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Hierarchical Vector Bases for Pyramid Cells

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***Abstract*—This presentation summarizes a very simple and straightforward new procedure to build hierarchical vector-bases for the pyramid that conform to those used on adjacent differently shaped cells (tetrahedra, hexahedron and triangular prisms). Our new curl- and divergence-conforming bases, together with the corresponding curls and divergences, have simple and easy to implement mathematical expressions. Results confirming faster convergence and avoidance of spurious modes/solutions will be reported at the conference.**

***Index Terms*—Electromagnetic fields, finite-element methods, method of moments, higher order vector elements, pyramidal elements, numerical analysis**

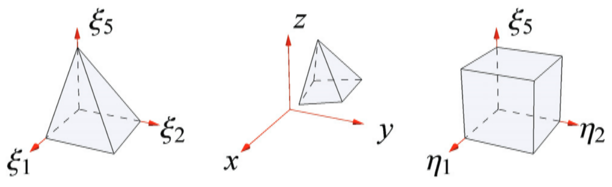

Fig. 1. The shape functions map the parent pyramid on the left to the child pyramid in the center. In the grandparent space $(\boldsymbol{\eta}, \xi_5)$ the shape functions and the basis functions take polynomial form, while the pyramid is described by the cubic cell shown on the right. Figure taken from [1].

I. Introduction

Successful three-dimensional (3D) electromagnetic codes must be able to model complicated geometries using higher-order vector basis functions on all four types of geometrical shapes: tetrahedra, hexahedron, prisms, and pyramids. No commercial code is able to do this because the scientific literature on pyramid bases has favored mathematical and theoretical aspects from whose results it is difficult to extract ready-to-use recipes for computational applications; see the discussion in the Introduction of [1] as well as [2]- [10] and references therein.

To overcome this difficulty, we have recently proposed in [1] a new paradigm for deriving hierarchical vector bases for quadrilateral-based pyramidal cells which prescribes to map the pyramid into a *grandparent* cube, like the one in Fig. 1 on the right, and to directly enforce the conformity of the vector bases with those discussed in [11] for tetrahedra, hexahedron and triangular prisms, which are differently shaped cells which, when attached to a pyramid, share an edge or an entire face with it.

This paradigm was used in [1] to derive arbitrarily high order curl-conforming bases for pyramids, demonstrating in the same paper the completeness of the obtained bases. However, as noted in [12], divergence-conforming bases for the pyramid can be derived using the same paradigm used in [1]. In both cases (*i.e.*, in the curl- and divergence-conforming case), the basis functions of the lowest possible polynomial order are the *historical* ones given in [4]. The lowest order bases (*i.e.*, the zero-order bases) have no volume-based basis functions and the number of basis functions is equal to the number of edges (8) in the curl-conforming case, and to the number of faces (5) in the divergence-conforming case. For higher orders, the curl-conforming volume-based functions have zero tangential component on all faces; instead, in the divergence-conforming case, the volume-based functions have zero normal component on all faces.

In the observer's space, also called child-space, a pyramid is described by suitable shape functions of the five *parent* variables $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$; see [1], [4] and Fig. 1. The difficulties in defining the vector bases of the pyramid are due to the fact that the shape functions and the vector basis functions take fractional (non-polynomial) form in the parent space. Therefore, for a pyramid

- It is difficult to define what is meant by *base order*,
- It is difficult to calculate the number of volume-based basis functions necessary to complete the base up to to a given order,
- It is difficult to find a simple technique to derive the volume-based basis functions (that is, the so-called bubbles).

As with the other hierarchical volumetric elements in [11], our new pyramid bases have four distinctive features:

(a) the vector basis functions are subdivided from the outset into different groups of volume, face and edge-based functions (the latter only exist for curl-conforming bases);
(b) each edge, face or volume-based basis function is obtained by using one generating scalar polynomial (in turn based on the given edge, face or volume of the cell) whose analytical expression involves all the dependent variables that describe the cell;
(c) the generating polynomials are either symmetric or anti-symmetric with respect to the variables that describe each edge and face of the cell, and are organized hierarchically;
(d) in each group, the generating polynomials are mutually orthogonal for inner product defined by the integral on the volume, the face or the edge of the cell.

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II. The Grandparent Space Is Where the Basis Vector-Functions Take on Polynomial Form

In [1], in addition to the parent coordinates $\boldsymbol{\xi} \equiv \{\xi_1,\ \xi_2,\ \xi_3,\ \xi_4,\ \xi_5\}$, we have defined and used four *scaled* coordinates

$$\eta_j \quad = \quad \frac{\xi_j}{1-\xi_5} \tag{1}$$

for $j = 1,\,2,\,3,\,4$, with dependence relations (for $\xi_5 \neq 1$)

$$\eta_1 + \eta_3 = 1; \quad \eta_2 + \eta_4 = 1. \tag{2}$$

and with

$$\boldsymbol{\nabla}\eta_j \quad = \quad \frac{\boldsymbol{\nabla}\xi_j + \eta_j \boldsymbol{\nabla}\xi_5}{1-\xi_5} \tag{3}$$

These coordinates transform surface integrals on the triangular face $\xi_j = 0$ of the parent pyramid (i.e. a triangular simplex) into integrals on a unit square, while volume integrals on the parent pyramid become integrals on the unit cube of the $(\boldsymbol{\eta}, \xi_5)$ grandparent space shown in Fig. 1 (see [1]).

However, as usual, the curl- and divergence-conforming vector basis functions are convenientely written in terms of the gradient vectors $\boldsymbol{\nabla}\xi_j$ (with $j = 1, 2, \ldots, 5$) or the unitary basis vectors $\left\{\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^5\right\}$, respectively [11].

Now, in the grandparent space $(\boldsymbol{\eta}, \xi_5)$, the shape functions take on polynomial form. Moreover, in the grandparent space $(\boldsymbol{\eta}, \xi_5)$, also the bases of the lowest possible order given in [4] take on polynomial form while they match the zero-order functions of adjacent, differently shaped elements on the boundary of the pyramid. It is therefore logical to state that these bases which do not contain any volume-based functions are of order zero [1], [11]. Furthermore, it is also reasonable to insist that all remaining higher-order basis functions take polynomial form in the grandparent space, if only for uniformity and energy considerations. In this regard, it should be noted that not only do we want polynomial basis vectors, but we also want that the curls of the curl-conforming functions and the divergences of the divergence-conforming functions take polynomial form.

Thus, to derive the vector basis functions, and in particular the volume-based vector functions, we use the following fundamental results

- In the grandparent space, the gradient of any linear combination of terms such as

  $$\eta_1^{\alpha}\, \eta_2^{\beta}\, \xi_5^{z} (1-\xi_5)$$

  takes a polynomial form;
- In the grandparent space, the curl of any linear combination of terms such as

  $$\eta_1^{\alpha}\, \eta_2^{\beta}\, \xi_5^{z} (1-\xi_5) \boldsymbol{\nabla}\xi_a$$

  takes a polynomial form; it is understood that the subscript a in $\boldsymbol{\nabla}\xi_a$ is 1, 2, or 5;
- In the grandparent space, the divergence of any linear combination of terms such as

  $$\eta_1^{\alpha}\, \eta_2^{\beta}\, \xi_5^{z} (1-\xi_5) \boldsymbol{\ell}^a$$

  takes a polynomial form; it is understood that the superscript a in $\boldsymbol{\ell}^a$ is 1, 2, or 5.

Of course, each term of these linear combinations can have different values of the exponents α, β, and z. The role that the factor $(1 - \xi_5)$ plays in obtaining the previous results it is also quite evident. Some of the zero-order vector functions contain this factor while others do not, and this must be kept in mind when constructing the generating scalar polynomials that multiply the zeroth order basis functions to build the higher order ones.

The paradigm introduced in [1] requires to derive polynomial edge-based and face-based basis functions that guarantee tangential or normal continuity on the boundaries of the pyramid, and this turns out to be not excessively complicated since we know the basis functions of all the other 3D elements other than the pyramid. On the contrary, what is complicated and in our opinion is the main problem that researchers have had so far in building the pyramid's bases is finding a simple and straightforward way to build the volume-based polynomial vector functions (at least, those of the lowest order), to then demonstrate the completeness of the entire family in the space described by the grandparent variables. Results confirming faster convergence and avoidance of spurious modes/solutions will be reported at the conference.

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