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ESTIMATES ON COUNTING FUNCTIONS ASSOCIATED TO SOME HYPERBOLIC OPERATORS AND SPECTRAL PROPERTIES

S. EL MANOUNI - Y. LAZAR

We study the distribution of the eigenvalues of a linear operator associated to a hyperbolic partial differential equation with periodic boundary conditions. Using some recent results concerning the distributions of the values of indefinite quadratic forms at integers, we are able to derive the equidistribution of the eigenvalues relatively to the Lebesgue measure with exact asymptotics. Also we provide an asymptotic lower bound in the rational case.

1. Introduction

Given an open bounded connected subset Ω of $\mathbb{R}^n$ with continuous boundary $\partial\Omega$ we consider the following boundary value problem of hyperbolic type corresponding to the wave equation with forcing term,

$$(Ep) \qquad \begin{cases} u_{tt} - \Delta u + \sigma u = f, & (t,x) \in \mathbb{R} \times \Omega, \quad (1.1) \\ u(t,x) = 0, & (t,x) \in \mathbb{R} \times \partial\Omega, \quad (1.2) \\ u(t+\tau,x) = u(t,x) & (t,x) \in \mathbb{R} \times \Omega. \quad (1.3) \end{cases}$$

Here σ is a real parameter and the forcing term $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is assumed to be τ-periodic with respect to t and square-integrable over $(0,\tau) \times \Omega$. In the sequel

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we are interested in the case when $\tau = 2\pi$, Ω is the n-dimensional parallelepiped $\prod_{i=1}^{n}(0, \alpha_i \pi) \subset \mathbb{R}^n$ ($n \geq 3$) and where the α_i's are positive real numbers ordered as follows,

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1} > \alpha_n > 0.$$

Let us denote by P the symmetric linear operator $P = \partial_t^2 - \Delta + \sigma$ associated to our boundary problem (Ep) above. As one can expect for linear problems, the solutions of (Ep) depend essentially on the properties of the eigenvalues of P acting on $L^2((0, 2\pi) \times \Omega)$. Due to the periodic condition and the shape of the domain, the eigenvalues of the operator P can be written in the form

$$\lambda_{k,l} = \left(\frac{l_1}{\alpha_1}\right)^2 + \ldots + \left(\frac{l_n}{\alpha_n}\right)^2 - k^2 + \sigma,$$

where $l_1, \ldots, l_n$ and k are positve integers (see e.g. §5 in [16]). It follows that the spectrum of P can be written $\mathrm{Spec}(P) = \sigma + \Gamma$ where

$$\Gamma = \left\{ \left(\frac{l_1}{\alpha_1}\right)^2 + \ldots + \left(\frac{l_n}{\alpha_n}\right)^2 - k^2 \mid (l, k) \in \mathbb{N}^n \times \mathbb{N} \right\}.$$

The set Γ is the most interesting part of the spectrum and it can be seen as the range of a quadratic form with signature $(n, 1)$ evaluated at the integers vectors $\mathbb{N}^n \times \mathbb{N}$. More precisely, let us set $\beta_i = \alpha_i^{-2}$ for each $1 \leq i \leq n$ and define the quadratic form Q_α in $n+1$-variables given by

$$Q_\alpha(x_1, \ldots, x_n, y) = \beta_1 x_1^2 + \ldots + \beta_n x_n^2 - y^2.$$

Thus $\Gamma = Q_\alpha(\mathbb{N}^{n+1})$ and therefore $\mathrm{Spec}(P) = \sigma + Q_\alpha(\mathbb{N}^{n+1})$. We are led to the study of the values of indefinite quadratic forms at integral vectors. Such a problem is quite difficult in number theory and a good understanding of the distributions of the values of such forms at integers has completed only very recently. Unexpectedly it occurs that Ergodic theory is the most suitable setting in order to treat the distribution of such forms. We review the most recent results regarding this theory in section §2. In this setting, Conrey and Simley proved in Lemma 2.2, [4] that the spectrum of P is dense provided that for for some i, β_i is not rational. The aim of this paper is to adress a quantitative version of this result, more precisely, we show that the equistribution of the spectrum of P holds when some ratio is irrational (Theorem 2.1 (i)). We provide also an asymptotic upper bound in the rational case ((Theorem 2.1 (ii))). The main tools used here come from recent developments in number theory and dynamical systems.

Related work- Weyl's law for the Laplacian operator

A Weyl Law is the term commonly used for any result which treats the statistical properties of the distributions of the eigenvalues of the Laplacian. We illustrate now why the distribution of the eigenvalues of the Laplacian in tori is related with counting the lattice points in ellipsoids.

In dimension 2, we can consider the classical eigenvalue problem $-\Delta_{\mathbb{T}^2} u = \lambda u$ where $\mathbb{T}^2$ is the flat torus $\mathbb{R}^2/l\mathbb{Z} \times l\mathbb{Z}$ with fundamental domain $\Omega = [0,l)^2$. The functions u are taken in $L^2(\Omega)$ and are assumed to satisfy the boundary conditions:

$$u(x_1+l, x_2+l) = u(x_1, x_2) \text{ for all } (x_1, x_2) \in \Omega.$$

The torus $\mathbb{T}^2$ is a smooth compact manifold without boundary with area $|\Omega| = l^2$. The family $(e_m(x)) = \left(e^{2i\pi(x,m)/l}\right)_{m\in\mathbb{Z}^3}$ is an orthonormal basis of eigenvalues of $L^2(\mathbb{T}^2)$ for $-\Delta_{\mathbb{T}^2}$ with eigenvalues:

$$\lambda_m = \frac{4\pi^2}{l^2}(m_1^2 + m_2^2) \quad (m \in \mathbb{Z}^3).$$

Let us focus on the asymptotic growth of the counting function of the eigenvalues less or equal than λ:

$$N(\lambda) = \sum_{m\in\mathbb{Z}^2} \mathbf{1}_{[0,\lambda]}(\lambda_m) = \#\{m \in \mathbb{Z}^2 \;\;:\;\; \lambda_m \leq \lambda\}.$$

One can write

$$N(\lambda) = \sum_{m\in\mathbb{Z}^2} \mathbf{1}_{\left[0,\frac{\lambda L^2}{4\pi^2}\right]}(m_1^2 + m_2^2) = \sum_{n \leq \frac{\lambda l^2}{4\pi^2}} r(n),$$

where $r(n)$ is the number of ways the positive integer n can be written as a sum of two integer squares. In view of this we can interpret $N(\lambda)$ as the number of lattice points in $\mathbb{Z}^2$ lying inside the circle of radius $\sqrt{\lambda}l/2\pi$. The exact asymptotic of the number of lattice points inside a circle is due to Gauss and it asserts that as $x \to \infty$

$$\sum_{n\leq x} r(n) = \pi x + o(x^{1/2}).$$

As a consequence as $\lambda \to \infty$, we obtain the Weyl Law for the 2-torus

$$N(\lambda) \sim \frac{|\Omega|}{4\pi}\lambda.$$

The generalization of the previous results to the Laplacian in the d-dimensional flat torus is still valid but the lattice points problem in dimension greater than two is more suitably solved using the trace formula for the heat kernel on $\mathbb{T}^d$. This powerful method yields the following asymptotic as $\lambda \to \infty$

$$N(\lambda) = \frac{|\Omega|}{(4\pi)^{d/2}\Gamma(1+d/2)}\lambda^{d/2} + o(\lambda^{d/2}).$$

For the historical developement of this topic and more details about Weyl's Law for the Laplacian, we refer the reader to [2], [9] and [10].

2. Weyl's Law type problem for hyperbolic linear operators

For the usual case corresponding to the Laplace operator, the Weyl Law reduces to a lattice point counting in ellipsoids as we have pointed above. For linear hyperbolic operators such as $P = \partial_t^2 - \Delta + \sigma$, the situation is much more complicated. The main reason is that we are reduced to a lattice point counting in hyperboloids rather than positive definite ellipsoids. This difference is illustrated by the choice of the counting function, indeed if we just want to count the number of eigenvalues less than some threshold as in the elliptic case, or more generally in some interval $[a,b]$ we would introduce the following counting function

$$N(a,b) = \sum_{m=(k,l)\in\mathbb{N}^{n+1}} \mathbf{1}_{[a,b]}(\lambda_m).$$

Geometrically, this is interpreted as

$$N(a,b) = \sum_{m=(k,l)\in\mathbb{N}^{n+1}} \mathbf{1}_{[a,b]}(Q_\alpha(m)+\sigma).$$

This function counts the number of integral vectors in $\mathbb{N}^{n+1}$ lying in the hyperbolic shell $\{x \in \mathbb{R}^{n+1} \mid a-\sigma \leq Q_\alpha(x) \leq b-\sigma\}$. Since the latter subspace is noncompact, the counting function $N(a,b)$ is always infinite which is obviously meaningless. In order to obtain a finite quantity, we may introduce the following counting function

$$N_T(a,b) = \sum_{m=(k,l)\in\mathbb{N}^{n+1},\|m\|\leq T} \mathbf{1}_{[a-\sigma,b-\sigma]}(Q_\alpha(m)).$$

This function counts the number of eigenvalues of P in the open interval (a,b) with integral index m lying in the euclidean ball of radius T. Remark the following facts,

• The function $l \in \mathbb{N}^n \mapsto \lambda_{l,k}$ is increasing for the lexicographic order.
• The function $k \in \mathbb{N} \mapsto \lambda_{l,k}$ is decreasing.
The indices (l,k) may be seen as the level of energy of wave of frequency vector in each direction given the α_i's. In the counting, we restrict the spectrum to the interval (a,b) and for the indices (l,k) such that $\|l\|^2 + k^2 < T^2$, the latter condition forces the levels of energy to be bounded.

This justifies to calling *Weyl's Law for the hyperbolic operator P* any estimate of the counting function $N_T(a,b)$ as T gets large with a,b fixed.
Our main result gives an exact asymptotic estimate in the irrational case and an upper bound for the rational case. As in the elliptic case, the asymptotic behaviour depends on the arithmetical properties of the sidelengths of our domain.

Our main result gives an exact asymptotic estimate in the irrational case and an upper bound for the rational case. As in the elliptic case, the asymptotic behaviour depends on the arithmetical properties of the sidelengths of our domain.

Theorem 2.1. *Let us consider the counting function associated to the operator P satisfying the boundary problem (Ep) for $n \geq 5$ given by*

$$N_T(a,b) = \left|\{(k,l) \in \mathbb{N}^{n+1} \cap B_T \ : \ a \leq \lambda_{k,l} \leq b\}\right|,$$

for each reals $a < b$ and $T > 0$. Then we have

1. *if $\alpha_i \notin \mathbb{Q}$ for every $1 \leq i < j \leq n$, then the spectrum of P, $\{\lambda_{k,l}\}$ is equidistributed for the Lebesgue measure of $\mathbb{R}$, that is,*

$$N_T(a,b) \sim \ c_{\alpha,n}(b-a)T^{n-1}$$

for some explicit constant $c_{\alpha,n}$.

2. *If $\alpha_i \in \mathbb{Q}$ for all $1 \leq i \leq n$, then we have the following upper asymptotic estimate*

$$N_T(a,b) \lessapprox CT^n$$

where the constant involved depends on n,a,b and the α_i's.

Remarks. (i) The assumption $n \geq 3$ cannot be weakened to $n \geq 2$ at least for the first part of the Theorem. This is due to the fact that for $n = 2$, Theorem 3.2 in §3, which is the main result we use in the case 1., does not hold for forms of signature $(2,1)$ (see [6]).

(ii) We could have consider another counting function by taking for instance the supremum norm $|m|_\infty$ instead of the euclidean norm $\|m\|$. In this case, other methods come into play such as the circle method but it gives less compelling results. Using recent developments due to Browning ([3]), it is possible to give sharp upper bounds for $N_T(0,0)$ which correspond to the multiplicity of the zero eigenvalue. For arbitrary a,b, it seems difficult to obtain sharp bounds for the counting function by using the same methods.

3. Values of indefinite quadratic forms at integral points

In this section, we review briefly the main arithmetical results we are going to use in order to prove Theorem 2.1. Let be given an indefinite nondegenerate quadratic form Q in n variables with real coefficients. In this section, we are interested with the distribution of the values of the set $Q(\mathbb{Z}^n)$ in $\mathbb{R}$. There are two opposite situations depending if Q is proportional or not to a rational quadratic form. If Q is proportional to a rational form, then the set $Q(\mathbb{Z}^n)$ is a discrete set in $\mathbb{R}$. In the other case if we suppose that Q is not proportional to a rational form, the situation is more complicated. It was conjectured by Oppenheim [15] that this set is dense in $\mathbb{R}$ (initially conjectured for $n \geq 5$) and this is a relatively recent result that this conjecture is true, due to G.A. Margulis [12]. In fact quite more is true, the set $Q(\mathbb{Z}^n)$ is equidistributed w.r.t. the Lebesgue measure. We recall the main results needed in the sequel. It is noteworthy to say that the proofs of all this results are based on the Ergodic theory of the action of the orthogonal group of the quadratic on the set of unimodular lattices in $\mathbb{R}^n$. A key element is the fact that such symmetry groups are generated by unipotent elements, roughly speaking, we have enough symmetry in order to ensure the density of the orbits in the space of lattices in $\mathbb{R}^n$. For more details about this topic, see Margulis' review in [18] and also [13].

3.1. Distribution of the values of irrational quadratic forms at integers

We say that a quadratic form Q is irrational if it is not proportional to a quadratic form with coefficients in $\mathbb{Q}$. It is easy to see that Q is irrational if and only if Q is equivalent to diagonal form $a_1x_1^2 + \ldots + a_nx_n^2$ with some ratio $a_i/a_j \notin \mathbb{Q}$ $(i \neq j)$. Indeed for diagonal forms, it is not suffcient that at least one a_i may be irrational, as we can see the form

$$Q(x) = \sqrt{2}x_1^2 + \ldots - \sqrt{2}x_n^2 = \sqrt{2}Q_0(x)$$

where $Q_0(x) = x_1^2 + x_1^2 + \ldots - x_n^2$ is obviously a rational form. It is sufficient that at least one a_i is irrational, when one of the a_i equals one. As we can see, if for instance $a_n = 1$ then

$$Q(x) = a_1x_1^2 + \ldots + a_{n-1}x_{n-1}^2 - x_n^2$$

is irrational as soon as one the a_i 's for $1 \leqslant i \leqslant n-1$ is irrational.

We have the following deep theorem which proves a conjecture of Oppenheim,

Theorem 3.1 (Margulis, [12])**.** *Let Q be a nondegenerate indefinite irrational quadratic form in $n \geq 3$ variables with real coefficients. Then the set $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

The ergodic theoretic proof of this density theorem leads naturally to the more general question of the uniform distribution of the values of such forms at integral points. As above, we consider Q an irrational indefinite real quadratic form in $n \geqslant 3$ variables, and for reals $a < b$, let $N^Q_{(a,b)}(T)$ denotes the number of integral points x in an euclidian ball of radius T with $a < Q(x) < b$. The Oppenheim conjecture is equivalent to the statement $N^Q_{(a,b)}(T) \to \infty$ when $T \to \infty$. Gauss' generalization of the circle lattice points problem in dimension $n \geq 3$ gives us that

$$|\{x \in \mathbb{Z}^n : \|x\| < T\}| \sim \mathrm{Vol}\{x \in \mathbb{R}^n : \|x\| < T\} \text{ as } T \to \infty$$

and suggests the following estimate as $T \to \infty$

$$N^Q_{(a,b)}(T) \sim \mathrm{Vol}\{x \in \mathbb{R}^n : a < Q(x) < b, \|x\| < T\}. \tag{1}$$

Calculations (see Lemma 3.8 (i), [6]) show that as $T \to \infty$,

$$\mathrm{Vol}\{x \in \mathbb{R}^n : a < Q(x) < b, \|x\| < T\} \sim c_Q(b-a)T^{n-2}. \tag{2}$$

The exact value of the constant is given by

$$c_Q = \int_{L \cap B} \frac{dA}{\|\nabla Q\|}, \tag{3}$$

with B the unit ball, L the light cone $Q = 0$ and dA the area element on L.

We propose to give an elementary proof of the volume estimate (2) using tools from geometric measure theory. The reason why the proof in [6] (Lemma 3.8 (i)) is more complicated is because the domain for which the volume is computed therein, is a star-shaped open set which clearly can fails to be a smooth manifold due to the eventual presence of cusps or corners. In the case of the euclidean ball, we have a smooth manifold and we are allowed to use the Coarea

formula (e.g. see §3.2.22, [8]). Let us recall this useful result.
Given a Riemannian manifold equipped with a volume form μ, the Coarea formula says that for any $f \in \mathcal{C}^\infty(M)$ and $g \in L^1(M,\mu)$

$$\int_M g(x)\|\nabla f(x)\|\mu(dx) = \int_0^\infty \int_{L_t} g(u)dA_t(y)dy, \tag{4}$$

where $L_t = f^{-1}(t)$ is the fiber of f at t and dA_t is the Riemannian volume form on L_t induced by μ.
Let us apply the Coarea formula (4) with $M = B_T$ be the euclidean ball of radius T with its usual volume form n-form ω, $f(x) = Q(x)$ and $g(x) = \mathbf{1}_{[a,b]}(Q(x))\|\nabla Q(x)\|^{-1}$. Clearly f is a smooth function on B_T and $g \in L^1(B_T)$ since $\|\nabla Q(x)\| \neq 0$ for a.e. $x \in B_T$. Hence,

$$\int_{B_T} \mathbf{1}_{[a,b]}(Q(x))\omega(dx) = \int_0^\infty \int_{L_t \cap B_T} \frac{\mathbf{1}_{[a,b]}(t)}{\|\nabla Q(y)\|} dA_t(y)dy, \tag{5}$$

where L_t is the level set $\{Q = t\}$ and dA_t is the surface area $(n-1)$-form on L_t induced by ω. Set

$$\Omega_T^{a,b} = \{v \in \mathbb{R}^n : a < Q(v) < b, \|v\| < T\} = \{v \in B_T \ : \ a < Q(v) < b\}.$$

We can write (5) as

$$\mathrm{vol}_\omega(\Omega_T^{a,b}) = \int_a^b \int_{L_t \cap B_T} \frac{dA_t(y)}{\|\nabla Q(y)\|}\, dt.$$

Using the change of variable $y = Tx$, we get

$$\mathrm{vol}_\omega(\Omega_T^{a,b}) = \int_a^b \int_{L_{t/T^2} \cap B} \frac{dA_t(Tx)}{\|\nabla Q(Tx)\|}\, dt = \left(\int_a^b \int_{L_{t/T^2} \cap B} \frac{dA_t(x)}{\|\nabla Q(x)\|}\, dt \right) T^{n-2}.$$

Furthermore, passing to the limit as T gets large, we have

$$\int_{L_{t/T^2} \cap B} \frac{dA_t(x)}{\|\nabla Q(x)\|} \sim_{T\to\infty} \int_{L_0 \cap B} \frac{dA_0(x)}{\|\nabla Q(x)\|}.$$

Finally, we obtain the required estimate (2)

$$\mathrm{vol}_\omega(\Omega_T^{a,b}) \sim_{T\to\infty} (b-a)T^{n-2} \int_{L_0 \cap B} \frac{dA_0(x)}{\|\nabla Q(x)\|}.$$

In general the computation of the volume confined between two level sets is a challenging problem which can be solved only in few cases. Very recently, Athreya and Margulis proposed in ([1], §3) a conjecture generalizing such kind of volume estimates when Q is replaced by a polynomial of degree $k \geq 2$. Namely they are interested in estimating the following quantity when $T \to \infty$,

$$\mathrm{Vol}\{v \in \mathbb{R}^n : a < P(v) < b, \|v\| < T\}.$$

It is expected that the following estimate occurs

$$\mathrm{Vol}\{v \in \mathbb{R}^n : a < P(v) < b, \|v\| < T\} \sim c_P(b-a)T^{n-k},$$

where

$$c_P = \int_{\{P=0\}\cap B} \frac{dA(x)}{\|\nabla P(x)\|}.$$

A serious problem which prevents such estimates to hold is that there is no guarantee $\|\nabla P(x)\| \neq 0$ a.e. on the hypersurface $\{P=0\}$ and in particular c_P cannot be always defined.

The following theorem shows that the values of irrational indefinite quadratic forms at integers are not only dense but also equidstributed for signature $(p, n-p)$ at least for $p \geq 3$. This is a highly nontrivial result due to Eskin-Margulis-Mozes (see [6], [7]) based on partial results previously obtained by Dani and Margulis ([5], Corollary 5 (i)).

Theorem 3.2 (see [6], Theorem 2.1)**.** *If Q is real quadratic form of signature $(p, n-p)$ $(p \geq 3)$ which is not proportional to a rational form then as $T \to \infty$*

$$|\{v \in \mathbb{Z}^n : a < Q(v) < b, \|v\| < T\}| \sim c_{Q,\Omega}(b-a)T^{n-2},$$

where $c_{Q,\Omega}$ is as in (3).

The previous asymptotics is still valid when the counting is restricted to the lattices points which do not lie on the hyperplanes sections. Indeed, those points do not contribute more than $o(T^{n-2})$, so discarding them has no effect on the asympotic main term as T goes to infinity. Let us explain this fact, more precisely for forms of signature $(n,1)$ in view of applying to Q_α.
Let I be a nonempty subset of $\{1,\ldots,n\}$ with $d \geqslant 1$ elements and define

$$N_T^{(I)}(a,b) = \{v \in \mathbb{Z}^n : a < Q(v) < b, \|v\| < T, v_i = 0, i \in I\}.$$

We divide the study of this counting function into three cases $d \geq 3$, $d = 1$ and $d = 2$, the aim is to show that as soon as $d \geq 1$ and $n \geq 5$,

$$N_T^{(I)}(a,b) =_{T\to\infty} o(T^{n-2}).$$

Case A. Assume $d \geq 3$, A crude upper bound for this quantity is just given by

$$N_T^{(I)}(a,b) \leq |\mathbb{Z}^{n-d} \cap B_T^{n-d}|$$

where B_T^{n-d} is the result of section of the n-dimensional euclidean ball B_T^n cutted out by the space of codimension d, $W_I := \bigcap_{i\in I} \ker l_i$. Since the number of lattice points of an euclidean ball is approximated by its volume as its radius tends to infinity we are granted that

$$N_T^{(I)}(a,b) = O(T^{n-d})$$

as T gets large. In particular, since $d \geq 3$, we get

$$N_T^{(I)}(a,b) = O(T^{n-3})$$

and then

$$N_T^{(I)}(a,b) = o(T^{n-2})$$

as T goes to infinity.

Case B. Assume $d = 1$. Since Q is an indefinite quadratic form of signature $(n,1)$ where $n \geq 5$ of the form

$$Q(v_1,\ldots,v_{n+1}) = a_1v_1^2 + \ldots + a_nv_n^2 - v_{n+1}^2$$

with $a_1,\ldots,a_n$ are positive real numbers. Let us define the linear form $l_i(v) = v_i$ for some $1 \leqslant i \leqslant n-1$, and the corresponding counting function associated to the restricted form $Q|_{\ker l_i}$

$$N_T^{(i)}(a,b) = \{v \in \mathbb{Z}^n : a < Q(v) < b, \|v\| < T, v_i = 0\}.$$

The the restricted form $Q|_{\ker l_i}$ takes the following form

$$Q|_{\ker l_i}(v) = Q(v_1,\ldots,v_i,0,v_{i+1},\ldots,v_{n+1}) = a_1v_1^2+\ldots+a_{i-1}v_{i-1}^2+a_{i+1}v_{i+1}^2+\ldots+a_nv_n^2-v_{n+1}^2.$$

Then $Q|_{\ker l_i}(v)$ is an indefinite form in n variables of signature indefinite with signature $(n-1,1)$. Thus, if one the a_j for $j \neq i$ is irrational and $n \geq 5$ then $Q|_{\ker l_i}(v)$ satisfies the hypothesis of Theorem 3.2 applies and we get

$$N_T^{(i)}(a,b) \sim_{T\to\infty} c(b-a)T^{n-3}.$$

In particular, $N_T^{(i)}(a,b) = o(T^{n-2})$ as T gets large.

Case C. Assume $d = 2$, and $I = \{i < j\}$.

$$Q|_{\ker l_i \cap \ker l_j}(v) = Q(v_1,\ldots,v_i,0,v_{i+1},\ldots,v_{n+1}) = a_1v_1^2+\ldots+a_{i-1}v_{i-1}^2+a_{i+1}v_{i+1}^2$$
$$+\ldots+a_{j-1}v_{j-1}^2+a_{j+1}v_{j+1}^2+\ldots+a_nv_n^2-v_{n+1}^2.$$

Then $Q|_{\ker l_i \cap \ker l_j}(v)$ is an indefinite form in $n-1$ variables of signature indefinite with signature $(n-2,1)$. Thus, since a_l is irrational for $l \notin I$ and $n \geq 5$

then $Q|_{\ker l_i \cap \ker l_j}(v)$ satisfies all the hypothesis of Theorem 3.2 and therefore we get

$$N_T^{(i,j)}(a,b) \sim_{T\to\infty} c(b-a)T^{n-4}.$$

In particular, $N_T^{(i,j)}(a,b) = o(T^{n-2})$ as T gets large.

To sum up, we have shown that the contribution of the lattice points with at least one zero coordinate in the counting can be neglicted in front of the main term, hence we deduce the following statement.

Corollary 3.3. *If Q is real diagonal quadratic form of signature $(n,1)$ $(n \geq 5)$ of the form*

$$Q(v_1,\ldots,v_{n+1}) = a_1 v_1^2 + \ldots + a_n v_n^2 - v_{n+1}^2$$

where all the a_i's are irrational positve real numbers then as $T \to \infty$,

$$|\{v \in \mathbb{N}^n : a < Q(v) < b, \|v\| < T\}| \sim c_{Q,\Omega}(b-a)T^{n-2}.$$

4. Proof of Theorem 2.1

The eigenvalues of the operator P are given by the double indexed real numbers of the following form

$$\lambda_{k,l} = \left(\frac{l_1}{\alpha_1}\right)^2 + \ldots + \left(\frac{l_n}{\alpha_n}\right)^2 - k^2 + \sigma$$

where $(l,k) \in \mathbb{N}^{n+1}$. Let us consider the counting function

$$N_T(a,b) = \sum_{m=(k,l)\in\mathbb{N}^{n+1}, \|m\|\leq T} \mathbf{1}_{[a,b]}(\lambda_{k,l}).$$

Case 1

Assume that α_i are irrational for $1 \leqslant i \leqslant n$, and let us introduce the quadratic form Q_α associated to the eigenvalues $\lambda_{k,l}$ of P,

$$Q_\alpha(x_1,\ldots,x_n,y) = \beta_1 x_1^2 + \ldots + \beta_n x_n^2 - y^2.$$

Then,

$$N_T(a,b) = \sum_{m=(k,l)\in\mathbb{N}^{n+1}, \|m\|\leq T} \mathbf{1}_{[a-\sigma,b-\sigma]}(Q_\alpha(m)).$$

where $\beta_i = \alpha_i^{-2}$. A crucial point is to interpret the eigenvalues of P as the values taken by the form Q_α at integers points up to an additive factor

$$\lambda_{k,l} = Q_\alpha(l_1, \ldots, l_n, k) + \sigma.$$

Then Q_α is an irrational quadratic form of signature $(n,1)$ with $n \geq 5$ and in particular Corollary 3.3 applies, that is, for any reals $a < b$ we have the following asymptotic estimate

$$\lim_{T\to\infty} \frac{N_T(a,b)}{T^{n-1}} = c_\alpha(b-a).$$

with

$$c_\alpha = \int_{L\cap B} \frac{dA}{\|\nabla Q_\alpha\|},$$

where L is the light cone $Q_\alpha = 0$ and B the unit ball in $\mathbb{R}^{n+1}$.

It remains to compute the constant c_α and this asks some computations. At first, we have

$$\nabla Q_\alpha(x) = (2\beta_1 x_1, \ldots, 2\beta_n x_n, -2y)$$

thus

$$\|\nabla Q_\alpha(x_1, x_2, \ldots, x_{n+1})\| = 2\sqrt{\beta_1^2 x_1^2 + \ldots + \beta_n^2 x_n^2 + x_{n+1}^2}.$$

Recall that the light cone L relative to Q_α is the hypersurface of equation

$$L : \beta_1 x_1^2 + \ldots + \beta_n x_n^2 - x_{n+1}^2 = 0.$$

Let us set the diagonal matrix of size $n+1$,

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n, 1)$$

and the standard quadric of signature $(n,1)$

$$L_0 : Q_0(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = 0.$$

Since $\beta_i = \alpha_i^{-2}$, it is clear that

$$L = \alpha L_0 = \{u \in L_0 \mid \alpha^{-1} u \in L_0\}.$$

If we remark that $Q_\alpha(x) = Q_0(\alpha^{-1}x)$, we obtain after substituting y by αx that

$$\int_{L\cap B} \frac{dA}{\|\nabla Q_\alpha\|} = \int_{\alpha L_0 \cap B} \frac{dA}{\|\nabla Q_\alpha(x)\|} = \frac{1}{|\det(\alpha)|} \int_{L_0 \cap \alpha^{-1} B} \frac{dA}{\|\nabla Q_0(y)\|}. \tag{6}$$

Let us consider the following change of coordinates corresponding to the polar coordinates in $\mathbb{R}^{n+1}$,

$$\begin{array}{rcl} f:\mathbb{R}_+\times S^{n-1}\times\mathbb{R} & \to & \mathbb{R}^{n+1} \\ (r,\xi,z) & \mapsto & (r\xi_1,\dots,r\xi_{n-1},r\xi_n,z). \end{array}$$

The condition $f(r,\xi,z)\in L_0$ is equivalent to $r^2\|\xi\|^2-z^2=0$, which forces the last coordinate z to be equal to $\pm r$. A parametrization of $L_0=L_0^+\cup L_0^-$ in this new system of coordinates given by

$$\begin{array}{rcl} f_{|L_0^\pm}:\mathbb{R}_+\times S^{n-1} & \to & \mathbb{R}^{n+1} \\ (r,\xi) & \mapsto & (r\xi_1,\dots,r\xi_{n-1},r\xi_n,\pm r) \end{array}$$

where

$$L_0^-=\{(r\xi,-r)\mid r>0,\ \xi\in S^{n-1}\}\text{ and }L_0^+=\{(r\xi,r)\mid r>0,\ \xi\in S^{n-1}\}.$$

The jacobian of $f_{|L_0^\pm}$ is given by

$$J_{f_{|L_0^\pm}}(r,\xi)=\det\begin{bmatrix} \xi_1 & r & 0 & \dots & 0 \\ \xi_2 & 0 & r & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & 0 & 0 & \dots & r \\ \pm1 & 0 & 0 & \dots & 0 \end{bmatrix}=\pm\, r^n$$

and then the pull-back of the volume form on $\mathbb{R}^{n+1}$ by $f_{|L_0}$ is given by

$$f_{|L_0}^*(dx_1\wedge dx_2\wedge\ldots\wedge dx_{n+1})=r^n dr\wedge d\xi_{|S^{n-1}}.$$

We use the parametrization of L_0 as defined above, in order to get

$$\begin{aligned}\int_{L_0\cap\alpha^{-1}B}\frac{dA}{\|\nabla Q_0\|}&=2\int_{L_0^+}\frac{1}{\sqrt{2r^2}}\mathbf{1}((r\xi,r)\in\alpha B)r^n dr\wedge d\xi_{|S^{n-1}} \\ &=\int_{L_0^+}\mathbf{1}((r\xi,r)\in\alpha B)\sqrt{2}r^{n-1}dr\wedge d\xi_{|S^{n-1}}.\end{aligned}$$

The cartesian parametrization of $\alpha^{-1}B$ is

$$\alpha^{-1}B=\{x\in\mathbb{R}^{n+1}\ :\ (\alpha_1x_1)^2+\ldots+(\alpha_nx_n)^2+x_{n+1}^2\le 1\}.$$

Using our new system of coordinates we can reparametrize it as

$$\alpha^{-1}B = \{(r,\xi) \in (0,1] \times S^{n-1} \ : \ (\alpha_1 r\xi_1)^2 + \ldots + (\alpha_n r\xi_n)^2 + r^2 \leq 1 \ \}$$
$$= \{(r,\xi) \in (0,1] \times S^{n-1} \ : \ (\alpha_1 \xi_1)^2 + \ldots + (\alpha_n \xi_n)^2 \leq \frac{1-r^2}{r^2} \ \}.$$

Hence,

$$\int_{L_0^+} \mathbf{1}((r\xi, r) \in \alpha^{-1}B)\sqrt{2}\, r^{n-1} dr \wedge d\xi_{|S^{n-1}}$$

$$= \int_{r=0^+}^{1} \int_{S^{n-1}} \mathbf{1}\left((r,\xi) : (\alpha_1\xi_1)^2 + \ldots + (\alpha_n\xi_n)^2 = \frac{1-r^2}{r^2} \right) \sqrt{2}\, r^{n-1} dr \wedge d\xi_{|S^{n-1}}.$$

Let us denote by $\sigma_{S^{n-1}}$ the measure area on the unit sphere S^{n-1}, using (6) we obtain,

$$\int_{L\cap B} \frac{dA}{\|\nabla Q_\alpha\|} = \frac{1}{|\det(\alpha)|} \int_{r=0^+}^{1} \sigma_{S^{n-1}}[\ \mathcal{E}_n(\frac{\sqrt{1-r^2}}{r}\alpha^{-1})]\sqrt{2}\, r^{n-1} dr, \qquad (7)$$

where $\mathcal{E}_n(\alpha^{-1})$ is the n-dimensional ellipsoid with half-lenght principal axes given by the diagonal elements of α^{-1} and $\mathcal{E}_n(R\alpha^{-1})$ is the ellipsoid obtained after multiplying the half-lenght principal axes of $\mathcal{E}_n(\alpha^{-1})$ by a factor $R > 0$. The measure area on the unit sphere S^{n-1}, understood as an hypersurface of $\mathbb{R}^n$, can be defined explicitely using the Lebesgue measure in $\mathbb{R}^{n-1}$. Given any Borel subset B in $\mathbb{R}^n$, we have

$$\sigma_{S^{n-1}}(B) = \frac{\lambda_{n-1}(B \cap S^{n-1})}{\lambda_{n-1}(S^{n-1})}.$$

Thus

$$\sigma_{S^{n-1}}\left(\mathcal{E}(\frac{\sqrt{1-r^2}}{r}\alpha^{-1}) \right) = \frac{\lambda_{n-1}(\ \mathcal{E}(\frac{\sqrt{1-r^2}}{r}\alpha^{-1}) \cap S^{n-1})}{\lambda_{n-1}(S^{n-1})}.$$

We remark that the intersection of the n-ellipsoid with the unit sphere is $(n-1)$-ellipsoid, indeed

$$\mathcal{E}_n(\frac{\sqrt{1-r^2}}{r}\alpha^{-1})\cap S^{n-1} = \{\xi \in S^{n-1} \ : \ (\alpha_1\xi_1)^2+\ldots+(\alpha_n\xi_n)^2 \leq \frac{1-r^2}{r^2} \ \}$$

$$= \{\xi \in \mathbb{R}^{n-1} \ : \ (\alpha_1\xi_1)^2+\ldots+\alpha_n^2(1-\xi_1^2-\xi_2^2-\ldots-\xi_{n-1}^2) \leq \frac{1-r^2}{r^2} \ \}$$

$$= \{\xi \in \mathbb{R}^{n-1} \ : (\alpha_1^2-\alpha_n^2)\xi_1^2+(\alpha_2^2-\alpha_n^2)\xi_2^2+\ldots+(\alpha_{n-1}^2-\alpha_n^2)\xi_{n-1}^2 \leq \frac{1-r^2}{r^2}-\alpha_n^2 \ \}.$$

Hence,

$$\mathcal{E}_n(\frac{\sqrt{1-r^2}}{r}\alpha^{-1})\cap S^{n-1} = \{\xi \in \mathbb{R}^{n-1} \ : \ \gamma_1^2\xi_1^2+\gamma_2^2\xi_2^2+\ldots+\gamma_{n-1}^2\xi_{n-1}^2 \leq \rho_n(r) \ \}$$

where $\rho_n(r) = \frac{1-r^2}{r^2} - \alpha_n^2$ and $\gamma_i = \frac{1}{\sqrt{\alpha_i^2-\alpha_n^2}}$ for $1 \leq i \leq n-1$. In other words,

$$\mathcal{E}_n(\frac{\sqrt{1-r^2}}{r}\alpha^{-1})\cap S^{n-1} = \ \mathcal{E}_{n-1}(\sqrt{\rho_n(r)}\gamma^{-1})$$

where[1] $\gamma = \mathrm{diag}(\frac{1}{\sqrt{\alpha_1^2-\alpha_n^2}},\ldots,\frac{1}{\sqrt{\alpha_{n-1}^2-\alpha_n^2}})$. It is important to note that this $(n-1)$-ellipsoid is nonempty if and only if $\rho_n(r) \geq 0$ which amounts to ask that $r \leq 1/\sqrt{1+\alpha_n^2}$.

Now we can write (7) as follows,

$$\int_{L\cap B} \frac{dA}{\|\nabla Q_\alpha\|} = \frac{|\det(\alpha)^{-1}|}{\lambda_{n-1}(S^{n-1})}\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}} \lambda_{n-1}(\mathcal{E}_{n-1}(\sqrt{\rho_n(r)}\gamma^{-1}))\sqrt{2}r^n dr. \quad (8)$$

A classical formula for the volume of a n-dimensional euclidean ellipsoid with scale R and half axis given by a diagonal matrix D is given by

$$\lambda_n(\mathcal{E}(RD)) = \frac{\pi^{n/2}R^n}{\det(D)\Gamma(\frac{n}{2}+1)} \quad .$$

[1]This matrix is well defined and invertible since we have assumed that all the α_i's are ordered so that $\alpha_i^2-\alpha_n^2 > 0$. Without this ordering there is the possibility that we obtain a hyperboloid instead of an ellipsoid, we want to avoid this complicated situtation.

Applying this to (8), we get

$$\int_{L\cap B}\frac{dA}{\|\nabla Q_\alpha\|}=\frac{\pi^{(n-1)/2}|\det(\gamma\alpha)|^{-1}}{\Gamma(\frac{n}{2}-1)\lambda_{n-1}(S^{n-1})}\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}}(\rho_n(r))^{\frac{n-1}{2}}\sqrt{2}r^n\,dr. \qquad (9)$$

The well-known formula for the surface area of the unit sphere

$$\lambda_{n-1}(S^{n-1})=\frac{2\pi^{n/2}}{\Gamma(n/2)}$$

yields

$$\int_{L\cap B}\frac{dA}{\|\nabla Q_\alpha\|}=\sqrt{\frac{2}{\pi}}\frac{|\det(\gamma\alpha)|^{-1}}{\Gamma(\frac{n}{2}-1)\Gamma(n/2)}\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}}(\rho_n(r))^{\frac{n-1}{2}}r^n\,dr. \qquad (10)$$

The integral in the RHS is improper at the singularity $r=0$, but it is easily removed since we have

$$\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}}(\rho_n(r))^{\frac{n-1}{2}}r^n\,dr=\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}}(\frac{1-r^2}{r^2}-\alpha_n^2)^{\frac{n-1}{2}}r^n\,dr.$$

$$\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}}(\frac{1-r^2}{r^2}-\alpha_n^2)^{\frac{n-1}{2}}r^n\,dr.$$

$$=\int_{r=0}^{1/\sqrt{1+\alpha_n^2}}\left(1-r^2(1+\alpha_n^2)\right)^{\frac{n-1}{2}}r\,dr.$$

$$=-\frac{1}{(n+1)(1+\alpha_n^2)}\ [\left(1-r^2(1+\alpha_n^2)\right)^{\frac{n+1}{2}}|_{r=0}^{1/\sqrt{1+\alpha_n^2}}.$$

Thus we arrive to a simple expression for the integral,

$$\int_{r=0^+}^{1/\sqrt{1+\alpha_n^2}}(\rho_n(r))^{\frac{n-1}{2}}r^n\,dr=\frac{1}{(n+1)(1+\alpha_n^2)}.$$

By definition we have $|\det(\gamma\alpha)|^{-1}=\prod_{1\le i\le n-1}\alpha_i(\alpha_i^2-\alpha_n^2)^{1/2}$ so finally we obtain the required constant,

$$c_{\alpha,n} = \int_{L\cap B} \frac{dA}{\|\nabla Q_\alpha\|} = \sqrt{\frac{2}{\pi}} \frac{\prod_{1\le i\le n-1} \alpha_i(\alpha_i^2-\alpha_n^2)^{1/2}}{(n+1)(1+\alpha_n^2)\Gamma(\frac{n}{2}-1)\Gamma(n/2)}.$$

To sum up, if some ratio α_j/α_k is irrational then

$$N_T(a,b) \sim \sqrt{\frac{2}{\pi}} \frac{\prod_{1\le i\le n-1} \alpha_i(\alpha_i^2-\alpha_n^2)^{1/2}}{(n+1)(1+\alpha_n^2)\Gamma(\frac{n}{2}-1)\Gamma(n/2)}(b-a)T^{n-1}$$

as $T \to \infty$.

Case 2

We are interested with the growth of $N_T(a,b)$ when T gets large, provided all β_i are all rational numbers. We develop the terms

$$N_T(a,b) = \sum_{m=(k,l)\in\mathbb{N}^{n+1},\|m\|\le T} \mathbf{1}_{[a-\sigma,b-\sigma]}(Q_\alpha(m)) = \sum_{m\in\mathbb{N}^{n+1},\|m\|\le T} \mathbf{1}_{[a-\sigma,b-\sigma]}(Q_\alpha(m))$$

$$= \sum_{m\in\mathbb{N}^{n+1},\|m\|\le T} \mathbf{1}_{[a-\sigma,b-\sigma]}(\beta_1 l_1^2 + \ldots + \beta_n l_n^2 - k^2),$$

so we get by slicing

$$N_T(a,b) = \sum_{1\le k\le T} \sum_{l\in\mathbb{N}^n,\|l\|\le\sqrt{T^2-k^2}} \mathbf{1}_{[a+k^2-\sigma,b+k^2-\sigma]}(\beta_1 l_1^2 + \ldots + \beta_n l_n^2). \quad (11)$$

Consider now the inner sum for a fixed integer k, this sum counts the number of lattice points with positive coordinates lying in the intersection of an euclidean ball of radius $(T^2-k^2)^{1/2}$ and an ellipsoid shell bounded by the ellipsoids $Q_\beta = a+k^2-\sigma$ and $Q_\beta = b+k^2-\sigma$, where $Q_\beta(l) = \beta_1 l_1^2 + \ldots + \beta_n l_n^2$.
Denote by $\mathcal{E}(r)$ the ellipsoid defined by $\{x \in \mathbb{R}^n \mid Q_\beta(x) \le r\}$. A classical result states that the number of lattice points in a ellipsoid is approximated by its volume (see e.g. [11], VI, §2, Thm 2),[2]

$$|\ \mathcal{E}(r) \cap \mathbb{Z}^n\ | = \mathrm{vol}_n(\mathcal{E}(r)) + O\left(r^{n-1}\right).$$

[2]It is possible to provide a better error bound but we prefer to avoid such complications.

Using the fact that the volume of the n-ellispoid $\mathcal{E}(r)$ behaves like r^n, we get

$$\mid \mathcal{E}(r)\cap \mathbb{Z}^n \mid = \mathrm{vol}_n(\mathcal{E}(r))\left(1+o(\frac{1}{r})\right). \tag{12}$$

The symmetry of the ellispoid and the ball, implies that

$$\mid \mathcal{E}(r)\cap \mathbb{N}^n \mid \sim \frac{1}{2^n}\mathrm{vol}_n(\mathcal{E}(r)) \tag{13}$$

as $r\to\infty$. Thus, the inner sum above in (11) can be reduced to a counting problem in elliptic shells,

$$\mid \mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap \mathbb{N}^n \mid = \sum_{l\in\mathbb{N}^n, \|l\|\le\sqrt{T-k^2}} \mathbf{1}_{[a+k^2-\sigma, b+k^2-\sigma]}(\beta_1 l_1^2+\ldots+\beta_n l_n^2)$$

where $a_k=\sqrt{a+k^2-\sigma}$ and $b_k=\sqrt{b+k^2-\sigma}$. Thus we obtain,

$$N_T(a,b)=\sum_{1\le k\le T} \mid \mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap \mathbb{N}^n \mid .$$

First we have

$$\begin{aligned}\mid \mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap \mathbb{N}^n \mid &= \sum_{l\in\mathbb{N}^n} \mathbf{1}_{\mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}}(l)\\ &= \sum_{l\in\mathbb{N}^n}\left(\mathbf{1}_{\mathcal{E}(b_k)}(l)-\mathbf{1}_{\mathcal{E}(a_k)}(l)\right)\mathbf{1}_{B_{\sqrt{T^2-k^2}}}(l).\end{aligned}$$

Using Cauchy-Schwarz's inequality we get

$$\mid \mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap \mathbb{N}^n \mid \le \left(\sum_{l\in\mathbb{N}^n}\left(\mathbf{1}_{\mathcal{E}(b_k)}(l)-\mathbf{1}_{\mathcal{E}(a_k)}(l)\right)^2\right)^{1/2}\left(\sum_{l\in\mathbb{N}^n}\mathbf{1}_{B_{\sqrt{T^2-k^2}}}(l)\right)^{1/2}.$$

Using the simple observation that $a_k\le b_k$ for each integer $|k|\le T$, we infer that $\mathcal{E}(b_k)$ contains $\mathcal{E}(a_k)$ and that

$$\left(\mathbf{1}_{\mathcal{E}(b_k)}(l)-\mathbf{1}_{\mathcal{E}(a_k)}(l)\right)^2=\mathbf{1}_{\mathcal{E}(b_k)}(l)+\mathbf{1}_{\mathcal{E}(a_k)}(l)-2\,\mathbf{1}_{\mathcal{E}(a_k)\cap\mathcal{E}(b_k)}(l)=\mathbf{1}_{\mathcal{E}(b_k)}(l)-\mathbf{1}_{\mathcal{E}(a_k)}(l).$$

Then we arrive to

$$\mid \mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap \mathbb{N}^n \mid \le \left(\sum_{l\in\mathbb{N}^n}\left(\mathbf{1}_{\mathcal{E}(b_k)}(l)-\mathbf{1}_{\mathcal{E}(a_k)}(l)\right)\right)^{1/2}\left(\sum_{l\in\mathbb{N}^n}\mathbf{1}_{B_{\sqrt{T^2-k^2}}}(l)\right)^{1/2}$$

$$\leq (\,|\,\mathcal{E}(b_k)\cap\mathbb{N}^n\,| - \,|\,\mathcal{E}(a_k)\cap\mathbb{N}^n\,|)^{1/2}\,|\,B_{\sqrt{T^2-k^2}}\cap\mathbb{N}^n\,|^{1/2}\,.$$

Using (13), we obtain

$$|\,\mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap\mathbb{Z}^n\,| \lessapprox (\,\mathrm{Vol}(\mathcal{E}(b_k)) - \,\mathrm{Vol}(\mathcal{E}(a_k)\,)^{1/2}\,\mathrm{Vol}(B_{\sqrt{T^2-k^2}})^{1/2}.$$

Classical volume formula for ellipsoids and euclidean balls yields

$$\mathrm{Vol}(\mathcal{E}(r)) = \frac{\pi^{n/2}r^n}{\det(Q_\beta)\Gamma(\frac{n}{2}+1)} \quad \text{and} \quad \mathrm{Vol}(B_r) = \frac{\pi^{n/2}r^n}{\Gamma(\frac{n}{2}+1)}.$$

From this we derive that

$$|\,\mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap\mathbb{N}^n\,| \lessapprox \frac{\pi^{n/4}}{\sqrt{\det(Q_\beta)\Gamma(\frac{n}{2}+1)}}\,(b_k^n - a_k^n)^{1/2}\,(T^2-k^2)^{n/4}.$$

Now let us focus on the factor involving a_k and b_k. Indeed we have

$$\begin{aligned} b_k^n - a_k^n &= (b+k^2-\sigma)^{n/2} - (a+k^2-\sigma)^{n/2} \\ &= k^n\left[\left(1+\frac{b-\sigma}{k^2}\right)^{n/2} - \left(1+\frac{a-\sigma}{k^2}\right)^{n/2}\right]. \end{aligned}$$

For k_0 large enough we have the first term Taylor expansion for $k \geq k_0$

$$b_k^n - a_k^n = k^n\left[\left(1+\frac{n(b-\sigma)}{2k^2}\right) - \left(1+\frac{n(a-\sigma)}{2k^2}\right) + o(\frac{1}{k^2})\right].$$

Which reduces to

$$b_k^n - a_k^n = k^n\left[\frac{n(b-a)}{2k^2} + o(\frac{1}{k^2})\right].$$

Then, for $k \geq k_0$

$$b_k^n - a_k^n = k^{n-2}\left[\frac{n(b-a)}{2} + o(1)\right].$$

For $k \geq k_0$, we get the following bound

$$b_k^n - a_k^n \lessapprox_{a,b,n} k^{n-2}.$$

So we arrive to

$$|\,\mathcal{E}(b_k)\backslash\mathcal{E}(a_k)\cap B_{\sqrt{T^2-k^2}}\cap\mathbb{Z}^n\,|\lessapprox \frac{\pi^{n/4}}{\sqrt{\det(Q_\beta)\Gamma(\frac{n}{2}+1)}}k^{(n-2)/2}(T^2-k^2)^{n/4},$$

and in terms of our counting function we get the following estimate

$$N_T(a,b)\lessapprox \frac{\pi^{n/4}}{\sqrt{\det(Q_\beta)\Gamma(\frac{n}{2}+1)}}\sum_{k_0\le k\le T}k^{\frac{n-2}{2}}(T^2-k^2)^{n/4}. \tag{14}$$

The last step consists into getting asymptotical upper bounds for the sum

$$\sum_{k_0\le k\le T}k^{\frac{n-2}{2}}(T^2-k^2)^{\frac{n}{4}}.$$

The previous sum can be wrriten,

$$\sum_{1\le k\le T}k^{\frac{n-2}{2}}(T^2-k^2)^{\frac{n}{4}}=T^{\frac{n-2}{2}}(T^2)^{\frac{n}{4}}\sum_{1\le k\le T}\left(\frac{k}{T}\right)^{\frac{n-2}{2}}\left(1-\left(\frac{k}{T}\right)^2\right)^{\frac{n}{4}}.$$

Or,

$$\sum_{1\le k\le T}k^{\frac{n-2}{2}}(T^2-k^2)^{\frac{n}{4}}=T^n\left[\frac{1}{T}\sum_{1\le k\le T}\left(\frac{k}{T}\right)^{\frac{n-2}{2}}\left(1-\left(\frac{k}{T}\right)^2\right)^{\frac{n}{4}}\right].$$

By performing Riemmian summation as $T\to\infty$, we get

$$\lim_{T\to\infty}\frac{1}{T}\sum_{1\le k\le T}\left(\frac{k}{T}\right)^{\frac{n-2}{2}}\left(1-\left(\frac{k}{T}\right)^2\right)^{\frac{n}{4}}=\int_0^1 x^{\frac{n-2}{2}}(1-x^2)^{n/4}dx.$$

The resulting integral I_n is a definite integral which is bounded by one, thus we get the following estimate for the sum as $T\to\infty$

$$\sum_{1\le k\le T}k^{n-1}(T^2-k^2)^{\frac{n}{4}}\lessapprox T^n.$$

As $T\to\infty$, one has

$$\sum_{k_0\le k\le T}k^{n-1}(T^2-k^2)^{\frac{n}{4}}\approx\sum_{1\le k\le T}k^{n-1}(T^2-k^2)^{\frac{n}{4}}-O(1)$$

Thus,

$$\sum_{k_0 \leq k \leq T} k^{n-1}(T^2-k^2)^{\frac{n}{4}} \lessapprox T^n.$$

Hence, the estimate 14 gives us that for T large

$$N_T(a,b) \lessapprox CT^n$$

where C depends on a, b, n and β .

□

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S. EL MANOUNI
Department of Mathematics and Statistics, Faculty of Sciences
Imam Mohammad Ibn Saud Islamic University (IMISU)
P. O. Box 90950, Riyadh 11623, Saudi Arabia
e-mail: `samanouni@imamu.edu.sa`

Y. LAZAR
Department of Mathematics and Statistics, Faculty of Sciences
Imam Mohammad Ibn Saud Islamic University (IMISU)
P. O. Box 90950, Riyadh 11623, Saudi Arabia
e-mail: `ylazar77@gmail.com`