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PERTURBED NONLINEAR ELLIPTIC NEUMANN PROBLEMS INVOLVING ANISOTROPIC SOBOLEV SPACES WITH VARIABLE EXPONENTS

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In this paper we study the existence of infinitely many weak solutions of the following perturbed Kirchhoff-type non-homogeneous Neumann problem

$$\begin{cases} -\sum_{i=1}^{N} M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) \\ +M_0\left(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)}\right)|u|^{p_0(x)-2}u = f(x,u)+g(x,u) \text{ in } \Omega, \\ \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\nu_i = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

by applying technical approach based on critical points theorem due to B. Ricceri in a reflexive anisotropic Sobolev spaces. We use some suitable assumptions on the right had side but without using log-Hölder continuous condition.

1. Introduction

In recent years, the anisotropic variable exponent Sobolev spaces have attracted the attention of many mathematicians, physicists and engineers. The impulse

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for this mainly come from their important applications in modelling real-world problems in electrorheological, magnetorheological fluids, elastic materials and image restoration, (see for example [9], [15], [16], [35], [38], [39]).

More recently, several authors (see e.g. [2], [10], [25]) have studied the anisotrop-ic quasi-linear elliptic equations with variable exponents, i.e. the quasi-linear elliptic equations involving the following $\vec{p}(\cdot)$-Laplacian

$$\Delta_{\vec{p}(\cdot)} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right). \tag{1}$$

It's clear that this $\vec{p}(\cdot)$-Laplace operator is a generalization of the $p(\cdot)$-Laplace operator

$$\Delta_{p(\cdot)} u = \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right). \tag{2}$$

We refer to [1], [37], [40] for the study of the $p(\cdot)$-Laplacian equations and the corresponding variational problems.

The $p(\cdot)$-Laplacian is a meaningful generalization of the p-Laplacian operator

$$\Delta_p u = \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right), \tag{3}$$

obtained in the case when p is a positive constant.

On the one hand, Ricceri [33], Anello and Cordaro [8] studied the existence of solutions for the following problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = b(x) f(u) + c(x) g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $a(x)$ is a positive function such that $a(\cdot) \in L^{\infty}(\Omega)$ with $a^- = \operatorname{ess\,inf}_{x \in \Omega} a(x) > 0$ and $p > N$. The existence of solutions of problem (4) was proved by applying Ricceri's variational principle (see [32]).

In [21], X. Fan, C. Ji treated the problem

$$\begin{cases} -\Delta_{p(\cdot)} u + a(x)|u|^{p(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

and they proved the existence of infinitely many solutions in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.

However, there are some non-homogeneous materials that have different behaviors in different space directions, hence the need for anisotropic spaces with variable exponent. Ahmed, Hjiaj, and Touzani have studied in [3] the Neumann $\vec{p}(\cdot)$-elliptic problem :

$$\begin{cases} -\Delta_{\vec{p}(\cdot)}u + a(x)|u|^{p_0(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

and they proved the existence of infinitely many weak solutions in the anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ under some hypotheses. For other related results, we refer to [5], [6], [14], [19].

On the other hand, much interest has been focused on the study of Kirchhoff type problems. More precisely, Kirchhoff studied the following model problem (see [26]) as an extension of d'Alembert's classical wave equation by considering changes in string length during vibrations

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2}, \tag{7}$$

where L is the length of the chord, h is the area of the cross section, E is the Young's modulus of the material, is the density and P_0 is the initial tension. A distinguishing feature of the Kirchhoff equation (7) is that the equation contains a non-local coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2$ of the kinetic energy $\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2$ on $[0,L]$, and hence the equation is no longer a point-wise identity. See also [11], [17], [23], [34], [36] for related topics.

The purpose of our paper is to investigate a class of Kirchhoff type problems involving operators in divergence form as follows:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} M_i \left(\int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx \right) \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right) \\ +M_0 \left(\displaystyle\int_\Omega \frac{1}{p_0(x)} |u|^{p_0(x)} \right) |u|^{p_0(x)-2}u = f(x,u) + g(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \nu_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where M_i, f and g define and satisfies some conditions detailed in Section 3.

In the Dirichlet case, A. Ourraoui in [29] have studied the problem (8) by assuming an Ambrosetti-Rabinowitz type condition and using techniques related to a Mountain pass theorem in the case where $g \equiv 0$. M. Avci, R. A. Mashiyev and B. Cekic in [12] have studied the same problem in the case where $g \equiv 0$,

$M_0 = M_2 = \cdots = M_N = M$ and the assumption $2 \leq p_i(x) \leq N$. Note that the hypotheses we adopt are totally different from the ones assumed in the papers just quoted.

It is no a surprise that the presence of Neumann conditions in an anisotropic non-homogeneous perturbed-Kirchhoff type problem make difficulties in the application of the Theorem 2.3 which is our main tool. To overpass these difficulties, we combine the classical techniques with the recent techniques that appeared when treating anisotropic problems with variable exponents.

This paper is organized as follows. In Section 2, we recall some basic facts about anisotropic variable exponent Sobolev spaces as well as Ricceri's variational principle. In Section 3, we state and prove our main results (Theorems 3.5 and 3.6), providing also some remarks and examples.

2. Preliminary results

Let Ω be a bounded domain in $\mathbb{R}^N$, we define:

$\mathcal{C}_+(\overline{\Omega}) = \{\text{measurable function}, p(\cdot) : \overline{\Omega} \longrightarrow \mathbb{R} \quad \text{such that} \quad 1 < p^- \leq p^+ < \infty\}$

where

$$p^- = \operatorname{ess\,inf}\{p(x) : x \in \overline{\Omega}\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x) : x \in \overline{\Omega}\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \longmapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx,$$

is finite, then

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\},$$

defines a norm in $L^{p(\cdot)}(\Omega)$ called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(\cdot)} + \dfrac{1}{p'(\cdot)} = 1$. Finally, we have the following Hölder type inequality

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \Big(\frac{1}{p^-} + \frac{1}{(p^-)'}\Big) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \tag{9}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We give the following result.

Proposition 2.1. *(See [18], [24]). If $u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

(i) $\|u\|_{p(\cdot)} < 1 \quad (respectively, = 1, > 1) \quad \Leftrightarrow \quad \rho_{p(\cdot)}(u) < 1 \quad (respectively, = 1, > 1)$,

(ii) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$,

(iii) $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^-}$.

Now, we define the Sobolev space with variable exponent by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We refer to [18] for the elementary properties of these spaces.
Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of our main problem.

Let $p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot)$ be $N+1$ variable exponents in $\mathcal{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(\cdot) = \Big\{p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot)\Big\}, D^0 u = u \text{ and } D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N,$$

and for all $x \in \overline{\Omega}$ we put

$$p_M(\cdot) = \max\Big\{p_0(\cdot), p_1(\cdot), \dots, p_N(\cdot)\Big\}$$

$$p_m(\cdot) = \min\Big\{p_0(\cdot), p_1(\cdot), ..., p_N(\cdot)\Big\}.$$

We define

$$\underline{p} = \min\Big\{p_0^-, p_1^-, p_2^-, \dots, p_N^-\Big\} \quad \text{then} \quad \underline{p} > 1, \tag{10}$$

and

$$\overline{p} = \max\Big\{p_0^+, p_1^+, p_2^+, \dots, p_N^+\Big\}. \tag{11}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{p}(\cdot)}(\Omega) = \Big\{u \in L^{p_0(\cdot)}(\Omega) : D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for all } i = 1, 2, \dots, N\Big\},$$

endowed with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{1,\vec{p}(\cdot)} = \|u\|_{L^{p_0(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}, \tag{12}$$

we refer to [13], [30], [31] for the constant exponent case.

We emphasize that the space $\left(W^{1,\vec{p}(\cdot)}(\Omega), \|\cdot\|_{1,\vec{p}(\cdot)}\right)$ is a reflexive Banach space (see [20]).
A more complete theory of anisotropic variable exponent Sobolev spaces may be obtained in [4], [7], [22], [27], [28].

Throughout this paper we assume that

$$\underline{p} > N. \tag{13}$$

Remark 2.2. Since $W^{1,\vec{p}(\cdot)}(\Omega)$ is continuously embedded in $W^{1,\underline{p}}(\Omega)$ and $W^{1,\underline{p}}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), thus $W^{1,\vec{p}(\cdot)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

Set

$$C_0 = \sup_{u \in W^{1,\vec{p}(\cdot)}(\Omega)\setminus\{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{1,\vec{p}(\cdot)}}. \tag{14}$$

Then C_0 is a positive constant.

Let us recall the following theorem obtained by B. Ricceri in [21], which will be applied to establish the existence of weak solutions for our main problem.

Theorem 2.3. *(See [21], Theorem 2.2). Let E be a reflexive real Banach space, and let $\Phi, \Psi : E \longrightarrow \mathbb{R}$ be two sequentially weakly lower semi-continuous and Gâteaux differentiable functionals. Assume also that Ψ is (strongly) continuous and satisfies $\lim_{\|u\|_E \to \infty} \Psi(u) = +\infty$. For each $\rho > \inf_E \Psi$, put*

$$\varphi(\rho) = \inf_{u \in \Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\rho[))_w}} \Phi(v)}{\rho - \Psi(u)}, \tag{15}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))_w}$ is the closure of $\Psi^{-1}(]-\infty,\rho[)$ for the weak topology. Then, the following conclusions hold

(a) *If there exist $\rho_0 > \inf_E \Psi$ and $u_0 \in E$ such that*

$$\Psi(u_0) < \rho_0, \tag{16}$$

and

$$\Phi(u_0) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\rho_0[))_w}} \Phi(v) < \rho_0 - \Psi(u_0), \tag{17}$$

then the restriction of $\Psi + \Phi$ to $\Psi^{-1}(]-\infty,\rho_0[)$ has a global minimum.

(b) *If there exists a sequence $(r_n)_n \subset \left(\inf_E \Psi, +\infty\right)$ with $r_n \to \infty$ and a sequence $(u_n)_n \subset E$ such that for each n*

$$\Psi(u_n) < r_n, \tag{18}$$

and

$$\Phi(u_n) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty, r_n[))_w}} \Phi(v) < r_n - \Psi(u_n), \tag{19}$$

and in addition,

$$\liminf_{\|u\| \to +\infty} (\Psi(u) + \Phi(u)) = -\infty, \tag{20}$$

then, there exists a sequence $(v_n)_n$ of local minima of $\Psi + \Phi$ such that $\Psi(v_n) \to +\infty$ as $n \to \infty$.

(c) *If there exists a sequence $(r_n)_n \subset (\inf_E \Psi, +\infty)$ with $r_n \to \inf_E \Psi$ and a sequence $(u_n)_n \subset E$ such that for each n the condition (18) and (19) are satisfied, and in addition,*

$$\text{every global minimizer of } \Psi \text{ is not a local minimizer of } \Phi + \Psi, \tag{21}$$

then, there exists a sequence $(v_n)_n$ of pairwise distinct local minimizers of $\Phi + \Psi$ such that $\lim_{n \to \infty} \Psi(v_n) = \inf_E \Psi$, and $(v_n)_n$ weakly converges to a global minimizer of Ψ.

3. Basic assumptions and main results

Throughout the paper, we assume that the following assumptions hold true:

Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary of class C^1, and let ν_i be the components of the outer normal unit vector on the boundary $\partial\Omega$.

Assume that $f, g : \Omega \times \mathbb{R} \longmapsto \mathbb{R}$ are Carathéodory functions satisfying,

$$\sup_{|t| \le r} | f(x,t) | \in L^1(\Omega), \qquad \text{and} \qquad \sup_{|t| \le r} | g(x,t) | \in L^1(\Omega) \qquad \text{for each} \quad r > 0. \tag{22}$$

We set

$$F(x,t) = \int_0^t f(x,s)ds, \quad \text{and} \quad G(x,t) = \int_0^t g(x,s)ds. \tag{23}$$

Assume the following assumptions:

(M1) $M_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous functions satisfy the condition

$$M_i(t) \ge m \text{ for all } t \ge 0,$$

where m is a positive constant.

(M2) There are constants $s^* > 1$ and $\lambda_1, \lambda_2 > 0$ such that

$$\widehat{M}_0(t) \leq \lambda_1 t^{s^*} + \lambda_2 \text{ for all } t \geq 0,$$

where $\widehat{M}_i(t) = \int_0^t M_i(s)ds.$

We define, for any $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, the functionals

$$J(u) = \sum_{i=1}^{N} \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) + \widehat{M_0}\left(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)} dx\right), \qquad (24)$$

$$\Psi(u) = J(u) - \int_\Omega G(x,u)dx \qquad and \qquad \Phi(u) = -\int_\Omega F(x,u)dx. \qquad (25)$$

We assume that G satisfies one of the following two conditions:

(G1) There are positive functions $\vartheta_0(\cdot), \vartheta_1(\cdot) \in L^1(\Omega)$ with $\vartheta_0 \neq 0$ such that

$$|G(x,t)| \leq \frac{m\vartheta_0(x)}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}\|\vartheta_0\|_{L^1(\Omega)}}|t|^{\underline{p}} + \vartheta_1(x)$$

almost everywhere $x \in \Omega$, and for any $t \geq 0$.

(G2) There are a constants $R > 0$, $\varepsilon \in (0,1)$ and $C_1 > 0$ such that

$$\text{for any } |t| \geq R, \quad \int_\Omega |G(x,t)|dx \leq (1-\varepsilon)\widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|t|^{p_0(x)} dx\right) + C_1,$$

almost everywhere x in Ω.

Remark 3.1. Assume the hypothesis **(M2)** and one of the assumptions **(G1)** and **(G2)**. Then there are positives constants d_1 and d_2 such that

$$\widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|\xi|^{p_0(x)} dx\right) - \int_\Omega G(x,\xi)dx \leq d_1|\xi|^{\overline{p}s^*} + d_2, \text{ for all } \xi \in \mathbb{R}. \quad (26)$$

Indeed, on the one hand, assumptions **(M2)** and **(G1)** implies

$$\begin{aligned}
&\widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|\xi|^{p_0(x)} dx\right) - \int_\Omega G(x,\xi)dx \\
&\leq \lambda_1\left(\int_\Omega \frac{1}{p_0(x)}|\xi|^{p_0(x)} dx\right)^{s^*} + \lambda_2 + \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}|\xi|^{\underline{p}} + \|\vartheta_1\|_{L^1(\Omega)} \\
&\leq \frac{\lambda_1}{\underline{p}^{s^*}}|\xi|^{\overline{p}s^*} + \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}|\xi|^{\underline{p}} + \|\vartheta_1\|_{L^1(\Omega)} + \lambda_2, \text{ for } \xi \text{ large enough .}
\end{aligned}$$

Then, we have (26) for $d_1 = \max\left(\frac{\lambda_1}{\underline{p}^{s^*}}, \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}\right)$ and $d_2 = \|\vartheta_1\|_{L^1(\Omega)} + \lambda_2$.

On the other hand, assumptions **(M2)** and **(G2)** implies

$$
\begin{aligned}
&\widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|\xi|^{p_0(x)}dx\right) - \int_\Omega G(x,\xi)dx \\
&\leq \widehat{M_0}\Big(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)}dx\Big) + (1-\varepsilon)\widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|t|^{p_0(x)}dx\right) + C_1 \\
&\leq (2-\varepsilon)\frac{\lambda_1}{\underline{p}^{s^*}}|\xi|^{\overline{p}s^*} + C_1 + \lambda_2(2-\varepsilon) \text{ for } \xi \text{ large enough .}
\end{aligned}
$$

Which ends the proof of (26) with $d_1 = (2-\varepsilon)\frac{\lambda_1}{\underline{p}^{s^*}}$ and $d_2 = C_1 + \lambda_2(2-\varepsilon)$.

Let us prove that the functional Ψ is coercive.

Lemma 3.2. *Assume that* **(M1)** *and* **(M2)** *hold and one of the condition* **(G1)** *or* **(G2)** *is satisfied. Then the functional* Ψ *is coercive.*

Proof. Assuming that the condition **(G1)** is satisfied, then for $\|u\|_{1,\vec{p}(\cdot)} \geq 1$, we have

$$
\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x,u)dx \\
&= \sum_{i=1}^N \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right) + \widehat{M_0}\left(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)}dx\right) \\
&\quad - \int_\Omega G(x,u)dx \\
&\geq \sum_{i=1}^N \frac{m}{p_i^+}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(\cdot)}^{\underline{p}} - 1\right) + \frac{m}{p_0^+}\left(\|u\|_{p_0(x)}^{\underline{p}} - 1\right) \\
&\quad - \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}\int_\Omega \frac{\vartheta_0(x)}{\|\vartheta_0\|_{L^1(\Omega)}} \mid u\mid^{\underline{p}} dx - \int_\Omega \vartheta_1(x)dx, \text{ (by } \textbf{(M1)}) \\
&\geq \frac{m}{\overline{p}(N+1)^{\underline{p}-1}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}\|u\|_{L^\infty(\Omega)}^{\underline{p}} \\
&\quad - \|\vartheta_1(\cdot)\|_{L^1(\Omega)} - N - 1 \\
&\geq \frac{m}{\overline{p}(N+1)^{\underline{p}-1}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - C_2 \qquad \text{(by (14))} \\
&\geq \frac{m}{2\overline{p}(N+1)^{\underline{p}-1}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - C_2. \qquad (27)
\end{aligned}
$$

Under the condition **(G2)** we have for $\|u\|_{1,\vec{p}(\cdot)} \geq 1$

$$\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x,u)dx \\
&= \sum_{i=1}^N \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right) + \widehat{M_0}\left(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)}dx\right) \\
&- (1-\varepsilon)\widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|t|^{p_0(x)}dx\right) - C_1 \\
&\geq \sum_{i=1}^N \frac{m}{p_i^+}\left(\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(\cdot)}^{\underline{p}} - 1\right) + \frac{m\varepsilon}{p_0^+}\left(\|u\|_{p_0(x)}^{\underline{p}} - 1\right) - C_1 \qquad (\text{by } \mathbf{(M1)}) \\
&\geq \frac{\min\{1,\varepsilon\}m}{\overline{p}(N+1)^{\underline{p}-1}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - N - m\varepsilon - C_1 \\
&\geq \frac{\min\{1,\varepsilon\}m}{\overline{p}(N+1)^{\underline{p}-1}}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - C_3. \qquad (28)
\end{aligned}$$

Thanks to (27)–(28) we conclude that Ψ is coercive. Moreover, there exist two positive constants α and β such that

$$\Psi(u) \geq \alpha\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} \quad \text{for all } \|u\|_{1,\vec{p}(\cdot)} \geq \beta. \qquad (29)$$

□

Definition 3.3. A measurable function $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is called a weak solution of the Neumann elliptic problem (8) if

$$\begin{aligned}
&\sum_{i=1}^N M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right)\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx \qquad (30) \\
+ \quad & M_0\left(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)}dx\right)\int_\Omega |u|^{p_0(x)-2}uvdx = \int_\Omega f(x,u)v(x)dx + \int_\Omega g(x,u)v(x)dx
\end{aligned}$$

for all $v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Definition 3.4. A function $F(x,t)$ satisfies the condition **(S)** if for each compact subset E of $\mathbb{R}$, there exists $\xi \in E$ such that

(S)

$$F(x,\xi) = \sup_{t\in E} F(x,t) \ \text{ for a.e. } x \in \Omega.$$

Now we are in the position to state our first main result.

Theorem 3.5. *Let the conditions* **(M1)**-**(M2)** *be satisfied, and let* **(G1)** *or* **(G2)** *hold. Moreover, let F satisfy the condition* **(S)**. *Moreover, we suppose that*

$$\liminf_{|\xi|\to+\infty} \widehat{M}_0\left(\int_\Omega\left(\frac{1}{p_0(x)}|\xi|^{p_0(x)}\right)dx\right) - \int_\Omega\Big(G(x,\xi)+F(x,\xi)\Big)dx = -\infty, \quad (31)$$

and there are positives sequences $(a_n)_n$ and $(b_n)_n$ such that

$$\lim_{n\to\infty} b_n = +\infty, \ \lim_{n\to\infty}\frac{a_n^{\overline{p}}}{b_n^{\underline{p}}} = 0. \quad (32)$$

Finally, we assume that there exist a positive function
$h(\cdot)\in L^1(\Omega)$ with $\|h(\cdot)\|_{L^1(\Omega)} = 1$ and suitable positives constants d_1 and d_2 such that for each n we have for almost every x in Ω

$$F(x,a_n) + h(x)\left(\alpha\left(\frac{b_n}{C_0}\right)^{\underline{p}s^*} - d_1 a_n^{\overline{p}s^*} - d_2\right) \geq \sup_{t\in[a_n,b_n]} F(x,t), \quad (33)$$

$$F(x,-a_n) + h(x)\left(\alpha\left(\frac{b_n}{C_0}\right)^{\underline{p}s^*} - d_1 a_n^{\overline{p}s^*} - d_2\right) \geq \sup_{t\in[-b_n,-a_n]} F(x,t), \quad (34)$$

where α is constant of coercivity defined in (29), $d_1 = \max\left(\frac{\lambda_1}{\underline{p}^{s^*}}, \frac{m}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}\right)$ and $d_2 = \|\vartheta_1\|_{L^1(\Omega)} + \lambda_2$ if we assume **(G1)** *and $d_1 = (2-\varepsilon)\frac{\lambda_1}{\underline{p}^{s^*}}$ and $d_2 = C_1 + \lambda_2(2-\varepsilon)$ if we assume* **(G2)**. *The last inequalities (33) and (34) are strict on a subset of Ω with positive measure.*
Then there is a sequence $(v_n)_n$ of local minima of $\Psi+\Phi$ such that $\lim_{n\to\infty}\Psi(v_n) = +\infty$. Consequently, the problem (8) admits an unbounded sequence of weak solutions.

Proof. Let Ψ, Φ be the functionals defined in (23)-(25).

Since $\underline{p} > N$, the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is continuous and compact. We can see that $J,\Phi,\Psi \in \mathcal{C}^1(W^{1,\vec{p}(\cdot)}(\Omega),\mathbb{R})$ (see [12], [29]) with the derivative given by

$$\begin{aligned}\langle\Psi'(u),v\rangle &= \sum_{i=1}^N M_i\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right)\int_\Omega\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx \\ &+ M_0\Big(\int_\Omega\frac{1}{p_0(x)}|u|^{p_0(x)}dx\Big)\int_\Omega|u|^{p_0(x)-2}uvdx - \int_\Omega g(x,u)v(x)dx,\end{aligned} \quad (35)$$

and

$$\langle\Phi'(u),v\rangle = -\int_\Omega f(x,u)v(x)dx, \quad \text{for all } (u,v)\in W^{1,\vec{p}(\cdot)}(\Omega). \quad (36)$$

Then $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution of (8) if and only if u is a critical point of $\Psi + \Phi$.

Let us prove that the functionals Ψ and Φ are sequentially weakly lower semi-continuous.

For $i = 0, \dots, N$ and any $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ define $J_i, H : W^{1,\vec{p}(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ by

$$J_i = \int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx, \quad \text{where} \quad \frac{\partial u}{\partial x_0} = u,$$

$$H(u) = -\int_\Omega G(x,u)dx.$$

Let $(u_n)_n$ be a sequence such that $u_n \rightharpoonup u$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Since J_i is convex, for any n we have

$$J_i(u) \le J_i(u_n) + \langle J_i'(u), u - u_n \rangle.$$

Passing to the limit in the above inequality with $n \to \infty$, we see that J_i is sequentially weakly lower semi-continuous. Then we have

$$\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx \le \liminf_{n \to +\infty} \int_\Omega \sum_{i=0}^N \frac{1}{p_i(x)} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} dx. \tag{37}$$

From (37) and since $\widehat{M_i}$ is continuous and monotone, we have

$$\begin{aligned}
\liminf_{n \to +\infty} J(u_n) &= \liminf_{n \to +\infty} \sum_{i=0}^N \widehat{M_i} \left(\int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} dx \right) \\
&\ge \sum_{i=0}^N \widehat{M_i} \left(\liminf_{n \to +\infty} \int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i(x)} dx \right) \\
&\ge \sum_{i=0}^N \widehat{M_i} \left(\int_\Omega \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx \right) \\
&\ge J(u),
\end{aligned} \tag{38}$$

namely, J is sequentially weakly lower semi-continuous.

On the other hand, by Remark 2.2 up to a sub-sequence we have $u_n \longrightarrow u$ in $C^0(\overline{\Omega})$. Hence,

$$u_n \longrightarrow u \text{ uniformelly in } \Omega,$$
$$k := \sup_{n \in \mathbb{N}} \|u_n\|_{L^\infty(\Omega)} < +\infty.$$

Therefore, $G(x, u_n(x)) \longrightarrow G(x, u(x))$ almost every x in Ω and $|G(x, u_n(x))| \le k \sup_{|s| \le k} |g(x,s)|$. Note that $\sup_{|s| \le k} |g(x,s)| \in L^1(\Omega)$ by (22). Thus, the dominated convergence theorem implies that $\lim_{n\to\infty} H(u_n) = H(u)$. So, the functional H is sequentially weakly continuous on $W^{1,\vec{p}(\cdot)}(\Omega)$, and hence, being $\Psi = J - H$, Ψ is sequentially weakly lower semi-continuous. In the same way (as in the case of the mapping H) we can show that Φ is sequentially weakly continuous.

For $\rho > \inf_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$, we define

$$\mathbf{K}(\rho) = \inf\Big\{\tau > 0 \text{ suh that } \Phi^{-1}(]-\infty,\rho[) \subset \overline{B(0,\tau)}\Big\}, \tag{39}$$

where $B(0,\tau) = \Big\{u \in W^{1,\vec{p}(\cdot)}(\Omega) : \|u\|_{1,\vec{p}(\cdot)} < \tau\Big\}$ and $\overline{B(0,\tau)}$ denotes the closure of $B(0,\tau)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ for the norm topology.

Owing to the fact that Ψ is coercive, we have $0 < \mathbf{K}(\rho) < +\infty$ for each $\rho > \inf\limits_{u\in W^{1,\vec{p}(\cdot)}(\Omega)} \Psi(u)$. In view of (29), we deduce that

$$\text{if } \Psi(u) < \alpha\|u\|^{\underline{p}}_{1,\vec{p}(\cdot)}, \text{ then } \|u\|_{1,\vec{p}(\cdot)} < \beta.$$

Thanks to (39), one has $\Psi^{-1}(]-\infty,\rho[) \subset \overline{B(0,\mathbf{K}(\rho))}$ and so $\overline{(\Psi^{-1}(]-\infty,\rho[))_w} \subset \overline{B(0,\mathbf{K}(\rho))}$.
Using (14), we get $\|u\|_{L^\infty(\Omega)} \le C_0\|u\|_{1,\vec{p}(\cdot)}$. Then,

$$\overline{B(0,\mathbf{K}(\rho))} \subset \big\{u \in C(\overline{\Omega}) : \|u\|_{L^\infty(\Omega)} \le C_0\mathbf{K}(\rho)\big\},$$

which yields

$$\inf_{v\in\overline{(\Psi^{-1}(]-\infty,\rho[))_w}} \Phi(v) \ge \inf_{\|v\|_{1,\vec{p}(\cdot)}\le\mathbf{K}(\rho)} \Phi(v) \ge \inf_{\|v\|_{L^\infty(\Omega)}\le C_0\mathbf{K}(\rho)} \Phi(v). \tag{40}$$

Let $\tau \ge \alpha\beta^{\underline{ps^*}}$ and $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ be such that $\Psi(u) < \tau$. When $\|u\|_{1,\vec{p}(\cdot)} \ge \beta$, by (29), one has

$$\tau > \Psi(u) \ge \alpha\|u\|^{\underline{p}}_{1,\vec{p}(\cdot)},$$

which implies that $\|u\|_{1,\vec{p}(\cdot)} \le \left(\frac{\tau}{\alpha}\right)^{\frac{1}{\underline{ps^*}}}$. When $\|u\|_{1,\vec{p}(\cdot)} < \beta$, it is clear that $\|u\|_{1,\vec{p}(\cdot)} \le \left(\frac{\tau}{\alpha}\right)^{\frac{1}{\underline{ps^*}}}$.

By the definition of $\mathbf{K}(\tau)$, we have

$$\mathbf{K}(\tau) \le \left(\frac{\tau}{\alpha}\right)^{\frac{1}{\underline{ps^*}}}. \tag{41}$$

Since $F(x,\cdot)$ satisfies condition **(S)**, for each n, there exists $\xi_n \in [-a_n, a_n]$ such that

$$F(x,\xi_n) = \sup_{t\in[-a_n,a_n]} F(x,t) \quad \text{a.e. in } \Omega. \tag{42}$$

Now, in order to satisfy (b) of Theorem 2.3, take as u_n the constant function whose value is ξ_n and $\rho_n = \alpha \left(\frac{b_n}{C_0}\right)^{\underline{p}s^*}$, then $\lim_{n\to\infty} \rho_n \to +\infty$, and thanks to (41) we obtain

$$\mathbf{K}(\rho_n) \leq \frac{b_n}{C_0} \qquad \text{then} \qquad C_0\mathbf{K}(\rho_n) \leq b_n. \tag{43}$$

By (26), one has

$$\begin{aligned} e_n &= \widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}|\xi_n|^{p_0(x)}dx\right) - \int_\Omega G(x,\xi_n)dx \\ &\leq d_1|\xi_n|^{\overline{p}s^*} + d_2 \leq d_1|a_n|^{\overline{p}s^*} + d_2. \end{aligned}$$

It follows from (32) that for n large enough,

$$d_1|a_n|^{\overline{p}s^*} + d_2 < \alpha\left(\frac{b_n}{C_0}\right)^{\underline{p}s^*} = \rho_n,$$

and consequently $e_n < \rho_n$, that is (18) holds. Without loss of generality, we may assume that (18) holds for all n.

From (33)-(34) and (42), we obtain

$$F(x,\xi_n) + h(x)(\rho_n - e_n) \geq \sup_{|t|\leq b_n} F(x,t) \ \text{ a.e. in } \ \Omega, \tag{44}$$

and the inequality (44) is strict on a subset of Ω with positive measure. Using (43) and (44), we obtain (19) and (20) follows directly from (31).

Therefore, all hypotheses of Theorem 2.3 (b) are satisfied, then the proof of the Theorem 3.5 is concluded. □

Our second result is the following theorem.

Theorem 3.6. *Assume that* **(M1)-(M2)** *hold. Suppose that*

$$G(x,t) \leq 0 \ \text{ for } \ t \in \mathbb{R} \ \text{ and } \ \text{a.e. } \ x \in \Omega, \tag{45}$$

and that there exist two positive constants δ and ε such that

$$-G(x,t) \leq \delta|t|^{\underline{p}} \ \text{ for } \ |t| \leq \varepsilon \ \text{ and } \ \text{a.e. } \ x \in \Omega, \tag{46}$$

Moreover, let the functional F satisfy the condition **(S)** *and*

$$\limsup_{|\xi|\to 0} \frac{\displaystyle\int_\Omega F(x,\xi)dx + \int_\Omega G(x,\xi)dx}{|\xi|^{\underline{p}}} > \widehat{M}_0\left(\int_\Omega \frac{1}{p_0(x)}dx\right). \tag{47}$$

Suppose that $(a_n)_n$ and $(b_n)_n$ be two positives sequences such that

$$\lim_{n\to\infty} b_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{a_n^{\underline{p}}}{b_n^{\overline{p}}} = 0, \tag{48}$$

and there exists a positive function $h \in L^1(\Omega)$ with $\|h\|_{L^1(\Omega)} = 1$ such that for each n and almost every x in Ω, we have

$$F(x,a_n) + h(x)\left(d_3\left(\frac{b_n}{C_0}\right)^{\overline{p}s^*} - d_4 a_n^{\underline{p}s^*} - \lambda_2\right) \geq \sup_{t\in[a_n,b_n]} F(x,t), \tag{49}$$

$$F(x,-a_n) + h(x)\left(d_3\left(\frac{b_n}{C_0}\right)^{\overline{p}s^*} - d_4 a_n^{\underline{p}s^*} - \lambda_2\right) \geq \sup_{t\in[-b_n,-a_n]} F(x,t), \tag{50}$$

with $d_3 = \dfrac{m}{\overline{p}(N+1)^{\overline{p}-1}}$ and $d_4 = \max\left(\frac{\lambda_1}{\underline{p}^{s^*}}, \delta|\Omega|\right)$, the inequalities (49) and (50) are strict on a subset of Ω with positive measure.

Then there exists a sequence $(v_n)_n$ of pairwise distinct local minima of $\Psi + \Phi$ such that $v_n \to 0$ in $W^{1,\vec{p}(\cdot)}(\Omega)$. Consequently, the problem (8) admits a sequence of non-zero weak solutions which strongly converges to 0 in $W^{1,\vec{p}(\cdot)}(\Omega)$.

Proof. Let us verify all the hypotheses of Theorem 2.3 point (c). Using (45), for $\|u\|_{1,\vec{p}(\cdot)} \leq 1$ we have

$$\begin{aligned}
\Psi(u) &= J(u) - \int_\Omega G(x,u)dx \\
&= \sum_{i=1}^{N} \widehat{M_i}\left(\int_\Omega \frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx\right) + \widehat{M_0}\left(\int_\Omega \frac{1}{p_0(x)}|u|^{p_0(x)} dx\right) \\
&\geq d_3 \|u\|_{1,\vec{p}(\cdot)}^{\overline{p}} - C_4,
\end{aligned}$$

with $d_3 = \frac{m}{\overline{p}(N+1)^{\overline{p}-1}}$. Then, Ψ is coercive, $\inf_{W^{1,p(\cdot)}(\Omega,w)} \Psi = \Psi(0) = 0$ and 0 is the unique global minimizer of Ψ. Thanks to (47) we have

$$\begin{aligned}
&\limsup_{|\xi|\to 0} \{\Psi(\xi) + \Phi(\xi)\} \\
&= \limsup_{|\xi|\to 0} \left\{ \widehat{M_0}\left(\int_\Omega \frac{|\xi|^{p_0(x)}}{p_0(x)} dx\right) - \int_\Omega G(x,\xi)dx - \int_\Omega F(x,\xi)\, dx \right\} \\
&\leq \limsup_{|\xi|\to 0} \left\{ \widehat{M_0}\left(\int_\Omega \frac{1}{p_0(x)} dx\right)|\xi|^{\underline{p}} - \int_\Omega G(x,\xi)dx - \int_\Omega F(x,\xi)\, dx \right\} < 0,
\end{aligned}$$

that is, 0 is not a local minimizer of $\Psi+\Phi$; so (21) is satisfied.

For $r>0$ sufficiently small, the condition $\Psi(u)<r$ implies that $\|u\|_{1,\vec{p}(\cdot)}<\left(\frac{r}{d_3}\right)^{\frac{1}{\overline{p}s^*}}$, this shows that

$$\mathbf{K}(r)\leq\left(\frac{r}{d_3}\right)^{\frac{1}{\overline{p}s^*}}.$$

Now put $\rho_n=d_3\left(\frac{b_n}{C_0}\right)^{\overline{p}s^*}$ and take u_0 and u_n as constants values functions ξ_0 and ξ_n respectively in Theorem 2.3. Then

$$C_0\mathbf{K}(\rho_n)\leq b_n. \tag{51}$$

By (46) and **(M2)**, there exists a sequence $(\xi_n)_n\subset\mathbb{R}$ with $\xi_n\in[-a_n,a_n]$ such that for each a_n sufficiently small,

$$\begin{aligned}
e_n &= \widehat{M}_0\left(\int_\Omega\frac{1}{p(x)}|\xi_n|^{p(x)}dx\right)-\int_\Omega G(x,\xi_n)dx\\
&\leq \widehat{M}_0\left(\int_\Omega\frac{|\xi_n|^{p_0(x)}}{p_0(x)}dx\right)+\delta|\Omega||\xi_n|^{\underline{p}}\\
&\leq \frac{\lambda_1}{\underline{p}^{s^*}}|\xi_n|^{\underline{p}s^*}+\delta|\Omega||\xi_n|^{\underline{p}}+\lambda_2\\
&\leq d_4|\xi_n|^{\underline{p}}+\lambda_2\\
&\leq d_4|a_n|^{\underline{p}}+\lambda_2.
\end{aligned}\tag{52}$$

where $d_4=\max\left(\frac{\lambda_1}{\underline{p}^{s^*}},\delta|\Omega|\right)$.

It follows from (48) that for n large enough,

$$d_4|a_n|^{\underline{p}s^*}<d_3\left(\frac{b_n}{C_0}\right)^{\overline{p}s^*}=\rho_n+\lambda_2.$$

Then (18) is obtained.

Noting that $F(x,\cdot)$ satisfies condition **(S)**, for each n, there exists $\xi_n\in[-a_n,a_n]$ such that

$$F(x,\xi_n)=\sup_{t\in[-a_n,a_n]}F(x,t)\quad\text{a.e. in}\quad\Omega. \tag{53}$$

Then, thanks to (49) and (50) we can obtain that

$$F(x,\xi_n)+h(x)(\rho_n-e_n-\lambda_2)\geq\sup_{|t|\leq b_n}F(x,t)\ \ \text{a.e. in}\quad\Omega, \tag{54}$$

and the inequality (54) is strict on a subset of Ω with positive measure. Thanks to (51) and (54) we obtain (19). Therefore, the hypotheses of Theorem 2.3 (c) are satisfied.

Consequently, there exists a sequence $(v_n)_n$ of pairwise distinct local minima of $\Psi+\Phi$ such that $\Psi(v_n)\to 0$, which implies $\|v_n\|_{1,\vec{p}(\cdot)}\to 0$, which complete the proof. $\square$

Now we give some remarks and some examples which motivate our results.

Remark 3.7. The definition of our framework requires only the measurability of $p(\cdot)$ and we do not use Modular-Poincaré inequality which require the log-Hölder continuity of the exponent $p(\cdot)$, while the norm Poincaré inequality requires only the continuity of $p(\cdot)$, for more details we refer to [3], [18].

Remark 3.8. Observe that our results require the condition **(S)** on F (for more details on the condition **(S)** see [21], Remark 2.5), and on the function G we need one of the assumptions **(G1)** which is a growth condition on G and **(G2)** which gives a relation between $\widehat{M}_0$ and the integral of G.

Now, we present some examples to illustrate our results.

For $i\in [\![0,N]\!]$ we set $M_i(t)=(1+t)^{\theta_i-1}$ where $\theta_i>1$ the we have $M_i(t)\geq 1$ for all $t\geq 0$ and by using following classical result

$$(a+b)^{\theta}\leq 2^{\theta-1}(a^{\theta}+b^{\theta}),\quad \text{for all } a,b\geq 0 \text{ and } \theta\geq 1,$$

we get

$$\widehat{M}_0(t)=\frac{(1+t)^{\theta_0}}{\theta_0}\leq 2^{\theta_0-1}(1+t^{\theta_0}).$$

Which means that **(M1)** and **(M2)** hold with $m=1$, $s^*=\theta_0$ and $\lambda_1=\lambda_2=2^{\theta_0-1}$.

Proposition 3.9. *Let $g(x,t)=\frac{\underline{p}\vartheta_0(x)}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}|t|^{\underline{p}-1}$ where $\vartheta_0\in L^1(\Omega)$ is positive function with $\|\vartheta_0\|_{L^1(\Omega)}=1$, and let $f(x,t)\equiv\alpha(x)f_1(t)$, with $\alpha(\cdot)\in L^1(\Omega)$ be a positive function such that $\|\alpha(\cdot)\|_{L^1(\Omega)}=1$ and $f_1(\cdot)$ a continuous function with $f_1(t)=F_1'(t)$ and $F_1(-t)=F_1(t)$. Then, the following nonlinear perturbed Kirchhoff problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^{N}\left(1+\int_\Omega\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right)^{\theta_i-1}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)\\ +\left(1+\displaystyle\int_\Omega\frac{1}{p_0(x)}|u|^{p_0(x)}\right)^{\theta_0-1}|u|^{p_0(x)-2}u\\ =\alpha_1(x)f_1(t)+\frac{\underline{p}\vartheta_0(x)}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}}|u|^{\underline{p}-1} \text{ in } \Omega,\\ \displaystyle\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\nu_i=0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{55}$$

admits a sequence of weak solutions $(u_n)_n$ in $W^{1,\vec{p}(x)}(\Omega)$ such that

$$\lim_{n\to\infty}\|u_n\|_{1,\vec{p}(x)}=\infty.$$

Proof. It is clear that Ψ, is coercive. We have $F(x,t) = \alpha(x)F_1(t)$, Choose two positive sequences $(a_n)_n$ and $(b_n)_n$ such that $a_1 \geq 1$, $b_n^{\underline{p}} = 2n^2 a_n^{\overline{p}}$ and $a_{n+1} > b_n$ for every n. Define $F_1(a_n) = a_n^{\theta_0 \overline{p}+1}$ and $F_1(b_n)$ such that

$$F_1(a_n) < F_1(b_n) < \left(\frac{1}{2\overline{p}(N+1)^{\underline{p}-1}} \left(\frac{b_n}{C_0} \right)^{\underline{p}\theta_0} - d_1 |a_n|^{\overline{p}\theta_0} - d_2 \right) + F_1(a_n), \quad (56)$$

where $d_1 = \max\left(\frac{2^{\theta_0 - 1}}{\underline{p}^{\theta_0}}, \frac{1}{2(N+1)^{\underline{p}-1}\overline{p}C_0^{\underline{p}}} \right)$ and $d_2 = \|\vartheta_1\|_{L^1(\Omega)} + 2^{\theta_0 - 1}$.

Put $\rho_n = \frac{1}{2\overline{p}(N+1)^{\underline{p}-1}} \left(\frac{b_n}{C_0} \right)^{\underline{p}\theta_0}$ and $\xi_n = a_n$.

Since

$$\begin{aligned} &\widehat{M}_0 \left(\int_\Omega \frac{1}{p_0(x)} |a_n|^{p_0(x)} dx \right) - \int_\Omega F(x, a_n) dx \\ &\leq d_1 |a_n|^{\overline{p}\theta_0} + d_2 - \|\alpha(\cdot)\|_{L^1(\Omega)} a_n^{\overline{p}^+ + 1} \to -\infty \end{aligned}$$

as $n \to \infty$, then the conditions $(31) - (32)$ holds true. Taking $w_1(x) = \alpha(x)$, and in view of (56) we can obtain the conditions $(33) - (34)$.
The hypotheses of Theorem 3.5 are satisfied, then the problem (55) admits a sequence of weak solutions $(u_n)_n$ in $W^{1,\vec{p}(x)}(\Omega)$ such that $\lim_{n\to\infty} \|u_n\|_{1,\vec{p}(x)} = \infty$. $\square$

Remark 3.10. Observe that the function

$$G(x,t) = \frac{1}{2|\Omega|} \widehat{M}_0 \left(\int_\Omega \frac{1}{p_0(x)} |t|^{p_0(x)} dx \right) + C_1 \gamma(x),$$

where $\|\gamma\|_{L^1(\Omega)} = 1$ satisfies the assumption **(G2)**. Then, we get the same result as the previous example with a function satisfies **(G2)**.

Proposition 3.11. *Let $g(x,t) = -\underline{p}\mu(x)|t|^{\underline{p}-1}$ where $\mu(\cdot)$ is a positive function such that $\int_\Omega \mu(x)dx = 1$ and let f be the function defined in the previous proposition, and consider two positives sequences $(a_n)_n$ and $(b_n)_n$ such that $a_n^{\underline{p}\theta_0} = \frac{1}{n} b_n^{\overline{p}s^*}$ and $b_{n+1} < a_n$, with $F_1(0) = 0$, $F_1(a_n) = a_n^{\underline{p}\theta_0 + 1}$ and*

$$F_1(a_n) < F_1(b_n) < \left(\frac{1}{\overline{p}(N+1)^{\overline{p}-1}} \left(\frac{b_n}{C_0} \right)^{\overline{p}\theta_0} - d_3 |a_n|^{\underline{p}\theta_0} \right) + F_1(a_n). \quad (57)$$

Then, the following perturbed Kirchhoff problem

$$\begin{cases} -\sum_{i=1}^{N}\left(1+\int_{\Omega}\frac{1}{p_i(x)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\right)^{\theta_i-1}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) \\ +\left(1+\int_{\Omega}\frac{1}{p_0(x)}|u|^{p_0(x)}\right)^{\theta_0-1}|u|^{p_0(x)-2}u \\ =\alpha_1(x)f_1(t)-\mu(x)\underline{p}|u|^{\underline{p}-1} \ in\ \Omega, \\ \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\nu_i=0 \quad on \quad \partial\Omega, \end{cases} \tag{58}$$

admits a sequence of weak solutions $(u_n)_n$ in $W^{1,\vec{p}(x)}(\Omega)$ such that

$$\lim_{n\to\infty}\|u_n\|_{1,\vec{p}(x)}=0.$$

Proof. Set $\rho_n=\frac{1}{\overline{p}(N+1)^{\overline{p}-1}}\left(\frac{b_n}{C_0}\right)^{\underline{p}\theta_0}$ and $\xi_n=a_n$, we have

$$\begin{aligned}&\int_{\Omega}F(x,a_n)dx+\int_{\Omega}G(x,a_n)dx-\widehat{M}_0\left(\int_{\Omega}\frac{1}{p_0(x)}dx\right)|a_n|^{\underline{p}}\\ &>|a_n|^{\underline{p}\theta_0+1}-|a_n|^{\underline{p}}-\tfrac{\lambda|\Omega|}{\overline{p}}|a_n|^{\underline{p}}\longrightarrow 0 \ \ as\ \ n\to\infty,\end{aligned}$$

therefore (45)-(47) hold true. Using (57) we obtain the conditions (49)-(50).

Thus all the assumptions of Theorem 3.6 are satisfied, which completes the proof. ∎

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