},\alpha}) + b_1 + a\,y_{1,n+1,\alpha}) \\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}(5(b_1 + a\,y_{1,n,\alpha}) + 8(b_1 + a\,y_{1,n+\frac{1}{2},\alpha}) - (b_1 + a\,y_{1,n+1,\alpha}) \\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}(b_2 + a\,y_{2,n,\alpha} + 4(b_2 + a\,y_{2,n+\frac{1}{2},\alpha}) + b_2 + a\,y_{2,n+1,\alpha}) \\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}(5(b_2 + a\,y_{2,n,\alpha}) + 8(b_2 + a\,y_{2,n+\frac{1}{2},\alpha}) - (b_2 + a\,y_{2,n+1,\alpha}).
\end{aligned}$$

The last system can rewritten in the matrix form

$$A_1 Y_1 = R_1,\; A_2 Y_2 = R_2, \tag{4.6}$$

where

$$A_1 = \begin{pmatrix} 1-\frac{ah}{3} & \frac{ah}{24} \\ -\frac{2ah}{3} & 1-\frac{ah}{6} \end{pmatrix}, Y_1 = \begin{pmatrix} y_{1,n+\frac{1}{2},\alpha} \\ y_{1,n+1,\alpha} \end{pmatrix}, R_1 = \begin{pmatrix} \frac{h}{2} \\ h \end{pmatrix} b_1 + \begin{pmatrix} 1+\frac{5ah}{24} \\ 1+\frac{ah}{24} \end{pmatrix} y_{1,n,\alpha},$$

$$A_2 = \begin{pmatrix} 1-\frac{ah}{3} & \frac{ah}{24} \\ -\frac{2ah}{3} & 1-\frac{ah}{6} \end{pmatrix}, Y_2 = \begin{pmatrix} y_{2,n+\frac{1}{2},\alpha} \\ y_{2,n+1,\alpha} \end{pmatrix}, R_2 = \begin{pmatrix} \frac{h}{2} \\ h \end{pmatrix} b_2 + \begin{pmatrix} 1+\frac{5ah}{24} \\ 1+\frac{ah}{24} \end{pmatrix} y_{2,n,\alpha}.$$

Since $\det(A_1) = \frac{1}{12}a^2h^2 - \frac{1}{2}ah + 1 = \frac{(ah-3)^2+3}{12} \neq 0$, then

$$A_1^{-1} = \begin{pmatrix} \frac{12-2ah}{a^2h^2-6ah+12} & \frac{-ha}{2a^2h^2-12ah+24} \\ \frac{8ha}{a^2h^2-6ah+12} & \frac{12-4ah}{a^2h^2-6ah+12} \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} y_{1,n+\frac{1}{2},\alpha} \\ y_{1,n+1,\alpha} \end{pmatrix} = b_1\, C_1 + y_{1,n,\alpha}\, C_2,$$

where

$$C_1 = \begin{pmatrix} -\frac{3}{2}h\frac{ah-4}{a^2h^2-6ah+12} \\ 12\frac{h}{a^2h^2-6ah+12} \end{pmatrix} \textit{and } C_2 = \begin{pmatrix} -\frac{1}{16}\frac{7a^2h^2-192}{a^2h^2-6ah+12} \\ \frac{3}{2}\frac{a^2h^2+3ah+8}{a^2h^2-6ah+12} \end{pmatrix}.$$

Hence,

$$y_{1,n+1,\alpha} = \frac{12hb_1}{a^2h^2-6ah+12} + \frac{3}{2}\frac{a^2h^2+3ah+8}{a^2h^2-6ah+12}\, y_{1,n,\alpha}.$$

Let $\theta = \frac{12hb_1}{a^2h^2-6ah+12}$ and $\varphi = \frac{3}{2}\frac{a^2h^2+3ah+8}{a^2h^2-6ah+12}$. Then,

$$\begin{aligned}
y_{1,n+1,\alpha} &= \theta + \varphi\, y_{1,n,\alpha} \\
&= \theta + \varphi(\theta + \varphi\, y_{1,n-1,\alpha}) \\
&= \theta + \varphi\theta + \varphi^2(\theta + \varphi\, y_{1,n-2,\alpha}) \\
&\ \vdots \\
&= \varphi^{n+1} y_{1,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k.
\end{aligned}$$

Therefore,

$$y_{1,n+1,\alpha} = \varphi^{n+1} y_{1,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k$$

and similarly,

$$y_{2,n+1,\alpha} = \varphi^{n+1} y_{2,0,\alpha} + \theta \sum_{k=0}^{n} \varphi^k.$$

2) Let $a < 0$. Then, the fuzzy HBM1 becomes

$$\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + \frac{h}{6}\left(b_1 + a\, y_{2,n,\alpha} + 4(b_1 + a\, y_{2,n+\frac{1}{2},\alpha}) + b_1 + a\, y_{2,n+1,\alpha}\right) \\
y_{1,n+\frac{1}{2},\alpha} &= y_{1,n,\alpha} + \frac{h}{24}\left(5(b_1 + a\, y_{2,n,\alpha}) + 8(b_1 + a\, y_{2,n+\frac{1}{2},\alpha}) - (b_1 + a y_{2,n+1,\alpha})\right) \\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + \frac{h}{6}\left(b_2 + a\, y_{1,n,\alpha} + 4(b_2 + a\, y_{1,n+\frac{1}{2},\alpha}) + b_2 + a\, y_{1,n+1,\alpha}\right) \\
y_{2,n+\frac{1}{2},\alpha} &= y_{2,n,\alpha} + \frac{h}{24}\left(5(b_2 + a y_{1,n,\alpha}) + 8(b_2 + a\, y_{1,n+\frac{1}{2},\alpha}) - (b_2 + a y_{1,n+1,\alpha})\right).
\end{aligned}$$

The last system can rewritten in the matrix form

$$AY = R,$$

where

$$A = \begin{pmatrix} 1 & 0 & \frac{-ah}{3} & \frac{ah}{24} \\ 0 & 1 & -\frac{2ah}{3} & \frac{-ah}{6} \\ \frac{-ah}{3} & \frac{ah}{24} & 1 & 0 \\ -\frac{2ah}{3} & \frac{-ah}{6} & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} y_{1,n+\frac{1}{2},\alpha} \\ y_{1,n+1,\alpha} \\ y_{2,n+\frac{1}{2},\alpha} \\ y_{2,n+1,\alpha} \end{pmatrix},$$

$$R = \begin{pmatrix} \frac{h}{2} & 0 \\ h & 0 \\ 0 & \frac{h}{2} \\ 0 & h \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} 1 & \frac{5ah}{24} \\ 1 & \frac{ah}{6} \\ \frac{5ah}{24} & 1 \\ \frac{ah}{6} & 1 \end{pmatrix} \begin{pmatrix} y_{1,n,\alpha} \\ y_{2,n,\alpha} \end{pmatrix}.$$

Since

$$\det(A) = \frac{144a^4h^4 - 1728a^2h^2 + 20736}{20736} \neq 0,$$

then

$$A^{-1} = \begin{pmatrix} \frac{144}{a^4h^4-12a^2h^2+144} & -\frac{3a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{48ah-2a^3h^3}{a^4h^4-12a^2h^2+144} & -\frac{ah\left(a^2h^2+12\right)}{2a^4h^4-24a^2h^2+288} \\ \frac{48a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{144-12a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{8ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & -\frac{4ah\left(a^2h^2-6\right)}{a^4h^4-12a^2h^2+144} \\ \frac{48ah-2a^3h^3}{a^4h^4-12a^2h^2+144} & -\frac{ah\left(a^2h^2+12\right)}{2a^4h^4-24a^2h^2+288} & \frac{144}{a^4h^4-12a^2h^2+144} & -\frac{3a^2h^2}{a^4h^4-12a^2h^2+144} \\ \frac{8ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & -\frac{4ah\left(a^2h^2-6\right)}{a^4h^4-12a^2h^2+144} & \frac{48a^2h^2}{a^4h^4-12a^2h^2+144} & \frac{144-12a^2h^2}{a^4h^4-12a^2h^2+144} \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} y_{1,n+\frac{1}{2},\alpha} \\ y_{1,n+1,\alpha} \\ y_{2,n+\frac{1}{2},\alpha} \\ y_{2,n+1,\alpha} \end{pmatrix} = C_1 \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + C_2 \begin{pmatrix} y_{1,n,\alpha} \\ y_{2,n,\alpha} \end{pmatrix},$$

where

$$C_1 = \begin{pmatrix} \frac{72h-3a^2h^3}{a^4h^4-12a^2h^2+144} & \frac{36ah^2-3a^3h^4}{2a^4h^4-24a^2h^2+288} \\ \frac{12h\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & \frac{72ah^2}{a^4h^4-12a^2h^2+144} \\ \frac{36ah^2-3a^3h^4}{2a^4h^4-24a^2h^2+288} & \frac{72h-3a^2h^3}{a^4h^4-12a^2h^2+144} \\ \frac{72ah^2}{a^4h^4-12a^2h^2+144} & \frac{12h\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} \end{pmatrix} and\, C_2 = \begin{pmatrix} \frac{-a^4h^4+12a^2h^2+288}{2a^4h^4-24a^2h^2+288} & \frac{72ah-3a^3h^3}{a^4h^4-12a^2h^2+144} \\ \frac{a^4h^4+60a^2h^2+144}{a^4h^4-12a^2h^2+144} & \frac{12ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} \\ \frac{72ah-3a^3h^3}{a^4h^4-12a^2h^2+144} & \frac{-a^4h^4+12a^2h^2+288}{2a^4h^4-24a^2h^2+288} \\ \frac{12ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144} & \frac{a^4h^4+60a^2h^2+144}{a^4h^4-12a^2h^2+144} \end{pmatrix}.$$

Then,

$$y_{1,n+1,\alpha} = \theta_1 y_{1,n,\alpha} + \gamma_1 b_1 + \theta_2 y_{2,n,\alpha} + \gamma_2 b_2,$$

$$y_{2,n+1,\alpha} = \theta_2 y_{1,n,\alpha} + \gamma_2 b_1 + \theta_1 y_{2,n,\alpha} + \gamma_1 b_2,$$

where

$$\theta_1 = \frac{a^4h^4+60a^2h^2+144}{a^4h^4-12a^2h^2+144},\ \theta_2 = \frac{12ah\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144},$$

$$\gamma_1 = \frac{12h\left(a^2h^2+12\right)}{a^4h^4-12a^2h^2+144},\ \gamma_2 = \frac{72ah^2}{a^4h^4-12a^2h^2+144}.$$

4.2 Second Order Fuzzy Initial Value Problems

In this section, a proposed method of second order will be implemented to solve the second order fuzzy initial value problem. In addition, some theoretical results will be given.

Consider the following fuzzy initial value problem

$$y''(t) = f(t,y,y'), t \geq 0, \tag{4.7}$$

$$y(0) = \hat{a}, \tag{4.8}$$

$$y'(0) = \hat{b}. \tag{4.9}$$

Let the α-level of the solution $y(t)$, $\hat{a}$, $\hat{b}$ and the function $f(t,y,y')$ be given by

$$
\begin{aligned}
y(t,\alpha) &= [y_1(t,\alpha), y_2(t,\alpha)]\,,\\
y'(t,\alpha) &= [y'_1(t,\alpha), y'_2(t,\alpha)]\,,\\
y(0,\alpha) &= [a_1, a_2\,]\,,\\
y'(0,\alpha) &= [b_1, b_2]\,,\\
f(t,y,y',\alpha) &= [f_1(t,y(t,\alpha),y'(t,\alpha)), f_2(t,y(t,\alpha),y'(t,\alpha))]\,.
\end{aligned}
$$

Following the technique described in Chapter 3, the fuzzy HBM1 is given by

$$
\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + h\,y'_{1,n,\alpha} + \frac{h^2}{72}(9f_{1,n,\alpha} + 25f_{1,n+\frac{2}{5},\alpha} + 2f_{1,n+1,\alpha}),\\
y_{1,n+\frac{2}{5},\alpha} &= y_{1,n,\alpha} + \frac{2h}{5}\,y'_{1,n,\alpha} + \frac{2h^2}{1125}(27f_{1,n,\alpha} + 20f_{1,n+\frac{2}{5},\alpha} - 2f_{1,n+1,\alpha}),\\
y'_{1,n+1,\alpha} &= y'_{1,n,\alpha} + \frac{h}{36}(3f_{1,n,\alpha} + 25f_{1,n+\frac{2}{5},\alpha} + 8f_{1,n+1,\alpha}),\\
y'_{1,n+\frac{2}{5},\alpha} &= y'_{1,n,\alpha} + \frac{h}{225}(39f_{1,n,\alpha} + 55f_{1,n+\frac{2}{5},\alpha} - 4f_{1,n+1,\alpha}),\\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + h\,y'_{2,n,\alpha} + \frac{h^2}{72}(9f_{2,n,\alpha} + 25f_{2,n+\frac{2}{5},\alpha} + 2f_{2,n+1,\alpha}),\\
y_{2,n+\frac{2}{5},\alpha} &= y_{2,n,\alpha} + \frac{2h}{5}\,y'_{2,n,\alpha} + \frac{2h^2}{1125}(27f_{2,n,\alpha} + 20f_{2,n+\frac{2}{5},\alpha} - 2f_{2,n+1,\alpha}),\\
y'_{2,n+1,\alpha} &= y'_{2,n,\alpha} + \frac{h}{36}(3f_{2,n,\alpha} + 25f_{2,n+\frac{2}{5},\alpha} + 8f_{2,n+1,\alpha}),\\
y'_{2,n+\frac{2}{5},\alpha} &= y'_{2,n,\alpha} + \frac{h}{225}(39f_{2,n,\alpha} + 55f_{2,n+\frac{2}{5},\alpha} - 4f_{2,n+1,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
f_{1,n,\alpha} &= \min\{f(t_n,w,w') : w \in [y_{1,n,\alpha}, y_{2,n,\alpha}]\,, w' \in [y'_{1,n,\alpha}, y'_{2,n,\alpha}]\},\\
f_{2,n,\alpha} &= \max\{f(t_n,w,w') : w \in [y_{1,n,\alpha}, y_{2,n,\alpha}]\,, w' \in [y'_{1,n,\alpha}, y'_{2,n,\alpha}]\},\\
f_{1,n+\frac{2}{5},\alpha} &= \min\{f(t_{n+\frac{2}{5}},w,w') : w \in \left[y_{1,n+\frac{2}{5},\alpha}, y_{2,n+\frac{2}{5},\alpha}\right], w' \in \left[y'_{1,n+\frac{2}{5},\alpha}, y'_{2,n+\frac{2}{5},\alpha}\right]\},\\
f_{2,n+\frac{2}{5},\alpha} &= \max\{f(t_{n+\frac{2}{5}},w,w') : w \in \left[y_{1,n+\frac{2}{5},\alpha}, y_{2,n+\frac{2}{5},\alpha}\right], w' \in \left[y'_{1,n+\frac{2}{5},\alpha}, y'_{2,n+\frac{2}{5},\alpha}\right]\},\\
f_{1,n+1,\alpha} &= \min\{f(t_{n+1},w,w') : w \in [y_{1,n+1,\alpha}, y_{2,n+1,\alpha}]\,, w' \in [y'_{1,n+1,\alpha}, y'_{2,n+1,\alpha}]\},
\end{aligned}
$$

$$f_{2,n+1,\alpha} = \max\{f(t_{n+1}, w, w') : w \in [y_{1,n+1,\alpha}, y_{2,n+1,\alpha}], w' \in [y'_{1,n+1,\alpha}, y'_{2,n+1,\alpha}]\}.$$

In next theorem, the case when $f(t, y, y')$ is monotonic function of y and y' is studied.

Theorem 4.2.1. *Let $f(t, y, y')$ be increasing function in y and y'. Then the following are true.*

- *If $y > 0$ and $y' > 0$, then*

$$f_{1,n+j,\alpha} = f(t_{n+j}, y_{1,n+j,\alpha}, y'_{1,n+j,\alpha}),\ \ f_{2,n+j,\alpha} = f(t_{n+j}, y_{2,n+j,\alpha}, y'_{2,n+j,\alpha}), for\ \ j = 0,\ \frac{2}{5},\ 1.$$

- *If $y > 0$ and $y' < 0$, then*

$$f_{1,n+j,\alpha} = f(t_{n+j}, y_{1,n+j,\alpha}, y'_{2,n+j,\alpha}),\ \ f_{2,n+j,\alpha} = f(t_{n+j}, y_{2,n+j,\alpha}, y'_{1,n+j,\alpha}), for\ \ j = 0,\ \frac{2}{5},\ 1.$$

- *If $y < 0$ and $y' < 0$, then*

$$f_{1,n+j,\alpha} = f(t_{n+j}, y_{2,n+j,\alpha}, y'_{2,n+j,\alpha}),\ \ f_{2,n+j,\alpha} = f(t_{n+j}, y_{1,n+j,\alpha}, y'_{1,n+j,\alpha}), for\ \ j = 0,\ \frac{2}{5},\ 1.$$

- *If $y < 0$ and $y' > 0$, then*

$$f_{1,n+j,\alpha} = f(t_{n+j}, y_{2,n+j,\alpha}, y'_{1,n+j,\alpha}),\ \ f_{2,n+j,\alpha} = f(t_{n+j}, y_{1,n+j,\alpha}, y'_{2,n+j,\alpha}), for\ \ j = 0,\ \frac{2}{5},\ 1.$$

The proof of the theorem follows straight forward. The functions $f_{i,n+j,\alpha}$ for $i = 1, 2$, and $j = 0, \frac{2}{5}, 1$ can be generated for the decreasing case in similar way as in Theorem 4.2.1.

In the next theorem, the case when $f(t, y, y')$ is linear function of y and y' is investigated.

Theorem 4.2.2. *Let $\hat{c} = [c_1, c_2]$ be a fuzzy number, $a,\ b \in \mathbb{R}$ and $f(t,y,y') = ay' + by + \hat{c}$. Then the fuzzy system of HBM1 becomes as follow,*

1. *If $a \geq 0$, and $b \geq 0$, then*

$$Y_{n+1} = A_1^{-1}B_1\, y'_{j,n,\alpha} + A_1^{-1}C_1\, y_{j,n,\alpha} + A_1^{-1}D_1\, c_j,$$

for $j = 1$ and 2, where

$$A_1 = \begin{pmatrix} 1-\frac{8bh^2}{225} & \frac{4bh^2}{1125} & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} \\ -\frac{25bh^2}{72} & 1-\frac{2bh^2}{72} & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} \\ -\frac{55bh}{225} & \frac{4bh}{225} & 1-\frac{55ah}{225} & \frac{4ah}{225} \\ -\frac{25bh}{36} & -\frac{8bh}{36} & -\frac{25ah}{36} & 1-\frac{8ah}{36} \end{pmatrix},\ B_1 = \begin{pmatrix} \frac{6ah^2}{125}+\frac{2h}{5} \\ \frac{9ah^2}{72}+h \\ \frac{39ah}{225}+1 \\ \frac{3ah}{36}+1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \frac{6bh^2}{125}+1 \\ \frac{9bh^2}{72}+1 \\ \frac{39bh}{225} \\ \frac{3bh}{36} \end{pmatrix},\ D_1 = \begin{pmatrix} \frac{2h^2}{25} \\ \frac{h^2}{2} \\ \frac{2h}{5} \\ h \end{pmatrix},\ Y_{n+1} = \begin{pmatrix} y_{j,n+\frac{2}{5},\alpha} \\ y_{j,n+1,\alpha} \\ y'_{j,n+\frac{2}{5},\alpha} \\ y'_{j,n+1,\alpha} \end{pmatrix}$$

2. *If $a \geq 0$, and $b < 0$, then*

$$Y_{n+1} = A_2^{-1}B_2 \begin{pmatrix} y'_{1,n,\alpha} \\ y'_{2,n,\alpha} \end{pmatrix} + A_2^{-1}C_2 \begin{pmatrix} y_{1,n,\alpha} \\ y_{2,n,\alpha} \end{pmatrix} + A_2^{-1}D_2 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where

$$A_2 = \begin{pmatrix} 1 & 0 & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} & -\frac{8bh^2}{225} & \frac{4bh^2}{1125} & 0 & 0 \\ 0 & 1 & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} & -\frac{25bh^2}{72} & -\frac{2bh^2}{72} & 0 & 0 \\ 0 & 0 & 1-\frac{55ah}{225} & \frac{4ah}{225} & -\frac{55bh}{225} & \frac{4bh}{225} & 0 & 0 \\ 0 & 0 & -\frac{25ah}{36} & 1-\frac{8ah}{36} & -\frac{25bh}{36} & -\frac{8bh}{36} & 0 & 0 \\ -\frac{8bh^2}{225} & \frac{4bh^2}{1125} & 0 & 0 & 1 & 0 & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} \\ -\frac{25bh^2}{72} & -\frac{2bh^2}{72} & 0 & 0 & 0 & 1 & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} \\ -\frac{55bh}{225} & \frac{4bh}{225} & 0 & 0 & 0 & 0 & 1-\frac{55ah}{225} & \frac{4ah}{225} \\ -\frac{25bh}{36} & -\frac{8bh}{36} & 0 & 0 & 0 & 0 & -\frac{25ah}{36} & 1-\frac{8ah}{36} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \frac{6ah^2}{125}+\frac{2h}{5} & 0 \\ \frac{9ah^2}{72}+h & 0 \\ \frac{39ah}{225}+1 & 0 \\ \frac{3ah}{36}+1 & 0 \\ 0 & \frac{6ah^2}{125}+\frac{2h}{5} \\ 0 & \frac{9ah^2}{72}+h \\ 0 & \frac{39ah}{225}+1 \\ 0 & \frac{3ah}{36}+1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & \frac{6bh^2}{125} \\ 1 & \frac{9bh^2}{72} \\ 0 & \frac{39bh}{225} \\ 0 & \frac{3bh}{36} \\ \frac{6bh^2}{125} & 1 \\ \frac{9bh^2}{72} & 1 \\ \frac{39bh}{225} & 0 \\ \frac{3bh}{36} & 0 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} \frac{2h^2}{25} & 0 \\ \frac{h^2}{2} & 0 \\ \frac{2h}{5} & 0 \\ h & 0 \\ 0 & \frac{2h^2}{25} \\ 0 & \frac{h^2}{2} \\ 0 & \frac{2h}{5} \\ 0 & h \end{pmatrix}, Y_{n+1} = \begin{pmatrix} y_{1,n+\frac{2}{5},\alpha} \\ y_{1,n+1,\alpha} \\ y'_{1,n+\frac{2}{5},\alpha} \\ y'_{1,n+1,\alpha} \\ y_{2,n+\frac{2}{5},\alpha} \\ y_{2,n+1,\alpha} \\ y'_{2,n+\frac{2}{5},\alpha} \\ y'_{2,n+1,\alpha} \end{pmatrix}.$$

3. *If* $a \leq 0$, *and* $b > 0$, *then*

$$Y_{n+1} = A_3^{-1} B_3 \begin{pmatrix} y'_{1,n,\alpha} \\ y'_{2,n,\alpha} \end{pmatrix} + A_3^{-1} C_3 \begin{pmatrix} y_{1,n,\alpha} \\ y_{2,n,\alpha} \end{pmatrix} + A_3^{-1} D_3 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where

$$A_3 = \begin{pmatrix} 1-\frac{8bh^2}{225} & \frac{4bh^2}{1125} & 0 & 0 & 0 & 0 & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} \\ -\frac{25bh^2}{72} & 1-\frac{2bh^2}{72} & 0 & 0 & 0 & 0 & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} \\ -\frac{55bh}{225} & \frac{4bh}{225} & 1 & 0 & 0 & 0 & -\frac{55ah}{225} & \frac{4ah}{225} \\ -\frac{25bh}{36} & -\frac{8bh}{36} & 0 & 1 & 0 & 0 & -\frac{25ah}{36} & -\frac{8ah}{36} \\ 0 & 0 & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} & 1-\frac{8bh^2}{225} & \frac{4bh^2}{1125} & 0 & 0 \\ 0 & 0 & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} & -\frac{25bh^2}{72} & 1-\frac{2bh^2}{72} & 0 & 0 \\ 0 & 0 & -\frac{55ah}{225} & \frac{4ah}{225} & -\frac{55bh}{225} & \frac{4bh}{225} & 1 & 0 \\ 0 & 0 & -\frac{25ah}{36} & -\frac{8ah}{36} & -\frac{25bh}{36} & -\frac{8bh}{36} & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \frac{2h}{5} & \frac{6ah^2}{125} \\ h & \frac{9ah^2}{72} \\ 1 & \frac{39ah}{225} \\ 1 & \frac{3ah}{36} \\ \frac{6ah^2}{125} & \frac{2h}{5} \\ \frac{9ah^2}{72} & h \\ \frac{39ah}{225} & 1 \\ \frac{3ah}{36} & 1 \end{pmatrix}, \; C_3 = \begin{pmatrix} \frac{6bh^2}{125}+1 & 0 \\ \frac{9bh^2}{72}+1 & 0 \\ \frac{39bh}{225} & 0 \\ \frac{3bh}{36} & 0 \\ 0 & \frac{6bh^2}{125}+1 \\ 0 & \frac{9bh^2}{72} \\ 0 & \frac{39bh}{225} \\ 0 & \frac{3bh}{36} \end{pmatrix},$$

$$D_3 = \begin{pmatrix} \frac{2h^2}{25} & 0 \\ \frac{h^2}{2} & 0 \\ \frac{2h}{5} & 0 \\ h & 0 \\ 0 & \frac{2h^2}{25} \\ 0 & \frac{h^2}{2} \\ 0 & \frac{2h}{5} \\ 0 & h \end{pmatrix}, \; Y_{n+1} = \begin{pmatrix} y_{1,n+\frac{2}{5},\alpha} \\ y_{1,n+1,\alpha} \\ y'_{1,n+\frac{2}{5},\alpha} \\ y'_{1,n+1,\alpha} \\ y_{2,n+\frac{2}{5},\alpha} \\ y_{2,n+1,\alpha} \\ y'_{2,n+\frac{2}{5},\alpha} \\ y'_{2,n+1,\alpha} \end{pmatrix}.$$

4. *If* $a \leq 0$, *and* $b < 0$, *then*

$$Y_{n+1} = A_4^{-1}B_4 \begin{pmatrix} y'_{1,n,\alpha} \\ y'_{2,n,\alpha} \end{pmatrix} + A_4^{-1}C_4 \begin{pmatrix} y_{1,n,\alpha} \\ y_{2,n,\alpha} \end{pmatrix} + A_4^{-1}D_4 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where

$$A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{8bh^2}{225} & \frac{4bh^2}{1125} & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} \\ 0 & 1 & 0 & 0 & -\frac{25bh^2}{72} & -\frac{2bh^2}{72} & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} \\ 0 & 0 & 1 & 0 & -\frac{55bh}{225} & \frac{4bh}{225} & -\frac{55ah}{225} & \frac{4ah}{225} \\ 0 & 0 & 0 & 1 & -\frac{25bh}{36} & -\frac{8bh}{36} & -\frac{25ah}{36} & -\frac{8ah}{36} \\ -\frac{8bh^2}{225} & \frac{4bh^2}{1125} & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} & 1 & 0 & 0 & 0 \\ -\frac{25bh^2}{72} & -\frac{2bh^2}{72} & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} & 0 & 1 & 0 & 0 \\ -\frac{55bh}{225} & \frac{4bh}{225} & -\frac{55ah}{225} & \frac{4ah}{225} & 0 & 0 & 1 & 0 \\ -\frac{25bh}{36} & -\frac{8bh}{36} & -\frac{25ah}{36} & -\frac{8ah}{36} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} \frac{2h}{5} & \frac{6ah^2}{125} \\ h & \frac{9ah^2}{72} \\ 1 & \frac{39ah}{225} \\ 1 & \frac{3ah}{36} \\ \frac{6ah^2}{125} & \frac{2h}{5} \\ \frac{9ah^2}{72} & h \\ \frac{39ah}{225} & 1 \\ \frac{3ah}{36} & 1 \end{pmatrix}, C_4 = \begin{pmatrix} 1 & \frac{6bh^2}{125} \\ 1 & \frac{9bh^2}{72} \\ 0 & \frac{39bh}{225} \\ 0 & \frac{3bh}{36} \\ \frac{6bh^2}{125} & 1 \\ \frac{9bh^2}{72} & 1 \\ \frac{39bh}{225} & 0 \\ \frac{3bh}{36} & 0 \end{pmatrix},$$

$$D_4 = \begin{pmatrix} \frac{2h^2}{25} & 0 \\ \frac{h^2}{2} & 0 \\ \frac{2h}{5} & 0 \\ h & 0 \\ 0 & \frac{2h^2}{25} \\ 0 & \frac{h^2}{2} \\ 0 & \frac{2h}{5} \\ 0 & h \end{pmatrix}, Y_{n+1} = \begin{pmatrix} y_{1,n+\frac{2}{5},\alpha} \\ y_{1,n+1,\alpha} \\ y'_{1,n+\frac{2}{5},\alpha} \\ y'_{1,n+1,\alpha} \\ y_{2,n+\frac{2}{5},\alpha} \\ y_{2,n+1,\alpha} \\ y'_{2,n+\frac{2}{5},\alpha} \\ y'_{2,n+1,\alpha} \end{pmatrix}.$$

Proof. 1) Let $a \geq 0$ and $b \geq 0$. Then,

$$f(t_{n+j}, y_{i,n+j,\alpha}, y'_{i,n+j,\alpha}) = c_i + by_{i,n+j,\alpha} + ay'_{i,n+j,\alpha},$$

where $i = 1,2$ and $j = 0, \frac{2}{5}, 1$. Thus,

$$y_{1,n+1,\alpha} = y_{1,n,\alpha} + h\, y'_{1,n,\alpha} + \frac{h^2}{72}\Big(9(c_1 + by_{1,n,\alpha} + ay'_{1,n,\alpha}) + 25(c_1 + by_{1,n+\frac{2}{5},\alpha} + ay'_{1,n+\frac{2}{5},\alpha}) + 2(c_1 + by_{1,n+1,\alpha} + ay'_{1,n+1,\alpha})\Big),$$

$$y_{1,n+\frac{2}{5},\alpha} = y_{1,n,\alpha} + \frac{2h}{5}\, y'_{1,n,\alpha} + \frac{2h^2}{1125}\Big(27(c_1 + by_{1,n,\alpha} + ay'_{1,n,\alpha}) + 20(c_1 + by_{1,n+\frac{2}{5},\alpha} + ay'_{1,n+\frac{2}{5},\alpha}) - 2(c_1 + by_{1,n+1,\alpha} + ay'_{1,n+1,\alpha})\Big),$$

$$y'_{1,n+1,\alpha} = y'_{1,n,\alpha} + \frac{h}{36}\Big(3(c_1 + by_{1,n,\alpha} + ay'_{1,n,\alpha}) + 25(c_1 + by_{1,n+\frac{2}{5},\alpha} + ay'_{1,n+\frac{2}{5},\alpha}) + 8(c_1 + by_{1,n+1,\alpha} + ay'_{1,n+1,\alpha})\Big),$$

$$y'_{1,n+\frac{2}{5},\alpha} = y'_{1,n,\alpha} + \frac{h}{225}\Big(39(c_1 + by_{1,n,\alpha} + ay'_{1,n,\alpha}) + 55(c_1 + by_{1,n+\frac{2}{5},\alpha} + ay'_{1,n+\frac{2}{5},\alpha}) - 4(c_1 + by_{1,n+1,\alpha} + ay'_{1,n+1,\alpha})\Big),$$

$$y_{2,n+1,\alpha} = y_{2,n,\alpha} + h\, y'_{2,n,\alpha} + \frac{h^2}{72}\Big(9(c_2 + by_{2,n,\alpha} + ay'_{2,n,\alpha}) + 25(c_2 + by_{2,n+\frac{2}{5},\alpha} + ay'_{2,n+\frac{2}{5},\alpha}) + 2(c_2 + by_{2,n+1,\alpha} + ay'_{2,n+1,\alpha})\Big),$$

$$y_{2,n+\frac{2}{5},\alpha} = y_{2,n,\alpha} + \frac{2h}{5}\, y'_{2,n,\alpha} + \frac{2h^2}{1125}\Big(27(c_2 + by_{2,n,\alpha} + ay'_{2,n,\alpha}) + 20(c_2 + by_{2,n+\frac{2}{5},\alpha} + ay'_{2,n+\frac{2}{5},\alpha}) - 2(c_2 + by_{2,n+1,\alpha} + ay'_{2,n+1,\alpha})\Big),$$

$$y'_{2,n+1,\alpha} = y'_{2,n,\alpha} + \frac{h}{36}\Big(3(c_2 + by_{2,n,\alpha} + ay'_{2,n,\alpha}) + 25(c_2 + by_{2,n+\frac{2}{5},\alpha} + ay'_{2,n+\frac{2}{5},\alpha}) + 8(c_2 + by_{2,n+1,\alpha} + ay'_{2,n+1,\alpha})\Big),$$

$$y'_{2,n+\frac{2}{5},\alpha} = y'_{2,n,\alpha} + \frac{h}{225}\Big(39(c_2 + by_{2,n,\alpha} + ay'_{2,n,\alpha}) + 55(c_2 + by_{2,n+\frac{2}{5},\alpha} + ay'_{2,n+\frac{2}{5},\alpha}) - 4(c_2 + by_{2,n+1,\alpha} + ay'_{2,n+1,\alpha})\Big).$$

The last system can be rewritten in the matrix form as

$$A_1 Y_1 = B_1\, y'_{1,n,\alpha} + C_1\, y_{1,n,\alpha} + D_1\, c_1, \;\; A_1 Y_2 = B_1\, y'_{2,n,\alpha} + C_1\, y_{2,n,\alpha} + D_1\, c_2,$$

where

$$A_1 = \begin{pmatrix} 1-\frac{8bh^2}{225} & \frac{4bh^2}{1125} & -\frac{8ah^2}{225} & \frac{4ah^2}{1125} \\ -\frac{25bh^2}{72} & 1-\frac{2bh^2}{72} & -\frac{25ah^2}{72} & -\frac{2ah^2}{72} \\ -\frac{55bh}{225} & \frac{4bh}{225} & 1-\frac{55ah}{225} & \frac{4ah}{225} \\ -\frac{25bh}{36} & -\frac{8bh}{36} & -\frac{25ah}{36} & 1-\frac{8ah}{36} \end{pmatrix}, Y_1 = \begin{pmatrix} y_{1,n+\frac{2}{5},\alpha} \\ y_{1,n+1,\alpha} \\ y'_{1,n+\frac{2}{5},\alpha} \\ y'_{1,n+1,\alpha} \end{pmatrix}, Y_2 = \begin{pmatrix} y_{2,n+\frac{2}{5},\alpha} \\ y_{2,n+1,\alpha} \\ y'_{2,n+\frac{2}{5},\alpha} \\ y'_{2,n+1,\alpha} \end{pmatrix},$$

$$B1 = \begin{pmatrix} \frac{6ah^2}{125}+\frac{2h}{5} \\ \frac{9ah^2}{72}+h \\ \frac{39ah}{225}+1 \\ \frac{3ah}{36}+1 \end{pmatrix}, C_1 = \begin{pmatrix} \frac{6bh^2}{125}+1 \\ \frac{9bh^2}{72}+1 \\ \frac{39bh}{225} \\ \frac{3bh}{36} \end{pmatrix}, D_1 = \begin{pmatrix} \frac{2h^2}{25} \\ \frac{h^2}{2} \\ \frac{2h}{5} \\ h \end{pmatrix}.$$

Since

$det(A_1) = \frac{43740000a^2h^2+15309000abh^3-306180000ah+1458000b^2h^4-41553000bh^2+656100000}{656100000} \neq 0,$

then A_1^{-1} exist. Thus for $j = 1,2,$

$$\begin{pmatrix} y_{j,n+\frac{2}{5},\alpha} \\ y_{j,n+1,\alpha} \\ y'_{j,n+\frac{2}{5},\alpha} \\ y'_{j,n+1,\alpha} \end{pmatrix} = E_1\, y'_{j,n,\alpha} + E_2\, y_{j,n,\alpha} + E_3\, c_j,$$

where

$$E_1 = A_1^{-1}B_1 = \begin{pmatrix} \frac{2h\left(h\left(a\left(3abh^3+16\left(a^2+b\right)h^2-1200\right)-165bh\right)+4500\right)}{25\left(h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+900\right)} \\ \frac{h\left(h\left(186bh+a\left(-3abh^3-5\left(a^2+b\right)h^2+60\right)\right)+1800\right)}{2h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+1800} \\ \frac{22500-h\left(4h\left(8bh^2+225\right)a^2+3\left(b\left(2bh^2+135\right)h^2+500\right)a+3bh\left(24bh^2-125\right)\right)}{25\left(h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+900\right)} \\ \frac{h\left(5h\left(bh^2+36\right)a^2+3\left(bh^2+16\right)\left(bh^2+20\right)a+3bh\left(9bh^2+262\right)\right)+1800}{2h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+1800} \end{pmatrix},$$

$$E_2 = A_1^{-1}C_1 = \begin{pmatrix} \frac{6ab^2h^5+8b\left(4a^2-5b\right)h^4-75abh^3+375\left(4a^2+b\right)h^2-10500ah+22500}{25\left(h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+900\right)} \\ \frac{h\left(-5h\left(bh^2-24\right)a^2-3\left(b\left(bh^2+26\right)h^2+280\right)a+2bh\left(11bh^2+393\right)\right)+1800}{2h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+1800} \\ -\frac{2bh\left(bh^2+75\right)\left(3bh^2+16ah-60\right)}{25\left(h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+900\right)} \\ \frac{bh\left(bh^2+12\right)\left(3bh^2+5ah+150\right)}{2h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+1800} \end{pmatrix},$$

$$E_3 = A_1^{-1}D_1 = \begin{pmatrix} \frac{2h^2(ah-15)\left(3bh^2+16ah-60\right)}{25\left(h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+900\right)} \\ -\frac{h^2(ah-6)\left(3bh^2+5ah+150\right)}{2h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+1800} \\ -\frac{2h\left(bh^2+75\right)\left(3bh^2+16ah-60\right)}{25\left(h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+900\right)} \\ \frac{h\left(bh^2+12\right)\left(3bh^2+5ah+150\right)}{2h\left(60ha^2+21\left(bh^2-20\right)a+bh\left(2bh^2-57\right)\right)+1800} \end{pmatrix}.$$

Using a similar argument, the three other cases (2), (3), and (4) can be proven.

4.3 Numerical Results of FIVPs

In this section, a numerical examples will be presented to show the efficiency of the proposed methods. The two types of FIVPs will be studied which are linear, and non-linear problems for first and second-order fuzzy initial value problems, respectively.

Example 4.3.1. Consider the following linear first order fuzzy initial value problem

$$y'(x) = (x^2 \odot y(x)) \oplus (\hat{\gamma} \odot x^2), \quad y(0) = \hat{\gamma}$$

where α-level set for $\hat{\gamma} = (1\ \ 2\ \ 3)$ is $\gamma_\alpha = [1+\alpha, 3-\alpha]$ and, $h = 0.01$. Let $y(x) = [y_1(x,\alpha), y_2(x,\alpha)]$ be the fuzzy solution. By implementing α-level sets, the problem becomes

$$[y_1'(x,\alpha), y_2'(x,\alpha)] = (x^2 \odot [y_1(x,\alpha), y_2(x,\alpha)]) \oplus [x^2 \odot (1+\alpha), x^2 \odot (3-\alpha)],$$

$$[y_1(0,\alpha), y_2(0,\alpha)] = [1+\alpha, 3-\alpha].$$

Using HBM1 for the lower bound,

$$\begin{aligned} y_{1,n+1} &= y_{1,n} + \frac{h}{6}\left(x_n^2 y_{1,n} + x_n^2(1+\alpha) + 4\left(x_{n+\frac{1}{2}}^2 y_{1,n+\frac{1}{2}} + x_{n+\frac{1}{2}}^2(1+\alpha)\right) + \right. \\ &\qquad \left. x_{n+1}^2 y_{1,n+1} + x_{n+1}^2(1+\alpha)\right), \\ y_{1,n+\frac{1}{2}} &= y_{1,n} + \frac{h}{24}\left(5\left(x_n^2 y_{1,n} + x_n^2(1+\alpha)\right) + 8\left(x_{n+\frac{1}{2}}^2 y_{1,n+\frac{1}{2}} + x_{n+\frac{1}{2}}^2(1+\alpha)\right) - \right. \\ &\qquad \left. \left(x_{n+1}^2 y_{1,n+1} + x_{n+1}^2(1+\alpha)\right)\right). \end{aligned}$$

Let $x_n = nh,\ x_{n+\frac{1}{2}} = \frac{h}{2} + nh$ and $x_{n+1} = h + nh$. Then, the above system can be written in a matrix form as

$$Y_m = A^{-1}B\, y_{1,n} + A^{-1}C_1,$$

where

$$Y_m = \begin{pmatrix} y_{1,n+1} \\ y_{1,n+\frac{1}{2}} \end{pmatrix}, \quad A = \begin{pmatrix} 1 - \frac{h}{6}(h+nh)^2 & \frac{4h}{6}(\frac{h}{2}+nh)^2 \\ \frac{h}{24}(h+nh)^2 & 1 - \frac{8h}{24}(\frac{h}{2}+nh)^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 + \frac{h}{6}(nh)^2 \\ 1 + \frac{5h}{24}(nh)^2 \end{pmatrix},$$

and

$$C_1 = \begin{pmatrix} \frac{h}{6}(1+\alpha)\left((nh)^2 + 4(\frac{h}{2}+nh)^2 + (h+nh)^2\right) \\ \frac{h}{24}(1+\alpha)\left(5(nh)^2 + 8(\frac{h}{2}+nh)^2 - (h+nh)^2\right) \end{pmatrix}.$$

Similarly, HBM1 can be applied for the upper bound. Then,

$$Y_m = A^{-1}B\,y_{2,n} + A^{-1}C_2$$

where

$$Y_m = \begin{pmatrix} y_{2,n+1} \\ y_{2,n+\frac{1}{2}} \end{pmatrix}, \quad A = \begin{pmatrix} 1-\frac{h}{6}(h+nh)^2 & \frac{4h}{6}(\frac{h}{2}+nh)^2 \\ \frac{h}{24}(h+nh)^2 & 1-\frac{8h}{24}(\frac{h}{2}+nh)^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1+\frac{h}{6}(nh)^2 \\ 1+\frac{5h}{24}(nh)^2 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} \frac{h}{6}(3-\alpha)\left((nh)^2+4(\frac{h}{2}+nh)^2+(h+nh)^2\right) \\ \frac{h}{24}(3-\alpha)\left(5(nh)^2+8(\frac{h}{2}+nh)^2-(h+nh)^2\right) \end{pmatrix}.$$

The errors of approximation of $y_{1,n}$ and $y_{2,n}$ for $\alpha = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$, are given in Tables 4.1 and 4.2, respectively, where the exact solution is given by

$$[(2e^{\frac{x^3}{3}}-1)(1+\alpha),(2e^{\frac{x^3}{3}}-1)(3-\alpha)].$$

Table 4.1: The absolute error in approximating $y_{1,n}$ for $h = 0.01$.

$y_{1,n}$					
x \ α	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.1	1.38889×10^{-12}	1.73617×10^{-12}	2.08322×10^{-12}	2.4305×10^{-12}	2.77778×10^{-12}
0.2	5.56177×10^{-12}	6.95222×10^{-12}	8.34266×10^{-12}	9.7331×10^{-12}	1.11235×10^{-11}
0.3	1.25429×10^{-11}	1.56786×10^{-11}	1.88143×10^{-11}	2.195×10^{-11}	2.50857×10^{-11}
0.4	2.23879×10^{-11}	2.79847×10^{-11}	3.35818×10^{-11}	3.91789×10^{-11}	4.47757×10^{-11}
0.5	3.51532×10^{-11}	4.39417×10^{-11}	5.27298×10^{-11}	6.15179×10^{-11}	7.03064×10^{-11}
0.6	5.07772×10^{-11}	6.34715×10^{-11}	7.61657×10^{-11}	8.886×10^{-11}	1.01554×10^{-10}
0.7	6.87681×10^{-11}	8.59601×10^{-11}	1.03152×10^{-10}	1.20344×10^{-10}	1.37536×10^{-10}
0.8	8.74738×10^{-11}	1.09342×10^{-10}	1.31211×10^{-10}	1.53079×10^{-10}	1.74948×10^{-10}
0.9	1.02472×10^{-10}	1.2809×10^{-10}	1.53708×10^{-10}	1.79326×10^{-10}	2.04944×10^{-10}
1	1.03102×10^{-10}	1.28878×10^{-10}	1.54654×10^{-10}	1.80429×10^{-10}	2.06205×10^{-10}

Table 4.2: The absolute error in approximating $y_{2,n}$ for $h = 0.01$.

	$y_{2,n}$				
x \ α	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.1	4.16644×10^{-12}	3.81961×10^{-12}	3.47233×10^{-12}	3.12506×10^{-12}	2.77778×10^{-12}
0.2	1.66849×10^{-11}	1.52944×10^{-11}	1.3904×10^{-11}	1.25135×10^{-11}	1.11231×10^{-11}
0.3	3.76281×10^{-11}	3.44924×10^{-11}	3.13567×10^{-11}	2.8221×10^{-11}	2.50853×10^{-11}
0.4	6.71627×10^{-11}	6.15659×10^{-11}	5.59686×10^{-11}	5.03717×10^{-11}	4.47749×10^{-11}
0.5	1.05457×10^{-10}	9.66689×10^{-11}	8.78808×10^{-11}	7.90923×10^{-11}	7.03042×10^{-11}
0.6	1.52329×10^{-10}	1.39635×10^{-10}	1.26941×10^{-10}	1.14246×10^{-10}	1.01552×10^{-10}
0.7	2.06303×10^{-10}	1.89111×10^{-10}	1.71919×10^{-10}	1.54727×10^{-10}	1.37534×10^{-10}
0.8	2.6242×10^{-10}	2.40551×10^{-10}	2.18682×10^{-10}	1.96814×10^{-10}	1.74946×10^{-10}
0.9	3.07414×10^{-10}	2.81795×10^{-10}	2.56177×10^{-10}	2.3056×10^{-10}	2.04941×10^{-10}
1	3.09304×10^{-10}	2.83528×10^{-10}	2.57752×10^{-10}	2.31977×10^{-10}	2.06201×10^{-10}

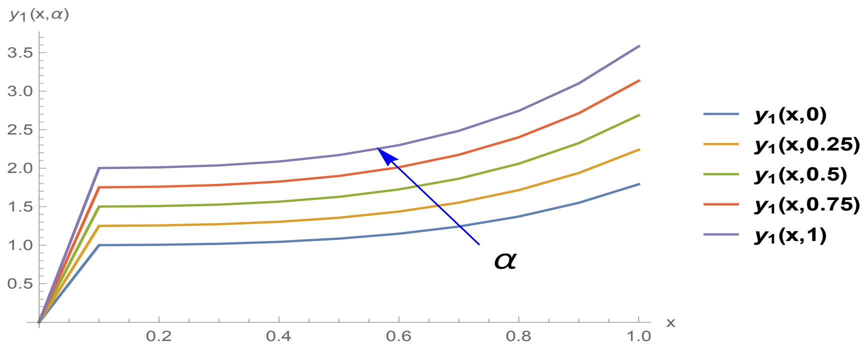


Figure 4.1: The approximate solution y_1 for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

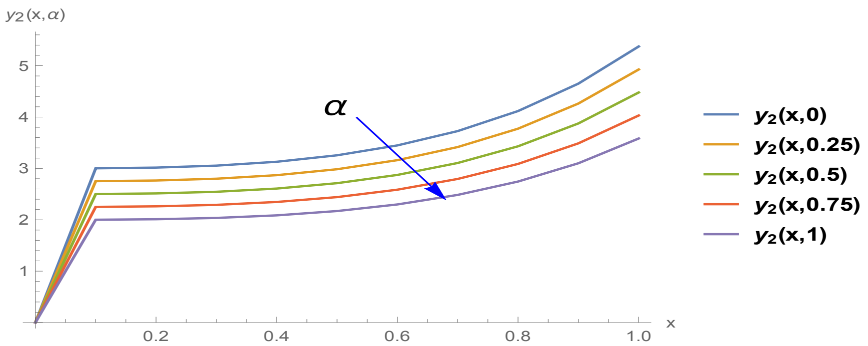


Figure 4.2: The approximate solution y_2 for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

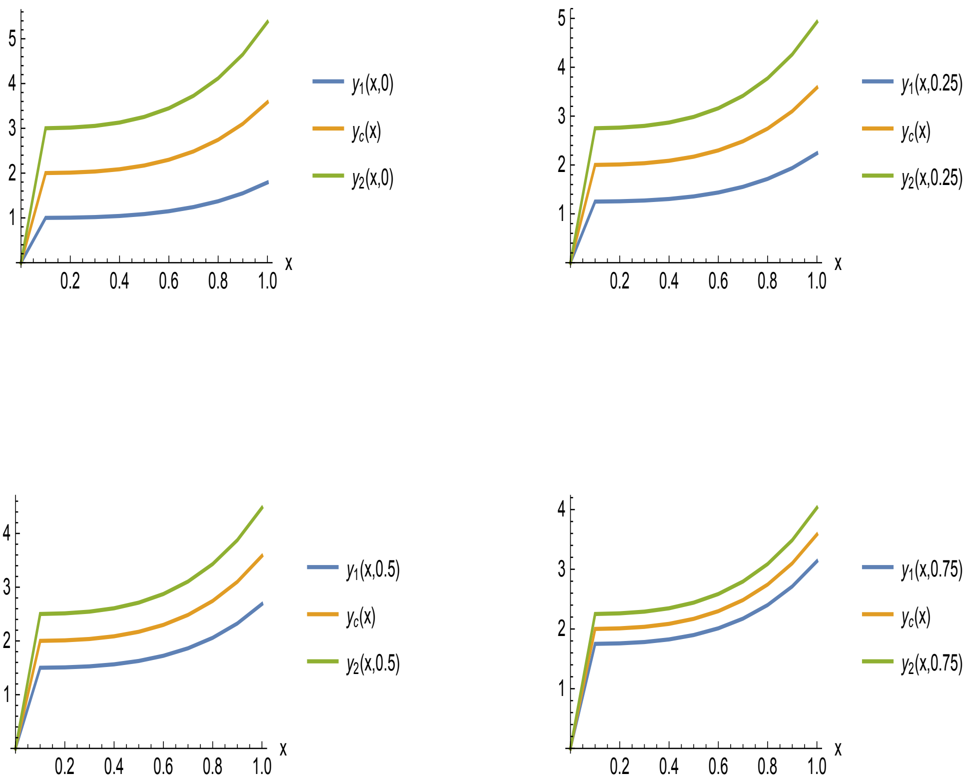


Figure 4.3: The crisp (y_c) and approximate solutions $(y_1,\ y_2)$, for different α s.

Example 4.3.2. Consider the nonlinear first order fuzzy initial value problem

$$y'(x) = y(x)^2 + x^2, \quad y(0) = \hat{\gamma}, \;\; x \geq 0,$$

where α-level set for $\hat{\gamma} = (-0.1\ \ 0\ \ 0.1)$ is $\hat{\gamma}_\alpha = [0.1(\alpha-1), 0.1(1-\alpha)]$, and $h = 0.1$. Let $y(x) = [y_1(x,\alpha), y_2(x,\alpha)]$ be a fuzzy solution. By implementing α-level sets, the problem becomes

$$[y_1'(x,\alpha), y_2'(x,\alpha)] = [y_1(x,\alpha), y_2(x,\alpha)]^2 + x^2, \;\; [y_1(0,\alpha), y_2(0,\alpha)] = [0.1(\alpha-1), 0.1(1-\alpha)],$$

where the exact solution is given by

$$y_k(x,\alpha)=\frac{(1-\alpha)J_{\frac{-3}{4}}(\frac{x^2}{2})\,\Gamma(\frac{1}{4})+(-1)^k 20xJ_{\frac{3}{4}}(\frac{x^2}{2})\,\Gamma(\frac{3}{4})}{(\alpha-1)J_{\frac{1}{4}}(\frac{x^2}{2})\,\Gamma(\frac{1}{4})+(-1)^k 20J_{\frac{-1}{4}}(\frac{x^2}{2})\,\Gamma(\frac{3}{4})},$$

for $k=1,\ 2, J_n(z)$ is the Bessel function of the first kind, and $\Gamma(z)$ is the Euler Gamma function. In Table 4.3, the absolute error of the results obtained by the current (HBM1) method and the ones obtained in [27] is presented.

Table 4.3: Absolute error for $y_{1,0.5}$ and $y_{2,0.5}$.

α \ $Error$	Er for $y_{1,0.5}$	Er for $y_{2,0.5}$	Er for $y_{1,0.5}$(HPM) in [27]	Er for $y_{2,0.5}$ (HPM) in [27]
0	5.57124×10^{-8}	5.43066×10^{-8}	1.63068×10^{-6}	1.99421×10^{-6}
0.2	5.6527×10^{-8}	5.58044×10^{-8}	1.03431×10^{-6}	1.27971×10^{-6}
0.4	5.72014×10^{-8}	5.68912×10^{-8}	6.33486×10^{-7}	8.09126×10^{-7}
0.6	5.77142×10^{-8}	5.76168×10^{-8}	3.51316×10^{-7}	4.87165×10^{-7}
0.8	5.80413×10^{-8}	5.80253×10^{-8}	1.34909×10^{-7}	2.50421×10^{-7}
1	5.81552×10^{-8}	5.81552×10^{-8}	5.46386×10^{-8}	5.46386×10^{-8}

Example 4.3.3. Consider the second order fuzzy linear initial value problem

$$y''=-y(x),\quad y(0)=0,\quad y'(0)=\hat{\gamma},$$

where $\gamma_\alpha=[0.9+0.1\alpha,1.1-0.1\alpha]$, and $h=0.1$. Let $y(x)=[y_1(x,\alpha),y_2(x,\alpha)]$ be a fuzzy solution and $y'(x)=[y_1'(x,\alpha),y_2'(x,\alpha)]$.

By implement the α-level sets, the problem will be

$$[y_1''(x,\alpha),y_2''(x,\alpha)]=[-y_2(x,\alpha),-y_1(x,\alpha)],\quad [y_1(0,\alpha),y_2(0,\alpha)]=0,$$

$$[y_1'(0,\alpha), y_2'(0,\alpha)] = [0.9+0.1\alpha, 1.1-0.1\alpha].$$

Using HBM1,

$$\begin{aligned}
y_{1,n+1,\alpha} &= y_{1,n,\alpha} + h\, y'_{1,n,\alpha} - \frac{h^2}{72}(9y_{2,n,\alpha} + 25y_{2,n+\frac{2}{5},\alpha} + 2y_{2,n+1,\alpha}),\\
y_{1,n+\frac{2}{5},\alpha} &= y_{1,n,\alpha} + \frac{2h}{5}\, y'_{1,n,\alpha} - \frac{2h^2}{1125}(27y_{2,n,\alpha} + 20y_{2,n+\frac{2}{5},\alpha} - 2y_{2,n+1,\alpha}),\\
y'_{1,n+1,\alpha} &= y'_{1,n,\alpha} - \frac{h}{36}(3y_{2,n,\alpha} + 25y_{2,n+\frac{2}{5},\alpha} + 8y_{2,n+1,\alpha}),\\
y'_{1,n+\frac{2}{5},\alpha} &= y'_{1,n,\alpha} - \frac{h}{225}(39y_{2,n,\alpha} + 55y_{2,n+\frac{2}{5},\alpha} - 4y_{2,n+1,\alpha}),\\
y_{2,n+1,\alpha} &= y_{2,n,\alpha} + h\, y'_{2,n,\alpha} - \frac{h^2}{72}(9y_{1,n,\alpha} + 25y_{1,n+\frac{2}{5},\alpha} + 2y_{1,n+1,\alpha}),\\
y_{2,n+\frac{2}{5},\alpha} &= y_{2,n,\alpha} + \frac{2h}{5}\, y'_{2,n,\alpha} - \frac{2h^2}{1125}(27y_{1,n,\alpha} + 20y_{1,n+\frac{2}{5},\alpha} - 2y_{1,n+1,\alpha}),\\
y'_{2,n+1,\alpha} &= y'_{2,n,\alpha} - \frac{h}{36}(3y_{1,n,\alpha} + 25y_{1,n+\frac{2}{5},\alpha} + 8y_{1,n+1,\alpha}),\\
y'_{2,n+\frac{2}{5},\alpha} &= y'_{2,n,\alpha} - \frac{h}{225}(39y_{1,n,\alpha} + 55y_{1,n+\frac{2}{5},\alpha} - 4y_{1,n+1,\alpha}).
\end{aligned}$$

The above system can be written in the matrix form

$$Y_m = A^{-1}By_m + A^{-1}Cy'_m,$$

where

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & \frac{2h^2}{72} & \frac{25h^2}{72} & 0 & 0\\
0 & 1 & 0 & 0 & -\frac{4h^2}{1125} & \frac{40h^2}{1125} & 0 & 0\\
0 & 0 & 1 & 0 & \frac{8h}{36} & \frac{25h}{36} & 0 & 0\\
0 & 0 & 0 & 1 & -\frac{4h}{225} & \frac{55h}{225} & 0 & 0\\
\frac{2h^2}{72} & \frac{25h^2}{72} & 0 & 0 & 1 & 0 & 0 & 0\\
-\frac{4h^2}{1125} & \frac{40h^2}{1125} & 0 & 0 & 0 & 1 & 0 & 0\\
\frac{8h}{36} & \frac{25h}{36} & 0 & 0 & 0 & 0 & 1 & 0\\
-\frac{4h}{225} & \frac{55h}{225} & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & -\frac{9h^2}{72}\\
1 & -\frac{54h^2}{1125}\\
0 & -\frac{3h}{36}\\
0 & -\frac{39h}{225}\\
-\frac{9h^2}{72} & 1\\
-\frac{54h^2}{1125} & 1\\
-\frac{3h}{36} & 0\\
-\frac{39h}{225} & 0
\end{pmatrix},$$

$$C = \begin{pmatrix} h & 0 \\ \frac{2h}{5} & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & h \\ 0 & \frac{2h}{5} \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad Y_m = \begin{pmatrix} y_{1,n+1} \\ y_{1,K+n} \\ y'_{1,n+1} \\ y'_{1,k+n} \\ y_{2,n+1} \\ y_{2,K+n} \\ y'_{2,n+1} \\ y'_{2,k+n} \end{pmatrix}, \quad y_m = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix}, \quad y'_m = \begin{pmatrix} y'_{1,n} \\ y'_{2,n} \end{pmatrix}.$$

The error in approximating $y_{1,n}$ and $y_{2,n}$ for $\alpha = 0,\ 0.25,\ 0.5,\ 0.75,\ 1$ are given in Table 4.4 and 4.5, respectively, where the exact solution is given by

$$[(0.1\alpha - 0.1)\sinh(x) + \sin(x), (0.1 - 0.1\alpha)\sinh(x) + \sin(x)].$$

Table 4.4: The absolute error in approximating $y_{1,n}$ for $h = 0.1$

$y_{1,n}$					
x \ α	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.1	4.50043×10^{-12}	4.39764×10^{-12}	4.29483×10^{-12}	4.19204×10^{-12}	4.08924×10^{-12}
0.2	2.49082×10^{-8}	2.56047×10^{-8}	2.63012×10^{-8}	2.69978×10^{-8}	2.76943×10^{-8}
0.3	7.40714×10^{-8}	7.6176×10^{-8}	7.82805×10^{-8}	8.03851×10^{-8}	8.24896×10^{-8}
0.4	1.46254×10^{-7}	1.50506×10^{-7}	1.54759×10^{-7}	1.59012×10^{-7}	1.63264×10^{-7}
0.5	2.39628×10^{-7}	2.46812×10^{-7}	2.53997×10^{-7}	2.61182×10^{-7}	2.68366×10^{-7}
0.6	3.51805×10^{-7}	3.62763×10^{-7}	3.73722×10^{-7}	3.8468×10^{-7}	3.95639×10^{-7}
0.7	4.79859×10^{-7}	4.95509×10^{-7}	5.11159×10^{-7}	5.26808×10^{-7}	5.42458×10^{-7}
0.8	6.20368×10^{-7}	6.41718×10^{-7}	6.63068×10^{-7}	6.84418×10^{-7}	7.05768×10^{-7}
0.9	7.6945×10^{-7}	7.9762×10^{-7}	8.2579×10^{-7}	8.5396×10^{-7}	8.82131×10^{-7}
1	9.22809×10^{-7}	9.59051×10^{-7}	9.95292×10^{-7}	1.03153×10^{-6}	1.06777×10^{-6}

Table 4.5: The absolute error in approximating $y_{2,n}$ for $h = 0.1$

$y_{2,n}$					
x \ α	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.1	3.67807×10^{-12}	3.78086×10^{-12}	3.88367×10^{-12}	3.98646×10^{-12}	4.08926×10^{-12}
0.2	3.04804×10^{-8}	2.97839×10^{-8}	2.90874×10^{-8}	2.83908×10^{-8}	2.76943×10^{-8}
0.3	9.09078×10^{-8}	8.88032×10^{-8}	8.66987×10^{-8}	8.45941×10^{-8}	8.24896×10^{-8}
0.4	1.80275×10^{-7}	1.76023×10^{-7}	1.7177×10^{-7}	1.67517×10^{-7}	1.63264×10^{-7}
0.5	2.97104×10^{-7}	2.8992×10^{-7}	2.82735×10^{-7}	2.75551×10^{-7}	2.68366×10^{-7}
0.6	4.39473×10^{-7}	4.28515×10^{-7}	4.17556×10^{-7}	4.06598×10^{-7}	3.95639×10^{-7}
0.7	6.05057×10^{-7}	5.89408×10^{-7}	5.73758×10^{-7}	5.58108×10^{-7}	5.42458×10^{-7}
0.8	7.91169×10^{-7}	7.69819×10^{-7}	7.48469×10^{-7}	7.27119×10^{-7}	7.05768×10^{-7}
0.9	9.94812×10^{-7}	9.66642×10^{-7}	9.38471×10^{-7}	9.10301×10^{-7}	8.82131×10^{-7}
1	1.21274×10^{-6}	1.1765×10^{-6}	1.14026×10^{-6}	1.10402×10^{-6}	1.06777×10^{-6}

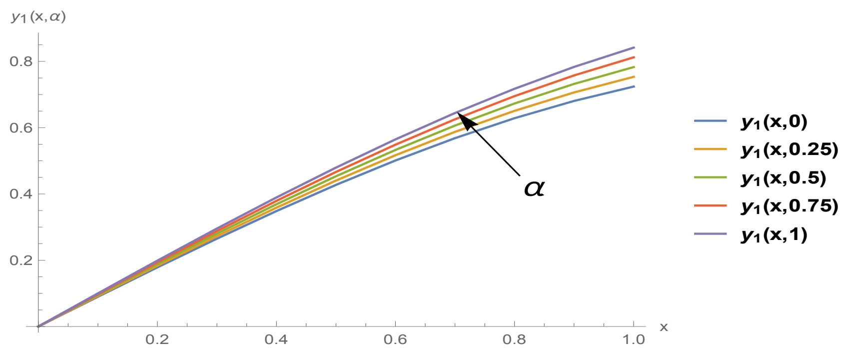


Figure 4.4: The approximate solution y_1 for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

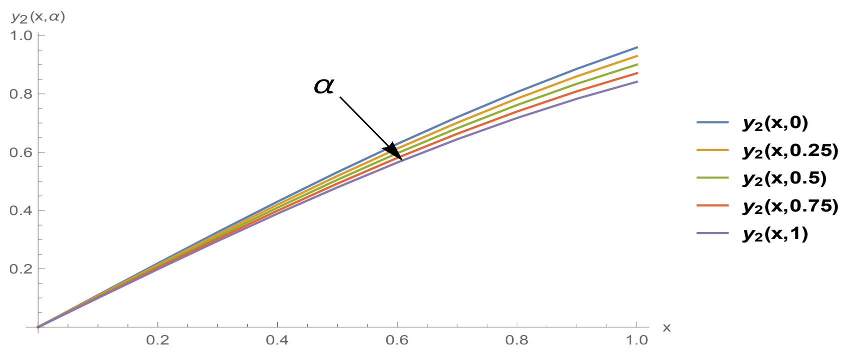


Figure 4.5: The approximate solution y_2 for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

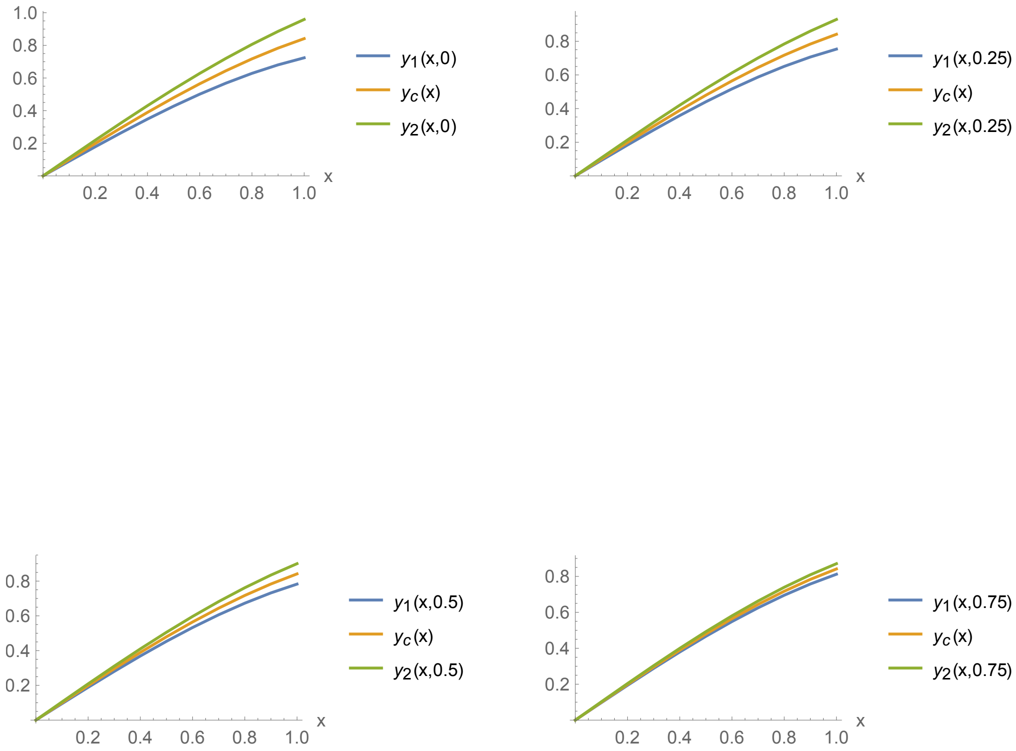


Figure 4.6: The crisp (y_c) and approximate solutions $(y_1,\ y_2)$, for α s.

Example 4.3.4. Consider the following nonlinear second order fuzzy initial value problem

$$y''(x) = -\left(y'(x)\right)^2, \quad y(0) = \hat{\gamma}_\alpha, \quad y'(0) = \hat{\theta}_\alpha, \quad 0 \le x \le 0.1,$$

where $\hat{\gamma}_\alpha = [\alpha, 2-\alpha]$, $\hat{\theta}_\alpha = [1+\alpha, 3-\alpha]$, and $h = 0.01$. Let $y(x) = [y_1(x,\alpha), y_2(x,\alpha)]$ be a fuzzy solution and $y'(x) = [y_1'(x,\alpha), y_2'(x,\alpha)]$. Using α-level sets, the problem will have the following form

$$[y_1''(x,\alpha), y_2''(x,\alpha)] = -\left([y_1(x,\alpha), y_2(x,\alpha)]\right)^2,$$

$$[y_1(0,\alpha), y_2(0,\alpha)] = [\alpha, 2-\alpha], \quad [y_1'(0,\alpha), y_2'(0,\alpha)] = [1+\alpha, 3-\alpha],$$

where the exact solution is given by

$$\left[\ln\left((\alpha e^{\alpha} + e^{\alpha})x + e^{\alpha}\right), \ln\left(\left(3e^{2-\alpha} - \alpha e^{2-\alpha}\right)x + e^{2-\alpha}\right)\right].$$

In Tables 4.6 and 4.7, the absolute error of the results obtained by the current (HBM1) method and the ones obtained in [11] is presented.

Table 4.6: The absolute error for $y_{1,0.1}$.

$y_{1,0.1}$				
α \ *Error*	Er for $y_{1,0.1}$	Er for h_1 HAM in [11]	Er for h_2 HAM in [11]	Er for OHAM in [11]
0	2.42502×10^{-10}	1.53529×10^{-7}	3.98956×10^{-8}	1.59889×10^{-10}
0.2	5.79712×10^{-10}	4.51332×10^{-7}	3.91122×10^{-10}	3.91122×10^{-10}
0.4	1.20466×10^{-9}	1.1207×10^{-6}	3.79328×10^{-9}	3.79328×10^{-9}
0.6	2.25974×10^{-9}	2.4597×10^{-6}	4.47279×10^{-8}	2.61947×10^{-9}
0.8	3.92067×10^{-9}	4.9128×10^{-6}	6.92457×10^{-8}	6.69669×10^{-8}
1	6.39707×10^{-9}	9.10987×10^{-6}	1.53606×10^{-7}	1.11097×10^{-8}

Table 4.7: The absolute error for $y_{2,0.1}$.

$y_{2,0.1}$				
α \ *Error*	Er for $y_{2,0.1}$	Er for h_1 HAM in [11]	Er for h_2 HAM in [11]	Er for OHAM in [11]
0	4.07084×10^{-8}	9.6735×10^{-5}	6.7076×10^{-6}	1.26440×10^{-7}
0.2	2.98335×10^{-8}	6.4822×10^{-5}	3.6619×10^{-6}	9.18889×10^{-8}
0.4	2.13238×10^{-8}	4.2133×10^{-5}	1.8656×10^{-6}	7.34552×10^{-8}
0.6	1.48046×10^{-8}	2.6432×10^{-5}	8.76047×10^{-7}	3.52946×10^{-8}
0.8	9.93257×10^{-9}	1.5907×10^{-5}	3.76985×10^{-7}	1.51728×10^{-8}
1	6.39707×10^{-9}	9.10987×10^{-6}	1.53606×10^{-7}	1.11097×10^{-8}

4.4 Concluding Remarks

In this section, the analysis of the results will be presented for first and second-order fuzzy initial value problems. Optimized one-step hybrid block methods have been proposed for solving fuzzy first and second-order initial value problems of ordinary differential equations. The methods are self-starting methods since they depend on the initial conditions only. The proposed methods are zero stable, have order 3, consistent, and thus they are convergent. Also, the method for order one IVPs is A-stable as illustrated by the regions of absolute stability in Figures 3.1. The numerical results show the efficiency of the current methods where high precision is achieved even when the only one-step point is used. For researchers who are interested to get more accuracy, they can use two or three off-step points. The absolute error in this case will be almost zero. Several examples, linear and nonlinear using the current methods are studied. From Tables 4.1, 4.2, 4.4, and 4.5, it is noted that the results are highly accurate with a small perturbation of errors. In Tables 4.3, 4.6, and 4.7, the obtained results with other ones obtained in [27] and [11] respectively are compared. It is remarked that proposed methods are better and more accurate than others in [27] and [11]. Besides, Figures 4.1 and 4.4 show the behavior of the lower bound solutions y_1 are increasing as α increases, and Figures 4.2 and 4.5, show the behavior of upper bound solutions y_2 is decreasing as α increases. Consequently, from these behaviors, the solutions can be concluded that are fuzzy. At the end, Figures 4.3 and 4.6, show that the crisp solutions are bounded by y_1 and y_2 and they become close to the crisp solution as α approaches to one.

For the future work, the boundary value problems will be investigated using HBM1 by applied Simple shooting method. In addition, an application for this method will be investigated such as eigenvalue problems as fuzzy Sturm-Liouville problems. Moreover, the delay fuzzy initial value problems will be investigated.

References

[1] Garrido, A. (2012). A brief history of fuzzy logic. BRAIN. Broad Research in Artificial Intelligence and Neuroscience, 3(1), 71-77.

[2] Buckley, J. J., & Eslami, E. (2002). An introduction to fuzzy logic and fuzzy sets. Springer Science & Business Media, (13).

[3] Raggi, M., Colosi, D., Flores, M., Anaya-Muñoz, V. H., García-Velázquez, L. M., & Torres-Miranda, A. (2019). An application of fuzzy logic to build ecological sympatry networks. Ecological Informatics, 53, 100978. https://doi.org/10.1016/j.ecoinf.2019.100978.

[4] Casasnovas, J., & Rossell, F. (2005). Averaging fuzzy biopolymers. Fuzzy Sets and Systems, 152(1) ,139-158. https://doi.org/10.1016/j.fss.2004.10.019.

[5] Guo, M., Xue, X., & Li, R. (2003). Impulsive functional differential inclusions and fuzzy population models. Fuzzy Sets and Systems, 138(3), 601-615. https://doi.org/10.1016/S0165-0114(02)00522-5.

[6] El Naschie, M. S. (2005). From experimental quantum optics to quantum gravity via a fuzzy Khler manifold. Chaos. Solitons and Fractals, 25(5), 969-977. https://doi.org/10.1016/j.chaos.2005.02.028.

[7] Bede, B. (2013). Fuzzy differential equations. Mathematics of Fuzzy Sets and Fuzzy Logic, 296, 171-191. https://doi.org/10.1007/978-3-642-35221-8_9.

[8] Ghanbari, M. (2012). Solution of the first order linear fuzzy differential equations by some reliable methods. Journal of fuzzy set valued analysis, 2012, 1-20. https://doi.org/10.5899/2012/jfsva-00126.

[9] Akın, Ö., Khaniyev, T., Oruç, Ö., & Türkşen, I. (2013). An algorithm for the solution of second order fuzzy initial value problems. Expert Systems with Applications, 40(3), 953-957. https://doi.org/10.1016/j.eswa.2012.05.052.

[10] Jameel, A. F., Ghoreishi, M., & Ismail, A. I. (2014). Approximate solution of high order fuzzy initial value problems. Journal of Uncertain Systems, 8(2), 149-160.

[11] Jameel, A. F., Shather, A. H., Anakira, N., Alomari, A. K., & Saaban, A. (2020). Comparison for the approximate solution of the second-order fuzzy nonlinear differential equation with fuzzy initial conditions. Mathematics and Statistics, 8(5), 527-534. https://doi.org/10.13189/ms.2020.080505.

[12] Jameel, A. F., Anakira, N. R., Shather, A. H., Saaban, A., & Alomari, A. K. (2020). Numerical algorithm for solving second order nonlinear fuzzy initial value problems. International Journal of Electrical and Computer Engineering (IJECE), 10(6), 6497-6506. https://doi.org/10.11591/ijece.v10i6.pp6497-6506.

[13] Rabiei, F., Ismail, F., Ahmadian, A., & Salahshour, S. (2013). Numerical solution of second-order fuzzy differential equation using improved runge-kutta Nystrom method. Mathematical Problems in Engineering, 2013, 1-10. https://doi.org/10.1155/2013/803462.

[14] Sindu Devi1, S., & Ganesan, K. (2016). Numerical solution of first order fuzzy differential equations by Simpson's Rule. British Journal of Mathematics & Computer Science, 18(1),1-13. https://doi.org/10.9734/BJMCS/2016/26060.

[15] Radhy, Z., Maghool, F., & Abed, A. (2017). Numerical solution of fuzzy differential equation (FDE). International Journal of Mathematics Trends and Technology (IJMTT), 52(9), 596-602.

[16] Alkasassbeh, M., & Omar, Z. (2017). Implicit one-step block hybrid third-derivative method for the direct solution of initial value problems of second-order ordinary differential equations. Journal of Applied Mathematics, 2017,1-8. https://doi.org/10.1155/2017/8510948.

[17] ElJaoui, E., Melliani, S., & Chadli, L. S. (2015). Solving second-order fuzzy differential equations by the fuzzy Laplace transform method. Advances in Difference Equations, 2015(1). https://doi.org/10.1186/s13662-015-0414-x.

[18] Mondal, S. P., Vishwakarma, D. K., & Saha, A. K. (2016). Solution of second order linear fuzzy difference equation by Lagrange's multiplier method. Journal of Soft Computing and Applications, 2016(1), 11-27. https://doi.org/10.5899/2016/jsca-00063.

[19] Mansouri, S. S., & Ahmady, N. (2012). A numerical method for solving nth-order fuzzy differential equation by using characterization theorem. Communications in Numerical Analysis, 2012, 1-12. https://doi.org/10.5899/2012/cna-00054.

[20] Lee, K. H. (2005). First course on fuzzy theory and applications. Springer Science & Business Media, 27. https://doi.org/10.1007/3-540-32366-X.

[21] Pirzada, U. M., & Vakaskar, D. C. (2017). Existence of Hukuhara differentiability of fuzzy-valued functions. Journal of the Indian Mathematical Society, 84(3-4), 239-254. arXiv preprint arXiv:1609.04748.

[22] Syam, M. I., & Al-Refai, M. (2020). A reliable method for first order delay equations based on the implicit hybrid method. Alexandria Engineering Journal, 59(4), 2677-2681. https://doi.org/10.1016/j.aej.2020.04.043.

[23] Fatunla, S. O. (1995). A class of block method for second order initial value problem. International journal of computer mathematics, 55:1-2, 119-133. https://doi.org /10.1080/00207169508804368.

[24] Syam, M. I., Alsuwaidi, A., Alneyadi, A., Al Refai, S., & Al Khaldi, S. (2018). An implicit hybrid method for solving fractional Bagley-Torvik boundary value problem. Mathematics, 6(7), 109. https://doi.org/10.3390/math6070109.

[25] Van Niekerk, F. D. (1987). Non-linear one-step methods for initial value problems. Computers & Mathematics with Applications, 13(4), 367-371. https://doi.org/10.1016/0898-1221(87)90004-6.

[26] Adee, S. O., Onumanyi, P, Sirisena, U. W, & Yahaya, Y. A. (2005). Note on starting the Numerov method more accurately by a hybrid formula of order four for initial-

value problem. Journal of Computational and Applied Mathematics, 175(2), 369-373. https://doi.org/10.1016/j.cam.2004.06.016.

[27] Jameel, A. F., & Ismail, A. I. M. (2015). Approximate solution of first order nonlinear fuzzy initial value problem with two different fuzzifications. Journal of Uncertain Systems, 9(3), 221-229.